On Neumann problem for non-symmetric second order partial differential operators of elliptic type

By Seizô Itô

§1. **Introduction.** Let M be an m-dimensional orientable manifold of class C^∞, and let D be a subdomain of M whose closure $\overline{D}$ is compact and whose boundary $S=\overline{D}-D$ consists of a finite number of $(m-1)$-dimensional simple hypersurfaces of class C^3. Let A be an elliptic differential operator of the following form:

$$Au(x)=\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left\{\sqrt{a(x)}\left[a^{ij}(x)\frac{\partial u(x)}{\partial x^j}-b^i(x)u(x)\right]\right\} \quad \text{for} \quad u\in C^2(D), \tag{1.1}$$

where $\|a^{ij}(x)\|$ and $\|b^i(x)\|$ $(1\leqq i, j\leqq m)$ are contravariant tensor of class C^2 on $\overline{D}$, $\|a^{ij}(x)\|$ is symmetric and strictly positive definite and $a(x)=\det\|a_{ij}(x)\|=\det\|a^{ij}(x)\|^{-1}$. We denote by dx and dS_ξ respectively the volume element in D and the hypersurface element on S with respect to the Riemannian metric defined by $\|a_{ij}(x)\|$. We also denote by $\dfrac{\partial u(\xi)}{\partial \boldsymbol{n}_\xi}$ and $\beta(\xi)$ respectively the outer normal derivative of the function $u(x)$ and the outer normal component of the 'vector' $\|b^i(x)\|$ at the point $\xi\in S$. The adjoint differential operator A^* of A is defined as follows:

$$A^*u(x)=\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left\{\sqrt{a(x)}\,a^{ij}(x)\frac{\partial u(x)}{\partial x^j}\right\}+b^i(x)\frac{\partial u(x)}{\partial x^i} \quad \text{for} \quad u\in C^2(D). \tag{1.1*}$$

We shall consider the second boundary value problem

$$\begin{cases} Au(x)=f(x) & \text{in } D \\ \dfrac{\partial u(\xi)}{\partial \boldsymbol{n}_\xi}-\beta(\xi)u(\xi)=\varphi(\xi) & \text{on } S \end{cases} \tag{1.2}$$

and also the adjoint problem

$$\begin{cases} A^*u(x)=f(x) & \text{in } D \\ \dfrac{\partial u(\xi)}{\partial \boldsymbol{n}_\xi}=\varphi(\xi) & \text{on } S \end{cases} \tag{1.2*}$$

where $f(x)$ and $\varphi(\xi)$ are functions continuous on $\overline{D}$ and on S respectively.

The fundamental solution $U(t, y, x)$ of the initial-boundary value problem of the parabolic equation:

$$\frac{\partial u}{\partial t}=Au+f \ (t>0,\ x\in D), \quad u|_{t=0}=u_0, \quad \frac{\partial u}{\partial n}-\beta u=\varphi \ \text{(on } S) \tag{1.3}$$

is constructed in [3] and is also the fundamental solution of the adjoint initial-boundary value problem:

$$(1.3^*) \qquad \frac{\partial u}{\partial t}=A^*u+f \ (t>0,\ x\in D), \quad u|_{t=0}=u_0, \quad \frac{\partial u}{\partial n}=\varphi \ (\text{on } S).$$

—— see [2] and [3].

In the present paper, we shall show that there exists a function $\omega(x)>0$ on $\overline{D}$ satisfying

$$(1.4) \qquad \int_D \omega(y)U(t, y, x)dy=\omega(x) \quad\text{and}\quad \int_D \omega(x)dx=1$$

and that

$$(1.5) \qquad K(y, x)=\int_0^\infty \{U(t, y, x)-\omega(x)\}dx$$

is well defined whenever $x, y\in\overline{D}$ and $x\neq y$, and $K(y, x)$ is a kernel function of the boundary value problem (1.2) and also that of (1.2*). Similar results in the case where $b^i(x)\equiv 0$ $(i=1,\cdots, m)$ (and accordingly (1.2*) is identical with (1.2)) are obtained in [4]; in this case $\omega(x)$ is constant. Corresponding results in the case of Dirichlet problem, or in the case where A and A^* are replaced by $A-c(x)$ and $A^*-c(x)$ respectively (here $c(x)$ is non-negative and not identically zero), are contained in [3; §10]; in these cases $\omega(x)\equiv 0$.

Our result on the relation between the invariant measure (see §2) and the second boundary value problems (Neumann problems) is somewhat interesting in the view-point of probability theory. In fact, N. Ikeda [1] has already obtained more general results on boundary value problems in two-dimensional domains by means of purely probabilistic method. The existence of invariant measure is closely related with mean ergodic theorems. In the present paper, we shall give the purely analytical proofs of the existence of invariant measure and the existence of solutions of the second boundary value problems.

§2. **Invariant measure.** Let $U(t, y, x)$ be the fundamental solution of initial-boundary value problems (1.3) and (1.3*). It is proved in [3] that

$$(2.1) \qquad U(t, y, x)>0 \text{ for any } t>0 \text{ and any } y, x\in\overline{D}$$

and

$$(2.2) \qquad \int_D U(t, y, x)dx=1 \text{ for any } t>0 \text{ and any } y\in D.$$

By definition, a bounded Borel measure μ on D is called an *invariant measure of the fundamental solution* $U(t, y, x)$ if

$$\mu(E)=\int_E dx \int_D U(t, y, x)d\mu(y) \qquad \text{for any Borel set } E\subset \overline{D}. \tag{2.3}$$

By virtue of (2.1) and (2.2), it follows from (2.3) that such measure μ is absolutely continuous with respect to dx and the density $\omega(x)$ satisfies that

$$\omega(x)>0 \quad \text{for any} \quad x \in \overline{D}, \tag{2.4}$$

$$\omega(x)=\int_D \omega(y)U(t, y, x)dy \qquad \text{for any} \quad t>0 \text{ and any } x \in \overline{D}. \tag{2.5}$$

and accordingly, by means of the properties of the fundamental solution stated in [3], that $\omega \in C^2(D)\cap C^1(\overline{D})$ and

$$A\omega=0 \quad \text{in} \quad D \quad \text{and} \quad \frac{\partial \omega}{\partial \boldsymbol{n}}-\beta\omega=0 \quad \text{on } S. \tag{2.6}$$

LEMMA 2.1. *Let μ be a bounded Borel measure on D, and assume that $\int_D A^*h(x)d\mu(x)=0$ for any $h \in C^2(\overline{D})$ satisfying $\frac{\partial h}{\partial \boldsymbol{n}}=0$ on S. Then μ is an invariant measure of $U(t, y, x)$.*

PROOF. For any continuous function f on $\overline{D}$, the function

$$h(t, y)=\int_D U(t, y, x)f(x)dx$$

satisfies that

$$\frac{\partial h}{\partial t}=A^*h \ \ (\text{on } (0, \infty)\times \overline{D}), \quad \frac{\partial h}{\partial \boldsymbol{n}}=0 \quad (\text{on } (0, \infty)\times S)$$

and that $\lim_{t\downarrow 0} h(t, y)=f(y)$ boundedly in D. Hence, from the assumption of this lemma, we obtain that

$$\frac{\partial}{\partial \tau}\int_D h(\tau, y)\, d\mu(y)=\int_D A^*h(\tau, y)d\mu(y)=0.$$

Integrating both sides of this equality in $0<\tau<t$ and using Fubini's theorem, we get

$$\int_D f(x)dx\int_D U(t, y, x)\, d\mu(y)-\int_D f(y)d\mu(y)=0.$$

Hence μ is an invariant measure since f is an arbitrary continuous function on $\overline{D}$.

LEMMA 2.2. *Let u be a continuous function on $\overline{D}$, and assume that $\int_D Ah(x)u(x)dx=0$ holds for any $h\in C^2(D)\cap C^1(\overline{D})$ such that $\int_D |Ah(x)|dx<\infty$ and $\frac{\partial h}{\partial \boldsymbol{n}}-\beta h=0$ on S. Then u is constant on $\overline{D}$.*

PROOF. By means of the similar argument to the proof of Lemma 1, we may show that $\frac{\partial}{\partial \tau}\int_D U(\tau, y, x)u(x)dx=0$ and accordingly that

$$u(y)=\int_D U(t, y, x)u(x)dx \quad \text{for any} \quad t>0 \quad \text{and any} \quad y \in D.$$

Hence $u(y)$ must be constant on $\bar{D}$ (otherwise, (2.1) and (2.2) imply that $\int_D U(t, y_0, x)u(x)dx<u(y_0)$ at any maximizing point y_0 of u).

THEOREM 1. *There exists one and only one (up to a constant factor) invariant measure of the fundamental solution* $U(t, y, x)$.

PROOF. Let $\mathfrak{D}$ be a countable set which is dense in $C(\bar{D})$ with respect to the norm $\|f\|_\infty=\max_{x\in \bar{D}}|f(x)|$ and satisfies that $\alpha f+\beta g \in \mathfrak{D}$ for any $f, g \in \mathfrak{D}$ and any rational numbers α and β. We put

$$F_n^f(y)=\frac{1}{n}\int_0^n dt\int_D U(t, y, x)f(x)dx \quad \text{for} \quad f \in C(\bar{D}) \quad \text{and} \quad n=1, 2, \cdots.$$

Then we may easily see that the family of functions $\{F_n^f; f\in\mathfrak{D}, n=1, 2, \cdots\}$ is uniformly bounded and equi-continuous. Hence, by Ascoli-Arzelà's theorem, we may choose a subsequence $\{n'\}$ of natural numbers such that $F(f; y)=\lim_{n'\to\infty} F_{n'}^f(y)$ exists for any $f\in\mathfrak{D}$ and the convergence is uniform in $y\in\bar{D}$, and accordingly $F(f; y)$ is continuous in $y\in\bar{D}$. For any $h \in C^2(D)\cap C^1(\bar{D})$ such that $\int_D |Ah(x)|dx<\infty$ and $\frac{\partial h}{\partial \boldsymbol{n}}-\beta h=0$ on S, we have

$$\begin{aligned}\left|\int_D Ah(y)\cdot F_n^f(y)dy\right| &= \frac{1}{n}\left|\int_0^n dt\int_D h(y)dy\int_D \frac{\partial U(t, y, x)}{\partial t}f(x)dx\right| \\ &= \frac{1}{n}\left|\int_D h(y)dy\int_D U(n, y, x)f(x)dx-\int_D h(y)f(y)dy\right| \\ &\leqq \frac{2\|f\|_\infty}{n}\int_D |h(y)|dy.\end{aligned}$$

Letting $n=n'\to\infty$, we obtain $\int_D Ah(y)\cdot F(f; y)dy=0$. Hence, by Lemma 2.2, $F(f; y)$ is independent of y; we hereafter denote the value by $F(f)$. Then $|F(f)|\leqq\|f\|_\infty$ and $F(\alpha f+\beta g)=\alpha F(f)+\beta F(g)$ for any $f, g \in \mathfrak{D}$ and any rational numbers α and β, and $F(f)\geqq 0$ if $f \in \mathfrak{D}$ and $f(x)\geqq 0$ on $\bar{D}$. Hence F is extended to a positive and bounded linear functional on $C(\bar{D})$, and accordingly there exists a bound Borel measure μ on $\bar{D}$ such that

$$F(f)=\int_D f(x)d\mu(x) \quad \text{for any} \quad f\in C(\bar{D}).$$

Furthermore we may easily show that $\lim_{n'\to\infty} F_{n'}^f(y)=F(f)$ for any $f\in C(\bar{D})$. If $h\in C^2(\bar{D})$ and $\partial h/\partial n=0$ on S, we have

$$|F_n^{A^*h}(y)|=\left|\frac{1}{n}\int_0^n dt\int_D \frac{\partial U(t,y,x)}{\partial t}A^*h(x)dx\right|\leq\frac{2\|A^*h\|_\infty}{n}.$$

Letting $n=n'\to\infty$, we get $F(A^*h)=0$, namely $\int_D A^*h(x)d\mu(x)=0$. Hence, by Lemma 2.1, μ is an invariant measure of the fundamental solution $U(t, y, x)$.

In order to show the uniqueness of invariant measure, it suffices to prove that, if

$$\omega_j(x)=\int_D \omega_j(y)U(t,y,x)dy \quad \text{and} \quad \omega_j(x)>0 \quad \text{on} \quad \bar{D} \quad \text{for} \quad j=1,2, \tag{2.7}$$

then $\omega_1(x)=\kappa\omega_2(x)$ for some positive constant κ (see (2.4) and (2.5)). We put $\kappa=\min_{x\in\bar{D}}\omega_1(x)/\omega_2(x)$ and $q(x)=\omega_1(x)-\kappa\omega_2(x)$. Then $\kappa>0$ and $q(x_0)=0$ at some point $x_0\in\bar{D}$. Hence we obtain from (2.7) that

$$\int_D q(y)U(t,y,x_0)dy=0.$$

This equality and (2.1) imply that $q(y)\equiv 0$ on $\bar{D}$, q.e.d.

§ 3. **Kernel function of the boundary value problems.** Let $\omega(x)$ be the density of the invariant measure μ (stated in § 2) such that $\int_D \omega(x)dx=1$. Then the function

$$V(t,y,x)=U(t,y,x)-\omega(x) \tag{3.1}$$

satisfies the 'semi-group property'

$$V(t+s,y,x)=\int_D V(t,y,z)V(s,z,x)dz; \tag{3.2}$$

this fact may be proved by virtue of (2.2), (2.5) and the semi-group property of the fundamental solution $U(t, y, x)$ (stated in [3]).

Hereafter we shall denote by div and ∇ respectively the divergent-operator and the gradient-operator with respect to the metric defined by $\|a_{ij}(x)\|$, and by $\boldsymbol{b}(x)$ the vector field $\|b^i(x)\|$. Then

$$Au=\operatorname{div}(\nabla u)-\operatorname{div}(u\boldsymbol{b}),\ A^*u=\operatorname{div}(\nabla u)+(\boldsymbol{b}\cdot\nabla u) \text{ in } D \text{ and } \beta(\xi)=(\boldsymbol{b}(\xi)\cdot\boldsymbol{n}_\xi) \text{ on } S \tag{3.3}$$

where $(\ \cdot\ )$ denotes the 'inner product'. We put $p(x)=\log\omega(x)$, and define

$$\Delta_\omega u=\omega^{-1}\operatorname{div}(\omega\nabla u),\ A_\omega u=\Delta_\omega u-([\boldsymbol{b}-\nabla p]\cdot\nabla u). \tag{3.4}$$

Then, by simple computation, we obtain that

$$\omega^{-1}\operatorname{div}\{\omega(\nabla p-\boldsymbol{b})\}=\omega^{-1}A\omega=0 \quad \text{in } D, \quad \frac{\partial p}{\partial \boldsymbol{n}}=(\boldsymbol{b}\cdot\boldsymbol{n}) \quad \text{on } S \tag{3.5}$$

and accordingly that

$$A_\omega u=\omega^{-1}A(\omega u). \tag{3.6}$$

We denote the real function space $L^2(D, \mu)$ simply by L^2_μ, and define

$$(u, v)_\mu=\int_D u(x)v(x)d\mu(x) \quad \text{and} \quad \|u\|_\mu=(u, u)_\mu^{\frac{1}{2}}$$

for any $u, v\in L^2_\mu$. Since Δ_ω is formally self-adjoint with respect to the measure μ, there exists a system of eigenvalues and eigenfunctions $\{\lambda_n, \psi_n(x)\,;\ n=0, 1, 2, \cdots\}$ of the equation $\Delta_\omega\psi=-\lambda\psi$ associated with the boundary condition $\partial\psi/\partial\boldsymbol{n}=0$, such that

$$\lambda_0=0<\lambda_1\leqq\lambda_2\leqq\cdots\leqq\lambda_n\leqq\cdots, \quad \lim_{n\to\infty}\lambda_n=\infty \tag{3.7}$$

and

$$\{\psi_n\} \text{ is a complete orthonormal system in } L^2_\mu. \tag{3.8}$$

Using these notions, we prove the following

LEMMA 3.1. *If* $u\in C^2(D)\cap C^1(\overline{D})$, $\Delta_\omega u\in L^2_\mu$, $\partial u/\partial\boldsymbol{n}=0$ (on S) *and* $(u, 1)_\mu=0$, *then* $\|\nabla u\|_\mu^2\geqq\lambda_1\|u\|_\mu^2$.

PROOF. It follows from the assumption that u and $\Delta_\omega u$ are expressible in the form: $u=\sum_{n=1}^{\infty}\alpha_n\psi_n$ and $\Delta_\omega u=-\sum_{n=1}^{\infty}\lambda_n\alpha_n\psi_n$. Hence we have

$$\|\nabla u\|_\mu^2=-(\Delta_\omega u, u)_\mu=\sum_{n=1}^{\infty}\lambda_n\alpha_n^2\geqq\lambda_1\|u\|_\mu^2.$$

LEMMA 3.2. *Assume that* $v=v(t, x)$ *is continuous on* $[0, \infty)\times D$ *and satisfies* $\partial v/\partial t=A_\omega v$ *in* $(0, \infty)\times D$ *and that* $v(t, \cdot)$ *satisfies the assumption in Lemma* 3.1 *for any fixed* $t>0$. *Then*

$$\lambda_1\int_0^\infty\|v\|_\mu dt\leqq\|v_0\|_\mu<\infty, \qquad \textit{where } v_0=v(0, x). \tag{3.9}$$

PROOF. By means of (3.4), (3.5) and the assumption of this lemma, we may show that $(([\boldsymbol{b}-\nabla p]\cdot\nabla v), v)_\mu=0$ and accordingly, by Lemma 3.1,

$$\|v\|_\mu\frac{d\|v\|_\mu}{dt}=\frac{1}{2}\frac{d\|v\|_\mu^2}{dt}=\left(\frac{\partial v}{\partial t}, v\right)_\mu=(A_\omega v, v)_\mu=-\|\nabla v\|_\mu^2\leqq-\lambda_1\|v\|_\mu^2,$$

which implies $\lambda_1\|v\|_\mu\leqq-\dfrac{d\|v\|_\mu}{dt}$. Hence

$$\lambda_1\int_0^T\|v\|_\mu dt\leqq\|v_0\|_\mu-\|v(T, \cdot)\|_\mu\leqq\|v_0\|_\mu<\infty \qquad \text{for any } T>0.$$

Letting $T\to\infty$ in the above inequality, we obtain (3.9).

THEOREM 2. *$K(y, x)=\int_0^\infty \{U(t, y, x)-\omega(x)\}dt$ is well defined whenever $y, x\in\bar{D}$ and $y\neq x$, and $\int_0^\infty dt\int_D |U(t, y, x)-\omega(x)|dx\leqq M$ for a suitable constant M.*

PROOF. By means of (2.6) and properties of $U(t, y, x)$ stated in [3], we may show that, for any fixed $y\in\bar{D}$, the function

$$v(t, x)=V(t+1, y, x)\omega(x)^{-1}\equiv U(t+1, y, x)\omega(x)^{-1}-1 \text{ (see (3.1))}$$

satisfies all assumptions of Lemma 3.2. Hence

$$\lambda_1\int_0^\infty\left\{\int_D |V(t+1, y, x)|^2\frac{dx}{\omega(x)}\right\}^{\frac{1}{2}}dt\leqq\left\{\int_D |V(1, y, x)|^2\frac{dx}{\omega(x)}\right\}^{\frac{1}{2}}<\infty.$$

Accordingly, by means of (3.2) and Schwarz's inequality, we have

$$\begin{aligned}\lambda_1\int_0^\infty &|V(t+2, y, x)|dt\\ &\leqq\lambda_1\int_0^\infty\left\{\int_D |V(t+1, y, z)|^2\frac{dz}{\omega(z)}\right\}^{\frac{1}{2}}\left\{\int_D |V(1, z, x)|^2\omega(z)dz\right\}^{\frac{1}{2}}dt\\ &\leqq\left\{\int_0 |V(1, y, z)|^2\frac{dz}{\omega(z)}\right\}^{\frac{1}{2}}\left\{\int_D |V(1, z, x)|^2\omega(z)dz\right\}^{\frac{1}{2}}\leqq M_1\end{aligned}$$

for some constant M_1. On the other hand, it is clear from the construction of $U(t, y, x)$ (stated in [3]) that $\int_0^2 U(t, y, x)dt<\infty$ whenever $y, x\in\bar{D}$ and $y\neq x$. From these facts and (2.2), the assertion of Theorem 2 follows immediately.

THEOREM 3. i) *If the boundary value problem* (1.2) *has a solution, then*

$$\int_D f(x)dx=\int_S \varphi(\xi)dS_\xi. \tag{3.10}$$

ii) *If the boundary value problem* (1.2*) *has a solution, then*

$$\int_D f(x)\omega(x)dx=\int_S \varphi(\xi)\omega(\xi)dS_\xi. \tag{3.10*}$$

This theorem may be proved by means of Green's formula and by (3, 3), (3.4) and (3.5).

THEOREM 4. *Assume that $f(x)$ and $\varphi(\xi)$ are Hölder-continuous on $\bar{D}$ and on S respectively. Then :—*

i) *Under the condition* (3.10), *any function of the form*

$$u(x)=-\int_D f(y)K(y, x)dy+\int_S \varphi(\xi)K(\xi, x)dS_\xi+c\omega(x) \tag{3.11}$$

(c being an arbitrary constant) is a solution of the boundary value problem (1.2), *and the difference of any two solutions of* (1.2) *is a constant multiple of* $\omega(x)$.

ii) *Under the condition* (3.10*), *any function of the form*

$$u(y)=-\int_D K(y, x)f(x)dx+\int_S K(y, \xi)\varphi(\xi)dS_\xi+c \tag{3.11*}$$

(c being an arbitrary constant) is a solution of the boundary value problem (1.2*), *and the difference of any two solutions of* (1.2*) *is constant.*

PROOF. We shall prove that the function $u(x)$ given by (3.11) is a solution of (1.2) in the case where $\varphi(x)\equiv 0$; the proof of general case may be achieved by similar technics used in the proof of Theorem 2 in [4].

We put

$$u(x)=-\int_D f(y)K(y, x)dy$$

and

$$v(t, x)=-\int_0^t d\tau\int_D f(y)V(\tau, y, x)dy.$$

Then it follows from (3.2) and Theorem 2 that

$$\begin{aligned}&|v(t, x)-u(x)|\\ &\leqq\int_D |f(y)|dy\int_t^\infty d\tau\int_D |V(\tau-1, y, z)|\cdot|V(1, z, x)|dz\to 0 \quad (\text{as } t\to\infty)\end{aligned} \tag{3.10}$$

uniformly in x. Since $\int_D f(x)dx=\int_S \varphi(\xi)dS_\xi=0$, we have

$$v(t, x)=-\int_0^t d\tau\int_D f(y)U(\tau, y, x)dy,$$

and hence

$$\frac{\partial v}{\partial t}=Av-f \text{ in } (0, \infty)\times D, \quad \frac{\partial v}{\partial n}-\beta v=0 \text{ on } S \tag{3.11}$$

and

$$\begin{aligned}&v(t+s, x)\\ &=-\int_0^t d\tau\int_D f(y)U(\tau, y, x)dy-\int_0^s d\tau\int_D\int_D f(y)U(\tau, y, z)U(t, z, x)dydz\\ &=v(t, x)-\int_D v(s, z)U(t, z, x)dz.\end{aligned} \tag{3.12}$$

Letting $s\to\infty$ in (3.12), we get

$$u(x)=v(t, x)+\int_D u(z)U(t, z, x,)dz \qquad \text{by (3.10).}$$

Hence, by means of (3.11), we get $\partial u/\partial \boldsymbol{n}-\beta u=0$ on S and

$$0=\frac{\partial u}{\partial t}=Av(t, x)-f(x)+A\left\{\int_D u(z)U(t, z, x)dz\right\}=Au(x)-f(x) \quad \text{in } D.$$

This result and (2.6) imply the first part of the assertion i).

To prove the second part of i), it suffices to show that, if $Au=0$ in D and $\partial u/\partial \boldsymbol{n}-\beta u=0$ on S, then $u=c\omega$ for some constant c. If we put $v=\omega^{-1}u$, then, by virtue of (3.4), (3.5) and (3.6), we have

$$0=(\omega^{-1}Au, u)=(\omega^{-1}A(\omega v), \omega v)=(\Delta_\omega v-([\boldsymbol{b}-\nabla p]\cdot\nabla v), v)_\mu=-\|\nabla v\|_\mu^2.$$

Hence $v(x)\equiv c$ on D for some constant c, and accordingly $u=c\omega$.

The assertion ii) may be proved similarly.

University of Tokyo

References

[1] N. Ikeda: On the construction of two-dimensional diffusion processes satisfying Wentzell's boundary conditions and its application to boundary value problems, Mem. Coll. Sci., Univ. Kyoto, Ser. A, **33** (1961), 367-427.

[2] S. Itô: A boundary value problem of partial differential equations of parabolic type, Duke Math. J., **24** (1957), 299-312.

[3] S. Itô: Fundamental solutions of parabolic differential equations and boundary value problems, Jap. J. Math. **27** (1957), 55-102.

[4] S. Itô: On Neumann problem for Laplace-Beltrami operators, Proc. Japan Acad., **37** (1961), 267-272.

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