On the embedding of topological rings into quotient rings

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It is well known that a commutative integral domain is embeddable in the quotient field. For non-commutative cases, if a ring is a principal ideal domain without zero divisors, it is also embeddable in the quotient division ring. More general cases are investigated by Ore [6] and Asano [1]. Asano's result is very elegant and includes all of the preceding results. But, when one treats topological rings, the situation becomes more complicated. Our aim of this paper is to give a criterion of embeddability of topological rings into quotient topological rings, whose topology is, in a sense, canonically given by the original rings. In § 1 we reconstruct Asano's quotient ring by the ordinary method for constructing quotient field, so that the resulting system fits to introduce the topology. Here, the method is more or less similar to that of [7] and [4]. In § 2 we generalize the results of [5] to non-commutative rings. The embeddability and the weakly embeddability are investigated.

§ 1. Algebraic part.

Let R be an associative ring with at least one non-zero divisor, where a non-zero divisor means a non-zero element with no right and left zero divisor. Then the totality H^* of all non-zero divisors in R forms a multiplicative semigroup.

DEFINITION 1. *Let H be a subsemigroup of H^*. Then an extension ring S of R is called a left quotient ring with the denominator system H if the following conditions are satisfied:*

1) *S has the unit element* 1,

2) *every element of H is regular in S (i.e. for any $a \in H$, there exists the inverse element a^{-1} of a in S),*

3) *for any element s in S, there exists an element a in H such that as is in R.*

The purpose of this section is to give an alternative proof of the following theorem proved by Asano;

THEOREM 1 (Asano). *A left quotient ring of R with the denominator system H exists if and only if the following condition* (A) *is satisfied;*

(A) *To any element a $(\in H)$ and any element x $(\in R)$ there exist elements a' $(\in H)$ and x' $(\in R)$ such that $x'a = a'x$,* *Moreover, such a left quotient ring of R is uniquely determined by R and H within isomorphisms.*

We will prove this theorem by several steps.

1. Necessity of Condition (A) is easily proved. For any element a $(\in H)$ and x $(\in R)$, $s=xa^{-1}$ is an element of S. By Condition 3) of Definition 1, there exists an element a' in H such that $a's$ is in R. Let $x'=a's$, then

$$x'=a'xa^{-1} \quad \text{i.e.} \quad x'a=a'x\,.$$

And Condition (A) holds.

2. Let a, b be elements of R. If there exists an element $b'\in R$ such that $b=b'a$, then b' is called *a left factor of a in b.*

An element $c\in R$ is called *a left common multiple of x, y* $(x, y\in R)$ if $c=y'x=x'y$ $(x', y'\in R)$.

In this paper, instead of a left common multiple, we simply say *a common multiple* because we consider only left common multiples. By Condition (A) any two elements a, b of H have a common multiple in H.

Proposition 1. *If c_0 $(\in H)$, c $(\in R)$ are common multiples of a, b $(\in H)$, and $c_0=b_0a=a_0b$, $c=b'a=a'b$, then any solution of the equation*

$$b_0x=a_0y$$

satisfies necessarily the equation

$$b'x=a'y\,.$$

Proof. By Condition (A) we can choose elements r $(\in R)$, h $(\in H)$ such that $rc_0=hc$. Then, from the relations

$$rb_0a=ra_0b=hb'a=ha'b\,,$$

we obtain

$$rb_0=hb'\,,\qquad ra_0=ha'\,,$$

since a, b are non-zero divisors. Obviously the equation $b_0x=a_0y$ yields the following equations

$$rb_0x=ra_0y \rightarrow hb'x=ha'y \rightarrow b'x=a'y\,.$$

Now consider the product set $H\times R$ of H and R. An element (b, y) $(\in H\times R)$ is called equivalent to (a, x) $(\in H\times R)$, in symbol $(a, x)\sim(b, y)$, if and only if there exists in H a common multiple $c_0=b_0a=a_0b$ of a, b such that

$$b_0x=a_0y\,.$$

Obviously, by Prop. 1, the equivalence $(a, x)\sim(b, y)$ does not depend on the choice of a common multiple c_0 ($\in H$) of a, b.

PROPOSITION 2. i) $(a, x)\sim(a, x')$ *if and only if* $x=x'$.
ii) *If* $ra\in H$, *then* $(ra, rx)\sim(a, x)$. *If* $r\in H$ *and* $(ar, xr)\sim(br, yr)$, *then* $(a, x)\sim(b, y)$.
iii) $(c, c)\sim(c', c')$, $(c, 0)\sim c', 0)$ *for arbitrary elements* c, c' *of* H.
iv) *If* $(a, x)\sim(b, y)$ *and* $b'a=a'b$ ($\in R$), *then*

$$b'x=a'y\ .$$

PROOF. i) $a^2=aa$ ($\in H$) is a common multiple of a and a. Hence, from $(a, x)\sim(a, x')$, $ax=ax'$ whence $x=x'$. Conversely, if $x=x'$, then $a^2=aa$, $ax=ax'$, and hence,

$$(a, x)\sim(a, x')\ .$$

ii) Since $a(ra)=(ar)a$ ($\in H$) and $a(rx)=(ar)x$,

$$(ra, rx)\sim(a, x)\ .$$

Let $b'ar=a'br$ ($a'\in H$) be a common multiple of ar and br. Then, we have $b'xr=a'yr$ since $(ar, xr)\sim(br, yr)$. Since $r\in H$, we obtain $b'a=a'b$ and $b'x=a'y$, i.e. $(a, x)\sim(b, y)$.
iii) Let c_0 ($\in H$) be a common multiple of c and c', and $c_0=d'c=dc'$. Then $(c, c)\sim(c', c')$. Further, since $d'0=d0=0$,

$$(c, 0)\sim(c', 0)\ .$$

iv) Let c_0 ($\in H$) be a common multiple of a, b. If b_0, a_0 denote the left factors of a, b in c_0 respectively, then we obtain from $(a, x)\sim(b, y)$ and $c_0=b_0a=a_0b$,

$$b_0x=a_0y\ .$$

Since by assumption $b'a=a'b$ the equality $b'x=a'y$ follows from Prop. 1.

The relation $\sim$ defined above is reflexive (see Prop. 2. i)) and symmetric. We shall show that the relation $\sim$ is transitive.

To this end, let $(a_1, x_1)\sim(a_2, x_2)$, $(a_2, x_2)\sim(a_3, x_3)$. Then there exist elements a_1', a_2' and a_2'', a_3'' such that

$$a_2'a_1=a_1'a_2\ (\in H),\ a_2'x_1=a_1'x_2\ ;$$
$$a_3''a_2=a_2''a_3\ (\in H),\ a_3''x_2=a_2''x_3\ .$$

Consider a common multiple

$$b''(a_1'a_2)=b'(a_3''a_2)\in H$$

of $a_1'a_2, a_3''a_2$. Then

$$(b''a_2')a_1=b''a_1'a_2=b'a_3''a_2=(b'a_2'')a_3\ (\in H)\ ,$$

whence $(b''a_2')a_1=(b'a_2'')a_3$ $(\in H)$.

On the other hand, from $(b''a_1')a_2=(b'a_3'')a_2$, we obtain $b''a_1'=b'a_3''$. Hence $(b''a_2')x_1=(b''a_1')x_2=(b'a_3'')x_2=(b'a_2'')x_3$,

$$\text{i.e.}\quad (b''a_2')x_1=(b'a_2'')x_3\,.$$

By $(b''a_2')a_1=(b'a_2'')a_3$, we obtain $(a_1, x_1)\sim(a_3, x_3)$.

3. The subset of $H\times R$ whose elements are all equivalent to a given element of $H\times R$ is called *a class of equivalent elements*. Let S be the set of all classes of equivalent elements of $H\times R$ and φ be the canonical mapping of $H\times R$ onto S by which every element of $H\times R$ is mapped onto the class containing this element. Then we shall define the addition, the subtraction and the multiplication of S as follows:

Let s_1 and s_2 be elements of S and $(a_1, x_1)\in s_1$, $(a_2, x_2)\in s_2$. Then we set:

$$s_1+s_2=\varphi((c, a_2'x_1+a_1'x_2))\qquad\text{(addition)},$$
$$s_1-s_2=\varphi((c, a_2'x_1-a_1'x_2))\qquad\text{(subtraction)},$$

where $c=a_2'a_1=a_1'a_2$ means a common multiple of a_1, a_2 in H.

$$s_1s_2=\varphi((a'a_1, x'x_2))\qquad\text{(multiplication)},$$

where $a'x_1=x'a_2$ with $a'\in H$, $x'\in R$.

These definitions do not depend on the choice of representative from s_1 and s_2, but are uniquely determined by s_1 and s_2. For, let $(a_1, x_1)\sim(b_1, y_1)$, $(a_2, x_2)\sim(b_2, y_2)$, $c_1=a_2'a_1=a_1'a_2\in H$ and $c_2=b_2'b_1=b_1'b_2\in H$. If $c=r_1c_1=r_2c_2\in H$ is a common multiple of c_1, c_2, then

$$(r_1a_2')a_1=(r_1a_1')a_2=(r_2b_2')b_1=(r_2b_1')b_2\ (\in H)\,.\qquad (1).$$

By Prop. 2. ii),

$$(c_1, a_2'x_1+a_1'x_2)\sim(r_1c_1, r_1(a_2'x_1+a_1'x_2))\sim(c, r_1(a_2'x_1+a_1'x_2))\,,$$
$$(c_2, b_2'y_1+b_1'y_2)\sim(c, r_2(b_2'y_1+b_1'y_2))\,.$$

However, from (1), $(a_1, x_1)\sim(b_1, y_1)$ and $(a_2, x_2)\sim(b_2, y_2)$, we have $r_1a_2'x_1=r_2b_2'y_1$, $r_1a_1'x_2=r_2b_1'y_2$, whence $r_1(a_2'x_1+a_1'x_2)=r_2(b_2'y_1+b_1'y_2)$. This proves

$$(c_1, a_2'x_1+a_1'x_2)\sim(c_2, b_2'y_1+b_1'y_2)\,,$$

that is, s_1+s_2 is uniquely determined by s_1, s_2.

Similarly for s_1-s_2.

Again let $(a_1, x_1)\sim(b_1, y_1)$ and $(a_2, x_2)\sim(b_2, y_2)$ and let

$$a'x_1=x'a_2\ (a'\in H)\,,\quad b'y_1=y'b_2\ (b'\in H)\,.\qquad (2).$$

Consider the elements $(a'a_1, x'x_2)$ and $(b'b_1, y'y_2)$. For a common multiple $c=r_1(a'a_1)=r_2(b'b_1)$ $(\in H)$ of $a'a_1, b'b_1$,

$$(a'a_1, x'x_2)\sim(c, r_1x'x_2),$$
$$(b'b_1, y'y_2)\sim(c, r_2y'y_2). \qquad (3).$$

Since $(a_1, x_1)\sim(b_1, y_1)$ and $c=(r_1a')a_1=(r_2b')b_1$, we obtain

$$(r_1a')x_1=(r_2b')y_1 \quad \text{(by Prop. 2. iv)).}$$

Further, by (2), $r_1a'x_1=r_1x'a_2$, $r_2b'y_1=r_2y'b_2$, therefore $(r_1x')a_2=(r_2y')b_2$. From the last relation it follows $r_1x'x_2=r_2y'y_2$ since $(a_2, x_2)\sim(b_2, y_2)$. By (3), we obtain

$$(a'a_1, x'x_2)\sim(b'b_1, y'y_2).$$

This proves that the product s_1s_2 is uniquely determined by s_1 and s_2.

4. The commutative law and the associative law of addition will be easily proved. *Associative law of multiplication:* We set

$$[\varphi((a, x))\varphi((b, y))]\varphi((c, z))=\varphi((b'a, x'y))\varphi((c, z))=\varphi((c''b'a, wz)),$$

where $b'x=x'b$ $(b'\in H)$ and $c''x'y=wc$ $(c''\in H)$. On the other hand, we set

$$\varphi((a, x))[\varphi((b, y))\varphi((c, z))]=\varphi((a, x))\varphi((c'b, y'z))=\varphi((da, x''y'z)),$$

where $c'y=y'c$ $(c'\in H)$ and $dx=x''c'b$ $(d\in H)$. We shall show

$$(c''b'a, wz)\sim(da, x''y'z).$$

To this end, let $fc''b'a=gda\in H$. Then $fc''b'x=gdx$. Hence, $fc''x'b=gx''c'b$ by $b'x=x'b$ and $dx=x''c'b$, whence $fc''x'y=gx''c'y$. By $c''x'y=wc$ and $c'y=y'c$, we obtain from $fc''x'y=gx''c'y : fwc=gx''y'c$, whence $fwz=gx''y'z$. Therefore, if $fc''b'a=gda\in H$, then $fwz=gx''y'z$. This proves $(c''b'a, wz)\sim(da, x''y'z)$.
Distributive law: Let $d=c'b=b'c$ $(\in H)$ and $d'x=x'd$ $(d'\in H)$. Then

$$\begin{aligned}&\varphi((a, x))[\varphi((b, y))+\varphi((c, z))]=\varphi((a, x))\varphi((d, c'y+b'z))\\&=\varphi((d'a, x'(c'y+b'z)))=\varphi((d'a, x'c'y))+\varphi((d'a, x'b'z))\\&=\varphi((a, x))\varphi((b, y))+\varphi((a, x))\varphi((c, z)),\end{aligned}$$

since $d'x=x'c'b=x'b'c$. Further, let $c_1'x=x'c$ $(c_1'\in H)$, $c_2'y=y'c$ $(c_2'\in H)$ and $d=d_1c_1'a=d_2c_2'b\in H$. Then

$$\begin{aligned}&[\varphi((a, x))+\varphi((b, y))]\varphi((c, z))=\varphi((d, d_1c_1'x+d_2c_2'y))\varphi((c, z))\\&=\varphi((d, d_1x'c+d_2y'c)\varphi((c, z))=\varphi((d, (d_1x'+d_2y')z))\\&=\varphi((c_1'a, x'z))+\varphi((c_2'b, y'z))=\varphi((a, x))\varphi((c, z))+\varphi((b, y))\varphi((c, z)).\end{aligned}$$

It is clear that S is an associative ring. Further we can easily prove that, for any element $a \in H$, $\varphi((a, a))$ is the unit element of S and $\varphi((a, 0))$ is the zero element of S. Now we fix an element c of H. Then the mapping $x \to x^* = \varphi((c, cx))$ maps R isomorphically onto the subring $R^* = \{\varphi((c, cx))\,;\, x \in R\}$ of S, and H is also isomorphic to $H^* = \{\varphi((c, ca))\,;\, a \in H\}$. If $\varphi((c, ca)) \in H^*$, then $\varphi((ca, c))\varphi((c, ca)) = \varphi((ca, ca))$ and $\varphi((c, ca))\varphi((ca, c)) = \varphi((c, c))$, that is, any element $\varphi((c, ca))$ of H^* has $\varphi((ca, c))$ as its inverse in S. Further, for any element $\varphi((a, ax)) \in S$, $\varphi((c, ca))\varphi((a, x)) = \varphi((c, cx)) \in R^*$.

As usual we will identify $\varphi((a, ax))$ with x. Then R and H may be considered as subrings of S and, in this sense, S satisfies all conditions of left quotient ring of R with the denominator system H.

5. Now let T be any left quotient ring of R with the denominator system H and t an element of T. Then there exists an element $a \in H$ such that $at \in R$. We set $f(t) = \varphi((a, at))$. Then, $\varphi((a, at))$ does not depend on the choice of $a \in H$. Therefore $f(t)$ is uniquely determined by t.

Conversely, let $s = \varphi((a, x))$ be an element of S. We set $t = a^{-1}x$, where a^{-1}, x are considered as elements of T. Then $f(t) = \varphi((a, x)) = s$. It can be easily seen that $t = a^{-1}x$ is uniquely determined by s.

Further we can easily show that the mapping f is a ring isomorphism of T onto S.

§ 2. Topological part.

Let R be a topological ring (by definition, R is a Hausdorff space) and let H be a multiplicative semigroup of non-zero divisors of R. Further, we assume that Condition (A) of § 1 is satisfied. Accordingly, there exists the (algebraic) left quotient ring S of R with the denominator system H. We introduce in $H \times R$ the equivalence relation $\sim$ as in § 1, and φ shall denote the canonical mapping of $H \times R$ onto S. Now we fix an element $c \in H$, and consider the set R^* of elements $x^* = \varphi((c, cx))$ where x runs over all elements of R (x^* is uniquely determined by x but does not depend on the choice of $c \in H$). R^* is a subring of S algebraically isomorphic to R. The product set $H \times R$ is a topological space as a subspace of $R \times R$. As well known, the totality of $U \times V$, where U, V are open neighbourhoods in H and R respectively, forms a basis of the open sets in $H \times R$. $U \times V$ is called *an open neighbourhood* in $H \times R$, and if $(a, x) \in U \times V$, $U \times V$ is called *an open neighbourhood of* (a, x). An open set in $H \times R$ is a union of finite or infinite number of open neighbourhoods in $H \times R$.

DEFINITION 2. *We say the relation $\sim$ in $H \times R$ is open, if, for any open subset W of $H \times R$, $\varphi^{-1}(\varphi(W))$ is always open.*

We assume now that the relation $\sim$ is open. Then, for any open neighbourhood

$U_a \times V_x$ of (a, x), $\varphi^{-1}(\varphi(U_a \times V_x))$ is open. Hence, any element (b, y) equivalent to (a, x) is contained in $\varphi^{-1}(\varphi(U_a \times V_x))$, and there exists an open neighbourhood $U_b \times V_y$ of (b, y) contained in $\varphi^{-1}(\varphi(U_a \times V_x))$. Since $\varphi(U_b \times V_y) \subset \varphi(U_a \times V_x)$, every element of $U_b \times V_y$ is equivalent to some element of $U_a \times V_x$. In this case, we say that *$U_b \times V_y$ is equivalent to a subset of $U_a \times V_x$.*

Conversely, let (b, y) be an arbitrary element equivalent to (a, x) and $U_a \times V_x$ an open neighbourhood of (a, x). If there exists a suitable open neighbourhood $U_b \times V_y$ of (b, y) which is equivalent to a subset of $U_a \times V_x$, then for any open neighbourhood $U \times V$, $\varphi^{-1}(\varphi(U \times V))$ is open. For, let (b, y) be an element of $\varphi^{-1}(\varphi(U \times V))$. Then, $\varphi((b, y)) \in \varphi(U \times V)$; therefore, there is an element $(a, x) \in U \times V$ which is equivalent to (b, y). Since $U \times V$ can be considered as an open neighbourhood of (a, x), there exists an open neighbourhood $U_b \times V_y$ of (b, y) which is equivalent to a subset of $U \times V$. This shows $U_b \times V_y \subset \varphi^{-1}(\varphi(U \times V))$, and so $\varphi^{-1}(\varphi(U \times V))$ is open.

Let $W = \bigcup_{\lambda \in \Lambda} U_\lambda \times V_\lambda$ be an open set of $H \times R$ where $U_\lambda \times V_\lambda$ means an open neighbourhood of $H \times R$. Then $\varphi^{-1}(\varphi(W)) = \bigcup_{\lambda \in \Lambda} \varphi^{-1}(\varphi(U_\lambda \times V_\lambda))$. Accordingly, the relation $\sim$ is open if and only if $\varphi^{-1}(\varphi(U \times V))$ is open where $U \times V$ means an arbitrary open neighbourhood of $H \times R$.

Therefore we have proved the following proposition:

PROPOSITION 3. *The relation $\sim$ is open if and only if, for any two equivalent elements (a, x), (b, y) and for any open neighbourhood $U_a \times V_x$ of (a, x) ($U_b' \times V_y'$ of (b, y)), there exists a suitable open neighbourhood $U_b \times V_y$ of (b, y) ($U_a' \times V_x'$ of (a, x)) which is equivalent to a subset of $U_a \times V_x$ ($U_b' \times V_y'$).*

Now let $(a, x) \sim (b, y)$ and $a = cb$. Then $x = cy$ by Prop. 2. ii) and i). To any open neighbourhood $U_a \times V_x$ of (a, x), there exist a neighbourhood U_b of b in H and a neighbourhood V_y of y in R such that

$$cU_b \subset U_a, \quad cV_y \subset V_x. \tag{1}$$

If $(b', y') \in U_b \times V_y$, there exists by (1) elements $a' \in U_a$ and $x' \in V_x$ such that

$$cb' = a' \ (\in H), \quad cy' = x'.$$

Therefore, $(b', y') \sim (cb', cy') = (a', x') \in U_a \times V_x$. Accordingly, $U_b \times V_y$ is equivalent to a subset of $U_a \times V_x = U_{cb} \times V_{cy}$.

In case $a = cb$, $x = cy$, the converse also holds by Prop. 3, if the relation $\sim$ is open, that is, for any open neighbourhood $U_b \times V_y$, there has to exist an open neighbourhood $U_a \times V_x = U_{cb} \times V_{cy}$ of (a, x) equivalent to a subset of $U_b \times V_y$.

LEMMA 1. *The relation $\sim$ is open if and only if the following condition holds: for any $(a, x) \in H \times R$ and any c with $ca \in H$, and for any open neighbour-*

hood $U_a \times V_x$ *of* (a, x), *there exists an open neighbourhood* $U_{ca} \times V_{cx}$ *of* (ca, cx) *which is equivalent to a subset of* $U_a \times V_x$.

PROOF. We have proved that this condition is necessary when the relation $\sim$ is open. To prove this converse, let $(a, x) \sim (b, y)$. Then it is sufficient to show that, to any open neighbourhood $U_a \times V_x$ of (a, x), there exists an open neighbourhood $U_b \times V_y$ of (b, y) which is equivalent to a subset of $U_a \times V_x$. Consider a common multiple $b'a = a'b$ $(\in H)$ of a and b. Then

$$(a, x) \sim (b'a, b'x) = (a'b, a'y) \sim (b, y) .$$

Therefore, by assumption, there exists an open neighbourhood $U_{b'a} \times V_{b'x}$ of $(b'a, b'x) = (a'b, a'y)$ which is equivalent to a subset of $U_a \times V_x$. Since $U_{b'a} \times V_{b'x}$ is an open neighbourhood of $(a'b, a'y)$ and $(a'b, a'y) \sim (b, y)$, there exists an open neighbourhood $U_b \times V_y$ which is equivalent to a subset of $U_{b'a} \times V_{b'x}$. Accordingly, $U_b \times V_y$ is equivalent to some subset of $U_a \times V_x$.

When R satisfies the first axiom of countability, the condition of Lemma 1 can be stated as follows: For any convergent sequence $(b_n, y_n) \to (ca, cx)$, there exists a sequence $(a_n, x_n) \to (a, x)$ such that $(b_n, y_n) \sim (a_n, x_n)$ for all n.

COROLLARY. *If, for any open set* V *of* R *and* $c \in H$, cV *is always an open set, then the relation* $\sim$ *is open.*

PROOF. If $U_a = H \cap V_a$, where V_a is an open neighbourhood of a in R, we can set $U_{ca} = H \cap cV_a$, $V_{cx} = cV_x$ (V_x is an open neighbourhood of x in R) in Lemma 1.

DEFINITION 3. *Let* S *be the algebraic left quotient ring of a topological ring* R *with the denominator system* H. *If the following conditions are satisfied, then* R *is called embeddable* (or *weakly embeddable*) *in* S:

1) S *is a topological ring with respect to some topology and every element of* H' *has the continuous inverse in* S,

2) *the mapping* $x \to x'$ *of* R *onto* $R' \subset S$ *is a topological* (or *continuous*) *isomorphism, where* R' *is considered as the relative subspace of* S.

LEMMA 2. *Assume that* R *is embeddable in the left quotient ring* S *of* R. *Then, the relation* $\sim$ *is open, if* $(H')^{-1} \subset \bar{H}'$, *where* $\bar{H}'$ *means the closure of* H' *in* S.

PROOF. Since R is embeddable in S, R is homemorphic to the subring R' of S. Therefore we will identify R with R' and H with H'.

Let $U_a \times V_x$ be an open neighbourhood of (a, x). Then there exists a neighbourhood W of 0 in S such that $(a + W) \cap H \subset U_a$ and $(x + W) \cap R \subset V_x$. For an element $c \in H$, we select a neighbourhood W_1 of 0 in S satisfying

$$c^{-1} W_1 + W_1 ca + W_1 W_1 \subset W \text{ and } c^{-1} W_1 + W_1 cx + W_1 W_1 \subset W ,$$

and we set $U_{ca}=(ca+W_1)\cap H$, $V_{cx}=(cx+W_1)\cap R$. By the assumption, $U=H\cap(c^{-1}+W)$ is not empty. If $(b,y)\in U_{ca}\times V_{cx}$ and $d\in U$, then $(b,y)\sim(db,dy)$. Further (db,dy) can be written in the form

$$\begin{aligned}&((c^{-1}+w)(ca+w_1),(c^{-1}+w)(cx+w_2))\\&=(a+c^{-1}w_1+wca+ww_1,x+c^{-1}w_2+wcx+ww_2)\\&=(a+u_1,x+u_2),\end{aligned}$$

where $w,w_1,w_2\in W_1$ and $u_1,u_2\in W$. On the other hand, d is in H, and so

$$a+u_1\in(a+W)\cap H\subset U_a,\ x+u_2\in(x+W)\cap R\subset V_x.$$

This proves that $U_{ca}\times V_{cx}$ is equivalent to a subset of $U_a\times V_x$. Therefore the relation $\sim$ is open by Lemma 1.

COROLLARY. *If R has no zero-divisors and densely embeddable in S, the relation $\sim$ is open.*

PROOF. We set $H=R\backslash\{0\}$. Then $H^a\supset S\backslash\{0\}$, and hence $(H^a)^{-1}\subset H^a$.

Now we can introduce in the algebraic left quotient ring S of R a topology such that φ becomes a continuous mapping of $H\times R$ onto the topological space S. Let T_1 and T_2 be two such topologies on S. If any open set of T_1 is always an open set of T_2, we say that the topology T_1 is weaker than that of T_2 (or T_2 is stronger than T_1) and denote by $T_1>T_2$. We consider the set of all topologies of S with respect to which φ is continuous. Then this set is partially ordered with respect to the relation "$>$". Let $T_\alpha>T_\beta>\cdots$ be a linearly ordered chain in this partially ordered set. Then φ is also continuous with the topology T of S, whose base of open sets is $\{\bigcap_{i=1}^{n}V_{\alpha_i};\ V_{\alpha_i}$ is an open set of T_{α_i} and n is finite$\}$ (the intersection topology of $\{T_\alpha\}$). And for any T_α in this chain, T is stronger than T_α. Hence, by Zorn's lemma, there exists in this partially ordered set a strongest topology. Further, a strongest one is unique. For, if T and T' are two such strongest ones, then φ is continuous with respect to the intersection topology of them and it is stronger than T and T'. Therefore it is equal to one of them, say, to T. Then $T>T'$ and the maximality of T' imples that $T=T'$.

LEMMA 3. *Let S be endowed with the strongest topology with respect to which φ is continuous. Then φ becomes an open mapping of $H\times R$ onto S if and only if the relation $\sim$ is open.*

PROOF. Assume that the relation $\sim$ is open. If V is an open set in a topological space S and φ is continuous with respect to the topology of S, then $U=\varphi^{-1}(V)$ is an open set in $H\times R$. Since $V=\varphi(\varphi^{-1}(V))=\varphi(U)$, every open set in S is of the form $\varphi(U)$, where U is an open set in $H\times R$. We define in S a topology whose base of open sets is $\{\varphi(W)\}$, where W runs over all open subsets of $H\times R$. This topology of S is clearly stronger than any other topology of S with respect to which φ is continuous. Then φ is an open mapping of $H\times R$ onto S with this strongest topology and further φ is continuous since by assumption $\varphi^{-1}(\varphi(W))$ is open.

Conversely, if φ is open and continuous, then $\varphi^{-1}(\varphi(W))$ is an open set of $H\times R$ for any open set W of $H\times R$. And the relation $\sim$ is open.

DEFINITION 4. *An element* $(a, x)\in H\times R$ *is said to be right transitive at* $(b, y)\in H\times R$, *if*

$$(a, x)\sim(ba, ya).$$

PROPOSITION 4. 1). *(a, x) is right transitive at (b, y) if and only if $ya=bx$.* 2). *Let (a, x) be right transitive at (b, y). Then (a, x) is right transitive at (b', y') if and only if $(b, y)\sim(b', y')$.*

PROOF. 1) If $ya=bx$ then $(a, x)\sim(ba, bx)=(ba, ya)$.

Conversely, let $(a, x)\sim(ba, ya)$. Since $(a, x)\sim(ba, bx)$, we obtain $(ba, ya)\sim(ba, bx)$, whence $bx=ya$ by Prop. 2, i).

2) By assumption, $(a, x)\sim(ba, ya)$. If (a, x) is right transitive at (b', y'), then $(a, x)\sim(b'a, y'a)$. Hence $(ba, ya)\sim(b'a, y'a)$, whence $(b, y)\sim(b', y')$ by Prop. 2, ii).

Conversely, let $(b, y)\sim(b', y')$. Then $(ba, ya)\sim(b'a, y'a)$. Since $(a, x)\sim(ba, ya)$ and $(ba, ya)\sim(b'a, y'a)$, we obtain $(a, x)\sim(b'a, y'a)$. By definition, (a, x) is right transitive at (b', y').

REMARK. Given an element (a, x) of $H\times R$, there exist by (A) elements $b\in H$ and $y\in R$ such that $ya=bx$. By Prop. 4. 1), (a, x) is right transitive at (b, y), that is, every element of $H\times R$ is right transitive at some element of $H\times R$.

DEFINITION 5. *Let (a, x) and (b, y) be elements of $H\times R$. Then (a, x) is said to be continuously right transitive at (b, y), if, to any open neighbourhood $U_b\times V_y$ of (b, y), there exists an open neighbourhood $U_a\times V_x$ of (a, x) such that every element of $U_a\times V_x$ is right transitive at some element of $U_b\times V_y$.*

PROPOSITION 5. *Let (a, x) be continuously right transitive at (b_1, y_1) and (a, x) right transitive at (b_2, y_2). Then (a, x) is also continuously right transitive at (b_2, y_2), if the relation $\sim$ is open.*

PROOF. By assumption and Prop. 4. 2), $(b_1, y_1)\sim(b_2, y_2)$. Since the relation $\sim$ is open, there exists to any open neighbourhood $U_{b_2}\times V_{y_2}$ of (b_2, y_2) an open neighbourhood $U_{b_1}\times V_{y_1}$ of (b_1, y_1) such that $U_{b_1}\times V_{y_1}$ is equivalent to a subset of $U_{b_2}\times V_{y_2}$.

By assumption there exists an open neighbourhood $U_a\times V_x$ of (a, x) such that every element of $U_a\times V_x$ is right transitive at some element of $U_{b_1}\times V_{y_1}$. By Prop. 4. 2), every element of $U_a\times V_x$ is right transitive at some element of

$U_{b_2}\times V_{y_2}$.

DEFINITION 6. *$H\times R$ is called continuously right transitive, if every element of $H\times R$ is continuously right transitive at some element of $H\times R$.*

LEMMA 4. *$H\times R$ is continuously right transitive,*

1). *if R is commutative, or*

2). *if φ is a continuous open mapping of $H\times R$ onto a topologized left quotient ring S of R with denominator system H, where S has the continuous inverse for every element of $H^\sharp$.*

PROOF. 1). Since R is commutative, $(a,x)\sim(a^2,ax)\sim(a^2,xa)$, that is, (a,x) is right transitive at (a,x). Therefore, to any element (a,x) of $H\times R$, we can choose as $b=a$ and $y=x$, so that $U_a=U_b$ and $V_x=V_y$ satisfy the condition of continuous transitivity.

2). We denote by $R^\sharp$ the set of all elements $x^\sharp=\varphi((c,cx))$ where x runs over all elements of the given ring R and c means a fixed element of H.

Let $x^\sharp=\varphi((c,cx))$ be an element of $R^\sharp$ and $U_c\times V_{cx}$ an open neighbourhood of $(c,cx)\in H\times R$. Since φ is an open mapping of $H\times R$ onto S, $\varphi(U_c\times V_{cx})=W_{x^\sharp}$ is an open neighbourhood of $x^\sharp$ in S. Since there exists an open neighbourhood V_x of $x\in R$ such that $cV_x\subset V_{cx}$, it is clear that

$$(V_x)^\sharp=\{\varphi((c,cy))\,;\ y\in V_x\}\subset W_{x^\sharp}\,.$$

Let (a,x) be an element of $H\times R$ and $bx=ya$, where $b\in H$, $y\in R$ (these elements b,y exist by Condition (A)). Then we can easily prove that

$$b^\sharp x^\sharp=\varphi((c,cb))\,\varphi((c,cx))=(bx)^\sharp \ \text{and}\ y^\sharp a^\sharp=(ya)^\sharp\,.$$

From $bx=ya$, we obtain $b^\sharp x^\sharp=y^\sharp a^\sharp$.

If $U_b\times V_y$ is an open neighbourhood of (b,y), $\varphi(U_b\times V_y)$ is an open neighbourhood of $\varphi((b,y))=\varphi((cb,c))\,\varphi((c,cy))=\varphi((c,cb))^{-1}\varphi((c,cy))=(b^\sharp)^{-1}y^\sharp=x^\sharp(a^\sharp)^{-1}$. By the contiunity of multiplication and inverse operation in S, there exist open neighbourhoods $W_{x^\sharp}$ of $x^\sharp$ and $W_{a^\sharp}$ of $a^\sharp$ such that

$$W_{x^\sharp}(W_{a^\sharp})^{-1}\subset\varphi(U_b\times V_y)\,.$$

Since φ is a continuous mapping of $H\times R$ onto S, we can choose open neighbourhoods V_a $(\subset R)$ of a and V_x $(\subset R)$ of x such that $\varphi(\{c\}\times cV_a)\subset W_{a^\sharp}$, $\varphi(\{c\}\times cV_x)\subset W_{x^\sharp}$.

If we set $U_a=V_a\cap H$, then every element of the open neighbourhood $U_a\times V_x$ of (a,x) is right transitive at some element of $U_b\times V_y$. For, let (a',x') be any element of $U_a\times V_x$. Then $x'^\sharp(a'^\sharp)^{-1}\in W_{x^\sharp}(W_{a^\sharp})^{-1}\subset\varphi(U_b\times V_y)$. Further $x'^\sharp(a'^\sharp)^{-1}=\varphi((c,cx'))\,\varphi((ca',c))=\varphi((hc,rc))$, if we choose elements $h\in H$, $r\in R$ such that $hcx'=rca'$. Therefore there exists in $U_b\times V_y$ an element (b',y') which satisfies $(b',y')\sim(hc,rc)$. Then

$$(b'a',y'a')\sim(hca',rca')\sim(hca',hcx') \quad (\text{by } hcx'=rca')$$
$$\sim(a',x')\,.$$

This proves that every element of $U_a\times V_x$ is right transitive at some element of $U_b\times V_y$.

LEMMA 5. *If $H\times R$ is continuously right transitive, then every class of equivalent elements in $H\times R$ is a closed set in $H\times R$.*

PROOF. We denote by $[(b,y)]$ the class containing (b,y). Let (a,x) be an element of the closure $\overline{[(b,y)]}$ of $[(b,y)]$ in $H\times R$. Then, for any open neighbourhood $U_a\times V_x$ of (a,x),

$$(U_a\times V_x)\cap[(b,y)]\neq\phi \quad (\text{the empty set})\,.$$

Suppose now $(a,x)\bar{\in}[(b,y)]$. Then

$$a'y\neq b'x$$

for any common multiple $a'b=b'a$ with $a'\in H$ and $b'\in R$. Since R is a Hausdorff space, there exist in R open neighbourhoods $V_{a'y}$ of $a'y$ and $V_{b'x}$ of $b'x$ such that

$$V_{a'y}\cap V_{b'x}=\phi\,.$$

Further, there exist in R open neighbourhoods $V_{a'}$, $V_{b'}$, V_x and V_y of a',b',x and y respectively such that

$$V_{a'}V_y\subset V_{a'y}\,,\quad V_{b'}V_x\subset V_{b'x}\,.$$

Observing $a'b=b'a$, we obtain $(a,b)\sim(a'a,b'a)$, that is, (a,b) is right transitive at (a',b'). Since R is continuously right transitive, there exists to an open neighbourhood $U_{a'}\times V_{b'}$ of (a',b') with $U_{a'}=V_{a'}\cap H$ an open neighbourhood $U_a^{(0)}\times V_b^{(0)}$ of (a,b) such that every element of $U_a^{(0)}\times V_b^{(0)}$ is right transitive at some element of $U_{a'}\times V_{b'}$. Since $U_a^{(0)}\times V_x$ is an open neighbourhood of (a,x), $(U_a^{(0)}\times V_x)\cap[(b,y)]$ contains an element, say, (a_1,x_1). Then

$$(a_1,x_1)\sim(b,y)\,.$$

As $(a_1,b)\in U_a^{(0)}\times V_b^{(0)}$, there exists in $U_{a'}\times V_{b'}$ an element (a_0',b_0') such that

$$(a_1,b)\sim(a_0'a_1,b_0'a_1)\,,$$
$$\text{i.e.}\quad b_0'a_1=a_0'b \quad (a_0'\in H)\,.$$

Since $(a_1,x_1)\sim(b,y)$ and $b_0'a_1=a_0'b$, we obtain

$$b_0'x_1=a_0'y\,.$$

Hence $b_0'x_1=a_0'y\in V_{a'}V_y\cap V_{b'}V_x\subset V_{a'y}\cap V_{b'x}=\phi$. This gives a contradiction. Therefore $(a, x) \in \overline{[(b, y)]}$, if (a, x) is a limiting element of $[(b, y)]$.

By Lemma 4. 1), $H \times R$ is continuously right transitive if R is commutative. Therefore, we obtain the following Corollary:

COROLLARY. *In case R is commutative, every class of equivalent elements in $H \times R$ is a closed set.*

THEOREM 2. *Let R be a topological ring having the algebraic left quotient ring S with denominator system H and let H be an open subset of R. Then, if the relation $\sim$ is open and $H \times R$ is continuously right transitive, S becomes a topological ring under the strongest topology with respect to which φ is continuous, and every element of H^* has the continuous inverse in S. Futher R is weakly embeddable in S.*

PROOF. Since H is an open subset of R, every open set U in H is also open in R. We shall show the continuity of operations in S.

Continuity of subtraction: Let $r=\varphi((a, x))$, $s=\varphi((b, y))$ and $t=r-s$. Then $t=\varphi((c, b'x-a'y))$, where $c=b'a=a'b$ $(a' \in H)$. To any neighbourhood W_t of t in S, there exists in $H \times R$ a suitable neighbourhood $U_c \times V_{b'x-a'y}$ of $(c, b'x-a'y)$ satisfying $\varphi(U_c \times V_{b'x-a'y}) \subset W_t$. Take neighbourhoods U_a', U_b', $U_{a'}$, $U_{b'}$, V_x and V_y of a, b, a', b', x and y respectively such that $U_{b'} U_a' \subset U_c$, $U_{a'} U_b' \subset U_c$ and $U_{b'} V_x - U_{a'} V_y \subset V_{b'x-a'y}$. For $U_{a'}$ and $U_{b'}$, select U_a'' and U_b'' such that every element of $U_a'' \times U_b''$ is right transitive at some element of $U_{a'} \times U_{b'}$ and let $U_a = U_a' \cap U_a''$ and $U_b = U_b' \cap U_b''$. Then $W_r=\varphi(U_a \times V_x)$ and $W_s=\varphi(U_b \times V_y)$ are neighbourhoods of r and s respectively and $W_r - W_s \subset W_t$.

Continuity of multiplication: Let $t=rs$, then $t=\varphi((b'a, x'y))$, where $b'x=x'b$ $(b' \in H)$. Let $\varphi(U_{b'a} \times V_{x'y}) \subset W_t$ and let $U_{b'}$, U_a, $V_{x'}$ and V_y be neighbourhoods of b', a, x' and y respectively such that $U_{b'} U_a \subset U_{b'a}$ and $V_{x'} V_y \subset V_{x'y}$. For $U_{b'} \times V_{x'}$, choose $U_b \times V_x$ such that every element of $U_b \times V_x$ is right transitive at some element of $U_{b'} \times V_{x'}$ and let $W_r=\varphi(U_a \times V_y)$ and $W_s=\varphi(U_b \times V_y)$. It is easily seen that

$$W_r W_s \subset \varphi(U_{b'a} \times V_{x'y}) \subset W_t .$$

Continuity of inverse: Let $r=\varphi((a, b))$ and $s=r^{-1}=\varphi((b, a))$. Further, let W_s be a neighbourhood of s in S and $W_s \supset \varphi(U_b \times U_a)$, where U_a and U_b are neighbourhoods of a and b in H (and so in R). Then $W_r=\varphi(U_a \times U_b)$ is a neighbourhood of r and $W_r^{-1} \subset W_s$.

Since equivalence classes are closed sets (Lemma 5) and φ is an open mapping (Lemma 3), every element of S is a closed set in S. By the continuity of operations the topology of S is Hausdorff.

Now, weak embeddability is equivalent to the continuity of the mapping $x \to x^*$ of R into S. Let $W_{x^*}=\varphi(U_c \times V_{cx})$ be a neighbourhood of x^* in S and V_x a neighbourhood of x in R such that $cV_x\subset V_{cx}$, then

$$(V_x)^{\natural}=\{\varphi((c, cy));\ y\in V_x\}\subset W_{x^{\natural}},$$

that is, the mapping $x\to x^{\natural}$ is continuous.

LEMMA 6. *If R is weakly embeddable in S for a suitable topology of S, φ is continuous on $H\times R$.*

PROOF. Let $r=\varphi((a, x))\in S$, $a^{\natural}=\varphi((c, ca))$ and $x^{\natural}=\varphi((c, cx))$. Then $r=(a^{\natural})^{-1}x^{\natural}$. Therefore, for a neighbourhood W_r of r, there exist in S neighbourhoods $W_{a^{\natural}}$ and $W_{x^{\natural}}$ of $a^{\natural}$ and $x^{\natural}$ respectively such that $W_{a^{\natural}}^{-1}W_{x^{\natural}}\subset W_r$. Then, there exist U_a and V_x satisfying $\varphi(\{c\}\times cU_a)\subset W_{a^{\natural}}$ and $\varphi(\{c\}\times cV_x)\subset W_{x^{\natural}}$. Since $U_a\times V_x$ is an open neighbourhood of (a, x) in $H\times R$ and

$$\varphi(U_a\times V_x)=\varphi(\{c\}\times cU_a)^{-1}\varphi(\{c\}\times cV_x)\subset(W_{a^{\natural}})^{-1}W_{x^{\natural}}\subset W_r,$$

φ is continuous.

THEOREM 3. *Let T_0 be the strongest topology of S (the algebraic left quotient ring of R with denominator system H), under which φ is continuous, and T some topology of S. Assume that S becomes the topological ring P_0, P with respect to T_0, T respectively and that P_0, P have the continues inverse for every element $a^{\natural}\in H^{\natural}$ respectively. Then, if R is weakly embeddable in P_0 and embeddable in P, R is also embeddable in P_0.*

PROOF. Since φ is continuous under the topology T of P (Lemma 6), the topology T_0 is stronger than T. Therefore the topology of $R^{\natural}$ induced by T_0 is stronger than that of $R^{\natural}$ induced by T. The latter is equivalent to the original topology of R (by definition of embeddability). On the other hand, since R is (in algebraic sense) isomorphically and continuously mapped onto the topological ring $R^{\natural}$ with the induced topology of P_0, the topology of $R^{\natural}$ is not stronger than that of R. And so $R^{\natural}$ is also homeomorphic to R, which proves that R is embeddable in P_0.

THEOREM 4. *Let R be embeddable in S, where S is the topological ring with respect to the topology T_0 in Theorem 3. Further, let R' be a subring of S containing $R^{\natural}$ where the topology of R' is the induced one of S and let H' be an open subsemigroup of R' consisting of regular elements in S and containing $H^{\natural}$. Then, R' has the algebraic left quotient ring S' with denominator system H' and $S=S'$ in the algebraic sense. Further, let φ' be the canonical mapping of $H'\times R'$ onto S' and let S' be endowed with the strongest topology under which φ' is continuous. Then S' is homeomorphic to S, that is, we can identify S' with S.*

PROOF. It is easily shown that S is the algebraic left quotient ring of R' with denominator system H'. Condition 1) of Definition 1 is evident. Since

H' is a subset of regular elements in S, Condition 2) is also clear. Condition 3) follows from $H^s \subset H'$.

Since H' and R' are subsets with the induced topology of $S, \varphi' : H' \times R' \to S$ (the topology of S is T_0) is continuous. The topology T_0 of S is the strongest one under which φ' is continuous. Otherwise, since there exists a topology T_1 of S stronger than T_0, and $\varphi'|H^s \times R^s$ (restriction of φ' on $H^s \times R^s$) induces φ of $H \times R$, T_1 is a topology of S under which φ is continuous. This contradicts the fact that S is endowed with the strongest topology under which φ is continuous.

THEOREM 5. *If R is algebraically equal to its left quotient ring S with denominator system H and H is open in R, and if $H \times R$ is continuously right transitive, then the topology of R can be weakened so that it becomes a topological ring with the continuous inverse for every element of H.*

PROOF. Let $c \in H$ and V be an open set in R, then, as the inverse image of V by the continuous mapping $x \to c^{-1}x$, $cV = \{x ; c^{-1}x \in V\}$ is an open set. Therefore the relation $\sim$ is open by the Corollary of Lemma 1. By Theorem 2, S becomes a topological ring and every element of H^s has the continuous inverse. Since R is weakly embeddable in S, this topology of S gives on R^s $(=S)$ the one which is not stronger than that of R.

THEOREM 6. *Let R be a topological ring and H its open subset which forms a multiplicative semigroup. If every element of H has the continuous inverse in R and R is equal to its left quotient ring S of R with denominator system H, then $R=S$ as topological rings, where the topology of S is the strongest one under which φ is continuous.*

PROOF. By the proof of Theorm 5, the relation $\sim$ is open. Further $\varphi : H \times R \to S = R$ is open and continuous. Therefore $H \times R$ is continuously right transitive by Lemma 4. 2). By Theorm 2, S becomes a topological ring with continuous inverses for elements in H^s. On the other hand, R is embeddable in R itself with the original topology. Hence R is embeddable in S by Theorem 3, which proves $R=S$ as topological rings.

In the above discussion, if the relation $\sim$ is not open, then we can only say that the topology of R^s is not stronger than that of R (Bourbaki [2] p. 56, Ex. 8). However there exists a topological ring in which the relation $\sim$ is not open ([5] p. 809).

In the remainder of this paper, we shall discuss a special type of embedding. We need to refer to the circle composition and Q-ring of Jacobson. For any elements x, y in R, we define $x \circ y = x + y - xy$ (the circle composition). If $x \circ y = 0$, then x (y) is called a left (right) quasi-inverse of y (x). And if x is right and left quasi-inverse of y, x is called the quasi-inverse of y and y is called a quasi-regular element. If the set of all quasi-regular elements in R forms an open set in R, R is called a Q-ring. It is well known (cf. Lemma 4. of [6]) that, when R has the identity 1, the inverse mapping $x \to x^{-1}$ is continuous in R if and only if R is a Q-ring and the quasi-inverse mapping $x \to x^{(0)}$ ($x^{(0)}$ means the quasi-inverse of x) is continuous at 0.

DEFINITION 5. *A topological ring R is openly embeddable in the left quotient ring S of R if R is embeddable in the topological ring S endowed with a suitable topology and R^{ϕ} is an open subset of S.*

THEOREM 7 (Warner [8]). *Let R be a non-discrete topological ring without proper zero divisor and $H=R^*$ (the set of all non-zero divisors in R). Then R is openly embeddable in S if and only if R satisfies the following conditions:* 1) *for any neighbourhood V of 0 and $a \in R^*$, aV and Va are also neighbourhoods of 0 in R,* 2) *R is a Q-ring with continuous quasi-inverses.*

PROOF. If R is openly embeddable in S, S is a topological division ring with continuous inverses. Therefore S is a Q-ring with continuous quasi-inverses and, for any open set V in S and $a \in S^*$, aV and Va are also open. Since R^{ϕ} is open in S, these facts also hold in R. And so the conditions are necessary.

Assume now that the conditions 1), 2) are satisfied. Then we shall prove that R is openly embeddable in the topological ring S with the strongest topology under which φ is continuous. For any $a, b \in R^*$, Ra and Rb are neighbourhoods of 0 in R. Since R is not discrete, $Ra \cap Rb \neq \{0\}$. Accordingly there exist in R^* b' and a' such that $b'a=a'b$. Therefore, by Theorm 1, R has the algebraic left quotient ring S with denominator system R^*. By 1) it is easily seen that the relation $\sim$ is open. Now we shall prove that R is weakly embeddable in the topological division ring S whose topology is the strongest one under which the mapping φ of $R^* \times R$ onto S is continuous. To this end, it suffices to show (see Theorem 2) that $R^* \times R$ is continuously right transitive. Let (a, b) be right transitive at (a', b') and let $V_{a'}$ and $V_{b'}$ be neighbourhoods of a' and b' respectively. By suitable choice of open neighbourhoods V_1, V_2 of 0 there hold:

$$V_{b'} \supset b' - V_1 b' , \qquad V_{a'} \supset a' - V_2 a' .$$

Since R is a Q-ring with continuous quasi-inverses, there exists a neighbourhood V_2' of 0 such that $(V_2')^{(0)} \subset V_2$ (here $(V_2')^{(0)}$ means the set of quasi-inverses of all elements of V_2'). Then $V_2' \circ V_1$ is also an open neighbourhood of 0. Consequently, by the continuity of quasi- inverses, there exist open neighbourhoods U_1 and U_2 of 0 satisfying $U_2 \circ U_1^{(0)} \subset V_2' \circ V_1$. Further we select neighbourhoods U_1' and U_2' of 0 such that $U_2 a' \supset a' U_2'$ and $U_1 b' \supset b' U_1'$, and let

$$U_a = a - U_1' a , \qquad U_b = b - U_2' b .$$

Then, every element of $U_a\times U_b$ is right transitive at some element of $V_{a'}\times V_{b'}$. For, let $(a_1, b_1)\in U_a\times U_b$, then $a_1=a-u_1'a$ $(u_1'\in U_1)$ and $b_1=b-u_2'b$ $(u_2'\in U_2)$. Since $U_2a'\supset a'U_2'$ and $U_1b'\supset b'U_1'$, there exist $u_1\in U_1$ and $u_2\in U_2$ such that $a'u_2'=u_2a'$ and $b'u_1'=u_1b'$. Further, from $u_2\circ u_1^{(0)}\in U_2\circ U_1^{(0)}\subset V_2'\circ V_1$ ($u_1^{(0)}$ means the quasi-inverse of u_1), we can find $v_1\in V_1$ and $v_2'\in V_2'$ such that $u_2\circ u_1^{(0)}=v_2'\circ v_1$. Let v_2 be the quasi-inverse of v_2'. Then $v_2\in V_2$. If we set $b_1'=b'-v_1b'\in V_{b'}$ and $a_1'=a'-v_2a'\in V_{a'}$, then

$$\begin{aligned} b_1'a_1&=(b'-v_1b')(a-u_1'a)=b'a-v_1b'a-b'u_1'a+v_1b'u_1'a\\ &=b'a-v_1b'a-u_1b'a+v_1u_1b'a=b'a-(v_1\circ u_1)b'a\,. \end{aligned}$$

Since $u_2\circ u_1^{(0)}=v_2'\circ v_1$, we obtain

$$(v_2')^{(0)}\circ(u_2\circ u_1^{(0)})\circ u_1=(v_2')^{(0)}\circ(v_2'\circ v_1)\circ u_1\,.$$

Therefore

$$\begin{aligned} (v_2')^{(0)}\circ(u_2\circ u_1^{(0)})\circ u_1&=((v_2')^{(0)}\circ u_2)\circ(u_1^{(0)}\circ u_1)=(v_2')^{(0)}\circ u_2\,,\\ (v_2')^{(0)}\circ(v_2'\circ v_1)\circ u_1&=((v_2')^{(0)}\circ v_2')\circ(v_1\circ u_1)=v_1\circ u_1\,. \end{aligned}$$

Hence, by $b'a=a'b$,

$$\begin{aligned} b_1'a_1&=a'b-((v_2')^{(0)}\circ u_2)a'b=a'b-(v_2\circ u_2)a'b\\ &=a'b-v_2a'b-u_2a'b+v_2u_2a'b=a'b-v_2a'b-a'u_2'b+v_2a'u_2'b\\ &=(a'-v_2a')(b-u_2'b)=a_1'b_1\,, \end{aligned}$$

that is, (a_1, b_1) is right transitive at (a_1', b_1'). Therefore R is weakly embeddable in S.

In order to prove that R is openly embeddable, it is sufficient to show that, for any neighbourhood W_x of x in R, there exist open neighbourhoods U_c of c and V_{cx} of cx $(c\in R)$ satisfying $\varphi(U_c\times V_{cx})\subset\varphi(\{c\}\times cW_x)$, because $\varphi(U_c\times V_{cx})$ is an open set in S. Let W be a neighbourhood of 0 such that $x-Wx\subset W_x$. By the continuity of quasi-inverse at 0, there exist neighbourhoods U, V of 0 satisfying $U^{(0)}\circ V\subset W$, where $U^{(0)}$ means the set of quasi-inverses of all elements of U. Let $U_c=c-cU$ and $V_{cx}=cx-cVx$, then U_c and V_{cx} satisfy the above condition.

THEOREM 8. *Let R be a ring without proper zero divisors and have the left quotient ring S with the denominator system $R^*=R\backslash\{0\}$. Further we assume that any non-zero right ideal and left ideal of R always contains a non-zero two-sided ideal. Then, by the topology T of R, whose base of open sets is the family of all non-zero (two-sided) idals of R, R is openly embeddable if and only if R is not semi-simple.*

PROOF. If J_1 and J_2 are two non-zero ideals, then to any non-zero $a\in J_1$ and $b\in J_2$, there exist by assumption elements a', b' of R^* such that $b'a=a'b\in J_1\cap J_2$, that is, $J_1 \cap J_2 \neq \{0\}$. Therefore, the topology T of R is not discrete.

Necessity: If R is openly embeddable, then every non-zero iedal is open. By Theorem 7, R is a Q-ring and so the set Q of all quasi-regular elements forms an open set. Hence, there exists an ideal $I \neq \{0\}$ such that $I \subset Q$. Then the radical N of R contains I, in other words, R is not semi-simple.

Sufficiency: To show that T is Hausdorff, it is sufficient to show that the intersection Z of all the non-zero ideals in R is $\{0\}$. If $Z \neq \{0\}$, let a be a non-zero element of Z. Then, as Ra^2 is a non-zero left ideal and every non-zero left ideal contains some non-zero ideal of $R, Z \subset Ra^2$ so that there exists $b \in R$ such that $a = ba^2$. If I is any non-zero left ideal of R and x is an arbitrary element of R, then $a \in Z \subset I$ and $xa = xba^2$, therefore $x = xba \in I$. Hence R is the only non-zero left ideal of R. Similarly, we can show that R has no proper right ideal and so R becomes a division ring. But it contradicts the hypothesis that R is not semi-simple. Hence, by the continuity of operations in R, the topology T is Hausdorff. The radical $N \neq \{0\}$ and so R is a Q-ring. Further, for any non-zero ideal $I \subset N$, $x \in I$ implies $x^{(0)} = xx^{(0)} - x \in I$, where $x^{(0)}$ means the quasi-inverse of x. Therefore the quasi-inverse mapping is continuous. And Theorem 8 follows from Theorem 7.

COROLLARY. *Let R be an algebraic ring without proper zero divisors. Assume that every non-zero right ideal and every non-zero left ideal of R contain at least one non-zero two-sided ideal. Then R is openly embeddable in S, if R is a compact ring with respect to the topology T in Theorem* 8.

PROOF. We note that every non-zero right (left) ideal of R is open and closed in the topological ring R since it contains by assumption a non-zero two-sided ideal. We will show that R satisfies the ascending chain condition for left ideals. For, let $J_1 \subset J_2 \subset \cdots$ be an ascending chain of non-zero left ideals. Then $J = \bigcup_{i=1}^{\infty} J_i$ is also a non-zero left ideal of R. Since R is compact, J is compact and $\{J_i\}$ is an open covering of the compact set J. Therefore there exists in $\{J_i\}$ a finite number of ideals $J_{n_1}, \cdots, J_{n_r}$ such that $J = \bigcup_{i=1}^{r} J_{n_i}$ and hence J coincides with the maximal ideal of $J_{n_1}, \cdots, J_{n_r}$. Therefore R satisfies Condition (A) (Cohn [3] Theorem 6.1). Further, since R is not a field, it has the non-zero radical (Warner [8], Theorem 4), so that Corollary follows from Theorm.

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