A generalization of secondary composition and applications

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Introduction

Professor H. Toda introduced in [1] a construction *the secondary composition*, denoted by $\{\alpha, \beta, \gamma\}$, to give the generators of homotopy groups of spheres. He suggested in [13] that this construction can be generalized into a *higher construction*.

The purpose of the first part of this paper (§1~§8) is to give an explicit description of this higher construction. To distinguish between these two constructions, I shall call $\{\alpha, \beta, \gamma\}$ *the first derived composition*, and $\{\alpha, \beta, \gamma, \delta\}$ *the second derived composition*. Most properties of the first derived composition analogously hold for the second. To make this point clear, some of the properties of the first derived composition shall be proved again. In the second part of this paper (§9 ~§12), I shall give the generators of the 2-primary components of $\pi_q(SO(n))$, $\pi_q(SU(n))$, and $\pi_q(Sp(n))$ for $q \leqq 13$ as in [1], and I shall investigate the relations among them. (These results were announced in [16] and [17]). They are very complicated, but not so difficult.

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§1 Preliminaries

Throughout this paper, we shall use the notation.

$$V^n = \{(x_1, x_2, \cdots, x_n)\,;\ x_1^2 + x_2^2 + \cdots + x_n^2 \leqq 1\}$$
$$S^n = \{(x_1, x_2, \cdots, x_{n+1})\,;\ x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$
$$E_+^n = \{(x_1, x_2, \cdots, x_{n+1}) \in S^n\,;\ x_{n+1} \geqq 0\}$$
$$E_-^n = \{(x_1, x_2, \cdots, x_{n+1}) \in S^n\,;\ x_{n+1} \leqq 0\}$$
$$e^0 = (-1, 0, \cdots 0) \in S^n$$
$$I = [0, 1]$$

Let X be a CW-complex with basic point x^0. We shall denote by CX the cone over X ([2], p. 20), which is the topological product $I \times X$ with identifications: $(0, x) = x$, and $(1, x) = (t, x^0) = x^0$ for all $t \in I$. The identification map is denoted by

$$c_X : I \times X \to CX$$

Let Y be another CW-complex with basic point y^0, and let a map(1) $f : (X, x^0) \to$

(1) "map" means always a continuous map.

(Y, y^0) be given. The cone Cf over f is a map $(CX, x^0) \to (CY, y_0)$ which is given by

$$Cf(C_X(t, x)) = C_Y(t, f(x))$$

for all $x \in X$ and $t \in I$.

The cone Ce^n over an n-cell e^n of X is decomposed into an n-cell $\{0\} \times e^n = e^n$ and an $(n+1)$-cell $(0, 1) \times e^n$. The cone CF over the characteristic map $F : (V^n, S^{n-1}) \to (X, X_{n-1})$(1) of the cell e^n determines a map $CF : (CV^n, (CV^n)^\cdot \to (CX, (CX)_n)$, which maps CV^n-$(CV^n)^\cdot$ homeomorphically onto $(0, 1) \times e^n$. Let $f = \partial F = F | S^{n-1}$.

Then $(0, 1) \times e^n$ is attached to CX by a map which represents the homotopy class $d(F, Cf) \in \pi_n(CX)$. Thus, CX is also a CW-complex, and X is a sub-complex of CX.

Let X be a CW-complex and $x^0 \in X$. The suspension space EX of X is the topological product $X \times V^1$ with identification: $(x, 0) = x$ and $(x, -1) = (x, 1) = (x^0, t) = x^0$ for all $x \in X$ and $t \in V^1$. The identification map is denoted by

$$d_X : X \times V^1 \to EX.$$

A map $f : (X, x^0) \to (Y, y^0)$ induces a map $Ef : (EX, x^0) \to (EY, y^0)$ given by

$$Ef : (d_X(x, t)) = d_Y(f(x), t)$$

for all $x \in X$ and $t \in V^1$. We identify X with $X \times \{0\}$.

The suspension space EX of X is clearly a CW-complex.

Since $(I \times X) \times V^1$ may be identified with $I \times (X \times V^1)$, we may identify $E(I \times X)$ with $I \times (EX)$, so that $E(CX)$ with $C(EX)$ i.e.

$$d_{CX}(C_X(t, x), s) = C_{EX}(t, d_X(x, s))$$

for all $x \in X$, $t \in I$, $s \in V^1$.

We shall use the notations:

$$\begin{aligned} &CCX = C^2X,\ EEX = E^2X, \\ &c_{CX}(s, c_X(t, x)) = c_X{}^2(s, t\, ; x) \quad x \in X,\ s, t \in I, \\ &d_{EX}(d_X(x, t), s) = d_x{}^2(x\, ; t, s) \quad x \in X,\ t, s \in V^1. \end{aligned}$$

More generally, C^nX, E^nX, c^n_X, d^n_X, C^nE^mX, etc.(2)

Let (X, x^0) and (Y, y^0) be CW-complexes with basic points, and let f be a map $(X, x^0) \to (Y, y^0)$. A space which is obtained from topological sum $Y \cup CX$ of Y and CX by identifying

$$f(x) = C_X(0, x)$$

(1) "X_n" indicates the n-skelton of X.

(2) Define $E^0X = X$.

for all $x \in X$ will be denoted by $Y \underset{f}{\cup} CX$. ([1], p. 13). In particular, if $X = S^n$, then CX is homeomorphic with V^{n+1}, so that $Y \underset{f}{\cup} CX$ is the space obtained by attaching Y an n-cell e^{n+1} with the attaching map f. Hence it will be denoted by $Y \underset{f}{\cup} e^{n+1}$ as usual. In general, $Y \underset{f}{\cup} CX$ is a CW-complex, if X and Y are CW-complexes. We may identify

$$E(Y \underset{f}{\cup} CX) = EY \underset{Ef}{\cup} CEX$$

Given a homotopy H: $(I \times X,\ I \times x^0) \to (Y,\ y^0)$ between $H_0 = f$ and $H_1 = g$, where $H_t(x)$ indicates $H(t,\ x)$, we define maps ψ: $Y \underset{f}{\cup} CX \to Y \underset{g}{\cup} CX$ and φ: $Y \underset{g}{\cup} CX \to Y \underset{f}{\cup} CX$ by

$$\psi / Y = id.$$

$$\psi c_X(t,\ x) = \begin{cases} H(1-2t,\ x) & 0 \leqq t \leqq 1/2, \quad x \in X \\ c_X(2t-1,\ x) & 1/2 \leqq t \leqq 1, \quad x \in X \end{cases}$$

and by

$$\varphi / Y = id.$$

$$\varphi c_X(t,\ x) = \begin{cases} H(2t,\ x) & 0 \leqq t \leqq 1/2, \quad x \in X \\ c_X(2t-1,\ x) & 1/2 \leqq t \leqq 1, \quad x \in X. \end{cases}$$

We see that $\varphi \cdot \psi \simeq id.$ and $\psi \cdot \varphi \simeq id.$, so that the homotopy type of $Y \underset{f}{\cup} CX$ does not depent on choice of representatives of the homotopy class α of f, which allows us the notation $Y \underset{\alpha}{\cup} CX$. ([1]. p. 13)

The space obtained from $Y \underset{\alpha}{\cup} CX$ by shrinking the subspace Y of $Y \underset{\alpha}{\cup} CX$ into a point x^0 is clearly homeomorphic with EX, so we denote the shrinking map by p: $(Y \underset{\alpha}{\cup} CX,\ Y) \to (EX,\ x^0)$, which is given by

$$(1.3) \qquad \begin{cases} p(Y) = x^0 \\ pc_X(t,\ x) = d_X(x,\ 2t-1) \end{cases}$$

for all $x \in X$ and $t \in I$.

Suppose we have a commutative diagram

$$\begin{array}{ccc} (X_0,\ x_0^0) & \xleftarrow{\ f\ } & (X_1,\ x_1^0) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ (Y_0,\ y_0^0) & \xleftarrow[\ g\ ]{} & (Y_1,\ y_1^0) \end{array}$$

where X_i and Y_i are CW-complexes and $x_i^0 \in X_i$, $y_i^0 \in Y_i$ $(i = 0,1)$.

Define a map $(h \cup k)$: $X_0 \underset{f}{\cup} CX_1 \to Y_0 \underset{g}{\cup} CY_1$ by

$$(1.4)\qquad \begin{cases}(h\cup k)/X_0=h\\ (h\cup k)c_{X1}(t,\ x)=c_{Y1}(t,\ kx) \quad x\in X,\ t\in I.\end{cases}$$

Now, suppose we have a homotopy commutative diagram

$$\begin{tikzcd}(X_0,\ x_0^0) \arrow[d, "h"'] & (X_1,\ x_1^0) \arrow[l, "f"'] \arrow[d, "k"] \\ (Y_0,\ y_0^0) & (Y_1,\ y_1^0) \arrow[l, "g"]\end{tikzcd} \quad \text{i.e. } h\circ f\simeq g\circ k,$$

and let $\{f\}=\alpha$, $\{g\}=\beta$, $\{h\}=\gamma$ and $\{k\}=\delta$[1]. We define a map

$$(\gamma\cup\delta): X_0\underset{\alpha}{\cup}CX_1 \longrightarrow Y_0\underset{\beta}{\cup}CY_1 \qquad \text{by}$$
$$(\gamma\cup\delta)=(1\cup k)\circ\psi\circ(h\cup 1),$$

where 1 indecates the idehtity map and ψ the homotopy equivalence $Y_0\underset{h\cdot f}{\cup}CX_1\simeq Y_0\underset{g\cdot k}{\cup}CX_1$.

In the diagram

$$\begin{tikzcd}X_0\underset{f}{\cup}CX \arrow[r, dashed, "\nu"] \arrow[d, "h\cup k"'] & EX_1 \arrow[d, "Ek"] \\ Y_0\underset{g}{\cup}CY_1 \arrow[r, "\nu"'] & EY_1\end{tikzcd}$$

commutativity holds.

Let $(X,\ x^0)$ and $(Y,\ y^0)$ be CW-complexes with basic points. Given maps f, $g:(E^nX,\ x^0)\longrightarrow(Y,\ y^0)$ $(n\geqq 1)$, the sum $f+g$ of f and g is a map $(E^nX,\ x^0)\longrightarrow(Y,\ y^0)$ defined by

$$(f+g)d_X^n(x;\ t_1,\cdots t_n)=\begin{cases}fd_X^n(x;t_1,\ \cdots,\ t_{n-1},\ 2t_n+1) & -1\leq t_n\leq 0\\ gd_X^n(x;t_1,\ \cdots,\ t_{n-1},\ 2t_n-1) & 0\leq t_n\leq 1\end{cases}$$

for all $x\in X$ and $t_i\in V^1(1\leq i\leq n-1)$. If $n\geqq 2$, $f+g\simeq g+f$. Let $f+_ig$ be the sum of f and g on the i-th coordinate, then

$$(1.7)\qquad f+_ig\simeq f+_jg \qquad (1\leq i,\ j\leq n).$$

Set of homotopy classes of maps $(E^nX,\ x^0)\longrightarrow(Y,\ y^0)$ is denoted by $\pi((E^nX,\ x^0),\ (Y,\ y^0))$, or simply by $\pi(X,\ Y)$, which is a group if $n\geqq 1$, and is an abelian group if $n\geqq 2$. In general, set of homotopy classes of maps $(X,\ A)\longrightarrow(Y,\ B)$ will be denoted by $\pi((X,\ A),\ (Y,\ B))$, where A and B are subspaces of X and Y respectively.

The shrinking map $p:(CX,\ X)\longrightarrow(EX,\ x^0)$ induces a one by one correspondence

$$p_*:\ \pi((EX,\ x^0),\ (Y,\ y^0))\longrightarrow\pi((CX,\ X),\ (Y,\ y^0)) \qquad ([2],\ \text{p. } 206)$$

Let $i_*:\ \pi(X,\ B)\longrightarrow\pi(X,\ Y)$, and

(1) $\{f\}$ indicates the homotopy class of f.

$$j_*: \pi((CX, X), (Y, y_0)) \longrightarrow \pi((CX, X), (Y, B))$$

be correspondences induced by the inclusion maps $i:(B, y^0) \longrightarrow (Y, y^0)$ and $j: (Y, y^0) \longrightarrow (Y, B)$, where $y^0 \in B \subset Y$. j_* may be regarded as $j_*: \pi(EX, Y) \longrightarrow \pi((CX, X)(Y, B))$.

Define an operation $\partial: \pi((CX, X), (Y, B)) \longrightarrow \pi(X, B)$ by restriction on X, as usual. i_*, j_* and ∂ are all homomorphisms if X is a suspension space.

A sequence of sets and correspondences

(1.8) $\pi(X, Y) \xleftarrow{i_*} \pi(X, B) \xleftarrow{\partial} \pi((CX, X), (Y, B)) \xleftarrow{j_*} \pi(EX, Y) \xleftarrow{i_*} \pi(EX, B)$

is exact in the sense that

$$\begin{cases} i_*^{-1}(0) = I_m\partial, \\ \partial^{-1}(0) = I_m j_*, \\ j_*^{-1}(0) = I_m i_*. \end{cases}$$

Let G be a group, G_1 and G_2 its subgroups, and let M_1 and M_2 be cosets of G_1 and G_2 in G, respectively. If M_1 and M_2 have a common element, we shall describe as $M_1 \sim M_2$. Note that this relation is not transitive.

If a map $f:(X, x^0) \longrightarrow (Y, y^0)$ is homotopic to a constant map, then it is extended to a map $A_f:(CX, x^0) \longrightarrow (Y, y^0)$, which gives a null homotopy of f. Let B_f be another null homotopy of f. Define a map

$$d(B_f, A_f): (EX, x^0) \longrightarrow (Y, y^0) \qquad \text{by}$$

$$d(B_f, A_f)d_X(x, t) = \begin{cases} A_f c_X(-t, x), & -1 \leqq t \leqq 0, \quad x \in X \\ B_f c_X(t, x), & 0 \leqq t \leqq 1, \quad x \in X \end{cases}$$

which is a generalization of "separation element" in the sense of [6]. Homotopy class of $d(B_f, A_f)$ is denoted by $\partial(B_f, A_f)$.

§2 Category of n-tuples

Let n be an integer $\geqq 1$. n-tuple (ν) is a set of CW-complexes $(X_i, x_i)(x_i \in X_i, 0 \leqq i \leqq n)$ and homotopy classes $\alpha_i \in \pi(X_i, X_{i-1})(1 \leqq i \leqq n)$, and is described as

$$(\nu): \quad X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \longleftarrow \cdots \longleftarrow X_{n-1} \xleftarrow{\alpha_n} X_n.$$

A representative (N) of (ν) is a set of representatives f_i of α_i $(1 \leqq i \leqq n)$, and is also described as

$$(N): \quad X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \longleftarrow \cdots \longleftarrow X_{n-1} \xleftarrow{f_n} X_n.$$

Let

$$(\nu'): \quad Y_0 \xleftarrow{\beta_1} Y_1 \xleftarrow{\beta_2} Y_2 \longleftarrow \cdots \longleftarrow Y_{n-1} \xleftarrow{\beta_n} Y_n$$

be another n-tuple. A set H of maps $h_i:(X_i, x_i)\longrightarrow(Y_i, y_i)$ $(0\leqq i\leqq n)$ is called a map $(\nu)\longrightarrow(\nu')$, if there exist representatives (N) and (N') of (ν) and (ν') such that the diagram

$$
(2.1)\qquad
\begin{array}{lccccccccccc}
(N): & X_0 & \xleftarrow{f_1} & X_1 & \xleftarrow{f_2} & X_2 & \longleftarrow & \cdots & \longleftarrow & X_{n-1} & \xleftarrow{f_n} & X_n \\
 & \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & & & \downarrow{\scriptstyle h_{n-1}} & & \downarrow{\scriptstyle h_n} \\
(N'): & Y_0 & \xleftarrow[g_1]{} & Y_1 & \xleftarrow[g_2]{} & Y_2 & \longleftarrow & \cdots & \longleftarrow & Y_{n-1} & \xleftarrow[g_n]{} & Y_n
\end{array}
$$

is homotopy commutative, i.e.

$$h_i\circ f_{i+1}\simeq g_{i+1}\circ h_{i+1}\quad(0\leqq i\leqq n-1).$$

[NOTE] If H is a map $(\nu)\longrightarrow(\nu')$, the diagram is homotopy commutative for any representatives of (ν) and (ν').

If, further, there exists a map $K:(\nu')\longrightarrow(\nu)$ such that for each i, $h_i\circ k_i\simeq id$, and $\kappa_i\circ h_i\simeq id$ $(0\leqq i\leqq n)$, we then say that (ν) and (ν') are homotopically equivalent $((\nu)\simeq(\nu'))$. The equivalence cleass of ν is denoted by $\{\nu\}$.

For example, if $X_i\simeq Y_i$ for some i, then n-tuples

$$(\nu):\quad X_0\xleftarrow{\alpha_1}X_1\longleftarrow\cdots\longleftarrow X_{i-1}\xleftarrow{\alpha_i}X_i\xleftarrow{\alpha_{i+1}}X_{i+1}\longleftarrow\cdots\longleftarrow X_{n-1}\xleftarrow{\alpha_n}X_n$$

and

$$(\nu'):\quad X_0\xleftarrow{\alpha_1}X_1\longleftarrow\cdots\longleftarrow X_{i-1}\xleftarrow{\varphi_*\alpha_i}Y_i\xleftarrow{\psi_*\alpha_{i+1}}X_{i+1}\longleftarrow\cdots\longleftarrow X_{n-1}\longleftarrow X_n$$

are homotopically equivalent, where $\psi:X_i\longrightarrow Y_i$ and $\varphi:Y_i\longrightarrow X_i$ are homotopy equivalences.

Suppose that for each n-tuple (ν), an $(n$-$k)$-tuple $(T\nu)$ is given and that for each map $H:(\nu)\longrightarrow(\nu')$, a map
$TH:(T\nu)\longrightarrow(T\nu')$ is given such that

(1) if $H:(\nu)\longrightarrow(\nu')$ is a homotopy equivalence, then so is $TH:(T\nu)\longrightarrow(T\nu')$.

(2) if $H:(\nu)\longrightarrow(\nu)$ is the identity, then so is TH.

We then say that the pair of functions $T\nu$, TH forms a (covariant) functor of degree k on the category of n-tuples with values in the category of $(n$-$k)$-tuples $(0\leqq k\leqq n-1)$.

T is a functor on the category of homotopy equivalence classes of n-tuples with values in that of $(n$-$k)$-tuples.

For example, the suspension operation EX, Ef forms a functor of degree 0. The n-tuple

$$(E\nu): \quad EX_0 \xleftarrow{E\alpha_1} EX_1 \xleftarrow{E\alpha_2} EX_2 \longleftarrow \cdots \longleftarrow EX_{n-1} \xleftarrow{E\alpha_n} EX_n$$

is called the suspension of the n-tuple (ν).

For each n-tuple

$$(\nu): \quad X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \longleftarrow \cdots \longleftarrow X_{n-1} \xleftarrow{\alpha_n} X_n,$$

define

$$(T\nu): \quad X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \longleftarrow \cdots \longleftarrow X_{n-2} \xleftarrow{\alpha_{n-1}\circ\alpha_n} X_n$$

and for each map $H: (\nu) \longrightarrow (\nu')$, define $TH: (T\nu) \longrightarrow (T\nu')$ by $TH=(h_1, h_2, \cdots, h_{i-2}, h_i)$, then T is a functor of degree 1.

Now, let n be an integer$\geqq 2$. *Null n-tuple* $(\bar{\nu})$ is an n-tuple such that $\alpha_i \circ \alpha_{i+1} = 0 (1 \leqq i \leqq n-1)$. *A representative* $(\bar{N})$ of $(\bar{\nu})$ is a set of representatives f_i of α_i $(1 \leqq i \leqq n)$ and null homotopies
$A_i: (CX_{i+1}, x_{i+1}^0) \longrightarrow (X_{i-1}, x_{i-1}^0)$ of $f_i \circ f_{i+1}$ $(1 \leqq i \leqq n-1)$, and is described

as $\quad (\bar{N}): \quad X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \longleftarrow \cdots \longleftarrow X_{n-1} \xleftarrow{f_n} X_n, \quad (A_1, A_2, \cdots, A_{n-1}).$

Consider a homotopy commutative diagram

$$\begin{tikzcd} X_0 \arrow[d, "h_0"'] & X_1 \arrow[l, "f_1"'] \arrow[d, "h_1"] & X_2 \arrow[l, "f_2"'] \arrow[d, "h_2"] \\ Y_0 & Y_1 \arrow[l, "g_1"] & Y_2 \arrow[l, "g_2"] \end{tikzcd}$$

such that $f_1 \circ f_2 \simeq 0$, and $g_1 \circ g_2 \simeq 0$. Let A and B be null homotopies of $f_1 \circ f_2$ and $g_1 \circ g_2$, respectively, and let C_t and D_t be homotopies such that $C_0 = g_1 \circ h_1$, $C_1 = h_0 \circ f_1$, $D_0 = g_2 \circ h_2$, and $D_1 = h_1 \circ f_2$, then, a null homotopy $G: (CX_2, x_2^0) \longrightarrow (Y_0, y_0^0)$ of $g_1 \circ g_2 \circ h_2$ is induced by $h_0 A$, as follows:

$$Gc_{X_2}(t, x) = \begin{cases} g_1 \circ D_{3t}(x), & 0 \leqq t \leqq 1/3, \quad x \in X_2, \\ C_{3t-1} \circ f_2(x), & 1/3 \leqq t \leqq 2/3, \quad x \in X_2, \\ h_0 A c_{X_2}(3t-2, x), & 2/3 \leqq t \leqq 1, \quad x \in X_2, \end{cases}$$

In the following, G will be denoted by $\widetilde{h_0 A}$. Similarly, a null homotopy $\underline{B \circ Ch_2}$ of $h_0 \circ f_1 \circ f_2$ is induced by $B \circ Ch_2$. It can easily be seen that

$$d(B \circ Ch_2, \widetilde{h_0 A}) \simeq d(\underline{B \circ Ch_2}, h_0 \circ A). \tag{2.2}$$

Let

$$(\bar{\nu}): \quad X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \longleftarrow \cdots \longleftarrow X_{n-1} \xleftarrow{\alpha_n} X_n$$

and

$$(\bar{\nu}'): \quad Y_0 \xleftarrow{\beta_1} Y_1 \xleftarrow{\beta_2} Y_2 \longleftarrow \cdots \longleftarrow Y_{n-1} \xleftarrow{\beta_n} Y_n$$

be null n-tuples. A set $\bar{H}$ of maps $h_i:(X_i,\ x_i^0)\longrightarrow(Y_i,\ y_i^0)$ $(0\leqq i\leqq n)$ is called a map $(\bar{\nu})\longrightarrow(\bar{\nu}')$, if there exist representatives

$$(N):\quad X_0\xleftarrow{f_1}X_1\xleftarrow{f_2}X_2\longleftarrow\cdots\longleftarrow X_{n-1}\xleftarrow{f_n}X_n,\quad(A_1,\ A_2,\ \cdots,\ A_{n-1}),$$

and

$$(N'):\quad Y_0\xleftarrow{g_1}Y_1\xleftarrow{g_2}Y_2\longleftarrow\cdots\longleftarrow Y_{n-1}\xleftarrow{g_n}Y_n,\quad(B_1,\ B_2,\ \cdots,\ B_{n-1}),$$

of $(\bar{\nu})$ and $(\bar{\nu}')$, respectively, such that

(1) homotopy commutativity holds in the diagram (2.1), and

(2) in the diagram

$$(2.3)\qquad \begin{array}{ccc} X_{i-1} & \xleftarrow{A_i} & CX_{i+1} \\ {\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle Ch_{i+1}} \\ Y_{i-1} & \xleftarrow[B_i]{} & CY_{i+1} \end{array}\qquad(1\leqq i\leqq n-1)$$

the following relation holds:

$$(2.4)\qquad \overline{h_i\circ A_i}\simeq B_i\circ Ch_{i+1}\quad(\text{or } h_i\circ A_i\simeq \underline{B_i\circ Ch_{i+1}}).^{(1)}$$

Homotopy equivalence is defined analogously. The equivalence class of $(\bar{\nu})$ is denoted by $\{\bar{\nu}\}$.

Functor T of degree k on the category of null n-tuples with values in the category of null $(n\text{-}k)$-tuples is also defined. Suspension operation EX, Ef also forms a functor of degree 0.

§3 Coextensions

Let $(N):\ X_0\xleftarrow{f_1}X_1\xleftarrow{f_2}X_2$, (A) be a representative of a null couple $(\bar{\nu}):\ X_0\xleftarrow{\alpha_1}X_1\xleftarrow{\alpha_2}X_2$.

Define a map $(f_1,\ A,\ f_2):(EX_2,\ x_2^0)\longrightarrow(X_0\underset{f}{\cup}CX_1,\ x_0^0)$ by

$$(3.1)\qquad (f_1,\ A,\ f_2)d_{X_2}(x,\ t)=\begin{cases}Ac_{X_2}(-t,\ x), & -1\leqq t\leqq 0,\quad x\in X_2\\ c_{X_1}(t,\ f_2(x)), & 0\leqq t\leqq 1,\quad x\in X_2\end{cases}$$

which is called a *coextension* of f_2 ([1], p. 13), and is sometimes denoted by $(f_1,\ f_2)$ or $\tilde{f}_2$ if there is no ambiguity.

Let $(N_1):\ X_0\xleftarrow{f_1^1}X_1\xleftarrow{f_2}X_2$ (A) be a representative of the null couple (ν), and let $f_1^0\simeq f_1^1$ with a homotopy $H_s:\ (X_1,\ x_1^0)\longrightarrow(X_0,\ x_0^0)$ $(s\in I)$ such that $H_0=f_1^0$ and $H_1=f_1^1$. Define a map $B:\ CX_2\longrightarrow X_0$ by

$$Bc_{X_2}(t,\ x)=\begin{cases}H_{2t}\circ f_2(x), & 0\leqq t\leqq 1/2,\qquad x\in X_2\\ Ac_{X_2}(2t-1,\ x), & 1/2\leqq t\leqq 1,\qquad x\in X_2\end{cases}$$

(1) In the following sections, we shall say that "homotopy commutativity holds in the diagram (2.3)", if the relation (2.4) holds.

then, $(\overline{N}_0): X_0 \xrightarrow{f_1^0} X_1 \xleftarrow{f_2} X_2$, (B) is also a representative of $(\bar{\nu})$.

Next, we define a homotopy $G_s : (EX_2, x_2^0) \longrightarrow (EX_2, x_2^0)$, $(s \in I)$ by

$$G_s d_{X_2}(x, t) = \begin{cases} d_{X_2}(x, (t-s)/(1+s)), & -1 \leqq t \leqq (s-1)/2, \quad x \in X_2, \\ d_{X_2}(x, t-s/2), & (s-1)/2 \leqq t \leqq s/2, \quad x \in X_2, \\ d_{X_2}(x, (2t-s)/(2-s)), & s/2 \leqq t \leqq 1, \quad x \in X_2. \end{cases}$$

We see that $G_0 = id.$ on EX_2, and that

$$\psi \circ (f_1^1, A, f_2) = (f_1^0, B, f_2) \circ G_1$$

where ψ is the homotopy equivalence $X_0 \underset{f_1^1}{\cup} CX_1 \longrightarrow X_0 \underset{f_1^0}{\cup} CX_1$ which is defined in §1. Thus, we have proved

LEMMA (3.2) *If $(\overline{N}_1)$ is a representative of $(\bar{\nu})$, and if $f_1^0 \simeq f_1^1$, then, there exists a representative $(\overline{N}_0)$ such thxt*

$$(f_1^1, A, f_2) \circ \psi \simeq (f_1^0, B, f_1).$$

Similarly, we have

LEMMA (3.3) Let $(N_1) : X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2^1} X_2$, (A) be a representative of the null couple $(\bar{\nu})$, and suppose $f_2^0 \simeq f_2^1$, then, there exists a representative $(\overline{N}_0): X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2^0} X_2$, (B) of $(\bar{\nu})$ such that

$$(f_1, A, f_2^1) \simeq (f_1, A, f_2^0).$$

PROOF. Let $H_s : (X_2, x_2^0) \longrightarrow (X_1, x_1^0)$ be a homotopy such that $H_0 = f_2^0$, $H_1 = f_2^1$, then, we define a null homotopy $B: CX_2 \longrightarrow X_0$ as follows:

$$Bc_{X_2}(t, x) = \begin{cases} f_1 \circ H_{2t}(x), & 0 \leqq t \leqq 1/2, \quad x \in X_2 \\ A \circ c_{X_2}(2t-1, x), & 1/2 \leqq t \leqq 1, \quad x \in X_2 \end{cases}$$

Next, we define a homotopy $G_s : (EX_2, x_2^0) \longrightarrow (X_0 \underset{f_1}{\cup} CX, x_0^0)$, $(s \in I)$ by

$$G_s d_{X_2}(x, \mathrm{t}) = \begin{cases} Ac_{X_2}(-t, x) & -1 \leqq t \leqq s-1, \quad x \in X_2 \\ Ac_{X_2}(-2t+s-1, x) & (s-1) \leqq t \leqq (s-1)/2, \quad x \in X_2 \\ f_1 \circ H_{s-2t}(x) & (s-1)/2 \leqq t \leqq 0, \quad x \in X_2 \\ c_{X_2}(t, H_s(x)) & 0 \leqq t \leqq 1, \quad x \in X_2 \end{cases}$$

then, we see that $G_1 = (f_1, A, f_2^1)$ and $G_0 = (f_1, B, f_2^0)$. q.e.d.

Let $X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2$, (A) be a representative of the null couple $(\bar{\nu})$, and let B be another null homotopy of $f_1 \circ f_2$. Define a homotopy $H_s : (EX_2, x_2^0) \longrightarrow (X_0 \cup CX_1, x_0^0)$ $(s \in I)$ by

$$H_s d_{X_2}(x, \mathrm{t}) = \begin{cases} A \circ c_{X_2}((-2t+s-1)/(1+s), x), & -1 \leqq t \leqq (s-1)/2, \quad x \in X_2 \\ c_{X_1}(1-s+2t, f_2(x)), & (s-1)/2 \leqq t \leqq 0, \quad x \in X_2 \\ c_{X_1}(1-s-2t, f_2(x)), & 0 \leqq t \leqq (1-s)/2, \quad x \in X_2 \\ Bc_{X_2}((2t+s-1)/(1+s), x), & (1-s)/2 \leqq t \leqq 1, \quad x \in X_2 \end{cases}$$

We see that H gives a homotopy between $(f_1, A, f_2)-(f_1, B, f_2)$ and $d(B, A)$.

Conversely, if a null homotopy A of $f_1 \circ f_2$ and an element $\delta \in \pi(EX_2, X_0)$ are given, then, there exists a null homotopy B of $f_1 \circ f_2$ such that $d(B, A) \in \delta$.

Together with Lemmata (3.2) and (3.3), we have proved

Proposition (3.4) *Set of all homotopy classes of coextensions* (3.1) *obtained from any representative of a null couple* $(\bar{\nu}): X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2$ *is a coset of the subgroup* $i_*\pi(EX_2, X_0)$ in $\pi(EX_2, X_0 \underset{\alpha_1}{\cup} CX_1)$.

This allows us the notation (α_1, α_2) or $\tilde{\alpha}_2$.

Let $(\bar{\nu}): X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2$, and $(\bar{\nu}'): Y_0 \xleftarrow{\beta_1} Y_1 \xleftarrow{\beta_2} Y_2$ be given, and let H: $(\bar{\nu}) \longrightarrow (\bar{\nu}')$ be a map i.e. there exist representatives $(\bar{N}): X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2$, (A) and $(\bar{N}'): Y_0 \xleftarrow{g_1} Y_1 \xleftarrow{g_2} Y_2$, (B) of $(\bar{\nu})$ and $(\bar{\nu}')$ such that homotopy commutativity holds in the diagrams

(3.5)
$$\begin{tikzcd} X_0 \arrow[d, "h_0"'] & X_1 \arrow[l, "f_1"'] \arrow[d, "h_1"] & X_2 \arrow[l, "f_2"'] \arrow[d, "h_2"] \\ Y_0 & Y_1 \arrow[l, "g_1"] & Y_2 \arrow[l, "g_2"] \end{tikzcd}, \qquad \begin{tikzcd} X_0 \arrow[d, "h_0"'] & CX_2 \arrow[l, "A"'] \arrow[d, "Ch_2"] \\ Y_0 & CY_2 \arrow[l, "B"] \end{tikzcd}$$
(c.f. (2.3)).

Proposition (3.6)

$$(\gamma_0 \cup \gamma_1) \circ (\alpha_1, \alpha_2) \sim (\beta_1, \beta_2) \circ E\gamma_2,$$

where $\gamma_i = \{h_i\} \in \pi(X_i, Y_i)$ $(0 \leqq i \leqq 2)$.

PROOF. If in the first diagram of (3.5), commutativity holds, then we can see that

$$(h_0 \cup h_1) \circ (f_1, A, f_2) = (g_1, h_0 \circ A, h_1 \circ f_2) = (g_1, h_0 \circ A, g_2 \circ h_2),$$
$$(g_1, B, g_2) \circ Eh_2 = (g_1, B \circ Ch_2, g_2 \circ h_2).$$

Hence, it follows that

$$(h_0 \cup h_1) \circ (f_1, A, f_2) - (g_1, B, g_2) \circ Eh_2 = d(B, Ch_2, h_0 \circ A) = 0.$$

Homotopy commutative diagram (3.5) is decomposed as follows:

$$\begin{tikzcd} Y_2 \arrow[d, "g_2"'] & X_2 \arrow[l, "h_2"'] \arrow[d, "g_2 h_2"] & X_2 \arrow[l, "1"'] \arrow[d, "h_1 \circ f_2"] & X_2 \arrow[l, "1"'] \arrow[d, "f_2"] & X_2 \arrow[l, "1"'] \arrow[d, "f_2"] & X_2 \arrow[l, "1"'] \arrow[d, "f_2"] \\ Y_1 \arrow[d, "g_1"'] & Y_1 \arrow[l] \arrow[d, "g_1"] & Y_1 \arrow[l] \arrow[d, "g_1"] & X_1 \arrow[l] \arrow[d, "g_1 \circ h_1"] & X_1 \arrow[l] \arrow[d, "h_0 \circ f_1"] & X_1 \arrow[l] \\ Y_0 & Y_0 \arrow[l, "1"] & Y_0 \arrow[l, "1"] & Y_0 \arrow[l, "1"] & Y_0 \arrow[l, "1"] & X_0 \arrow[l, "h_0"] \arrow[u, "f_1"'] \end{tikzcd}$$

The proposition follows from the diagram

$$\begin{tikzcd}
EY_2 \arrow[d, dotted, "{(g_1, g_2)}"'] & EX_2 \arrow[l, "Eh_2"'] \arrow[d, dotted, "{(g_1, g_2h_2)}"] & EX_2 \arrow[l, "1"'] \arrow[d, dotted, "{(g_1, h_1f_2)}"] & EX_2 \arrow[l, "1"'] \arrow[d, "{(g_1h_1, f_2)}"] & EX_2 \arrow[l, "1"'] \arrow[d, "{(h_0f_1, f_2)}"] & EX_2 \arrow[l, "1"'] \arrow[d, "{(f_1, f_2)}"] \\
Y_0 \cup_{g_1} CY_1 & Y_0 \cup_{g_1} CY_1 \arrow[l, "1"] & Y_0 \cup_{g_1} CY_1 \arrow[l, "1"] & Y_0 \cup_{g_1 \circ h_1} CX_1 \arrow[l, "(1 \cup h_1)"] & Y_0 \cup_{h_0 \circ f_1} CX_1 \arrow[l, "\varphi"] & X_0 \cup CX_1 \arrow[l, "(h_0 \cup 1)"]
\end{tikzcd}$$

where φ is the homotopy equivalence.

COROLLARY (3.7)

If $X_0 = Y_0$, $h_0 = id.$, *then* $(1 \cup \gamma_1) \circ (\alpha_1, \alpha_2) \supset (\beta_1, \beta_2) \circ E\gamma_2$.
If $X_2 = Y_2$, $h_2 = id.$, *then* $(\gamma_0 \cup \gamma_1) \circ (\alpha_1, \alpha_2) \subset (\beta_1, \beta_2)$
If $X_0 = Y_0$, $X_2 = Y_2$, *and* $h_0 = id.$, *then* $(1 \cup \gamma_1) \circ (\alpha_1, \alpha_2) = (\beta_1, \beta_2)$.

Let $(N^{(1)}) : X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2^{(1)}} EX_2$, $(A^{(1)})$ and $(\bar{N}^{(2)}) : X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2^{(2)}} EX_2$, $(A^{(2)})$ be representatives of null couples $(\bar{\nu}^{(1)}) : X_0 \xleftarrow{\alpha} X_1 \xleftarrow{\beta_1} EX_2$ and $(\bar{\nu}^{(2)}) : X_0 \xleftarrow{\alpha} X_1 \xleftarrow{\beta_2} EX_2$, respectively, and let $f_2^{(0)} = f_2^{(1)} + f_2^{(2)}$, $A^{(0)} = A^{(1)} + A^{(2)}$ i.e.

$$f_2^{(0)} d_{X_2}(x,\ t) = \begin{cases} f_2^{(1)} d_{X_2}(x,\ 2t+1), & -1 \leqq t \leqq 0, \quad x \in X_2, \\ f_2^{(2)} d_{X_2}(x,\ 2t-1), & 0 \leqq t \leqq 1, \quad x \in X_2, \end{cases}$$

$$A^{(0)} c_{EX_2}(s,\ d_{X_2}(x,\ t)) = \begin{cases} A^{(1)} c_{EX_2}(s,\ d_{X_2}(x,\ 2t+1)), & -1 \leqq t \leqq 0, \quad s \in I,\ x \in X_2 \\ A^{(2)} c_{EX_2}(s,\ d_{X_2}(x,\ 2t-1)), & 0 \leqq t \leqq 1, \quad s \in I,\ x \in X_2. \end{cases}$$

While, the map $(f_1,\ A^{(1)},\ f_2^{(1)}) + (f_1,\ A^{(2)},\ f_2^{(2)}) = g : E^2X_2 \longrightarrow X_0 \cup_{f_1} CX_1$ is given by (c.f. (1.7))

$$\begin{aligned} g d^2_{X_2}(x;\ t,\ s) &= \begin{cases} (f_1,\ A^{(1)},\ f_2^{(1)})(d^2_{X_2}(x;\ 2t+1,\ s)), & -1 \leqq t \leqq 0, \quad -1 \leqq s \leqq 1, \\ (f_1,\ A^{(2)},\ f_2^{(2)})(d^2_{X_2}(x;\ 2t-1,\ s)), & 0 \leqq t \leqq 1, \quad -1 \leqq s \leqq 1, \end{cases} \\ &= \begin{cases} A^{(1)} c_{EX_2}(-s,\ d_{X_2}(x,\ 2t+1)), & -1 \leqq t \leqq 0, \quad -1 \leqq s \leqq 1, \\ c_{X_1}(s,\ f_2^{(1)} d_{X_2}(x,\ 2t+1)), & -1 \leqq t \leqq 0, \quad 0 \leqq s \leqq 1, \\ A^{(2)} c_{EX_2}(-s,\ d_{X_2}(x,\ 2t-1)), & 0 \leqq t \leqq 1, \quad -1 \leqq s \leqq 0, \\ c_{X_1}(s,\ f_2^{(2)} d_{X_2}(x,\ d_{X_2}(x,\ 2t-1)), & 0 \leqq t \leqq 1, \quad 0 \leqq s \leqq 1, \end{cases} \\ &= \begin{cases} A^{(0)} c_{EX_2}(-s,\ d_{X_2}(x,\ t)), & -1 \leqq s \leqq 0, \quad -1 \leqq t \leqq 1, \\ c_{X_1}(s,\ f_2^{(0)} d_{X_2}(x,\ t)), & 0 \leqq s \leqq 1, \quad -1 \leqq t \leqq 1. \end{cases} \end{aligned}$$

for all $x \in X_2$. Thus, we have proved.

PROPOSITION (3.8) $\quad (\alpha,\ \beta_1) + (\alpha,\ \beta_2) \subset (\alpha,\ \beta_1 + \beta_2)$.

Let $(\overline{EN}) : EX_0 \xleftarrow{Ef_1} EX_1 \xleftarrow{Ef_2} EX_2$, (EA) be the suspension of a representative $(\bar{N}) : X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2$, (A) of a null couple $(\bar{\nu}) : X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2$. Then the coextension

$$(Ef_1,\ EA,\ Ef_2) : (E^2X_2,\ x_2^0) \longrightarrow (EX_0 \cup_{Ef_1} CEX_1,\ x_0^0)$$

is given by

$$(Ef_1,\ EA,\ Ef_2) d^2_{X_2}(x;\ t,\ s) = \begin{cases} EA \circ C_{EX_2}(-s,\ d_{X_2}(x,\ t)), & -1 \leqq s \leqq 0, \quad -1 \leqq t \leqq 1 \\ c_{EX_1}(s,\ d_{X_1}(f_2(x),\ t)), & 0 \leqq s \leqq 1, \quad -1 \leqq t \leqq 1 \end{cases}$$

for all $x\in X_2$. Since $c_{EX_1}(s, d_{X_1}(f_2(x), t))=d_{CX_1}(c_{X_1}(s, f_2(x)), t)$, it follows that

$$(Ef_1, EA, Ef_2)d^2_{X_2}(x ; t, s)=E(f_1, A, f_2)d^2_{X_2}(x ; s, t), \quad -1\leqq s, t\leqq 1$$

for all $x\in X_2$. Let $\rho: (E^2X, x^0)\longrightarrow(E^2X, x^0)$ be a map defined by $\rho d^2_{X_2}(x ; t, s)=d^2_{X_2}(x ; s, t)$, $-1\leqq t, s\leqq 1$, $x\in X_2$, which is a map of degree (-1) if X is a sphere. Then, we have proved

PROPOSITION (3.9) $E(\alpha_1, \alpha_2)\subset(E\alpha_1, E\alpha_2)\circ\rho$.

PROPOSITION (3.10) $p_*(\alpha, \beta)=E\beta$,

where p is a shrinking map: $(X_0\cup CX_1, X_0)\longrightarrow(EX_1, x^0_1)$. ([1] (1.18))

The proposition immediately follows from the definition (3.1).

§4 Extensions

Let (N): $X_0\xleftarrow{f_1}X_1\xleftarrow{f_2}X_2$, (A) be a representative of a null couple (ν): $X_0\xleftarrow{\alpha_1}X_1\xleftarrow{\alpha_2}X_2$. An extension $[f_1, A, f_2]: (X_1\underset{f_2}{\cup}CX_2, x^0_1)\longrightarrow(X_0, x^0_0)$ of the map f_1 over $X_1\underset{f_2}{\cup}CX_2$ is defined by

$$(4.1)\qquad \begin{cases}[f_1, A, f_2]\,|\,X_1=f_1, \\ [f_1, A, f_2]c_{X_2}(t, x)=Ac_{X_2}(t, x) \quad 0\leqq t\leqq 1, \ x\in X_2.\end{cases}$$

$[f_1, A, f_2]$ will be denoted by $[f_1, f_2]$ or $\bar{f}_1$ if there is no ambiguity.

LEMMA (4.2) *Let* $X_0\xleftarrow{f_1}X_1\xleftarrow{f^1_2}X_2$, (A) *be a representative of a null couple* (ν): $X_0\xleftarrow{\alpha_1}X_1\xleftarrow{\alpha_2}X_2$, *and let f^0_2 be another representative of α_2.*

Then, there exists a representative $X_0\xleftarrow{f_1}X_1\xleftarrow{f^0_2}X_2$, (B) *of* (ν) *such that* $[f_1, B, f^0_2]\circ\psi\simeq[f_1, A, f^1_2]$, *where* $\psi: X_1\underset{f^1_2}{\cup}f^1_2CX_2\longrightarrow X_1\underset{f^0_2}{\cup}CX_2$ *is the homotopy equivalence defined in* §1.

PROOF. Let $H_s: (X_2, x^0_2)\longrightarrow(X_1, x^0_1)$ $(s\in I)$ be a homotopy such that $H_0=f^0_2$ and $H_1=f^1_2$. Then, the null homotopy B of $f_1\circ f^0_2$ is defined as follows:

$$Bc_{X_2}(t, x)=\begin{cases}f_1\circ H_{2t}(x) & 0\leqq t\leqq 1/2, \ x\in X_2 \\ A\, c_{X_2}(2t-1, x) & 1/2\leqq t\leqq 1, \ x\in X_2,\end{cases}$$

and the homotopy $G_s: (X_1\underset{f_2}{\cup}CX_2, x^0_1)\longrightarrow(X_0, x^0_0)$ such that $G_0=[f_1, B, f^0_2]\circ\psi$ and $G_1=[f_1, A, f_2]$ is defined as

$$\begin{cases}G_s\,|\,X_1=f_1 \\ G_sc_{X_2}(t, x)=\begin{cases}f_1\circ H_{1-2t}(x) & 0\leqq t\leqq(1-s)/2 \\ f_1\circ H_{4t+3s-2}(x) & (1-s)/2\leqq t\leqq 3(1-s)/4 \\ A\, c_{X_2}((4t-3(1-s))/(1+3s), x) & (1-s)/4\leqq t\leqq 1\end{cases}\end{cases}$$

for all $x\in X_2$. q.e.d.

LEMMA (4.3) *Let* $X_0\xleftarrow{f^1_1}X_1\xleftarrow{f_2}X_2$, (A) *be a representative of a null couple* (ν): $X_0\xleftarrow{\alpha_1}X_1\xleftarrow{\alpha_2}X_2$, *and let f^0_1 be another representative of α_1, then, there exists* *a representative* $X_0 \xleftarrow{f_1^0} X_1 \xleftarrow{f_2} X_2$, (B) *of* $(\bar{\nu})$ *such that* $[f_1^0, B, f_2] \simeq [f_1^1, A, f_2]$.

PROOF. The null homotopy B of $f_1^0 \circ f_2$ is given by

$$Bc_{X_2}(t, x) = \begin{cases} H_{2t} \circ f_2(x), & 0 \leqq t \leqq 1/2, \ x \in X_2, \\ A\, c_{X_2}(2t-1, x), & 1/2 \leqq t \leqq 1, \ x \in X_2, \end{cases}$$

where $H_s : (X_1, x_1^0) \longrightarrow (X_0, x_0^0)$ be a homotopy such that $H_0 = f_1^0$ and $H_1 = f_1^1$, and the homotopy $G_s : (X_1 \underset{f_2}{\cup} CX_2, x_1^0) \longrightarrow (X_0, x_0^0)$ $(s \in I)$ such that $G_0 = [f_1^1, B, f_2]$ and $G_1 = [f_1^1, A, f_2]$, is given by

$$G_s | X_1 = H_s,$$

$$G_s c_{X_2}(t, x) = \begin{cases} H_{s+2t} \circ f_2(x), & 0 \leqq t \leqq (1-s)/2, & x \in X_2, \\ A\, c_{X_2}(s+2t-1, x), & (1-s)/2 \leqq t \leqq 1-s, & x \in X_2, \\ A\, c_{X_2}(t, x) & 1-s \leqq t \leqq 1, & x \in X_2. \end{cases}$$

q.e.d.

Thus the homotopy class of the extension does not depend on choice of representatives of α_1 and α_2. Classification of homotopy classes of the extensions was solved by W. D. Barcuss and M. G. Barratt in [3]. We state the results here.

First, let $X_2 = S^q$, so that $L = X_1 \underset{f_1}{\cup} CX = X_1 \underset{f_2}{\cup} e^{q+1}$, and let $X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} S^q$ be a representative of a null couple $(\bar{\nu}) : X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} S^q$.

Barcuss and Barratt introduced a homomorphism

$$\alpha_{f_1} : \pi_1(X_0^{X_1}, f_1) \longrightarrow \pi_{q+1}(X_0, x_0^0),$$

where $X_0^{X_1}$ is the space of continuous maps $(X_1, x_1^0) \longrightarrow (X_0, x_0^0)$ with compact open topology. The homomorphism α_{f_1} depends on the homotopy class α and the fixed map f_1. Let g_0 and g_1 be extensions $(L, x_1^0) \longrightarrow (X_0, x_0^0)$ of f_1 such that there exists a homotopy

$$\bar{H} : (I \times L, I \times x_1^0) \longrightarrow (X_0, x_0^0)$$

from g_0 to g_1. Then $H = \bar{H} | (I \times X_1, I \times x_1^0)$ determines an element $\{H\} \in \pi_1(X_0^{X_1}, f_1)$. They proved that the separation element $d(g_1, g_0)$ on the cell (e^{q+1}, x_1^0) belongs to $\alpha_f\{H\} \in \pi_{q+1}(X_0, x_0^0)$ and that

PROPOSITION (4.4) *The homotopy classes of the extensions* $\bar{f}_1 : X_1 \underset{f_2}{\cup} e^{q+1}, x_1^0) \longrightarrow (X_0, x_0^0)$ *of* $f_1 : (X_1, x_1^0) \longrightarrow (X_0, x_0^0)$ *are in one to one correspondence with the element of the cokernel of* α_{f_1} *i.e. of* $\pi_{q+1}(X_0, x_0^0)/\alpha_{f_1}\pi_1(X_0^{X_1}, f_1)$. ([3], Th. 3.2)

A conutable CW-complex with only vertex is called *a special complex*. ([4]). Let X_2 be a special complex and x_2^0 be its only vertex, then, CX_2 is also a special complex with only vertex x_2^0, which consists of the mutually disjoint cells e^{q_i+1} with attaching maps

$$g_i : (S^{q_i}, e^0) \longrightarrow ((CX_2)_{q_i}, x_2^0)$$

where $(CX_2)_n$ is the n-skelton of CX_2 i.e. the subcomplex of CX_2 whose cells are of dimensions $\leqq n$. Then each cell e^{q_i+1} is attached to X_1 by the attaching map $f_{2,i} \circ g_i : (S^{q_i}, e^0) \longrightarrow (X_1, x_1^0)$ (where $f_{2,i} = f_2 | (CX_2)_{q_i}$) which represents a homotopy class $\alpha_i \in \pi_{q_i}(X_1, x_1^0)$. Let $L = X_1 \underset{f_2}{\cup} CX_2$, and let $C(L, X_1) = \sum \pi_{q_i+1}(X_0 ; x_0)$, the direct sum. Sinnce $f_1 * \alpha_i = 0$ for all all i, the homomorphisms $(\alpha_i)_f$ together define

$$\alpha_{f_1} : \pi_1(X_0^{X_1}, f_1) \longrightarrow C(L, x_1^0)$$

such that the coordinate of $\alpha_{f_1}(\xi)$ in $\pi_{q_i+1}(X_0, x_0^0)$ is $(\alpha_i)_{f_1}(\xi)$ for $\xi \in (X_0^{X_1}, f_1)$. Then, Proposition (4.3) is generalized as

PROPOSITION (4.5) *The homtopy classes of the extensions* $\bar{f}_1 : (X_1 \underset{f_2}{\cup} CX, x_1^0) \longrightarrow (X_0, x_0^0)$ *of* $f_1 : (X_1, x_1^0) \longrightarrow (X_0, x_0^0)$ *are in one to one correspondence with the elements of the cokernel* α_{f_1} i.e. of $C(L, X_1)/\alpha_{f_1}\pi_1(X_0^{X_1}, f_1)$. ([3]. Th. 3.4)

Now a homotopy $G : (I \times X_1, I \times x_1^0) \longrightarrow (X_0, x_0^0)$ from f_1^0 to f_1^1 is equivalent to a path G' in $X_0^{X_1}$ from f_1^0 to f_1^1, which defines a homomorphism in the usual way of the homotopy groups based at f_1^1 into those based at f_1^0: we describe for this

$$G_\# : \pi_1(X_0^{X_1}, f_1^1) \longrightarrow (X_0^{X_1}, f_1^0).$$

It was proved that $\alpha_{f_1^1} = \alpha_{f_1^0} \circ G_\#$. ([3], Lemma 3)

In view of these Lemmata (4.2), (4.3) and Propositions (4.4), (4.5), we may denote by $[\alpha_1, \alpha_2]$ the set of all homotopy classes of extensions $(X_1 \underset{\alpha_2}{\cup} CX_2, x_1^0) \longrightarrow (X_0, x_0^0)$ of arbitrary representative of $\alpha_1 \in \pi(X_1, X_0)$.

Let $(\bar{\nu}) : X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2$, and $(\bar{\nu}') : Y_0 \xleftarrow{\beta_1} Y_1 \xleftarrow{\beta_2} Y_2$ be given, and let $\bar{H} : (\bar{\nu}) \longrightarrow (\bar{\nu}')$ be a map i.e. there exist representatives $(\bar{N}) : X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2$, (A) and $(\bar{N}') : Y_0 \xleftarrow{g_1} Y_1 \xleftarrow{g_2} Y_2$ (B) of $(\bar{\nu})$ and $(\bar{\nu}')$ such that homotopy commutivity holds in the diagrams

$$\text{(4.6)} \qquad \begin{array}{ccccc} X_0 & \xleftarrow{f_1} & X_1 & \xleftarrow{f_2} & X_2 \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} \\ Y_0 & \xleftarrow[g_1]{} & Y_1 & \xleftarrow[g_2]{} & Y_2 \end{array} \qquad \begin{array}{ccc} X_0 & \xleftarrow{A} & CX_2 \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle ch_2} \\ Y_0 & \xleftarrow[B]{} & CY_2 \end{array} \qquad \text{[c.f. (2.3)]}$$

PROPOSITION (4.7) $\qquad \gamma_0 \circ [\alpha_1, \alpha_2] \sim [\beta_1, \beta_2] \circ (\gamma_1 \cup \gamma_2)$

wkere $\gamma_i = \{h_i\} \in \pi(X_i, Y_i)$, $(0 \leqq i \leqq 2)$.

The proof is similar with that of (3.6).

Let $(\bar{N}^{(1)}) : X_0 \xrightarrow{f_1^{(1)}} EX_1 \xleftarrow{Ef_2} EX_2$, $(A^{(1)})$ and $(\bar{N}^{(2)}) : X_0 \xleftarrow{f_1^{(2)}} EX_1 \xleftarrow{Ef_2} EX_2$, $(A^{(2)})$ be representatives of null couples $(\bar{\nu}^{(1)}) : X_0 \xleftarrow{\alpha_1} EX_1 \xleftarrow{E\beta} EX_2$ and $(\bar{\nu}^{(2)}) : X_0 \xleftarrow{\alpha_2} EX_1 \xleftarrow{E\beta} EX_2$, respectively, and let $f_1^{(0)} = f_1^{(1)} + f_1^{(2)}$, $A^{(0)} = A^{(1)} + A^{(2)}$. Then, we see that

$$[f_1^{(1)}, A^{(1)}, Ef_2]+[f_1^{(2)}, A^{(2)}, Ef_2]=[f_1^{(0)}, A^{(0)}, Ef_2].$$

Hence, it follows that

PROPOSITION (4.8) $[\alpha_1, E\beta]+[\alpha_2, E\beta]\subset[\alpha_1+\alpha_2, E\beta]$.

Let $(\bar{EN}): EX_0\xleftarrow{Ef_1}EX_1\xleftarrow{Ef_2}EX_2$, (EA) be the suspension of a representative $(\bar{N}): X_0\xleftarrow{f_1}X_1\xleftarrow{f_2}X_2$, (A) of a null couple $(\bar{\nu}): X_0\xleftarrow{\alpha_1}X_1\xleftarrow{\alpha_2}X_2$. Then, we see that

$$E[f_1, A, f_2]=[Ef_1, EA, Ef_2].$$

Hence, we have

PROPOSITION (4.9) $E[\alpha_1, \alpha_2]\subset[E\alpha_1\ E\alpha_2]$.

§5 First derived compositions

Consider a representative $(\bar{N}): X_0\xleftarrow{f_1}X_1\xleftarrow{f_2}X_2\xleftarrow{f_3}X_3$, (A_1, A_2) of a null triple $(\bar{\nu}): X_0\xleftarrow{\alpha_1}X_1\xleftarrow{\alpha_2}X_2\xleftarrow{\alpha_3}X_3$. The composition

$$X_0\xleftarrow{\bar{f}_1}X_1\cup_{f_2} CX_2\xleftarrow{\tilde{f}_2}EX_3$$

of the coextension $\tilde{f}_3=(f_2, A_2, f_3)$ of f_3 followed by the extension $\bar{f}_1=[f_1, A_1, f_2]$ of f_1 is denoted by $\{f_1, A_1, f_2, A_2, f_3\}$, which is given by the formula

$$\{f_1, A_1, f_2, A_2, f_3\}d_{X_2}(x, t)=\begin{cases}f_1\circ A_2c_{X_3}(-t, x) & -1\leqq t\leqq 0,\\ A_1c_{X_2}(t, f_3(x)) & 0\leqq t\leqq 1,\end{cases}\tag{5.1}$$

for all $x\in X_3$.

PROPOSITION (5.2) *The set* $[\alpha_1, \alpha_2]\circ(\alpha_2, \alpha_3)$ *is a double coset of the subgroups* $\pi(EX_2, X_0)\circ E\alpha_3$ *and* $\alpha_1\circ\pi(EX_3, X_1)$ *in* $\pi(EX_3, X_0)$.

If $\pi(EX_3, X_0)$ *is abelian, in particular if* $X_3=EX_3$, *then it is a coset of the subgroup* $\pi(EX_2, X_0)\circ E\alpha_3+\alpha_1\circ\pi(EX_3, X_1)$ *in* $\pi(EX_3, X_0)$ ([1] p. 9).

The proof of the proposition was given by H. Toda in [1], and is omitted here.

DEFINITION (5.3) *The set* $[\alpha_1, \alpha_2]\circ(\alpha_2, \alpha_3)$ *is called the first derived composition, and is denoted by* $\{\alpha_1, \alpha_2, \alpha_3\}$.

[NOTE] In [1], notation $\{\alpha_1, \alpha_2, \alpha_3\}$ means the set $-[\alpha_1, \alpha_2]\circ(\alpha_2, \alpha_3)$, and is called *secondary composition.*

Let $(\bar{\nu}): X_0\xleftarrow{\alpha_1}X_1\xleftarrow{\alpha_2}X_2\xleftarrow{\alpha_3}X_3$ and $(\bar{\nu}'): Y_0\xleftarrow{\beta_1}Y_1\xleftarrow{\beta_2}Y_2\xleftarrow{\beta_3}Y_3$ be given, and let $\bar{H}: (\bar{\nu})\longrightarrow(\bar{\nu}')$ be a map, so that there exist representatives $(N): X_0\xleftarrow{f_1}X_1\xleftarrow{f_2}X_2\xleftarrow{f_3}X_3$, (A_1, A_2) and $(\bar{N'}): Y_0\xleftarrow{g_1}Y_1\xleftarrow{g_2}Y_2\xleftarrow{g_3}Y_3$, (B_1, B_2) of $(\bar{\nu})$ and $(\bar{\nu}')$, respec tively, such that homotopy commutativity holds in the diagrams

$$\begin{array}{ccccccc} X_0 & \xleftarrow{f_1} & X_1 & \xleftarrow{f_2} & X_2 & \xleftarrow{f_3} & X_3\\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3}\\ Y_0 & \xleftarrow[g_1]{} & Y_1 & \xleftarrow[g_2]{} & Y_2 & \xleftarrow[g_3]{} & Y_3\end{array}\qquad \begin{array}{ccc} X_{i-1} & \xleftarrow{A_i} & CX_{i+1}\\ \downarrow{\scriptstyle h_{i-1}} & & \downarrow{\scriptstyle Ch_{i+1}}\\ Y_{i-1} & \xleftarrow{B_i} & CY_{i-1}\,.\end{array}\tag{5.4}$$

$(i=1, 2)$. (c.f. (2.3))

It follows from (3.6) and (4.7) that

LEMMA (5.5) $h_0 \circ \{f_1, A_1, f_2, A_2, f_3\} \simeq \{g_1, B_1, g_2, B_2, g_3\} \circ Eh_3$

i.e. homotopy commutativity holds in the diagram

$$\begin{tikzcd}
X_0 \arrow[d, "h_0"'] & X_1 \cup_{f_2} CX_2 \arrow[l, "\tilde{f}_1"'] \arrow[d, dashed, "h_1 \cup h_2"] & EX_3 \arrow[l, "\tilde{f}_3"'] \arrow[d, "Eh_3"] \\
Y_0 & Y_1 \cup_{g_2} CY_2 \arrow[l, "\tilde{g}_1"] & EY_3 . \arrow[l, "\tilde{g}_3"]
\end{tikzcd}$$

PROPOSITION (5.6) *Suppose the following diagram is given:*

$$\begin{tikzcd}
 & X_1 \arrow[dl, "\alpha_1"'] \arrow[dd, "\gamma_1"] & & X_2 \arrow[ll, "\alpha_2"'] \arrow[dd, "\gamma_2"'] & \\
X_0 \arrow[r, phantom, "\circlearrowleft"] & {} \arrow[rr, phantom, "\circlearrowleft"] & & {} \arrow[r, phantom, "\circlearrowleft"] & X_3 , \arrow[ul, "\alpha_3"'] \arrow[dl, "\beta_3"] \\
 & Y_1 \arrow[ul, "\beta_1"] & & Y_2 \arrow[ll, "\beta_2"] &
\end{tikzcd}$$

where $\alpha_2 \circ \alpha_3 = 0$, *and* $\beta_1 \circ \beta_2 = 0$.

Then, it follows that $\{\alpha_1, \alpha_2, \alpha_3\} \sim \{\beta_1, \beta_2, \beta_3\}$

PROOF. Chose representatives $(\overline{N})$ ond $(\overline{N}')$ of the null triples

$$(\bar{\nu}): X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \text{ and } (\bar{\nu}'): X_0 \xleftarrow{\beta_1} Y_1 \xleftarrow{\beta_2} Y_2 \xleftarrow{\beta_3} X_3$$

as follows:

$$(\overline{N}): X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3, \ (\underline{A \circ Ch_2}, B)$$
$$(\overline{N}'): X_0 \xleftarrow{g_1} Y_1 \xleftarrow{g_2} Y_2 \xleftarrow{g_3} X_3, \ (\mathrm{A}, \overline{h_1 \circ B})$$

for any representatives of h_1 and h_2 of γ_1 and γ_2 respectively.

Then, $H = (id.\ h_1, h_2, id.)$ forms a map of $(\bar{\nu})$ into $(\bar{\nu}')$, so that the proposition follows from (5.5). q.e.d.

Let $(M): X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \leftarrow \cdots \leftarrow X_6 \xleftarrow{f_7} X_7$ be a representative of a 7-tuple $(\alpha): X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \leftarrow \cdots \leftarrow X_6 \xleftarrow{\alpha_7} X_7$ such that

$$\alpha_2 \circ \alpha_3 \circ \alpha_4 = 0 \quad \text{and} \quad \alpha_4 \circ \alpha_5 \circ \alpha_6 = 0.$$

Then, $H = (f_1, f_3, f_5, f_7)$ is a map of a null triple

$$(\bar{\nu}): X_1 \xleftarrow{\alpha_2 \circ \alpha_3} X_3 \xleftarrow{\alpha_4 \circ \alpha_5} X_5 \xleftarrow{\alpha_6 \circ \alpha_7} X_7$$

into a null triple

$$(\bar{\nu}'): X_0 \xleftarrow{\alpha_1 \circ \alpha_2} X_2 \xleftarrow{\alpha_3 \circ \alpha} X_4 \xleftarrow{\alpha_5 \circ \alpha_6} X_6 .$$

Indeed, we may chose the representatives $(\overline{N})$ and $(\overline{N}')$ of $(\bar{\nu})$ and $(\bar{\nu}')$, respectively as follows:

$$(\bar{N}):\ X_1 \xleftarrow{f_2\circ f_3} X_3 \xleftarrow{f_4\circ f_5} X_5 \xleftarrow{f_6\circ f_7} X_7,\ (A\circ Cf_5,\ B\circ Cf_7)$$

$$(N'):\ X_0 \xleftarrow{f_1\circ f_2} X_2 \xleftarrow{f_3\circ f_4} X_4 \xleftarrow{f_5\circ f_6} X_6,\ (f_1\circ A,\ f_3\circ B)$$

for any null homotopies A and B of $f_2\circ f_3\circ f_4$ and $f_4\circ f_5\circ f_6$, respectively.

Hence, it follows that

THEOREM (5.6) $\alpha_1\circ\{\alpha_2\circ\alpha_3,\ \alpha_4\circ\alpha_5,\ \alpha_6\circ\alpha_7\}\sim\{\alpha_1\circ\alpha_2,\ \alpha_3\circ\alpha_4,\ \alpha_5\circ\alpha_6\}\circ E\alpha_7$, *if* $\alpha_2\circ\alpha_3\circ\alpha_4=0$ *and* $\alpha_4\circ\alpha_5\circ\alpha_6=0$

In particular, taking $X_2=X_3,\ X_4=X_5,\ X_6=X_7,\ f_3=id.,\ f_5=id.,$ *and* $f_7=id.,$ then we have

COROLLARY (5.8) (i) $\alpha\circ\{\beta,\gamma,\delta\}\subset\{\alpha\circ\beta,\gamma,\delta\}$

Similaly we have

COROLLARY (5.8) (ii) $\{\alpha\circ\beta,\gamma,\delta\}\subset\{\alpha,\beta\circ\gamma,\delta\}$

(iii) $\{\alpha,\beta\circ\gamma,\delta\}\supset\{\alpha,\beta,\gamma\circ\delta\}$

(iv) $\{\alpha,\beta,\gamma\circ\delta\}\supset\{\alpha,\beta,\gamma\}\circ E\delta.$ ([1] Prop. 1, 2)

According to H. Toda [1], we denote by $\{\alpha_1,E^n\alpha_2,E^n\alpha_3\}_n\ (n\geqq 1)$ the first derived composition constructed from a null triple

$$(\nu):\ X_0 \xleftarrow{\alpha_1} E^nX_1 \xleftarrow{E^n\alpha_2} E^nX_2 \xleftarrow{E^n\alpha_3} E^nX_3$$

such that the null couple $E^nX_1 \xleftarrow{E^n\alpha_2} E^nX_2 \xleftarrow{E^n\alpha_3} E^nX_3$ is an n-fold suspension of a null couple $X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3$. Hence, $\{\alpha_1,E^n\alpha_2,E^n\alpha_3\}_n$ is a coset of the subgroup $\alpha_1\circ E^n\pi(EX_3,X_1)+\pi(E^{n+1}X_2,X_0)E^{n+1}\alpha_3$.

PROPOSITION (5.9)

(i) *For null triples* $X_0 \xleftarrow{\alpha} X_1 \xleftarrow{\beta} X_2 \xleftarrow{\gamma_i} EX_3\ (i=1,2)$, *it follows that*

$$\{\alpha,\beta,\gamma_1\}\pm\{\alpha,\beta,\gamma_2\}\supset\{a,\beta,\gamma_1\pm\gamma_2\}.$$

(ii) *For null triples* $X_0 \xleftarrow{\alpha} X_1 \xleftarrow{\beta_i} EX_2 \xleftarrow{E\gamma} EX_3\ (i=1,2)$, *it follows that*

$$\{\alpha,\beta_1,E\gamma\}\pm\{\alpha,\beta_2,E\gamma\}=\{\alpha,\beta_1\pm\beta_2,E\gamma\}.$$

(iii) *For null triples* $X_0 \xleftarrow{\alpha_i} EX_1 \xleftarrow{E\beta} EX_2 \xleftarrow{E\gamma} EX_3\ (i=1,2)$, *it follows that*

$$\{\alpha_1,E\beta,E\gamma\}_1\pm\{\alpha_2,E\beta,E\gamma\}_1\supset\{\alpha_1\pm\alpha_2,E\beta,E\gamma\}_1$$

The proof of the proposition is given in [1], but it can also be given by using (3.8) and (4.8) etc.

PROPOSITION (5.10) *For a null triple* $X_0 \xleftarrow{\alpha} X_1 \xleftarrow{\beta} X_2 \xleftarrow{\gamma} X_3$, *it follows that*

$$E\{\alpha,\beta,\gamma\}\subset\{E\alpha,E\beta,E\gamma\}\circ\rho$$

where $\rho\in\pi(E^2X,E^2X)$ *is defined by* $\rho d_X^2(x;t,s)=d_X^2(x;s,t)$ *for* $x\in X$, *and* $s,t\in V^1$, *which is of degree* -1, *if* X *is a sphere.* ([1], Prop. 1.3).

Proof. The proposition follows from (3.9) and (4.9).

Proposition (5.11) *Let* $(N): X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2$, (A) *be a representative of a null couple* $(\nu): X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2$, *then homotopy commutativity holds in the following diagram*:

$$\begin{tikzcd} X_0 \arrow[d, "i"'] & X_1 \cup_{f_2} CX_2 \arrow[l, "\bar{f}_1"'] \arrow[d, "p"] \\ X_0 \cup_{f_1} CX_1 & EX_2, \arrow[l, "-\tilde{f}_2"] \end{tikzcd}$$

where i: *is the inclusion map, and* p *is the shrinking map defined in* (1.3).

Proof. Define a map $h: (I\times X_1, I\times X_1^0) \longrightarrow (X_0 \cup_{f_1} CX_1, x_0^0)$ by

$$h(s, x) = c_{X_1}(s, x) \qquad s\in I,\ x\in X_1,$$

and a map $H: (I\times CX_2, I\times x_2^0) \longrightarrow (X_0 \cup_{f_1} CX_1, x_0^0)$ by

$$H(s, c_{X_2}(t, x)) = \begin{cases} Ac_{X_2}((s-2t)/(s-2), x), & s/2 \leqq t \leqq 1, \\ c_{X_1}(s-2t, f_2(x)), & 0 \leqq t \leqq s/2. \end{cases}$$

for all $s\in I$ and $x\in X_2$.

Consider a map $G: (I\times(X_1 \cup_{f_2} CX_2), I\times x_1^0) \longrightarrow (X_0 \cup_{f_2} CX_1, x_0)$ defined by

$$G \mid I\times X_1 = h \quad \text{and} \quad G \mid I\times CX_2 = H$$

then, we see that G is continuous. If we denote $G\mid\{s\}\times(X_1 \cup_{f_2} CX_2)$ by G_s for $s\in I$, we can see that

$$G_0 = i\circ\bar{f}_1 \quad \text{and} \quad G_1 = -\tilde{f}_2\circ p \qquad \text{q. e. d.}$$

Corollary (5.12) *Let* $(\nu): X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$ *be a null quadruple, then it follows that* $\alpha_1\circ\{\alpha_2, \alpha_3, \alpha_4\} = -\{\alpha_1, \alpha_2, \alpha_3\}\circ E\alpha_4$ ([1] Prop. 1.4)

Proof. Let $(N): X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} X_4$, (A_1, A_2, A_3) be a representative of the null couple (ν). In the diagram

$$\begin{tikzcd} X_0 \arrow[d, "id."'] & X_1 \arrow[l, "f_1"'] \arrow[d, "i"'] & X_2 \cup_{f_3} CX_3 \arrow[l, "\bar{f}_2"'] \arrow[d, "p"'] & EX_4 \arrow[l, "\tilde{f}_4"'] \arrow[d, "id."] \\ X_0 & X_1 \cup_{f_2} CX_2 \arrow[l, "f_1"] & EX_3 \arrow[l, "-\tilde{f}_3"] & EX_4 \arrow[l, "Ef_4"] \end{tikzcd}$$

homotopy commutativity holds by (5.11). It follows from (5.6) that $\alpha_1\circ\{\alpha_2, \alpha_3, \alpha_4\}$ and $-\{\alpha_1, \alpha_2, \alpha_3\}\circ E\alpha_4$ have a common element. And besides, these sets are cosets of the same subgroup $\alpha_1\circ\pi(EX_3, X_1)\circ E\alpha_4$ in $\pi(EX_4, X_0)$. g. e. d.

§ 6 Second derived compositions

Let $(N): X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} X_4$, (A_1, A_2, A_3) be a representative of a null quadruple $(\bar{\nu})$: $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$. Then, the triples

$$(N_1): X_0 \xleftarrow{\tilde{f}_1} X_1 \cup_{f_2} CX_2 \xleftarrow{\tilde{f}_3} EX_3 \xleftarrow{Ef_4} EX_4, \quad \text{and}$$

$$(N_2): X_0 \xleftarrow{f_1} X_1 \xleftarrow{\tilde{f}_2} X_2 \cup_{f_3} CX_3 \xleftarrow{\tilde{f}_4} EX_4$$

are called *the first derived triples* of $(\bar{N})$, where $\tilde{f}_1 = [f_1, A_1, f_2]$, $\tilde{f}_2 = [f_2, A_2, f_3]$, $\tilde{f}_3 = (f_2, A_2, f_3)$, and $\tilde{f}_4 = (f_3, A_3, f_4)$.

DEFINITION (6.1) A representative $(\bar{N})$ is called *admissible*, if $\tilde{f}_1 \circ \tilde{f}_3 \simeq 0$ in (N_1), and $\tilde{f}_2 \circ \tilde{f}_3 \simeq 0$ in (N_2), and a null quadruple $(\bar{\nu})$ is called *admissible*, if it contains an admissible representative.

Let $(\bar{\nu})$: $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$ be a null quadruple such that $0 \in \{\alpha_1\ \alpha_2\ \alpha_3\}$, and $0 \in \{\alpha_2, \alpha_3, \alpha_4\}$, and let

$$(M_1): X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3, \ (A_1, A_2), \quad \text{and}$$

$$(M_2): X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} X_4, \ (B_2, B_3),$$

be representatives of null triples $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3$, and $X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$, respectively, such that

$$\{f_1, A_1, f_2, A_2, f_3\} \simeq 0, \quad \text{and} \quad \{f_2, B_2, f_3, B_3, f_4\} \simeq 0.$$

Denote by G_1 and G_2 the subgroups of $\pi(EX_3, X_1)$ such that

$$(6.2) \qquad \alpha_1 \circ G_1 \subset \pi(EX_2, X_0) \circ E\alpha_3, \quad \text{and} \quad G_2 \circ E\alpha_4 \subset \alpha_2 \circ \pi(EX_3, X_1).$$

Note that for any unll homotopy A_2' of $f_2 \circ f_3$, there exixts a null homotopy A_1' of $f_1 \circ f_2$ such that $\{f_1, A_1', f_2, A_2', f_3\} \simeq 0$, if and only if $\partial(A_2', A_2) \in G_1$, and there exists a null homotopy A_3' of $f_3 \circ f_4$ such that $\{f_2, A_2', f_3, A_3', f_4\} \simeq 0$, if and only if $\partial(A_2', B_2) \in G_2$. Hence, we have a sufficient condition that a null quadruple $(\bar{\nu})$ is admissible:

PROPOSITION (6.3) Let $(\bar{\nu})$: $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$ *be a null quadruple such that* $0 \in \{\alpha_1, \alpha_2, \alpha_3\}$, *and* $0 \in \{\alpha_2, \alpha_3, \alpha_4\}$. *If* $G_1 + G_2 = \pi(EX_3, X_1)$, *then, the null quadruple* $(\bar{\nu})$ *is admissible.*

Indeed, for any representatives (M_1), and (M_2) as above, $\partial(B_2, A_2) = \gamma_1 + \gamma_2$, for some $\gamma_i \in G_i$ $(i=1, 2)$. Let A_2' be a null homotopy of $f_2 \circ f_3$ such that $\partial(A'_2, A_2) = \gamma_1$, then $\partial(A_2', B_2) = \partial(A_2', A_2) + \partial(A_2, B_2) = -\gamma_2$.

Now, let $(\bar{N})$: $X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} X_4$, (A_1, A_2, A_3) be a representative of a null quadruple $(\bar{\nu})$: $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$. Then, from (5.12) it follows that homotopy commutativity holds in the diagram

$$\begin{tikzcd}
X_0 \arrow[d, "id."'] & X_1 \arrow[l, "f_1"'] \arrow[d, "i"] & X_2 \underset{f_3}{\cup} CX_3 \arrow[l, "\bar{f}_2"'] \arrow[d, "p"'] & EX_4 \arrow[l, "\tilde{f}_4"'] \arrow[d, "id."] \\
X_0 & X_1 \underset{f_2}{\cup} CX_2 \arrow[l, "\bar{f}_1"] & EX_3 \arrow[l, "-\tilde{f}_3"] & EX_4 . \arrow[l, "Ef_4"]
\end{tikzcd}$$

If $(\bar{N})$ is admissible, then, both of the derived triples are null triples. Let B and D be null homotopies of $-(\bar{f}_1 \circ \tilde{f}_3)$ and $\bar{f}_2 \circ \tilde{f}_4$, respectively, and consider the null triples

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{\bar{f}_2} X_2 \underset{f_3}{\cup} CX_3 \xleftarrow{\tilde{f}_4} EX_4, \ (\overline{B \circ Cp}, D), \quad \text{and}$$
$$X_0 \xleftarrow{\bar{f}_1} X_1 \underset{f_2}{\cup} CX_2 \xleftarrow{\tilde{f}_3} EX_3 \xleftarrow{Ef_4} EX_4, \ (B, \widetilde{i \circ D})$$

where $\overline{B \circ Cp}$ and $\widetilde{i \circ D}$ are the null homotopies induced by $B \circ Cp$ and $i \circ D$, respectively (cf. § 2). Then, it follows that

$$\{f_1, \overline{BCp}, \bar{f}_2, D, \tilde{f}_4\} \simeq \{\bar{f}_1, B, -\tilde{f}_3, \widetilde{i \circ D}, Ef_4\}.$$

For a fixed representative $(\bar{N})$, the set of all homotopy classes of $\{\bar{f}_1, \overline{B \circ Cp}, \bar{f}_2, D, f_4\}$ is a coset of a subgroup

$$F = \alpha_1 \circ \pi(EX_4, X_1) + \pi(E^2X_3, X_0) \circ E^2\alpha_4$$

in $\pi(E^2X_4, X_0)$[1]. Similarly, the set of all homotopy classes of $\{\bar{f}_1, B, -\tilde{f}_3, \widetilde{i \circ C}, Ef_4\}$ is a coset of the subgroup F in $\pi(E^2X_4, X_0)$. And, these two cosets coincide with each other. Denote it by $\gamma(A_1, A_2, A_3)$. It is obvious that $\gamma(A_1', A_2, A_3) = \gamma(A_1, A_2, A_3) = \gamma(A_1, A_2, A_3')$. Hence, $\gamma(A_1, A_2, A_3) = \gamma(A_1', A_2, A_3')$.

DEFINITION (6.4) *Union of the cosets of F which are obtained from all the admissible representatives of* $(\bar{\nu})$: $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$ *is called the second derived composition, and is denoted by* $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

Let (N): $X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} X_4$, (A_1', A_2', A_3') be another admissible representative of (ν). Then, $\delta(A_2', A_2) = G_1 \cap G_2$. Hence, we have the following.

PROPOSITION (6.5) *Let* (ν): $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$ *be a null quadruple such that* $0 \in \{\alpha_1, \alpha_2, \alpha_3\}$, and $0 \in \{\alpha_2, \alpha_3, \alpha_4\}$.

(i) *If* $G_1 + G_2 = \pi(EX_3, X_1)$ *(direct), then,* $(\bar{\nu})$ *is admissible, and* $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ *is a coset of* F.

(ii) *If* $G_1 + G_2 = \pi(EX_3, X_1)$, $\alpha_1 \circ (G_1 \cap G_2) = 0$, *and* $(G_1 \cap G_2) \circ E\alpha_4 = 0$, *then,* $(\bar{\nu})$ *is admissible, and*

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \lambda + \mu_i + F,$$

(1) We restrict the null homotopies of $f_1 \circ \bar{f}_2$ to those which are induced by null homotopies of $-(\bar{f}_1 \ \tilde{f}_3)$.

where $\mu_i\in\{\alpha_1, G_1\cap G_2, E\alpha_4\}$, *and* $\lambda\in\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

DEFINITION (6.6) *Let* $(\nu): X_0\xleftarrow{\alpha_1}X_1\xleftarrow{\alpha_2}X_2\xleftarrow{\alpha_3}X_3\xleftarrow{\alpha_4}X_4$, *and* $(\nu'): Y_0\xleftarrow{\beta_1}Y_1\xleftarrow{\beta_2}Y_2\xleftarrow{\beta_3}Y_3\xleftarrow{\beta_4}Y_4$ *be admissible quadruples. A map* $\vec{H}: (\nu)\longrightarrow(\nu')$ *is called admissible, if*

(1) *there exist admissible representatives*

$$(\bar{N}): X_0\xleftarrow{f_1}X_1\xleftarrow{f_2}X_2\xleftarrow{f_3}X_3\xleftarrow{f_4}X_4,\ (A_1, A_2, A_3),\ \textit{and}$$
$$(\bar{N}'): Y_0\xleftarrow{g_1}Y_1\xleftarrow{g_2}Y_2\xleftarrow{g_3}Y_3\xleftarrow{g_4}Y_4,\ (B_1, B_2, B_3).$$

of (ν) and (ν') *such that homotopy commutativity holds in the diagram* (2.1) *and* (2.3), *and if*

(2) *there exist null homotopies* C_1, D_1, C_2, *and* D_2 *of* $\bar{f}_1\circ\tilde{f}_3\simeq0$, $\bar{g}_1\circ\tilde{g}_3\simeq0$, $\bar{f}_2\circ\tilde{f}_4\simeq0$, *and* $\bar{g}_2\circ\tilde{g}_4\simeq0$, *respectively, such that*

$$\overline{h_0\circ C_1}\simeq D_1\circ CEh_3, \quad\text{and}\quad \overline{h_1\circ C_2}\simeq D_2\circ CEh_4.$$

PROPOSITION (6.7) *If a map* $\vec{H}: (\nu)\longrightarrow(\nu')$ *is admissible, then it follows that* $\gamma_0\circ\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}\sim\{\beta_1, \beta_2, \beta_3, \beta_4\}\circ E^2\gamma_4$ where $\gamma_i=h_i$ $(0\leqq i\leqq4)$.

PROOF If follows from the condition (1) and from (5.5) that homotopy commutativity holds in the diagram

$$\begin{array}{ccccccc}
X_0 & \xleftarrow{f_1} & X_1 & \xleftarrow{\bar{f}_2} & X_2\cup_{f_3}CX_3 & \xleftarrow{Ef_4} & EX_4\\
\downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2\cup h_3} & & \downarrow{\scriptstyle Eh_4}\\
Y_0 & \xleftarrow[g_1]{} & Y_1 & \xleftarrow[\bar{g}_2]{} & Y_2\cup_{g_3}CY_3 & \xleftarrow[Eg_4]{} & EY_4
\end{array}$$

Hence, by the condition (2) and by (5.5), we have the proposition. q. e. d.

THEOREM (6.8) *Let* $X_0\xleftarrow{\alpha_1}X_1\xleftarrow{\alpha_2}X_2\longleftarrow\cdots\longleftarrow X_8\xleftarrow{\alpha_9}X_9$ *be 9-tuple such that*

(1) $\alpha_i\circ\alpha_{i+1}\circ\alpha_{i+2}=0$ *for* $i=2, 4$ *and* 6,

(2) $0\in\{\alpha_2, \alpha_3\circ\alpha_4, \alpha_5\circ\alpha_6\}$, $0\in\{\alpha_4, \alpha_5\circ\alpha_6, \alpha_7\circ\alpha_8\}$,

and (3) $G_1+G_2=\pi(EX_7, X_4)$, *where* G_1 *and* G_2 *be the subgroups of* $\pi(EX_7, X_4)$ *such that* $\alpha_2\circ\alpha_3\circ G_1\subset\pi(EX_5, X_2)\circ E\alpha_5\circ E\alpha_6$, *and* $G_2\circ E\alpha_7\circ E\alpha_8\subset\alpha_4\circ\alpha_5\circ\pi(EX_9, X_5)$. *Then it follows that*

$\alpha_1\circ\{\alpha_2\circ\alpha_3, \alpha_4\circ\alpha_5, \alpha_6\circ\alpha_7, \alpha_8\circ\alpha_9\}\sim\{\alpha_1\circ\alpha_2, \alpha_3\circ\alpha_4, \alpha_5\circ\alpha_6, \alpha_7\circ\alpha_8\}\circ E\alpha_9$.

PROOF. It is very easy to see that the null quadruples

$$(\nu_1): X_1\xleftarrow{\alpha_2\circ\alpha_3}X_3\xleftarrow{\alpha_4\circ\alpha_5}X_5\xleftarrow{\alpha_6\circ\alpha_7}X_7\xleftarrow{\alpha_8\circ\alpha_9}X_8,\ \text{and}$$
$$(\nu_2): X_0\xleftarrow{\alpha_1\circ\alpha_2}X_2\xleftarrow{\alpha_3\circ\alpha_4}X_4\xleftarrow{\alpha_5\circ\alpha_6}X_6\xleftarrow{\alpha_7\circ\alpha_8}X_8$$

have the admissible representatives as follows:

$$(\bar{N}_1): X_1\xleftarrow{f_2f_3}X_3\xleftarrow{f_4f_5}X_5\xleftarrow{f_6f_7}X_7\xleftarrow{f_8f_9}X_9,\ (A_1\circ Cf_5, A_2\circ Cf_7, A_3\circ Cf_9)\ \text{and}$$

$$(\bar{N}_2):\ X_0 \xleftarrow{f_1 f_2} X_2 \xleftarrow{f_3 f_4} X_4 \xleftarrow{f_5 f_6} X_6 \xleftarrow{f_7 f_8} X_8,\ (f_1\circ A_1, f_3\circ A_2, f_5\circ A_3),$$

where A_i is a null homotopy of $\alpha_{2i}\circ\alpha_{2i+1}\circ\alpha_{2i+2}$ for $i=1, 2$ and 3.

Consider a map $\bar{H}=(f_1, f_3, f_5, f_7, f_9):(\bar{\nu}_1)\longrightarrow(\bar{\nu}_2)$, which clearly satisfies the condition (1) of (6.5). Let B_1 and B_2 be null homotopies of $\{f_2, A_1, f_3\circ f_4, f_3\circ A_2, f_5\circ f_6\}$ and $\{f_4, A_2, f_5\circ f_6, f_5\circ A_3, f_7\circ f_8\}$, respectivity, then, commutativity holds in the diagrams

$$\begin{array}{ccc} X_1 & \xleftarrow{B_1\cdot CEf_7} & CEX_7 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle CEf_7} \\ X_0 & \xleftarrow[f_1\cdot B_1]{} & CEX_6 \end{array}, \qquad \begin{array}{ccc} X_3 & \xleftarrow{B_2\cdot CEf_9} & CEX_9 \\ {\scriptstyle f_3}\downarrow & & \downarrow{\scriptstyle CEf_9} \\ X_2 & \xleftarrow[f_3\cdot B_2]{} & CEX_8 \end{array}.$$

This means that $\bar{H}$ is admissible. Hence, the theorem follows from (5.6).

In particular, if $X_2=X_3$, $X_4=X_5$, $X_6=X_7$, $X_8=X_9$, $\alpha_3=id.$, $\alpha_5=id.$, $\alpha_7=id.$, and $\alpha_9=id.$, it follows that

COROLLARY (6.9) (i) $\alpha\circ\{\beta,\gamma,\delta,\varepsilon\}\subset\{\alpha\circ\beta,\gamma,\delta,\varepsilon\}$.

Similarly, we have

(ii) $\{\alpha\circ\beta,\gamma,\delta,\varepsilon\}\subset\{\alpha,\beta\circ\gamma,\delta,\varepsilon\}$,

(iii) $\{\alpha,\beta\circ\gamma,\delta,\varepsilon\}=\{\alpha,\beta,\gamma\circ\delta,\varepsilon\}$,

(iv) $\{\alpha,\beta,\gamma\circ\delta,\varepsilon\}\supset\{\alpha,\beta,\gamma,\delta\circ\varepsilon\}$,

(v) $\{\alpha,\beta,\gamma,\delta\circ\varepsilon\}\supset\{\alpha,\beta,\gamma,\delta\}\circ E^2\varepsilon$.

PROPOSITION (6.10)

(i) *For admissible quadruples* $X_0\xleftarrow{\alpha}X_1\xleftarrow{\beta}X_2\xleftarrow{\gamma}X_3\xleftarrow{\delta_i}EX_4$ $(i=1,2)$, *it follows that* $\{\alpha,\beta,\gamma,\delta_1\}\pm\{\alpha,\beta,\gamma,\delta_2\}\supset\{\alpha,\beta,\gamma,\delta_1\pm\delta_2\}$.

(ii) *For admissible quadruples* $X_0\xleftarrow{\alpha}X_1\xleftarrow{\beta}X_2\xleftarrow{\gamma_i}EX_3\xleftarrow{\delta}EX_4$ $(i=1,2)$, *it follows that* $\{\alpha,\beta,\gamma_1,\delta\}\pm\{\alpha,\beta,\gamma_2,\delta\}=\{\alpha,\beta,\gamma_1\pm\gamma_2,\delta\}$.

(iii) *For admissible quadruples* $X_0\xleftarrow{\alpha}X_1\xleftarrow{\beta_i}EX_2\xleftarrow{E\gamma}EX_3\xleftarrow{E\delta}EX_4$ $(i=1,2)$, *it follows that* $\{\alpha,\beta_1,E\gamma,E\delta\}\pm\{\alpha,\beta_2,E\gamma,E\delta\}=\{\alpha,\beta_1\pm\beta_2,E\gamma,E\delta\}$, *if* $\gamma\circ\delta=0$.

(iv) *For admissible quadruples* $X_0\xleftarrow{\alpha_i}EX_1\xleftarrow{E\beta}EX_2\xleftarrow{E\gamma}EX_3\xleftarrow{E\delta}EX_4$ $(i=1,2)$, *it follows that* $\{\alpha_1,E\beta,E\gamma,E\delta\}\pm\{\alpha_2,E\beta,E\gamma,E\delta\}\supset\{\alpha_1\pm\alpha_2,E\beta,E\gamma,E\delta\}$ *if* $\beta\circ\gamma=0$ *and* $\gamma\circ\delta=0$.

PROOF (iii). Let $X_0\xleftarrow{f_1}X_1\xleftarrow{f_2^{(i)}}X_2\xleftarrow{Ef_3}EX_3\xleftarrow{Ef_4}EX_4$, $(A_1^{(i)}, A_2^{(i)}, EA_3)$ $(i=1,2)$, be admissible representatives of $X_0\xleftarrow{\alpha_1}X_1\xleftarrow{\beta_i}EX_2\xleftarrow{E\gamma}EX_3\xleftarrow{E\delta}EX_4$ $(i=1,2)$, where A_3 is a null homotopy of $f_3\circ f_4$. Then,
$\{f_1, f_2^{(1)}, Ef_3, Ef_4\}\pm\{f_1, f_2^{(2)}, Ef_3, Ef_4\}=\{f_1,\bar{f}_2^{(1)},\widetilde{Ef}_4\}\pm\{f_1,\bar{f}_2^{(2)},\widetilde{Ef}_4\}=\{f_1, f_2^{(1)}\pm\bar{f}_1^{(2)}, \widetilde{Ef}_4\}$ (Since, $\widetilde{Ef}_4$ is a suspension element (c.f. 3.9))$=\{f_1,\overline{f_2^{(1)}\pm f_2^{(2)}}, Ef_4\}$ (where $\overline{f_2^{(1)}\pm f_2^{(2)}}=[f_2^{(1)}\pm f_2^{(2)}, A_2^{(1)}\pm A_2^{(2)}, f_3])=\{f_1, f_2^{(1)}\pm f_2^{(2)}, f_3, f_4\}$.

The proofs of the other three are left to the reader.

PROPOSITION (6.11) *$E\{\alpha, \beta, \gamma, \delta\} \subset \{E\alpha, E\beta, E\gamma, E\delta\} \circ (E\rho \circ \rho)$ where $(E\rho \circ \rho): E^3X_4 \longrightarrow E^3X_4$ is a map defined by*

$$(E\rho \circ \rho) d^3{}_{X_4}(x\,;\ t_1, t_2, t_3) = d^3{}_{X_4}(x\,;\ t_3, t_1, t_2),\ x \in X_4,\ t_1, t_2, \in V^1.$$

If X_4 is a sphere, then, $(E\rho \circ \rho)$ is a map of degree $+1$.

PROOF. Let $X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} X_4$, (A_1, A_2, A_3) be an admissible representative of $X_0 \xleftarrow{\alpha} X_1 \xleftarrow{\beta} X_2 \xleftarrow{\gamma} X_3 \xleftarrow{\delta} X_4$. Then,

$$\begin{aligned} E\{f_1, [f_2, A_2, f_3], (f_3, A_3, f_4)\} &= \{Ef_1, E[f_2, A_2, f_3], E(f_3, A_3, f_4)\} \circ \rho && (5.10) \\ &= \{Ef_1, [Ef_2, EA_2, Ef_3], (Ef_3, EA_3, Ef_4) \circ \rho\} \circ \rho && (3.7) \\ &= \{Ef_1, [Ef_2, EA_2, Ef_3], (Ef_3, EA_3, Ef_4)\} \circ (E\rho \circ \rho). \end{aligned}$$

Since, for each fixed representative

$$E(\alpha \circ \pi(E^2X_4, X_1) + \pi(E^2X_3, X_0) \circ E^2\delta) \subset E\alpha \circ \pi(E^3X_4, EX_1) + \pi(E^3X_3, EX_0) \circ E^3\delta),$$

the proposition holds. q. e. d.

Now, let $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4 \xleftarrow{\alpha_5} X_5$ be a null 5-tuple such that $0 \in \{\alpha_i, \alpha_{i+1}, \alpha_{i+2}\}$ $(i=1, 2, 3)$, and let G_1, G_2, and G_3, G_4 be the subgroups of $\pi(EX_3, X_1)$ and $\pi(EX_4, X_2)$, respectively, which are defined in (6.2). Denote by $\overline{G}_2$ and $\overline{G}_3$ the subgroups of G_2 and G_3, respectively, such that $\overline{G}_2 \circ E\alpha_4 = 0$ and $\alpha_2 \circ \overline{G}_3 = 0$.

PROPOSITION (6.12) *If, either* (1) $G_1 + G_2 = \pi(EX_3, X_1)$, $\overline{G}_3 + G_4 = \pi(EX_4, X_2)$, *or* (2) $G_1 + \overline{G}_2 = \pi(EX_3, X_1)$, $G_3 + G_4 = \pi(EX_4, X_2)$, *then, it follows that*

$$\alpha_1 \circ \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\} \sim \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} E^2\alpha_5.$$

PROOF. Null quadruples $(\bar{\nu}_1)$: $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$, and $(\bar{\nu}_2)$: $X_1 \xleftarrow{\alpha_1} X_2 \xleftarrow{\alpha_2} X_3 \xleftarrow{\alpha_3} X_4 \xleftarrow{\alpha_4} X_5$ have the admissible representatives as follows:

$$\begin{aligned} (\overline{N}_1): \ & X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} X_4,\ (A_1, A_2, A_3) \\ (\overline{N}_2);\ & X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} X_4 \xleftarrow{f_5} X_5,\ (A_1, A_2, A_3) \end{aligned}$$

Indeed, in the case (1), for a representative $(\overline{N}_1)$ of $(\bar{\nu}_1)$ and a representative $X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} X_4 \xleftarrow{f_5} X_5$, (B_3, B_4), $\partial(A_3, B_3) = \gamma_3 + \bar{\gamma}_4$ for some $\bar{\gamma}_3 \in G_3$, $\gamma_4 \in G_4$. Let A_3' be a null homotopy of $f_3 \circ f_4$ such that $\partial(A_3', A_3) = \bar{\gamma}_3$. But, (A_1, A_2, A_3') still is an admissible representative of $(\bar{\nu}_1)$, because $\alpha_2 \circ \bar{\gamma}_3 = 0$. Then, $\partial(A_3', B_3) = \partial(A_3', A_3) + \partial(A_3, B_3) = -\gamma_4 \in G_4$. Hence, there exists a null homotopy A_4 of $f_4 \circ f_5$ such that (A_2, A_3', A_4) is admissible. The same argument holds in the case (2). It follows from (5.12) that

$$\begin{aligned} \{f_1, [f_2, A_2, f_3], (f_3, A_3, f_4)\} \circ E^2 f_5 &= -f_1 \circ \{[f_2, A_2, f_3], (f_3, A_3, f_4), Ef_5\} \\ &= f_1 \circ \{[f_2, A_2, f_3], -(f_3, A_3, f_4), Ef_5\}. \end{aligned}$$

Hence, $\alpha_1 \circ \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \circ E^2\alpha_5$ have a common element. q.e.d.

PROPOSITION (6.13) (i) *Let* (ν): $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4 \xleftarrow{\alpha_5} X_5$ *be a 5-tuple obtained by combining a null couple* $(\bar{\nu}_1)$: $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2$ *and a null triple* $(\bar{\nu}_2)$: $X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4 \xleftarrow{\alpha_4} X_5$. *Assume that* $0 \in \alpha_2 \circ \{\alpha_3, \alpha_4, \alpha_5\}$, *and that* $G_1 + G_2 = \pi(EX_4, X_1)$, *where* G_1 *and* G_2 *be the subgroups of* $\pi(EX_4, X_1)$, *which are defined in* (6.2) *for the null quadruple* (μ): $X_0 \xleftarrow{\alpha_1} X_1 \xrightarrow{\alpha_2 \alpha_3} X_3 \xleftarrow{\alpha_4} X_4 \xleftarrow{\alpha_5} X_5$, *then, there is an element* $\lambda \in \{\alpha_3, \alpha_4, \alpha_5\}$ *such that* $\alpha_2 \circ \lambda = 0$, and that $\{\alpha_1, \alpha_2, \lambda\} \sim \{\alpha_1, \alpha_2 \circ \alpha_3, \alpha_4, \alpha_5\}$.

(*Briefly*, $\{\alpha_1, \alpha_2, \{\alpha_3, \alpha_4, \alpha_5\}\} \sim \{\alpha_1, a_2 \circ \alpha_3, a_4, a_5\}$).

PROOF. Let $(\bar{N}_1)$: $X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2, (A)$, be a representative of $(\bar{\nu}_1)$, and let $(\bar{N}_2)$: $X_2 \xrightarrow{f_3} X_3 \xrightarrow{f_4} X_4 \xrightarrow{f_5} X_5$, (B_1, B_2) be a representative of $(\bar{\nu}_2)$ such that $\{f_3, B_1, f_4, B_2, f_5\} \in \lambda$. Note that $\{f_1, A \circ Cf_3, f_2 \circ f_3, f_2 \circ B_1, f_4\} \simeq 0$. Hence, it follows that $\{f_1, [f_2 \circ f_3, f_2 \circ B_1, f_4], (f_4, B_2, f_5)\} = \{f_1, f_2 \circ [f_3, B_1, f_4], (f_4, B_2, f_5)\} = \{f_1, f_2, [f_3, B_1, f_4] \circ (f_4, B_2, f_5)\}$. (5.8 (iii)).

Hence, $\{\alpha_1, \alpha_2 \circ \alpha_3, \alpha_4, \alpha_5\}$ and $\{\alpha_1, \alpha_2, \lambda\}$ have a common element. Similarly, we have

PROPOSITION (6.13) (ii) *Let* (ν): $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4 \xleftarrow{\alpha_5} X_5$ *be a 5-tuple obtained by combining a null triple* $(\bar{\nu}_1)$: $X_0 \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3$ *and a null couple* (ν_2): $X_3 \xleftarrow{\alpha_4} X_4 \xleftarrow{\alpha_5} X_5$. *Assume that* $0 \in \{\alpha_1, \alpha_2, \alpha_3\} \circ E\alpha_4$ *and* $G_1 + G_2 = \pi(EX_4, X_1)$, *where* G_1 *and* G_2 *are the subgroups of* $\pi(EX_4, X_1)$, *which are defined in* (6.2). *Then, there exists an element* $\tau \in \{\alpha_1, \alpha_2, \alpha_3\}$ *such that* $\tau \circ E\alpha_4 = 0$, *and* $-\{\tau, E\alpha_4, E\alpha_5\} \sim \{\alpha_1, \alpha_2, \alpha_3 \circ \alpha_4, \alpha_5\}$.

(*Briefly*, $-\{\{\alpha_1, \alpha_2, \alpha_3\}, E\alpha_4, E\alpha_5\} \sim \{\alpha_1, \alpha_2, \alpha_3 \circ \alpha_4, \alpha_5\}$).

Now, let (ν): $X_0 \underset{\alpha_1}{\cup} CX_1 \xleftarrow{i} X_0 \xleftarrow{\alpha_1 \cdot \alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$ be an admissible quadruple. Then, commutativity holds in the diagram

$$
\begin{tikzcd}
X_0 \underset{\alpha_1}{\cup} CX_1 \arrow[d, "p"'] & X_0 \underset{\alpha_1 \alpha_2}{\cup} CX_2 \arrow[l, "\tilde{i}"'] \arrow[d, "p"] & EX_3 \arrow[l, "\bar{\alpha}_3"'] \arrow[d, "id."] & EX_4 \arrow[l, "E\alpha_4"'] \arrow[d, "id."] \\
EX_1 & EX_2 \arrow[l, "E\alpha_2"] & EX_3 \arrow[l, "E\alpha_2"] & EX_4 \arrow[l, "E\alpha_4"]
\end{tikzcd}
$$

where $\tilde{i} = (id.) \cup \alpha_2$, which is considered to be the homotopy class of an extension of the inclusion map i. Since $p_* \circ i_* = 0$, and since $p^* \pi(E^2X_3, X_0 \underset{\alpha_1}{\cup} CX_1) \subset \pi(E^2X_3, EX_1)$, it follows that

PROPOSITION (6.14) $-p_*\{i, \alpha_1 \circ \alpha_2, \alpha_3, \alpha_4\} \subset \{E\alpha_2, E\alpha_3, E\alpha_4\}$ *for an admissible quadruple* (ν).

§ 7 Generalized Hopf-homomorphism

Let f be a map $X \longrightarrow \Omega(Y)$, where $\Omega(Y)$ indicates the space of loops on Y

(with compact-open topology).

Define a map $\Theta f: (EX, x^0) \longrightarrow (Y, y^0)$ by

(7.1) $$(\Theta f)d_X(x, t)=f(x)((1+t)/2),\ x\in X,\ -1\leqq t\leqq 1$$

The correspondence $f\leftrightarrow\Theta f$ is one to one, and hence it induces a one to one correspondence

$$\Theta: \pi(X, \Omega(Y))\longrightarrow\pi(EX, Y)$$

which is an isomorphism if X is a suspension space. ([1], (1.10))

We can easily verify that

PROPOSITION (7.2) *For $\alpha\in\pi(X_1, \Omega(Y))$ and $\beta\in\pi(X_2, X_1)$, the following relation holds:* $\Theta(\alpha\circ\beta)=\Theta\alpha\circ E\beta$.

Given an n-tuple

$$(\nu): \Omega(Y)\xleftarrow{\alpha_1}X_1\xleftarrow{\alpha_2}X_2\longleftarrow\cdots\longleftarrow X_{n-1}\xleftarrow{\alpha_n}X_n$$

we associate an n-tuple

$$(\Theta\nu): Y\xleftarrow{\Theta\alpha_1}EX_1\xleftarrow{E\alpha_2}EX_2\longleftarrow\cdots\longleftarrow EX_{n-1}\xleftarrow{E\alpha_n}EX_n.$$

To a represntative

$$(N): \Omega(Y)\xleftarrow{f_1}X_1\xleftarrow{f_2}X_2\longleftarrow\cdots\xleftarrow{f_{n-1}}X_{n-1}\xleftarrow{f_n}X_n,\ (A_1, A_2, \cdots, A_{n-1})$$

of a null n-tuple $(\bar\nu)$, we associate a representative

$$(\overline{\Theta N}): Y\xleftarrow{\Theta f_1}EX_1\xleftarrow{Ef_2}EX_2\longleftarrow\cdots\longleftarrow EX_{n-1}\xleftarrow{Ef_n}EX_n,\ (\Theta A_1, EA_2, \cdots, EA_{n-1})$$

of $(\overline{\Theta\nu})$.

LEMMA (7.3) *Let $(\overline{\Theta\nu}): Y\xleftarrow{\Theta\alpha_1}EX_1\xleftarrow{E\alpha_2}EX_2$ be the associated null couple of $(\bar\nu): \Omega(Y)\xleftarrow{\alpha_1}X_1\xleftarrow{\alpha_2}X_2$. Then it follows that*

$$\Theta[\alpha_1, \alpha_2]=[\Theta\alpha_1, E\alpha_2].$$

This follows directly from the definitions.

PROPOSITION (7.4) *Let $(\overline{\Theta\nu}): Y\xleftarrow{\Theta\alpha_1}EX_1\xleftarrow{E\alpha_2}EX_2\xleftarrow{E\alpha_3}EX_3$ be the associated null tripe of $(\bar\nu): \Omega(Y)\xleftarrow{\alpha_1}X_1\xleftarrow{\alpha_2}X_2\xleftarrow{\alpha_3}X_3$, then, it follows that $\Theta\{\alpha_1, \alpha_2, \alpha_3\}=\{\Theta\alpha_1, E\alpha_2, E\alpha_3\}_1\circ\rho$.*

PROOF. $\Theta\{\alpha_1, \alpha_2, \alpha_3\}=\Theta([\alpha_1, \alpha_2]\circ(\alpha_2, \alpha_3))=\Theta[\alpha_1, \alpha_2]\circ E(\alpha_2, \alpha_3)$
$=[\Theta\alpha_1, E\alpha_2]\circ(E\alpha_2, E\alpha_3)\circ\rho=\{\Theta\alpha_1, E\alpha_2, E\alpha_3\}\circ\rho$

While, $\{\Theta\alpha_1, E\alpha_2, E\alpha_3\}_1$ is a coset of the subgroup

$$\Theta\alpha_1\circ E\pi(EX_2, X_1)+\pi(E^2X_2, Y)\circ E^2\alpha_3=\Theta(\alpha_1\circ\pi(EX_3, X_1)+\pi(EX_2, \Omega(Y))\circ E\alpha_3).$$

Hence, the proposition holds. q. e. d.

We denote by $\{\alpha_1, E^n\alpha_2, E^n\alpha_3, E^n\alpha_4\}_n$ $(n\geqq 1)$, the second derived composition constructed from an admissible quadruple $X_0 \xleftarrow{\alpha_1} E^nX_1 \xleftarrow{E^n\alpha_2} E^nX_2 \xleftarrow{E^n\alpha_3} E^nX_3 \xleftarrow{E^n\alpha_4} E^nX_4$ such that the null triple $E^nX_1 \xleftarrow{E^n\alpha_2} E^nX_2 \xleftarrow{E^n\alpha_3} E^nX_3 \xleftarrow{E^n\alpha_4} E^nX_4$ is an n-fold suspended null triple of $X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$, so that it is a union of cosets of the subgroup $\alpha_1 \circ E^n\pi(E^2X_4, X_1)+\pi(E^{n+2}X_3, X_0)\circ E^{n+2}\alpha_4$.

Let $(\Theta\nu)$: $Y \xleftarrow{\Theta\alpha_1} EX_1 \xleftarrow{E\alpha_2} EX_2 \xleftarrow{E\alpha_3} EX_3 \xleftarrow{E\alpha_4} EX_4$ be the associated null quadruple of a null quadruple (ν): $\Omega(Y) \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$. If (ν) is admissible, then so is $(\Theta\nu)$.

DEFINITION (7.5) *The associated null quadruple* $(\overline{\Theta\nu})$ *is called admissibe, if it contains an admissible representative as follows*:

$$Y \xleftarrow{f_1} EX_1 \xleftarrow{Ef_2} EX_2 \xleftarrow{Ef_3} EX_3 \xleftarrow{Ef_4} EX_4,\ (A_1,\ EA_2,\ EA_3).$$

Let G'_1 and G'_2 be subgroups of $\pi(EX_3, X_1)$ such that $\alpha_1\circ EG'_1 \subset \pi(EX_2, EY)\circ E^2\alpha_3$, and $E(G'_2\circ E\alpha_4)\subset E(\alpha_2\circ\pi(EX_4, X_2))$. A sufficient condition that the associated null quadruple $(\Theta\nu)$ is admissible is given as follows:

$$\begin{aligned}(7.6)\qquad & 0\in\{\Theta\alpha_1,\ E\alpha_2,\ E\alpha_3\},\quad 0\in\{\alpha_2,\ \alpha_3,\ \alpha_4\},\ \text{and}\\ & EG'_1+EG'_2=E\pi(EX_3,\ X_1).\end{aligned}$$

PROPOSITION (7.7) *Under the condition* (7.6), *it follows that*

$$\Theta\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}=\{\Theta\alpha_1,\ E\alpha_2,\ E\alpha_3,\ E\alpha_4\}_1\circ(E\rho\circ\rho)$$

where $(E\rho\circ\rho)$ *is the map defind in* (6.11).

PROOF. *Since* $(\overline{\nu})$ *is adminissible, let*

$$(\overline{N}):\ \Omega(Y) \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} X_4,\ (A_1,\ A_2,\ A_3)$$

be its admissible representative. It follows that

$$\begin{aligned}&\Theta\{f_1,\ [f_2,\ f_3],\ (f_3,\ f_4)\}=\{\Theta f_1,\ E[f_2,\ f_3],\ E(f_3,\ f_4)\}\circ\rho\\ &=\{\Theta f_1\ [Ef_2\ Ef_3],\ (Ef_3,\ Ef_4)\}\circ(E\rho\circ\rho).\end{aligned}$$

Hence, $\Theta\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\{\Theta\alpha_1, E\alpha_2, E\alpha_3, E\alpha_4\}_1\circ(E\rho\circ\rho)$ have a common element. For a fixed representative $(\overline{N})$, these two sets are cosets of the same subgroup $\Theta(\alpha_1\circ\pi(E^2X_4, X_1)+\pi(E^2X_3, \Omega(Y))\circ E^2\alpha_4)=\Theta\alpha_1\circ E\pi(E^2X_4, X_1)+\pi(E^3X_3, Y)\circ E^3\alpha_4$. Moreover, the admissible representatives of $(\overline{\nu})$ and $(\overline{\Theta\nu})$ are in one to one correspondence. Hence, the proposition holds. q. e. d.

In [4], I. M. James introduced the concept of reduced product space Y_∞ of a special complex Y i.e. a countable CW-complex with only one vertex y^0 (c.f. §4). He defined the canonical map

Φ: $Y_\infty \longrightarrow \Omega(EY)$, and he proved that Φ induces isomorphism of homotopy groups: Φ_*: $\pi_q(Y_\infty)\longrightarrow\pi_q(\Omega(EY))$, for all q. That is to say, Y_∞ and $\Omega(EY)$ have the same homotopy type, and hence $\Phi_*: \pi(X, Y_\infty) \longrightarrow \pi(X, \Omega(EY))$, is one to one for arbitrary finite CW-complex X. ([1], (2.1)

Define $\phi = \Theta\Phi_*: \pi(X, Y_\infty) \longrightarrow \pi(EX, EY)$ for any finite CW-complex X and special complex Y. ϕ is also one to one correspondence, and is an isomorphism if X is a suspended space.

In the following, CW-complexes X_i, $(i=1, 2, \cdots)$ are always assumed to be finite, and Y to be a special complex such that y^0 is its only vertex.

Consider the following n-tuples:

$$\begin{aligned}
(\nu):&\quad Y_\infty \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \longleftarrow \cdots \longleftarrow X_{n-1} \xleftarrow{\alpha_n} X_n\\
(\Phi\nu):&\quad \Omega(EY) \xleftarrow{\Phi\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \longleftarrow \cdots \longleftarrow X_{n-1} \xleftarrow{\alpha_n} X_n\\
(\phi\nu):&\quad EY \xleftarrow{\phi\alpha_1} EX_1 \xleftarrow{E\alpha_2} EX_2 \longleftarrow \cdots \longleftarrow EX_{n-1} \xleftarrow{E\alpha_n} EX_n
\end{aligned}$$

where the (n-1)-tuple $EX_1 \xleftarrow{E\alpha_2} EX_2 \longleftarrow \cdots \longleftarrow EX_{n-1} \xleftarrow{E\alpha_n} EX_n$, is the sespension of the (n-1)-tuple $X_1 \xleftarrow{\alpha_2} X_2 \longleftarrow \cdots \longleftarrow X_{n-1} \xleftarrow{\alpha_n} X_n$. Hence, if one of these three n-tuples is an n-tuple, so are the other two. We shall call each of these three the associated null n-tuple of any of the other two.

The following two propositions hold by (7.4) and (7.7).

PROPOSITION (7.8) *Let* $(\phi\nu): EY \xleftarrow{\phi\alpha_1} EX_1 \xleftarrow{E\alpha_2} EX_2 \xleftarrow{E\alpha_3} EX_3$ *be the associated null triple of a null triple* (ν) $Y_\infty \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3$, *then, it follows that* $\phi\{\alpha_1, \alpha_2, \alpha_3\} = \{\phi\alpha_1, E\alpha_2, E\alpha_3\}_1 \circ \rho$.

PROPOSITION (7.9) *Let* $(\phi\nu): EY \xleftarrow{\alpha_1} EX_1 \xleftarrow{E\alpha_2} EX_2 \xleftarrow{E\alpha_3} EX_3 \xleftarrow{E\alpha_4} EX_4$ *be the associated null quadruple of a null quadruple* $(\nu): Y_\infty \xleftarrow{\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4$ *Assume the condition* (7.6) *(substituting* ϕ *for* Θ*), then, it follows that*

$$\phi\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \phi\{\alpha_1, E\alpha_2, E\alpha_3, E\alpha_4\} \circ (E\rho \circ \rho).$$

Now, m-dimensional sphere S^m is considered as a special complex with only vertex e^0. Consider a mapping $h': (S_2^m, S^m) \longrightarrow (S^{2m}, e^0)$ defined by shrinking the subcomplex S^m of $S_2^m = S^m \cup e^{2m}$ into the point e^0, and let $h: (S_\infty^m, S^m) \longrightarrow (S_\infty^{2m}, e^0)$ be the canonical extensions of h' ([14]).

Generalized Hopf-homomorphism $H: \pi(EX, S^{m+1}) \longrightarrow (EX, S^{2m+1})$ is defined by

$$H = \phi \circ h \circ \phi^{-1} \tag{7.10}$$

for any finite CW-complex X ([1], [4]).

PROPOSITION (7.11) *If* $\alpha \in \pi(EX_1, S^{m+1})$, $\beta \in \pi(X_2, X_1)$ *and* $\gamma \in \pi_m(S^r)$, *it follows that*

$$\begin{aligned}
&H(\alpha \circ E\beta) = H(\alpha) \circ E\beta\\
&H(E\gamma \circ \alpha) = E(\gamma \times \gamma) \circ H(\alpha)\\
&H(E\alpha) = 0
\end{aligned}$$

where, in general, $\alpha\times\beta$ *indicates the reduced join of* α *and* β.

PROPOSITION (7.12) *Let* $(\nu): S^{m+1}\xleftarrow{\alpha_1} EX_1 \xleftarrow{\alpha_2} EX_2 \xleftarrow{\alpha_3} EX_3$ *be a null triple, then it follows that*

$$H\{\alpha_1,\ E\alpha_2,\ E\alpha_3\}_1 \subset \{H(\alpha_1),\ E\alpha_2,\ E\alpha_3\}_1.$$

Proofs of the propositions (7.11) and (7.12) are given in [1], so that it is omitted here.

PROPOSITION (7.13) *Let* $(\nu): S^{m+1}\xleftarrow{\alpha_1} EX_1 \xleftarrow{E\alpha_2} EX_2 \xleftarrow{E\alpha_3} EX_3 \xleftarrow{E\alpha_4} EX_4$ *be a null quadruple such that the condition (7.6) (substituting* ϕ *for* Θ*) holds. Then, it follows that*

$$H\{\alpha_1,\ E\alpha_2,\ E\alpha_3,\ E\alpha_4\}_1 \subset \{H(\alpha_1),\ E\alpha_2,\ E\alpha_3,\ E\alpha_4\}_1.$$

Proof. It follows from the conditions that the null quadruple (ν) is admissible as well as its associated null quadruple $(\phi^{-1}\nu): S^m_\infty \xleftarrow{\phi^{-1}\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X$. Hence, we have

$$\begin{aligned}
H\{\alpha_1,\ E\alpha_2,\ E\alpha_3,\ E\alpha_4\}_1 &= \phi\circ h\circ\phi^{-1}\{\alpha_1,\ E\alpha_2,\ E\alpha_3,\ E\alpha_4\}_1 && (7.10)\\
&= \phi\circ h\circ\{\phi^{-1}\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}\circ(\rho\circ E\rho) && (7.9)\\
&\subset \phi\{h\circ\phi^{-1}\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}\circ(\rho\circ E\rho) && (6.9)\\
&= \{\phi\circ h\circ\phi^{-1}\alpha_1,\ E\alpha_2,\ E\alpha_3,\ E\alpha_4\}_1\circ(E\rho\circ\rho)\circ(\rho\circ E\rho) && (7.9)\\
&= \{H(\alpha_1),\ E\alpha_2,\ E\alpha_3,\ E\alpha_4\}_1. && \text{q.e.d.}
\end{aligned}$$

Let Y be a CW-complex, and A a subcomplex of Y, and suppose that there is given a map $h;\ (Y, A)\longrightarrow(Z, z^0)$, where Z is a CW-complex and $z^0\in Z$. Consider the exact sequence of (1.8)

$$\pi(X_2,\ Y)\xleftarrow{i_*}\pi(X_2,\ A)\xleftarrow{\partial}\pi((CX_2,\ X_2),\ (Y,\ A))\xleftarrow{j_*}\pi(EX_2,\ Y)\xleftarrow{i_*}\pi(EX_2,\ A)$$

where X is a CW-complex.

Let $\alpha_1\in\pi(X_1,\ A)$ and $\alpha_2\in\pi(X_2,\ X_1)$ such that $i_*(\alpha_1\circ\alpha_2)=0$, and let $Y\xleftarrow{i\cdot f_1} X_1 \xleftarrow{f_2} X_2$, (A) be a representative of the null couple $Y\xleftarrow{i_*\alpha_1} X_1 \xleftarrow{\alpha_2} X_2$. Then there exists a map $g;\ (EX_2,\ x_2^0)\longrightarrow(z,\ Z^0)$ such that commutativity holds in the diagram

$$\begin{tikzcd}
(Y,\ A) \arrow[d, "h"'] & (X_1\cup_{f_2} CX_2,\ X_1) \arrow[l, "{[if_1,\ f_2]}"'] \arrow[d, "p"] & (CX_2,\ X_2) \arrow[l, "F"'] \arrow[dl, "p"] \\
(Z,\ z^0) & (EX_2,\ x_2^0) \arrow[l, "g"] &
\end{tikzcd}$$

where F is defined by the identity map $CX_2\longrightarrow CX_2$. Since $[if_1,\ f_2]\circ F|X_2=f_1\circ f_2$ it follows that g represents an element $\beta\in h_*\circ\partial^{-1}(\alpha_1\circ\alpha_2)$.

Let $(\nu):\ Y\xleftarrow{i_*\alpha_1} X_1\xleftarrow{\alpha_2} X_2\xleftarrow{\alpha_3} X_3$ be a null triple, then, commutativity holds in the diagram

$$\begin{tikzcd}
Y \arrow[d, "h"'] & X_1 \cup_{\alpha_2} CX_2 \arrow[l, "i_*\alpha_1"'] \arrow[d, "p_*"] & EX_3 \arrow[l, "\alpha_3"'] \arrow[d, "id."] \\
Z & EX_2 \arrow[l, "\hat{p}"] & EX_3 \arrow[l, "E\alpha_3"]
\end{tikzcd}$$

Hence, $h_*\{i_*\alpha_1, \alpha_2, \alpha_3\} \sim h_* \circ \hat{\partial}^{-1}(\alpha_1, \alpha_2) \circ E\alpha_3$. Morever, these two sets are cosets of the same subgroup

$$\begin{aligned} & h_* \circ \{i_*\alpha_1 \circ \pi(EX_3, X_1) + \pi(EX_2, Y) \circ E\alpha_3\} \\ & = h_* \pi(EX_2, Y) \circ E\alpha_3, \end{aligned}$$

because, $h_* \circ i_*\alpha_1 = h_* \circ j_* \circ i_*\alpha_1 = 0$. Thus, we have proved

PROPOSITION (7.14) $h_*\{i_*\alpha_1, \alpha_2, \alpha_3\} = h_* \circ \hat{\partial}^{-1}(\alpha_1 \circ \alpha_2) \circ E\alpha_3$.

Similarly, let $(\mathcal{L}): Y \xleftarrow{i_*\alpha_1} X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_2$ be a null quadruple which contains an admissible representative $(\bar{N}): Y \xleftarrow{if_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} X_4$. (A_1, A_2, A_3).

In the diagram

$$\begin{tikzcd}
Y \arrow[d, "h"'] & Y_1 \cup_{f_2} CX_2 \arrow[l, "if_1"'] \arrow[d, "p"] & EX_3 \arrow[l, "f_3"'] \arrow[d, "id."] & EX_4 \arrow[l, "Ef_4"'] \arrow[d, "id."] \\
Z & EX_2 \arrow[l, "g"] & EX_3 \arrow[l, "Ef_3"] & EX_4 \arrow[l, "Ef_4"]
\end{tikzcd}$$

commutativity holds, and it can easily be seen that

PROPOSITION (7.15) $-h_* \circ \{i_*\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset \{h_* \circ \hat{\partial}^{-1}(\alpha_1 \circ \alpha_2), E\alpha_3, E\alpha_4\}$.

The map $\phi: \pi_q(S^m_\infty) \sim \pi_{q+1}(S^{m+1})$ induces an isomorphism of the homotopy sequence of (S^m_∞, S^m) onto the suspension sequence ([5]) of S^m i.e. commutativity holds in the diagram :

$$\begin{tikzcd}
\pi_{q-1}(S^m) \arrow[d, "\phi"'] & \pi_q(S^m_\infty, S^m) \arrow[l, "\partial"'] \arrow[d, "\phi"] & \pi_q(S^m_\infty) \arrow[l, "j_*"'] \arrow[d, "\phi"] & \pi_q(S^m) \arrow[l, "i_*"'] \arrow[d, "\phi"] \\
\pi_{q-1}(S^m) & \pi_{q+1}(S^{m+1}; E_+, E_-) \arrow[l, "\partial"] & \pi_{q+1}(S^{m+1}) \arrow[l, "k_*"] & \pi_q(S^m) \arrow[l, "E"]
\end{tikzcd}$$

in which, ϕ is identity on $\pi_r(S^m)$ $(r \geqq 1)$.

Hence, it means that

$$(7.16) \qquad \begin{cases} \text{(a)} & \hat{\partial} = \partial \circ \phi \\ \text{(b)} & \phi \circ j_* = k_* \circ \phi \\ \text{(c)} & \phi \circ i_* = E \end{cases}$$

Let $S^{m+1} \xleftarrow{E\alpha_1} EX_1 \xleftarrow{E\alpha_2} EX_2$ be a null couple such that $\alpha_1 \circ \alpha_2 \neq 0$, and let $S^m_\infty \xleftarrow{i_*\alpha_1} X_1 \xleftarrow{E\alpha_2} X_2$ be its associated null couple.

Applying (7.14) and (7.15) by substituting as $(Y, A) = (S^m_\infty, S^m)$, $(Z, z^0) = (S^{2m}_\infty, e^0)$ and $h: (S^m_\infty, S^m) \longrightarrow (S^{2m}_\infty, e^0)$, we have the following propositions.

PROPOSITION (7.17) Let $S^{m+1} \xleftarrow{E\alpha_1} EX_1 \xleftarrow{E\alpha_2} S^{q+1} \xleftarrow{E\alpha_3} EX_3$ be a null triple such that $\alpha_2 \circ \alpha_3 = 0$, then, it follows that

$$H\{E\alpha_1,\ E\alpha_2,\ E\alpha_3\}_1 = H\partial^{-1}(\alpha_1 \circ \alpha_2) \circ E^2\alpha_3 \circ \rho$$

where H and ∂ are homomorphisms as follows:

$$\begin{tikzcd} \pi_q(S^m) & \pi_{q+2}(S^{m+1};\ E_+,\ E_-) \arrow[l, "\partial"'] \arrow[d, "H"] & \pi_{q+2}(S^{m+1}) \arrow[l, "j_*"'] \arrow[dl, dashed, "H"] \\ & \pi_{q+2}(S^{2m+1}) & \end{tikzcd} \qquad ([1]).$$

PROPOSITION (7.18) *Let* $S^{m+1} \xleftarrow{E\alpha_1} EX_1 \xleftarrow{E\alpha_2} S^{q+1} \xleftarrow{E\alpha_3} EX_3 \xleftarrow{E\alpha_4} EX_4$ *be a null quadruple such that the condition* (7.6) *holds. Then, it follows that*

$$H\{E\alpha_1,\ E\alpha_2,\ E\alpha_3,\ E\alpha_4\}_1 \subset -\{H\partial^{-1}(\alpha_1 \circ \alpha_2),\ E^2\alpha_3,\ E^2\alpha_4\} \circ (E\rho).$$

PROOF. $H\{E\alpha_1,\ E\alpha_2,\ E\alpha_3,\ E\alpha_4\}_1 = \phi h\{i_*\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\} \circ (\rho \circ E\rho)$ (7.10)

$\subset -\phi\{h \circ \partial^{-1}(\alpha_1 \circ \alpha_2),\ E\alpha_3,\ E\alpha_4\} \circ (\rho \circ E\rho)$ (7.15)

$= -\{\phi \circ h \circ \phi^{-1} \circ \partial^{-1}(\alpha_1 \circ \alpha_2),\ E^2\alpha_3,\ E^2\alpha_4\} \circ (\rho^2 \circ E\rho)$ (7.8)

$= -\{H \circ \partial^{-1}(\alpha_1 \circ \alpha_2),\ E^2\alpha_3,\ E^2\alpha_4\} \circ (E\rho).$ q.e.d.

§ 8 Boundary operator

Throughout this section, we assume that X and X_i $(i=1, 2, \cdots)$ be finite CW-complexes. Let Y be a fibre space over X_0 with a fibre F and a projection p': $(Y, F) \longleftarrow (X_0, x_0^0)$. Take the basic point y^0 of Y such that $y^0 \in F$. Consider a diagram

$$\begin{tikzcd} \pi(X,\ Y) & \pi(X,\ F) \arrow[l, "i_*"'] & \pi((CX,\ X),\ (Y,\ F)) \arrow[l, "\partial"'] \arrow[d, "p_*'"] & \pi(EX,\ Y) \arrow[l, "j_*"'] \arrow[dl, dashed, "p_*"] & \pi(EX,\ F) \arrow[l, "i_*"'] \\ & & \pi(EX,\ X_0) & & \end{tikzcd}$$

If follows from the covering homotopy theorem that p_*' is a one to one correspondence. We define Δ: $\pi(EX,\ X_0) \longrightarrow \pi(X,\ F)$ by

$$\Delta = \partial \circ p_*'^{-1}.$$

The exact sequence

$$\pi(X,\ Y) \xleftarrow{i_*} \pi(X,\ F) \xleftarrow{\Delta} \pi(EX,\ X_0) \xleftarrow{p_*} \pi(EX,\ Y) \xleftarrow{i_*} \pi(EX,\ F) \longleftarrow \cdots$$

is called the homotopy sequence of the fibre space Y.

Let f_1: $(EX_1,\ x_1^0) \longrightarrow (X_0,\ x_0^0)$ be a map. Since p_*' is one to one, there exists a map G: $(CX_1,\ X_1) \longrightarrow (Y,\ F)$ such that commutativity holds in the diagram

$$\begin{array}{ccc}(Y,\ F) & \xleftarrow{G} & (CX_1,\ X_1)\\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p}\\ (X_0,\ x_0^0) & \xleftarrow[f_1]{} & (EX_1,\ x_1^0)\end{array}$$

PROPOSITION (8.1) $\qquad \Delta(\alpha_1\circ E\alpha_2)=\Delta\alpha_1\circ\alpha_2$

PROOF. Let $f_1\in\alpha_1$, $f_2\in\alpha_2$, and consider a commutative diagram

$$\begin{array}{ccccc}(Y,\ F) & \xleftarrow{G} & (CX_1,\ X_1) & \xleftarrow{Cf_2} & (CX_2,\ X_2)\\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p}\\ (X_0,\ x_0^0) & \xleftarrow[f_1]{} & (EX_1,\ x_1^0) & \xleftarrow[Ef_2]{} & (EX_2,\ x_2^0)\end{array}$$

$\partial G=G\mid X_1$ represents $\Delta\alpha_1$. While $\partial(G\circ Cf_2)=G\circ Cf_2\mid X_2=\partial G\circ f_2$ represents $\Delta(\alpha_1\circ E\alpha_2)$. q.e.d.

Let $(\nu):\ X_0\xleftarrow{\alpha_1}EX_1\xleftarrow{E\alpha_2}EX_2\xleftarrow{E\alpha_3}EX_3$ be a null triple such that $\alpha_2\circ\alpha_3=0$, then the triple $(\Delta\nu):\ F\xleftarrow{\Delta\alpha_1}X_1\xleftarrow{\alpha_2}X_2\xleftarrow{\alpha_3}X_3$ is also a null triple.

PROPOSITION (8.2) $\quad \Delta\{\alpha_1,\ E\alpha_2,\ E\alpha_3\}_1\subset\{\Delta\alpha_1,\ \alpha_2,\ \alpha_3\}$

PROOF. Let $X_0\xleftarrow{f_1}EX_1\xleftarrow{Ef_2}EX_2\xleftarrow{Ef_3}EX_3$, $(A,\ EB)$ be a representative of (ν). Let $G:(CX_1,\ X_1)\longrightarrow(Y,\ F)$ be a map such that $p'\circ G=f_1\circ p$. It follows from the covering homotopy theorem that there exists a homotopy D_t of $G\circ Cf_2$ such that $p'\circ D_t=A_t\circ p$, where $A_t(x)=Ac(t,\ x)$ for $t\in I$, and $x\in EX_2$. Then, D_1 maps CX_2 into F. Define a homotopy $h_s:(CX_2,\ x_2^0)\longrightarrow(F,\ y^0)$ by

$$h_sc(t,\ x)=D_1c((1-s)t+s,\ x),\quad s,\ t\in I,\ x\in X_2,$$

and define a null homotopies D_t' of $G\circ Cf_2$ and A_t' of $f_1\circ Ef_2$ by

$$\begin{cases}D_t'=D_{2t} & 0\leqq t\leqq 1/2,\\ D_t'=h_{2t-1} & 1/2\leqq t\leqq 1,\end{cases}$$

$$A_t'=\begin{cases}A_{2t} & 0\leqq t\leqq 1/2,\\ x_0^0 & 1/2\leqq t\leqq 1.\end{cases}$$

It follows that

$$p'\circ D_t'=A_t'\circ p \qquad \text{for}\quad t\in I,$$

and that $D_t'\mid X_2=\partial D_t'$ defines a null homotopy of $\partial G\circ f_2$.

Consider a null triple $Y\xleftarrow{G}CX_1\xleftarrow{Cf_2}CX_2\xleftarrow{Cf_3}CX_3$, $(D',\ CB)$ which defines a representative of $(\Delta\nu):\ F\xleftarrow{\partial G}X_1\xleftarrow{f_2}X_2\xleftarrow{f_3}X_3$, $(\partial D',\ B)$. Since

$$\partial\{G,\ D',\ Cf_2,\ CB,\ Cf_3\}=\{\partial G,\ \partial D',\ f_2,\ B,\ f_3\}$$

and since $\qquad p'\circ\{G,\ D',\ Cf_2,\ CB,\ Cf_3\}=\{f_1,\ A',\ Ef_2,\ EB,\ Ef_3\}\circ p,$

it follows that $\Delta\{\alpha_1,\ E\alpha_2,\ E\alpha_3\}_1\sim\{\Delta\alpha_1,\ \alpha_2,\ \alpha_3\}$. The proposition follows from the fact that $\qquad \Delta\pi(E^2X_2,\ X_0)\subset\pi(EX_2,\ F)$. q.e.d.

Let $(\nu): X_0 \xleftarrow{\alpha_1} EX_1 \xleftarrow{E\alpha_2} EX_2 \xleftarrow{E\alpha_3} EX_3 \xleftarrow{E\alpha_4} EX_4$ be a null quadruple such that $\alpha_2\circ\alpha_3=0$, $\alpha_3\circ\alpha_4=0$. Denote by G_1' the subgroup of $\pi(E^2X_3, EX_1)$ such that $\alpha_1\circ G_1'\subset\pi(E^2X_2, EX_0)\circ E^2\alpha_3$, and by G_2' the subgroup of $\pi(EX_3, X_1)$ such that $G_2'\circ E\alpha_4\subset\alpha_2\circ\pi(EX_4, X_2)$.
Assume that

$$(8.3)\qquad \begin{cases} 0\in\{\alpha_1, E\alpha_2, E\alpha_3\}_1, & 0\in\{\alpha_2, \alpha_3, \alpha_4\}, \\ G_1'+EG_2'=\pi(E^2X_3, EX_1), & \end{cases}$$

then, it follows that the null quadruple $(\Delta\nu): F\xleftarrow{\Delta\alpha_1} X_1\xleftarrow{\alpha_2} X_2\xleftarrow{\alpha_3} X_3\xleftarrow{\alpha_4} X_4$ is admissible as well as (ν).

PROPOSITION (8.4) *Under the condition* (8.3), *it follows that*

$$\Delta\{\alpha_1, E\alpha_2, E\alpha_3, E\alpha_4\}_1\subset\{\Delta\alpha_1, \alpha_2, \alpha_3, \alpha_4\}\circ\rho$$

where $\rho: E^2X_4\longrightarrow E^2X_4$ *is the map defined in* (3.9).

PROOF. It follows from the condition (8.3) that the null quadruple (ν) contains an admissible representative such that

$$X_0\xleftarrow{f_1} EX_1\xleftarrow{Ef_2} EX_2\xleftarrow{Ef_3} EX_3\xleftarrow{Ef_4} EX_4,\ (A_1, EA_2, EA_3).$$

Hence, it follows that

$$\begin{aligned} &\Delta\{f_1, (Ef_2, EA_2, Ef_3), Ef_3, EA_3, Ef_4\}_1 \\ &=\Delta(\{f_1, E(f_2, A_2, f_3), E[f_3, A_3, f_4]\}\circ E\rho) \\ &\subset\{\Delta f_1, (f_2, A_2, f_3), [f_3, A_3, f_4]\}\circ\rho. \end{aligned}$$

q.e.d.

§ 9 Notations and main results

In the following sections, denote by R_n $(n\geqq 2)$ the special orthogonal group, U_n $(n\geqq 1)$ the special unitary group, and Sp_n $(n\geqq 1)$ the symplectic group. Let

$$i^{m,n}: R_n\longrightarrow R_m,\ i'^{m,n}: U_n\longrightarrow U_m,\ i''^{m,n}: Sp_n\longrightarrow Sp_m,\ (n\geqq m),$$

and $\quad l^{2n}: Sp_n\longrightarrow U_{2n},\ k^{2n}: U_n\longrightarrow R_{2n}\ (n\geqq 1)$

be the inclusion maps. Let us denote the projections of the bundles R_n, U_n, and Sp_n by

$$p: R_n\longrightarrow S^{n-1},\quad p': U_n\longrightarrow S^{2n-1},\quad p'': Sp_n\longrightarrow S^{4n-1}$$

Now, let G_n be one of these classical groups. A generator of $\pi_q(G_n)$ is denoted by g_q^n and is called original if

(1) $g_q^n\notin i^{n,n-1}\circ\pi_q(G_{n-1})$

(2) g_q^n is not represented as $g_q^n=\alpha\circ\beta$ for any $\alpha\in\pi_{q-r}(G_n)$ and $\beta\in\pi_q(S^{q-r})$ $(q>r)$.

Denote $i^{m,n}\circ g^n_q$ by $g^n_q(m)$ for $m\geqq n$, where $g^n_q(n)=g^n_q$.

For the generators of the homotopy groups of spheres, we shall use the same notations as in [1].

Let Z, Q, and C denote the field of complex numbers, algebras of quaternions, and of Caylay numbers over the field of real numbers, respectively. Then the spheres S^1, S^2, S^3, S^6, and S^7 are represented as follows:

$$S^1=\{z\in Z:\ z\bar z=1\},\quad S^3=\{p\in Q:\ q\bar q=1\},\quad S^7=\{c\in C,\ c\bar c=1\}$$
$$S^2=\{q=x_1k+x_2j+x_3i+x_4\in S^3\ ;\ x_4=0\}$$
$$S^6=\{c=(q,\ q')\in S^7\ ;\ x_8=0\},\ \text{where}\ q'=x_5k+x_6j+x_7i+x_8.$$

Define the maps

$$(9.1)\qquad \begin{cases}\tau'^1_1:\ S^1\longrightarrow U_1,\ \tau^2_1:\ S^1\longrightarrow R_2,\ \tau''^1_3:\ S^3\longrightarrow Sp_1,\ \tau'^2_3:\ S^3\longrightarrow U_2,\\ \tau^4_3:\ S^3\longrightarrow R_4,\ \tau^8_7:\ S^7\longrightarrow R_8,\ \lambda^3_3:\ S^3\longrightarrow R_3,\ \lambda^7_7:\ S^7\longrightarrow R_7\end{cases}$$

$$\tau'^1_1(z)(z')=zz'\quad (z,\ z'\in S^1),\quad \tau^2_1=k^2\circ\tau'^1_1,$$
$$\tau''^1_3(q)(q')=qq'\quad (q,\ q'\in S^3),\quad \tau'^2_3=l^2\circ\tau''^1_3,\ \tau^4_3=k^4\circ\tau'^2_3,$$
$$\tau^8_7(c)(c')=cc'\quad (c,\ c'\in S^7),$$
$$\lambda^3_3(q)(q')=qq'\bar q\quad (q\in S^3,\ q'\in S^2),$$
$$\lambda^7_7(c)(c')=cc'\bar c\quad (c\in S^7,\ c'\in S^6).$$

It is well known that

$$(9.2)\qquad \begin{cases}p_*\tau^2_1=\iota_1,\quad p_*\tau^4_3=\iota_3,\quad p_*\tau^8_7=\iota_7,\\ p_*\lambda^3_3=\eta_2,\quad p^*\lambda^7_7=\eta_6,\end{cases}$$

We shall denote their homotopy classes by the same notations.

In the following sections, we always consider 2-primary components of groups. For simplicity, we shall denote e. g. $\pi_q(G_n)$ to mean the 2-primary components $\pi_q(G_n\,;2)$ of $\pi_q(G_n)$; and use the terms such as *equal, isomorphic,* in the sense of C_2([2], § 10). Denote by ${}^{(2)}Z_n$ the 2-primary component of the cyclic group Z_n, and by ${}^{(2)}m$ the 2-primary component of an integer m.

Now, we shall state our main results.

PROPOSITION (9.3) *The original generators of* $\pi_q(Sp_n)$ $(q\leqq 13)$ *are given as follows:*

$$\tau''^1_3,\ \omega''^2_7\in\{\tau''^1_3(2),\ \nu',\ 4\iota^6\},\quad \bar\gamma''^2_{10}\in\{\tau''^1_3(2),\ \nu',\ \nu_6\},$$

$\omega''^3_{11}\in\{i''^{3,2},\ \gamma''^2_{10},\ 8\iota_{10}\}$, *where* $\bar\gamma''^2_{10}=m\gamma''^2_{10}$ *for some odd integer* m.

PROPOSITION (9.4) *The original generators of* $\pi_q(U_m)$ $(q\leqq 13)$ *are given as follows:*

$$\tau'^1_1,\ \tau'^2_3,\ \omega'^3_5\in\{\tau'^2_3(3),\ \eta_3,\ 2\iota_4\},\quad u^3_{10}\in\{\tau'^2_3(3),\ \eta_3,\ \nu_4\circ\eta^2_5\},$$

$$u^3_{11}\in\{\tau'^2_3(3),\ \eta_3,\ \nu^2_4\},\quad u^3_{12}\in\{\omega'^3_5,\ 4\nu_5,\ \nu_8\},$$
$$\omega'^4_7\in\{i'^{4,3},\ \tau'^2_3(3)\circ\nu',\ 2\iota_6\},\ \gamma'^4_8\in\{i'^{4,3}_3,\ \tau'^2_3(3)\circ\nu',\ \eta_6\},$$
$$\omega'^5_9\in\{i'^{5,4},\ \gamma'^4_8,\ 8\iota_8\},\ l^4\circ\gamma''^2_{10},\ l_6\circ\omega''^3_{11},\ u^5_{12}\in\{i'^{5,4},\ \gamma'^4_8,\ 4\nu_8\},$$
$$\gamma'^6_{12}\in\{i'^{6,5},\ i'^{5,4}\circ l^4\circ\gamma''^2_{10},\ \eta_{10}\},\ \omega'^7_{13}\in\{i'^{7,6},\ \gamma'^6_{12},\ 16\iota_{12}\},$$

PROPOSITION (9.5) *The original generators of* $\pi_q(R_n)$ $(q\leqq 13)$ *are given as follows:*

$$\tau^2_1,\ \lambda^3_3,\ k^6\circ\omega'^3_5,\quad r^5_7\in\{\tau^4_3(5),\ \nu',\ 4\iota_6\},$$
$$r^6_7\in\{i^{6,5},\ \tau^4_3(5)\circ\eta_3,\ \eta^2_4\},\quad r^8_6\in\{i^{6,5},\ \tau^4_3(5)\circ\eta_3,\ \nu_4\},$$
$$\lambda^7_7,\ \tau^8_7,\ r^{10}_9\in\{i^{10,9},\ \gamma^9_8,\ 2\iota_8\}\ \textit{where}\ \gamma^9_8=\tau^8_7(9)\circ\eta_7+r^6_8(9),$$
$$r^5_{10}\in\{\tau^4_3(5),\ \nu',\ \nu_6\},\ k^6\circ u^3_{10},\quad r^7_{11}\in\{i^{7,6},\ 2r^6_8,\ \eta_8,\ 4\iota_9\},$$
$$k^6\circ u^3_{12},\quad r^{11}_{12}\in\{i^{11,10},\ \gamma^{10}_9,\ \eta^2_9\},\ k^{12}\circ\gamma'^6_{12},$$
$$\gamma^{14}_{13}\in\{i^{14,12},\ k^{12}\circ\gamma'^6_{12},\ 2\iota_{12}\}.$$

§ 10 $\pi_q(Sp(n)\,;\,2)$ $q\leqq 13$

Consider the exact sequence

$$\pi_{4n+2}(Sp_{n+1})\xleftarrow{i_*'}\pi_{4n+2}(Sp_n)\xleftarrow{\Delta}\pi_{4n+3}(S^{4n+3})\xleftarrow{p_*'}\pi_{4n+3}(Sp_{n+1})$$

From periodicity of the stable homotopy groups of symplectic groups [8], it follows that $\pi_{4n+2}(Sp_{n+1})=0$ and $\pi_{4n+3}(Sp_{n+1})=Z$. Hence,

(10.1) $\pi_{4n+2}(Sp_n)$ *is a cyclic group generated by* $\Delta\iota_{4n+3}$.

Denote $\Delta\iota_{4n+3}$ by γ''^n_{4n+2} or simply by γ'', which is the characteristic class of the bundle Sp_{n+1} in the sense of [7], and we shall see later (§ 11) that γ''^n_{4n+2} is of order $\geqq^{(2)}(2n+1)!$.

Assume that $d\gamma''^n_{4n+2}=0$ for some integer $d>0$. According to the cellular decomposition of symplectic groups [8],

(10.2) $Sp_n\cup_{\gamma''}e^{4n+3}$ *is a subcomplex of* Sp_{n+1} *such that commutativity holds in the diagram*

$$\begin{tikzcd}
Sp_{n+1} \arrow[dr, "p''"'] & Sp_n\cup_{\gamma''}e^{4n+3} \arrow[l] \arrow[d, "p"] \\
 & S^{4n+3}
\end{tikzcd}$$

where p *is the map defined in* (3.1).

Consider a null couple $Sp_n\xleftarrow{\gamma''}S^{4n+2}\xleftarrow{d\iota}S^{4n+2}$, then the coextension $(\gamma'',\ d\iota)$ satisfies $p_*(\gamma'',\ d\iota)=d\iota_{4n+3}$. $(\gamma'',\ d\iota)$ has only one element because $i_*'\pi_{4n+2}(Sp_n)=0$ (c.f. (3.4)).

(10.3) *Denote* $(\gamma''^n_{4n+2},\ d\iota_{4n+2})$ *by* ω''^{n+1}_{4n+3}, *which generates* $\pi_{4n+3}(Sp_{n+1})$ *for* $n\geqq 1$.

Since the identity map of $Sp_n \underset{\gamma''}{\cup} e^{4n+3}$ on itself is an extension of the enclusion map $Sp_n \longrightarrow Sp_n \underset{\gamma''}{\cup} e^{4n+3}$, it follows that

(10.4) $$\omega''^{n+1}_{4n+3} \in \{i''^{n+1,n}, \gamma''^{n}_{4n+2}, d\iota_{4n+2}\}.$$

Define an integer $r(n)$ by $p_*'\gamma''^{n}_{4n+2} = r(n)\nu_{4n-1}$ for $n \geqq 2$, and $r(n) \neq 0$ ($r(n)$ will be computed in § 11). Then it follows that

$$\gamma''^{n-1}_{4n-2} \circ r(n)\nu_{4n-2} = \Delta(r(n)\nu_{4n-1}) = \Delta p_*'\gamma''^{n}_{4n+2} = 0.$$

Consider a null couple $Sp_{n-1} \xleftarrow{\gamma''} S^{4n-2} \xleftarrow{r(n)\nu} S^{4n+1}$, then the coextension $(\gamma'', r(n)\nu)$ satisfies $p_*'(\gamma'', r(n)\nu) = r(n)\nu_{4n-1}$.

(10.5) $$\begin{cases} \gamma''^{n}_{4n+2} \in (\gamma''^{n-1}_{4n-2}, r(n)\nu_{4n-2}) \\ \gamma''^{n}_{4n+2} \in \{i''^{n,n-1}, \gamma''^{n-1}_{4n-2}, r(n)\nu_{4n-2}\}. \end{cases}$$

Now, consider the exact sequence

(10.6) $$\begin{aligned} 0 \longleftarrow \pi_{4n+3}(Sp_n) &\xleftarrow{\Delta} \pi_{4n+4}(S^{4n+3}) \xleftarrow{p_*'} \pi_{4n+4}(Sp_{n+1}) \xleftarrow{i_*'} \pi_{4n+4}(Sp_n) \\ &\xleftarrow{\Delta} \pi_{4n+5}(S^{4n+3}) \xleftarrow{p_*'} \pi_{4n+5}(Sp_{n+1}) \xleftarrow{i_*'} \pi_{4n+5}(Sp_n) \\ &\xleftarrow{\Delta} \pi_{4n+6}(S^{4n+3}) \xleftarrow{p_*'} \pi_{4n+6}(Sp_{n+1}) \xleftarrow{i_*'} \pi_{4n+6}(Sp_n) \xleftarrow{\Delta} \pi_{4n+7}(S^{4n+3}) \end{aligned}$$

According to [1],

$$\pi_{4n+4}(S^{4n+3}) = Z_2 = (\eta_{4n+3}),\ \pi_{4n+5}(S^{4n+3}) = Z_2 = (\eta^2_{4n+3})$$
$$\pi_{4n+6}(S^{4n+3}) = Z_8 = (\nu_{4n+3}), \text{ and } \pi_{4n+7}(S^{4n+3}) = 0,$$

Let n be odd, then it follows from [9],

$$\pi_{4n+4}(Sp_{n+1}) = 0,\ \pi_{4n+5}(Sp_{n+1}) = 0, \text{ and } \pi_{4n+6}(Sp_{n+1}) = (\gamma''_{4n+6})$$

Hence

(10.7) $$\begin{cases} \pi_{4n+3}(Sp_n) = Z_2 = (\gamma''_{4n+2} \circ \eta_{4n+2}), \\ \pi_{4n+4}(Sp_n) = Z_2 = (\gamma''_{4n+2} \circ \eta^2_{4n+2}), \\ \pi_{4n+5}(Sp_n) = 0 \quad \text{(c.f. (11.11)), and} \\ \pi_{4n+6}(Sp_n) \approx \pi_{4n+6}(Sp_{n+1}) \end{cases}$$

Consider the exact sequence,

(10.8) $$\begin{aligned} \pi_{4n+7}(S^{4n+3}) &\xleftarrow{p_*'} \pi_{4n+7}(Sp_{n+1}) \xleftarrow{i_*'} \pi_{4n+7}(Sp_n) \xleftarrow{\Delta} \pi_{4n+8}(S^{4n+3}) \\ &\xleftarrow{p_*'} \pi_{4n+8}(Sp_{n+1}) \xleftarrow{i_*'} \pi_{4n+8}(Sp_n) \xleftarrow{\Delta} \pi_{4n+9}(S^{4n+3}) \xleftarrow{p_*'} \pi_{4n+9}(Sp_{n+1}) \end{aligned}$$

According to [1],

$$\pi_{4n+7}(S^{4n+3}) = 0,\ \pi_{4n+8}(S^{4n+3}) = 0,\ \pi_{4n+9}(S^{4n+3}) = Z_2 = (\nu^2_{4n+3}),$$

Hence, it follows that

(10.9) $$\pi_{4n+7}(Sp_n)\approx\pi_{4n+7}(Sp_{n+1}) \text{ for } n\geqq 1.$$

Consider the element $\gamma''^{n+1}_{4n+6}\circ\eta_{4n+6}\in i''^{n+1,n}\circ\{\gamma''^{n}_{4n+2},\ r(n+1)\nu_{4n+2},\ \eta_{4n+5}\}$, then $p''_*\{\gamma''_{4n+2},\ r(n+1)\nu_{4n+2},\ \eta_{4n+5}\}\sim\{r(n)\nu_{4n-1},\ r(n+1)\nu_{4n+2},\ \eta_{4n+5}\}$ for $n\geqq 2$. It follows from (11.11) that $r(n)r(n+1)=\pm 2$ *for* $n\equiv 0$ *or* 1 *mod* 4. Since $\varepsilon\in\{\nu^2,\ 2\iota,\ \eta\}$ by [1], it follows that

(10.10) *there exists an element* $s^n_{4n+7}\in\{\gamma''^n_{4n+2},\ m\nu_{4n+2},\ \eta_{4n+5}\}$ *such that* $p''_*s^n_{2n+7}=\varepsilon_{4n-1}$, *and* $s^n_{4n+7}(n+1)=\gamma''^{n+1}_{4n+6}\circ\eta_{4n+6}$ *for* $n\equiv 0$ *or* 1 mod 4 ($n\geqq 4$), *where* $m=2$ *or* 1 *according as* $n\equiv 0$ *or* 1 mod 4.

(10.11) $\pi_{4n+7}(Sp_n)=Z_2$ *for even* n, *and*

it is generated by s^n_{4n+7} *for* $n\equiv 0$ mod 4.

Since $\pi_{4n+9}(Sp_n)=0$ for even n, and since $\gamma''^n_{4n+2}\circ\nu^2_{4n+2}$ is of order 2 ([1]), it follows that

(10.12) $\pi_{4n+8}(Sp_n)=Z_2+Z_2$ *for even* n, *and*

it is generated by $s^n_{4n+7}\circ\eta_{4n+7}$ *and* $\gamma''^n_{4n+2}\circ\nu^2_{4n+2}$, *for* $n\equiv 0$ mod 4.

Let n be odd, then $\gamma''_{4n+6}\circ\nu_{4n+6}\in\pi_{4n+9}(Sp_{n+1})$, and $p''_*(\gamma''^{n+1}_{4n+6}\circ\nu_{4n+6})=\nu^2_{4n+3}$, so that $\Delta(\nu^2_{4n+3})=0$, i.e. i'' is a monomorphism. Hence substituting n for $n+1$, we have

(10.13) $$\begin{cases}\gamma''^n_{4n+2}\circ\eta_{4n+2}=s^{n-1}_{3n+3}(n)\neq 0,\\ \gamma''^n_{4n+2}\circ\eta^2_{4n+2}=s^{n-1}_{4n+3}(n)\circ\eta_{4n+3}\neq 0 \text{ for } n\equiv 2 \text{ mod } 4.\end{cases}$$

Since $\gamma''^n_{4n+2}\circ\nu_{4n+2}$ is of order 4 or 8 by (11.11), and since $\pi_{4n+4}(Sp_{n+1})=Z_2$, $\pi_{4n+5}(Sp_{n+1})=Z_2$, and $\pi_{4n+6}(Sp_{n+1})=(\gamma''^{n+1}_{4n+6})$ it follows that

(10.14) $$\begin{cases}\textit{for } n\equiv 2 \text{ mod } 4,\\ \pi_{4n+3}(Sp_n)=Z_2,\\ \pi_{4n+4}(Sp_n)=Z_2+Z_2,\\ \pi_{4n+5}(Sp_n)=Z_2+Z_4 \text{ or } Z_2+Z_8,\\ \pi_{4n+6}(Sp_n) \textit{ is a cyclic group of order } d/2 \textit{ or } d, \textit{ where } d \textit{ is}\\ \textit{the order of } \gamma''^{n+1}_{4n+6}.\end{cases}$$

Hence, from (10.8) it follows that if $n\equiv 1$ mod 4 ($n\geqq 5$)

(10.15) $$\begin{cases}\pi_{4n+7}(Sp_n)=Z_2=(S_{4n+7}),\\ \pi_{4n+8}(Sp_n)=Z_2+Z_2\ni S^n_{4n+7}\circ\eta_{4n+7}\end{cases}$$

Now, we shall calculate $\pi_q(Sp_n)$ for low dimensional cases.(1) The map τ''^1_3 (defined in (9.1)) is a homeomorphism, hence (10.16) $\tau''^1_{3*}:\pi_q(S^3)\approx\pi_q(Sp_1)$ ($q\geqq 1$). Thus, we have

(1) H. Toda announced the results in [13].

$$(10.17)\quad \begin{cases}\pi_1(Sp^n)=0,\ \pi_2(Sp_n)=0,\ \pi_3(Sp_n)=(\tau''^1_3(n)) & (n\geqq 1),\\ \pi_4(Sp_n)=Z_2=(\tau''^1_3(n)\circ\eta_3) & (n\geqq 1),\\ \pi_5(Sp_n)=Z_2=(\tau''^1_3(n)\circ\eta_3^2) & (n\geqq 1).\end{cases}$$

Since $\pi_6(Sp_1)=Z_4=(\tau''^1_3\circ\nu')$, it follows that

$$(10.18)\quad \gamma''^1_6=\pm\tau''^1_3\circ\nu',\ \pi_6(Sp_n)=0 \quad (n\geqq 2)$$

It follows from (10.4) that

$$(10.19)\quad \begin{cases}\pi_7(Sp_n)=Z=(\omega''^2_7(n)) \ (n\geqq 2), \text{ where}\\ \omega''^2_7\in\{\tau''^1_3(2),\ \nu',\ 4\iota_6\} \text{ mod } 4\pi_7(Sp_2), \text{ and}\\ p''\omega''^2_7=4\iota_7\end{cases}$$

Note that $\pi_8(Sp_n)=0$ $(n\geqq 2)$, $\pi_9(Sp_n)=0$ $(n\geqq 1)$ and $\pi_{10}(Sp_1)=0$, hence by the homotopy sequence of Sp_2, p'': $\pi_{10}(Sp_2)\approx\pi_{10}(S_7)$. It follows from (10.5) and (11.11) that $\gamma''^2_{10}\in\{i''^{2,1},\ \gamma''^1_6,\ m\nu_6\}$ for some integer $m=\pm 1$ or ± 3. Let $\bar{\gamma}''^2_{10}\in\{\tau''^1_3(2),\ \nu',\ \nu_6\}$ mod $(\omega''^2_7\circ\nu_7)$, then, $\bar{\gamma}''^2_{10}$ also generates $\pi_{10}(Sp_2)$.

$$(10.20)\quad \begin{cases}\pi_{10}(Sp_2)=Z_8=(\gamma''^2_{10}),\\ \bar{\gamma}''^2_{10}\in\{\tau''^1_3(2),\ \nu',\ \nu_6\} \text{ mod } 4\pi_{10}(Sp_2),\\ \omega''^2_7\circ\nu_7=4\gamma''^2_{10}=4\bar{\gamma}''^2_{10}.\end{cases}$$

Since, $\{\nu',\ \nu_6,\ \eta_9\}=\varepsilon_3$ ([1], (7.6)), and $\{\nu',\ 2\nu_6,\ \nu_9\}=\varepsilon'$ ([1], p. 58), it follows that

$$(10.21)\quad \gamma''^2_{10}\circ\eta_{10}=\tau''^1_3(2)\circ\varepsilon_3,\ 2\gamma''^{12}_{10}\circ\nu_{10}=\tau''^1_3(2)\circ\varepsilon'$$

Note that $\pi_{10}(Sp_n)=0$ $(n\geqq 3)$.

It follows from (10.14) that

$$(10.22)\quad \begin{cases}\pi_{11}(Sp_2)=Z_2=(\tau''^1_3(2)\circ\varepsilon_3),\\ \pi_{12}(Sp_2)=Z_2+Z_2=(\tau''^1_3(2)\circ\varepsilon_3\circ\eta_{11})+(\tau''^1_3(2)\circ\mu_3),\\ \pi_{13}(Sp_2)=Z_2+Z_4 \text{ or } Z_2+Z_8=(\tau''^1_3(2)\circ\mu_3\circ\eta_{12})+(\gamma''^2_{10}\circ\nu_{10}).\end{cases}$$

It follows from (10.4) that

$$(10.23)\quad \begin{cases}\pi_{11}(Sp_n)=Z=(\omega''^3_{10}(n)) & (n\geqq 3),\\ \omega''^3_{11}\in\{i''^{3,2},\ \gamma''^2_{10},\ 8\iota_{10}\} \text{ mod } 8\pi_{11}(Sp_3), \textit{ and}\\ p''\omega''^3_{11}=8\iota_{11}.\end{cases}$$

It follows directly that

$$(10.24)\quad \begin{cases}\pi_{12}(Sp_n)=Z_2=(\tau''^1_3(n)\circ\mu_3) & (n\geqq 3),\\ \pi_{13}(Sp_n)=Z_2=(\tau''^1_3(n)\circ\mu_3) & (n\geqq 3).\end{cases}$$

In § 11, we shall show that $l^6\circ\omega''^3_{11}\circ\eta_{11}\neq 0$. Hence, we have

$$(10.25)\quad \omega''^3_{11}\circ\eta_{11}=\tau''^1_3(3)\circ\mu_3.$$

§ 11 $\pi_q(SU(n))$ $q \leqq 13$

Consider the exact sequence

$$\pi_{2n}(U_{n+1}) \xleftarrow{i_*} \pi_{2n}(U_n) \xleftarrow{\Delta} \pi_{2n+1}(S^{2n+1}) \xleftarrow{p'_*} \pi_{2n+1}(U_{n+1})$$

From periodicity of the stable homotopy groups of unitary groups [9], it follows that $\pi_{2n}(U_{n+1})=0$ and $\pi_{2n+1}(U_{n+1})=Z$. It is well known that the homomorphism p_*' is of degree $n!$ [9], so that

$$\pi_{2n}(U_n) =^{(2)} Z_n! = (\Delta \iota_{2n+1}). \tag{11.1}$$

Denote $\Delta\iota_{2n+1}$ by γ'^n_{2n} or simply by γ'^n or by γ', which is the characteristic class of the bundle U_{n+1} [7]. Hence

$$\begin{cases} \gamma'^n_{2n} = i'^{n,\, n-1} \circ l^{n-1} \circ \gamma''^k_{2n} & \textit{for } n=2k+1 \\ p'_*\gamma'^n_{2n} = \eta_{2n-1} & \textit{for } \text{even } n. \end{cases} \tag{11.2}$$

Note that γ''^n_{4n+2} is of order $\geqq^{(2)} (2n+1)!$ (c.f. § 10).

According to the cellular decomposition of unitary groups [8],

(11.3) *$U_n \underset{\gamma'}{\cup} e^{2n+1}$ is a sub-complex of U_{n+1} such that commutativity holds in the diagram*

$$\begin{tikzcd} U_{n+1} \arrow[dr, "p'"'] & U_n \underset{\gamma'}{\cup} e^{2n+1} \arrow[l] \arrow[d, "p"] \\ & S^{2n+1} \end{tikzcd}$$

where p is the map defined in (3.1).

Consider a null couple $U_n \xleftarrow{\gamma'} S^{2n} \xleftarrow{d\iota} S^{2n}$, where $d=^{(2)} n!$, then the coextension $(\gamma', d\iota)$ satisfies $p_*(\gamma', d\iota)=d\iota_{2n+1}$. $(\gamma', d\iota)$ is a coset of the subgroup $i \cdot \pi_{2n+1}(U_n)$, but since p_*' is a monomorphism, $i_*'\pi_{2n+1}(U_n)=0$.

(11.4) *$(\gamma'_{2n}, d\iota_{2n})$ consists of a single element, which we denote by ω''^{n+1}_{2n+1} for $n \geqq 1$.*

Since the identity map of $U_n \underset{\gamma'}{\cup} e^{2n+1}$ onto itself is an extension of the inclusion map $U_n \longrightarrow U_n \underset{\gamma'}{\cup} e^{2n+1}$, substituting n for $n+1$, we have

(11.5) $\omega'^n_{2n-1} \in \{i'^{n,\, n-1}, \gamma'^{n-1}_{2n-2}, d\iota_{2n-2}\}$ *for* $n \geqq 2$, *where* $d=^{(2)}(n-1)!$

Consider the exact sequence

$$\pi_{2n-1}(U_{n-1}) \xleftarrow{\Delta} \pi_{2n}(S^{2n-1}) \xleftarrow{p_*'} \pi_{2n}(U_n).$$

Since $\gamma'^{n-1} \circ \eta = (\Delta\iota)\eta = \Delta\eta = \Delta p_*'\gamma'$ for even n, the coextension (γ'^{n-1}, η) exists, and $p_*'(\gamma'^{n-1}, \eta) = \eta_{2n-1}$. Hence,

(11.6)
$$\begin{cases} \gamma'^{n}_{2n} \in (\gamma'^{n-1}_{2n-2},\ \eta_{2n-2}), \text{ or} \\ \gamma'^{n}_{2n} \in \{i'^{n,n-1},\ \gamma'^{n-1}_{2n-2},\ \eta_{2n-2}\} \textit{ for even } n. \end{cases}$$

It is well known that [10]

(11.7)
$$\pi_{2n+1}(U_n) = \begin{cases} Z_2 = (\gamma'_{2n} \circ \eta_{2n}) & \textit{for even } n, \\ 0 & \textit{for odd } n. \end{cases}$$

(11.8)
$$\begin{cases} \pi_{2n+2}(U_n) = {}^{(2)}Z_{(n+1)!} + Z_2 = (l^n \circ \gamma''^{k}_{2n+2}) + (\gamma'_{2n} \circ \eta^2_{2n}) & \textit{for } n=2k\ (k \geqq 2), \\ \pi_{2n+2}(U_n) = {}^{(2)}Z_{(n+1)!}/2 & \textit{for odd } n. \end{cases}$$

Let $d(n)$ be the integer defined as follows:

(11.9)
$$d(n) = \begin{cases} 1 & \text{if } n \equiv 3,\ 7 & \text{mod } 8, \\ 2 & \text{if } n \equiv 1,\ 2,\ 6 & \text{mod } 8, \\ 4 & \text{if } n \equiv 4,\ 5 & \text{mod } 8, \\ 8 & \text{if } n \equiv 0 & \text{mod } 8. \end{cases}$$

According to [14] and [15],

(11.10)
$$\begin{cases} \pi_{2n+3}(U_n) = Z_{d(n)} \quad (n \geqq 2), \ \textit{where } Z_{d(n)} = Z/d(n)Z \\ \pi_{2n+4}(U_n) = \begin{cases} {}^{(2)}Z_{(n+2)! \times d(n)/16} & \textit{for even } n \geqq 4, \\ {}^{(2)}Z_{(n+2)! \times d(n)/4} & \textit{for odd } n \geqq 3. \end{cases} \\ \pi_{2n+6}(U_n) = \begin{cases} {}^{(2)}Z_{(n+3)! \times d(n+1)/4} + Z_2 & \textit{for } n \equiv 0 \text{ mod } 4, \\ {}^{(2)}Z_{(n+3)! \times d(n+1)/4} & \textit{for } n \equiv 2 \text{ mod } 4\ (n \geqq 6), \\ {}^{(2)}Z_{(n+3)! \times d(n+1)/16} + Z_2 & \textit{for } n \equiv 1 \text{ mod } 4\ (n \geqq 5), \\ {}^{(2)}Z_{(n+3)! \times d(n+1)/16} & \textit{for } n \equiv 3 \text{ mod } 4, \end{cases} \\ \pi_{2n+5}(U_n) = Z_{d(n+1)} \quad (n \geqq 3), \\ \pi_{2n+7}(U_n) = \begin{cases} Z_{d(n+2)} + Z_{t(n)} & \textit{for } n \equiv 0,\ 1 \text{ mod } 4\ (n \geqq 6), \\ Z_{d(n+2)/2} + Z_{t(n)} & \textit{for } n \equiv 2,\ 3 \text{ mod } 4\ (n \leqq 4). \end{cases} \end{cases}$$

where $t(n)$ is the integer defined as follows:

$$t(n) = \begin{cases} 16 & \textit{for } n \equiv 0,\ 1 \text{ mod } 8, & n \equiv 10 \text{ mod } 16, \\ 8 & \textit{for } n \equiv 4,\ 5 \text{ mod } 8, & n \equiv 2 \text{ mod } 16, \\ 4 & \textit{for } n \equiv 6 \text{ mod } 8, \\ 2 & \textit{for } n \equiv 7 \text{ mod } 8, \\ 2 & \textit{for } n \equiv 3 \text{ mod } 16, \\ 4 & \textit{for } n \equiv 11 \text{ mod } 32, \\ 8 & \textit{for } n \equiv 27 \text{ mod } 64, \\ 16 & \textit{for } n \equiv 59 \text{ mod } 64. \end{cases}$$

Consider the exact sequence

$$0 \longleftarrow \pi_{2n+1}(U_{n-1}) \overset{\Delta}{\longleftarrow} \pi_{2n+2}(S^{2n-1}) \overset{p_*'}{\longleftarrow} \pi_{2n+2}(U_n).$$

Since $\pi_{2n+2}(S^{2n-1})=Z_8$ $(n\geqq 3)$, $\pi_{2n+1}(U_{n-1})=0$, Z_2, or Z_4 according as $n\equiv 0 \bmod 4$, $n\equiv 2 \bmod 8$, or $n\equiv 6 \bmod 8$, respectively, and since $p_*'\circ l^{2k}=p_*''$ it follows that

(11.11) $$p_*''\gamma''_{4k+2}=r(k)\nu_{4k-1} \quad (k\geqq 2),$$

where $$r(k)=\begin{cases} \pm 1 \text{ or } \pm 3 & \text{for } k\equiv 0 \mod 2, \\ \pm 2 & \text{for } k\equiv 1 \mod 4, \\ 4 \text{ or } 0 & \text{for } k\equiv 3 \mod 4. \end{cases}$$

Let n be odd. Since $d(n-1)\nu_{2n-1}\in Imp_*'$,
$\gamma'^{n-1}\circ d(n-1)\nu=\Delta(d(n-1)\nu)=0$. Consider a null couple $U_{n-1} \overset{\gamma'}{\longleftarrow} S^{2n-2} \overset{d(n-1)\nu}{\longleftarrow} S^{2n+1}$, then the coextension $(\gamma', d(n-1)\nu)$ satisfies $p_*(\gamma', d(n-1)\nu)=d(n-1)\nu_{2n-1}$. Hence, as a generator of $\pi_{2n+2}(U_n)$ for odd $n\geqq 5$, we can chose an element

(11.12) $$\begin{cases} u^n_{2n+2}\in\{i'^{n,n-1}, \gamma'^{n-1}_{2n-2}, d(n-1)\nu_{2n-2}\} & \text{such that} \\ p_*'u^n_{2n+2}=d(n-1)\nu_{2n-1}, & \text{and} \\ u^n_{2n+2}(n+1)=2\gamma'^{n+1}_{2n+2}. \end{cases}$$

In the case $n=3$,

(11.13) $$\omega'^3_5\circ\nu_5\in\pi_8(U_3) \textit{ and } p_*'(\omega'^3_5\circ\nu_5)=2\nu_5.$$

It is easily observed that

(11.14) $\pi_{2n+3}(U_n)$ *is generated by* $\gamma'^n_{2n}\circ\nu_{2n}$ *for* $n\geqq 3$.

Now, we describe the homotopy sequence of the bundle U_n as follows:

$$(\mathrm{II}^n_q):\ \pi_q(S^{2n-1}) \overset{p_*'}{\longleftarrow} \pi_q(U_n) \overset{i_*'}{\longleftarrow} \pi_q(U_{n-1}) \overset{\Delta}{\longleftarrow} \pi_{q+1}(S^{2n-1}).$$

It follows from the exact sequence $(\mathrm{II}^{n+1}_{2n+5})$ that

$$\pi_{2n+5}(U_n)\approx\pi_{2n+5}(U_{n+1})=Z_{d(n+1)}.$$

If $n=4k$ $(k\geqq 1)$, we consider an element $l^n\circ\gamma''^{2k}\circ\nu\in\pi_{2n+5}(U_n)$, which satisfies $p_*'(l^n\circ\gamma''^{2k}\circ\nu)=\nu^2_{2n-1}$, and $i'^{n+1,n}\circ l^n\circ\gamma''^{2k}\circ\nu=\gamma'^{n+1}\circ\nu$. Consider the exact sequence (II^n_{2n+4}). If $n\equiv 3 \bmod 4$, it is as follows:

$$(\mathrm{II}^n_{2n+4}):\ 0 \overset{p_*'}{\longleftarrow} Z_{d(n+2)!\times d(n)/4} \overset{i_*'}{\longleftarrow} Z_{d(n+2)!d(n)/4} \overset{\Delta}{\longleftarrow} \pi_{2n+5}(S^{2n-1}).$$

Hence, $\Delta\nu^2_{2n-1}=\gamma'^{n-1}\circ\nu=0$. The coextension $(\gamma'^{n-1}, \nu^2)\subset\pi_{2n+5}(U_n)$ satisfies $p_*(\gamma'^{n-1}, \nu^2)=\nu^2_{2n-1}$.

Summarizing, we have

(11.15) $$\pi_{2n+5}(U_n)=Z_{d(n+1)}=\begin{cases} (l^n\circ\gamma''^{2k}_{2n+2}\circ\nu_{2n+2}) & \textit{for } n=4k \\ (u^n_{2n+5}) & \textit{for } n\equiv 3 \bmod 4 \end{cases}$$

$$\begin{aligned} \textit{where}\quad & u^n_{2n+5}\in\{i'^{n,n-1},\ \gamma'^{n-1}_{2n-2},\ \nu^2_{2n-2}\}\quad \textit{such that}\\ & p_*'u^n_{2n+5}=\nu^2_{2n-1},\\ & u^n_{2n+5}(n+1)=\gamma'^{n+1}_{2n+2}\circ\nu_{2n+2}. \end{aligned}$$

Note that if $n\equiv1 \bmod 4$, then $\pi_{2n+5}(U_n)=i_*'\pi_{2n+1}(U_{n-1})$, and that if $n\equiv2 \bmod 4$, then $\pi_{2n+5}(U_n)=0$.

Let $n=8k+1$, then the exact sequence (Π^n_{2n+5}) is as follows:

$$(\Pi^n_{2n+5}):\ Z_2\xleftarrow{p_*'}Z_2\xleftarrow{i_*'}Z_2+Z_{16}\xleftarrow{\Delta}Z_{16}.$$

Consider an element $l^{n-1}\circ s^{4k}_{2n+5}\in\pi_{2n+5}(U_{n-1})$, then it satisfies $p_*'(l^{n-1}\circ s^{4k}_{2n+5})=\varepsilon_{2n-3}$.

Now, let $n=4k+3$ ($k\geqq1$), then the exact sequence (Π^n_{2n+5}) becomes as follows:

$$(\Pi^n_{2n+5}):\ Z_2\xleftarrow{p_*'}Z_4\xleftarrow{i_*'}Z_2+Z_{t(n-1)}\xleftarrow{\Delta}Z_{16}.$$

Since $\gamma'^{n-1}\circ2\nu=0$, we consider the first derived composition $\{\gamma'^{n-1},\ 2\nu,\ \nu\}$, which satisfies that $p'_*\{\gamma'^{n-1},\ 2\nu,\ \nu\}=\{\eta,\ 2\nu,\ \nu\}=\varepsilon$. Hence, there exists an element $u^{n-1}_{2n+5}\in\{\gamma'^{n-1},\ 2\nu,\ \nu\}$ such that $p_*'u_*{}'^{n-1}_{2n+5}=\varepsilon_{2n-3}$, and that $u^{n-1}_{2n+5}(n)=2u_{2n+5}^n$. Summarizing, we have

$$(11.16)\quad \pi_{2n+7}(U_n)=Z_2+Z_{t(n)}=\begin{cases}(l^n\circ s^{4k}_{2n+7})+(\gamma'^{n-1}\circ\sigma_{2n-2}) & \textit{for } n=8k\ (k\geqq1),\\ (u^n_{2n+7})+(\gamma'^{n-1}\circ\sigma_{2n-2}) & \textit{for } n\equiv2 \bmod 4\ (n\geqq6),\end{cases}$$

$$\begin{aligned} \textit{where}\quad & u^n_{2n+7}\in\{\gamma'^n_{2n},\ \nu_{2n},\ \nu_{2n+3}\}\ \textit{such that}\\ & p_*'u^n_{2n+7}=\varepsilon_{2n-1},\\ & u^n_{2n+7}(n+1)=2u^{n+1}_{2n+5}. \end{aligned}$$

It follows from the exact sequence (Π^n_{2n+4}) that for $n\geqq4$, $\pi_{2n+4}(U_n)=i'_*\pi_{2n+4}(U_{n-1})$. The exact sequence (Π^n_{2n+6}) and (Π^{n+1}_{2n+6}) are as follows: for $n\geqq5$

$$(\Pi^n_{2n+6}):\ 0\longleftarrow Z_{t(n-1)}\xleftarrow{\Delta}Z_{16}\xleftarrow{p_*'}\pi_{2n+6}(U_n)\xleftarrow{i_*'}\pi_{2n+6}(U_{n-1})\longleftarrow Z_2+Z_2,$$

$$(\Pi^{n+1}_{2n+6}):\ 0\longleftarrow {}^{(2)}Z_{(n+3)!\times d(n+1)/4}\xleftarrow{i_*'}\pi_{2n+6}(U_n)\xleftarrow{\Delta}Z_2,\ \text{for even } n\geqq4,$$

$$0\longleftarrow {}^{(2)}Z_{(n+3)!\times d(n+1)/16}\xleftarrow{i_*'}\pi_{2n+6}(U_n)\xleftarrow{\Delta}Z_2,\ \text{for odd } n\geqq3.$$

If $t_{(n-1)}=0 \bmod 16$, then $\gamma'^{n-1}\circ t(n-1)\sigma_{2n-2}=\Delta(t_{(n-1)}\sigma_{2n-1})=0$, hence we consider the coextension $(\gamma'^{n-1},\ t_{(n-1)}\sigma)\subset\pi_{2n+6}(U_n)$, which satisfies
$p_*(\gamma'^{n-1},\ t(n-1)\sigma)=t(n-1)\sigma_{2n-1}$.
Hence, the generators of $\pi_{2n+6}(U_n)$ are as follows:

$$(11.17)\quad \pi_{2n+6}(U_n)=\begin{cases}{}^{(2)}Z_{(n+3)!\times d(n+1)/4}+Z_2=(u^n_{2n+6})+(\gamma'^n\circ\nu^2_{2n}) & \textit{for } n\equiv0 \bmod 4,\\ {}^{(2)}Z_{(n+3)!\times d(n+1)/4}=(u^n_{2n+6}) & \textit{for } n\equiv6 \bmod 8,\\ {}^{(2)}Z_{(n+3)!\times d(n+1)/16}+Z_2=(u^n_{2n+6})+(\gamma'^n\circ\nu^2_{2n}) & \textit{for } n\equiv5 \bmod 8,\\ {}^{(2)}Z_{(n+3)!\times d(n+1)/16}=(u^n_{2n+6}) & \textit{for } n\equiv3 \bmod 4,\end{cases}$$

expect for $n\equiv 11$ mod 16, $n\equiv 60$ mod 64, *and for* $n\leqq 4$, *where*
$u^n_{2n+6}\in\{i'^{n,n-1}, \gamma'^{n-1}_{2n-2}, t(n-1)\sigma_{2n-2}\}$, $p'_* u^n_{2n+6}=t(n-1)\sigma_{2n-2}$.

The exact sequence Π^{n+2}_{2n+6} is as follows:

$$(\Pi^{n+2}_{2n+6}):\ 0\xleftarrow{\Delta} Z_{8/d(n+1)}\xleftarrow{p_*'} Z_{(n+3)!}+Z_2\xleftarrow{i_*'} Z_{(n+3)!d(n+1)/4}\longleftarrow 0,\ \text{for even } n\geqq 6,$$
$$0\xleftarrow{\Delta} Z_{8/d(n+1)}\xleftarrow{p_*'} Z_{(n+3)!/2}\xleftarrow{i_*'} Z_{(n+3)!d(n+1)/16}\longleftarrow 0,\ \text{for odd } n\geqq 5.$$

Hence, if follows that for $n\geqq 6$,

$$(11.18)\quad u^n_{2n+6}(n+2)=\begin{cases} d(n+1)u^{n+2}_{2n+6}, & \textit{for odd } n, \textit{ except for } n\equiv 1 \text{ mod } 8,\ n\equiv 11 \text{ mod } 16,\\ m\cdot l^{n+2}\circ\gamma''^{2k}_{2n+6}+\gamma'^{n+2}_{2n+4}\circ\eta^2_{2n+4}, & \textit{for } n=8k-2\ (k\geqq 2),\ m=\pm 1 \text{ or } \pm 3,\\ \pm 2\ l^{n+2}\circ\gamma''^{4k+1}_{2n+6}+\gamma'^{n+2}_{2n+4}\circ\eta^2_{2n+4}, & \textit{for } n=8k\ (k\geqq 1),\\ 4\ l^{n+2}\circ\gamma''^{4k+3}_{2n+6}+\gamma'^{n+2}_{2n+4}\circ\eta^2_{2n+4}, & \\ \textit{or } 4\ l^{n+2}\circ\gamma''^{4k+3}_{2n+6}, & \textit{for } n=8k+4\ (k\geqq 1),\ n\neq 60 \text{ mod } 64. \end{cases}$$

Now, we shall find out generators of $\pi_q(U_n)$ for low dimensional cases.(1) It is well known that

$$(11.19)\quad \begin{cases} \pi_1(U_n)=Z=(\tau'^1_1(n)) & \textit{for } n\geqq 1,\\ \pi_2(U_n)=0 & \textit{for } n\geqq 1,\\ \pi_q(U_1)=0 & \textit{for } q\geqq 2,\\ \pi_q(U_2)=\tau'^2_{3*}\pi_q(S^3) & \textit{for } q\geqq 2,\\ \pi_3(U_n)=Z=(\tau'^2_3(n)) & \textit{for } n\geqq 2. \end{cases}$$

It follows from the exact sequence (Π^3_4) that

$$(11.20)\quad \gamma'^2_4=\tau'^2_3\circ\eta_3$$

(11.21) *There exist original generatprs of* $\pi_q(U_3)$ *as follows:*

(i) $\omega'^3_5\in\pi_5(U_3)$, $\omega'^5_3\in\{\tau'^2_3(3), \eta_3, 2\iota_4\}$ mod $2\pi_5(U_3)$
$p_*'\omega'^3_5=2\iota_5$,

(ii) $u^3_{10}\in\pi_{10}(U_3)$, $u^3_{10}\in\{\tau'^2_3(3), \eta_3, \nu_4\circ\eta^2_7\}$ mod 0, $p_*'u^3_{10}=\nu_5\circ\eta^2_8$,

(iii) $u^3_{11}\in\pi_{11}(U_3)$, $u^3_{11}\in\{\sigma'^2_3(3), \eta_3, \nu^2_4\}$ mod $(\tau'^2_3(3)\circ\varepsilon_3)$,
$p_*'u^3_{11}=\nu^2_5$, $2u^3_{12}=\tau'^2_3(3)\circ\varepsilon_3$.

(iv) $u^3_{12}\in\pi_{12}(U_3)$, $u^3_{12}\in\{\omega'^3_5, 4\nu_5, \nu_8\}$ mod 0,
$p{\cdot}u^3_{12}=\sigma'''$, $2u^3_{12}(3)\circ\tau^{12}_3(3)\circ\mu_3$.

(i)~(iii) follow directly from (11.20) and from

$$(11.22)\quad \omega'^3_5\circ\eta_5=\tau'^2_3(3)\circ\nu',\quad \omega'^3_5\circ\nu^2_5=\tau'^2_3(3)\circ\varepsilon_3.$$

Indeed, $\omega'^3_5\circ\eta_5\in\tau'^2_3(3)\circ\{\eta_3, 2\iota_4, \eta_4\}$, which consists of a uniqe element $\tau'^2_3(3)\circ\nu'$ and

(1) $\pi_q(U_n)$ are calculated by H. Toda for $q\leqq 15$ in [13].

$\omega'^3_5\circ\nu^2_5=\{\omega'^3_5\circ\eta_5,\ \nu_6,\ \eta_9\}=\{\tau'^2_3(3)\circ\nu',\ \nu_6,\ \eta_9\}=\tau'^2_3(3)\circ\{\nu',\ \nu_6,\ \nu_9\}=\tau'^2_3(3)\circ\varepsilon_3$ ([1], (7.6)). Next we shall prove (iv). Consider a null quadruple

$$(\bar{\alpha}):\ U_3\xleftarrow{i'}U_2\xleftarrow{\gamma'^2_4}S^4\xleftarrow{8\nu}S^7\xleftarrow{\nu}S^{10}.$$

It is very easy to see that $\{i'^{3,2}\gamma'^2_4,\ 8\nu_4\}=0$, and $\{\gamma'^2_4,\ 8\nu_4,\ \nu_7\}=0$. Hence $(\bar{\alpha})$ is admissible. $i'^{3,2}\circ\pi_{12}(U_2)+\pi_q(U_3)\circ\nu_9=i'^{3,2}\circ\pi_2(U_2)=(\tau'^2_3(3)\circ\mu_3)$, because $\pi_q(U_3)=0$. Since $\{i'^{3,2},\ \tau'^2_3\circ\nu'\circ\eta^2_6,\ \nu_8\}\ni 0$, it follows from (6.5) (ii) that the second derived composition $\{i'^{3,2},\ \gamma'^2_4,\ 8\nu_4,\ \nu_7\}$ is a coset of the subgroup generated by $\tau'^2_3(3)\circ\mu_3$. It is observed that $U_3\xleftarrow{i'}U_2\xleftarrow{\gamma'^2\circ\nu}S^7\xleftarrow{8\iota}S^7\xleftarrow{\nu}S^{10}$ is also admissible, and that $\{i'^{3,2},\ \gamma'^2_4\circ\nu_4,\ 8\iota_7\}=\{i'^{3,2},\ \gamma'^2_4,\ 8\nu_4,\ \nu_7\}$ by (6.9) (iii). It follows from (6.14) that $p_*'\{i'^{3,2},\gamma'^2_4\circ\nu_4,8\iota_7,\nu_7\}=\{\nu_5,\ 8\iota_8,\ \nu_8\}=(\sigma''')$ [1]. Let $u^3_{12}\in\pi_{12}(U_3)$ be an element such that $u^3_{12}\in\{i'^{3,2},\ \gamma'^2_4,\ 8\nu_4,\ \nu_7\}$, then $p_*'u^3_{12}=\sigma'''$. Since $\pi_{12}(U_3)=Z_4$, $\{i'^{3,2},\gamma'^2_4,8\nu_4,\nu_7\}$ consists of two elements, $+u^3_{12}$ and $-u^3_{12}$, and $2u^3_{12}=\tau'^2_3(3)\circ\mu_3$.
While, $\{i'^{3,2},\ \gamma'^2_4,\ 8\nu_4,\ \nu_7\}=\{\tau'^2_3(3),\ \eta_3,\ (2\iota_4)\circ(4\nu_4),\ \nu_7\}\sim\pm\{\omega'^3_5,\ 4\nu_5,\ \nu_8\}$ by (i) and (6.13) (ii).
(Note that $\omega'^3_5\circ4\nu_5=\omega'^3_5\circ\eta^3_5=\tau'^2_3(3)\circ\nu'\circ\eta^2_6=\tau'^2_3(3)\circ\eta_3\circ\nu_4\circ\eta_7=i'^{3,2}\circ\gamma'^2_4\circ\nu_4\circ\eta_7=0$, and that $\omega'^3_5\circ\pi_{12}(S^5)+\pi_9(U_3)\circ\nu_9=(\omega'^3_5\circ\sigma''')=\omega'^3_5\circ\{\nu_5,\ 8\iota_8,\ \nu_8\}=\{\omega'^3_5,\ 4\nu_5,\ 2\iota_8\}\circ\nu_9=0$ (by using (5.7)).

Hence, we determine the signature of u^3_{12} so that $u^3_{12}\in\{\omega'^3_5,\ 4\nu_5,\ \nu_8\}$. q.e.d.

We have proved that

$$\omega'^3_5\circ\sigma'''=0. \tag{11.23}$$

It follows from the exact sequences (Π^4_5), and (Π^2_6) that

$$\begin{cases}\pi_5(U_n)=Z=(\omega'^3_5(n)) & \text{for } n\geqq 3,\\ \pi_6(U_3)=Z_2=(\tau'^2_3\circ\nu'),\\ \gamma'^3_6=\tau'^2_3\circ\nu'.\end{cases} \tag{11.24}$$

Generators of $\pi_q(U_n)$ ($q\leqq 13$, $n\geqq 4$) are obtained by (11.5), (11.6) and (11.12), which are listed as follows:

$$\begin{cases}\omega'^4_7,\ \omega'^5_9,\ \omega'^6_{11},\ \omega'^7_{13},\ \gamma'^4_8,\ \gamma'^6_{12},\ \gamma'^4_8\circ\eta_8,\ \gamma'^6_{12}\circ\eta_{12},\\ \omega'^3_5\circ\nu_5,\ l^4\circ\gamma''^2_{10},\ u^5_{12},\ l^4\circ\gamma''^2_{10}\circ\nu_{10},\ \textit{which satisfy}:\\ p_*'\omega'^4_7=2\iota_7,\ p_*'\omega'^5_9=8\iota_9,\ p_*'\omega'^6_{11}=8\iota_{11},\ p_*'\omega'^7_{13}=16\iota_{13},\\ p_*'\gamma'^4_8=\eta_7,\ p_*'\gamma'^6_{12}=\eta_{11},\ p_*'(l^4\circ\gamma''^2_{10})=\nu_7,\ p_*'u^5_{12}=4\nu_9,\\ \omega'^3_5(4)\circ\nu_5=2\gamma'^4_8,\ u^5_{12}(6)=2\gamma'^6_{12}.\end{cases} \tag{11.25}$$

Comparing (11.23) with (10.19) and (10.23), it follows that

$$l^4\circ\omega''^2_7=2\omega'^4_7,\quad l^6\circ\omega''^3_{11}=\omega'^6_{11}. \tag{11.26}$$

It is easy to see that

$$(11.27)\qquad \begin{cases} u^3_{10}(4)=\gamma'^4_8\circ\eta_8^2+4l^4\circ\gamma''^2_{10} \\ u^3_{12}(5)=2u^5_{12} \end{cases} \qquad \text{(c.f. 11.18)}$$

Now, we shall prove the following

(11.28)

(i) $\omega'^4_7\circ\eta_7=4\gamma'^4_8$, (ii) $\omega^5_9\circ\eta'_9=4i'^{5,4}\circ l^4\circ\gamma''^2_{10}$

(iii) $u^3_{10}\circ\eta_{10}=0$, (iv) $u^3_{11}\circ\eta_{11}=0$,

(v) $\omega'^5_9\circ\nu_9=2u^5_{12}$, (vi) $\omega'^3_5\circ\varepsilon_5=\tau'^2_3(3)\circ\varepsilon'$,

(vii) $u^3_{10}\circ\nu_{10}=0$, (viii) $u^3_{12}\circ\eta_{12}=0$,

(ix) $\omega'^4_7\circ\nu^2_7=2l^4\circ\gamma^2_{10}\circ\nu_{10}=\tau'^2_3(4)\circ\varepsilon'$,

(x) $u^5_{12}\circ\eta_{12}=2i'^{5,4}\circ l^4\gamma''^2_{10}\circ\nu_{10}$,

(xi) $\omega'^6_{11}\circ\eta_{11}=4u^5_{12}(6)$, (xii) $\gamma'^4_8\circ\nu_8=\pm u^3_{11}(4)$,

(i′) $\omega'^4_7\in\{i'^{4,3},\ \tau'^2_3(3)\circ\nu',\ 2\iota_6)$ mod $2\pi_7(U_4)$,

(ii′) $\omega'^5_9\in\{i'^{5,4},\ \gamma'^4_8,\ 8\iota_8\}$ mod $8\pi_9(U_5)$,

(iii′) $\omega'^6_{11}\in\{i'^{6,4},\ l^4\circ\gamma''^2_{10},\ 8\iota_{10}\}$ mod $8\pi_{11}(U_6)$,

(iv′) $\omega'^7_{13}\in\{i'^{7,6},\ \gamma'^6_{12},\ 16\iota_{12}\}$ mod $16\pi_{13}(U_7)$,

(v′) $\gamma'^4_8\in\{i'^{4,3},\ \tau'^2_3(3)\circ\nu',\ \eta_6\}$ mod $(\omega'^3_5(4)\circ\nu_5)$,

(vi′) $\gamma'^6_{12}\in\{i'^{6,4},\ l^4\circ\gamma''^2_{10},\ \eta_{10}\}$ mod $(2u^5_{12}(6))$,

(vii′) $u^5_{12}\in\{i'^{5,4},\ \gamma'^4_8,\ 4\nu_8\}$ mod $(u^3_{12}(5))$.

PROOF. (i′)~(iv′) follow directly from (11.5).

(i) $\omega'^4_7\circ\eta_7=\{i'^{4,3},\ \tau'^2_3(3)\circ\nu',\ 2\iota_6\}\circ\eta_7=i'^{4,3}\circ\{\tau'^2_3(3),\ 2\nu',\ \eta_6\}$

$=i'^{4,3}\circ\{\tau'^2_3(3),\ \eta_3,\ 4\nu_4\}=i'^{4,3}\circ\{\tau'^2_3(3),\ \eta_3,\ 2\iota_4\}\circ(2\nu_5)$

$=\omega'^3_5(4)\circ(2\nu_5)=4\gamma'^4_8$ (by 11.25).

Note that each of these sets consists of a single element. In the following, the reader is expected to examine.

(ii) $\omega'^5_9\circ\eta_9=\{i'^{5,4},\ \gamma'^4_8,\ 8\iota_8\}\circ\eta_9=i'^{5,4}\circ\{4\gamma'^4_8,\ 2\iota_8,\ \eta_8\}$

$=\omega'^4_7(5)\circ\}\eta_7,\ 2\iota_8,\ \eta_8\}=\omega'^4_7(5)\circ(2\nu_7)$

$=i'^{5,4}\circ l^4\circ\omega''^2_7\circ\nu_7$ (by 11.26)

$=4i'^{5,4}\circ l^4\circ\gamma''^2_{10}$ (by 10.20).

(iii) $u^3_{10}(4)\circ\eta_{10}=\gamma'^5_8\circ\eta^3_8+4l^4\circ\gamma''^2_{10}\circ\eta_{10}$ (by 11.27)

$=4\gamma'^4_8\circ\eta^3_8+4l^4\circ\tau''^1_3(2)\circ\varepsilon_3$ (by 10.21)

$=\omega'^4_7\circ\eta_7\circ\nu_8+4\tau'^2_3(4)\circ\varepsilon_3=0.$

Since $i_*'^{4,3}$: $\pi_{10}(U_3)\longrightarrow\pi_{10}(U_4)$ is a monomorphism, it follows that $u^3_{10}\circ\eta_{10}=0$.

(iv) $u^3_{11}\circ\eta_{11}=\{\tau'^2_3(3),\ \eta_3,\ \nu^2_4\}\circ\eta_{11}=\tau'^2_3(3)\circ\{\eta_3\circ\nu_4,\ \nu_7,\ \eta_{10}\}$

$=\tau'^2_3(3)\circ\nu'\circ\{\eta_6,\ \nu_7,\ \eta_{10}\}=\tau'^2_3(3)\circ\nu'\circ\nu^2_6=0$ (by [1], Lemma 5.12).

(v) $\omega'^5_9\circ\nu_9=\{i'^{5,4},\ \gamma'^4_8,\ 8\iota_8\}\circ\nu_9=\{i'^{5,4},\ 2\gamma'^4_8,\ 4\iota_8\}\circ\nu_9$

$=\{i'^{5,4},\ \omega'^3_5(4)\circ\nu_5,\ 4\iota_8\}=\nu_9$ (by 11.25)

$\sim i'^{5,3}\circ\{\omega'^3_5,\ 4\nu_5,\ \nu_8\}=u^3_{12}(5)=2u^5_{12}$.

(vi) $\omega'^3_5\circ\varepsilon_5=\omega'^3_5\circ\{\eta_5,\ 2\nu,\ \nu_9\}_2=\{\omega'^3_5\circ\eta_5,\ 2\nu_6,\ \nu_9\}_2$
$\{\tau'^3_3(3)\circ\nu',\ 2\nu_6,\ \nu_9\}_2=\tau'^3_3(3)\circ\varepsilon'$.

(vii) $u^3_{10}(4)\circ\nu_{10}=\gamma'^4_8\circ\eta^2_8\circ\nu_{10}+4l^4\circ\gamma''^2_{10}\circ\nu_{10}=0$, since $\pi_{13}(U_4)=Z_4$ by (11.10). Considering the exact sequence (II^4_{13}), we see that $i'^{4,3}_*:\pi_{13}(U_3)\longrightarrow\pi_{14}(U_4)$ is a monomorphism. Hence $u^3_{10}\circ\nu_{10}=0$.

(viii) $u^3_{12}\circ\eta_{12}=\{\omega'^3_5,\ 4\nu_5,\ \nu_8\}\circ\eta_{12}=\omega'^3_5\circ\{\eta^3_5,\ \nu_8,\ \eta_{11}\}$
$=\omega'^3_5\circ\eta^2_5\circ\nu^2_7=0$.

(ix) Since $p'(\omega'^4_7\circ\nu_7)=3\nu_7$, $\omega'^4_7\circ\nu_7=2l^4\circ\gamma''^2_{10}$ or $2l^4\circ\gamma''^2_{10}+u^3_{10}(4)$. Since $u^3_{10}\circ\nu_{10}=0$, it follows that
$\omega'^4_7\circ\nu^2_7=2l^4\circ\gamma''^2_{10}\circ\nu_{10}=\tau''^2_3(4)\circ\varepsilon'$ (by 10.21).

(v′) and (vii′) follow from (11.6), (11.12) and from the facts that

$$i'^{4,3}_*\circ\pi_8(U_3)+\pi_7(U_4)\circ\eta_7=(\omega'^3_5(4)\circ\nu_5)+(\omega'^4_7\circ\eta_7=(\omega'^3_5(4)\circ\nu_5),$$
$$i'^{5,4}_*\pi_{12}(U_4)+\pi_9(U_5)\circ(4\nu_9)=(u^3_{12}(5))+(4\omega'^5_9\circ\nu_9)=(u^3_{12}(5)).$$

(x) $u^5_{12}\circ\eta_{12}=\{i'^{5,4},\ \gamma'^4_8,\ 4\nu_8\}\circ\eta_{12}=i'^{5,4}\circ\{4\gamma'^4_8,\ \nu_8,\ \eta_{11}\}$
$=i'^{5,4}\circ\{\omega'^4_7\circ\eta_7,\ \nu_8,\ \eta_{11}\}$ (by (i))
$=\omega'^4_7(5)\circ\nu^2_7=2i'^{5,4}\circ l^4\circ\gamma''^2_{10}\circ\nu_{10}$.

(xi) $\omega'^6_{11}\circ\eta_{11}=\{i'^{6,4},\ l^4\circ\gamma''^2_{10},\ 8\iota_{10}\}\circ\eta_{11}$
$=i'^{6,5}\circ\{4i'^{5,4}\circ l^4\circ\gamma''^2_{10},\ 2\iota_{10},\ \eta_{10}\}$
$=i'^{6,5}\circ\{\omega'^5_9\circ\eta_9,\ 2\iota_{10},\ \eta_{10}\}$ (by (ii))
$=i'^{6,5}\circ\omega'^5_9(2\nu_9)=4u^5_{12}(6)$ (by (v)).

(xii) Since $2\gamma'^4_8\circ\nu_8=\omega'^3_5(4)\circ\nu^2_5$ (by 11.25)
$=2u^3_{11}(4)$, (by 11.22 and 11.21 (iii))
hence, $\gamma'^4_8\circ\nu_8=\pm u^3_{11}(4)$.

(vi′) follows from (11.6) and from the fact that

$$i'^{6,4}\circ\pi_{12}(U_4)+\pi_{11}(U_6)\circ\eta_{11}=(u^3_{12}(6))+(\omega'^6_{11}\circ\eta_{11})=2u^5_{12}(6)$$

(by (11.27) and (xi)). q.e.d.

§ 12 $\pi_q(SO(n))$ $q\leqq 13$

Let $\Delta:\pi_q(S^n)\longrightarrow\pi_{q-1}(R_n)$ be the boundary operator of the homotopy sequence of the bundle R_{n+1}. Denote $\Delta\iota_n$ by γ^n_{n-1} or simply by γ^n or by γ, which is the characteristic class of R_{n+1}. It is well known that [7].

$$(12.1)\qquad\begin{cases}\gamma^n_{n-1}=i^{n,n-1}\circ k^{n-1}\circ\gamma'^k_{n-1} & \textit{for}\quad n=2k+1,\\ p_*\gamma^n_{n-1}=2\iota_{n-1} & \textit{for even } n,\\ \gamma^n\neq 0\quad \textit{if}\ n\neq 3,7.\end{cases}$$

According to the celluar decomosition of real orthognal groups [8],

(12.2) *$R_n \underset{\gamma}{\cup} e^n$ is a subcomplex of R_{n+1} such that commutativity holds in the diagram*

$$\begin{tikzcd} R_{n+1} & R_n \underset{\gamma}{\cup} e^n \arrow[l] \arrow[dl, "p"] \\ S^n \arrow[from=1-1, "p"'] & \end{tikzcd}$$

If $R_n \xleftarrow{\gamma} S^{n-1} \xleftarrow{\alpha} S^m$ is a null couple, the coextension (γ^n, α) satisfies $p_*(\gamma^n, \alpha) = E\alpha$. Hence, there exists an element $\beta \in \{i^{n+1,n}, \gamma^n, \alpha\}$ such that $p_*\beta = E\alpha$.

$\pi_q(R_n)$ is calculated in [12] for $q \leqq 15$. In the following, we shall give the generators of $\pi_q(R_n)$ $q \leqq 13$, and investigate the behaviour under composition with the generators of the homotopy groups of spheres.

It is well known that [7]

$$(12.3) \qquad \begin{cases} \pi_1(R_2) = Z = (\tau_1^2),\ \pi_1(R_n) = Z_2 = (\tau_1^2(n)) \ \textit{for}\ n \geqq 3, \\ \pi_q(R_2) = 0 \ \textit{for}\ q \geqq 2,\ \pi_2(R_n) = 0 \ \textit{for}\ n \geqq 2, \\ \pi_q(R_3) = \lambda^3_{3*}\pi_q(S^3) \ \textit{for}\ q \geqq 3, \\ \pi_q(R_4) = \tau^4_{3*}\pi_q(S^3) + \lambda_3^3(4)_*\pi_q(S^3) \ \textit{for}\ q \geqq 3, \\ \gamma_3^4 = 2\tau_3^4 + \lambda_3^4(4). \end{cases}$$

Let $\Delta : \pi_{q+1}(S^4) \longrightarrow \pi_q(R_4)$ be the boundary operator, then $\Delta(\eta_4) = \lambda_3^3(4) \circ \eta_3$, $\Delta(\varepsilon_4) = \lambda_3^3(4) \circ \varepsilon_3$, $\Delta(\mu_4) = \lambda_3^3(4) \circ \mu_3$, $\Delta(E\varepsilon') = \lambda_3^3(4) \circ \varepsilon' + 2\tau_3^5 \circ \varepsilon'$, hence it follows that

$$(12.4) \qquad \begin{cases} \lambda_3^3(5) \circ \eta_3 = 0,\ \lambda_3^3(5) \circ \varepsilon_3 = 0,\ \lambda_3^3(5) \circ \mu_3 = 0, \\ \lambda_3^3(5) \circ \varepsilon' = 2\tau_3^4(5) \circ \varepsilon'. \end{cases}$$

Since $\pi_6(R_5) = 0$, we have

$$(12.5) \qquad \lambda_3^3(5) \circ \nu' = \tau_3^4(5) \circ \nu' = 0.$$

Let $q : Sp_2 \longrightarrow R_5$ and $q' : U_4 \longrightarrow R_6$ be the projections of the well known coverings, then commutativity holds in the diagrams

$$(12.6) \qquad \begin{tikzcd} & U_4 \arrow[dl, "k^6"'] \arrow[d, "q'"] & Sp_2 \arrow[l, "k^4"'] \arrow[d, "q"] & Sp_1 \arrow[l, "i''^{2,1}"'] \arrow[d, "k^4 \circ \lambda^2"] \\ R_8 & R_6 \arrow[l, "i^{8,6}"] & R_5 \arrow[l, "i^{6,5}"] & R_4 \arrow[l, "i^{5,4}"] \end{tikzcd} \qquad \begin{tikzcd} U_4 \arrow[d, "q'"'] & U_3 \arrow[l, "i'^{4,3}"'] \arrow[dl, "k^5"] \\ R_6 & \end{tikzcd}$$

The original operators of $\pi_q(R_5)$ and $\pi_q(R_6)$ are obtained by the isomorphisms $q_* : \pi_q(Sp_2) \longrightarrow \pi_q(R_5)$ and $q'_* : \pi_q(U_4) \longrightarrow \pi_q(R_1)$ as follows:

$$(12.7) \qquad \begin{cases} r_7^5 = q_*(\omega''^2_7) \in \{\tau_3^4(5), \nu', 4\iota_6\} \bmod 4\ \pi_7(R_5), \\ r_{10}^5 = q_*(\omega''^2_{10}) \in \{\tau_3^4(5), \nu', \nu_6\} \bmod 4\ \pi_{10}(R_5), \\ k^6 \circ \omega'^3_5 = q'_*(\omega'^3_5(4)), \\ r_7^6 = q'_*(\omega'^4_7) \in \{i^{6,5}, \tau_3^4(5) \circ \eta_3, \eta_4^2\} \bmod 2\ \pi_7(R_6), \\ r_8^6 = q'_*(\gamma'^4_8) \in \{i^{6,5}, \tau_3^4(5) \circ \eta_3, \nu_4\} \bmod 2\ \pi_8(R_6), \\ k^6 \circ u_{10}^3 = q'_*(u_{10}^3(4)), \\ k^6 \circ u_{12}^3 = q'_*(u_{12}^3(4)). \end{cases}$$

Note that by (12.1) and (11.8) we have

(12.8) $\qquad \gamma_4^5=\tau_3^4(5)\circ\eta_3$, hence $\tau_3^4(6)\circ\eta_3=0$.

Proof of (12.7). Since

$$p_*:\pi_7(R_6)\longrightarrow\pi_7(S^5) \text{ and } p_*:\pi_8(R_6)\longrightarrow\pi_8(S^5)$$

are eqimorphisms, $p_*r_7^6=\eta_5^2$ and $p_*r_8^6=m\nu_5$ for some odd m.
While, $p_*\{i^{6,5},\ \tau_3^4(5)\circ\eta_3,\ \eta_4^2\}=\eta_5^2$ and $p_*\{i^{6,5},\ \tau_3^4(5)\circ\eta_3,\ \nu_4\}=\nu_5$.
It follows from (11.26), and (11.25) that

(12.9) $\qquad r_7^5(6)=2r_7^6,\ k^6\circ\omega_5'^3\circ\nu_5=2r_8^6.$

Hence, $i^{6,5}\circ\pi_7(R_5)+\pi_5(R_6)\circ\eta_5^2=2\pi_7(R_6)$, and $i^{6,5}\circ\pi_8(R_6)+\pi_5(R_6)\circ\nu_5=2\pi_8(R_6)$. Thus $\{i^{6,5},\ \tau_3^4(5)\circ\eta_3,\ \eta_4^2\}$ and $\{i^{6,5},\ \tau_3^4(5)\circ\eta_3,\ \nu_4\}$ are cosets of the subgroups $2\pi_7(R_6)$ and $2\pi_8(R_6)$.
The other parts of (12.7) are obvious.
We shall show that

(12.10)
$$\begin{cases} p_*r_7^5=4\nu_4,\ p_*r_{10}^5=\nu_4^2,\ p_*k^6\circ\omega_5'^3=2\iota_5,\\ p_*r_7^6=\eta_5^2,\ p_*r_8^6=\nu_5,\ p_*k^6\circ u_{10}^3=\nu_5\circ\eta_8^2\\ p_*k^6\circ u_{12}^3=\sigma'''. \end{cases}$$

Since $\pi_6(R_5)=0$, it follows from the exact seqoence

$$\pi_6(R_5)\longleftarrow\pi_6(R_4)\xleftarrow{\Delta}\pi_7(S^4)$$

that $\Delta\nu_4=\tau_3^4\circ\nu'$ or $\tau_3^4\circ\nu'+\lambda_3^3(4)\circ\nu'$. Since $4\nu'=0$ and $\nu'\circ\nu_6=0$, it follows from (7.14) that $p^*\{\tau_3^4(5),\ \nu',\ 4\iota_6\}=\Delta^{-1}(\tau_3^4\circ\nu')\circ(4\iota_7)=4\nu_4$, and $p_*\{\tau_3^4(5),\ \nu',\ \nu_6\}=\Delta^{-1}(\tau_3^4\circ\nu')\circ\nu_7=\nu_4^2$.
The other parts of (12.10) are obvious.

The other generators of $\pi_q(R_n)$ $(n=5, 6)$ which do not belong to $i_*^{n,n-1}\pi_q(R_{n-1})$ $(n=5, 6)$ are obtained by composition. They are listed as follows:

(12.11) $\qquad r_{10}^5\circ\nu_{10},\ r_8^6\circ\eta_8,\ r_8^6\circ\nu_8.$

We *shall show that*:

(12.12)
$$\begin{cases} \text{(i)}\ \ \gamma_5^6=k^6\circ\omega_5'^3, & \text{(ii)}\ \ k^6\circ\omega_5'^3\circ\eta_5=0,\ r_7^5\circ\eta_7=0,\\ \text{(iii)}\ \ r_7^5\circ\nu_7=4r_{10}^5, & \text{(iv)}\ \ r_7^6\circ\eta_7=4r_8^6,\\ \text{(v)}\ \ r_7^6\circ\nu_7=\pm2r_{10}^5, & \text{(vi)}\ \ r_8^6\circ\eta_8^2=k^6\circ u_{10}^3+4r_{10}^5(6).\\ \text{(vii)}\ \ r_{10}^5\circ\eta_{10}=\tau_3^4(5)\circ\varepsilon_3, & \text{(viii)}\ \ 2r_8^6=\gamma_5^6\circ\nu_5,\\ \text{(ix)}\ \ 2r_8^6\circ\nu_8=\tau_3^4(6)\circ\varepsilon_3, & \text{(x)}\ \ 2k^6\circ u_{12}^3=\tau_3^4(6)\circ\mu_3,\\ \text{(xi)}\ \ 2r_{10}^5\circ\nu_{10}=\tau_3^4(5)\circ\varepsilon', & \text{(xii)}\ \ k^6\circ u_{11}^3=\pm r_8^6\circ\nu_8. \end{cases}$$

PROOF. (i) Consider the exact sequence

$$\pi_5(S^5)\xleftarrow{p_*}\pi_5(R_6)\xleftarrow{i_*}\pi_5(R_5)\xleftarrow{\Delta}\pi_6(S^5)$$
$$\Delta\pi_6(S^5)=(\Delta\eta_5)=(\tau_3^4(5)\circ\eta_3^2)=\pi_5(R_5) \qquad \text{(by 12.8)}.$$

Hence i_* is a monomorphism. While $p_*(k^6\circ\omega'^3_5)=2\iota_5=p_*\gamma^6_5$, so that $k^6\circ u^3_5=\gamma^6_5$.

(ii) follows from the fact that $\pi_6(R_6)=0$ and $\pi_8(R_5)=0$.

(iii) follows from (10.20) and (12.7).

(iv) follows from (11.28) (i) and (12.7).

(v) follows from the fact that $2r^6_7\circ\nu_7=q'_*(2\omega'^4_7\circ\nu_7)=q'_*(l^4\circ\omega''^2_7\circ\nu_7)=4q_*(\gamma''^2_{10})$ $=4r^5_{10}$ by using (11.26), (10.20) and (12.7).

(vi) follows from (11.27) and (12.7).

(vii) follows from (10.21) and (12.6).

(viii) follows from (11.25) and (12.7).

(ix) follows from (11.25), (11.22) and (12.6).

(x) follows from (11.21) (iv) and (12.6).

(xi) follows from (10.21) and (12.6).

(xii) $k^6\circ u^3_{11}=q'_*(u^3_{11}(4))=\pm q'_*(\gamma'^4_8\circ\nu_8)=\pm r^6_8\circ\nu_8$ (11.28, (xii)).

Note that $k^6\circ u^3_{11}$ is not an original element.

In the exact sequence ([1], (2.11))

$$\pi_9(S^5)\xleftarrow{\partial}\pi_{11}(S^{11})\xleftarrow{H}\pi_{11}(S^6)\xleftarrow{E}\pi_{10}(S^5),$$

$\partial(\iota_{11})=\nu_5\circ\eta_8$ and E is a monomorphism ([1], p. 45).

It follows from (7.17) that $H\{\nu_6,\eta_9,2\iota_{10}\}_5=\pm 2\iota_{11}$, so that $[\iota_6,\iota_6]\in\{\nu_6,\eta_9,2\iota_{10}\}_6$ mod $2\pi_{11}(S^6)$, since $H[\iota_6,\iota_6]=\pm 2\iota_{11}$.

Now, in the exact sequence

$$0\longleftarrow\pi_{10}(R_7)\xleftarrow{i^*}\pi_{10}(R_6)\xleftarrow{\Delta}\pi_{11}(S^6)\xleftarrow{p^*}\pi_{11}(R_7),$$

$\pi_{10}(R_7)=Z_8$, $\pi_{10}(R_6)=Z_8+Z_2$, hence

(12.13) $$\begin{cases}\text{Im } p_*=(2[\iota_6,\iota_6])\in\{\nu_6,\eta_8,4\iota_{10}\},\\ \Delta[\iota_6,\iota_6]=k^6\circ u^3_{10}+4r^6_{10}.\end{cases}$$

Now, we shall show that

(12.14) $$\begin{cases}\lambda^7_7\in\{i^{7,6},k^6\circ\omega'^3_5,\eta_5\} \text{ mod } 2\pi_7(R_7),\\ 2\lambda^7_7=\pm r^6_7(7),\ \lambda^7_7\circ\nu_7=mr^5_{10}(7)\ \textit{for some odd } m.\end{cases}$$

(12.15) *There exists an orginal element*

$$r^7_{11}\in\{i^{7,6},2r^6_8,\eta_8,4\iota_9\}\ \textit{such that}\ p_*r^7_{11}=2[\iota_6,\iota_6].$$

PROOF OF 12.14.

$$k^6\circ\omega'^3_5=\gamma^6_5,\ \text{hence}\ p_*\{i^{7,6},k^6\circ\omega'^3_5,\eta_5\}=\eta_6.$$

It follows from $p_*(2\lambda^7_7)=0$ that $2\lambda^7_7=\pm r^6_7(7)$, and hence $p_*^{-1}(0)=2\pi_7(R_7)$. Thus, we have $\lambda^7_7\in\{i^{7,6},k^6\circ\omega'^3_5,\eta_5\}$ mod $2\pi_7(R_7)$, $2\lambda^7_7\circ\nu_7=\pm r^5_{10}(6)$ by (12.12 (v)).

Hence $\lambda_7^7\circ\nu_7$ generates $\pi_{10}(R_7)$. q.e.d.

PROOF OF 12.15.

Consider a null quadruple

$$R_7 \xleftarrow{i} R_6 \xleftarrow{2r_8^6} S^8 \xleftarrow{\eta} S^9 \xleftarrow{4\iota} S^9 \tag{12.16}$$

Note that $2r_8^6(7)=i^{7,6}\circ\gamma_5^6\circ\nu_5=0$ by (12.12) (viii).

Since $i^{7,6}\pi_{10}(R_6)=\pi_{10}(R_7)$, $\{i^{7,6}, 2r_8^6, \eta_8\}=\pi_{10}(R_7)\ni 0$.

While, $\{2r_8^6, \eta_8, 4\iota_9\}\supset r_8^6\circ\{2\iota_8, \eta_8, 2\iota_9\}\circ(2\iota_{10})=0$.

Let G_1 and G_2 be the subgroups of $\pi_{10}(R_6)$ such that

$$i^{7,6}\circ G_1\subset\pi_9(R_7)\circ\eta_9 \text{ and } G_2\circ(4\iota_{10})\subset 2r_8^6\circ\pi_{10}(S^8)=0,$$
$$i.e.\ G_1=(k^6\circ u_{10}^3)+(4r_{10}^5(6)),\ G_2=(k^6\circ u_{10}^3)+(2r_{10}^5(6)).$$

Hence, in this case, we can not apply (6.3), but I assert that the null quadruple (12.16) is admissible as follows. Let B be a null homotopy of $(2\iota_8)\circ\eta_8$, and let $A_2=r_8^6\circ B$. Since $\{2\iota_8, \eta_8, 4\iota_9\}=0$, $\{2r_8^6, A_2, \eta_8, A_3, 4\iota_9\}\simeq 0$ for any null homotopy A_3 of $\eta_8\circ(4\iota_9)$. While $2\{r_8^6(7), 2\iota_8, \eta_8\}=r_8^6(7)\circ\{2\iota_8, \eta_8, 2\iota_9\}=r_8^6(7)\circ\eta_8^2$ (c.f. [1], p. 84) $=i^{7,6}k^6\circ u_{10}^3+4r_{10}^5(7)=0$ or $4r_{10}^5(7)$ (c.f. 12.12 (vi) and 12.13)

Hence, $\{r_8^6(7), 2\iota_8, \eta_8\}\subset(2r_{10}^5(7))\subset i^{7,6}\circ G_2$, i.e. for any null homotopy A_1 of $r_8^6(7)\circ(2\iota_8)$ $=i^{7,6}\circ 2r_8^6$ $\{i^{7,6}, A_1, 2r_8^6, A_2, \eta_8\}=i^{7,6}\circ\alpha$ for some $\alpha\in G_2$. Let A_2' be a null homotopy of $(2r_8^6)\circ\eta_8$ such that $\partial(A_2', A_2)=\alpha$, then we have $\{i^{7,6}, A_1, 2r_8^6, A'_2, \eta_8\}=\{i^{7,6}, A_2, 2r_8^6, A_2, \eta_8\}-i^{7,6}\circ\partial(A_2', A_2)=0$, and $\{2r_8^6, A_2', \eta_8, A_3, 4\iota_9\}=\{2r_8^6, A_2', \eta_8, A_3, 4\iota_9\}-\partial(A_2', A_2)\circ(4\iota_{10})=0$. Thus, (12.16) is admissible. It is a union of cosets of the subgroup $(r_8^6(7)\circ\nu_8)+4\pi_{11}(R_7)$.

It follows from (6.14) that $-p_*\{i^{7,6}, 2r_8^6, \eta_8, 4\iota_9\}=-p_*\{i^{7,6}, \gamma^6\circ\nu_5, \eta_8, 4\iota_9\}\subset\{\nu_6, \eta_9, 4\iota_{10}\}$.

Hence, there exists an element $r_{11}^7\in\{i^{7,6}, 2r_8^6, \eta_8, 4\iota_9\}$ such that $p_*r_{11}^7=2[\iota_6, \iota_6]$.

The other generators of $\pi_q(R_7)$ which do not belong to $i^{7,6}\circ\pi_q(R_7)$ are obtained as follows:

$$\lambda_7^7\circ\eta_8,\ \lambda_7^7\circ\eta_8^2. \tag{12.17}$$

All the generators of $\pi_q(R_8)$ are obtained by the formula

$$\pi_q(R_8)=\tau_{7*}^8\pi_q(S^7)+\lambda_7^7(8)_*\pi(S^7). \tag{12.18}$$

It is well known that

$$\gamma_7^8=2\tau_7^8+\lambda_7^7(8). \tag{12.19}$$

It follows that

$$\lambda_7^7(9)=2\tau_7^8(9). \tag{12.20}$$

We shall show that

$$(12.21)\qquad \begin{cases} \text{(i)} & k^8\circ\omega'^4_7=2\tau^8_7+2\lambda^7_7(8), \\ \text{(ii)} & k^8\circ\gamma'^4_8=\tau^8_7\circ\eta_7+r^6_8(8), \\ \text{(iii)} & \gamma^9=\tau^8_7(9)\circ\eta_7+r^6_8(9),\ \ r^6_8(10)=\tau^8_7(10)\circ\eta_7, \\ \text{(iv)} & k^8\circ l^4\circ\bar{\gamma}''^2_{10}=\tau^8_7\circ\nu_7+r^5_{10}(8), \\ \text{(v)} & r^5_{10}(9)=\pm 2\tau^8_7(9)\circ\nu_7. \end{cases}$$

PROOF. (i) $k^8\circ\omega'^4_7=\tau^8_7p_*(k^8\circ\omega'^4_7)+i^{8,6}\circ q'(\omega'^4_7)=\tau^8_7(2\iota_7)+i^{8,6}\circ(r^6_7)=2\tau^8_7+2\lambda^7_7(8)$.

(ii) $k^8\circ\gamma'^4_8=\tau^8_7p_*(k^8\circ\gamma'^4_8)+i^{8,6}\circ q'(\gamma'^4_8)=\tau^8_7\circ\eta_7+r^6_8(8)$.

(iii) follows from (12.1).

(iv) $k_8\circ l^4\circ\bar{\gamma}''^2_{10}=\tau^8_7\circ p_*(k^8\circ l^4\circ\bar{\gamma}''^2_{10})+i^{8,5}q_*(\bar{\gamma}''^2_{10})=\tau^8_7(\nu_7)+r^5_{10}(8)$.

(v) $\gamma^8_7\circ\nu_7=2\tau^8_7\circ\nu_7+\lambda^7_7(8)\circ\nu_7=2\tau^8_7\circ\nu_7+mr^5_{10}(8)$ for some odd m, hence, $r^5_{10}(9)=2m\,\tau^8_7(9)=\pm 2\tau^8_7\circ\nu_7$. q.e.d.

$$(12.22)\qquad \begin{cases} \textit{The original element of } \pi_q(R_{10})(q\leqq 13) \textit{ is } \gamma^{10}_8\in\{i^{10,9},\gamma^9_8,2\iota_8\}, \\ \pi_{12}(R_{10})=Z_4=(\gamma^{10}_9\circ\nu_9), \quad \textit{and} \\ \gamma^{10}_9\circ\eta_9=\tau^8_7(10)\circ\nu_7. \end{cases}$$

(12.22) follows from the homotopy sequence of the bundle R_{11}.

(12.23) *There exists an original element* $r^{11}_{12}\in\{i^{11,10},\gamma^{10}_9,\eta^2_9\}$ mod 0, *such that* $p_*r^{11}_{12}=\eta^2_{10}$.

Indeed, $\gamma^{10}_9\circ\eta^2_9=\tau^8_7(10)\circ\nu_7\circ\eta_{10}=0$, and $i^{11,10}\pi_{12}(R_{10})+\pi_{10}(R_{11})\circ\eta^2_{10}=(i^{11,10}\circ\gamma^{10}_9\circ\nu_9)+(\tau^8_7(11)\circ\nu_7\circ\eta^2_{10})=0$.

$p_*r^{11}_{12}=\eta^2_{10}$ is obvoious by (12.2). q.e.d.

The other generaters of $\pi_q(R_n)$ $(11\leqq n\leqq 13)$ which do not belong to $i^{m,n-1}\circ\pi_q(R_{n-1})$ are listed as follows:

$$(12.24)\qquad \gamma^{12}_{11},\ k^{12}\circ\gamma'^6_{12},\ r^{11}_{12}\circ\eta_{12},\ k^{12}\circ\gamma'^6_{12},\ \gamma^{14}_{13}.$$

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References

[1] H. Toda: Composition Method in homotopy groups, Princeton, 1962.
[2] S. T. Hu: Homotopy theory, Academic Press, 1959.
[3] W. D. Barcus & M. G. Barratt: On the homotopy classification of the extensions of a fixed map, Trans. Amer. Math. Soc. **88**, (1958), 57-74.
[4] I. M. James: Reduced product spaces, Ann. Math. **62** (1955), 170-197.
[5] I. M. James: On the suspension triad, Ann. Math. **63** (1956), 407-429.
[6] I. M. James: On spaces with a multiplication, Pacific Jour. Math. **7** (1957), 1083-1100.
[7] N. E. Steenrod: The topology of fibre bundles, Princton, 1951.
[8] N. E. Steenrod: Cohomology operations, Princeton, 1962.
[9] R. Bott: The stable homotopy of the classical groups, Proc. Nat. Acad. Sci. U. S. A., **43** (1957), 933-935.
[10] M. A. Kervaire: Some nonstable homotopy groups of Lie grotps, Illinois Jour. Math., **4** (1960), 161-169.
[11] H. Toda: A topological proof of theorems of Bott and Borel-Hirzebruch for homotopy groups of unitary groups, Memoirs Coll. Sci. Univ. Kyoto, **32** (1959), 103-120.
[12] H. Toda: Sur les groupes d'homotopie des sphères, C. R. (Paris), **240** (1955), 42-44.
[13] H. Toda: Quelques tables des groupes d'homotopy des groupes de Lie, C. R. (Paris), **241** (1955), 922-923.
[14] H. Matunaga: On the groups $\pi_{2n+i}(U(n))$ for $i=3, 4$, and 5. Errata and Correction, Memoirs Fac. Sci. Kyushu Univ. **15** (1961), 72-81.
[15] H. Matunaga: Applications of functional cohomology operations to the calculus of $\pi_{2n+1}(U(n))$ for $i=6$ and 7, $n\geqq 4$, Memoirs Fac. Sci. Kyushu Univ. **17** (1963), 29-62.
[16] K. Ôguchi: 2-primary components of the homotopy groups of some Lie groups, Proc. Japan Acad., **38** (1962), 235-238.
[17] K. Ôguchi: 2-primary components of the homotopy groups of sphers and Lie groups, ibid, 619-620.

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