# *Several remarks on projective representations of finite groups*

By Nagayoshi IWAHORI and Hideya MATSUMOTO

Dedicated to Professor Masao SUGAWARA

## Introduction

We shall consider in this note some questions about the projective representations of a finite group and related problems. In § 1, we shall extend the Hochschild-Serre's exact sequence [3]

$$1 \to \mathrm{Hom}(\mathfrak{H}, \Omega) \to \mathrm{Hom}(\mathfrak{G}, \Omega) \to \mathrm{Hom}(\mathfrak{A}, \Omega) \to H^2(\mathfrak{H}, \Omega) \to H^2(\mathfrak{G}, \Omega)$$

associated to a central group extension of a group $\mathfrak{H}$ by an abelian group $\mathfrak{A}$, $1 \to \mathfrak{A} \to \mathfrak{G} \to \mathfrak{H} \to 1$, where $\mathfrak{G}, \mathfrak{H}$ and $\mathfrak{A}$ act on an abelian group $\Omega$ trivially, to get the following exact sequence:

$$\cdots \to \mathrm{Hom}(\mathfrak{A}, \Omega) \to H^2(\mathfrak{H}, \Omega) \to H^2(\mathfrak{G}, \Omega) \to H^2(\mathfrak{A}, \Omega) \times P(\mathfrak{G}, \mathfrak{A}; \Omega).$$

Here $P(\mathfrak{G}, \mathfrak{A}; \Omega)$ means the group of all pairings $\mathfrak{G} \times \mathfrak{A} \to \Omega$. Then we note that the last term can be replaced, preserving the exactness of the sequence, by simpler groups if $\Omega$ satisfies some conditions. For example, if $\Omega$ is infinitely divisible, then we can replace the last term by $P(\mathfrak{G}, \mathfrak{A}; \Omega)$. Also, if $\mathrm{Hom}(\mathfrak{H}, \Omega) = 1$, then the last term can be replaced by $H^2(\mathfrak{A}, \Omega)$. Then considering the subgroup $H^2(\mathfrak{G}, \Omega)^{\natural} = \{c \in H^2(\mathfrak{G}, \Omega);\ c|_{\mathfrak{A}} = 1\}$, we shall give another exact sequence:

$$\cdots \to H^2(\mathfrak{H}, \Omega) \to H^2(\mathfrak{G}, \Omega)^{\natural} \to P(\mathfrak{H}, \mathfrak{A}; \Omega) \to H^3(\mathfrak{H}, \Omega).$$

In § 2, using the results in § 1, we shall consider the question of the existence of a closed representation group of a given finite group $\mathfrak{H}$. More precisely, this question means the following. As was shown by Schur [4], for any finite group $\mathfrak{H}$, there exists a finite group $\mathfrak{G}$ and a surjective homomorphism $\varphi: \mathfrak{G} \to \mathfrak{H}$ with the following properties:

1) $(\mathfrak{G}, \varphi)$ is a central group extension of $\mathfrak{H}$, i.e. the kernel $\mathfrak{A}$ of $\varphi$ is a central subgroup of $\mathfrak{G}$.

2) For any projective representation $\rho$ of $\mathfrak{H}$, i.e. for any homomorphism

$$\rho: \mathfrak{H} \to PGL(n, C) = GL(n, C)/\mathfrak{z}, \quad (\mathfrak{z} \text{ is the center of } GL(n, C)),$$

there exists a linear representation $\tilde{\rho}$ of $\mathfrak{G}$, i.e. a homomorphism

$$\tilde{\rho}: \mathfrak{G} \to GL(n, C)$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{\tilde{\rho}} & GL(n, C) \\ \varphi \downarrow & & \downarrow \pi \\ \mathfrak{H} & \xrightarrow{\rho} & PGL(n, C) \end{array} \qquad (\pi \text{ is the natural projection}).$$

Such a central extension $(\mathfrak{G}, \varphi)$ is called by Schur a *sufficient central extension* (hinreichend ergänzte Gruppe) of $\mathfrak{H}$. (In the terminology of K. Yamazaki [7], $(\mathfrak{G}, \varphi)$ is of *surjective type.*) A sufficient central extension is called a *representation group* of $\mathfrak{H}$ (Darstellungsgruppe, Schur [4]) if the order of $\mathfrak{G}$ is the least among all sufficient central extensions of $\mathfrak{H}$. A group $\mathfrak{H}$ is called *closed* (abgeschlossen, Schur [4]) if $H^2(\mathfrak{H}, C^*)=1$, where $\mathfrak{H}$ acts on $C^*$ trivially. Now for each central group extension $1 \to \mathfrak{A} \to \mathfrak{G} \to \mathfrak{H} \to 1$, there is associated the *transgression map* $\tau : \mathrm{Hom}(\mathfrak{A}, C^*) \to H^2(\mathfrak{H}, C^*)$ appearing in the Hochschild-Serre's exact sequence. In this case, $\tau$ can be defined as follows (cf. Schur [1]): let $u : \mathfrak{H} \to \mathfrak{G}$ be any section of the extension, i.e. any map $\mathfrak{H} \to \mathfrak{G}$ such that $\varphi \circ u = \mathrm{id}$. Let $\chi \in \mathrm{Hom}(\mathfrak{A}, C^*)$ and $P, Q \in \mathfrak{H}$. Then the map $(P, Q) \to \chi(A_{P,Q})$, $A_{P,Q} = u(P) \cdot u(Q) \cdot u(PQ)^{-1} \in \mathfrak{A}$, is a 2-cocycle of $\mathfrak{H}$ and its cohomology class $\tau(\chi)$ is independent of the choice of the section $u$. Now, using the transgression map $\tau$, a criterion for the central group extension $(\mathfrak{G}, \varphi)$ to be sufficient or to be a representation group is obtained (Schur [4], see also K. Yamazaki [7]). Namely, $(\mathfrak{G}, \varphi)$ is sufficient if and only if $\tau$ is surjective, and $(\mathfrak{G}, \varphi)$ is a representation group of $\mathfrak{H}$ if and only if $\tau$ is bijective. We shall prove in §2 that if $\mathfrak{H}$ coincides with its commutator group, then any representation group of $\mathfrak{H}$ is closed. (In this case, any two representation groups of $\mathfrak{H}$ are isomorphic to each other, see Schur [5] or K. Yamazaki [7]). We may conjecture that for any finite group $\mathfrak{H}$ there exists at least one closed representation group of $\mathfrak{H}$. We shall give at the end of §2 few examples of groups $\mathfrak{H}$ for which $\mathfrak{H} \neq [\mathfrak{H}, \mathfrak{H}]$ and some representation group of $\mathfrak{H}$ is closed.

In §3, we shall construct an obstruction cocycle for the extension of a linear representation $\rho$ of a given group $\mathfrak{N}$ to a linear representation $\tilde{\rho}$ of a group $\mathfrak{G}$ which contains $\mathfrak{N}$ as a normal subgroup. If $\rho$ is extendable to $\mathfrak{G}$, $\rho$ is clearly self-conjugate, i.e. $\rho \sim \rho^g$ for any $g$ in $\mathfrak{G}$ where $\rho^g(n) = \rho(gng^{-1})$ for any $n$ in $\mathfrak{N}$. Thus we associate to each self-conjugate linear representation $\rho$ of $\mathfrak{N}$ an element $c(\rho)$ in $H^2(\mathfrak{H}, C^*)$ where $\mathfrak{H} = \mathfrak{G}/\mathfrak{N}$. Then it is proved that $\rho$ is extendable to $\mathfrak{G}$ if and only if $c(\rho)=1$. Using this obstruction, we shall prove that if $\mathfrak{G}$ is finite and if the orders of $\mathfrak{N}$ and $\mathfrak{H}$ are relatively prime, then any self-conjugate linear representation of $\mathfrak{N}$ is extendable to $\mathfrak{G}$.

In §4, we shall consider a particular case of §3 where $\mathfrak{G}$ is finite and $\mathfrak{H} = \mathfrak{G}/\mathfrak{N}$

is abelian. Then $\mathfrak{H}$ acts naturally on the set $R(\mathfrak{N})$ of all classes of equivalent irreducible linear representations of $\mathfrak{N}$ over C and $\hat{\mathfrak{H}}=\mathrm{Hom}(\mathfrak{H}, C^*)$ acts on the set $R(\mathfrak{G})$ of all classes of equivalent irreducible linear representations of $\mathfrak{G}$. It is shown that there exists a natural bijective map between the quotient sets $R(\mathfrak{N})/\mathfrak{H}$ and $R(\mathfrak{G})/\hat{\mathfrak{H}}$. This is also valid in case where $\mathfrak{G}$ is a compact group and $\mathfrak{N}$ is a closed normal subgroup with finite abelian quotient group. Using a property of this bijection, a result of R. Frucht [2] about the uniqueness of irreducible projective representation with the given factor set for the case where the group is finite abelian will be reproduced as a corollary.

In § 5 we shall give miscellaneous examples about the existence of a central simple algebra extensions of a finite group $\mathfrak{H}$ (see for the definition of an algebra extension of $\mathfrak{H}$, K. Yamazaki [7]).

*Acknowledgement.* In preparing this note, the authors got many advantages by the discussion with Mr. K. Yamazaki, to whom we express our thanks here.

**§ 1.** Let $1\rightarrow\mathfrak{A}\rightarrow\mathfrak{G}\xrightarrow{\varphi}\mathfrak{H}\rightarrow 1$ be a group extension of a group $\mathfrak{H}$ such that the kernel $\mathfrak{A}$ of the homomorphism $\varphi$ is contained in the center of $\mathfrak{G}$. Let $\Omega$ be any abelian group (written multiplicatively) and we consider the cohomology groups of $\mathfrak{G}, \mathfrak{A}, \mathfrak{H}$ under the trivial action on $\Omega$. Then there exists an exact sequence of Hochschild-Serre [3]:

$$1\rightarrow H^1(\mathfrak{H}, \Omega)\xrightarrow{i_1} H^1(\mathfrak{G}, \Omega)\xrightarrow{r_1} H^1(\mathfrak{A}, \Omega)\xrightarrow{\tau} H^2(\mathfrak{H}, \Omega)\xrightarrow{i_2} H^2(\mathfrak{G}, \Omega),$$

where $i_1, i_2$ are the inflation maps and $r_1$ is the restriction map and the transgression map $\tau$ is given as follows: let $u:\mathfrak{H}\rightarrow\mathfrak{G}$ be any section of the group extension $\varphi:\mathfrak{G}\rightarrow\mathfrak{H}$, i.e. $\varphi\circ u=$identity, and $A:\mathfrak{H}\times\mathfrak{H}\rightarrow\mathfrak{A}$ the factor set of the extension associated to $u$, i.e.

$$A(P, Q)=u(P)u(Q)u(PQ)^{-1} \quad \text{for } P, Q \text{ in } \mathfrak{H}.$$

Let $\chi\in H^1(\mathfrak{A}, \Omega)=\mathrm{Hom}(\mathfrak{A}, \Omega)$. Then $\chi\circ A$ is a $\Omega$-valued 2-cocycle on H and the cohomology class $\tau(\chi)$ of $\chi\circ A$ is independent of the choice of the section $u$. Now we denote by $P(\mathfrak{G}, \mathfrak{A}; \Omega)$ the abelian group consisting of all pairings $G\times A\rightarrow\Omega$. (Note that a map $f:\mathfrak{G}\times\mathfrak{A}\rightarrow\Omega$ is called a *pairing* if

$$f(XY, A)=f(X, A)f(Y, A),$$
$$f(X, AB)=f(X, A)f(X, B)$$

for any $X, Y$ in $\mathfrak{G}$, and for any $A, B$ in $\mathfrak{A}$. The product of $f_1, f_2$ in $P(\mathfrak{G}, \mathfrak{A}; \Omega)$ is defined by $(f_1 f_2)(X, A)=f_1(X, A)f_2(X, A)$ ($X$ in $\mathfrak{G}$, $A$ in $\mathfrak{A}$)).

Let $\theta_1: H^2(\mathfrak{G}, \Omega)\rightarrow H^2(\mathfrak{A}, \Omega)$ be the restriction map. We now define a map

$\theta_2: \mathrm{H}^2(\mathfrak{G}, \Omega) \to \mathrm{P}(\mathfrak{G}, \mathfrak{A}; \Omega)$ as follows: let $f$ be any $\Omega$-valued 2-cocycle of $\mathfrak{G}$, then the map $\theta_2 f: \mathfrak{G} \times \mathfrak{A} \to \Omega$ defined by

$$\theta_2 f: (X, A) \to \langle X, A\rangle_f = f(X, A) f(A, X)^{-1} \quad (X \text{ in } \mathfrak{G},\ A \text{ in } \mathfrak{A})$$

is a pairing of $\mathfrak{G} \times \mathfrak{A}$ into $\Omega$. In fact, since $f$ is a 2-cocycle, we have $f(X, YZ) \cdot f(Y, Z) = f(X, Y) f(XY, Z)$ for any $X, Y, Z$ in $\mathfrak{G}$. Thus we get

$$\begin{aligned}
\langle XY, A\rangle_f &= f(XY, A) f(A, XY)^{-1} \\
&= f(X, Y)^{-1} f(X, YA) f(Y, A)(f(X, Y)^{-1} f(A, X) f(AX, Y))^{-1} \\
&= f(X, YA) f(Y, A) f(A, X)^{-1} f(AX, Y)^{-1} \\
&= f(X, AY) f(Y, A) f(A, X)^{-1} f(XA, Y)^{-1} \quad (\text{since } AY = YA \text{ and } XA = AX) \\
&= f(A, Y)^{-1} f(X, A) f(XA, Y) \cdot f(Y, A) \cdot f(A, X)^{-1} f(XA, Y)^{-1} \\
&= \langle X, A\rangle_f \cdot \langle Y, A\rangle_f,
\end{aligned}$$

also similarly we have $\langle X, AB\rangle_f = \langle X, A\rangle_f \cdot \langle X, B\rangle_f$ ($X$ in $\mathfrak{G}$, $A, B$ in $\mathfrak{A}$).
It is easy to check that two cohomologous cocycles give the same pairing. Thus there is induced a homorphism $\theta_2: \mathrm{H}^2(\mathfrak{G}, \Omega) \to \mathrm{P}(\mathfrak{G}, \mathfrak{A}; \Omega)$. Now we define the homomorphism $\theta: \mathrm{H}^2(\mathfrak{G}, \Omega) \to \mathrm{H}^2(\mathfrak{A}, \Omega) \times \mathrm{P}(\mathfrak{G}, \mathfrak{A}, \Omega)$ by $\theta = \theta_1 \times \theta_2$.

PROPOSITION 1.1. *The sequence* $\mathrm{H}^2(\mathfrak{H}, \Omega) \xrightarrow{i_2} \mathrm{H}^2(\mathfrak{G}, \Omega) \xrightarrow{\theta} \mathrm{H}^2(\mathfrak{A}, \Omega) \times \mathrm{P}(\mathfrak{G}, \mathfrak{A}; \Omega)$ *is exact.*

PROOF. Clearly we have $\mathrm{Im}(i_2) \subset \mathrm{Ker}(\theta)$. Now let us show $\mathrm{Im}(i_2) \supset \mathrm{Ker}(\theta)$. Let $f$ be any $\Omega$-valued 2-cocycle of $\mathfrak{G}$ such that the cohomology class $[f]$ of $f$ is in $\mathrm{Ker}(\theta)$, i.e. the restriction $\theta_1 f$ of $f$ on $\mathfrak{A}$ is cohomologous to 1 and $\theta_2 f = 1$. Then there is a map $g_0: \mathfrak{A} \to \Omega$ such that $f(A, B) = g_0(A) g_0(AB)^{-1} g_0(B)$ ($A, B$ in $\mathfrak{A}$). We may assume that $f$ is normalized, i.e. $f(X, 1) = f(1, X) = 1$ for any $X$ in $\mathfrak{G}$. Let $u: \mathfrak{H} \to \mathfrak{G}$ be any section such that $u(1) = 1$, and we define a map $g: \mathfrak{G} \to \Omega$ by $g(Au(P)) = g_0(A) f(A, u(P))^{-1}$ for $A$ in $\mathfrak{A}$, $P$ in $\mathfrak{H}$. (Note that every $X$ in $\mathfrak{G}$ can be uniquely expressed as $X = Au(P)$, $A$ in $\mathfrak{A}$, $P$ in $\mathfrak{H}$). Obviously $g|_{\mathfrak{A}} = g_0$. Now let us consider the coboundary $\delta g$ of $g$. We have

$$\begin{aligned}
\delta g(A, u(P)) &= g(A) g(Au(P))^{-1} g(u(P)) \\
&= g_0(A) g_0(A)^{-1} f(A, u(P)) \\
&= f(A, u(P)), \\
\delta g(u(P), A) &= g(u(P)) g(u(P)A)^{-1} g(A) \\
&= g_0(A)^{-1} f(A, u(P)) g_0(A) \\
&= f(A, u(P)) \\
&= f(u(P), A) \qquad (\text{since } \theta_2 f = 1).
\end{aligned}$$

Hence $f_1 = f(\delta g)^{-1}$ satisfies $f_1(A, u(P)) = f_1(u(P), A) = 1$ for any $A$ in $\mathfrak{A}$, $P$ in $\mathfrak{H}$. Also, clearly we have $f_1(A, B) = 1$ for any $A, B$ in $\mathfrak{A}$. Thus we have

$$f_1(A, Bu(P)) = f_1(B, u(P))^{-1} f_1(A, B) f_1(AB, u(P)) = 1$$
$$f_1(Au(P), B) = f_1(A, u(P))^{-1} f(A, u(P)B) f(u(P), B) = 1$$

for any $A, B$ in $\mathfrak{A}$, $P$ in $\mathfrak{H}$. Hence, for any $A, B$ in $\mathfrak{A}$, $P, Q$ in $\mathfrak{H}$, we get

$$\begin{aligned} f_1(Au(P), Bu(Q)) &= f_1(A, u(P))^{-1} f_1(A, Bu(P)u(Q)) f_1(u(P), Bu(Q)) \\ &= f_1(u(P), Bu(Q)) \\ &= f(u(Q), B)^{-1} f(u(P), u(Q)) f(u(P)u(Q), B) \\ &= f(u(P), u(Q)). \end{aligned}$$

Thus the cohomology class $[f]$ of $f$ is in $\mathrm{Im}(i_2)$, Q.E.D.

Now we shall consider some particular case of Prop. 1.1 where the abelian group $\Omega$ is infinitely divisible, i.e. for any positive integer $m$, the map $\alpha \to \alpha^m$ is a surjective map from $\Omega$ onto $\Omega$. In this case we have the following

LEMMA 1.2. *Let $\mathfrak{A}$ be a finitely generated abelian group and $\Omega$ be an infinitely divisible abelian group. If a $\Omega$-valued 2-cocyle $f$ of $\mathfrak{A}$ satisfies $f(A, B) = f(B, A)$ for any $A, B$ in $\mathfrak{A}$, then $f$ is a coboundary.*

PROOF. see K. Yamazaki [7, § 2.3.]

REMARK. By using Zorn's Lemma, this lemma is also valid for any abelian group $\mathfrak{A}$.

Using this lemma, it is immediate that in Prop. 1.1, if $\Omega$ is infinitely divisible and $\mathfrak{A}$ is finitely generated, then for $c$ in $\mathrm{H}^2(\mathfrak{G}, \Omega)$, $\theta_2(c) = 1$ implies $\theta_1(c) = 1$. Hence we get the

PROPOSITION 1.3. *If $\Omega$ is infinitely divisible and $\mathfrak{A}$ is finitely generated, then the sequence*

$$\mathrm{H}^2(\mathfrak{H}, \Omega) \xrightarrow{i_2} \mathrm{H}^2(\mathfrak{G}, \Omega) \xrightarrow{\theta_2} \mathrm{P}(\mathfrak{G}, \mathfrak{A}; \Omega)$$

*is exact.*

Next we consider the case where $\mathrm{H}^1(\mathfrak{H}, \Omega) = 1$ in Proposition 1.1. Assume that $\theta_1([f]) = 1$ for given $[f]$ in $\mathrm{H}^2(\mathfrak{G}, \Omega)$, then $\theta_2 f(A, B) = 1$ for all $A, B$ in $\mathfrak{A}$. Thus $\theta_2 f$ induces a pairing $(\mathfrak{G}/\mathfrak{A}) \times \mathfrak{A} \to \Omega$. Now since $\mathrm{Hom}(\mathfrak{H}, \Omega) = 1$ by our assumption, we have $\theta_2 f(X, A) = 1$ for all $X$ in $\mathfrak{G}$, $A$ in $\mathfrak{A}$. Thus $\theta_1([f]) = 1$ implies $\theta_2([f]) = 1$ and we have

PROPOSITION 1.4. *If $\mathrm{H}^1(\mathfrak{H}, \Omega) = 1$, then the sequence*

$$\mathrm{H}^2(\mathfrak{H}, \Omega) \xrightarrow{i_2} \mathrm{H}^2(\mathfrak{G}, \Omega) \xrightarrow{\theta_1} \mathrm{H}^2(\mathfrak{A}, \Omega)$$

*is exact.*

Now denote the kernel of $\theta_1 : \mathrm{H}^2(\mathfrak{G}, \Omega) \to \mathrm{H}^2(\mathfrak{A}, \Omega)$ by $\mathrm{H}^2(\mathfrak{G}, \Omega)^s$. Then $\theta_2$ induces a homomorphism $\theta : \mathrm{H}^2(\mathfrak{G}, \Omega)^s \to \mathrm{P}(\mathfrak{H}, \mathfrak{A}; \Omega)$ as we have seen above. Also it is obvious that the sequence

$$H^2(\mathfrak{H}, \Omega) \xrightarrow{i_2} H^2(\mathfrak{G}, \Omega)^{\varepsilon} \xrightarrow{\theta} P(\mathfrak{H}, \mathfrak{A}; \Omega)$$

is exact. We define a homomorphism $\rho: P(\mathfrak{H}, \mathfrak{A}; \Omega) \to H^3(\mathfrak{H}, \Omega)$ as follows: let $u: \mathfrak{H} \to \mathfrak{G}$ be a map such that $u(1)=1$, $\varphi \circ u = \mathrm{id}.$ and $A: \mathfrak{H} \times \mathfrak{H} \to \mathfrak{A}$ be the associated factor set: $u(P)u(Q) = A(P, Q)u(PQ)$, for $P, Q$ in $\mathfrak{H}$. For $f$ in $P(\mathfrak{H}, \mathfrak{A}; \Omega)$, put

$$(\rho f)(P, Q, R) = f(P, A(Q, R)) \qquad \text{for } P, Q, R \text{ in } \mathfrak{H}.$$

It is easily checked that $\rho f$ is a 3-cocycle and its cohomology class is independent of the choice of the section $u$. Thus $\rho$ induces a homomorphism $\rho: P(\mathfrak{H}, \mathfrak{A}; \Omega) \to H^3(\mathfrak{H}, \Omega)$.

THEOREM 1.5. *Let* $1 \to \mathfrak{A} \to \mathfrak{G} \xrightarrow{\varphi} \mathfrak{H} \to 1$ *be a central extension of a group* $\mathfrak{H}$ *by an abelian group* $\mathfrak{A}$. *Let* $\Omega$ *be an arbitrary abelian group. Then the sequence*

$$1 \to H^1(\mathfrak{H}, \Omega) \xrightarrow{i_1} H^1(\mathfrak{G}, \Omega) \xrightarrow{r_1} H^1(\mathfrak{A}, \Omega)$$
$$\xrightarrow{\tau} H^2(\mathfrak{H}, \Omega) \xrightarrow{i_2} H^2(\mathfrak{G}, \Omega)^{\varepsilon} \xrightarrow{\theta} P(\mathfrak{H}, \mathfrak{A}; \Omega) \xrightarrow{\rho} H^3(\mathfrak{H}, \Omega)$$

*is exact.*

PROOF. We have only to show that the sequence $H^2(\mathfrak{G}, \Omega)^{\varepsilon} \xrightarrow{\theta} P(\mathfrak{H}, \mathfrak{A}, \Omega) \xrightarrow{\rho} H^3(\mathfrak{H}, \Omega)$ is exact. Let $[f]$ be an element in $H^2(\mathfrak{G}, \Omega)^{\varepsilon}$. Then there is a map $g_0: \mathfrak{A} \to \Omega$ such that $f(A, B) = g_0(A)g_0(AB)^{-1}g_0(B)$ for all $A, B$ in $\mathfrak{A}$. We may assume $f$ is normalized, i.e. $f(X, 1) = f(1, X) = 1$ for any $X$ in $\mathfrak{G}$. Define an extension of $g_0$ to a map $g: \mathfrak{G} \to \Omega$ by $g(Au(p)) = g_0(A)f(A, u(P))^{-1}$ for $A$ in $\mathfrak{A}$, $P$ in $\mathfrak{H}$. Then as in Proposition 1.1, $f_1 = f(\delta g)^{-1}$ satisfies

$$(1) \qquad \begin{cases} f_1(A, X) = 1 & \text{for } A \text{ in } \mathfrak{A},\ X \text{ in } \mathfrak{G} \\ f_1(Au(P), B) = \psi(P, B) & \text{for } A, B \text{ in } \mathfrak{A},\ P \text{ in } \mathfrak{H} \\ f_1(Au(P), Bu(Q)) = \psi(P, B) f_1(u(P), u(Q)) & \text{for } A, B \text{ in } \mathfrak{A},\ P,\ Q \text{ in } \mathfrak{H}, \end{cases}$$

where $\psi = \theta f \in P(\mathfrak{H}, \mathfrak{A}; \Omega)$. The cocycle condition for $f_1$:

$$f_1(u(P), u(Q)) f_1(u(P)u(Q), u(R)) = f_1(u(P), u(Q)u(R)) f_1(u(Q), u(R))$$

implies by (1)

$$(2) \qquad \psi(P, A(Q, R)) f_1(u(P), u(QR)) f_1(u(Q), u(R)) = f_1(u(P), u(Q)) f_1(u(PQ), u(R)).$$

Thus, defining a 2-cochain $\tilde{f}_1: \mathfrak{H} \times \mathfrak{H} \to \Omega$ by $\tilde{f}_1(P, Q) = \tilde{f}_1(u(P), u(Q))$, we obtain $\rho(\psi) = (\delta \tilde{f}_1)^{-1}$. Thus we get $\mathrm{Im}(\theta) \subset \mathrm{Ker}(\rho)$.

Conversely let $\psi \in P(\mathfrak{H}, \mathfrak{A}; \Omega)$ be in $\mathrm{Ker}(\rho)$. Then there is a map $\tilde{f}_1: \mathfrak{H} \times \mathfrak{H} \to \Omega$ such that $\rho(\psi) = (\delta \tilde{f}_1)^{-1}$. Define $f_1: \mathfrak{G} \times \mathfrak{G} \to \Omega$ by $f_1(Au(P), Bu(Q)) = \psi(P, B) \tilde{f}_1(P, Q)$ for $A, B$ in $\mathfrak{A}$ and $P, Q$ in $\mathfrak{H}$. Then it is easy to check that $f_1$ is a 2-cocycle and

$[f_1]\in H^2(\mathfrak{G}, \Omega)^\varepsilon$. Moreover we have $\theta f_1=\varphi$. Thus $\mathrm{Ker}(\rho)\subset \mathrm{Im}(\theta)$, Q.E.D.

**§2.** As an application of §1, we shall prove the

THEOREM 2.1. *Let $\mathfrak{H}$ be a finite group and $\mathfrak{G}$ a representation group of $\mathfrak{H}$. Then $H^2(\mathfrak{G}, C^*)$ is isomorphic to a subgroup of $P(\mathfrak{H}, \mathfrak{M}; C^*)$, where $\mathfrak{M}=H^2(\mathfrak{H}, C^*)$.*

PROOF. Since $\mathfrak{G}$ is a representation group of $\mathfrak{H}$, $\mathfrak{G}$ is a central extension of $\mathfrak{H}$ by the abelian group $\mathfrak{M}$, $1\to\mathfrak{M}\to\mathfrak{G}\to\mathfrak{H}\to 1$ (cf. Schur [4] or K. Yamazaki [7]). Thus we may utilize the exact sequence of §1 (Proposition 1.3)

$$H^1(\mathfrak{M}, C^*)\xrightarrow{\tau} H^2(\mathfrak{H}, C^*)\xrightarrow{i_2} H^2(\mathfrak{G}, C^*)\xrightarrow{\theta_2} P(\mathfrak{G}, \mathfrak{M}; C^*).$$

The map $\tau$ is bijective since $\mathfrak{G}$ is a representation group of $\mathfrak{H}$ (cf. K. Yamazaki [7]). Hence $i_2$ is a zero-map and $\theta_2$ is an injective map. Now since $\tau$ is injective, $[G, G]\supset M$ (cf. Schur [4]). Thus we have the natural isomorphism: $P(\mathfrak{G}, \mathfrak{M}; C^*)\cong P(\mathfrak{G}/\mathfrak{M}, \mathfrak{M}; C^*)=P(\mathfrak{H}, \mathfrak{M}; C^*)$, Q.E.D.

COR. 2.2. *Let $\mathfrak{H}$ be a finite group and $\mathfrak{M}=H^2(\mathfrak{H}, C^*)$. If $P(\mathfrak{H}, \mathfrak{M}; C^*)=1$, then any representation group $\mathfrak{G}$ satisfies $H^2(\mathfrak{G}, C^*)=1$.*

REMARK. Let $\mathfrak{H}/[\mathfrak{H}, \mathfrak{H}]=Z_{e_1}\times\cdots\times Z_{e_r}$, $\mathfrak{M}=Z_{f_1}\times\cdots\times Z_{f_s}$ be direct product decompositions of $\mathfrak{H}/[\mathfrak{H}, \mathfrak{H}]$, $\mathfrak{M}$ into cyclic factors respectively. Then the order of $P(\mathfrak{H}, \mathfrak{M}; C^*)$ is equal to $\prod_{i=1}^{r}\prod_{j=1}^{s}(e_i, f_j)$, where $(e_i, f_j)$ denotes the greatest common divisor of $e_i, f_j$. This is the upper bound given by Schur [5] for the number of non-isomorphic representation groups of $\mathfrak{H}$. Thus, if $P(\mathfrak{H}, \mathfrak{M}; C^*)=1$, $\mathfrak{H}$ has only one representation group up to an isomorphism.

Let us exhibit some examples of a finite group $\mathfrak{H}$ for which $P(\mathfrak{H}, \mathfrak{M}; C^*)\neq 1$ but still there exists a representation group $\mathfrak{G}$ such that $H^2(\mathfrak{G}, C^*)=1$.

EXAMPLE 1. $\mathfrak{H}=\mathfrak{D}_n=$the dihedral group of order $2n$. $\mathfrak{H}$ is generated by $P$, $Q$ together with the fundamental relations $P^2=Q^n=1$, $PQP^{-1}=Q^{-1}$. Now consider a group $\mathfrak{G}$ generated by $\bar{P}$, $\bar{Q}$ together with the fundamental relations $\bar{P}^2=\bar{Q}^n$, $\bar{Q}^{2n}=1$, $\bar{P}\bar{Q}\bar{P}^{-1}=\bar{Q}^{-1}$. $\mathfrak{G}$ is the generalized quaternion group of order $4n$. Then $\bar{P}\to P$, $\bar{Q}\to Q$ defines a central extension $\mathfrak{G}$ of $\mathfrak{H}$. It is easy to see that, if $n$ is even, $\mathfrak{G}$ is a representation group of $\mathfrak{H}$ and $H^2(\mathfrak{G}, C^*)=1$. Here $P(\mathfrak{H}, \mathfrak{M}; C^*)=Z_2\times Z_2$.

EXAMPLE 2. $\mathfrak{H}=\mathfrak{S}_n=$the symmetric group of degree $n$. Then one of the two representation groups given by Schur [6] has trivial 2-cohomology group. Here $P(\mathfrak{H}, \mathfrak{M}; C^*)=Z_2$.

**§3.** In this section, let $\mathfrak{N}$ be a normal subgroup of a group $\mathfrak{G}$ and we denote by $\mathfrak{H}$ the factor group $\mathfrak{G}/\mathfrak{N}$ and by $\pi$ the canonical homomorphism $\mathfrak{G}\xrightarrow{\pi}\mathfrak{H}$. Let $\rho:\mathfrak{N}\to GL(m, C)$ be an irreducible linear representation of degree $m$ of $\mathfrak{N}$. Then for any $g$ in $\mathfrak{G}$, a representation $\rho^g:\mathfrak{N}\to GL(m, C)$ is defined by $\rho^g(x)=\rho(gxg^{-1})$

($x$ in $\mathfrak{H}$). The rep. $\rho$ is called to be self-conjugate if for any $g$ in $\mathfrak{G}$, $\rho^g$ is equivalent to $\rho$. Now let us consider the question when an irreducible linear rep. $\rho:\mathfrak{N}\to GL(m, \mathrm{C})$ can be extended to a linear rep. $\tilde{\rho}:\mathfrak{G}\to GL(m, \mathrm{C})$ of $\mathfrak{G}$. Clearly, if $\rho$ is extendable to a rep. of $\mathfrak{G}$, then $\rho$ is self-conjugate. Thus we shall construct an obstruction cohomology class $c_\rho$ in $\mathrm{H}^2(\mathfrak{H}, \mathrm{C}^*)$ for any irreducible, self-conjugate linear representation $\rho$ of $\mathfrak{N}$. Let $\rho$ be such a rep. of $\mathfrak{N}$. Then for each $g$ in $\mathfrak{G}$, there exists an element $X_g$ in $GL(m, \mathrm{C})$ such that $X_g\rho(x)X_g^{-1}=\rho^g(x)$ ($x$ in $\mathfrak{N}$). Let us choose $X_g$ as follows: let $\mathfrak{G}=\bigcup_i \mathfrak{N}g_i=\bigcup_i g_i\mathfrak{N}$ be a coset decomposition of $\mathfrak{G}$ w.r.t. $\mathfrak{N}$ ($g_1=1$). Choose for each $g_i$ the matrix $X_{g_i}$ such that $X_{g_i}\cdot\rho(x)\cdot X_{g_i}^{-1}=\rho^{g_i}(x)$ for any $x$ in $\mathfrak{N}$. ($X_{g_1}=1$). Then for $g=ag_i$ in $\mathfrak{N}g_i$, put $X_g=\rho(a)X_{g_i}$ ($a$ in $\mathfrak{N}$). Then it is easy to check that the map $g\to X_g$ constructed above satisfies the following (1)~(4).

(1) $\qquad \rho^g(x)=X_g\rho(x)X_g^{-1}$ (for $a$ in $\mathfrak{N}$, $g$ in $\mathfrak{G}$)

(2) $\qquad X_a=\rho(a)$ (for $a$ in $\mathfrak{N}$)

(3) $\qquad X_{ag}=\rho(a)\cdot X_g$ (for $a$ in $\mathfrak{N}$, $g$ in $\mathfrak{G}$)

(4) $\qquad X_{ga}=X_g\cdot\rho(a)$ (for $a$ in $\mathfrak{N}$, $g$ in $\mathfrak{G}$)

We shall call a map $g\to X_g$ from $\mathfrak{G}$ into $GL(m, \mathrm{C})$ a *section* for the homomorphism $\rho:\mathfrak{N}\to GL(m, \mathrm{C})$ if it satisfies (1)~(4) above. Now let $g\to X_g$ and $g\to Y_g$ be two sections for $\rho$. Then $\rho^g(x)=X_g\cdot\rho(x)\cdot X_g^{-1}=Y_g\cdot\rho(x)Y_g^{-1}$ ($x$ in $\mathfrak{N}$) and the irreducibility of $\rho$ imply the existence of a map $g\to\zeta_g$ from $\mathfrak{G}$ into $\mathrm{C}^*$ such that $\zeta_gX_g=Y_g$ for any $g$ in $\mathfrak{G}$. Clearly $\zeta_a=1$ ($a$ in $\mathfrak{N}$) and $\zeta_{ag}=\zeta_g$ ($a$ in $\mathfrak{N}$, $g$ in $\mathfrak{G}$).

Now let $g\to X_g$ be a section for an irreducible linear rep. $\rho:\mathfrak{N}\to GL(m, \mathrm{C})$. Then for any $x, y$ in $\mathfrak{G}$ there exists a scalar $c_{x,y}$ in $\mathrm{C}^*$ such that $X_xX_y=c_{x,y}X_{xy}$. In fact $X_{xy}\rho(a)X_{xy}^{-1}=\rho(xyay^{-1}x^{-1})=X_x\cdot\rho(yay^{-1})X_x^{-1}=X_xX_y\rho(a)X_y^{-1}X_x^{-1}$ implies that $X_{xy}^{-1}X_xX_y$ commutes with any $\rho(a)$, $a$ in $\mathfrak{N}$. Thus by Schur's lemma, the existence of $c_{x,y}$ is established. It is easy to see that the map $(x, y)\to c_{x,y}$ from $\mathfrak{G}\times\mathfrak{G}$ into $\mathrm{C}^*$ is a $\mathrm{C}^*$-valued 2-cocycle of $\mathfrak{G}$. Also we have

(5) $\qquad c_{ax,by}=c_{x,y}$ ($a, b$ in $\mathfrak{N}$, $x, y$ in $\mathfrak{G}$).

In fact, $X_{ax}X_{by}=c_{ax,by}X_{axby}$ and (3) imply that

$$\rho(a)X_x\cdot\rho(b)X_y=c_{ax,by}X_{a\cdot xbx^{-1}\cdot xy}=c_{ax,by}\cdot\rho(axbx^{-1})X_{xy}\,,$$

i.e.
$$\rho(a)X_x\cdot\rho(b)X_x^{-1}\cdot X_xX_y=c_{ax,by}\cdot\rho(a\cdot xbx^{-1})X_{xy}\,,$$

hence we have

$$\rho(a)\cdot\rho(xbx^{-1})c_{x,y}X_{xy}=c_{ax,by}\cdot\rho(axbx^{-1})X_{xy}\,,$$

which implies (5) immediately. Thus the cocycle $(x, y)\to c_{x,y}$ of $\mathfrak{G}$ induces a 2-cocycle $c^*:(u, v)\to c^*_{u,v}$ of $\mathfrak{H}$ such that $c^*_{\pi(x),\pi(y)}=c_{x,y}$ ($x, y\in\mathfrak{G}$). Now we shall verify that the cohomology class $[c^*]$ of $c^*$ is independent of the section $g\to X_g$ for $\rho$.

In fact, if $g \to Y_g$ is another section for $\rho$, then, as we noticed above, there is a map $g \to \zeta_g$ from $\mathfrak{G}$ into $\mathbf{C}^*$ such that

$$Y_g = \zeta_g \cdot X_g \quad (g \text{ in } \mathfrak{G}),\ \zeta_a = 1 \ (a \text{ in } \mathfrak{N}),$$
$$\zeta_{ag} = \zeta_g \quad (a \text{ in } \mathfrak{N},\ g \text{ in } \mathfrak{G}).$$

Now, let $(x, y) \to r_{x,y}$ be the 2-cocycle of $\mathfrak{G}$ defined by the section $g \to Y_g$. Then it is easy to see $r_{x,y} = \frac{\zeta_x \zeta_y}{\zeta_{xy}} c_{x,y}$ (for any $x, y$ in $\mathfrak{G}$). Since $\zeta_{ag} = \zeta_g$ ($a$ in $\mathfrak{N}$, $g$ in $\mathfrak{G}$), $g \to \zeta_g$ induces a map $u \to \zeta_u^*$ from $\mathfrak{H}$ into $\mathbf{C}^*$ such that $\zeta_{\pi(x)}^* = \zeta_x$ ($x$ in $\mathfrak{G}$). Then we have $r_{u,v}^* = \frac{\zeta_u^* \zeta_v^*}{\zeta_{uv}^*} c_{u,v}^*$ (for any $u, v$ in $\mathfrak{H}$), i.e. $[c^*] = [r^*]$. This cohomology class will be denoted by $c_\rho$ and we shall call it the *obstruction class* for $\rho$.

PROPOSITION 3.1. *An irreducible linear rep.* $\rho : \mathfrak{N} \to GL(m, \mathbf{C})$ *can be extended to* $\mathfrak{G}$ *if and only if* $c_\rho = 1$.

PROOF. If $\rho$ can be extended to a linear rep. $\sigma : \mathfrak{G} \to GL(m, \mathbf{C})$, then as a section for $\rho$, we can take $X_g = \sigma(g)$ ($g$ in $\mathfrak{G}$). It is obvious then $c_\rho = 1$. Conversely, suppose $c_\rho = 1$. Then, using above notations $g \to X_\rho$, $c_{x,y}$, $c_{u,v}^*$, for $\rho$, there exists a map $u \to \zeta_u^*$ from $\mathfrak{H}$ into $\mathbf{C}^*$ such that $c_{u,v}^* = \frac{\zeta_u^* \zeta_v^*}{\zeta_{uv}^*}$. Define a map $g \to \zeta_g$ from $\mathfrak{G}$ into $\mathbf{C}^*$ by $\zeta_g = \zeta_{\pi(g)}^*$. Since $c_{1,v}^* = c_{v,1}^* = 1$, we have $\zeta_1^* = 1$, $\zeta_a = 1$ ($a$ in $\mathfrak{N}$). Moreover, it is easy to see that $\zeta_{ag} = \zeta_g$ ($a$ in $\mathfrak{N}$, $g$ in $\mathfrak{G}$). Then the map $g \to Y_g = \frac{1}{\zeta_g} X_g$ is easily seen to be a section for $\rho$. Moreover, we have $Y_x Y_y = Y_{xy}$ for any $x, y$ in $\mathfrak{G}$. Thus the map $g \to Y_g$ is a homomorphism from $\mathfrak{G}$ into $GL(m, \mathbf{C})$ extending $\rho$.

COR. 3.2. *Let* $\mathfrak{N}$ *be a normal subgroup of a finite* $\mathfrak{G}$ *such that the order of* $\mathfrak{N}$ *is relatively prime to the index of* $\mathfrak{N}$ *in* $\mathfrak{G}$. *Then any self-conjugate irreducible linear representation of* $\mathfrak{N}$ *can be extended to a linear rep. of* $\mathfrak{G}$.

PROOF. Let $\rho$ be a self-conjugate irreducible linear rep. of $\mathfrak{N}$. Let $e$ be the order of the obstruction class $c_\rho$ in $H^2(\mathfrak{H}, \mathbf{C}^*)$, $\mathfrak{H} = \mathfrak{G}/\mathfrak{N}$. Then, $e$ is a divisor of the order of $\mathfrak{H}$ (cf. Schur [4]). Also, $e$ is a divisor of the degree of $\rho$ (cf. Schur [4]). Hence $e$ is a divisor of the order of $\mathfrak{N}$. Thus, we have $e = 1$ and $c_\rho = 1$, q.e.d.

REMARK. Let $\mathfrak{N}$ be a normal subgroup of a group $\mathfrak{G}$ and $\rho : \mathfrak{N} \to GL(m, \mathbf{C})$ be a self-conjugate irreducible linear representation of $\mathfrak{N}$. If the obstruction class $c_\rho$ is of order $e$, then we see by a similar argument as in the proof of Proposition 3.1. that the representation $\rho \otimes \cdots \otimes \rho$ ($e$-times) can be extended to a representation of $\mathfrak{G}$. For example, let $\mathfrak{G}/\mathfrak{N} \cong \mathfrak{S}_n$. Here $H^2(\mathfrak{S}_n, \mathbf{C}^*) = 1$ or $\mathbf{Z}_2$ according to $n \leqq 3$ or $n \geqq 4$ (cf. Schur [6]). Thus, for any self-conjugate irreducible representation $\rho$ of $\mathfrak{N}$, $\rho \otimes \rho$ can be extended to a representation of $\mathfrak{G}$.

§4. Let $\mathfrak{G}$ be a finite group and $\mathfrak{N}$ a normal subgroup of $\mathfrak{G}$ with *abelian* quotient group $\mathfrak{H}=\mathfrak{G}/\mathfrak{N}$. We denote for a group $\mathfrak{G}$ by $R(\mathfrak{G})$ the set of all irreducible representation classes of $\mathfrak{G}$. We can also regard $R(\mathfrak{G})$ as the set of all classes of isomorphic simple $C[\mathfrak{G}]$-modules, where $C[\mathfrak{G}]$ is the group algebra of $\mathfrak{G}$ over $C$. Then $\mathfrak{H}$ acts on the set $R(\mathfrak{N})$ as follows: let $\mathfrak{m}$ be a simple $C[\mathfrak{N}]$-module and $g$ in $\mathfrak{G}$. Then a simple $C[\mathfrak{N}]$-module $\mathfrak{m}^g$ is defined as follows: as a module $\mathfrak{m}^g=\mathfrak{m}$. For $x$ in $\mathfrak{N}$, a new action of $x$ on $\mathfrak{m}$ is defined by $x\circ m=gxg^{-1}m$. Them $\mathfrak{m}^g$ is also a simple $C[\mathfrak{N}]$-module. Clearly $\mathfrak{m}_1\cong\mathfrak{m}_2$ implies $\mathfrak{m}_1^g\cong\mathfrak{m}_2^g$. Thus $\mathfrak{G}$ acts on $R(\mathfrak{N})$. However it is obvious that if $g_1, g_2$ in $\mathfrak{G}$, $g_1\equiv g_2$ (mod. $\mathfrak{N}$), then for any simple $C[\mathfrak{N}]$-module $\mathfrak{m}$, we have $\mathfrak{m}^{g_1}\cong\mathfrak{m}^{g_2}$ (as $C[\mathfrak{N}]$-modules). Thus, the action of $\mathfrak{G}$ on $R(\mathfrak{N})$ induces an action of $\mathfrak{H}$ on $R(\mathfrak{N})$.

Next we shall define an action of $\hat{\mathfrak{H}}=\mathrm{Hom}(\mathfrak{H}, C^*)$ on $R(\mathfrak{G})$. Let $\mathfrak{n}$ be a simple $C[\mathfrak{G}]$-module and $f$ in $\hat{\mathfrak{H}}$. Then $C$ is a $C[\mathfrak{G}]$-module by $g\cdot z=f(\pi(g))z$ ($g$ in $\mathfrak{G}$, $z$ in $C$) where $\pi:\mathfrak{G}\to\mathfrak{H}$ is the canonical homomorphism. Hence $\mathfrak{n}^f=\mathfrak{n}\otimes_C C$ is also a $C[\mathfrak{G}]$-module which is easily seen to be simple. Obviously $\mathfrak{n}_1\cong\mathfrak{n}_2$ implies $\mathfrak{n}_1^f\cong\mathfrak{n}_2^f$. Thus $\hat{\mathfrak{H}}$ acts on $R[\mathfrak{G}]$. We denote for $[\mathfrak{n}]$ in $R(\mathfrak{G})$ the isotropy group of $[\mathfrak{n}]$ in $\hat{\mathfrak{H}}$ by $\hat{\mathfrak{H}}[\mathfrak{n}]: \hat{\mathfrak{H}}[\mathfrak{n}]=\{f\in\hat{\mathfrak{H}};\ [\mathfrak{n}]^f=[\mathfrak{n}]\}=\{f\in\hat{\mathfrak{H}};\ \mathfrak{n}^f\cong\mathfrak{n}\}$. Also we denote for $[\mathfrak{m}]$ in $R(\mathfrak{N})$ the isotropy group of $[\mathfrak{m}]$ in $\mathfrak{H}$ by $\mathfrak{H}[\mathfrak{m}]$. Since $\hat{\mathfrak{H}}$ is abelian, we have $\hat{\mathfrak{H}}_{[\mathfrak{n}]}=\hat{\mathfrak{H}}_{[\mathfrak{n}]^f}$ for any $[\mathfrak{n}]$ in $R(\mathfrak{G})$ and for any $f$ in $\hat{\mathfrak{H}}$. Similarly, we have $\mathfrak{H}_{[\mathfrak{m}]}=\mathfrak{H}_{[\mathfrak{m}]^h}$ for any $[\mathfrak{m}]$ in $R(\mathfrak{N})$ and for any $h$ in $\mathfrak{H}$.

Let $\tilde{\mathfrak{N}}$ be a $C[\mathfrak{G}]$-module and $\mathfrak{n}$ be a simple $C[\mathfrak{G}]$-module. Let $\tilde{\mathfrak{N}}=\mathfrak{n}_1+\cdots+\mathfrak{n}_r$ be a direct sum decomposition of $\tilde{\mathfrak{N}}$ into simple $C[\mathfrak{G}]$-modules $\mathfrak{n}_1,\cdots,\mathfrak{n}_r$. Then we denote by $(\tilde{\mathfrak{N}}:\mathfrak{n})$ the number of $\mathfrak{n}_i$ which is isomorphic to $\mathfrak{n}$ as $C[\mathfrak{G}]$-modules. The number $(\tilde{\mathfrak{N}}:\mathfrak{n})$ is independent of the decomposition $\tilde{\mathfrak{N}}=\mathfrak{n}_1+\cdots+\mathfrak{n}_r$. Denote by $\chi_{\tilde{\mathfrak{N}}}, \chi_{\mathfrak{n}}$ the characters associated to the $C[\mathfrak{G}]$-modules $\tilde{\mathfrak{N}}, \mathfrak{n}$ respectively. Then it is well known that

$$(\tilde{\mathfrak{N}}:\mathfrak{n})=\frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}}\chi_{\tilde{\mathfrak{N}}}(g)\overline{\chi_{\mathfrak{n}}(g)}=\frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}}\chi_{\tilde{\mathfrak{N}}}(g)\,\chi_{\mathfrak{n}}(g)$$

where $|\mathfrak{G}|$ is the order of $\mathfrak{G}$.

Similarly, if $\mathfrak{M}$, $\mathfrak{m}$ are $C[\mathfrak{N}]$-modules and if $\mathfrak{m}$ is simple, $(\mathfrak{M}:\mathfrak{m})$ is defined. Now, if $\mathfrak{n}$ is a simple $C[\mathfrak{G}]$-module and if $\mathfrak{m}$ is a simple $C[\mathfrak{N}]$-module, the following formula is also well known as Frobenius reciprocity:

$$(\mathfrak{m}^*:\mathfrak{n})=(\mathfrak{n}|_{\mathfrak{N}}:\mathfrak{m}),$$

where $\mathfrak{m}^*=C[\mathfrak{G}]\otimes_{C[\mathfrak{N}]}\mathfrak{m}$ is the *induced* $C[\mathfrak{G}]$-module and $\mathfrak{n}|_{\mathfrak{N}}$ is the $C[\mathfrak{N}]$-module obtained naturally from the $C[\mathfrak{G}]$-module $\mathfrak{n}$.

LEMMA 4.1. *Let* $\mathfrak{n}$ *be a simple* $C[\mathfrak{G}]$*-module and* $\mathfrak{m}$ *a simple* $C[\mathfrak{N}]$*-module. Then*

(i) $$(\mathfrak{n}|_{\mathfrak{N}} : \mathfrak{m}) = (\mathfrak{n}|_{\mathfrak{N}} : \mathfrak{m}^h), \quad \text{for any } h \text{ in } \mathfrak{H}.$$

(ii) $$(\mathfrak{m}^* : \mathfrak{n}) = (\mathfrak{m}^* : \mathfrak{n}^f), \quad \text{for any } f \text{ in } \hat{\mathfrak{H}}.$$

PROOF. (i) $(\mathfrak{n}|_{\mathfrak{N}} : \mathfrak{m}) = \frac{1}{|\mathfrak{N}|} \sum_{n \in \mathfrak{N}}' \chi_{\mathfrak{n}}(n)\overline{\chi_{\mathfrak{m}}(n)}$, $(\mathfrak{n}|_{\mathfrak{N}} : \mathfrak{m}^h) = \frac{1}{|\mathfrak{N}|} \sum_{n \in \mathfrak{N}}' (\chi_{\mathfrak{n}}(n)\overline{\chi_{\mathfrak{m}}(gng^{-1})})$ where $g$ is an element in $\mathfrak{G}$ such that $\pi(g) = h$. Thus, since $\chi_{\mathfrak{n}}(gng^{-1}) = \chi_{\mathfrak{n}}(n)$, we have

$$(\mathfrak{n}|_{\mathfrak{N}} : \mathfrak{m}^h) = \frac{1}{|\mathfrak{N}|} \sum_{n \in \mathfrak{N}}' \chi_{\mathfrak{n}}(gng^{-1})\overline{\chi_{\mathfrak{m}}(gng^{-1})} = (\mathfrak{n}|_{\mathfrak{N}} : \mathfrak{m}).$$

(ii) By Frobenius reciprocity, we have $(\mathfrak{m}^* : \mathfrak{n}) = (\mathfrak{n}|_{\mathfrak{N}} : \mathfrak{m})$ and $(\mathfrak{m}^* : \mathfrak{n}^f) = (\mathfrak{n}^f|_{\mathfrak{N}} : \mathfrak{m})$. Now clearly we have $\mathfrak{n}^f|_{\mathfrak{N}} \cong \mathfrak{n}|_{\mathfrak{N}}$, which completes the proof.

Now we denote by $R(\mathfrak{G})/\hat{\mathfrak{H}}$ the *quotient set* (the orbit set) obtained from $R(\mathfrak{G})$ by identifying two points in $R(\mathfrak{G})$ in the same orbit of $\hat{\mathfrak{H}}$. Similarly we denote by $R(\mathfrak{N})/\mathfrak{H}$ the quotient set obtained from $R(\mathfrak{N})$ by identifying two points in $R(\mathfrak{N})$ in the same orbit of $\mathfrak{H}$. We denote by $\varphi : R(\mathfrak{G}) \to R(\mathfrak{G})/\hat{\mathfrak{H}}$ and $\psi : R(\mathfrak{N}) \to R(\mathfrak{N})/\mathfrak{H}$ the canonical projections respectively.

Now let us call $[\mathfrak{n}]$ in $R(\mathfrak{G})$ and $[\mathfrak{m}]$ in $R(\mathfrak{N})$ are *incident* if $(\mathfrak{n}|_{\mathfrak{N}} : \mathfrak{m}) \geqq 1$. By Frobenius reciprocity, $[\mathfrak{n}]$ and $[\mathfrak{m}]$ are incident if and only if $(\mathfrak{m}^* : \mathfrak{n}) \geqq 1$. For $h$ in $\mathfrak{H}$, $f$ in $\hat{\mathfrak{H}}$, by lemma 4.1. we have $(\mathfrak{n}|_{\mathfrak{N}} : \mathfrak{m}) = (\mathfrak{n}|_{\mathfrak{N}} : \mathfrak{m}^h) = ((\mathfrak{m}^h)^* : \mathfrak{n}) = ((\mathfrak{m}^h)^* : \mathfrak{n}^f)$. Hence $[\mathfrak{n}]$ and $[\mathfrak{m}]$ are incident if and only if $[\mathfrak{n}]^f$ and $[\mathfrak{m}]^h$ are incident. Thus we may define an incidence relation between elements $\alpha$ in $R(\mathfrak{G})/\hat{\mathfrak{H}}$ and $\beta$ in $R(\mathfrak{N})/\mathfrak{H}$. Namely $\alpha$ and $\beta$ are called to be incident if there exist incident $[\mathfrak{n}] \in R(\mathfrak{G})$, $[\mathfrak{m}] \in R(\mathfrak{N})$ such that $\varphi([\mathfrak{n}]) = \alpha$, $\psi([\mathfrak{m}]) = \beta$. Our purpose here is to establish the following two facts:

(I) For any $\alpha \in R(\mathfrak{G})/\hat{\mathfrak{H}}$, there exists one and only one $\beta \in R(\mathfrak{N})/\mathfrak{H}$ such that $\alpha$ and $\beta$ are incident.

(II) For any $\beta \in R(\mathfrak{N})/\mathfrak{H}$, there exists one and only one $\alpha \in R(\mathfrak{G})/\hat{\mathfrak{H}}$ such that $\alpha$ and $\beta$ are incident.

Now (I) is nothing but a theorem of Clifford [1]. In fact, let $\alpha \in R(\mathfrak{G})/\hat{\mathfrak{H}}$ and $\alpha = \varphi([\mathfrak{n}])$. $\mathfrak{n}|_{\mathfrak{N}}$ is decomposed into a direct sum of simple $C[\mathfrak{N}]$-modules: $\mathfrak{n}|_{\mathfrak{N}} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_r$. By Clifford's theorem, there exist, $g_1, g_2, \cdots, g_r$ in $\mathfrak{G}$ such that $\mathfrak{m}_i = g_i \mathfrak{m}_1$ $(i = 1, \cdots, r)$. Then it is easy to see $\mathfrak{m}_i \cong \mathfrak{m}_1^{g_i}$ $(i = 1, \cdots, r)$. In other words, for any $[\mathfrak{m}]$ in $R[\mathfrak{N}]$ such that $(\mathfrak{n}|_{\mathfrak{N}} : \mathfrak{m}) \geqq 1$, we have $\psi([\mathfrak{m}]) = \psi([\mathfrak{m}_1])$. Thus $\beta$ exists uniquely and is given by $\beta = \psi([\mathfrak{m}_1])$.

To prove (II), we need several lemmas.

LEMMA 4.2. *Let* $h \in \mathfrak{H}$, $[\mathfrak{n}] \in R(\mathfrak{G})$. *Then*

$$\sum_{f\in\hat{\mathfrak{H}}_{[\mathfrak{n}]}}' f(h)=\frac{1}{|\mathfrak{N}|}\sum_{x\in\pi^{-1}(h)}'|\chi_{\mathfrak{n}}(x)|^2.$$

PROOF. For any C-valued function $F$ on $\mathfrak{G}$, we have easily

$$\frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}}' F(g)=\frac{1}{|\mathfrak{H}|}\sum_{h\in\mathfrak{H}}\frac{1}{|\mathfrak{N}|}\sum_{x\in\pi^{-1}(h)}F(x).$$

Applying this equality to the function $F(g)=f(\pi(g))|\chi_{\mathfrak{n}}(g)|^2$, we get

$$\frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}}' f(\pi(g))|\chi_{\mathfrak{n}}(g)|^2=\frac{1}{|\mathfrak{H}|}\sum_{h\in\mathfrak{H}}'\frac{1}{|\mathfrak{N}|}\sum_{x\in\pi^{-1}(h)}' f(\pi(x))|\chi_{\mathfrak{n}}(x)|^2$$
$$=\frac{1}{|\mathfrak{H}|}\sum_{h\in\mathfrak{H}}\frac{1}{|\mathfrak{N}|}f(h)\sum_{x\in\pi^{-1}(h)}'|\chi_{\mathfrak{n}}(x)|^2.$$

Since $f(\pi(g))|\chi_{\mathfrak{n}}(g)|^2=\chi_{\mathfrak{n}^f}(g)\overline{\chi_{\mathfrak{n}}(g)}$, the left hand side is $=1$ (if $\mathfrak{n}^f\cong\mathfrak{n}$), $=0$ (if $\mathfrak{n}^f\not\cong\mathfrak{n}$). Thus, multiplying $f(h')$ on both sides and summing up w.r.t. $f$ in $\hat{\mathfrak{H}}$ we get

$$\sum_{f\in\hat{\mathfrak{H}}_{[\mathfrak{n}]}}' f(h')=\frac{1}{|\mathfrak{H}|}\sum_{h\in\mathfrak{H},\, f\in\hat{\mathfrak{H}}}f(h)f(h')S_h$$

where $S_h=\frac{1}{|\mathfrak{N}|}\sum_{x\in\pi^{-1}(h)}'|\chi_{\mathfrak{n}}(x)|^2$. Thus by orthogonality relations of characters, we obtain

$$\sum_{f\in\hat{\mathfrak{H}}_{[\mathfrak{n}]}}' \overline{f(h')}=\sum_{h\in\mathfrak{H}}'\delta_{h,h'}S_h=S_{h'}.$$

Since $\overline{S_{h'}}=S_{h'}$, Lemma 4.2. is proved.

LEMMA 4.3. *Let* $[\mathfrak{n}]\in R(\mathfrak{G})$ *and* $[\mathfrak{m}]\in R(\mathfrak{N})$ *be incident. Then*

$$|\hat{\mathfrak{H}}_{[\mathfrak{n}]}|\cdot|\mathfrak{H}_{[\mathfrak{m}]}|=(\mathfrak{m}^*:\mathfrak{n})^2\cdot|\mathfrak{H}|.$$

PROOF. Put $h=1$ in lemma 4.1. Then we have

$$|\hat{\mathfrak{H}}_{[\mathfrak{n}]}|=\frac{1}{|\mathfrak{N}|}\sum_{n\in\mathfrak{N}}'|\chi_{\mathfrak{n}}(n)|^2.$$

Now let $\mathfrak{n}|_{\mathfrak{N}}=\mathfrak{m}_1+\cdots+\mathfrak{m}_s$ $(\mathfrak{m}=\mathfrak{m}_1)$ be a direct sum decomposition of $\mathfrak{n}|_{\mathfrak{N}}$ into simple $C[\mathfrak{N}]$-modules. Let $k=(\mathfrak{m}^*:\mathfrak{n})=(\mathfrak{n}|_{\mathfrak{N}}:\mathfrak{m})$, $r=[\mathfrak{H}:\mathfrak{H}_{[\mathfrak{m}]}]$ and $\mathfrak{H}=\bigcup_{i=1}^{r}\mathfrak{H}_{[\mathfrak{m}]}h_i$ be a coset decomposion. Then $s=kr$ and we may assume that $\mathfrak{m}_i\cong\mathfrak{m}_{i+k}\cong\mathfrak{m}_{i+2k}\cong\cdots\cong\mathfrak{m}_{i+(r-1)k}$ $(i=1,\cdots,r)$ and $\mathfrak{m}_1,\cdots,\mathfrak{m}_r$ are not isomorphic to each other. Then $\chi_{\mathfrak{n}}|_{\mathfrak{N}}=k(\chi_{\mathfrak{m}_1}+\cdots+\chi_{\mathfrak{m}_r})$, and using the orthogonality relations of characters, we get

$$|\hat{\mathfrak{H}}_{[\mathfrak{n}]}|=k^2r=(\mathfrak{m}^*:\mathfrak{n})^2[\mathfrak{H}:\mathfrak{H}_{[\mathfrak{m}]}],$$

which is to be proved.

COR. 4.4. *$\mathfrak{n}|_{\mathfrak{N}}$ is simple if and only if $\hat{\mathfrak{H}}_{[\mathfrak{n}]}=1$. In this case $\mathfrak{H}_{[\mathfrak{m}]}=\mathfrak{H}$ where* $\mathfrak{m}=\mathfrak{n}|_{\mathfrak{N}}$.

PROOF. $\mathfrak{n}|_{\mathfrak{N}}$ is simple if and only if $k=r=1$ using above notations, which is equivalent to $\hat{\mathfrak{H}}_{[\mathfrak{n}]}=1$. If this is the case, then $r=1$ implies $\mathfrak{H}=\mathfrak{H}_{[\mathfrak{m}]}$, Q.E.D.

LEMMA 4.5. *Let $[\mathfrak{m}]\in R(\mathfrak{N})$ and $\mathfrak{H}=\bigcup_{i=1}^{r}\mathfrak{H}_{[\mathfrak{m}]}h_i$ be a coset decomposion, then*

$$x_{\mathfrak{m}^*}(x)=\begin{cases}0\,, & \text{for } x\notin\mathfrak{N},\\ |\mathfrak{H}_{[\mathfrak{m}]}|\cdot(\chi_1(x)+\cdots+\chi_r(x)), & \text{for } x\in\mathfrak{N},\end{cases}$$

*where* $\mathfrak{m}^*=C[\mathfrak{G}]\underset{C[\mathfrak{N}]}{\otimes}\mathfrak{m}$ *and* $\chi_i=\chi_{\mathfrak{m}^{h_i}}$ $(i=1,\cdots,r)$.

PROOF. This is immediate using the following well known formula for $\chi_{\mathfrak{m}^*}$:

$$\chi_{\mathfrak{m}^*}(x)=\frac{1}{|\mathfrak{N}|}\sum_{y\in\mathfrak{G}}\chi^0_{\mathfrak{m}}(yxy^{-1})=\frac{1}{|\mathfrak{N}|}\sum_{y\in\mathfrak{G}}\chi_{\mathfrak{m}^y}(x),$$

where $\chi^0_{\mathfrak{m}}$ is a function on $\mathfrak{G}$ defined by

$$\chi^0_{\mathfrak{m}}(z)=\begin{cases}0\quad, & \text{for } z\notin\mathfrak{N}\\ \chi_{\mathfrak{m}}(z), & \text{for } z\in\mathfrak{N}.\end{cases}$$

COR. 4.6. *Let $[\mathfrak{m}]\in R(\mathfrak{N})$. Then*

$$\frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}}|\chi_{\mathfrak{m}^*}(g)|^2=|\mathfrak{H}_{[\mathfrak{m}]}|\,.$$

PROOF. The left hand side is equal by lemma 4.5 and by lemma 4.3. to

$$\frac{1}{|\mathfrak{G}|}|\mathfrak{H}_{[\mathfrak{m}]}|^2\cdot r\cdot|\mathfrak{N}|=\frac{|\mathfrak{N}|}{|\mathfrak{G}|}\cdot|\mathfrak{H}_{[\mathfrak{m}]}|^2\cdot\frac{|\mathfrak{H}|}{|\mathfrak{H}_{[\mathfrak{m}]}|}=|\mathfrak{H}_{[\mathfrak{m}]}|.$$

LEMMA 4.7. *Let $[\mathfrak{m}]\in R(\mathfrak{N})$ and $[\mathfrak{n}]$ be a simple $C[\mathfrak{G}]$-submodule of $\mathfrak{m}^*=C[\mathfrak{G}]\underset{C[\mathfrak{N}]}{\otimes}\mathfrak{m}$. Then for any simple $C[\mathfrak{G}]$-submodule $\mathfrak{n}'$ of $\mathfrak{m}^*$, there exists an element $f$ in $\hat{\mathfrak{H}}$ such that $\mathfrak{n}'\cong\mathfrak{n}^f$.*

PROOF. Let $\mathfrak{m}^*=\mathfrak{n}_1+\cdots+\mathfrak{n}_t$ $(\mathfrak{n}=\mathfrak{n}_1)$ be a direct sum decomposition of $\mathfrak{m}^*$ into simple $C[\mathfrak{G}]$-modules. Using lemma 4.1 we may assume that

(i) $\mathfrak{n}_1,\cdots,\mathfrak{n}_s$, are not isomorphic to each other, where $s=[\hat{\mathfrak{H}}:\hat{\mathfrak{H}}_{[\mathfrak{n}]}]$,

(ii) $\mathfrak{n}_i\cong\mathfrak{n}_{i+k}\cong\mathfrak{n}_{i+2k}\cong\cdots\cong\mathfrak{n}_{i+(s-1)k}$ $(i=1,\cdots,s)$, where $k=(\mathfrak{m}^*:\mathfrak{n})$,

(iii) $\mathfrak{n}_i\cong\mathfrak{n}^{f_i}$ $(i=1,\cdots s)$, where $\hat{\mathfrak{H}}=\bigcup_{i=1}^{s}\hat{\mathfrak{H}}_{[\mathfrak{n}]}f_i$ is a coset decomposition.

(iv) $\mathfrak{n}_i\ncong\mathfrak{n}_j$ for any $i\leqslant ks, j>ks$.

Then of course we have $ks\leqslant t$. To complete the proof, it is enough to show that $ks=t$.

Now, by cor. 4.6, we have

$$|\mathfrak{H}_{[\mathfrak{m}]}|=\frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}}|\chi_{\mathfrak{m}^*}(g)|^2=\frac{1}{|\mathfrak{G}|}\sum_{i=1}^{t}\sum_{g\in\mathfrak{G}}|\chi_{\mathfrak{n}}(g)|^2\geqslant\frac{1}{|\mathfrak{G}|}\sum_{i=1}^{sk}\sum_{g\in\mathfrak{G}}|\chi_{\mathfrak{n}_i}(g)|^2$$

$$=\frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}}\Big|\sum_{i=1}^{ks}\chi_{\mathfrak{n}_i}(g)\Big|^2+\frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}}\Big|\sum_{j>ks}\chi_{\mathfrak{n}_j}(g)\Big|^2\geqslant\frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}}\Big|\sum_{i=1}^{ks}\chi_{\mathfrak{n}_i}(g)\Big|^2=k^2s=|\mathfrak{H}_{[\mathfrak{m}]}|$$

by lemma 4.3. Hence we obtain $t=ks$, Q.E.D.

By lemma 4.7, the proposition (II) raised above was proved. Thus, we established a bijective mapping $R(\mathfrak{G})/\hat{\mathfrak{H}}\to R(\mathfrak{N})/\mathfrak{H}$ which associates to each $\alpha$ in $R(\mathfrak{G})/\hat{\mathfrak{H}}$ a uniquely determined incident element $\beta$ in $R(\mathfrak{N})/\mathfrak{H}$.

COR. 4.8. *Let* $[\mathfrak{m}]\in R(\mathfrak{N})$. *Then* $\mathfrak{m}^*=C[\mathfrak{G}]\underset{C[\mathfrak{N}]}{\otimes}\mathfrak{m}$ *is simple if and only if* $\mathfrak{H}_{[\mathfrak{m}]}=1$. *In this case* $\hat{\mathfrak{H}}_{[\mathfrak{n}]}=\hat{\mathfrak{H}}$, *where* $\mathfrak{n}=\mathfrak{m}^*$.

PROOF. Obvious by the proof of lemma 4.7.

Now we shall give some other properties of the bijection $R(\mathfrak{G})/\hat{\mathfrak{H}}\to R(\mathfrak{N})/\mathfrak{H}$.

LEMMA 4.9. *Let* $[\mathfrak{n}]\in R(\mathfrak{G})$ *and* $[\mathfrak{m}]\in R(\mathfrak{N})$ *be incident. Then* $\mathfrak{H}^{\perp}_{[\mathfrak{m}]}\subset\hat{\mathfrak{H}}_{[\mathfrak{n}]}$, *where* $\mathfrak{H}^{\perp}_{[\mathfrak{m}]}=\{f\in\hat{\mathfrak{H}};\ f(h)=1$ *for all* $h$ *in* $\mathfrak{H}_{[\mathfrak{m}]}\}$.

PROOF. Let $\pi^{-1}(\mathfrak{H}_{[\mathfrak{m}]})=\mathfrak{G}_0$. Then there exists an element $[\mathfrak{k}]$ in $R(\mathfrak{G}_0)$ which is incident with both $[\mathfrak{n}]$ and $[\mathfrak{m}]$. In fact any simple $C[\mathfrak{G}_0]$-submodule of $\mathfrak{n}|_{\mathfrak{G}_0}$ can be taken as $\mathfrak{k}$. Denote the factor group $\mathfrak{G}/\mathfrak{G}_0=\mathfrak{H}/\mathfrak{H}_{[\mathfrak{m}]}$ by $\mathfrak{K}$. Then we have $\mathfrak{K}_{[\mathfrak{k}]}=1$. In fact, let $g\in\mathfrak{G}$ satisfy $\mathfrak{k}^g\cong\mathfrak{k}$. Then $\chi_{\mathfrak{k}}(x)=\chi_{\mathfrak{k}}(gxg^{-1})$ for any $x$ in $\mathfrak{G}$, hence for any $x$ in $\mathfrak{N}$. On the other hand, since $\mathfrak{m}^{g'}=\mathfrak{m}$ for any $g'$ in $\mathfrak{G}_0$, we have $\chi_{\mathfrak{k}}|_{\mathfrak{N}}=m\cdot\chi_{\mathfrak{m}}$, where $m=(\mathfrak{k}|_{\mathfrak{N}}:\mathfrak{m})$. Thus $m\chi_{\mathfrak{m}}(x)=m\chi_{\mathfrak{m}}(gxg^{-1})$ for any $x$ in $\mathfrak{N}$. Hence $\mathfrak{m}^g\cong\mathfrak{m}$, and we get $\pi(g)\in\mathfrak{H}_{[\mathfrak{m}]}$, i.e. $g\in\mathfrak{G}_0$. This means however $\mathfrak{K}_{[\mathfrak{k}]}=1$.

Now applying lemma 4.3 for the pair $\mathfrak{G}\supset\mathfrak{G}_0$ and $\mathfrak{k}$, we obtain from $\mathfrak{K}_{[\mathfrak{k}]}=1$ that $\hat{\mathfrak{K}}_{[\mathfrak{n}]}=\hat{\mathfrak{K}}$ and $(\mathfrak{n}|_{\mathfrak{K}}:\mathfrak{k})=1$. Let $\gamma\in\mathfrak{H}^{\perp}_{[\mathfrak{m}]}$. Then $\gamma$ may be regarded as an element of the character group of $\mathfrak{H}/\mathfrak{H}_{[\mathfrak{m}]}$, i.e. $\gamma$ may be regarded as an element of $\hat{\mathfrak{K}}$. Then $\hat{\mathfrak{K}}_{[\mathfrak{n}]}=\hat{\mathfrak{K}}$ implies that $\mathfrak{n}^{\gamma}\cong\mathfrak{n}$. Hence $\gamma\in\hat{\mathfrak{H}}_{[\mathfrak{n}]}$. Thus $\mathfrak{H}^{\perp}_{[\mathfrak{m}]}\subset\hat{\mathfrak{H}}_{[\mathfrak{n}]}$ is proved.

Now in general $\mathfrak{H}^{\perp}_{[\mathfrak{m}]}=\hat{\mathfrak{H}}_{[\mathfrak{n}]}$ is *false*. By lemma 4.3, we have

COR. 4.10. *Let* $[\mathfrak{n}]\in R(\mathfrak{G})$ *and* $[\mathfrak{m}]\in R(\mathfrak{N})$ *be incident. Then*

$$[\hat{\mathfrak{H}}_{[\mathfrak{n}]}:\mathfrak{H}^{\perp}_{[\mathfrak{m}]}]=(\mathfrak{m}^*:\mathfrak{n})^2\,.$$

*Thus we have* $\hat{\mathfrak{H}}_{[\mathfrak{n}]}=\mathfrak{H}^{\perp}_{[\mathfrak{m}]}$ *if and only if* $(\mathfrak{m}^*:\mathfrak{n})=1$.

EXAMPLE. Let $\mathfrak{G}$ be the dihedral group $\mathfrak{D}_4$ of order 8, i.e. $\mathfrak{G}$ is generated by $a, b$ together with the fundamental relations $a^2=b^4=1$, $aba^{-1}=b^{-1}$. Let $\mathfrak{N}$ be the commutator group of $\mathfrak{G}$. Then $\mathfrak{N}$ is the center of $\mathfrak{G}$ and $\mathfrak{N}$ is a cyclic group of order $2:\mathfrak{N}=\{1, b^2\}$. Let $\mathfrak{N}$ acts on $\mathfrak{m}=C$ by $b^2\xi=-\xi$ ($\xi$ in C). Then it is easy to see that the induced module $\mathfrak{m}^*$ is decomposed into a direct sum of simple $C[\mathfrak{G}]$-modules as follows: $\mathfrak{m}^*=\mathfrak{n}_1+\mathfrak{n}_2$, $\mathfrak{n}_1\cong\mathfrak{n}_2$. Hence $(\mathfrak{m}^*:\mathfrak{n}_1)=2$. Since $[\mathfrak{G}:\mathfrak{N}]=4$, $\dim\mathfrak{n}_1=2$. Since $\mathfrak{G}$ has only one irreducible representation of degree 2, we have

$\hat{\mathfrak{H}}_{[\mathfrak{n}]}=\hat{\mathfrak{H}}$. On the other hand, since $\mathfrak{N}$ is the center of $\mathfrak{G}$, we have $\mathfrak{H}_{[\mathfrak{m}]}=\mathfrak{H}$. Thus we have $\mathfrak{H}_{[\mathfrak{m}]}^{\perp}=\{1\}\neq\hat{\mathfrak{H}}_{[\mathfrak{n}]}$.

Now we shall give a condition for $\mathfrak{H}$ where we have $(\mathfrak{m}^*:\mathfrak{n})=1$ for all incident pair $[\mathfrak{n}]$ in $R(\mathfrak{G})$, $[\mathfrak{m}]$ in $R(\mathfrak{N})$.

LEMMA 4.11. *Let $\mathfrak{H}=\mathfrak{G}/\mathfrak{N}$ be a cyclic group and $[\mathfrak{m}]$ in $R(\mathfrak{N})$. Then $\mathfrak{H}_{[\mathfrak{m}]}=\mathfrak{H}$ if and only if the associated homomorphism $\rho:\mathfrak{N}\to GL(\mathfrak{m})$ (the linear representation of $\mathfrak{N}$) can be extended to a homomorphism $\mathfrak{G}\to GL(\mathfrak{m})$.*

PROOF. Clearly if the homomorphism $\rho:\mathfrak{N}\to GL(\mathfrak{m})$ can be extended to a homomorphism $\tilde{\rho}:\mathfrak{G}\to GL(\mathfrak{m})$, then $\mathfrak{H}_{[\mathfrak{m}]}=\mathfrak{H}$. Conversely if $\mathfrak{H}_{[\mathfrak{m}]}=\mathfrak{H}$, then $\rho$ is self-conjugate. Now since $\mathfrak{H}$ is cyclic, we have $H^2(\mathfrak{H},C^*)=1$ (cf. Schur [4]). Thus the obstruction cohomology class of $\rho$ is 1. Hence $\rho$ can be extended to a homomorphism $\mathfrak{G}\to GL(\mathfrak{m})$ (cf. § 3).

LEMMA 4.12. *Let $\mathfrak{H}=\mathfrak{G}/\mathfrak{N}$ be a cyclic group. Then for any incident pair $[\mathfrak{n}]$ in $R(\mathfrak{G})$, $[\mathfrak{m}]$ in $R(\mathfrak{N})$, we have $(\mathfrak{m}^*:\mathfrak{n})=1$. Thus also we have $\mathfrak{H}_{[\mathfrak{m}]}^{\perp}=\hat{\mathfrak{H}}_{[\mathfrak{n}]}$.*

PROOF. Let $\pi^{-1}(\mathfrak{H}_{[\mathfrak{m}]})=\mathfrak{G}_0$ and take an $[\mathfrak{k}]$ in $R(\mathfrak{G}_0)$ which is incident with both $[\mathfrak{n}]$ and $[\mathfrak{m}]$ as in the proof of lemma 4.9. Them $\mathfrak{m}^g\cong\mathfrak{m}$ (for all $g$ in $\mathfrak{G}_0$) implies that the associated homomorphism $\rho:\mathfrak{N}\to GL(\mathfrak{m})$ can be extended to a homomorphism $\tilde{\rho}:\mathfrak{G}_0\to GL(\mathfrak{m})$ (lemma 4.11). In this manner, $\mathfrak{m}$ has also a $C[\mathfrak{G}_0]$-module structure. When $\mathfrak{m}$ is regarded as a $C[\mathfrak{G}_0]$-module, we write $\mathfrak{m}$ as $\mathfrak{m}_0$. Then $[\mathfrak{m}_0]$ in $R(\mathfrak{G}_0)$ and $[\mathfrak{m}]$ in $R(\mathfrak{N})$ are incident. Hence by the uniqueness (II), there is an element $\gamma$ in $\hat{\mathfrak{L}}$, such that $\mathfrak{k}\cong\mathfrak{m}_0^{\gamma}$, where $\mathfrak{L}=\mathfrak{G}_0/\mathfrak{N}\cong\mathfrak{H}_{[\mathfrak{m}]}$, $\hat{\mathfrak{L}}=\mathrm{Hom}(\mathfrak{L},C^*)$. Now, since $(\mathfrak{m}_0:\mathfrak{m})=1$, we have $(\mathfrak{k}:\mathfrak{m})=1$ and $\mathfrak{k}|\mathfrak{N}\cong\mathfrak{m}$. Now let us show that $(\mathfrak{n}|\mathfrak{G}_0:\mathfrak{k})=1$ and $\mathfrak{K}_{[\mathfrak{k}]}=1$, where $\mathfrak{K}=\mathfrak{G}/\mathfrak{G}_0=\mathfrak{H}/\mathfrak{H}_{[\mathfrak{m}]}$. In fact, if $g\in\mathfrak{G}$ satisfies $\mathfrak{k}^g\cong\mathfrak{k}$, then $\chi_{\mathfrak{k}}(gxg^{-1})=\chi_{\mathfrak{k}}(x)$ for any $x$ in $\mathfrak{G}_0$. Hence $\chi_{\mathfrak{k}}(gxg^{-1})=\chi_{\mathfrak{k}}(x)$ for any $x$ in $\mathfrak{N}$, i.e. $(\mathfrak{k}|\mathfrak{N})^g\cong\mathfrak{k}|\mathfrak{N}$, i.e. $\mathfrak{m}^g\cong\mathfrak{m}$. Hence $\pi(g)\in\mathfrak{H}_{[\mathfrak{m}]}$, i.e. $g\in\mathfrak{G}_0$, i.e. $\mathfrak{K}_{[\mathfrak{k}]}=1$. Therefore, $\mathfrak{k}^*=C[\mathfrak{G}]\underset{C[\mathfrak{G}_0]}{\otimes}\mathfrak{k}$ is a simple $C[\mathfrak{G}]$-module by Cor. 4.8 and $\mathfrak{k}^*\cong\mathfrak{n}$ by the uniqueness. Thus $(\mathfrak{n}|\mathfrak{G}_0:\mathfrak{k})=(\mathfrak{k}^*:\mathfrak{n})=1$.

Now let $\mathfrak{n}|\mathfrak{G}_0=\mathfrak{k}_1+\cdots+\mathfrak{k}_s$ $(\mathfrak{k}=\mathfrak{k}_1)$ be a direct sum decomposition of $\mathfrak{n}|\mathfrak{G}_0$ into simple $C[\mathfrak{G}_0]$-modules. By $(\mathfrak{n}|\mathfrak{G}_0:\mathfrak{k})=1$, we have $\mathfrak{k}_i\not\cong\mathfrak{k}_j$ for any $1\leqslant i\neq j\leqslant s$. Since there exist $g_i$ in $\mathfrak{G}$ $(i=1,\cdots,s)$ such that $\mathfrak{k}_i\cong\mathfrak{k}^{g_i}$ $(i=1,\cdots,s)$, we see that $g_ig_j^{-1}\notin\mathfrak{G}_0$ for any $1\leqslant i\neq j\leqslant s$ and that every $\mathfrak{k}_i|\mathfrak{N}$ is a simple $C[\mathfrak{N}]$-module. To complete the proof, it is thus enough to show $\mathfrak{k}_i|\mathfrak{N}\not\cong\mathfrak{k}_j|\mathfrak{N}$ (for any $1\leqslant i\neq j\leqslant s$). Suppose $\mathfrak{k}_i|\mathfrak{N}\cong\mathfrak{k}_j|\mathfrak{N}$. Then $\mathfrak{k}^{g_i}|\mathfrak{N}\cong\mathfrak{k}^{g_j}|\mathfrak{N}$. Hence $(\mathfrak{k}|\mathfrak{N})^{g_i}\cong(\mathfrak{k}|\mathfrak{N})^{g_j}$, i.e. $\mathfrak{m}^{g_i}\cong\mathfrak{m}^{g_j}$. Hence $\pi(g_ig_j^{-1})$ is in $\mathfrak{H}_{[\mathfrak{m}]}$, i.e. $g_ig_j^{-1}$ is in $\mathfrak{G}_0$. Then $i=j$ as we have seen above, Q.E.D.

Let us finally resume our results above in a theorem as follows:

THEOREM 4.13. *Let $\mathfrak{G}$ be a finite group and $\mathfrak{N}$ be a normal subgroup of $\mathfrak{G}$ such that the factor group $\mathfrak{H}=\mathfrak{G}/\mathfrak{N}$ is abelian.* (i) *Then for any simple* $C[\mathfrak{G}]$-

*module* $\mathfrak{n}$, *there is a simple* $C[\mathfrak{N}]$*-module* $\mathfrak{m}$ *such that* $(\mathfrak{n}|\mathfrak{N}:\mathfrak{m})=(\mathfrak{m}^*:\mathfrak{n})\geqslant 1$ (*where* $\mathfrak{m}^*=C[\mathfrak{G}]\underset{C[\mathfrak{N}]}{\otimes}\mathfrak{m}$). $\mathfrak{m}$ *is unique up to the action of* $\mathfrak{H}$. *Also for any simple* $C[\mathfrak{N}]$*-module* $\mathfrak{m}$, *there is a simple* $C[\mathfrak{G}]$*-module* $\mathfrak{n}$ *such that* $(\mathfrak{m}^*:\mathfrak{n})\geqslant 1$. $\mathfrak{n}$ *is unique up to the action of* $\hat{\mathfrak{H}}=\mathrm{Hom}(\mathfrak{H}, C^*)$. *Thus there is a natural bijection between* $R(\mathfrak{G})/\hat{\mathfrak{H}}$ *and* $R(\mathfrak{N})/\mathfrak{H}$. (ii) *If* $[\mathfrak{n}]\in R(\mathfrak{G})$, $[\mathfrak{m}]\in R(\mathfrak{N})$ *correspond each other* (i.e. if $(\mathfrak{m}^*:\mathfrak{n})\geqslant 1$), *then* $|\hat{\mathfrak{H}}_{[\mathfrak{n}]}|\cdot|\mathfrak{H}_{[\mathfrak{m}]}|=(\mathfrak{m}^*:\mathfrak{n})^2|\mathfrak{H}|$. *Also* $\hat{\mathfrak{H}}_{[\mathfrak{n}]}\supset\mathfrak{H}_{[\mathfrak{m}]}^{\perp}$ *and* $[\hat{\mathfrak{H}}_{[\mathfrak{n}]}:\mathfrak{H}_{[\mathfrak{m}]}^{\perp}]=(\mathfrak{m}^*:\mathfrak{n})^2$, *where* $\mathfrak{H}_{[\mathfrak{m}]}^{\perp}$ *is the annihilator of* $\mathfrak{H}_{[\mathfrak{m}]}$ *in* $\hat{\mathfrak{H}}$. (iii) *If* $\mathfrak{H}$ *is cyclic, then for any corresponding pair* $[\mathfrak{n}]$ *in* $R(\mathfrak{G})$, $[\mathfrak{m}]$ *in* $R(\mathfrak{N})$, *we have* $(\mathfrak{m}^*:\mathfrak{n})=1$ *and* $\hat{\mathfrak{H}}_{[\mathfrak{n}]}=\mathfrak{H}_{[\mathfrak{m}]}^{\perp}$.

REMARK. If $\mathfrak{G}$ is a compact topological group and $\mathfrak{N}$ is a closed normal subgroup for which $\mathfrak{G}/\mathfrak{N}$ is finite and abelian, then the proofs of Theorem 4.13 is easily checked to be valid when we consider finite-dimensional irreducible representations of $\mathfrak{G}$ and $\mathfrak{N}$.

As a corollary to Theorem 4.13, we shall prove the following theorem of Frucht [2]. (cf. also K. Yamazaki [7] Th. 6.1, Cor.)

THEOREM 4.14. (Frucht [2]). *Let* $\mathfrak{H}$ *be a finite abelian group and* $c$ *in* $H^2(\mathfrak{H}, C^*)$. *Then there exists one and only one* (*up to equivalence*) *irreducible projective representation of* $\mathfrak{H}$ *which has* $c$ *as its factor set.*

PROOF. Let $\mathfrak{G}$ be a representation group of $\mathfrak{H}$. Then there is a central subgroup $\mathfrak{N}$ of $\mathfrak{G}$ such that $\mathfrak{G}/\mathfrak{N}=\mathfrak{H}$ and the transgression map $\hat{\mathfrak{N}}=\mathrm{Hom}(\mathfrak{N}, C^*)\to H^2(\mathfrak{H}, C^*)$ is bijective. Now let $\chi$ be the element in $\hat{\mathfrak{N}}$ corresponding to $c$ in $H^2(\mathfrak{H}, C^*)$. Then it is easy to see that any irreducible projective representation of $\mathfrak{H}$ with factor set $c$ is obtained by an irreducible linear representation $\rho$ of $\mathfrak{G}$ which is incident with $\chi^*$ (=the induced representation of $\mathfrak{G}$ by $\chi$), i.e. $(\chi^*:\rho)>0$. Thus it is enough to show the following: let $\rho_1$, $\rho_2$ be two irreducible linear representations of $\mathfrak{G}$ such that $(\chi^*:\rho_1)>0$, $(\chi^*:\rho_2)>0$, then $\rho_1$ and $\rho_2$ induce on $\mathfrak{H}$ equivalent projective representations. Now by Theorem 4.13, there exists an element $f$ in $\hat{\mathfrak{H}}$ such that $\rho_1^f\cong\rho_2$ i.e. $\rho_1\otimes(f\circ\pi)\cong\rho_2$ where $\pi:\mathfrak{G}\to\mathfrak{H}$ is the canonical homomorphism. Since $f\circ\pi$ is a representation of $\mathfrak{G}$ of degree 1, $\rho_1$ and $\rho_2$ induce equivalent projective representations of $\mathfrak{H}$, Q.E.D.

§5. Let $\mathfrak{H}$ be a finite group and $k$ be a field. Let $r:\mathfrak{H}\times\mathfrak{H}\to k^*$ be a $k^*$-valued 2-cocycle of $\mathfrak{H}$ (under the trivial action of $\mathfrak{H}$ on $k^*$). Then the vector space $\mathcal{A}=\sum_{P\in\mathfrak{H}} k\cdot u_P$ (where $\{u_P; P\in\mathfrak{H}\}$ is a base of $\mathcal{A}$) becomes an associative algebra over $k$ w.r.t. the multiplication $u_P u_Q=r_{P,Q}u_{PQ}$. (the crossed-product!). This algebra will be denoted by $\mathcal{A}(\mathfrak{H}, k, r)$ or simply by $\mathcal{A}(r)$. If the cocycles $r$ and $r'$ are cohomologous, then $\mathcal{A}(r)\cong\mathcal{A}(r')$ as is easily seen. Thus we may define

$\mathcal{A}(c)$ for $c$ in $H^2(\mathfrak{H}, k^*)$. (cf. K. Yamazaki [7; §4]) for the detail of the properties of the algebra $\mathcal{A}(c)$, which was called as an algebra extension of $\mathfrak{H}$). In K. Yamazaki [7: §6], for an abelian group $\mathfrak{H}$, the condition for the existence of $c$ in $H^2(\mathfrak{H}, k^*)$ such that $\mathcal{A}(c)$ is central simple is given. We shall give in this section some examples of non-abelian $\mathfrak{H}$ which has an element $c$ in $H^2(\mathfrak{H}, k^*)$ such that $\mathcal{A}(c)$ is central simple. In all our examples $\mathfrak{H}$ is solvable. Thus we may raise the following question: let $\mathfrak{H}$ be a finite group. Let $\mathcal{A}(c)$ be central simple for some $c$ in $H^2(\mathfrak{H}, C^*)$. Is $\mathfrak{H}$ then solvable?

Now let $p$ be the characteristic of the field $k$. If $p \nmid |\mathfrak{H}|$, then as in the case of the group algebra $k[\mathfrak{H}]$, $\mathcal{A}(c)$ is semi-simple for any $c$ in $H^2(\mathfrak{H}, k^*)$. (cf. K. Yamazaki [7; §4]). Hence in such a case, $\mathcal{A}(c)$ is central simple over $k$ if and only if the center of $\mathcal{A}(c)$ coincides with $k$.

Now, if $\mathcal{A}(c)$ is central simple for $c$ in $H^2(\mathfrak{H}, k^*)$, $\mathcal{A}(c) \otimes_k \bar{k}$ is also a central simple algebra over $\bar{k}$, where $\bar{k}$ is the algebraic closure of $k$. It is easy to see that if we denote by $\bar{c}$ the image of $c$ under the canonical homomorphism $H^2(\mathfrak{H}, k^*) \to H^2(\mathfrak{H}, \bar{k}^*)$, we have $\mathcal{A}(c) \otimes_k \bar{k} \cong \mathcal{A}(\mathfrak{H}, \bar{k}, \bar{c})$. Thus, when we seek for the structure of $\mathfrak{H}$, we may consider the case where $k$ is algebraically closed. We also note that if $\mathcal{A}(c)$ is central simple for some $c$ in $H^2(\mathfrak{H}, k^*)$, then the order of $\mathfrak{H}$ is a square of some positive integer $m : |\mathfrak{H}| = m^2$. Now,

LEMMA 5.1. *Let $\mathfrak{H}$ be a finite group and $k$ an algebraically closed field of characteristic $p$, where $p \nmid |\mathfrak{H}|$. Then $\mathcal{A}(\mathfrak{H}, k, c)$ is central simple for some $c$ in $H^2(\mathfrak{H}, k^*)$ if and only if*

i) $|\mathfrak{H}| = m^2$ *for some positive integer $m$, and*

ii) $\mathfrak{H}$ *has an irreducible projective representation of degree $m$.*

PROOF. Necessity. Let $\mathcal{A}(c)$ be central simple for some $c$. Then i) is clear and $\mathcal{A}(c) \cong M_m(k)$ (the total matric algebra of degree $m$ over $k$), since $k$ is algebraically closed. If $\mathcal{A}(c) = \sum_{P \in \mathfrak{H}} k u_P$, $u_P u_Q = r_{P,Q} u_{PQ}$ ($r_{P,Q}$ in $k^*$), then the map $P \to u_P$ from $\mathfrak{H}$ into $\mathcal{A}(c)$ induces an irreducible projective representation of $\mathfrak{H}$ over a minimal left ideal of $\mathcal{A}(c)$. Sufficiency. Let $|\mathfrak{H}| = m^2$ and $T : \mathfrak{H} \to GL(m, k)$ be a map which induces an irreducible projective representation of $\mathfrak{H}$. Then if $r$ is the factor set of $T$, we see easily that $\mathcal{A}(r) \cong M_m(k)$ (cf. K. Yamazaki [7; §5]).

EXAMPLE 1. $\mathfrak{H} = \mathfrak{A}_4 \times \mathbf{Z}_3$ ($\mathfrak{A}_4$ = the alternating group of degree 4, $\mathbf{Z}_3$ = the cyclic group of order 3). Here $|\mathfrak{H}| = 36$, $m = 6$. Now by $H^2(\mathfrak{A}_4 \times \mathbf{Z}_3, C^*) \cong H^2(\mathfrak{A}_4, C^*) \times H^2(\mathbf{Z}_3, C^*) \times P(\mathfrak{A}_4, \mathbf{Z}_3, C^*)$ (cf. K. Yamazaki [7; §2.2]), $H^2(\mathfrak{A}_4, C^*) = \mathbf{Z}_2$ (cf. Schur [6]), $H^2(Z_3, C^*) = 1$ (cf. Schur [4]) imply that $H^2(\mathfrak{A}_4 \times \mathbf{Z}_3, C^*) \cong \mathbf{Z}_2 \times P(\mathfrak{A}_4, \mathbf{Z}_3, C^*)$. Now $\mathfrak{A}_4/[\mathfrak{A}_4, \mathfrak{A}_4] \cong \mathbf{Z}_3$, hence $P(\mathfrak{A}_4, \mathbf{Z}_3, C^*) \cong \mathbf{Z}_3$. Thus we have

$$H^2(\mathfrak{A}_4 \times \mathbf{Z}_3, C^*) \cong \mathbf{Z}_6 .$$

Let $c$ be any generator of $H^2(\mathfrak{A}_4\times Z_3, C^*)$. Then since $c$ is of order 6, the degree $d$ of any irreducible projective representation $\rho$ of $\mathfrak{A}_4\times Z_3$ which has $c$ as its factor set is divisible by 6. (cf. Schur [4]). On the other hand, since $|\mathfrak{H}|=36$, $d=6$. Thus $\mathcal{A}(c)$ is central simple.

REMARK. In example 1, we may construct a central simple $\mathcal{A}(c)$ for $k=Q(\omega)$, where $Q$ is the field of rational numbers and $\omega=\exp\frac{2\pi i}{3}$. Because we can construct an element $c$ in $H^2(\mathfrak{H}, Q(\omega)^*)$ such that the image $\bar{c}$ of $c$ in the homomorphism $H^2(\mathfrak{H}, Q(\omega)^*)\to H^2(\mathfrak{H}, C^*)$ is of order 6.

EXAMPLE 2. $\mathfrak{H}=\mathfrak{D}_4\times\mathfrak{A}$, $\mathfrak{A}\simeq Z_2$, ($\mathfrak{D}_4$=the dihedral group of order 8). Here $|\mathfrak{H}|=16$, $m=4$. $H^2(\mathfrak{H}, C^*)\simeq H^2(\mathfrak{D}_4, C^*)\times P(\mathfrak{D}_4, \mathfrak{A}, C^*)$. Now $H^2(\mathfrak{D}_4, C^*)\simeq Z_2$ (cf. Schur [5]), $\mathfrak{D}_4/[\mathfrak{D}_4, \mathfrak{D}_4]\simeq\mathfrak{D}_2\simeq Z_2\times Z_2$, hence $H^2(\mathfrak{H}, C^*)\simeq Z_2\times(Z_2\times Z_2)$. Let $\mathfrak{A}=\{1, Y\}$, $\mathfrak{Z}=[\mathfrak{D}_4, \mathfrak{D}_4]=\{1, Z\}$=the center of $\mathfrak{D}_4$, $H^2(\mathfrak{D}_4, C^*)=\{1, c_0\}$. By Theorem 2.1, the homomorphism $\theta: H^2(\mathfrak{D}_4, C^*)\to P(\mathfrak{D}_4/\mathfrak{Z}, \mathfrak{Z}; C^*)$ is injective. Take an element $X$ in $\mathfrak{D}_4$ such that $\theta c_0(X, Z)\neq 1$. Then take an element $\varphi\neq 1$ in $P(\mathfrak{D}_4, \mathfrak{A}; C^*)$ such that $\varphi(X, Y)=1$. Then let $c$ be the element in $H^2(\mathfrak{H}, C^*)$ which corresponds to $(c_0, \varphi)$ by the isomorphism $H^2(\mathfrak{H}, C^*)\simeq H^2(\mathfrak{D}_4, C^*)\times P(\mathfrak{D}_4, \mathfrak{A}; C^*)$. Then it is easy to show that the center of the algebra $\mathcal{A}(c)=\mathcal{A}(\mathfrak{H}, C, c)$ is of dimension 1. Thus $\mathcal{A}(c)$ is central simple. Also in this case $c$ can be constructed in $H^2(\mathfrak{H}, Q^*)$. Thus, there is also an element $c$ in $H^2(\mathfrak{H}, Q^*)$ for which $\mathcal{A}(\mathfrak{H}, Q, c)$ is a central simple algebra over $Q$.

REMARK. Analogous construction is possible also for $\mathfrak{H}=\mathfrak{D}_4\times\mathfrak{D}_4$, and we obtain a central simple $\mathcal{A}(\mathfrak{H}, Q, c)$.

University of Tokyo

## References


[1] A. H. Clifford, Representations induced in an invariant subgroup. Ann. Math., **38** (1937), 533-550.

[2] R. Frucht, Über die Darstellung endlicher Abelscher Gruppen durch Kollineationen. Crelles Journal, **166** (1931), 16-29.

[3] G. Hochschild and J. P. Serre, Cohomology of group extensions. Trans. Amer. Math. Soc., **74** (1953), 110-134.

[4] I. Schur, Über die Darstellung der endlichen Gruppen durch gebrochene Substitutionen. Crelles Journal, **127** (1904), 20-50.

[5] ——— Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen. Crelles Journal, **132** (1907), 85-137.

[6] ——— Über die Darstellung der symmetrischen und der alternierenden Gruppen durch gebrochene lineare Substitutionen, Crelles Journal, **166** (1911), 155-250.

[7] K. Yamazaki, On projective representations and ring extensions of finite groups. In this journal.


(Received October 26, 1963)