On traces of Hecke operators[1]

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Let G be the group of all $g=(g^{(1)},\cdots,g^{(n)})$ with $g^{(i)}\in GL(2\boldsymbol{R})$ and $\mathfrak{H}_n$ the set of all $z=(z^{(1)},\cdots,z^{(n)})$ with $z^{(i)}\in \boldsymbol{C}$, $\operatorname{Im} z^{(i)}\neq 0$. We consider G as a group of transformations in $\mathfrak{H}_n$, putting

$$gz=(g^{(1)}z^{(1)},\cdots,g^{(n)}z^{(n)})$$
$$g^{(i)}z^{(i)}=\frac{a^{(i)}z^{(i)}+b^{(i)}}{c^{(i)}z^{(i)}+d^{(i)}},\quad g^{(i)}=\begin{pmatrix}a^{(i)} & b^{(i)}\\ c^{(i)} & d^{(i)}\end{pmatrix}.$$

Let Γ be a subgroup of G operating on $\mathfrak{H}_n$ discontinuously and satisfying (A1), (A2) in §1. Let α be an element in G such that Γ and $\alpha\Gamma\alpha^{-1}$ are commensurable. Let χ be a unitary representation of the subgroup of G generated by Γ and α. Let $\{k_i\}_{i=1}^{n}$ be the set of positive integers. Under a certain condition on χ ((R1) in §1) we shall define the space of cusp forms of type $(\Gamma,\{k_i\},\chi)$, and associate the double coset $\Gamma\alpha\Gamma$ with a linear transformation $\mathfrak{T}(\Gamma\alpha\Gamma)$ in this space. The trace of $\mathfrak{T}(\Gamma\alpha\Gamma)$ can be calculated by means of Selberg's trace formula (Selberg [8, 9]).

§1 is concerned with preliminary statements. In §§2–3 an explicit formula for the trace of $\mathfrak{T}(\Gamma\alpha\Gamma)$ will be given (Theorem 1). In §4 we shall apply Theorem 1 to the operator $\mathfrak{T}(\mathfrak{q})$ defined in Shimura [7] giving a formula for the trace of $\mathfrak{T}(\mathfrak{q})$. This will be carried out by following Eichler [3, 4].

Notation. $\boldsymbol{Z},\boldsymbol{Q},\boldsymbol{R},\boldsymbol{C},\boldsymbol{K}$ denote the ring of rational integers, the field of rational numbers, the field of real numbers, the field of complex numbers, the division ring of quaternions over $\boldsymbol{R}$, respectively. If R is a ring, R^*, $M_n(R)$ denote the group of all invertible elements in R, the ring of all matrices of degree n with coefficients in R, respectively.

§1. An operator of Hecke.

1.1. Let $G=GL(2\boldsymbol{R})\times\cdots\times GL(2\boldsymbol{R})$ be the product of n copies of $GL(2\boldsymbol{R})$. An element of G will be written in the form

$$g=(g^{(1)},\cdots,g^{(n)})$$

with $g^{(i)}\in GL(2\boldsymbol{R})$. Let $\mathfrak{H}_n$ be the set of all $z=(z^{(1)},\cdots,z^{(n)})$ with $z^{(i)}\in\boldsymbol{C}$, $\operatorname{Im} z^{(i)}\neq 0$. Putting

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$$
(1)\qquad \begin{aligned} g(z) &= (g^{(1)}z^{(1)}, \cdots, g^{(n)}z^{(n)}), \\ g^{(i)}z^{(i)} &= \frac{a^{(i)}z^{(i)}+b^{(i)}}{c^{(i)}z^{(i)}+c^{(i)}}, \quad g^{(i)} = \begin{pmatrix} a^{(i)} b^{(i)} \\ c^{(i)} d^{(i)} \end{pmatrix} \end{aligned}
$$

for $g \in G$, $z \in \mathfrak{H}_n$, we consider G as a group of transformations in $\mathfrak{H}_n$. g induces the identity transformation in $\mathfrak{H}_n$ if and only if g is contained in the center $Z(G)$ of G.

Let Γ be a subgroup of G operating discontinuously on $\mathfrak{H}_n$. ι being the canonical homomorphism of G onto $G/Z(G)$, this is equivalent to saying that $\iota(\Gamma)$ is discrete in $\iota(G)$. Let G^0 be the group of all $g \in G$ such that $\det g^{(i)} > 0$ $(1 \leqq i \leqq n)$ and set $\Gamma^0 = \Gamma \cap G^0$, $Z(\Gamma) = \Gamma \cap Z(G)$. It is assumed throughout this parper that

(Al) *$\iota(\Gamma^0)$ is an irreducible subgroup of $\iota(G^0)$ such that $\iota(G^0)/\iota(\Gamma^0)$* is of finite *measure.*

We first prove the following:

LEMMA 1.1. *Let G', G'' be partial factors of G such that $G = G' \times G''$, $G \neq G'$, $G \neq G''$. Write an element in G in the form $g = (g', g'')$ with $g' \in G'$, $g'' \in G''$. Let g_0 be an element in G such that Γ and $g_0 \Gamma g_0^{-1}$ are comensurable. If $\iota(g_0) \neq 1$, then we have $\iota(g_0') \neq 1$, $\iota(g_0'') \neq 1$.*

PROOF. Suppose that $\iota(g_0'') = 1$. $\iota(\Gamma g_0 \Gamma)$ is a discrete subset of $\iota(G)$ since it is the finite union of the cosets of $\iota(\Gamma)$. Let $\iota(G')^0$ be the connected component of the identity in $\iota(G')$. Let U' be an open neighbor-hood of the identity in $\iota(G')^0$ such that

$$
(2)\qquad \iota(\Gamma g_0 \Gamma) \cap U' \iota(g_0) U'^{-1} = \{\iota(g_0)\}.
$$

Let ξ be an arbitrary element in U'. By our assumption (Al) and by virtue of [1, Corollary 4. 3], there exists a sequence $\{\gamma_\nu\}$ of elements in Γ such that $\iota(\gamma_\nu')$ converges to ξ. Since U' is open, we may assume that $\iota(\gamma_\nu') \in U'$ for all ν. Then it follows from (2) that $\iota(\gamma_\nu' g_0 \gamma_\nu'^{-1}) = \iota(g_0)$ for all ν and hence that $\xi \iota(g_0) \xi^{-1} = \iota(g_0)$. Now U' generates $\iota(G')^0$. Consequently, $\iota(g_0)$ commutes with all elements in $\iota(G')^0$ and hence $\iota(g_0) = 1$. This is a contradiction.

1.2. Hereafter we shall assume, besides (Al), that

(A2) *$\iota(\Gamma^0)$ satisfies the assumption (F) in [6].*

We fix once and for all an element α in G such that $\alpha \Gamma \alpha^{-1}$ is commensurable with Γ and denote by Γ' the subgroup of G generated by Γ and α. Let χ be a representation of Γ' by unitary matrices. In the case where $\iota(G^0)/\iota(\Gamma^0)$ is not compact, we assume that

(R1) *the kernel Γ_χ* of χ in Γ *is of finite index in Γ.*

Let $k_1, \cdots, k_n$ be positive integers. In the same notation as in (1) we put

(3) $$j(g, z)=\prod_{i=1}^{n}(c^{(i)}z^{(i)}+d^{(i)})^{-k_i}|\det g^{(i)}|^{\frac{k_i}{2}}$$

By a cusp form of type $(\Gamma, \{k_i\}, \chi)$ we understand a function $f(z)$ on $\mathfrak{H}_n$ taking values in the representation space of χ, which satisfies the following conditions:

(S1) $f(z)$ is holomorphic on each connected component of $\mathfrak{H}_n$.

(S2) $f(\gamma z)=j(\gamma, z)^{-1}\chi(\gamma)f(z)$ for $\gamma\in\Gamma$.

(S3) In case $\iota(G^0)/\iota(\Gamma^0)$ is not compact, $f(z)$ is regular at every parabolic point x of Γ_χ and the constant term in the Fourier expansion of f at x vanishes (cf. [6, §4]).[2)]

The set of all such $f(z)$ is denoted by $S(\Gamma, \{k_i\}, \chi)$ or simply by S. For the reason stated in [7, §3.3] we lose no generality by assuming that

(R2) $\chi(\varepsilon)=\prod_{i=1}^{n}(\operatorname{sgn}\varepsilon^{(i)})^{k_i}$ *for* $\varepsilon\in Z(\Gamma)$.

We now define a linear transformation $\mathfrak{T}(\Gamma\alpha\Gamma)$ in S. Let $\Gamma\alpha\Gamma=\bigcup_{\nu=1}^{d}\alpha_\nu\Gamma$ be a disjoint sum. For $f\in S$ we set

(4) $$(\mathfrak{T}(\Gamma\alpha\Gamma)f)(z)=\sum_{\nu=1}^{d}j(\alpha_\nu^{-1}, z)\chi(\alpha_\nu)f(\alpha_\nu^{-1}).$$

We shall calculate the trace of $\mathfrak{T}(\Gamma\alpha\Gamma)$ in the following section.

§2. Selberg's trace formula.

2.1. Let $\mathfrak{K}_1,\cdots,\mathfrak{K}_{2^n}$ be the connected components of $\mathfrak{H}_n$. Each $\gamma\in\Gamma$ induces a permutation of $\{\mathfrak{K}_\nu\}_{\nu=1}^{2^n}$ and this permutation is the identity if and only if $\gamma\in\Gamma^0$. Therefore the quotient group Γ/Γ^0 is identified with a subgroup of permutations of $\{\mathfrak{K}_\nu\}_{\nu=1}^{2^n}$. We fix a subset, say $\{\mathfrak{K}_1,\cdots,\mathfrak{K}_f\}$, of $\{\mathfrak{K}_\nu\}_{\nu=1}^{2^n}$ such that every $\mathfrak{K}_\nu$ is mapped by the elements in Γ/Γ^0 to one and the only one of $\{\mathfrak{K}_\mu\}_{\mu=1}^{f}$. If F_μ is a fundamental domain of Γ_0 in $\mathfrak{K}_\mu$, the union

$$F=\bigcup_{\mu=1}^{f}F_\mu$$

is obviously a fundamental domain of Γ in $\mathfrak{H}_n$. By (A2) we may assume that F_μ is of the form described in the assumption (F), or that F is given in the following way.

x being a parabolic point of Γ, let $\Gamma_x^{(1)}$ be the group of all $\gamma\in\Gamma$ leaving x fixed and Γ_x the group consisting of all parabolic tranoformations in $\Gamma_x^{(1)}\cap\Gamma^0$.[3)] Let $x_\nu(1\leq\nu\leq s)$ be a complete system of Γ^0-inequivalent parabolic points of Γ. Taking a $\rho_\nu\in G$ such that $\rho_\nu x_\nu=\infty$, we put

2) By a parabolic point of Γ we understand a parabolic point of $\iota(\Gamma^0)$.

3) If $g\in G^0$, we say that g is elliptic, hyperbolic, parabolic, or mixed according as $\iota(g)$ is of the corresponding type. cf. [6, §1].

$$U_\nu'=\{\rho_\nu^{-1}z;\ \prod_{i=1}^{n}|\mathrm{Im}\, z^{(i)}|>d_\nu\}\frown(\bigcup_{\mu=1}^{f}\mathfrak{K}_\mu),$$

d_ν being a suitable positive number. Let V_ν' be a fundamental domain of $(\Gamma^0)_{x_\nu}^{(1)}$ in U_ν'. Then F is of the form

$$F=F_0'\cup V_1'\cup\cdots\cup V_s',$$

where F_0' is relatively compact in $\mathfrak{H}_n$.

For our later use it is convenient to group together all the V_ν' such that x_ν are Γ-equivalent. Suppose $x_1,\cdots,x_a$ are all the Γ-equivalent points to x_1: $\gamma_\nu x_1=x_\nu$ $(1\leqq\nu\leqq a)$ with $\gamma_\nu\in\Gamma$. For any $\gamma\in\Gamma$, γx_1 is a parabolic point of Γ so that there exist a $\delta\in\Gamma^0$ and a $x_\nu(1\leqq\nu\leqq s)$ such that $\gamma x_1=\delta x_\nu$. We have necessarily $1\leqq\nu\leqq a$. Hence γ is written in the form $\gamma=\delta\gamma_\nu\varepsilon$ with $\varepsilon\in\Gamma_{x_1}^{(1)}$. It follows that the permutation of $\{\mathfrak{K}_\nu\}_{\nu=2}^{2n}$ induced by Γ are all obtained from the elements in $\gamma_\nu\Gamma_{x_1}^{(1)}(1\leqq\nu\leqq a)$. It is then easy to see that the union of $V_\nu'(1\leqq\nu\leqq a)$ is Γ-equivalent to V_1 up to a relatively compact set in $\mathfrak{H}_n$, where, for each ν, V_ν is a fundamental domain of $\Gamma_{x_\nu}^{(1)}$ in

$$U_\nu=\{\rho_\nu^{-1}z;\ \prod_{i=1}^{n}|\mathrm{Im}\, z^{(i)}|>d_\nu\}.$$

Therefore, if we assume after reordering the indices that $\{x_\nu\}_{\nu=1}^{t}$ is a complete system of Γ-inequivalent parabolic points of Γ, F is written as

$$F=F_0\cup V_1\cup\cdots\cup V_t$$

with a relatively compact subset F_0 in $\mathfrak{H}_n$.

2.2. Let (u, v) be the inner product in the representation space of χ such that

$$(\chi(\gamma)u,\chi(\gamma)v)=(u,v)\ \text{ for }\gamma\in\Gamma.$$

Put $\|u\|=(u,u)^{1/2}$. For $z, z'\in\mathfrak{H}_n$, we put

$$(5)\qquad k(z,z')=\begin{cases}\displaystyle\prod_{i=1}^{n}\left(\frac{z^{(i)}-\overline{z'^{(i)}}}{2\sqrt{-1}}\right)^{-k_i} & \text{if } z, z' \text{ are in the same connected component of } \mathfrak{H}_n,\\ 0 & \text{otherwise.}\end{cases}$$

Then we have

$$k(z,z')=\overline{k(z',z)}$$

$$k(gz,gz')=k(z,z')j(g,z)^{-1}\overline{j(g,z')}^{-1}\sigma(g)$$

for $g\in G$, where $\sigma(g)=\prod_{i=1}^{n}(\mathrm{sgn}\det g^{(i)})^{k_i}$. Call $H^2(\Gamma,\{k_i\},\chi)$ (resp: $H^\infty(\Gamma,\{k_i\},\chi)$) the space of all functions f on $\mathfrak{H}_n$ satisfying (S1), (S2) and

$$\|f\|_2=\left[\int_F |k(z,z)|^{-1}|f(z)|^2 dz\right]^{1/2}<\infty$$

$$(\text{resp}:\ \|f\|_\infty=\sup_{z\in F}|k(z,z)|^{-1/2}|f(z)|<\infty).$$

Here we have put

$$(6)\qquad dz=\prod_{i=1}^{n}\frac{dx^{(i)}dy^{(i)}}{y^{(i)2}},\quad z^{(i)}=x^{(i)}+\sqrt{-1}\,y^{(i)}.$$

$H^2(\Gamma,\{k_i\},\chi)$ (resp: $H^\infty(\Gamma,\{k_i\},\chi)$) forms a Banach space with respect to $\|\ \|_2$ (resp: $\|\ \|_\infty$), and we have $H^\infty(\Gamma,\{k_i\},\chi)\subset H^2(\Gamma,\{k_i\},\chi)$. By an analogue of [6, Lemmas 8, 9 and Theorem 10] we conclude that $S(\Gamma,\{k_i\},\chi)$ coincides with $H^\infty(\Gamma,\{k_i\},\chi)$ and that this one is a closed subspace of $H^2(\Gamma,\{k_i\},\chi)$. Put

$$a(\{k_i\})=(4\pi)^{-n}\prod_{i=1}^{n}(k_i-1),$$

$$K(z,z')=\sum_{\gamma\in\Gamma,\gamma \bmod Z(\Gamma)}{}^{'}k(z,\gamma z')j(\gamma,z')\sigma(\gamma)\chi(\gamma)$$

for $z,z'\in\mathfrak{H}_n$. Then it follows from [10, Exposé 8, Théorème 1 and Exposé 10, Théorème 8] that, if $k_i>2(1\leqq i\leqq n)$,

$$f(z)\to(Kf)(z)=a(\{k_i\})\int_F\frac{K(z,z')f(z')}{k(z',z')}dz'$$

is an operator of Hibert-Schmidt type in $H^2(\Gamma,\{k_i\},\chi)$ and $Kf=f$ if and only if $f\in H^\infty(\Gamma,\{k_i\},\chi)$.

From now on we assume $k_i>2(1\leqq i\leqq 2)$. Now, in the notation in §2.1, we have

$$\mathfrak{T}(\Gamma\alpha\Gamma)f(z)=a(\{k_i\})\int_F\sum_{\nu=1}^{d}\frac{j(\alpha_\nu^{-1},z)\chi(\alpha_\nu)K(\alpha_\nu^{-1}z,z')}{k(z',z')}f(z')dz'$$

$$=a(\{k_i\})\int_F\left[\sum_{\nu=1}^{d}\sum_{\gamma\in\Gamma,\gamma \bmod Z(\Gamma)}{}^{'}\frac{k(z,\alpha_\nu\gamma z')j(\alpha_\nu\gamma,z')}{k(z',z')}\sigma(\alpha_\nu\gamma)\chi(\alpha_\nu\gamma)f(z')\right]dz'.$$

Consequently we have

$$\operatorname{tr}\mathfrak{T}(\Gamma\alpha\Gamma)=a(\{k_i\})\int_F\left[\sum_{g\in\Gamma\alpha\Gamma,g \bmod Z(\Gamma)}{}^{'}\frac{k(z,gz)j(g,z)}{k(z,z)}\operatorname{tr}\chi(g)\right]dz.$$

2.3. Before going further we have to prove a few lemmas which are analogues of [6, Lemma 12]. For the sake of simplicity we write $B=\Gamma\alpha\Gamma$. Here we are interested only in the case where Γ contains parabolic transformations. If x is a parabolic point of Γ, we put

$$B_x^{(1)}=\{g\in B;\ gx=x\},$$

$$B_x=\{g\in B_x^{(1)};\ g \text{ is parabolic}\}.$$

In the following lemmas it is assumed that ∞ is a parabolic point of Γ.

LEMMA 2.1. *The notation being the same as in* (1), *there exists a positive constant κ such that*

$$\prod_{i=1}^{n}\left|\frac{c^{(i)2}}{\det g^{(i)}}\right|\geqq\kappa$$

for all $g\in B-B_\infty^{(1)}$.

PROOF. We remark first that, if $g\Gamma g^{-1}$ is commensurable with Γ and if x is a parabolic point of Γ, then gx is also a parabolic point of Γ. Let $B=\cup\alpha_\nu\Gamma$ be a disjoint sum and let $g=\alpha_\nu\gamma$ be an element of B. Since $\alpha_\nu^{-1}(\infty)$ is a parabolic point of Γ by the above remark, we can apply [6, Lemma 5] to our case putting $x_1=\infty$, $x_2=\alpha_\nu^{-1}(\infty)$. The proof there shows that if

$$|c^{(i)2}(\det g^{(i)})^{-1}\mu_1^{(i)}\mu_2^{(i)}|<1(1\leqq i\leqq n)$$

for $\mu_\nu\in M_{x_\nu}$, $\mu_\nu\neq 0\,(\nu=1,2)$, then we have $g\in B_\infty^{(1)}$. Therefore, our lemma holds by virtue of Minkovski's theorem, if we take $d(M_{x_1})^{-1}\prod_{i=1}^{i}|\mu_2^{(i)}|^{-1}$ for κ, μ_2 being any non-zero element of M_{x_2}.

LEMMA 2.2. *Let D be a compacts subset of $\mathfrak{H}_n$. There exists a constant M such that*

$$\prod_{i=1}^{n}|\mathrm{Im}\,(g^{(i)}z^{(i)})|<M$$

for all $g\in B-B_\infty^{(1)}$, $z\in D$.

PROOF. Since

$$|\mathrm{Im}\,z^{(i)}\cdot\mathrm{Im}\,(g^{(i)}z^{(i)})|\leqq|\det g^{(i)}c^{(i)-2}|,$$

this follows from Lemma 2.1.

LEMMA 2.3. *For $\varepsilon>0$, we have*

$$\sum_g\prod_{i=1}^{n}\frac{1}{c^{(i)2}(c^{(i)2}+1)^{\varepsilon}}<\infty,$$

g running over all the representatives of $\Gamma_\infty\backslash(B-B_\infty^{(1)})/\Gamma_\infty$.

PROOF. Let D be a compact subset of $\mathfrak{H}_n$. Writting D' for the union of all gD with $g\in B-B_\infty^{(1)}$, we get

$$\int_D\left[\sum_{g\in\Gamma_\infty^{(1)}\backslash B-B_\infty^{(1)}}\prod_{i=1}^{n}\left|\frac{y^{(i)}\det g^{(i)}}{(c^{(i)}z^{(i)}+d^{(i)})^2}\right|^{1+\varepsilon}\right]dz\leqq l\int_{\Gamma_\infty^{(1)}\backslash D'}\prod_{i=1}^{n}|y^{(i)}|^{1+\varepsilon}dz,$$

where l is the number of $\xi\in\iota(B^{-1}B)$ such that $\xi D\cap D\neq\phi$. Since D' is contained in the set of all $z\in\mathfrak{H}_n$ such that $\prod_{i=1}^{n}|y^{(i)}|<M$, the integral on the right hand side exists. It follows in particular that the series in the above inequality converges for all $z\in\mathfrak{H}_n$. Hence Lemma 2.3 is proved in exactly the same way as in the proof of [6, Lemma 12].

LEMMA 2.4. *For $\varepsilon>0$, we have*

$$\sum_{g}{}' \prod_{i=1}^{n}\left(\frac{|\det g^{(i)}|^{1/2}}{|a^{(i)}+d^{(i)}|}\right)^{\varepsilon}<\infty,$$

g running over all the representatives of $B_{\infty}^{(1)}/\Gamma_{\infty}$.

PROOF. We use the notation in the proof of Lemma 2.1. If $g=\alpha_\nu\gamma$ is an element of $B_{\infty}^{(1)}$, we may assume that $\alpha_\nu\in B_{\infty}^{(1)}$, replacing α_ν by $\alpha_\nu\gamma$. By doing so for all cosets $\alpha_\nu\Gamma$ containing an element of $B_{\infty}^{(1)}$, we get

$$B_{\infty}^{(1)}=\bigcup_{\alpha_\nu(\infty)=\infty}\alpha_\nu\Gamma_{\infty}^{(1)}$$

Then, the lemma follows from [6, Lemma 12, 2°].

2.4. We set

$$I(z)=\prod_{i=1}^{n}|\operatorname{Im} z^{(i)}|,$$

$$j_0(g, z)=\prod_{i=1}^{n}(c^{(i)}z^{(i)}+d^{(i)})^{-2}|\det g^{(i)}|$$

On account of Lemmas 2.3, 2.4, we can proceed just as in [6, No. 14] and obtain

$$\begin{aligned}a(\{k_i\})^{-1}\operatorname{tr}\mathfrak{T}(\Gamma\alpha\Gamma)=&\sum_{\substack{g \bmod Z(\Gamma)\\ g\in B-C}}\int_{F}\frac{k(z, gz)j(g, z)}{k(z, z)}\operatorname{tr}\chi(g)dz\\&+\lim_{s\to+0}\sum_{\nu=1}^{t}\sum_{\substack{g \bmod Z(\Gamma)\\ g\in B_{x_\nu}^{(1)}-Z(B)}}\left[\int_{F-V_\nu}\frac{k(z, gz)j(g, z)}{k(z, z)}\operatorname{tr}\chi(g)dz+\int_{V_\nu}\frac{k(z, gz)j(g, z)\operatorname{tr}\chi(g)}{I(z)^s|j_0(\rho_\nu, z)|^s k(z, z)}dz\right].\end{aligned}$$

Here we have put $Z(B)=B\cap Z(G)$, $C=\bigcup_{\nu=1}^{t}B_{x_\nu}^{(1)}-Z(B)$.

We now classify the elements in B with respect to the following equivalence relation:

$$(7)\qquad g\sim g'\Longleftrightarrow g'=\varepsilon\gamma g\gamma^{-1}\ \text{ for } \gamma\in\Gamma,\ \varepsilon\in Z(\Gamma).$$

The class containing g is denoted by $[g]$. Let $\Gamma(g)$ be the group of all $\gamma\in\Gamma$ such that $\gamma g\gamma^{-1}=\varepsilon g$ for some $\varepsilon\in Z(\Gamma)$ and F_g a fundamental domain of $\Gamma(g)$ in $\mathfrak{H}_n$. Let $\{g_0\}$ be a full system of representatives of the above equivalence classes in B and, for each g_0, $\{\delta\}$ a system of representatives of $\Gamma/\Gamma(g_0)$. We set

$$F_{g_0}^{*}=F_{g_0}-\bigcup_{\nu=1}^{t}\Big(\bigcup_{\delta g_0\delta^{-1}\in B_{x_\nu}^{(1)}}\delta^{-1}V_\nu\Big).$$

In this notation we have

$$
\begin{aligned}
a(\{k_i\})^{-1}\operatorname{tr}\mathfrak{T}(\Gamma\alpha\Gamma) &= \sum_{g_0:[g_0]\cap\mathfrak{C}=\phi}{}' \int_{F_{g_0}} \frac{k(z, g_0 z) j(g_0, z)}{k(z, z)} \operatorname{tr}\chi(g_0)\,dz \\
(8) \qquad &+ \lim_{s\to+0} \sum_{g_0:[g_0]\cap\mathfrak{C}\neq\phi}{}' \Bigg[\int_{F^*_{g_0}} \frac{k(z, g_0 z) j(g_0, z)}{k(z, z)} \operatorname{tr}\chi(g_0)\,dz \\
&+ \sum_{\nu=1}^{l}{}' \sum_{\delta g_0 \delta^{-1}\in B^{(1)}_{\kappa_\nu}}{}' \int_{\delta^{-1}V_\nu} \frac{k(z, g_0 z) j(g_0, z)\operatorname{tr}\chi(g_0)}{I(z)^s |j_0(\rho_\nu\delta, z)|^s k(z, z)}\,dz \Bigg].
\end{aligned}
$$

Remark. By virtue of [10, Exposé 10, No. 6]

$$\frac{K(z, z')}{|k(z, z)|^{1/2}|k(z', z')|^{1/2}}$$

is bounded on $\mathfrak{H}_n\times\mathfrak{H}_n$. Therefore,

$$\sum_{\nu=1}^{d}{}' \frac{|j(\alpha_\nu, z)|\chi(\alpha_\nu)K(\alpha_\nu^{-1}z, z)}{|k(z, z)|}$$

is bounded on $\mathfrak{H}_n$. It follows that all the integrals in (8) are absolutely convergent.

§3. An explicit formula for $\operatorname{tr}\mathfrak{T}(\Gamma\alpha\Gamma)$.

3.1. In this section we shall calculate the integrals in (8). Since $k(z, g_0 z)=0$ if $g_0\notin G^0$, it is enough to consider those g_0 contained in $G^0\cap B$. By Lemma 1.1 such a g_0 is of one of the following types.

i) $g_0\in Z(B)$. ii) g_0 is elliptic iii) g_0 is hyperbolic and no fixed point of g_0 is a parabolic point of Γ. iv) g_0 is hypabolic and one of the fixed points of g_0 is a parabolic point of Γ. v) g_0 is parabolic. vi) g_0 is mixed.

Remark. Put $\Gamma^0(g_0)=\Gamma(g_0)\cap\Gamma^0$. If g_0 is of type i), we have $\iota(\Gamma^0(g_0))=\iota(\Gamma^0)$. If g_0 is of type ii), $\iota(\Gamma^0(g_0))$ is a finite abelian group. In other case $\iota(\Gamma^0(g_0))$ is a free abelian group. It is of rank $n-r$ except for the case where g_0 is of type iv), r being the number of $g_0^{(i)}$ such that $g_0^{(i)}$ is elliptic. If g_0 is of type iv), $\iota(\Gamma^0(g_0))$ is of rank $n-1$.

This is a consequence of absolute convergence of the integrals in (8), as we see by writing out these integrals explicitly. (c.f. [6, §5]).

In particular, the fixed point of g_0 of type v) is necessarily a parabolic point of Γ, for all the elements in $\Gamma_0(g_0)$ are parabolic transformations having the same fixed point as g_0.

3.2. Case i). Suppose that $Z(B)\neq\phi$ and let g_0 be an element in $Z(B)$. Then $B=\Gamma g_0\Gamma=g_0\Gamma$ and $Z(B)=g_0 Z(\Gamma)$. Consequently, $Z(B)$ consists of a single equivalence class. We have

$$\int_{F_{g_0}} = \int_F \prod_{i=1}^n (\operatorname{sgn} g_0^{(i)})^{k_i} \cdot \operatorname{tr} \chi(g_0) dz$$

(9) $$= \prod_{i=1}^n (\operatorname{sgn} g_0^{(i)})^{k_i} \cdot \operatorname{tr} \chi(g_0) v(F).$$

Case ii). Let η_i, ζ_i be the eigenvalues of $g_0^{(i)}$ and suppose that we have

$$\frac{g_0^{(i)} z^{(i)} - z_0^{(i)}}{g_0^{(i)} z^{(i)} - \bar{z}_0^{(i)}} = \eta_i \cdot \zeta_i^{-1} \frac{z^{(i)} - z_0^{(i)}}{z^{(i)} - \bar{z}_0^{(i)}} \ (1 \leqq i \leqq n).$$

Here z_0 is the fixed point of g_0 with $\operatorname{Im} z_0^{(i)} > 0 \ (1 \leqq i \leqq n)$. If $\mathfrak{K}_\nu$ is defined by

(10) $$\begin{aligned} &\operatorname{Im} z^{(i)} > 0 \qquad (1 \leqq i \leqq p) \\ &\operatorname{Im} z^{(j)} < 0 \qquad (p+1 \leqq j \leqq n), \end{aligned}$$

we have

$$\int_{F_{g_0} \cap \mathfrak{K}_\nu} = \frac{(-1)^{n-p} \operatorname{tr} \chi(g_0)}{a(\{k_i\})[\Gamma(g_0) : Z(\Gamma)]} \prod_{i=1}^n (\det g_0^{(i)})^{1-\frac{k_i}{2}} \cdot \prod_{i=1}^p \frac{\zeta_i^{k_i-1}}{\eta_i - \zeta_i} \cdot \prod_{j=p+1}^n \frac{\eta_j^{k_j-1}}{\eta_j - \zeta_j}.$$

Summing up the above equality for $\nu = 1, 2, \cdots, 2^n$, we obtain

(11) $$\int_{F_{g_0}} = \frac{(-1)^n \operatorname{tr} \chi(g_0)}{a(\{k_i\})[\Gamma(g_0) : Z(\Gamma)]} \times \prod_{i=1}^n \frac{\eta_i^{k_i-1} - \zeta_i^{k_i-1}}{\eta_i - \zeta_i} \cdot (\det g_0^{(i)})^{1-\frac{k_i}{2}}.$$

Case iii). By a calculation similar to the calculation in [6, No. 19], we get

$$\int_{F_{g_0}} = 0.$$

3.3. Case iv). In view of the argument in [6, No. 20] we may assume that g_0 leaves each of ∞ and 0 fixed, and that both of them are parabolic points of Γ. By the remark in the beginning of [6, No. 20] any fixed point of g_0 other than ∞ and 0 cannot be a parabolic point of Γ. In the notation in §2.1, let us suppose that x_ν and $x_{\nu'}$ are Γ-equivalent to ∞ and 0, respectively. Put $x_\nu = \varepsilon(\infty)$, $x_{\nu'} = \varepsilon'(0)$ with $\varepsilon, \varepsilon' \in \Gamma$. If $\delta g_0 \delta^{-1} \in B_{x_\mu}^{(1)}$ for $1 \leqq \mu \leqq t$, $\delta^{-1} x_\mu$ is a parablic point of Γ which is left fixed by g_0. Hence $\delta^{-1} x_\mu$ is either ∞ or 0; accordingly δ is contained in $\Gamma_{x_\nu}^{(1)} \varepsilon$ or in $\Gamma_{x_{\nu'}}^{(1)} \varepsilon'$. Therefore it is enough to calculate the integrals

(12) $$\int_{F_{g_0} \cap E} \frac{k(z, g_0 z) j(g_0, z) \operatorname{tr} \chi(g_0)}{I(z)^s |j_0(\rho_\nu \varepsilon, z)|^s k(z, z)} dz,$$

(13) $$\int_{F_{g_0} \cap E} \frac{k(z, g_0 z) j(g_0, z) \operatorname{tr} \chi(g_0)}{I(z)^s |j_0(\rho_{\nu'} \varepsilon', z)|^s k(z, z)} dz,$$

(14) $$\int_{F_{g_0}^*} \frac{k(z, g_0 z) j(g_0, z)}{k(z, z)} \operatorname{tr} \chi(g_0) dz,$$

where $E=\varepsilon^{-1}\Gamma_{x_\nu}^{(1)}V_\nu=\{z;\prod_{i=1}^n|\mathrm{Im}\,z^{(i)}|>\kappa\}$, $E'=\varepsilon'^{-1}\Gamma_{x_{\nu'}}^{(1)}V_{\nu'}=\{z;\prod_{i=1}^n|\mathrm{Im}\,z^{(i)}|^{-2}>\kappa'\}$, $F_{g_0}^*=F_{g_0}-E-E'$, κ, κ' being suitable positive numbers. We now consturct F_{g_0} in the following way. Fix, say, $\mathfrak{K}_1,\cdots,\mathfrak{K}_g$ such that every $\mathfrak{K}_\nu(1\leqq\nu\leqq 2^n)$ is mapped by $\Gamma(g_0)$ to one and the only one of $\mathfrak{K}_1,\cdots,\mathfrak{K}_g$. By the remark in §3.1 $\Gamma^0(g_0)$ is generated as a group of transformations in $\mathfrak{H}_n$ by $n-1$ independent elements $\gamma_1,\cdots,\gamma_{n-1}:(\gamma_j z^{(i)})=\lambda^{(i)}z^{(i)}$ $(1\leqq i\leqq n,\ 1\leqq j\leqq n-1))$. Set $l_j^{(i)}=\log\lambda_j^{(i)}$ $(1\leqq i\leqq n,\ 1\leqq j\leqq n-1)$ and $l_n^{(i)}=1/n(1\leqq i\leqq n)$. For $z\in\mathfrak{H}_n$, write $z^{(i)}=r^{(i)}e^{\sqrt{-1}\theta^{(i)}}$ and $\log\gamma^i=u_1l_1^{(i)}+\cdots+u_nl_n^{(i)}$ with $u_i\in\boldsymbol{R}$. We can take for F_{g_0} the set of all $z\in\bigcup_{\nu=1}^{i}\mathfrak{K}_\nu$ such that $0<u_i<1$ $(1\leqq i\leqq n-1)$, $-\infty<u_n<\infty$. As it is proved in [6, No. 20], the integrals (12), (13) vanish. Now, let $\mathfrak{K}_\nu$ be given in (10). Writing

$$g_0^{(i)}=\begin{pmatrix}a^{(i)} & 0\\ 0 & d^{(i)}\end{pmatrix},$$

we have

$$\begin{aligned}\int_{F_{g_0}^*\mathfrak{K}_\nu} &=\mathrm{tr}\,\chi(g_0)|\det(l_j^{(i)})|\prod_{i=1}^n\frac{(2\sqrt{-1})^{k_i}|\det g_0^{(i)}|^{\frac{k_i}{2}}}{d^{(i)k_i}}\int\cdots\int\left[\int_{\log\varepsilon'^{-1}\Pi|\sin\theta^{(i)}|}^{\log\varepsilon\Pi|\sin\theta^{(i)}|^{-1}}du_n\right.\\ &\quad\times\prod_{i=1}^n\frac{(\sin\theta^{(i)})^{k_i-2}}{(e^{\sqrt{-1}\theta^{(i)}}-a^{(i)}d^{(i)-1}e^{-\sqrt{-1}\theta^{(i)}})^{k_i}}d\theta^{(1)}\cdots d\theta^{(n)}\\ &=\mathrm{tr}\,\chi(g_0)\,|\det(l_j^{(i)})|\cdot\prod_{i=1}^n\frac{(2\sqrt{-1})^{k_i}|\det g_0^{(i)}|^{\frac{k_i}{2}}}{d^{(i)k_i}}\int\cdots\int\log(\kappa\kappa'\prod_{i=1}^n|\sin\theta^{(i)}|^{-2})\\ &\quad\times\prod_{i=1}^n\frac{(\sin\theta^{(i)})^{k_i-2}}{(e^{\sqrt{-1}\theta^{(i)}}-a^{(i)}d^{(i)-1}e^{-\sqrt{-1}\theta^{(i)}})^{k_i}}d\theta^{(1)}\cdots d\theta^{(n)}.\end{aligned}\tag{15}$$

The integral on the right hand side is extended over $0<\theta<\pi(1\leqq i\leqq p)$, $\pi<\theta^{(i)}<2\pi$ $(p+1\leqq i\leqq n)$. By [6, (30)] we see that, if $n>1$, the integral (15) vanishes for each ν.

Suppose that $n=1$. In this case we write simply $k_1=k$, $a^{(1)}=a$, $d^{(1)}=d$. Since $\Gamma^0(g_0)=Z(\Gamma)$, we can assume $F_{g_0}=\mathfrak{K}_1\cup\mathfrak{K}_2$ or $\mathfrak{K}_1$ according as $[\Gamma(g_0):Z(\Gamma)]=1$ or 2. By a direct calculation we get

$$\int_{F_g^*}=-\frac{8\pi}{k-1}\,\frac{\mathrm{tr}\,\chi(g_0)(\det g_0)^{1-\frac{k}{2}}(\mathrm{Min}\{|a|,|d|\})^{k-1}}{[\Gamma(g_0):Z(\Gamma)]|a-b|}\tag{16}$$

3.4. Case v). All $g\in G_{x_\nu}^{(1)}$ are written in the form

$$\rho_\nu^{(i)}g^{(i)}\rho_\nu^{(i)-1}=\begin{pmatrix}a^{(i)} & b^{(i)}\\ 0 & d^{(i)}\end{pmatrix}(1\leqq i\leqq n).\tag{17}$$

We have $g\in G_{x_\nu}$ if and only if $a^{(i)}=d^{(i)}(1\leqq i\leqq n)$. Put

$$\lambda(g)=(\lambda(g^{(1)}),\cdots,\lambda(g^{(n)})),$$

$$\mu(g)=(\mu(g^{(1)}),\cdots,\mu(g^{(n)}))$$

with $\lambda(g^{(i)})=a^{(i)}d^{(i)-1}$, $\mu(g^{(i)})=b^{(i)}d^{(i)-1}$.

We now state

LEMMA 3.1. *Set* $N_{x_\nu}=\{\mu(g);g\in B_{x_\nu}\}$, $M_{x_\nu}=\{\mu(g);g\in\Gamma_{x_\nu}\}$, $\Lambda_{x_\nu}=\{\lambda(g);g\in\Gamma_{x_\nu}^{(1)}\}$. *Then* M_{x_ν} *is a discrete subgroup of* $\boldsymbol{R}^n$ *of rank* n. Λ_{x_ν} *is a discrete subgroup of* $(\boldsymbol{R}^*)^n$ *of rank* $n-1$ *and we have* $\prod_{i=1}^n|\lambda(g^{(i)})|=1$ *for all* $\lambda(g)\in\Lambda_{x_\nu}$. N_{x_ν} *is the union of a finite number of cosets of* M_{x_ν}. *If* $\mu(g)\in N_{x_\nu}$, $\lambda(g_1)\in\Lambda_{x_\nu}$, *then*

$$\lambda(g_1)\mu(g)=(\lambda(g_1^{(1)})\mu(g^{(1)}),\cdots,\lambda(g_1^{(n)})\mu(g^{(n)}))$$

is contained in N_{x_ν}.

PROOF. The first two statements follow from [6, Theorem 3]. Since we have

$$\mu(g\,g')=\mu(g)+\mu(g')\ \ (g,\in B_{x_\nu},\ g'\in\Gamma_{x_\nu}),$$
$$\mu(g_1g\,g_1^{-1})=\lambda(g_1)\mu(g)\ \ (g\in B_{x_\nu},\ g_1\in\Gamma_{x_\nu}^{(1)}),$$

the other statements follow from the definition.

We classify the elements in N_{x_ν} putting $\mu(g)$, $\mu(g')\in N_{x_\nu}$ into the same class if $\mu(g)=\lambda(g_1)\mu(g')$ with $\lambda(g_1)\in\Lambda_{x_\nu}$. The class of $\mu(g)$ is denoted by $\overline{\mu(g)}$.

LEMMA 3.2. *Let* L_ν *be a complete system of inequivalent elements in* B_{x_ν}. *Then* $\overline{\mu(g)}$ $(g\in L_\nu)$ *runs oves all the classes in* $\dot{N}_{x_\nu}$, *each of which being repeated the same number of times.*

PROOF. Let g, g' be elements in B_{x_ν}. We have $\overline{\mu(g)}=\overline{\mu(g')}$ if and only if $g=\delta\gamma g'\gamma^{-1}$ with $\gamma\in\Gamma$, $\delta\in Z(G)$. If this is the case, we get $\delta B=B$, for $B=\Gamma g\Gamma=\Gamma g'\Gamma$. Let Z_1 be the group of all $\delta\in Z(G)$ such that $\delta B=B$. Let $B\cdot B^{-1}=\Gamma\alpha\Gamma\alpha^{-1}\Gamma$ $=\bigcup_{\nu=1}^{c}\delta_\nu\Gamma$ be a disjoint sum. We can assume that $\delta_\nu\in Z(G)$ if $\delta_\nu\Gamma\cap Z(G)\neq 0$. Then $BB^{-1}\cap Z(G)$ is the union of $\delta_\nu Z(\Gamma)$ such that $\delta_\nu\in Z(G)$. It follows that $Z(\Gamma)$ is a subgroup of finite index, say e_1, in Z_1. It is then clear that $\overline{\mu(g)}$ $(g\in L_\nu)$ takes every class in N_{x_ν} exactly e_1 times.

By the remark in §3.1, $L_\nu(1\leqq\nu\leqq t)$ jointly form a complete system of inequivalent elements of type v) in B.

Fix one of the x_ν's, say x_1, and assume that $x_1=\infty$, $\rho_1=1$. Let g be an element in $B_{x_1}\cdot\Gamma(g)$ is generated as a group of transformations in $\mathfrak{H}_n$ by n independent elements $\gamma_1,\cdots,\gamma_n$. Write $z^{(i)}=x^{(i)}+\sqrt{-1}\,y^{(i)}$ and $x^{(i)}=v_1\mu(\gamma_1^{(i)})+\cdots+v_n\mu(\gamma_n^{(i)})$ with $v_i\in\boldsymbol{R}$. Then, the set of all $z\in\mathfrak{H}_n$ such that $0<v_i<1(1\leqq i\leqq n)$ forms a fundamental domain F_g of $\Gamma(g)$. Set

$$d(g)=|\det(\mu(\gamma_j^{(i)}))|. \tag{18}$$

Besides, we put

$$m(g)=\prod_{i=1}^{n}|\mu(g^{(i)})|. \tag{19}$$

It is to be noted that $d(g)$ and F_g do not depend on g so long as g is in B_{x_1}. Put $E=\{z\in\mathfrak{H}_n\,;\,\prod_{i=1}^{n}|\operatorname{Im} z^{(i)}|>d_1\}$. Then, the contribution of $g\in L_1$ to (8) is equal to

$$\begin{aligned} w&=\lim_{s\to+0}\sideset{}{'}\sum_{g\in L_1}\left[\int_{F_g^0}\frac{k(z,\,gz)j(gz)}{k(z,\,z)}\operatorname{tr}\chi(g)dz+\int_{F_g\frown E}\frac{k(z,\,gz)j(gz)}{I(z)^s k(z,\,z)}\operatorname{tr}\chi(g)dz\right]\\ &=\lim_{s\to+0}\sideset{}{'}\sum_{g\in L_1}\int_{F_g}\frac{k(z,\,gz)j(gz)}{I(z)^s k(z,\,z)}\operatorname{tr}\chi(g)dz\\ &=\frac{(-1)^n}{a(\{k_i\})(2\pi)^n}\lim_{s\to+0}\left(e^{\frac{\pi}{2}s\sqrt{-1}}-e^{-\frac{\pi}{2}s\sqrt{-1}}\right)^n\sideset{}{'}\sum_{g\in L_1}\prod_{i=1}^{n}(\operatorname{sgn} a^{(i)})^{k_i}\cdot\frac{d(g)\operatorname{tr}\chi(g)}{m(g)^{1+s}}. \end{aligned}$$

Now, Lemmas 3.1, 3.2 imply that the series in the last equality has at most a pole of order 1 at $s=0$. It follows that $w=0$ if $n>1$. If $n=1$, putting $k_1=k$, $a^{(1)}=a$, we can write

$$w=-\frac{2\pi}{k-1}\lim_{s\to0}\,s\sideset{}{'}\sum_{g\in L_1}(\operatorname{sgn} a)\operatorname{tr}\chi(g)\left(\frac{d(g)}{m(g)}\right)^{1+s}. \tag{20}$$

3.5. Case vi). We have

$$\int_{F_{g_0}}=0$$

by the same argument as in [6, No. 22].

We state the result as

THEOREM 1. *If $k_i>2\ (1\leq i\leq n)$, the trace of $\mathfrak{T}(\Gamma\alpha\Gamma)$ is given in the following formulas.*

i) $n>1$.

$$\begin{aligned}\operatorname{tr}\mathfrak{T}(\Gamma\alpha\Gamma)&=v(F)\operatorname{tr}\chi(g_0)\prod_{i=1}^{n}(\operatorname{sgn} g_0^{(i)})^{k_i}\left(\frac{k_i-1}{4\pi}\right)\\ &\quad+\sideset{}{'}\sum_{g\in\mathfrak{G}_1}\frac{(-1)^n\operatorname{tr}\chi(g)}{[\Gamma(g):Z(\Gamma)]}\prod_{i=1}^{n}\frac{\zeta(g^{(i)})^{k_i-1}-\eta(g^{(i)})^{k_i-1}}{\zeta(g^{(i)})-\eta(g^{(i)})}(\det g^{(i)})^{1-\frac{k_i}{2}}.\end{aligned}$$

ii) $n=1$. *In this case we write k for k_1.*

$$\begin{aligned}\operatorname{tr}\mathfrak{T}(\Gamma\alpha\Gamma)&=v(F)\operatorname{tr}\chi(g_0)(\operatorname{sgn} g_0)^{k}\left(\frac{k-1}{4\pi}\right)\\ &\quad-\sideset{}{'}\sum_{g\in\mathfrak{G}_1}\frac{\operatorname{tr}\chi(g)}{[\Gamma(g):Z(\Gamma)]}\cdot\frac{\zeta(g)^{k-1}-\eta(g)^{k-1}}{\zeta(g)-\eta(g)}(\det g)^{1-\frac{k}{2}}\\ &\quad-\sideset{}{'}\sum_{g\in\mathfrak{G}_2}\frac{2\operatorname{tr}\chi(g)}{[\Gamma(g):Z(\Gamma)]}\cdot\frac{(\operatorname{Min}\{|\zeta(g)|,\,|\eta(g)|\})^{k-1}}{|\zeta(g)-\eta(g)|}(\det g)^{1-\frac{k}{2}}\end{aligned}$$

$$-\lim_{s\to 0}\frac{s}{2}\sum_{g\in\mathfrak{G}_2}'(sgn\,\zeta(g))\,\mathrm{tr}\,\chi(g)\left(\frac{d(g)}{m(g)}\right)^{1+s}.$$

Here g_0 is an arbitrary element in $\Gamma\alpha\Gamma\cap Z(G)$. $\mathfrak{G}_1$ (resp : $\mathfrak{G}_2$; $\mathfrak{G}_3$) *is a complete system of inequivalent elliptic elements* (resp : *hyperbolic elements leaving a parabolic point of Γ fixed; parabolic elements*) *in $\Gamma\alpha\Gamma$ with respect to the equivalence relation* (7). *$\Gamma(g)$ is the group of all $\gamma\in\Gamma$ such that*

$$g=\varepsilon\gamma g\gamma^{-1}\quad \textit{for some } \varepsilon\in Z(\Gamma).$$

$v(F)$ denotes the volume of a fundmental domain of Γ in $\mathfrak{H}_n$ relative to dz (see (6)). *For $g\in GL(2\boldsymbol{R})$ $\zeta(g)$, $\eta(g)$ denote the eigenvalues of g. Let g be a parabolic element in $\Gamma\alpha\Gamma$ and x the fixed point of g. Let ρ be an element in G such that $\rho x=\infty$. Then, $d(g)$, $m(g)$ are defined by* (17)–(19) *substituting ρ for ρ_ν.*[4]

§4. A formula for $\mathrm{tr}\,\mathfrak{T}(\mathfrak{q})$.

4.1. Let A be an indefinite quaternion algebra over a totally real number field Φ of degree m over $\boldsymbol{Q}$. Writing $\Phi^{(i)}(1\leqq i\leqq m)$ for the completion of Φ with respect to the infinite valuation $\mathfrak{p}_{\infty_i}$ of Φ, we get

$$A\otimes_Q\boldsymbol{R}=A^{(1)}\oplus\cdots\oplus A^{(m)},$$
$$A^{(i)}=A\otimes_\Phi\Phi^{(i)}.$$

For $\alpha\in A$, $\alpha^{(i)}$ is defined by $\alpha=\sum_{i=1}^{m}{}'\alpha^{(i)}$ with $\alpha^{(i)}\in A^{(i)}$ and for every $\alpha\in A$ (resp : $A^{(i)}$) the reduced norm of α from A to Φ (resp : from $A^{(i)}$ to $\Phi^{(i)}$) is simply denoted by $N(\alpha)$. We assume once and for all that $A^{(i)}=M_2(\boldsymbol{R})$ for $1\leqq i\leqq n$ and $A^{(i)}=\boldsymbol{K}$ for $n+1\leqq i\leqq m$.

We denote by $\mathfrak{g}$ and E_0 the ring of all integers in Φ and the group of all units in $\mathfrak{g}$, respectively. Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{g}$. We put

$$A_\mathfrak{p}=A\otimes_\Phi\Phi_\mathfrak{p},$$

$\Phi_\mathfrak{p}$ being the completion of Φ with respect to $\mathfrak{p}$. Denote by $\mathfrak{g}_\mathfrak{p}$ the valuation ring in $\Phi_\mathfrak{p}$. For a normal $\mathfrak{g}$-lattice $\mathfrak{M}$ in A, we write $\mathfrak{M}_\mathfrak{p}$ for the $\mathfrak{g}_\mathfrak{p}$ module in $A_\mathfrak{p}$ generated by $\mathfrak{M}$. $N(\mathfrak{M})$ denotes the norm of $\mathfrak{M}$.

Let $\mathfrak{O}$ be a maximal order in A and Γ the group of all units in $\mathfrak{O}$. The projection from A to $\sum_{i=1}^{n}{}'A^{(i)}$ maps Γ isomorphically onto a subgroup of $GL(2\boldsymbol{R})\times\cdots\times GL(2\boldsymbol{R})$ (n times), which is again denoted by Γ. Then Γ satisfies our assumption (A1), (A2). $\iota(G^0)/\iota(\Gamma^0)$ is not compact if and only if $A=M_2(\Phi)$.

Let $\mathfrak{A}$ be an integral two sided $\mathfrak{O}$-ideal. Let $\Delta(\mathfrak{A})$ be the set of all $\alpha\in A^*$ such

4) $d(g)/m(g)$ does not depend on a choice of ρ.

that α is a unit in $\mathfrak{O}_{\mathfrak{p}}$ for all $\mathfrak{P}$ dividing $N(\mathfrak{A})$. Let ρ be a unitary representation of $(\mathfrak{O}/\mathfrak{A})^*$ and φ_i $(n+1\leqq i\leqq m)$ a unitary representation of K^*. Since $(\mathfrak{O}/\mathfrak{A})^*$ is the direct product of $(\mathfrak{O}_{\mathfrak{p}}/\mathfrak{A}_{\mathfrak{p}})^*(\mathfrak{P}|N(\mathfrak{A}))$, ρ may be considered as a representation of $\Delta(\mathfrak{A})$ in a natural manner. We put

$$\chi(\alpha)=\rho(\alpha)\otimes\varphi_{n+1}(\alpha^{(n+1)})\otimes\cdots\otimes\varphi_m(\alpha^{(m)}) \tag{21}$$

for $\alpha\in\Delta(\mathfrak{A})$. Then χ satisfies our assumption (R 1). We assume that χ satisfies also $(R, 2)$. We can now define the linear transformation $\mathfrak{T}(\Gamma\alpha\Gamma)$ in $S(\Gamma, \{k_i\}, \chi)$ for any $\alpha\in\Delta(\mathfrak{A})$. If $\mathfrak{q}$ is an integral ideal in $\mathfrak{g}$ prime to $N(\mathfrak{A})$, we put

$$\mathfrak{T}(\mathfrak{q}, \mathfrak{O})=\sum{}'\mathfrak{T}(\Gamma\alpha\Gamma), \tag{22}$$

the sum being extended over all the double cosets $\Gamma\alpha\Gamma$ such that $\alpha\mathfrak{O}$ is an integral right $\mathfrak{O}$-ideal of norm $\mathfrak{q}$.

4.2. Let $B(\mathfrak{q})$ be the union of all the double cosets $\Gamma\alpha\Gamma$ appearing in (22). It is clear that we have

$$B(\mathfrak{q})=\{\alpha\in\mathfrak{O},\ N(\alpha)\mathfrak{g}=\mathfrak{q}\}. \tag{23}$$

Hence $B(\mathfrak{q})\neq\phi$ only if $\mathfrak{q}$ is a principal ideal, and $Z(B(\mathfrak{q}))\neq 0$ only if $\mathfrak{q}$ is of the form $\mathfrak{q}=q_0^2\mathfrak{g}$ with $q_0\in\mathfrak{g}$. Fixing such a q_0, we get $Z(B(\mathfrak{q}))=q_0E_0$.

We define $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ in the same way as in Theorem 1, taking $B(\mathfrak{q})$ for $\Gamma\alpha\Gamma$. In order to obtain an explicit formula for $\mathrm{tr}\,\mathfrak{T}(\mathfrak{q}, \mathfrak{O})$, we want to determine $\mathfrak{E}_1$ and, besides, $\mathfrak{E}_2$, $\mathfrak{E}_3$ in case $n=1$.

First we are going to determine $\mathfrak{E}_1$. Since the following argument is quite analogous to the argument in [6, §6], we shall omit the details.

Let J be the set of all elliptic elements in $B(\mathfrak{q})$ and α an element in J. Let $\Phi(\alpha)$ denote the subfield of A generated by α over Φ and put $\mathfrak{o}=\Phi(\alpha)\cap\mathfrak{O}$. $\Phi(\alpha)$ is then a totally imaginary maximal subfield of A and $\mathfrak{o}$ is an order in $\Phi(\alpha)$. Let α' be another element in J and define $\mathfrak{o}'$ as above by means of α'. If $\alpha'=\varepsilon\gamma\alpha\gamma^{-1}$ for $\gamma\in\Gamma$, $\varepsilon\in E_0$, we have $\mathfrak{o}'=\gamma\mathfrak{o}\gamma^{-1}$. Let $\tilde{\Omega}$ be the set of all subrings $\mathfrak{o}$ of A with the following properties.

1° $K=\Phi(\mathfrak{o})$ is a totally imaginary maximal subfield of A.

2° $\mathfrak{o}=K\cap\mathfrak{O}$.

Let $\mathfrak{d}$ be the discriminant of A over Φ. Then, we obtain

Lemma 4.1. *Let $\tilde{\Omega}_0$ be the set of all $\mathfrak{o}$ (taken up to isomorphisms) with the following properties.*

1° *$\mathfrak{o}$ is an order in a totally imaginary quadratic extension of Φ, in which all the prime divisors of $\mathfrak{d}$ do not split.*

2° *the conductor of $\mathfrak{o}$ is prime to $\mathfrak{d}$.*

Then $\tilde{\Omega}_0$ forms a full system of representatives of the isomorphism classes in $\tilde{\Omega}$.

The proof is the same as that of [6, Lemma 19]. It is to be noted that, if $\mathfrak{o}\in\tilde{\Omega}_0$, $\mathfrak{o}$ can be embedded in A so that we have $\mathfrak{o}=\Phi(\mathfrak{o})\frown\mathfrak{O}$, and hence $\tilde{\Omega}_0$ can be thought of as a subset of $\tilde{\Omega}$.

We say that $\mathfrak{o}_1, \mathfrak{o}_2\in\tilde{\Omega}$ are conjugate if there is a $\gamma\in\Gamma$ such that $\mathfrak{o}_2=\gamma\mathfrak{o}_1\gamma^{-1}$. Fixing an $\mathfrak{o}$ in $\tilde{\Omega}_0$, we now count the number of the conjugate classes contained in the isomorpeism class of $\mathfrak{o}$. Let $\mathfrak{o}_1=\mu_1\mathfrak{o}\mu_1^{-1}(\mu_1\in A^*)$ be an element in $\tilde{\Omega}$ isomorphic to $\mathfrak{o}$. By [4, Satz 7] we have

$$\mathfrak{O}\mu_1=\mathfrak{M}\mathfrak{a},$$

where $\mathfrak{M}$ is a two sided ideal of $\mathfrak{O}$ and $\mathfrak{a}$ is an ideal of $\mathfrak{o}$ (the word 'an ideal of $\mathfrak{o}$' or '$\mathfrak{o}$-ideal' should be understood in the sense stated in [4, §3]).

Put $K=\Phi(\mathfrak{o})$. Let T be the group of all the two sided $\mathfrak{O}$-ideals and T' the subgroup of T consisting of all two sided ideals generated by $\mathfrak{o}$-ideals. Let $\{\mathfrak{M}\}$ be a full system of representatives in T of T/T' and $\{\mathfrak{a}\}$ a full system of representatives of the ideal classes in $\mathfrak{o}$. We take and fix an element ι in A such that the automorphism $x\rightarrow\iota x\iota^{-1}$ of A induces on K the isomorphism of K over Φ which is not the identity. In this notation we can attach to the conjugate class of $\mathfrak{o}_1$ two couples $(\mathfrak{M}_0, \mathfrak{a}_0)$, $(\mathfrak{M}_1, \mathfrak{a}_1)$ which are defined by

$$\mathfrak{O}\mu_1\iota^\nu=\mathfrak{M}_\nu\mathfrak{a}_\nu\xi,\ \xi\in K,\ \nu=0,1.$$

$(\mathfrak{M}_0, \mathfrak{a}_0)$ and $(\mathfrak{M}_1, \mathfrak{a}_1)$ coincide if and only if we have

$$\mathfrak{O}\mu_1\iota=\mathfrak{O}\mu_1\xi \text{ for some } \xi\in K. \tag{24}$$

Conversely, let $\mathfrak{M}, \mathfrak{a}$ be as above and assume that $\mathfrak{M}\mathfrak{a}$ is a principal ideal of $\mathfrak{O}$: $\mathfrak{M}\mathfrak{a}=\mathfrak{O}\mu_1$. Put $\mathfrak{o}_1=\mu_1\mathfrak{o}\mu_1^{-1}$. By [4, Satz 7] we have $\mathfrak{o}_1\in\Omega$, and the conjugate class of $\mathfrak{o}_1$ is uniquely determined by $(\mathfrak{M}, \mathfrak{a})$. Now, if (24) holds, the conjugate class of $\mathfrak{o}_1$ shall be conuted with a multiplicity 1/2. It turns out that the number of the conjugate classes (in the above sense) contained in the isomorphism class of $\mathfrak{o}$ is equal to

$$\frac{h(\mathfrak{o})}{2h}\prod_{\mathfrak{p}\mid\mathfrak{d}}\left(1-\left(\frac{\mathfrak{o}}{\mathfrak{p}}\right)\right), \tag{25}$$

where h is the class number of A, $h(\mathfrak{o})$ is the class number of $\mathfrak{o}$ and $\left(\frac{\mathfrak{o}}{\mathfrak{p}}\right)$ stands for the Artin symbol $\left(\frac{K}{\mathfrak{p}}\right)$.

4.3. $\mathfrak{o}_1$ being as above, let $E(\mathfrak{o}_1)$ be the group of all units in $\mathfrak{o}_1$. Let α, α' be elements in $\mathfrak{o}_1\cap J$. If $\alpha'=\varepsilon\gamma\alpha\gamma^{-1}$ with $\gamma\in\Gamma$, $\varepsilon\in E_0$, we have $\Phi(\alpha')=\gamma\Phi(\alpha)\gamma^{-1}$, and hence $\mu_1K\mu_1^{-1}=\gamma\mu_1K\mu_1^{-1}\gamma^{-1}$. Therefore, $\mu_1^{-1}\gamma\mu_1=\xi$ or $\iota\xi$ with $\xi\in K$. If (24) does not hold, we must have $\mu_1^{-1}\gamma\mu_1=\xi$, hence $\gamma\in\mathfrak{o}_1$, $\varepsilon\alpha=\alpha'$. In this case we see also that $\Gamma(\alpha)=E(\mathfrak{o}_1)$. Assume that (24) holds. There exists a $\gamma_1\in\Gamma$ such that $\mu_1\iota=\gamma_1\mu_1\xi_1$ with $\xi_1\in K$. Then, we have $\gamma\in\mathfrak{o}_1$ or $\gamma\in\gamma_1\mathfrak{o}_1$; $\alpha'=\varepsilon\alpha$ or $\alpha'=\varepsilon\gamma_1\alpha\gamma_1^{-1}$. It follows that

$$\Gamma(\alpha)=\begin{cases} E(\mathfrak{o}_1)\cup\gamma_1 E(\mathfrak{o}_1)) \\ \qquad\qquad \text{if } \alpha=\pm\gamma_1\alpha\gamma_1^{-1} \\ E(\mathfrak{o}_1) \text{ otherwise.}\end{cases}$$

Note that we have $[\Gamma(\alpha):E(\mathfrak{o}_1)]=2$ in the first case.

4.4. Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{g}$. If $\mathfrak{p}$ is prime to $\mathfrak{d}$, every two sided $\mathfrak{O}_\mathfrak{p}$-ideal is a power of $\mathfrak{p}\mathfrak{O}_\mathfrak{p}$. Consequently, if $N(\mathfrak{N})$ is prime to $\mathfrak{d}$ for an integral two sided $\mathfrak{O}$-ideal $\mathfrak{N}$, $\mathfrak{N}$ is contained in T'. We may therefore assume that all $\mathfrak{M}$ in $\{\mathfrak{M}\}$ are integral two sided $\mathfrak{O}$-ideals such that every prime divisor of $N(\mathfrak{M})$ divides $\mathfrak{d}$. Then, if $N(\mathfrak{A})$ is prime to $\mathfrak{d}$, $N(\mathfrak{M})$ is prime $N(\mathfrak{A})$ for all $\mathfrak{M}$. We assume also that all $\mathfrak{a}$ in $\{\mathfrak{a}\}$ are integral $\mathfrak{o}$-ideals such that prime to $N(\mathfrak{A})$.

Suppose that $\mathfrak{M}\mathfrak{a}=\mathfrak{O}\mu_1$ and $\mathfrak{o}_1=\mu_1\mathfrak{o}\mu_1^{-1}$. The elements in $\mathfrak{o}\cap J$ are in one-to-one correspondence with the elements in $\mathfrak{o}_1\cap J$ by $\alpha\to\alpha_1=\mu_1\alpha\mu_1^{-1}$. Since μ_1 is contained in $\Delta(\mathfrak{A})$ by our choice of $\{\mathfrak{M}\}$, $\{\mathfrak{a}\}$, we have $\rho(\alpha_1)=\rho(\alpha)$. For the sake of simplicity we write

$$\Psi(\alpha)=\operatorname{tr}\chi(\alpha)\prod_{i=1}\frac{\eta(\alpha^{(i)})^{k_i-1}-\zeta(\alpha^{(i)})^{k_i-1}}{\eta(\alpha^{(i)})-\zeta(\alpha^{(i)})}N(\alpha^{(i)})^{1-\frac{k_i}{2}} \tag{26}$$

for $\alpha\in J$. It is then obvious that $\Psi(\alpha_1)=\Psi(\alpha)$.

4.5. By the consideration in §4.3, if $\{\alpha_1\}$ denotes a representative system of the equivalence classes in $\mathfrak{o}_1\cap J$, we have

$$\sum_{\alpha_1}\frac{\Psi(\alpha_1)}{[\Gamma(\alpha):E_0]}=\binom{1}{2}\sum_{\substack{\alpha \bmod E_0\\ \alpha\in\mathfrak{o}_1\cap J}}\frac{\Psi(\alpha)}{[E(\mathfrak{o}_1):E_0]}\,.$$

The factor $\binom{1}{2}$ oppears only if (24) holds. Together with the results in §§4.2, 4.4, we get

$$\sum_{\alpha\in\mathfrak{G}_1}\frac{\Psi(\alpha)}{[\Gamma(\alpha):E_0]}=\frac{1}{2h}\sum_{\mathfrak{o}\in\mathfrak{S}_0}\frac{h(\mathfrak{o})}{[E(\mathfrak{o}):E_0]}\sum_{\mathfrak{p}|\mathfrak{d}}\left(1-\left(\frac{\mathfrak{o}}{\mathfrak{p}}\right)\right)\sum_{\alpha \bmod E_0,\,\alpha\in\mathfrak{o}\cap J}\Psi(\alpha).$$

4.6. In this section we assume that $n=1$. $\mathfrak{G}_2$, $\mathfrak{G}_3$ are empty if A is a division algebra. Therefore, we restrict ouselves to the case where $A=M_2(\boldsymbol{Q})$. We can assume $\mathfrak{O}=M_2(\boldsymbol{Z})$; then $\mathfrak{A}$ is written in the form $\mathfrak{A}=N\mathfrak{O}$ with $N\in\boldsymbol{Z}$.[5] In this case there exists only one equivalence class of para bolic points of Γ, which is

5) If $A=M_2(\Phi)$, any maximal order in A is isomorphic to the ring of all $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $a, d\in\mathfrak{g}$, $c\in\mathfrak{a}$, $b\in\mathfrak{a}^{-1}$, $\mathfrak{a}$ being a certain member of a given representative system of the ideal classes in $\mathfrak{g}$. Cf. [2].

represented by ∞.

LEMMA 4.2. *Write $\mathfrak{q}=qZ$ with $q\in Z$, $q>0$. Then we can take for $\mathfrak{E}_2$ the set of all α such that*

$$\alpha=\begin{pmatrix} a & b \\ 0 & d \end{pmatrix},\ a,\ b,\ d\in Z$$

$$ad=q,\ 0<a<d,\ 0\leq b\leq\frac{d-a}{2}.$$

Furthermore, we have

$$[\Gamma(\alpha):\ E_0]=\begin{cases} 2 & 2b\equiv 0 \bmod (a-d) \\ 1 & \textit{otherwise}. \end{cases}$$

PROOF. Let $\alpha_1=\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}$, $\alpha_2=\begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}$ be elements of type iv) in $B(qZ)$. Let $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of Γ. If $\varepsilon'\gamma\alpha_1\gamma^{-1}=\gamma_2$ for $\varepsilon'=\pm 1$, we have

$$\varepsilon\varepsilon' a=ada_1-c(ab_1+bd_1),$$
$$\varepsilon\varepsilon' b_2=-ada_1+a(ab_1+bd_1),$$
$$\varepsilon\varepsilon' d_2=-bca_1+a(cb_1+dd_1),$$
$$c(da_1-cb_1-dd_1)=0.$$

Here $\varepsilon=ad-bc=\pm 1$. Consequently, if $c=0$, then we have $\varepsilon' a_2=a_1$, $\varepsilon' d_2=d_1$; if $da_1-ca_1-dd_1=0$, then we have $\varepsilon' a_2=d_1$, $\varepsilon' d_2=a_1$.

Suppose that, for given α_1 and α_2, we have $\varepsilon' a_2=d_1$, $\varepsilon' d_2=a_1$. Put $e=(d_1-a_1, b_1)$ and find a, $b\in Z$ such that

$$(d_1-a_1)b+d_1a=e.$$

Putting $d_1=b_1/e$, $c=-(d_1-a_1)/e$, we get $da_1-cb_1-dd_1=0$, $ab-bc=1$. Hence, after replacing α_1 by a suitable element equivalent to α_1, we can assume that $\varepsilon' a_2=a_1$, $\varepsilon' d_2=d_1$. Then, it is easy to see that α_1 is equivalent to α_2 if and only if $b_1\equiv\pm b_2 \bmod (a_1-d_1)$. Putting $\alpha_1=\alpha_2$, we see also that there exists a $\gamma\in\Gamma(a_1)$, $\gamma\neq 1$ if and only if $b_1\equiv -b_1 \bmod (a_1-d_1)$. Therefore Lemma 4.2 follows.

4.7. Under the same assumptions as in §4.6, $\mathfrak{E}_3$ is not empty if and only if $\mathfrak{q}$ is of the form $\mathfrak{q}=q_0^2Z$ with $q_0\in Z$, $q_0>0$. This being so, we have

LEMMA 4.3. *We can take for $\mathfrak{E}_3$ the set of all α such that*

$$\alpha=\begin{pmatrix} q_0 & b \\ 0 & q_0 \end{pmatrix},\ b\in Z,\ b>0.$$

Furthermore, $d(a)/m(a)=q_0/b$ for all α.

The proof is so easy that we omit it. Since $\rho(\alpha)$ depend only on $b \bmod N$, we see that the contribution of $\alpha\in\mathfrak{E}_3$ to $\mathrm{tr}\,\mathfrak{T}(\mathfrak{q}, \mathfrak{O})$ is

$$-\lim_{s\to 0}\frac{s}{2}\sum_{\alpha\in\mathfrak{C}_s}{}' \operatorname{tr}\chi(\alpha)\left(\frac{d(\alpha)}{m(\alpha)}\right)^{1+s}$$
$$=-\frac{q_0}{2N}\sum_{0\leq b<N}{}' \operatorname{tr}\rho\begin{pmatrix} q_0 & b\\ 0 & q_0\end{pmatrix}.$$

4.8. Summing up, we obtain

THEOREM 2. *Suppose $k_i>2(1\leqq i\leqq n)$. Let $\mathfrak{d}$ be the discriminant of A over Φ. If $N(\mathfrak{A})$ is prime to $\mathfrak{d}$, the trace of $\mathfrak{T}(\mathfrak{q}, \mathfrak{O})$ is given in the following formulas.*

i) *A is either a division algebra or $M_2(\Phi)$ with $\Phi\neq\boldsymbol{Q}$.*

$$\operatorname{tr}\mathfrak{T}(\mathfrak{q}, \mathfrak{O})=\delta(\mathfrak{q})v(F)\operatorname{tr}\chi(q_0)\prod_{i=1}^{n}\left(\frac{k_i-1}{4\pi}\right)(\operatorname{sgn} q_0^{(i)})^{k_i}$$
$$+\frac{(-1)^n}{2h}\sum_{\mathfrak{o}\in\tilde{\Omega}_0}{}'\frac{h(\mathfrak{o})\prod_{\mathfrak{p}\mid\mathfrak{d}}\left(1-\left(\frac{\mathfrak{o}}{\mathfrak{p}}\right)\right)}{[E(\mathfrak{o}):E_0]}\sum_{\substack{\alpha\in J(\mathfrak{o})\\ \alpha \bmod E_0}}{}'\Psi(\alpha).$$

Here h is the class number of A. $\tilde{\Omega}_0$ and $\Psi(\alpha)$ are defined in Lemma 4.1 *and in* (26), *respectively. $J(\mathfrak{o})$ is the set of all $\alpha\in\mathfrak{o}$ such that $\alpha\notin\Phi$, $N(\alpha)\mathfrak{g}=\mathfrak{p}$. $h(\mathfrak{o})$, $E(\mathfrak{o})$ denote the class number of $\mathfrak{o}$, the group of all units in $\mathfrak{o}$, respectively. $\left(\frac{\mathfrak{o}}{\mathfrak{p}}\right)$ denote the Artin symbol $\left(\frac{K}{\mathfrak{p}}\right)(K=\Phi(\mathfrak{o}))$. $\delta(\mathfrak{q})=1$ if $\mathfrak{q}=q_0{}^2\mathfrak{g}$ for some $q_0\in\mathfrak{g}$ and otherwise $\delta(\mathfrak{p})=0$.*

ii) *$A=M_2(\boldsymbol{Q})$. In this case we put $\mathfrak{O}=M_2(\boldsymbol{Z})$, $\mathfrak{A}=N\mathfrak{O}$, $\mathfrak{q}=q\boldsymbol{Z}(N, q\in\boldsymbol{Z})$.*

$$\operatorname{tr}\mathfrak{T}(q\boldsymbol{Z}, \mathfrak{O})=\delta(q)v(F)\operatorname{tr}\chi(q_0)\frac{k-1}{4\pi}-\frac{1}{2}\sum_{\mathfrak{o}\in\tilde{\Omega}_0}{}'\frac{h(\mathfrak{o})\prod_{\mathfrak{p}\mid\mathfrak{d}}\left(1-\left(\left(\frac{\mathfrak{o}}{\mathfrak{p}}\right)\right)\right)}{[E(\mathfrak{o}):E_0]}\sum_{\substack{\alpha\in J(\mathfrak{o})\\ \alpha \bmod E_0}}{}'\Psi(\alpha)$$
$$-q^{1-\frac{k}{2}}\sum_{\substack{ad=q,\, 0<a<\sqrt{q}\\ 0\leq b<d-a}}{}'\frac{a^{k-1}}{d-a}\operatorname{tr}\rho\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}-\delta(q)\frac{q_0}{2N}\sum_{0\leq b<N}{}'\operatorname{tr}\rho\begin{pmatrix} q_0 & b\\ 0 & q_0\end{pmatrix}.$$

The notation is the same as in i). *If $q=q_0^2$, q_0 is supposed to be positive.*

COROLLARY. *$\operatorname{tr}\mathfrak{T}(\mathfrak{q}, \mathfrak{O})=0$ if $\mathfrak{q}$ is not a principal ideal of the form $q\mathfrak{g}$, q being a totally positive element in $\mathfrak{g}$.*

Remark. 1) If ρ is the identity representation, our formula ii) coincides with the formula given in [8, p. 85] up to a factor $q^{1-k/2}$. 2) Though $\tilde{\Omega}_0$ is not a finite set, there exist only a finite number of $\mathfrak{o}\in\tilde{\Omega}_0$ such that $J(\mathfrak{o})\neq\phi$ for a given $\mathfrak{q}$. 3) Apparently, $\operatorname{tr}\mathfrak{T}(\mathfrak{q}, \mathfrak{O})$ does not depend on $\mathfrak{O}$. However, it might not be the case if ρ is not the identity representation, for an embedding of $\mathfrak{o}\in\tilde{\Omega}_0$ in A is restricted by a condition $\mathfrak{o}=\Phi(\mathfrak{o})\cap\mathfrak{O}$.

4.9. Let Γ_1 be the group of all $\gamma\in\Gamma$ with $N(\gamma)=1$. By [3, Satz 5] we have $[\Gamma:E_0\Gamma_1]=2^m/[E_0:E_0']$. Here E_0' is the group of all $\varepsilon\in E_0$ such that $\varepsilon^{(i)}>0$

$(n+1 \leqq i \leqq m)$. By [6, (53)] we get

$$v(F)=\frac{2^{n-m+1}D_0^{3/2}h_0\zeta_0(2)}{\pi^{2m-n}h}\prod_{\mathfrak{p}|\mathfrak{d}}(N_{\Phi/Q}\mathfrak{p}-1), \tag{27}$$

where D_0, h_0, $\zeta_0(s)$ denote the discriminant of Φ over $\boldsymbol{Q}$, class number of Φ, the zetafunction of Φ, respectively.

4.10. Let $\varphi_\lambda(1 \leqq \lambda \leqq h)$ be representatives of the equivalence classes of right $\mathfrak{O}$-ideals. Let $\mathfrak{O}_\lambda$ be the left order of φ_λ and Γ_λ the group of all units in $\mathfrak{O}_\lambda$. If we take $\varphi_{\lambda\mathfrak{p}}=\mathfrak{O}_\mathfrak{p}$ for all $\mathfrak{p}$ dividing $N(\mathfrak{A})$, Γ_λ is contained in $\Delta(\mathfrak{A})$. Therefore we can define $\mathfrak{T}(\mathfrak{q}, \mathfrak{O}_\lambda)$ for each λ. Let $\mathfrak{T}(\mathfrak{q})$ be the linear transformation defined in [7, §3]. It is immediately seen that

$$\operatorname{tr}\mathfrak{T}(\mathfrak{q})=\sum_{i=1}^{h}\operatorname{tr}\mathfrak{T}(\mathfrak{q}, \mathfrak{O}_\lambda). \tag{28}$$

Therefore, we obtain a formula for $\operatorname{tr}\mathfrak{T}(\mathfrak{q})$ by Theorem 2.

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