On the Hilbert-Siegel modular group and abelian varieties II

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Introduction

Let F be a totally real algebraic number field of finite degree r over $\boldsymbol{Q}$, $\mathfrak{o}$ the ring of integers of F. We denote by G the subgroup of $GL(2n, F)$, n being any natural number, defined as follows:

$$G=\{\sigma\in GL(2n, F) | \sigma J {}^t\sigma=m(\sigma)J,\ m(\sigma)\in F\}\ ,$$

where J is the matrix $\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ with 1_n=the unit matrix of degree n. (G may be called the symplectic group of order n with similitudes in F.)

Let $\mathfrak{p}$ be any finite or infinite prime spot of F and $F_{\mathfrak{p}}$ the $\mathfrak{p}$-completion of F. Let $\mathfrak{o}_{\mathfrak{p}}$ be the ring of integers in $F_{\mathfrak{p}}$. We define

$$G_{\mathfrak{p}}=\{\sigma_{\mathfrak{p}}\in GL(2n, F_{\mathfrak{p}}) | \sigma_{\mathfrak{p}} J {}^t\sigma_{\mathfrak{p}}=m(\sigma_{\mathfrak{p}})J,\ m(\sigma_{\mathfrak{p}})\in F_{\mathfrak{p}}\}\ ,$$
$$\mathfrak{U}_{\mathfrak{p}}=\{\sigma_{\mathfrak{p}}\in GL(2n, \mathfrak{o}_{\mathfrak{p}}) | \sigma_{\mathfrak{p}} J {}^t\sigma_{\mathfrak{p}}=m(\mathfrak{p})J,\ m(\sigma_{\mathfrak{p}}): \mathfrak{p}\text{-unit}\}$$

and denote by $\boldsymbol{J}_G$ the idelization of G, i.e., the restricted direct product of $\{G_{\mathfrak{p}}\}_{\mathfrak{p}\leqslant\infty}$ with respect to $\{\mathfrak{U}_{\mathfrak{p}}\}_{\mathfrak{p}\leqslant\infty}$. (For infinite $\mathfrak{p}$, we have $G_{\mathfrak{p}}=\mathfrak{U}_{\mathfrak{p}}$.) We put as usual $\boldsymbol{J}_G=\boldsymbol{J}_{G,\infty}\times\boldsymbol{J}_{G,0}$ with infinite and finite parts $\boldsymbol{J}_{G,\infty}$, $\boldsymbol{J}_{G,0}$ of $\boldsymbol{J}_G$ respectively, and also $\mathfrak{U}_0=\prod_{\mathfrak{p}<\infty}\mathfrak{U}_{\mathfrak{p}}$.

Two $\mathfrak{o}$-lattices $\mathfrak{M}$, $\mathfrak{N}$ of the $2n$-dimensional row vector space $\mathfrak{V}=\mathfrak{V}(2n, F)$ over F will be called G-equivalent if there exists $\sigma\in G$ such that $\mathfrak{M}\sigma=\mathfrak{N}$. The $G_{\mathfrak{p}}$-equivalence of two $\mathfrak{o}_{\mathfrak{p}}$-lattices $\mathfrak{M}_{\mathfrak{p}}$, $\mathfrak{N}_{\mathfrak{p}}$ of $\mathfrak{U}_{\mathfrak{p}}=\mathfrak{U}(2n, F_{\mathfrak{p}})$ will be similarly defined. $\mathfrak{M}$, $\mathfrak{N}$ are said to belong to the same genus if the $\mathfrak{p}$-completions $\mathfrak{M}_{\mathfrak{p}}$, $\mathfrak{N}_{\mathfrak{p}}$ are $G_{\mathfrak{p}}$-equivalent for all $\mathfrak{p}$. Then it will be proved in the first part of §1, that every genus of $\mathfrak{o}$-lattices consists of just h G-equivalence classes, where h is the class number of F (Theorem 1).

According to this result, $\boldsymbol{J}_{G,0}$ can be decomposed into double cosets in the following form:

$$(*)\qquad \boldsymbol{J}_{G,0}=\bigcup_{\lambda=1}^{h} Gx_{\lambda}\mathfrak{U}_0\ , \qquad x_{\lambda}=(\cdots, x_{\lambda,\mathfrak{p}}, \cdots)_{\mathfrak{p}<\infty}\ ,$$

where $x_{1,\mathfrak{p}}=1_n$, hence the ideal $(m(x_1))=\bigcap_{\mathfrak{p}}(F\cap m(x_{1,\mathfrak{p}}))$ of F is just $\mathfrak{o}$.

We define

$$\mathfrak{U}_\lambda = x_\lambda \mathfrak{U}_0 x_\lambda^{-1}, \qquad \mathfrak{L}_{\lambda,\mathfrak{p}} = \mathfrak{B}(2n, \mathfrak{g}_\mathfrak{p}) x_{\lambda,\mathfrak{p}}^{-1}$$

for every $\mathfrak{p}$, λ and put

$$\mathfrak{L}_\lambda = \bigcap_{\mathfrak{p}} (\mathfrak{L}_{\lambda,\mathfrak{p}}) \cap \mathfrak{B}(2n, F))$$

$$\Gamma_\lambda = \{\sigma \in GL(2n, F) | \mathfrak{L}_\lambda \sigma = \mathfrak{L}_\lambda, \sigma J {}^t\sigma = J\}$$

for every λ. Γ_1 is the Hilbert-Siegel modular group and Γ_λ the Hilbert-Siegel para-modular group of type $\mathfrak{L}_\lambda$ for $\lambda > 1$.

We shall then prove a so-called approximation theorem (Proposition 7.3): $\mathfrak{U}_\mu \alpha \mathfrak{U}_\lambda = \mathfrak{U}_\mu \alpha \Gamma_\lambda = \Gamma_\mu \alpha \mathfrak{U}_\lambda$ for $\alpha \in G$. Now the Hecke ring $\mathfrak{R}$ attached to G is defined with $J_{G,0}$ and $\mathfrak{U}_0$ and the transformation sets $\mathfrak{R}_{\lambda\mu}$ are defined with Γ_λ, Γ_μ and G following Shimura [1]. It will be proved that $\mathfrak{R}$ is commutive.

The approximation theorem establishes a connection between $\mathfrak{R}$ and $\mathfrak{R}_{\lambda\mu}$. (Proposition 8.1.) In § 2 we consider the polarized abelian varieties of type $\mathfrak{g}$ with data $\mathfrak{M}$, U, V and P constructed in [4].

Let $\boldsymbol{D} = \boldsymbol{D}_\infty \times \boldsymbol{D}_0$ be an element of $\boldsymbol{J}_G$; we may write

$$\boldsymbol{D}_\infty = (\boldsymbol{D}^{(1)}, \cdots, \boldsymbol{D}^{(r)}), \qquad \boldsymbol{D}^{(i)} = \left(\begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & D^{(1)} \end{pmatrix}, \cdots, \begin{pmatrix} A^{(r)} & B^{(r)} \\ C^{(r)} & D^{(r)} \end{pmatrix}\right),$$

Let η be an element of F and $m(\boldsymbol{D}_\infty)\eta > 0$ (totally positive), i.e., $m(\boldsymbol{D}^{(i)})\eta^{(i)} > 0$ for $i = 1, \cdots, r$, where $\eta^{(i)}$ is the i-th conjugate of η.

Now we put

$$(**) \qquad \mathfrak{M} = \bigcap_{\mathfrak{p} < \infty} (\mathfrak{B}(2n, \mathfrak{g}_\mathfrak{p}) \boldsymbol{D}_\mathfrak{p}^{-1} \cap \mathfrak{B})$$

$$\begin{pmatrix} U \\ V \end{pmatrix} = \left(\begin{pmatrix} \sqrt{-1} A^{(1)} + B^{(1)} \\ \sqrt{-1} C^{(1)} + D^{(1)} \end{pmatrix}, \cdots, \begin{pmatrix} \sqrt{-1} A^{(r)} + B^{(r)} \\ \sqrt{-1} C^{(r)} + D^{(r)} \end{pmatrix}\right) \quad \text{and} \quad P = J.$$

Then we can show that the following conditions are satisfied:

$$\eta^{(i)-1} ({}^tU^{(i)}\, {}^tV^{(i)}) {}^tJ^{-1} \begin{pmatrix} U^{(i)} \\ V^{(i)} \end{pmatrix} = 0 \qquad \text{for every } i,$$

$$-\sqrt{-1}\, \eta^{(i)-1} ({}^tU^{(i)}\, {}^tV^{(i)}) {}^tJ^{-1} \begin{pmatrix} \overline{U}^{(i)} \\ \overline{V}^{(i)} \end{pmatrix} > 0 \qquad \text{for every } i.$$

Therefore we can construct a polarized abelian variety with these data $\mathfrak{M}$, U, V, η, and J and furthermore, conversely we can show that every such abelian variety can be constructed in this way from some element $\boldsymbol{D}$ of $\boldsymbol{J}_G$ (Theorem 5).

We shall show furthermore that isomorphism classes of these abelian varieties and double cosets $G \backslash \boldsymbol{J}_G / \mathfrak{K}$ are in bijective correspondence, where

$$\mathfrak{K}_\infty = \left\{ \boldsymbol{D}_\infty = \left(\begin{pmatrix} -S^{(1)} & T^{(1)} \\ T^{(1)} & S^{(1)} \end{pmatrix}, \cdots, \begin{pmatrix} S^{(r)} & T^{(r)} \\ -T^{(r)} & S^{(r)} \end{pmatrix}\right) \in \boldsymbol{J}_{G,\infty} \right\}$$

and

$$\mathfrak{K}=\mathfrak{K}_{\infty}\times\mathfrak{U}_0 . \qquad \text{(Theorem 6.)}$$

In § 3, we define h abelian varieties A_λ by means of x_λ $(\lambda=1,\cdots,h)$ in (*), i.e., $\mathfrak{M}$ defining A_λ is $\mathfrak{L}_\lambda$ (c. f. (**)), and we put $A=A_1\times\cdots\times A_h$. Then we shall show that every element of the Hecke ring defines an isogeny of A (Theorem 7).

In [4], we considered a parametrization of a system of polarized abelian varieties by means of elements of the Hilbert-Siegel upper half-plane, which amounts to considering the maximal compact sub-group of $J_{G,\infty}$. Now that we are dealing with $\mathfrak{K}_\infty$ amounts to saying that a parametrization of a similar system of abelian varieties by means of elements of the following symmetric space (c. f., [1], [3]) is obtained:

$$\mathscr{H}(n,r)=\{Z=(Z^{(1)},\cdots,Z^{(r)})\,|\,Z^{(i)}:$$
$$n\text{-}n \text{ complex symmetric matrix and } \operatorname{Im} Z^{(i)}>0 \text{ or } <0\}.$$

$\mathscr{H}(n,r)$ consists of 2^r connected components, each of which is upper or lower Siegel's half-plane.

Many of the ideas developed in this paper were suggested to the author by Professor G. Shimura in two lectures on "Siegel modular groups" at Komaba (campus of the College of General Education of Tokyo University) and on "Arithmetic of Algebraic groups" at Hongo (campus of Faculty of Science of Tokyo University), particularly in the former lecture, and also in the seminar of Professors I. Satake and G. Shimura. The approximation theorem was first formulated by Shimura and proved for our case by me in the seminar; it was then proved in more general case in Hongo lecture by Shimura. In his Komaba lecture, Shimura defined the Hecke ring, constructed abelian varieties attached to Siegel modular groups and proved that every element of the Hecke ring defines an isogeny of a certain abelian variety. This result is yet unpublished, but his recent paper [1] gives allied results on the unit groups of indefinite quaternion algebra over a totally real algebraic number field. In [1], Shimura developes the theory of Dirichlet series attached to this kind of groups, proves Euler product formula and functional equations. These results belong to holomorphic theory of automorphic forms, but he suggests that an analogous theory would be also valid for non-holomorphic automorphic forms which are eigenfunctions of invariant differential operators.

The author is intending to pursue the study of our group G in case $n=1$. In a subsequent paper we shall define a certain class of invariant differential operators on $J_{G,\infty}$ generalizing Laplacians, and study some eigenfunctions of these operators, which are not always holomorphic and have $\prod_{i=1}^{r}(c^{(i)}z^{(i)}+d^{(i)})^{n_i}(c^{(i)}\bar{z}^{(i)}+d^{(i)})^{m_i}$, n_i, m_i

$\in \boldsymbol{Z}$, as automorphic factors. Their "Mellin transform" defines Dirichlet series which represents "formal Dirichlet series" with Hecke operators as coefficients. We shall consider the "Euler product formula" and the "functional equation" for this "Dirichlet series".

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§ 1. Hecke ring

1. Let F be an algebraic number field of finite degree r over $\boldsymbol{Q}$ and $\mathfrak{g}$ the ring of integers in F. Denote by $\mathfrak{V}=\mathfrak{V}(m, F)$ the row vector space of dimension m over F. By a lattice in $\mathfrak{V}$, we shall mean a free $\boldsymbol{Z}$-submodule $\mathfrak{L}$ of $\mathfrak{V}$ of rank mr such that $F\mathfrak{L}=\mathfrak{V}$. If $\mathfrak{g}\mathfrak{L}\subset\mathfrak{L}$, we call $\mathfrak{L}$ a $\mathfrak{g}$-lattice.

Let $\mathfrak{L}$ be a $\mathfrak{g}$-lattice. Then it is well-known that $\mathfrak{L}$ has a basis $\{x_i\}$ such that

$$\mathfrak{L}=\mathfrak{g}x_1+\cdots+\mathfrak{g}x_{m-1}+\mathfrak{a}x_m \qquad (*)$$

where $\mathfrak{a}$ is an ideal of F.

For two $\mathfrak{g}$-lattices $\mathfrak{L}$, $\mathfrak{M}$ in $\mathfrak{V}$, we say that $\mathfrak{L}$ is equivalent to $\mathfrak{M}$ if there exists a regular element σ of the ring $\mathcal{M}(\mathfrak{V})$ of endomorphisms of $\mathfrak{V}$, such that $\mathfrak{L}\cdot\sigma=\mathfrak{M}$. Suppose that $\mathfrak{L}$ is of the form (*) and also $\mathfrak{M}$ is of the following form, with a suitable basis $\{y_i\}$ and an ideal $\mathfrak{b}$ in F

$$\mathfrak{M}=\mathfrak{g}y_1+\cdots+\mathfrak{g}y_{m-1}+\mathfrak{b}y_m\,.$$

Then, we know that $\mathfrak{L}$ is equivalent to $\mathfrak{M}$ if and only if $\mathfrak{a}$ and $\mathfrak{b}$ belong to one and the same ideal class in F.

Let $\mathfrak{L}$ be a $\mathfrak{g}$-lattice. Put $\mathcal{M}_{\mathfrak{g}}(\mathfrak{L})$ (ring of endomorphisms of $\mathfrak{L}$ over $\mathfrak{g}$) $=\mathfrak{O}(\mathfrak{L})$, then $\mathfrak{O}(\mathfrak{L})$ is a maximal order in $\mathcal{M}(\mathfrak{V})$. $\mathfrak{O}(\mathfrak{L})$ is called the right order of $\mathfrak{L}$. Let $\mathfrak{A}$ be a $\mathfrak{g}$-lattice in $\mathcal{M}(\mathfrak{V})$, i.e., a free Z-submodule of $\mathcal{M}(\mathfrak{V})$ of rank m^2r such that $F\mathfrak{A}=\mathcal{M}(\mathfrak{V})$. Put

$$\mathfrak{O}_l=\{\sigma\in\mathcal{M}(\mathfrak{V})\,|\,\sigma\mathfrak{A}\subset\mathfrak{A}\} \quad\text{and}\quad \mathfrak{O}_r=\{\sigma\in\mathcal{M}(\mathfrak{V})\,|\,\mathfrak{A}\sigma\subset\mathfrak{A}\}\,.$$

The $\mathfrak{O}_l$ and $\mathfrak{O}_r$ are orders and called a left and right order of $\mathfrak{A}$, respectively. We know that $\mathfrak{O}_l$ is maximal if $\mathfrak{O}_r$ is so and vice versa. Such a $\mathfrak{g}$-lattice $\mathfrak{A}$ in $\mathcal{M}(\mathfrak{V})$ is called normal.

$\mathfrak{L}$, $\mathfrak{M}$ be $\mathfrak{g}$-lattices in $\mathfrak{V}$. Put $\mathfrak{A}=\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{L}, \mathfrak{M})$ (module of homomorphisms of $\mathfrak{L}$ into $\mathfrak{M}$ over $\mathfrak{g}$), then $\mathfrak{A}$ is a normal $\mathfrak{g}$-lattice in $\mathcal{M}(\mathfrak{V})$ and its left and right orders are $\mathfrak{O}(\mathfrak{L})$ and $\mathfrak{O}(\mathfrak{M})$, respectively. Let $\mathfrak{O}$ be a maximal order in $\mathcal{M}(\mathfrak{V})$ and $\mathfrak{A}$ a $\mathfrak{g}$-lattice in $\mathcal{M}(\mathfrak{V})$ whose left order is $\mathfrak{O}$ (called a left $\mathfrak{O}$-lattice in $\mathcal{M}(\mathfrak{V})$). Let $\mathfrak{L}$ be a $\mathfrak{g}$-lattice in $\mathfrak{V}$ such that $\mathfrak{O}(\mathfrak{L})=\mathfrak{O}$. Put $\mathfrak{L}\mathfrak{A}=\mathfrak{M}$, then $\mathfrak{M}$ is a $\mathfrak{g}$-lattice in $\mathfrak{V}$ and $\mathfrak{A}=\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{L},\mathfrak{M})$. Therefore, for fixed $\mathfrak{L}$, left $\mathfrak{O}$-lattices $\mathfrak{A}$ in $\mathcal{M}(\mathfrak{V})$ and $\mathfrak{g}$-lattices $\mathfrak{M}$ in $\mathfrak{V}$ are in one-to-one correspondence. We say that, for two left $\mathfrak{O}$-lattices $\mathfrak{A}$, $\mathfrak{B}$ in $\mathcal{M}(\mathfrak{V})$, $\mathfrak{A}$ is equivalent to $\mathfrak{B}$ if there exists a regular element σ of $\mathcal{M}(\mathfrak{V})$ such that

$$\mathfrak{A}\sigma = \mathfrak{B}\,.$$

By this equivalence relation, we can classify the set of left $\mathfrak{O}$-lattices in $\mathcal{M}(\mathfrak{V})$ for a fixed maximal order $\mathfrak{O}$ in $\mathcal{M}(\mathfrak{V})$. Then the number of classes is equal to h (the class number of F).

Now, $\mathfrak{V}=\mathfrak{V}(2n,F)=Fx_1+\cdots+Fx_m$. Every $\sigma\in\mathcal{M}(\mathfrak{V})$ defines a matrix $\sigma=(\sigma_{ij})$ by putting $x_i\sigma=\sum_j\sigma_{i,j}x_j$. So we identify $\mathcal{M}(\mathfrak{V})$ with the algebra $\mathcal{M}_m(F)$ of all m-m matrices in F. Hence the set of all regular elements $GL(\mathfrak{V})$ is identified with the general linear group $GL(m,F)$ over F.

LEMMA 1.1. *Let $\mathfrak{L}$ be a $\mathfrak{g}$-lattice in $\mathfrak{V}=\mathfrak{V}(2n,F)$. Let $P(\ ,\ )$ be an F-bilinear non-degenerate alternating form on $\mathfrak{V}$. Then, $\mathfrak{L}$ has the following basis $\{x_i,y_i\}$;*

$$\mathfrak{L}=\mathfrak{g}x_1+\cdots+\mathfrak{g}x_n+\mathfrak{a}_1y_1+\cdots+\mathfrak{a}_ny_n\,,$$
$$P(x_i,y_j)=\delta_{ij}\,,\qquad P(x_i,x_j)=P(y_i,y_j)=0$$

the $\mathfrak{a}_i$ are ideals in F such that $\mathfrak{a}_1\supset\cdots\supset\mathfrak{a}_n$ and are uniquely determined by $\mathfrak{L}$ and $P(\ ,\ )$.

The ideals $\mathfrak{a}_i$ are called elementary divisors of $\mathfrak{L}$. A basis $\{x_i,y_i\}$ of $\mathfrak{V}$ such that

$$P(x_i,y_j)=\delta_{ij}\,,\qquad P(x_i,x_j)=P(y_i,y_j)=0$$

is called a canonical basis of $\mathfrak{V}$.

We say that a $\mathfrak{g}$-lattice $\mathfrak{L}$ is *maximal* if the elementary divisors of $\mathfrak{L}$ are equal to one and the same ideal $\mathfrak{a}$; so $\mathfrak{L}$ can be expressed in the following form using a canonical basis $\{x_i,y_i\}$

$$\mathfrak{L}=\mathfrak{g}x_1+\cdots+\mathfrak{g}x_n+\mathfrak{a}y_1+\cdots+\mathfrak{a}y_n\,.$$

We denote, for ideals $\mathfrak{b}_1,\cdots,\mathfrak{b}_{2n}$ in F,

$$[\mathfrak{b}_1,\cdots,\mathfrak{b}_{2n}]=\{(b_1,\cdots,b_{2n})\in\mathfrak{V}(2n,F)\,|\,b_i\in\mathfrak{b}_i\}\,.$$

2. Let $\{G_i\}$ be a collection of locally compact groups for $i\in I$, $I=I_\infty\cup I_0$ a set of indices with $I_\infty=(i_1,\cdots,i_s)$ a finite subset of I, $\cup$ denoting a direct union. Let H_i be an open compact sub-group of G_i for every $i\in I_0$, and $G_i=H_i$ for $i=i_1,\cdots,i_s$. Put

$$G=\prod{}'G_i=\{x=(\cdots,x_i,\cdots)\in\prod G_i\,|\,x_i\in H_i \text{ except for a finite number of } i\}$$

and

$$H = \prod H_i .$$

We denote

$$G = G_\infty \times G_0 \qquad \text{with } G_\infty = \prod_{i \in I_\infty} G_i , \quad G_0 = \prod_{i \in I_0}{}' G_i ,$$

and

$$H = H_\infty \times H_0 \qquad \text{with } H_\infty = G_\infty , \quad H_0 = \prod_{i \in I_0}{}' H_i .$$

We introduced the direct product topology into H, so that H becomes locally compact, and the topology into G so that G/H becomes discrete. This topology is uniquely determined and G becomes locally compact. We call G with this topology the *restricted direct product* of $\{G_i\}_{i \in I}$.

Let F and $\mathfrak{g}$ be as in the preceeding section. For a prime ideal $\mathfrak{p}$ (finite or infinite) of F, we denote by $F_\mathfrak{p}$, $\mathfrak{g}_\mathfrak{p}$ the $\mathfrak{p}$-completion of F and $\mathfrak{g}$, respectively. Let $\boldsymbol{A}_F$ and $\boldsymbol{J}_F$ be the adele ring (ring of valuation vectors) and the idele group of F: namely $\boldsymbol{A}_F$ is the restricted direct product of $\{F_\mathfrak{p}\}_{\mathfrak{p} \leqslant \infty}$ with respect to $\{\mathfrak{g}_\mathfrak{p}\}_{\mathfrak{p} < \infty}$. $F_\mathfrak{p}{}^*$ denoting the multiplicative group of $F_\mathfrak{p}$, $\boldsymbol{J}_F$ is the restricted direct product of $\{F_\mathfrak{p}{}^*\}_{\mathfrak{p} \leqslant \infty}$ with respect to $\{\mathfrak{u}_\mathfrak{p}\}_{\mathfrak{p} < \infty}$ where $\mathfrak{u}_\mathfrak{p}$ is the group of $\mathfrak{p}$-units. (If $\mathfrak{p}$ is infinite, then $\mathfrak{u}_\mathfrak{p} = F_\mathfrak{p}{}^*$.) Put $\boldsymbol{A}_\mathfrak{p} = \prod \mathfrak{g}_\mathfrak{p}$ and $\boldsymbol{J}_\mathfrak{u} = \prod \mathfrak{u}_\mathfrak{p}$.

As above we decompose

$$\boldsymbol{A}_F = \boldsymbol{A}_{F,\infty} \times \boldsymbol{A}_{F,0} , \qquad \boldsymbol{J}_F = \boldsymbol{J}_{F,\infty} \times \boldsymbol{J}_{F,0} ,$$
$$\boldsymbol{A}_\mathfrak{g} = \boldsymbol{A}_{\mathfrak{g},\infty} \times \boldsymbol{A}_{\mathfrak{g},0} \quad \text{and} \quad \boldsymbol{J}_\mathfrak{u} = \boldsymbol{J}_{\mathfrak{u},\infty} \times \boldsymbol{J}_{\mathfrak{u},0} ,$$

where $\boldsymbol{J}_{F,\infty} = \boldsymbol{J}_{\mathfrak{u},\infty}$ and $\boldsymbol{A}_{F,\infty} = \boldsymbol{A}_{\mathfrak{g},\infty}$.

We identify as usual the principal adele and idele with F and F^* respectively, and denote them again by F and F^*.

For an element $\mathfrak{y} = (\mathfrak{y}_\mathfrak{p})$ of $\boldsymbol{J}_F$, $\bigcap_{\mathfrak{p} < \infty} (\mathfrak{y}_\mathfrak{p} \cdot \mathfrak{g}_\mathfrak{p} \cap F)$ is an ideal of F, denoted by $(\mathfrak{y})$. This ideal is determined up to the multiplication by elements of $\boldsymbol{J}_\mathfrak{u}$. Conversely, for an ideal $\mathfrak{a}$ of F, there exists an element $\mathfrak{y}$ of $\boldsymbol{J}_F$ such that $\mathfrak{a} = (\mathfrak{y})$. Hence we have an isomorphism (with topology)

$$\mathfrak{G}(F) \cong \boldsymbol{J}_F / \boldsymbol{J}_\mathfrak{u}$$

where $\mathfrak{G}(F)$ is the ideal group of F. Furthermore, denoting the ideal class group by $\mathfrak{C}(F)$, we have the isomorphism

$$\mathfrak{C}(F) \cong F^* \backslash \boldsymbol{J}_F / \boldsymbol{J}_\mathfrak{u} .$$

The order of $\mathfrak{C}(F)$ is h, the ideal class number of F.

Let $\mathfrak{L}$ be a $\mathfrak{g}$-lattice in $\mathfrak{V} = \mathfrak{V}(m, F)$. Put $\mathfrak{L}_\mathfrak{p} = \mathfrak{g}_\mathfrak{p} \cdot \mathfrak{L}$. Then $\mathfrak{L}_\mathfrak{p}$ is a $\mathfrak{g}_\mathfrak{p}$-lattice in

$\mathfrak{B}(m, F_{\mathfrak{p}})$. $\mathfrak{L}_{\mathfrak{p}}$ is called the $\mathfrak{p}$-completion of $\mathfrak{L}$.

The following Lemma is well-known, but for readers, we shall state its proof.

LEMMA 2.1. *For every* $\mathfrak{p}$, *let* $\mathfrak{N}^{(\mathfrak{p})}$ *be a* $\mathfrak{g}_{\mathfrak{p}}$*-lattice in* $\mathfrak{B}(m, F_{\mathfrak{p}})$ *and* $\mathfrak{M}$ *be a* $\mathfrak{g}$*-lattice in* $\mathfrak{B}$. *Then there exists a* $\mathfrak{g}$*-lattice* $\mathfrak{N}$ *such that* $\mathfrak{N}_{\mathfrak{p}}=\mathfrak{N}^{(\mathfrak{p})}$ *for every* $\mathfrak{p}$ *if and only if* $\mathfrak{M}_{\mathfrak{p}}=\mathfrak{N}^{(\mathfrak{p})}$ *for almost all* $\mathfrak{p}$.

PROOF*. We shall first define an ideal $\beta(\mathfrak{M})$ of F for every torsion $\mathfrak{g}$-module $\mathfrak{M}$. Let

$$\mathfrak{M}=\mathfrak{M}_0\supset\mathfrak{M}_1\supset\cdots\supset\mathfrak{M}_t=\{0\}, \qquad \mathfrak{M}_{i-1}/\mathfrak{M}_i: \quad \text{simple}$$

be a composition series of $\mathfrak{M}$. Then $\mathfrak{M}_{i-1}/\mathfrak{M}_i$ is isomorphic to $\mathfrak{g}/\mathfrak{p}_i$ with a prime ideal $\mathfrak{p}_i$ in F. Put $\beta(\mathfrak{M})=\mathfrak{p}_1\cdots\mathfrak{p}_t$. Then we see that

$$\beta(\mathfrak{M}/\mathfrak{N})_{\mathfrak{p}} = \beta(\mathfrak{M}_{\mathfrak{p}}/\mathfrak{N}_{\mathfrak{p}})$$

holds for any $\mathfrak{g}$-modules $\mathfrak{M}$, $\mathfrak{N}$.

Proof of "only if" part. Since $\beta(\mathfrak{M}+\mathfrak{N}/\mathfrak{M})_{\mathfrak{p}}=\beta(\mathfrak{M}_{\mathfrak{p}}+\mathfrak{N}_{\mathfrak{p}}/\mathfrak{M}_{\mathfrak{p}})$, we see that $\mathfrak{M}_{\mathfrak{p}}=\mathfrak{N}_{\mathfrak{p}}=\mathfrak{N}^{(\mathfrak{p})}$ for almost all $\mathfrak{p}$.

Proof of "if" part. Put $\mathfrak{N}=\bigcap_{\mathfrak{p}}\mathfrak{N}^{(\mathfrak{p})}$. Then there exist α, β such that $\alpha\mathfrak{N}^{(\mathfrak{p})}\subset\mathfrak{M}_{\mathfrak{p}}$ and $\beta\mathfrak{M}_{\mathfrak{p}}\subset\mathfrak{N}^{(\mathfrak{p})}$ for all $\mathfrak{p}$. Hence $\alpha\mathfrak{N}\subset\mathfrak{M}$ and $\beta\mathfrak{M}\subset\mathfrak{N}$. Thus $\mathfrak{N}$ is a $\mathfrak{g}$-lattice.

We shall show $\mathfrak{N}_{\mathfrak{p}}=\mathfrak{N}^{(\mathfrak{p})}$. It is obvious that $\mathfrak{N}_{\mathfrak{p}}\subset\mathfrak{N}^{(\mathfrak{p})}$. Conversely, for $x\in\mathfrak{N}^{(\mathfrak{p})}$, there exists $\alpha\neq0$ in $\mathfrak{g}$ such that $\alpha x\in\mathfrak{M}$. Let $(\alpha)=\mathfrak{p}^r\cdot\mathfrak{q}$ with $(\mathfrak{p},\mathfrak{q})=1$. Take an element β of $\mathfrak{g}$ such that $\beta\equiv1(\mathfrak{p})$ and $\beta\equiv0(\mathfrak{q})$. Then $\beta x\in\mathfrak{N}^{(\mathfrak{p})}$ and $\beta x=(\beta/\alpha)\alpha x$ where $\beta/\alpha\in\mathfrak{q}_{\mathfrak{p}'}$ for $\mathfrak{p}'$ such that $\mathfrak{p}'|\mathfrak{q}$ and $\alpha x\in\mathfrak{M}$. Hence $\beta x\in\mathfrak{N}_{\mathfrak{p}'}\subset\mathfrak{N}^{(\mathfrak{p}')}$. For $\mathfrak{p}'|\mathfrak{q}$, we see

$$\beta x\in\mathfrak{N}_{\mathfrak{p}'}\subset\mathfrak{N}^{(\mathfrak{p}')} \qquad \text{for } x\in\mathfrak{N}_{\mathfrak{p}'}.$$

Hence $\beta x\in\mathfrak{N}$. Since $\beta\equiv1(\mathfrak{p})$, we see $x\in\mathfrak{N}_{\mathfrak{p}}$. Hence

$$\mathfrak{N}^{(\mathfrak{p})} = \mathfrak{N}_{\mathfrak{p}}. \qquad \text{q. e. d.}$$

The following Lemma is also well-known.

LEMMA 2.2.

$$\bigcap_{\mathfrak{p}<\infty} (\mathfrak{L}_{\mathfrak{p}}\cap\mathfrak{B}) = \mathfrak{L}.$$

In the following, $\mathfrak{L}_0$ will denote the "finite part" of $\mathfrak{L}$. $\mathfrak{L}$ is determined by $\mathfrak{L}_0$ by Lemma 2.1 and Lemma 2.2.

3. Let $\mathfrak{B}=\mathfrak{B}(2n, F)$ and $P(x, y)$ be an F-bilinear non-degenerate alternating form defined on $\mathfrak{B}$. Put

$$G(\mathfrak{B}, P)=\{\sigma\in GL(\mathfrak{B})\,|\,P(x\sigma, y\sigma)=m(\sigma)P(x, y) \quad \text{for } x. y\in\mathfrak{B} \text{ with some } m(\sigma)\in F^*\},$$

*) This proof is due to Shimura's Seminar on "arithmetics of algebras."

and

$$G^0(\mathfrak{V}, P)=\{\sigma\in G(\mathfrak{V}, P)|m(\sigma) \text{ is a unit in } F\}.$$

$m(\sigma)$ is called the *multiplicator* of σ. Further, for a $\mathfrak{g}$-lattice $\mathfrak{L}$ in $\mathfrak{V}$, put

$$\Gamma(\mathfrak{L}, P)=\{\sigma\in G(\mathfrak{V}, P)|\mathfrak{L}\sigma\subset\mathfrak{L}\}$$

and

$$\Gamma^0(\mathfrak{L}, P)=\{\sigma\in G(\mathfrak{V}, P)|\mathfrak{L}\sigma=\mathfrak{L}\}.$$

Now by Lemma 1.1, the matrix of $P(x, y)$ can be transformed to $J=\begin{pmatrix} 0 & 1_n \\ -1_n & 0\end{pmatrix}$ by choosing a canonical basis. Then, we write $G(\mathfrak{V}, J)$ and $G^0(\mathfrak{V}, J)$ instead of $G(\mathfrak{V}, P)$ and $G^0(\mathfrak{V}, P)$, respectively. Similarly, we use notations $\Gamma(\mathfrak{L}, J)$ and $\Gamma^0(\mathfrak{L}, J)$ for $\mathfrak{L}$.

Two $\mathfrak{g}$-lattices $\mathfrak{M}$, $\mathfrak{N}$ in $\mathfrak{V}$ will be called $G(\mathfrak{V}, P)$-equivalent ($G^0(\mathfrak{V}, P)$-equivalent) if there exists $\sigma\in G(\mathfrak{V}, P)$ ($\in G^0(\mathfrak{V}, P)$) such that $\mathfrak{M}\sigma=\mathfrak{N}$. The $\Gamma(\mathfrak{L}, P)$- and $\Gamma^0(\mathfrak{L}, P)$-equivalence for $\mathfrak{M}, \mathfrak{N}\subset\mathfrak{L}$ and $G(\mathfrak{V}_\mathfrak{p}, P_\mathfrak{p})$-, $G^0(\mathfrak{V}_\mathfrak{p}, P_\mathfrak{p})$- and $\Gamma^0(\mathfrak{L}_\mathfrak{p}, P_\mathfrak{p})$-equivalence for $\mathfrak{g}_\mathfrak{p}$-lattices $\mathfrak{M}_\mathfrak{p}$, $\mathfrak{N}_\mathfrak{p}$ in $\mathfrak{V}_\mathfrak{p}$ will be similarly defined. If $\mathfrak{M}_\mathfrak{p}\sigma_\mathfrak{p}=\mathfrak{N}_\mathfrak{p}$, we denote by $m(\mathfrak{M}:\mathfrak{N})$ the ideal in F determined by the idele $(\cdots, m(\sigma_\mathfrak{p}), \cdots)$ of F.

4.

LEMMA 4.1. *Let $\mathfrak{r}$ be a principal ideal domain and k its quotient field. Let $P(x, y)$ be a k-bilinear non-degenerate alternating form defined on $\mathfrak{V}=\mathfrak{V}(2n, k)$. Let $\mathfrak{L}$, $\mathfrak{M}$ be maximal $\mathfrak{r}$-lattices in $\mathfrak{V}$ such that*

$$\mathfrak{L}=\mathfrak{r}u_1+\cdots+\mathfrak{r}u_n+\mathfrak{r}v_1+\cdots+\mathfrak{r}v_n,$$
$$P(u_i, v_j)=\delta_{ij}, \qquad P(u_i, u_j)=P(v_i, v_j)=0$$

and

$$\mathfrak{M}=\mathfrak{r}u_1'+\cdots+\mathfrak{r}u_n'+a\mathfrak{r}v_1'+\cdots+a\mathfrak{r}v_n',$$
$$P(u_i', v_j')=\delta_{ij}, \qquad P(u_i', u_j')=P(v_i', v_j')=0.$$

Then $\mathfrak{L}$, $\mathfrak{M}$ have the following basis $\{x_i, y_j\}$;

$$\mathfrak{L}=\mathfrak{r}x_1+\cdots+\mathfrak{r}x_n+\mathfrak{r}y_1+\cdots+\mathfrak{r}y_n,$$
$$\mathfrak{M}=\alpha_1\mathfrak{r}x_1+\cdots+\alpha_n\mathfrak{r}x_n+\beta_1\mathfrak{r}y_1+\cdots+\beta_n\mathfrak{r}y_n,$$
$$P(x_i, y_j)=\delta_{ij}, \qquad P(x_i, x_j)=P(y_i, y_j)=0$$

where

$$\alpha_i\beta_i=a \quad \textit{and} \quad \alpha_1\mathfrak{r}\supset\alpha_2\mathfrak{r}\supset\cdots\supset\alpha_n\mathfrak{r}\supset\beta_n\mathfrak{r}\supset\cdots\supset\beta_2\mathfrak{r}\supset\beta_1\mathfrak{r}.$$

This Lemma is well-known and rewritten as follows.

LEMMA 4.2. *$\mathfrak{r}$, k being as above, let σ be an element of $GL(2n, \mathfrak{r})$ such that*

$\sigma J^t\sigma=aJ$ with $a\in\mathfrak{r}$. Then there exist elements u, v of $Sp(n,\mathfrak{r})=\{u\in GL(2n,\mathfrak{r})$ $|uJ^tu=J\}$ with the following properties:

$$u\sigma v=\operatorname{diag}(\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_n),$$
$$\alpha_i,\beta_i\in\mathfrak{r}, \qquad \alpha_i\beta_i=a$$

and

$$\alpha_1\mathfrak{r}\supset\cdots\supset\alpha_n\mathfrak{r}\supset\beta_1\mathfrak{r}\supset\cdots\supset\beta_n\mathfrak{r}.$$

$\operatorname{diag}(\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_n)$ is called the diagonal form of σ.

For every $\mathfrak{p}$, $\mathfrak{g}_\mathfrak{p}$ is a principal ideal domain, so putting $\mathfrak{r}=\mathfrak{g}_\mathfrak{p}$, and $k=F_\mathfrak{p}$, we can apply the Lemma to our case. It is easily seen that a $\mathfrak{g}$-lattice $\mathfrak{L}$ is maximal if and only if $\mathfrak{g}_\mathfrak{p}$-lattice $\mathfrak{L}_\mathfrak{p}$ is maximal for every $\mathfrak{p}$, where $P(x,y)$ on $\mathfrak{V}(2n,F)$ is extended to a form $P_\mathfrak{p}(x,y)$ on $\mathfrak{V}(2n,F_\mathfrak{p})$ for every $\mathfrak{p}$.

We put

$$\Theta=\{\mathfrak{L}|\text{ maximal }\mathfrak{g}\text{-lattice in }\mathfrak{V}(2n,F)\text{ such that }\mathfrak{V}(2n,\mathfrak{g})\supset\mathfrak{L}\}.$$

Then by the maximality of $\mathfrak{L}_\mathfrak{p}$, the matrix of $P_\mathfrak{p}(x,y)$ is $a_\mathfrak{p}J$, $a_\mathfrak{p}\in\mathfrak{g}_\mathfrak{p}$ and $\mathfrak{L}_\mathfrak{p}=[\mathfrak{g}_\mathfrak{p},\cdots,\mathfrak{g}_\mathfrak{p},a_\mathfrak{p}\mathfrak{g}_\mathfrak{p},\cdots,a_\mathfrak{p}\mathfrak{g}_\mathfrak{p}]$ with a suitable canonical basis.

LEMMA 4.3. *A $\mathfrak{g}$-lattice $\mathfrak{L}$ belongs to Θ if and only if, for every $\mathfrak{p}$, $\mathfrak{V}(2n,\mathfrak{g}_\mathfrak{p})$ and $\mathfrak{L}_\mathfrak{p}$ are $\Gamma(\mathfrak{V}(2n,\mathfrak{g}_\mathfrak{p}),J)$-equivalent.*

PROOF.

$$\mathfrak{V}(2n,\mathfrak{g}_\mathfrak{p})=\mathfrak{g}_\mathfrak{p}x_1+\cdots+\mathfrak{g}_\mathfrak{p}x_1+\mathfrak{g}_\mathfrak{p}y_1+\cdots+\mathfrak{g}_\mathfrak{p}y_n$$

and the matrix of $P_\mathfrak{p}(x,y)$ is J. Since $\mathfrak{L}$ is contained in Θ,

$$\mathfrak{L}_\mathfrak{p}=\mathfrak{g}_\mathfrak{p}x_1'+\cdots+\mathfrak{g}_\mathfrak{p}x_n'+a_\mathfrak{p}\mathfrak{g}_\mathfrak{p}y_1'+\cdots+a_\mathfrak{p}\mathfrak{g}_\mathfrak{p}y_n' \qquad a_\mathfrak{p}\in\mathfrak{g}_\mathfrak{p}$$

and the matrix of $P_\mathfrak{p}(x',y')$ is $a_\mathfrak{p}J$.

Take a matrix $\sigma_\mathfrak{p}$ such that

$$x_i'=x_i\sigma_\mathfrak{p}=\sum_j x_j\sigma_{\mathfrak{p},i,j} \quad\text{and}\quad y_i'=y_i\sigma_\mathfrak{p}=\sum_j y_j\sigma_{\mathfrak{p},i,j}$$

then $\sigma_\mathfrak{p}\in\Gamma(\mathfrak{V}(2n,\mathfrak{g}_\mathfrak{p}),J)$, $m(\sigma_\mathfrak{p})=a_\mathfrak{p}$ and $\mathfrak{V}(2n,\mathfrak{g}_\mathfrak{p})\sigma_\mathfrak{p}=\mathfrak{L}_\mathfrak{p}$. The inverse is obvious.

Now put $\mathfrak{V}(\mathfrak{g})=\mathfrak{V}(2n,\mathfrak{g})$, so $\mathfrak{V}(\mathfrak{g})_\mathfrak{p}=\mathfrak{V}(2n,\mathfrak{g})_\mathfrak{p}$ can be identified with $\mathfrak{V}(\mathfrak{g}_\mathfrak{p})=\mathfrak{V}(2n,\mathfrak{g}_\mathfrak{p})$. Let $\mathfrak{L}$ be an element of Θ. Then by Lemma 4.3, for every $\mathfrak{p}$ there exists $\sigma_\mathfrak{p}\in\Gamma(\mathfrak{V}(\mathfrak{g}_\mathfrak{p}),J)$ stch that $\mathfrak{V}(\mathfrak{g}_\mathfrak{p})\sigma_\mathfrak{p}=\mathfrak{L}_\mathfrak{p}$ and $\sigma_\mathfrak{p}J^t\sigma_\mathfrak{p}=m(\sigma_\mathfrak{p})J$. By Lemma 4.2, there exist $u_\mathfrak{p},v_\mathfrak{p}\in\Gamma^0(\mathfrak{V}(\mathfrak{g}_\mathfrak{p}),J)$ such that

$$u_\mathfrak{p}\sigma_\mathfrak{p}v_\mathfrak{p}=\operatorname{diag}(\alpha_{1,\mathfrak{p}},\cdots,\alpha_{n,\mathfrak{p}},\beta_{1,\mathfrak{p}},\cdots,\beta_{n,\mathfrak{p}}),$$
$$\alpha_{1,\mathfrak{p}}\mathfrak{g}_\mathfrak{p}\supset\cdots\supset\alpha_{n,\mathfrak{p}}\mathfrak{g}_\mathfrak{p}\supset\beta_{n,\mathfrak{p}}\mathfrak{g}_\mathfrak{p}\supset\cdots\supset\beta_{1,\mathfrak{p}}\mathfrak{g}_\mathfrak{p}$$

and

$$\alpha_{i,\mathfrak{p}}\beta_{i,\mathfrak{p}} = m(\sigma_{\mathfrak{p}}) .$$

Then we write

$$(\mathfrak{V}(\mathfrak{g}):\mathfrak{L})_{\mathfrak{p}}=(\alpha_{1,\mathfrak{p}},\cdots,\alpha_{n,\mathfrak{p}},\beta_{1,\mathfrak{p}},\cdots,\beta_{n,\mathfrak{p}})$$

and

$$(\mathfrak{V}(\mathfrak{g}):\mathfrak{L})=(\cdots,(\mathfrak{V}(\mathfrak{g}):\mathfrak{L})_{\mathfrak{p}},\cdots) .$$

For two elements $\mathfrak{L}$, $\mathfrak{M}$ of Θ such that $\mathfrak{L}\supset\mathfrak{M}$, $(\mathfrak{L}:\mathfrak{M})$ is also defined: namely since $\mathfrak{V}(\mathfrak{g}_{\mathfrak{p}})\supset\mathfrak{L}_{\mathfrak{p}}\supset\mathfrak{M}_{\mathfrak{p}}$, there exist elements $\sigma_{\mathfrak{p}}$, $\tau_{\mathfrak{p}}$ such that $\mathfrak{V}(\mathfrak{g}_{\mathfrak{p}})\sigma_{\mathfrak{p}}=\mathfrak{L}_{\mathfrak{p}}$ and $\mathfrak{L}_{\mathfrak{p}}\tau_{\mathfrak{p}}=\mathfrak{M}_{\mathfrak{p}}$. Put $\mathfrak{M}'\sigma_{\mathfrak{p}}=\mathfrak{M}_{\mathfrak{p}}$. Then $\mathfrak{V}(\mathfrak{g}_{\mathfrak{p}})\sigma_{\mathfrak{p}}\tau_{\mathfrak{p}}\sigma^{-1}=\mathfrak{M}'_{\mathfrak{p}}$, so as above, we can take $u_{\mathfrak{p}}$, $v_{\mathfrak{p}}$ such that $u_{\mathfrak{p}}(\sigma_{\mathfrak{p}}\tau_{\mathfrak{p}}\sigma_{\mathfrak{p}})^{-1}v_{\mathfrak{p}}=\mathrm{diag}\,(\alpha_{1,\mathfrak{p}},\cdots,\alpha_{n,\mathfrak{p}},\beta_{1,\mathfrak{p}},\cdots,\beta_{n,\mathfrak{p}})$.

Hence we can write by a suitable canonical basis $\{x_i, y_i\}$

$$\mathfrak{V}(\mathfrak{g})_{\mathfrak{p}}=\mathfrak{g}_{\mathfrak{p}}x_1+\cdots+\mathfrak{g}_{\mathfrak{p}}x_n+\mathfrak{g}_{\mathfrak{p}}y_1+\cdots+\mathfrak{g}_{\mathfrak{p}}y_n$$

and

$$\mathfrak{M}'_{\mathfrak{p}}=\alpha_{1,\mathfrak{p}}\mathfrak{g}_{\mathfrak{p}}x_1+\cdots+\alpha_{n,\mathfrak{p}}\mathfrak{g}_{\mathfrak{p}}x_n+\beta_{1,\mathfrak{p}}\mathfrak{g}_{\mathfrak{p}}y_1+\cdots+\beta_{n,\mathfrak{p}}\mathfrak{g}_{\mathfrak{p}}y_n .$$

Then putting $x'_i=x_i\sigma_{\mathfrak{p}}$ and $y'_i=y_i\sigma_{\mathfrak{p}}$, we have

$$\mathfrak{L}_{\mathfrak{p}}=\mathfrak{g}_{\mathfrak{p}}x'_1+\cdots+\mathfrak{g}_{\mathfrak{p}}x'_n+\mathfrak{g}_{\mathfrak{p}}y'_1+\cdots+\mathfrak{g}_{\mathfrak{p}}y'_n$$

and

$$\mathfrak{M}_{\mathfrak{p}}=\alpha_{1,\mathfrak{p}}\mathfrak{g}_{\mathfrak{p}}x'_1+\cdots+\alpha_{n,\mathfrak{p}}\mathfrak{g}_{\mathfrak{p}}x'_n+\beta_{1,\mathfrak{p}}\mathfrak{g}_{\mathfrak{p}}y'_1+\cdots+\beta_{n,\mathfrak{p}}\mathfrak{g}_{\mathfrak{p}}y'_n .$$

This means that the diagonal form of $\tau_{\mathfrak{p}}$ with respect to $\{x'_i, y'_i\}$ is

$$\mathrm{diag}\,(\alpha_{1,\mathfrak{p}},\cdots,\alpha_{n,\mathfrak{p}},\beta_{1,\mathfrak{p}},\cdots,\beta_{n,\mathfrak{p}}) .$$

Thus $(\mathfrak{L};\mathfrak{M})_{\mathfrak{p}}$ is defined as $(\alpha_{1,\mathfrak{p}},\cdots,\alpha_{n,\mathfrak{p}},\beta_{1,\mathfrak{p}},\cdots,\beta_{n,\mathfrak{p}})$.

Furthermore we see

$$\mathfrak{L}_{\mathfrak{p}}/\mathfrak{M}_{\mathfrak{p}}=\mathfrak{V}(\mathfrak{g})_{\mathfrak{p}}\sigma_{\mathfrak{p}}/\mathfrak{M}'_{\mathfrak{p}}\sigma_{\mathfrak{p}}\cong\mathfrak{V}(\mathfrak{g})_{\mathfrak{p}}/\mathfrak{M}'_{\mathfrak{p}} .$$

$\nu_{\mathfrak{p}}$ being the normal exponential valuation at every $\mathfrak{p}$, we define the invariant $\mathfrak{p}$-factor of $\mathfrak{L}\in\Theta$ by the set

$$\mathrm{inv}\,(\mathfrak{V}(\mathfrak{g}):\mathfrak{L})_{\mathfrak{p}}=\{\mathfrak{p}^{\nu_{\mathfrak{p}}(\alpha_{1,\mathfrak{p}})},\cdots,\mathfrak{p}^{\nu_{\mathfrak{p}}(\alpha_{n,\mathfrak{p}})},\mathfrak{p}^{\nu_{\mathfrak{p}}(\beta_{1,\mathfrak{p}})},\cdots,\mathfrak{p}^{\nu_{\mathfrak{p}}(\beta_{n,\mathfrak{p}})}\} .$$

Put

$$\mathrm{inv}\,(\mathfrak{V}(\mathfrak{g}):\mathfrak{L})=(\cdots,\mathrm{inv}\,(\mathfrak{V}(\mathfrak{g}):\mathfrak{L})_{\mathfrak{p}},\cdots)$$

and call it the invariant of $\mathfrak{V}(\mathfrak{g})/\mathfrak{L}$. In the same way, for two $\mathfrak{g}$-lattices $\mathfrak{L}$, $\mathfrak{M}$ in Θ such that $\mathfrak{L}\supset\mathfrak{M}$, we can define $\mathrm{inv}\,(\mathfrak{L}:\mathfrak{M})$ using

$$(\mathfrak{L}:\mathfrak{M})_{\mathfrak{p}}=(\alpha_{1,\mathfrak{p}},\cdots,\alpha_{n,\mathfrak{p}},\beta_{1,\mathfrak{p}},\cdots,\beta_{n,\mathfrak{p}}) .$$

LEMMA 4.4. *Let $\mathfrak{L}$, $\mathfrak{M}$ be two $\mathfrak{g}$-lattices in $\mathfrak{V}$. Then $\mathfrak{L}$ and $\mathfrak{M}$ are $G(\mathfrak{V}, J)$-exuivalent if and only if $\mathfrak{L}_\mathfrak{p}$ and $\mathfrak{M}_\mathfrak{p}$ are $G(\mathfrak{V}_\mathfrak{p}, J)$-equivalent for every $\mathfrak{p}$ and the ideal $m(\mathfrak{L}:\mathfrak{M})$ is principal.*

PROOF. "Only if" part is obvious. "If" part is proved as follows.

By Lemma 2.1,

$$\mathfrak{L}=\mathfrak{g}x_1+\cdots+\mathfrak{g}x_n+\mathfrak{a}_1y_1+\cdots+\mathfrak{a}_ny_n$$

and the matrix of $P(x, y)$ is J on $\mathfrak{L}$ with suitable basis $\{x_i, y_i\}$, and

$$\mathfrak{M}=\mathfrak{g}u_1+\cdots+\mathfrak{g}u_n+\mathfrak{b}_1v_1+\cdots+\mathfrak{b}_nv_n$$

and the matrix of $P(u, v)$ is J on $\mathfrak{M}$.

Then

$$\mathfrak{L}_\mathfrak{p}=\mathfrak{g}_\mathfrak{p}x_1+\cdots+\mathfrak{g}_\mathfrak{p}x_n+\mathfrak{a}_{1,\mathfrak{p}}y_1+\cdots+\mathfrak{a}_{n,\mathfrak{p}}y_n.$$

Put $x'_i=x_i\tau_\mathfrak{p}$ and $y'_i=y_i\cdot\tau_\mathfrak{p}$, then

$$\begin{aligned}\mathfrak{M}_\mathfrak{p}=\mathfrak{L}_\mathfrak{p}\tau_\mathfrak{p}&=\mathfrak{g}_\mathfrak{p}x'_1+\cdots+\mathfrak{g}_\mathfrak{p}x'_n+\mathfrak{a}_{1,\mathfrak{p}}y'_1+\cdots+\mathfrak{a}_{n,\mathfrak{p}}y'_n\\&=\mathfrak{g}_\mathfrak{p}x'_1+\cdots+\mathfrak{g}_\mathfrak{p}x'_n+\gamma_\mathfrak{p}\mathfrak{a}_{1,\mathfrak{p}}y'_1/\gamma_\mathfrak{p}+\cdots+\gamma_\mathfrak{p}\mathfrak{a}_{n,\mathfrak{p}}y'_n/\gamma_\mathfrak{p}\end{aligned}$$

where

$$\gamma_\mathfrak{p}=m(\tau_\mathfrak{p})=m(\mathfrak{L}_\mathfrak{p}:\mathfrak{M}_\mathfrak{p})$$

and the matrix of $P(x, y)$ with respect to the basis $\{x'_i, y'_i/\gamma_\mathfrak{p}\}$ is J. By assumption,

$$\bigcap_\mathfrak{p}(\gamma_\mathfrak{p}\mathfrak{g}_\mathfrak{p}\cap\mathfrak{g})=(\gamma_\mathfrak{p})$$

is a principal ideal. Put $(\gamma)=(\gamma_\mathfrak{p})$ with $\gamma\in\mathfrak{g}$. Hence by the uniqueness of the elementary divisors $\mathfrak{a}_i$, $\mathfrak{b}_i$, we have

$$\gamma\mathfrak{a}_i=\mathfrak{b}_i \qquad \text{for every } i.$$

Hence we have

$$\begin{aligned}\mathfrak{M}&=\mathfrak{g}u_1+\cdots+\mathfrak{g}u_n+\mathfrak{b}_1v_1+\cdots+\mathfrak{b}_nv_n\\&=\mathfrak{g}u_1+\cdots+\mathfrak{g}u_n+\gamma\mathfrak{a}_1v_1+\cdots+\gamma\mathfrak{a}_nv_n\,.\end{aligned}$$

Take a matrix σ such that

$$x_i\sigma=u_i \quad \text{and} \quad y_i\sigma=\gamma v_i\,,$$

then

$$\sigma J{}^t\sigma=\gamma J \qquad \gamma=m(\sigma) \quad \text{and} \quad \mathfrak{L}\sigma=\mathfrak{M}\,. \qquad \text{Q. E. D.}$$

For two $\mathfrak{g}$-lattices in $\mathfrak{V}$, we say that they belong to the same *genus* if their $\mathfrak{g}$-completions are $G(\mathfrak{V}_\mathfrak{p}, J)$-equivalent for every $\mathfrak{p}$, and they belong to the same *class* if they are $G(\mathfrak{V}, J)$-equivalent.

Now, for a given ideal $\mathfrak{a}$ in F, we can construct $\mathfrak{g}$-lattices $\mathfrak{L}$, $\mathfrak{M}$ such that $\mathfrak{a}=m(\mathfrak{L}:\mathfrak{M})$, by Lemma 2.2 and Lemma 4.1. Therefore by the above Lemma 4.4, and the isomorphism $\mathfrak{C}(F)\cong F^*\backslash J_F/J_u$, we obtain the following

THEOREM 1. *The number of classes contained in a genus is the ideal class number h of F.*

5. We shall define Hecke ring attached to $G(\mathfrak{V}, J)$ in the next section, so we shall quote the results of G. Shimura [1] in this section to prepare for it.

Let G be a group and Γ_λ a subgroup of G for every λ in Λ, Λ being a set of indices.

Let Ω be a subset of G with the following properties:

(HI) Ω is a semi-group containing Γ_λ for every $\lambda\in\Lambda$.

(HII) For every σ of Ω and for every $\lambda, \mu\in\Lambda$, $\sigma\Gamma_\lambda\sigma^{-1}$ is commensurable with Γ_μ.

Denote by $\mathfrak{R}_{\mu\lambda}$ the $\boldsymbol{Z}$-module generated by $\Gamma_\mu\sigma\Gamma_\lambda$ for $\sigma\in\Omega$; so an element of $\mathfrak{R}_{\mu\lambda}$ is of the form

$$\sum_{k} c_k(\Gamma_\mu\sigma_k\Gamma_\lambda)$$

with $c_k\in\boldsymbol{Z}$ and $\sigma_k\in\Omega$.

We define a bilinear mapping of $\mathfrak{R}_{\nu\mu}\times\mathfrak{R}_{\mu\lambda}$ into $\mathfrak{R}_{\nu\lambda}$ as follows.

Put

$$\bar{\sigma}=\Gamma_\mu\sigma\Gamma_\lambda=\bigcup_i \sigma_i\Gamma_\lambda \qquad \text{(disjoint)}$$

and

$$\bar{\tau}=\Gamma_\nu\tau\Gamma_\mu=\bigcup_j \tau_j\Gamma_\mu \qquad \text{(disjoint)}.$$

For every element $\rho\in\Gamma_\nu\tau\Gamma_\mu\sigma\Gamma_\lambda$, the numbers of (i, j) such that $\tau_j\sigma_i\Gamma_\lambda=\rho\Gamma_\lambda$ is determined only depending on $\bar{\sigma}$, $\bar{\tau}$ and $\bar{\rho}=\Gamma_\nu\rho\Gamma_\lambda$ and not depending on the choices of $\{\sigma_i\}$, $\{\tau_j\}$ and ρ. This number is denoted by $\mu(\bar{\tau}\cdot\bar{\sigma}; \bar{\rho})$. Put

$$\bar{\tau}\cdot\bar{\sigma}=\sum\mu(\bar{\tau}\cdot\bar{\sigma}; \bar{\rho})\bar{\rho},$$

where the sum is extended over all $\bar{\rho}=\Gamma_\nu\rho\Gamma_\lambda$ contained in $\Gamma_\nu\tau\Gamma_\mu\sigma\Gamma_\lambda$. We extend this product $\cdot$ by linearity to a bilinear mapping of $\mathfrak{R}_{\lambda\mu}\times\mathfrak{R}_{\mu\lambda}$ into $\mathfrak{R}_{\nu\lambda}$. Then, it is seen that for $\bar{\rho}\in\mathfrak{R}_{\nu\kappa}$, $\bar{\sigma}\in\mathfrak{R}_{\mu\lambda}$ and $\tau\in\mathfrak{R}_{\nu\mu}$,

$$\bar{\tau}\cdot(\bar{\sigma}\cdot\bar{\rho})=(\bar{\tau}\cdot\bar{\sigma})\cdot\bar{\rho}$$

holds. In particular, $\mathfrak{R}_{\lambda\lambda}=\mathfrak{R}_\lambda=\mathfrak{R}(\Gamma_\lambda; \Omega)$ is an associative ring for every λ.

6. Let J_G be the idelization of $G=G(\mathfrak{V}, J)$; namely the restricted direct product $\prod'_{\mathfrak{p}\leq\infty} G(\mathfrak{V}, J)_\mathfrak{p}$ with respect to the system of locally compact subgroups $\{\Gamma^0(\mathfrak{V}, J)_\mathfrak{p}\}$.

Put $\mathfrak{U}_0 = \prod_{\mathfrak{p}<\infty} \Gamma^0(\mathfrak{V}(\mathfrak{g}), J)_{\mathfrak{p}}$. We can regard J_G and $\mathfrak{U}_0$ as $G(\mathfrak{V}(2_n, A_F), J)$ and $\Gamma^0(\mathfrak{V}(2n, A_{\mathfrak{g},0}), J)$ respectively. Denote by $J_{G,\infty}$ and $J_{G,0}$ the infinite and finite parts of J_G, respectively.

For any $a \in J_{G,0}$, $\mathfrak{U}_0$ and $a\mathfrak{U}_0 a^{-1}$ are commensurable. Hence we can define $\mathfrak{R} = \mathfrak{R}(\mathfrak{U}_0, J_{G,0})$ which is called the Hecke ring attached to G.

PROPOSITION 6.1. *The ring* $\mathfrak{R}$ *is commutative.*

PROOF. By Proposition 1.2 in Shimura [2], it is sufficient to show that there exists an anti-automorphism $a \to a^*$ of the group $J_{G,0}$ such that $(\mathfrak{U}_0 a \mathfrak{U}_0)^* = \mathfrak{U}_0 a \mathfrak{U}_0$ for every a. To prove this, we may assume that a belongs to $\Gamma(\mathfrak{V}(2n, A_{\mathfrak{g},0}), J)$. Then there exist u, v of $\Gamma^0(\mathfrak{V}(2n, A_{\mathfrak{g},0}), J)$ such that

$$uav = \operatorname{diag}(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n)$$

where $\alpha_i \beta_i = m(a)$ and $\alpha_1 A_{\mathfrak{g},0} \supset \cdots \supset \alpha_n A_{\mathfrak{g},0} \supset \beta_n A_{\mathfrak{g},0} \supset \cdots \supset \beta_1 A_{\mathfrak{g},0}$.

We define a^* by $a^* = m(a)a^{-1}$. Then, $m(a^*) = m(a)$ holds. In fact, by definition,

$$P_0(xa, ya) = m(a)P_0(x, y)$$

and $P_0(xa^*, ya^*) = m(a^*)P_0(x, y)$. Hence, $P_0(xa^*, ya^*) = P_0(xm(a)a^{-1}, ym(a)a^{-1}) = m(a)^2 P_0(xa^{-1}, ya^{-1})$ and $m(a^*)P_0(x, y) = m(a)^2 P_0(xa^{-1}, ya^{-1})$. Put $xa^{-1} = x'$, $ya^{-1} = y'$, then we have

$$m(a^*)P_0(x'a, y'a) = m(a)^2 \cdot P_0(x', y') \quad \text{and} \quad m(a^*)m(a)P_0(x', y') = m(a)^2 P_0(x', y')$$

and so $m(a^*) = m(a)$. Next we have $(a^*)^* = a$. In fact, $(a^*)^* = m(a^*)a^{*-1} = m(a)m(a)^{-1}a = a$. Thus $a \to a^*$ is an involution.

Now we may assume that $a = \operatorname{diag}(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n)$. Then $a^* = m(a)a^{-1} = \operatorname{diag}(\beta_1, \cdots, \beta_n, \alpha_1, \cdots, \alpha_n)$, hence there exist elements u', v' of $\Gamma^0(\mathfrak{V}(2n, A_{\mathfrak{g},0}), J) = \mathfrak{U}_0$ such that

$$u'a^*v' = \operatorname{diag}(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots \beta_n).$$

So we have $(\mathfrak{U}_0 a \mathfrak{U}_0)^* = \mathfrak{U}_0 a \mathfrak{U}_0$ for every $a \in J_{G,0}$. Q. E. D.

For $\rho \in J_{G,0}$, put $T(\rho) = \mathfrak{U}_0 \cdot \rho \cdot \mathfrak{U}_0$. Let $T(\rho) = \bigcup_{i=1}^{d} \mathfrak{U}_0 \rho_i$ be disjoint sum, where d means the number of left cosets of $T(\rho)$ by $\mathfrak{U}_0$. This number d will be denoted by $d[T(\rho)]$ in the following.

PROPOSITION 6.2*. *$d[T(\rho)]$ is the number of $\mathfrak{L}_0 \in \Theta_0$ such that*

$$(\mathfrak{V}(\mathfrak{g}) : \mathfrak{L})_{\mathfrak{p}} = (\alpha_{1,\mathfrak{p}}, \cdots, \alpha_{n,\mathfrak{p}}, \beta_{1,\mathfrak{p}}, \cdots, \beta_{n,\mathfrak{p}}),$$

where Θ_0 is the finite part of Θ defined in section 4 *and* $\operatorname{diag}(\alpha_{1,\mathfrak{p}}, \cdots, \alpha_{n,\mathfrak{p}}, \beta_{1,\mathfrak{p}}, \cdots, \beta_{n,\mathfrak{p}})$ *is the diagonal form of $\rho_{\mathfrak{p}}$.*

*) These Propositions and their proofs are the same as that given by Shimura in his Komaba lecture "On Siegel modular group." For a subsequent papar, we shall state here.

PROOF. For $\mathfrak{U}_0 \cdot \rho_i$, put $\mathfrak{V}(\mathfrak{g})_0 \cdot \rho_i = \mathfrak{L}_{0,i}$. Then $\mathfrak{L}_{0,i}$ belongs to Θ_0 and

$$(\mathfrak{V}(\mathfrak{g}):\mathfrak{L})_{\mathfrak{p}} = (\alpha_{1,\mathfrak{p}}, \cdots, \alpha_{n,\mathfrak{p}}, \beta_{1,\mathfrak{p}}, \cdots, \beta_{n,\mathfrak{p}}) .$$

If $\mathfrak{V}(\mathfrak{g})_0 \cdot \rho_i = \mathfrak{V}(\mathfrak{g}_0) \cdot \rho_j$ holds, then we have $\mathfrak{V}(\mathfrak{g})_0 \cdot \rho_i \cdot \rho_j^{-1} = \mathfrak{V}(\mathfrak{g})_0$ and so $\rho_i \cdot \rho_j^{-1}$ belongs to $\mathfrak{U}_0$. Hence, we have the proposition.

PROPOSITION 6.3*. *Let ρ, σ, τ be elements of $J_{G,0}$ and their diagonal forms be* $\operatorname{diag}(\alpha_{1,\mathfrak{p}}(\rho), \cdots, \alpha_{n,\mathfrak{p}}(\rho), \beta_{1,\mathfrak{p}}(\rho), \cdots, \beta_{n,\mathfrak{p}}(\rho))$, $\operatorname{diag}(\alpha_{1,\mathfrak{p}}(\sigma), \cdots, \alpha_{n,\mathfrak{p}}(\sigma), \beta_{1,\mathfrak{p}}(\sigma), \cdots, \beta_{n,\mathfrak{p}}(\sigma))$ *and* $\operatorname{diag}(\alpha_{1,\mathfrak{p}}(\tau), \cdots, \alpha_{n,\mathfrak{p}}(\tau), \beta_{1,\mathfrak{p}}(\tau), \cdots, \beta_{n,\mathfrak{p}}(\tau))$ *respectively. Put $T(\rho) = \mathfrak{U}_0 \rho \mathfrak{U}_0$, $T(\sigma) = \mathfrak{U}_0 \cdot \sigma \cdot \mathfrak{U}_0$ and $T(\tau) = \mathfrak{U}_0 \tau \mathfrak{U}_0$. Then, $\mu(T(\rho) \cdot T(\sigma); T(\tau))$ is the number of $\mathfrak{L}_0$ such that $\mathfrak{L}_0 \in \Theta_0$, $(\mathfrak{V}(\mathfrak{g}):\mathfrak{L})_{\mathfrak{p}} = (\alpha_{1,\mathfrak{p}}(\rho), \cdots, \alpha_{n,\mathfrak{p}}(\rho), \beta_{1,\mathfrak{p}}(\rho), \cdots, \beta_{n,\mathfrak{p}}(\rho))$ and $(\mathfrak{L}:\mathfrak{M})_{\mathfrak{p}} = (\alpha_{1,\mathfrak{p}}(\sigma), \cdots, \alpha_{n,\mathfrak{p}}(\sigma), \beta_{1,\mathfrak{p}}(\sigma), \cdots, \beta_{n,\mathfrak{p}}(\sigma))$, where $\mathfrak{M}_0 \in \Theta_0$ is fixed and $(\mathfrak{V}(\mathfrak{g}):\mathfrak{M})_{\mathfrak{p}} = (\alpha_{1,\mathfrak{p}}(\tau), \cdots, \alpha_{n,\mathfrak{p}}(\tau), \beta_{1,\mathfrak{p}}(\tau), \cdots, \beta_{n,\mathfrak{p}}(\tau))$.*

PROOF. Let $T(\rho) = \bigcup_{i=1}^{r} \mathfrak{U}_0 \cdot \rho_i$, and $T(\sigma) = \bigcup_{j=1}^{s} \mathfrak{U}_0 \cdot \sigma_j$ be disjoint sums. Put $\mathfrak{V}(\mathfrak{g})_0 \cdot \rho_i = \mathfrak{L}_{i,0}$ and $\mathfrak{U}_{i,0} = \rho_i^{-1} \cdot \mathfrak{U}_0 \cdot \rho_i$, which equals $\Gamma^0(\mathfrak{L}_{i,0}, J)$.

We have

$$\rho_i^{-1} \cdot \mathfrak{U}_0 \cdot \rho_i \cdot \rho_i^{-1} \cdot T(\sigma) \cdot \rho_i \cdot \rho_i^{-1} \mathfrak{U}_0 \rho_i = \bigcup_{j=1}^{s} \rho_i^{-1} \mathfrak{U}_0 \rho_i \rho_i^{-1} \sigma_j \rho_i .$$

Hence, putting $\rho_i^{-1} T(\sigma) \rho_i = T(\sigma)^i$ and $\rho_i^{-1} \sigma_j \rho_i = \sigma_j^i$, we have $\mathfrak{U}_{i,0} T(\sigma)^i \mathfrak{U}_{i,0} = \bigcup_{j=1}^{s} \mathfrak{U}_{i,0} \sigma_j^i$ and $\mathfrak{L}_{i,0} \sigma_j^i = \mathfrak{V}(\mathfrak{g})_0 \cdot \rho_i \cdot \rho_i^{-1} \sigma_j \rho_i = \mathfrak{V}(\mathfrak{g})_0 \sigma_j \rho_i$. Now, $T(\tau) = \mathfrak{U}_0 \tau \mathfrak{U}_0$ and $\mathfrak{M}_0 = \mathfrak{V}(\mathfrak{g})_0 \tau$. $\mu(T(\rho) \cdot T(\sigma) : T(\tau))$ is the number of (i, j) such that

$$\mathfrak{U}_0 \sigma_j \rho_i = \mathfrak{U}_0 \tau ,$$

which latter condition is equivalent with $\mathfrak{V}(\mathfrak{g})_0 \sigma_j \rho_i = \mathfrak{M}_0$, or with $\mathfrak{L}_{i,0} \sigma_j^i = \mathfrak{M}_0$. If we take such an (i, j), we obtain $\mathfrak{L}_{i,0}$ such that $\mathfrak{V}(\mathfrak{g})_0 \supset \mathfrak{L}_{i,0} \supset \mathfrak{M}_0$, $(\mathfrak{V}(\mathfrak{g}):\mathfrak{L})_{\mathfrak{p}} = (\alpha_{1,\mathfrak{p}}(\rho), \cdots, \alpha_{n,\mathfrak{p}}(\rho), \beta_{1,\mathfrak{p}}(\rho), \cdots, \beta_{n,\mathfrak{p}}(\rho))$, putting $\mathfrak{L}_{i,0} \cdot \sigma_j^i = \mathfrak{M}_0$. The diagonal form of σ_j^i coincides with that of σ. Hence $(\mathfrak{L}_{i,0} : \mathfrak{M}_0)_{\mathfrak{p}} = (\alpha_{1,\mathfrak{p}}(\sigma), \cdots, \alpha_{n,\mathfrak{p}}(\sigma), \beta_{1,\mathfrak{p}}(\sigma), \cdots, \beta_{n,\mathfrak{p}}(\sigma))$ and j is uniquely determined by giving one i. Conversely, if there exists $\mathfrak{L}_0 \in \Theta_0$ such that $\mathfrak{L}_0 \supset \mathfrak{M}_0$ and $(\mathfrak{L}:\mathfrak{M})_{\mathfrak{p}} = (\alpha_{1,\mathfrak{p}}(\sigma), \cdots, \alpha_{n,\mathfrak{p}}(\sigma), \beta_{1,\mathfrak{p}}(\sigma), \cdots, \beta_{n,\mathfrak{p}}(\sigma))$ and $(\mathfrak{V}(\mathfrak{g}):\mathfrak{L})_{\mathfrak{p}} = (\alpha_{1,\mathfrak{p}}(\rho), \cdots, \alpha_{n,\mathfrak{p}}(\rho), \beta_{1,\mathfrak{p}}(\rho), \cdots, \beta_{n,\mathfrak{p}}(\rho))$, then we have $\mathfrak{L}_0 = \mathfrak{V}(\mathfrak{g})_0 \cdot \rho_i$ for some i, and $\mathfrak{L}_0 \cdot \sigma_j^i = \mathfrak{M}_0$ with σ_j^i for the i. So we obtain our Proposition.

PROPOSITION 6.4. *Let ρ, σ be elements of $J_{G,0}$ and* $\operatorname{diag}(\alpha_1(\rho), \cdots, \alpha_n(\rho), \beta_1(\rho), \cdots, \beta_n(\rho))$, $\operatorname{diag}(\alpha_1(\sigma), \cdots, \alpha_n(\sigma), \beta_1(\sigma), \cdots, \beta_n(\sigma))$ *be their diagonal forms, respectively. Put $\tau = \rho\sigma$ and denote by* $\operatorname{diag}(\alpha_1(\rho\sigma), \cdots, \alpha_n(\rho\sigma), \beta_1(\rho\sigma), \cdots, \beta_n(\rho\sigma))$ *its diagonal form. Then for every i,*

$$\nu_{\mathfrak{p}}(\alpha_i(\rho\sigma)) \leqslant \nu_{\mathfrak{p}}(\alpha_i(\rho) \cdot \alpha_i(\sigma)) \quad \textit{and} \quad \nu_{\mathfrak{p}}(\beta_i(\rho\sigma)) \leqslant \nu_{\mathfrak{p}}(\beta_i(\rho) \cdot \beta_i(\sigma)) .$$

In particular, if $(m(\rho), m(\sigma)) = 1$, then for every i,

$$\nu_{\mathfrak{p}}(\alpha_i(\rho\sigma)) = \nu_{\mathfrak{p}}(\alpha_i(\rho) \cdot \alpha_i(\sigma)) \quad \textit{and} \quad \nu_{\mathfrak{p}}(\beta_i(\rho\sigma)) = \nu_{\mathfrak{p}}(\beta_i(\rho) \cdot \beta_i(\sigma)) .$$

PROOF. Let $\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}}=\mathfrak{g}_{\mathfrak{p}}x_1+\cdots+\mathfrak{g}_{\mathfrak{p}}x_n+\mathfrak{g}_{\mathfrak{p}}y_1+\cdots+\mathfrak{g}_{\mathfrak{p}}y_n$ with canonical basis $\{x_i, y_i\}$. Put

$$\mathfrak{L}_{\mathfrak{p}}=\alpha_{1,\mathfrak{p}}(\rho)\mathfrak{g}_{\mathfrak{p}}x_1+\cdots+\alpha_{n,\mathfrak{p}}(\rho)\mathfrak{g}_{\mathfrak{p}}x_n+\beta_{1,\mathfrak{p}}(\rho)\mathfrak{g}_{\mathfrak{p}}y_1+\cdots+\beta_{n,\mathfrak{p}}(\rho)\mathfrak{g}_{\mathfrak{p}}y_n\,,$$
$$\mathfrak{M}_{\mathfrak{p}}=\alpha_{1,\mathfrak{p}}(\tau)\mathfrak{g}_{\mathfrak{p}}x_1+\cdots+\alpha_{n,\mathfrak{p}}(\tau)\mathfrak{g}_{\mathfrak{p}}x_n+\beta_{1,\mathfrak{p}}(\tau)\mathfrak{g}_{\mathfrak{p}}y_1+\cdots+\beta_{n,\mathfrak{p}}(\tau)\mathfrak{g}_{\mathfrak{p}}y_n$$

and

$$\mathfrak{L}'_{\mathfrak{p}}=\alpha_{1,\mathfrak{p}}(\tau)\alpha_{1,\mathfrak{p}}(\sigma)^{-1}\mathfrak{g}_{\mathfrak{p}}x_1+\cdots+\alpha_{n,\mathfrak{p}}(\tau)\alpha_{n,\mathfrak{p}}(\sigma)^{-1}\mathfrak{g}_{\mathfrak{p}}x_n+\beta_{1,\mathfrak{p}}(\tau)\beta_{1,\mathfrak{p}}(\sigma)^{-1}\mathfrak{g}_{\mathfrak{p}}y_1+\cdots+\beta_{n,\mathfrak{p}}(\tau)\beta_{n,\mathfrak{p}}(\sigma)^{-1}\mathfrak{g}_{\mathfrak{p}}y_1\,.$$

Then,

$$(\mathfrak{B}(\mathfrak{g}):\mathfrak{L})_{\mathfrak{p}}=(\alpha_{1,\mathfrak{p}}(\rho),\cdots,\alpha_{n,\mathfrak{p}}(\rho),\beta_{1,\mathfrak{p}}(\rho),\cdots,\beta_{n,\mathfrak{p}}(\rho))\,,$$
$$(\mathfrak{B}(\mathfrak{g}):\mathfrak{M})_{\mathfrak{p}}=(\alpha_{1,\mathfrak{p}}(\tau),\cdots,\alpha_{n,\mathfrak{p}}(\tau),\beta_{1,\mathfrak{p}}(\tau),\cdots,\beta_{n,\mathfrak{p}}(\tau))$$

and

$$(\mathfrak{L}':\mathfrak{M})_{\mathfrak{p}}=(\alpha_{1,\mathfrak{p}}(\sigma),\cdots,\alpha_{n,\mathfrak{p}}(\sigma),\beta_{1,\mathfrak{p}}(\sigma),\cdots,\beta_{n,\mathfrak{p}}(\sigma))\,.$$

We have

$$\mathfrak{L}_{\mathfrak{p}}\cup\mathfrak{L}'_{\mathfrak{p}}=(\alpha_{1,\mathfrak{p}}(\rho)\mathfrak{g}_{\mathfrak{p}}\cup\alpha_{1,\mathfrak{p}}(\tau)\alpha_{1,\mathfrak{p}}(\sigma)^{-1}\mathfrak{g}_{\mathfrak{p}})x_1+\cdots+ (\beta_{1,\mathfrak{p}}(\rho)\mathfrak{g}_{\mathfrak{p}}\cup\beta_{1,\mathfrak{p}}(\tau)\beta_{1,\mathfrak{p}}(\sigma)^{-1}\mathfrak{g}_{\mathfrak{p}})y_1+\cdots$$

and

$$\mathfrak{L}_{\mathfrak{p}}\cap\mathfrak{L}'_{\mathfrak{p}}=(\alpha_{1,\mathfrak{p}}(\rho)\mathfrak{g}_{\mathfrak{p}}\cap\alpha_{1,\mathfrak{p}}(\tau)\alpha_{1,\mathfrak{p}}(\sigma)^{-1}\mathfrak{g}_{\mathfrak{p}})x_1+\cdots+ (\beta_{1,\mathfrak{p}}(\rho)\mathfrak{g}_{\mathfrak{p}}\cap\beta_{1,\mathfrak{p}}(\tau)\beta_{1,\mathfrak{p}}(\sigma)^{-1}\mathfrak{g}_{\mathfrak{p}})y_1+\cdots\,.$$

And we have

$$\mathfrak{L}_0\cup\mathfrak{L}'_0/\mathfrak{L}_0\cong\mathfrak{L}'_0/\mathfrak{L}'_0\cap\mathfrak{L}_0\,,$$

hence obtain the first part of the Proposition. If $(m(\rho), m(\sigma))=1$, we must have $\mathfrak{L}_0\cup\mathfrak{L}'_0=\mathfrak{L}_0$ and $\mathfrak{L}'_0=\mathfrak{L}_0\cap\mathfrak{L}'_0$. So $\mathfrak{L}_0$ contains $\mathfrak{L}'_0$. But, since this holds for $\mathfrak{L}_0$, $\mathfrak{L}'_0$ instead for $\mathfrak{L}'_0, \mathfrak{L}_0$, we have $\mathfrak{L}_0=\mathfrak{L}'_0$. Hence, there exists only one component $T(\tau)$ contained in $T(\rho)T(\sigma)$. By Proposition 6.3, it is seen that the multiplicity $\mu(T(\rho)\cdot T(\sigma);\, T(\tau))$ equals 1. Q. E. D.

PROPOSITION 6.5. *Let ρ be an element of $\boldsymbol{J}_{G,0}$ whose diagonal form is* $\mathrm{diag}\,(\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_n)$. *Put $T(\rho)=\mathfrak{U}_0\rho\,\mathfrak{U}_0$. Then, the diagonal form of* $\mathrm{diag}\,(\alpha,\cdots,\alpha)\cdot T(\rho)$ *is* $\mathrm{diag}\,(\alpha\alpha_1,\cdots,\alpha\alpha_n,\alpha\beta_1,\cdots,\alpha\beta_n)$.

PROPOSITION 6.6. *Let $\mathfrak{L}$, $\mathfrak{M}$ be two $\mathfrak{g}$-lattices in $\mathfrak{B}$ which belong to Θ. Then* $\mathrm{inv}\,(\mathfrak{B}(\mathfrak{g}):\mathfrak{L})=\mathrm{inv}\,(\mathfrak{B}(\mathfrak{g}):\mathfrak{M})$ *holds if and only if $\mathfrak{L}$ and $\mathfrak{M}$ are $\Gamma^0(\mathfrak{B}(\mathfrak{g}), J)$-equivalent.*

PROOF. If $\mathfrak{L}$ and $\mathfrak{M}$ are $\Gamma^0(\mathfrak{B}(\mathfrak{g}), J)$-equivalent, there exists an element σ of $\Gamma^0(\mathfrak{B}(\mathfrak{g}), J)$ such that $\mathfrak{L}\sigma=\mathfrak{M}$. Then for every $\mathfrak{p}$, we have $\mathfrak{L}_{\mathfrak{p}}\cdot\sigma=\mathfrak{M}_{\mathfrak{p}}$, where σ is considered as an element of $\Gamma^0(\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}}, J)=\mathfrak{U}_{\mathfrak{p}}$. Taking elements $\tau_{\mathfrak{p}}$, $\rho_{\mathfrak{p}}$ of $\Gamma(\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}}, J)$ such that $\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}}\cdot\tau_{\mathfrak{p}}=\mathfrak{L}_{\mathfrak{p}}$ and $\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}}\cdot\rho_{\mathfrak{p}}=\mathfrak{M}_{\mathfrak{p}}$, we have $\tau_{\mathfrak{p}}\sigma=\rho_{\mathfrak{p}}$, and so for every $\mathfrak{p}$, the diagonal forms of $\tau_{\mathfrak{p}}$, $\rho_{\mathfrak{p}}$ coincide. Hence, $\mathrm{inv}(\mathfrak{B}(\mathfrak{g}):\mathfrak{L})=\mathrm{inv}(\mathfrak{B}(\mathfrak{g}):\mathfrak{M})$.

Conversely, putting $\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}}\cdot\tau_{\mathfrak{p}}=\mathfrak{L}_{\mathfrak{p}}$ and $\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}}\cdot\rho_{\mathfrak{p}}=\mathfrak{M}_{\mathfrak{p}}$, let the diagonal form of $\tau_{\mathfrak{p}}$ and $\rho_{\mathfrak{p}}$ be one and the same $\mathrm{diag}(\alpha_{1,\mathfrak{p}},\cdots,\alpha_{n,\mathfrak{p}},\beta_{1,\mathfrak{p}},\cdots,\beta_{n,\mathfrak{p}})$. Then there exists an element $\sigma_{\mathfrak{p}}$ of $G^0(\mathfrak{B}_{\mathfrak{p}},J)$ such that $\mathfrak{L}_{\mathfrak{p}}\sigma_{\mathfrak{p}}=\mathfrak{M}_{\mathfrak{p}}$. By Lemma 4.4, there exists an element σ of $G^0(\mathfrak{B},J)$ such that $\mathfrak{L}\sigma=\mathfrak{M}$. Then, for every $\mathfrak{p}$, $\tau_{\mathfrak{p}}\sigma=\rho_{\mathfrak{p}}$, where $\tau_{\mathfrak{p}}$, $\rho_{\mathfrak{p}}$ belong to $\Gamma(\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}},J)$. Take an element $\alpha_{\mathfrak{p}}$ of $\mathfrak{g}_{\mathfrak{p}}$ so that $\sigma_1=\alpha_{\mathfrak{p}}\sigma$ is contained in $\Gamma^0(\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}},J)$. Putting $\alpha_{\mathfrak{p}}\rho_{\mathfrak{p}}=\rho_{1,\mathfrak{p}}$, we have $\tau_{\mathfrak{p}}\sigma_1=\rho_{1,\mathfrak{p}}$. Let $\mathrm{diag}(\alpha_1(\tau_{\mathfrak{p}}),\cdots,\alpha_n(\tau_{\mathfrak{p}}),\beta_1(\tau_1),\cdots,\beta_n(\tau_{\mathfrak{p}}))$ and $\mathrm{diag}(\alpha_1(\sigma_1),\cdots,\alpha_n(\sigma_1),\beta_1(\sigma_1),\cdots,\beta_n(\sigma_1))$ be the diagonal forms of $\tau_{\mathfrak{p}}$ and σ_1, respectively. By assumption, the diagonal form of $\rho_{1,\mathfrak{p}}$ is

$$\begin{aligned}&\mathrm{diag}(\alpha_{\mathfrak{p}},\cdots,\alpha_{\mathfrak{p}})\,\mathrm{diag}(\alpha_1(\tau_{\mathfrak{p}}),\cdots,\alpha_n(\tau_{\mathfrak{p}}),\beta_1(\tau_{\mathfrak{p}}),\cdots,\beta_n(\tau_{\mathfrak{p}}))\\ &=\mathrm{diag}(\alpha_{\mathfrak{p}}\alpha_1(\tau_{\mathfrak{p}}),\cdots,\alpha_{\mathfrak{p}}\alpha_n(\tau_{\mathfrak{p}}),\alpha_{\mathfrak{p}}\beta_1(\tau_{\mathfrak{p}}),\cdots,\alpha_{\mathfrak{p}}\beta_n(\tau_{\mathfrak{p}}))\,.\end{aligned}$$

By Proposition 6.4, it holds that for every i,

$$\nu(\alpha_{\mathfrak{p}})+\nu_{\mathfrak{p}}(\alpha_i(\tau_{\mathfrak{p}}))\leqslant\nu_{\mathfrak{p}}(\alpha_i(\tau_{\mathfrak{p}}))+\nu_{\mathfrak{p}}(\alpha_i(\sigma_1))$$

and

$$\nu_{\mathfrak{p}}(\alpha_{\mathfrak{p}})+\nu_{\mathfrak{p}}(\beta_i(\tau_{\mathfrak{p}}))\leqslant\nu_{\mathfrak{p}}(\beta_i(\tau_{\mathfrak{p}}))+\nu_{\mathfrak{p}}(\beta_i(\sigma_1))\,.$$

Hence, for every i, $\nu_{\mathfrak{p}}(\alpha_{\mathfrak{p}})\leqslant\nu_{\mathfrak{p}}(\alpha_i(\sigma_1))$ and $\nu_{\mathfrak{p}}(\alpha_{\mathfrak{p}})\leqslant\nu_{\mathfrak{p}}(\beta_i(\sigma_1))$. Then observing that $m(\sigma_1)=\alpha_{\mathfrak{p}}$, we have

$$\alpha_{\mathfrak{p}}=\alpha_i(\sigma_1)=\beta_i(\sigma_1)\,;$$

namely the diagonal form of σ_1 is $\mathrm{diag}(\alpha_{\mathfrak{p}},\cdots,\alpha_{\mathfrak{p}})$. Hence, the diagonal form of σ is $\mathrm{diag}(1,\cdots,1)$, which means that σ belongs to $\Gamma^0(\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}},J)$ for every $\mathfrak{p}$ and so belongs to $\Gamma^0(\mathfrak{B}(\mathfrak{g}),J)$. Q. E. D.

Now for fixed $\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}}$ we have a one-to-one correspondence between $\mathfrak{g}_{\mathfrak{p}}$-lattices $\mathfrak{L}_{\mathfrak{p}}$ and $\mathfrak{U}_{\mathfrak{p}}\sigma_{\mathfrak{p}}$, $\sigma_{\mathfrak{p}}\in\boldsymbol{J}_{G,\mathfrak{p}}$, by putting $\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}}\sigma_{\mathfrak{p}}=\mathfrak{L}_{\mathfrak{p}}$, hence $\mathfrak{U}_0\backslash\boldsymbol{J}_{G,0}$ is the genus to which $\mathfrak{B}(\mathfrak{g})$ belongs, and cosets $\mathfrak{U}_0\backslash\boldsymbol{J}_{G,0}/G$ are classes contained in that genus. Therefore we have the double coset-decomposition

$$\boldsymbol{J}_{G,0}=\bigcup_{\lambda=1}^{h}\mathfrak{U}_0x'_\lambda G\,.$$

7. In what follows, we shall use the decomposition

$$\boldsymbol{J}_{G,0}=\bigcup_{\lambda=1}^{h}Gx_\lambda\mathfrak{U}_0$$

for conveniences. This amounts to considering the identification of $\sigma_{\mathfrak{p}}\mathfrak{U}_{\mathfrak{p}}$ with $\mathfrak{L}_{\mathfrak{p}}=\mathfrak{B}(\mathfrak{g})_{\mathfrak{p}}\cdot\sigma_{\mathfrak{p}}^{-1}$ with $\mathfrak{B}(\mathfrak{g})$ fixed.

Now we define the set

$$\Theta_1=\{\mathfrak{L}\mid\text{ maximal }\mathfrak{g}\text{-lattice }\mathfrak{L}\text{ such that }\mathfrak{L}\supset\mathfrak{B}(\mathfrak{g})\}\,.$$

Let $\mathfrak{L}\supset\mathfrak{M}$ be two $\mathfrak{g}$-lattices in Θ_1. Then there exists an element $\sigma_{\mathfrak{p}}\in\boldsymbol{J}_{G,\mathfrak{p}}$ such that $\mathfrak{L}_\mathfrak{p}\cdot\sigma_\mathfrak{p}=\mathfrak{V}(\mathfrak{g})_\mathfrak{p}$ for every $\mathfrak{p}$. Put $\mathfrak{M}_\mathfrak{p}\sigma_\mathfrak{p}=\mathfrak{M}'_\mathfrak{p}$ and $\mathfrak{M}'=\bigcap_\mathfrak{p}(\mathfrak{M}'_\mathfrak{p}\cap\mathfrak{V})$. Then $\mathfrak{M}'$ belongs to Θ. We define $(\mathfrak{L}:\mathfrak{M})$ and $\mathrm{inv}(\mathfrak{L}:\mathfrak{M})$ as $(\mathfrak{V}(\mathfrak{g}):\mathfrak{M}')$ and $\mathrm{inv}(\mathfrak{V}(\mathfrak{g}):\mathfrak{M}')$, respectively.

PROPOSITION 7.1. *Let $\mathfrak{L}$, $\mathfrak{M}$ be two $\mathfrak{g}$-lattices in Θ_1. Then* $\mathrm{inv}(\mathfrak{L}:\mathfrak{V}(\mathfrak{g}))=\mathrm{inv}(\mathfrak{M}:\mathfrak{V}(\mathfrak{g}))$ *holds if and only if $\mathfrak{L}$ and $\mathfrak{M}$ are $\Gamma^0(\mathfrak{V}(\mathfrak{g}),J)$-equivalent.*

PROOF. "If" part. By assumption, there exists an element σ of $\Gamma^0(\mathfrak{V}(\mathfrak{g}),J)$ such that $\mathfrak{L}_\mathfrak{p}\sigma=\mathfrak{M}_\mathfrak{p}$ for every $\mathfrak{p}$. There exist integral elements $\tau_\mathfrak{p}$, $\rho_\mathfrak{p}$ of $\boldsymbol{J}_{G,\mathfrak{p}}$ such that

$$\mathfrak{L}_\mathfrak{p}\cdot\tau_\mathfrak{p}=\mathfrak{V}(\mathfrak{g})_\mathfrak{p} \quad\text{and}\quad \mathfrak{M}_\mathfrak{p}\rho_\mathfrak{p}=\mathfrak{V}(\mathfrak{g})_\mathfrak{p}\,.$$

Hence $\mathfrak{V}(\mathfrak{g})_\mathfrak{p}\cdot\rho_\mathfrak{p}^{-1}\cdot\sigma^{-1}=\mathfrak{V}(\mathfrak{g})_\mathfrak{p}\tau_\mathfrak{p}^{-1}$ and $\rho_\mathfrak{p}^{-1}\sigma^{-1}\tau_\mathfrak{p}\in\Gamma^0(\mathfrak{V}(\mathfrak{g})_\mathfrak{p},J)=\mathfrak{U}_\mathfrak{p}$. Since σ belongs to $\mathfrak{U}_\mathfrak{p}$ for every $\mathfrak{p}$, we see that

$$\tau_\mathfrak{p}\in\mathfrak{U}_\mathfrak{p}\rho_\mathfrak{p}\mathfrak{U}_\mathfrak{p}\,.$$

Hence diagonal forms of $\tau_\mathfrak{p}$ and $\rho_\mathfrak{p}$ coincide and $\mathrm{inv}(\mathfrak{L}:\mathfrak{V}(\mathfrak{g}))=\mathrm{inv}(\mathfrak{M}:\mathfrak{V}(\mathfrak{g}))$.

"Only if" part. For every $\mathfrak{p}$, there exist integral elements $\sigma_\mathfrak{p}$, $\tau_\mathfrak{p}$ such that $\mathfrak{L}_\mathfrak{p}\sigma_\mathfrak{p}=\mathfrak{V}(\mathfrak{g})_\mathfrak{p}$ and $\mathfrak{M}_\mathfrak{p}\cdot\tau_\mathfrak{p}=\mathfrak{V}(\mathfrak{g})_\mathfrak{p}$. Put $\mathfrak{L}'_\mathfrak{p}=\mathfrak{V}(\mathfrak{g})_\mathfrak{p}\cdot\sigma_\mathfrak{p}^*=\mathfrak{L}_\mathfrak{p}\cdot m(\sigma_\mathfrak{p})$ and $\mathfrak{M}'_\mathfrak{p}=\mathfrak{V}(\mathfrak{g})_\mathfrak{p}\cdot\tau_\mathfrak{p}^*=\mathfrak{M}_\mathfrak{p}m(\tau_\mathfrak{p})$. Then $\mathfrak{L}'_\mathfrak{p}$ and $\mathfrak{M}'_\mathfrak{p}$ are contained in $\mathfrak{V}(\mathfrak{g})_\mathfrak{p}$. By assumption and by Proposition 6.6 for $\mathfrak{L}'=\bigcap_\mathfrak{p}(\mathfrak{L}'_\mathfrak{p}\cap\mathfrak{V})$ and $\mathfrak{M}'=\bigcap_\mathfrak{p}(\mathfrak{M}'_\mathfrak{p}\cap\mathfrak{V})$ we can prove the existence of $\sigma\in\Gamma^0(\mathfrak{V}(\mathfrak{g}),J)$ such that $\mathfrak{L}'\sigma=\mathfrak{M}'$. By assumption $m(\sigma_\mathfrak{p})=m(\tau_\mathfrak{p})$ holds for every $\mathfrak{p}$, hence $\mathfrak{L}\sigma=\mathfrak{M}$ holds. Q. E. D.

Put $\mathfrak{L}_{\lambda,\mathfrak{p}}=\mathfrak{V}(\mathfrak{g})_\mathfrak{p}\cdot x_{\lambda,\mathfrak{p}}^{-1}$ and $\mathfrak{L}_\lambda=\bigcap_\mathfrak{p}(\mathfrak{L}_{\lambda,\mathfrak{p}}\cap\mathfrak{V})$. Here we put $x_1=(\cdots,1_{2n},\cdots)$ so that $\mathfrak{L}_1=\mathfrak{V}(\mathfrak{g})$.

Put

$$\mathfrak{U}_{\lambda,0}=x_\lambda\mathfrak{U}_0x_\lambda^{-1}$$

for every λ. Then we have

$$\mathfrak{U}_{\lambda,0}=\Gamma^0(\mathfrak{L}_{\lambda,0},J)\,.$$

Let

$$\mathfrak{O}_1=\begin{pmatrix}\mathfrak{g},\cdots\mathfrak{g}\\ \vdots\quad\ \vdots\\ \mathfrak{g},\cdots\mathfrak{g}\end{pmatrix},\mathfrak{O}_2,\cdots,\mathfrak{O}_h$$

be non-equivalent maximal orders in $\mathcal{M}(2n,F)$ and $\Gamma_\lambda=G^0(\mathfrak{V},J)\cap\mathfrak{O}_\lambda$ for every λ. (Here, $\Gamma_1=\Gamma^0(\mathfrak{g}),J)$.) We see easily that

$$\Gamma_\lambda=\Gamma^0(\mathfrak{V}_\lambda,J)=\bigcap_\mathfrak{p}(\mathfrak{U}_{\lambda,\mathfrak{p}}\cap G)\,.$$

Define

$$\Theta_\lambda = \{\mathfrak{L} \mid \text{maximal } \mathfrak{g}\text{-lattice such that } \mathfrak{L} \supset \mathfrak{L}_\lambda\}.$$

Let $\mathfrak{L} \supset \mathfrak{M}$ be two $\mathfrak{g}$-lattices in Θ_λ, so $\mathfrak{L}_\mathfrak{p} \supset \mathfrak{M}_\mathfrak{p} \supset \mathfrak{L}_{\lambda,\mathfrak{p}}$ for every $\mathfrak{p}$. There exists an element $\sigma_\mathfrak{p}$ of $G_\mathfrak{p}$ such that

$$\mathfrak{L}_\mathfrak{p} x_{\lambda,\mathfrak{p}} \sigma_\mathfrak{p} = \mathfrak{B}(\mathfrak{g})_\mathfrak{p}.$$

Put $\mathfrak{M}'_\mathfrak{p} = \mathfrak{M}_\mathfrak{p} x_{\lambda,\mathfrak{p}} \sigma_\mathfrak{p}$ and $\mathfrak{M}' = \bigcap_\mathfrak{p} (\mathfrak{M}'_\mathfrak{p} \cap \mathfrak{B})$. $\mathfrak{M}'$ is contained in Θ. We define $(\mathfrak{L} : \mathfrak{M})$ and $\mathrm{inv}(\mathfrak{L} : \mathfrak{M})$ as $(\mathfrak{B}(\mathfrak{g}) : \mathfrak{M}')$ and $\mathrm{inv}(\mathfrak{B}(\mathfrak{g}) : \mathfrak{M}')$, respectively.

By Proposition 7.1, we can show the following

PROPOSITION 7.2. *Let $\mathfrak{L}$, $\mathfrak{M}$ be two $\mathfrak{g}$-lattices in Θ_λ. Then* $\mathrm{inv}(\mathfrak{L} : \mathfrak{L}_\lambda) = \mathrm{inv}(\mathfrak{M} : \mathfrak{L}_\lambda)$ *holds if and only if $\mathfrak{L}$ and $\mathfrak{M}$ are Γ_λ-equivalent.*

Now we shall show the following approximation theorem.

PROPOSITION 7.3. *Let α be an element of G. Then*

$$\mathfrak{U}_\mu \alpha \mathfrak{U}_\lambda = \mathfrak{U}_\mu \alpha \Gamma_\lambda = \Gamma_\mu \alpha \mathfrak{U}_\lambda.$$

PROOF. $u_{\lambda,\mathfrak{p}}$ being an element of $\mathfrak{U}_{\lambda,\mathfrak{p}}$, put

$$\mathfrak{L}_{\lambda,\mathfrak{p}} x_{\lambda,\mathfrak{p}} x_{\mu\mathfrak{p}}^{-1} \alpha u_{\lambda,\mathfrak{p}} = \mathfrak{M}_\mathfrak{p}.$$

We may suppose that $\mathfrak{L}_{\lambda,\mathfrak{p}} \supset \mathfrak{M}_\mathfrak{p}$ for every $\mathfrak{p}$. Then, as $\mathfrak{L}_{\mu,\mathfrak{p}} \alpha u_{\lambda,\mathfrak{p}} = \mathfrak{M}_\mathfrak{p}$ holds, we have

$$\mathrm{inv}(\mathfrak{L}_\lambda : \mathfrak{L}_\mu \alpha) = \mathrm{inv}(\mathfrak{L}_\lambda : \mathfrak{M}).$$

Hence by Proposition 7.2, there exists an element σ_λ of Γ_λ such that $\mathfrak{L}_\mu \alpha \sigma_\lambda = \mathfrak{M}$. For every $\mathfrak{p}$, we have

$$\mathfrak{U}(\mathfrak{g})_\mathfrak{p} x_{\mu,\mathfrak{p}}^{-1} \alpha u_{\lambda,\mathfrak{p}} = \mathfrak{L}_{\mu,\mathfrak{p}} \alpha \sigma_\lambda = \mathfrak{B}(\mathfrak{g})_\mathfrak{p} x_{\mu,\mathfrak{p}}^{-1} \alpha \sigma_\lambda,$$

hence there exists an element $v_\mathfrak{p}$ of $\Gamma^0(\mathfrak{B}(\mathfrak{g})_\mathfrak{p}, J)$ such that

$$v_\mathfrak{p} x_{\mu,\mathfrak{p}}^{-1} \alpha \sigma_\lambda = x_{\mu,\mathfrak{p}}^{-1} \alpha u_{\lambda,\mathfrak{p}}.$$

Hence, $x_{\mu,\mathfrak{p}} v_\mathfrak{p} x_{\mu,\mathfrak{p}}^{-1} \alpha \sigma_\lambda = \alpha u_{\lambda,\mathfrak{p}}$ and $\alpha u_\mathfrak{p} \in \mathfrak{U}_\mu \alpha \sigma_\lambda$.

Therefore, $\mathfrak{U}_\mu \alpha \Gamma_\mu$ contains $\mathfrak{U}_\mu \alpha \mathfrak{U}_\lambda$. Since $\mathfrak{U}_\mu \alpha \Gamma_\kappa$ is contained in $\mathfrak{U}_\mu \alpha \mathfrak{U}_\kappa$, we have $\mathfrak{U}_\mu \alpha \Gamma_\lambda = \mathfrak{U}_\mu \alpha \mathfrak{U}_\lambda$.

Next, for $\alpha^* = m(\alpha)\alpha^{-1}$, we have in the same way $\mathfrak{U}_\lambda \alpha^* \mathfrak{U}_\mu = \mathfrak{U}_\lambda \alpha^* \Gamma_\mu$. Observing the proof of Proposition 6.1, we obtain $\mathfrak{U}_\mu \alpha \mathfrak{U}_\lambda = \Gamma_\mu \alpha \mathfrak{U}_\lambda$. Q. E. D.

8. x_λ and $\mathfrak{U}_\lambda$ being as in § 7, we can define $\mathfrak{R}_{\lambda\mu}$ with $\Gamma_1, \cdots, \Gamma_h$ and G, since they satisfy the conditions (HI, II) in § 5.

PROPOSITION 8.1. *Let β, α be elements of G. If $(\Gamma_\kappa \beta \Gamma_\lambda)(\Gamma_\lambda \alpha \Gamma_\mu) = \sum c_\gamma (\Gamma_\kappa \gamma \Gamma_\mu)$ then $T(x_\kappa^{-1} \beta x_\lambda) T(x_\lambda^{-1} \alpha x_\mu) = \sum c_\gamma T(x_\kappa^{-1} \gamma x_\mu)$.*

PROOF. Let $\Gamma_\lambda \alpha \Gamma_\mu = \bigcup_{i=1}^{d} \alpha_i \Gamma_\mu$ be a disjoint sum. Then by Proposition 7.1, and as Γ_μ is contained in $\mathfrak{U}_\mu$, we have

$$\mathfrak{U}_\lambda\alpha\mathfrak{U}_\mu=\Gamma_\lambda\alpha\mathfrak{U}_\mu=\bigcup_{i=1}^{d}\alpha_i\mathfrak{U}_\mu$$

and this is a disjoint sum. For, we have

$$x_\lambda\mathfrak{U}_0x_\lambda^{-1}\alpha x_\mu\mathfrak{U}_0x_\mu^{-1}=\bigcup_{i=1}^{d}\alpha_ix_\mu\mathfrak{U}_0x_\mu^{-1}\quad\text{and}\quad\mathfrak{U}_0(x_\lambda^{-1}\alpha x_\mu)\mathfrak{U}_0=\bigcup_{i=1}^{d}(x_\lambda^{-1}\alpha x_\mu)\mathfrak{U}_0\,.$$

If $x_\lambda^{-1}\alpha_ix_\mu\mathfrak{U}_0=x_\lambda^{-1}\alpha_jx_\mu\mathfrak{U}_0$, then $\alpha_ix_\mu\mathfrak{U}_0=\alpha_jx_\mu\mathfrak{U}_0$, hence $\alpha_i\mathfrak{U}_\mu=\alpha_j\mathfrak{U}_\mu$. Since $\Gamma_\mu=\bigcap_{\mathfrak{p}}\mathfrak{U}_{\mu,\mathfrak{p}}\cap G$, we have $\alpha_i\Gamma_\mu=\alpha_j\Gamma_\mu$ and $i=j$.

In the same way, $\Gamma_\kappa\beta\Gamma_\lambda=\bigcup_{j=1}^{f}\beta_j\Gamma_\lambda$ being disjoint, we have $\mathfrak{U}_0x_\kappa^{-1}\beta x_\lambda\mathfrak{U}_0=\bigcup_{j=1}^{f}x_\kappa^{-1}\beta_jx_\lambda\mathfrak{U}_0$ and this is a disjoint sum.

Now

$$\begin{aligned}T(x_\kappa^{-1}\beta x_\lambda)\cdot T(x_\lambda^{-1}\alpha x_\mu)&=\mathfrak{U}_0x_\kappa^{-1}\beta x_\lambda\mathfrak{U}_0\mathfrak{U}_0x_\lambda^{-1}\alpha x_\mu\mathfrak{U}_0=x_\kappa^{-1}\mathfrak{U}_\kappa\beta\mathfrak{U}_\lambda\alpha x_\mu\mathfrak{U}_0\\&=\bigcup_{j=1}^{f}x_\kappa^{-1}\beta_j\mathfrak{U}_\lambda\alpha x_\mu\mathfrak{U}_0=\bigcup_{i,j}x_\kappa^{-1}\beta_j\alpha_ix_\mu\mathfrak{U}_0\,.\end{aligned}$$

Let γ be an element of $\Gamma_\kappa\beta\Gamma_\lambda\alpha\Gamma_\mu$. If $\beta_j\alpha_i\Gamma_\mu=\gamma\Gamma_\mu$, then $\beta_j\alpha_i\mathfrak{U}_\mu=\gamma\mathfrak{U}_\mu$ and $\beta_j\alpha_ix_\mu\mathfrak{U}_0x_\mu^{-1}=\gamma x_\mu\mathfrak{U}_0x_\mu^{-1}$ and so

$$(x_\kappa^{-1}\beta_jx_\lambda)(x_\lambda^{-1}\alpha_ix_\mu)\mathfrak{U}_0=(x_\kappa^{-1}\gamma x_\mu)\mathfrak{U}_0\qquad\text{holds.}$$

Conversely, if the last equality holds, then we have $\beta_j\alpha_i\Gamma_\mu=\gamma\Gamma_\mu$. Hence by the proof of Proposition 6.3, the multiplicity of $\Gamma_\kappa\gamma\Gamma_\mu$ in $(\Gamma_\kappa\beta\Gamma_\lambda)(\Gamma_\lambda\alpha\Gamma_\mu)$ and the multiplicity of $T(x_\kappa^{-1}\gamma x_\mu)$ in $T(x_\kappa^{-1}\beta x_\lambda)T(x_\lambda^{-1}\alpha x_\mu)$ are the same, so we have Proposition.

Put $e=\mathrm{inv}\,(\mathfrak{L}:\mathfrak{M})$ with $\mathfrak{L}$, $\mathfrak{M}$ in Θ_λ. For every $\mathfrak{p}$, there exists an element $\tau_\mathfrak{p}\in\Gamma(\mathfrak{L}_{\lambda,\mathfrak{p}},J)$ such that $\mathfrak{L}_\mathfrak{p}\tau_\mathfrak{p}=\mathfrak{M}_\mathfrak{p}$. Now $(m(\tau_\mathfrak{p}))$ is an ideal in F and uniquely determined by e, so let it be denoted by $N(e)$. If $e=\mathrm{inv}\,(\mathfrak{L}_\lambda;\mathfrak{L}_\mu\alpha)$ with $\alpha\in G$, we denote

$$T(e)=\mathfrak{U}_0x_\mu^{-1}\alpha x_\lambda\mathfrak{U}_0\quad\text{and}\quad T_{\mu\lambda}(e)=\Gamma_\mu\alpha\Gamma_\lambda$$

where $T(e)$ was denoted by $T(x_\mu^{-1}\alpha x_\lambda)$ in the above. Then by Proposition 8.1 yields that if $T_{\kappa\lambda}(e)T_{\lambda\mu}(f)=\sum c_kT_{\kappa\mu}(g_k)$, then

$$T(e)T(f)=\sum c_kT(g_k)\,.$$

Let $\mathfrak{a}$ be an ideal in F. We define

$$T(\mathfrak{a})=\sum T(e)$$

where the sum is extended over all invariants e such that $N(e)=\mathfrak{a}$.

Let $T(e)=\mathfrak{U}_0\tau\mathfrak{U}_0=\mathfrak{U}_0x_\mu^{-1}\alpha x_\lambda\mathfrak{U}_0$ be an element of $\mathfrak{R}(\mathfrak{U}_0,G)$. Then $T(e)=x_\mu^{-1}\mathfrak{U}_\mu\alpha\mathfrak{U}_\lambda x_\lambda^{-1}$. For every $\mathfrak{p}$, we have

$$\mathfrak{L}_{\lambda,\mathfrak{p}}/\mathfrak{L}_{\mu,\mathfrak{p}}\alpha=\mathfrak{B}(\mathfrak{g})_\mathfrak{p}x_{\lambda,\mathfrak{p}}^{-1}/\mathfrak{B}(\mathfrak{g})_\mathfrak{p}x_{\mu,\mathfrak{p}}^{-1}\alpha\,.$$

Since $(m(\alpha))$ is principal, there exists an element σ of G such that $\mathfrak{L}_\lambda\sigma=\mathfrak{L}_\mu\alpha$. By Proposition 7.3, σ is contained in Γ_λ. Hence

$$\Gamma_\mu\alpha\sigma^{-1}\Gamma_\lambda=\Gamma_\mu\alpha\Gamma_\lambda$$

and e defines $T_{\mu\lambda}(e)=\Gamma_\mu\alpha\Gamma_\lambda$. By Proposition 8.1, we see that $\begin{pmatrix} T_{1,\lambda_1}(e) \\ \vdots \\ T_{h,\lambda_h}(e)\end{pmatrix}$ is the representation of $T(e)$.

Here we notice that for given μ, λ is uniquely determined, observing the ideal class of $\mathfrak{a}$.

§ 2. Construction of abelian varieties

We shall define the space $\mathcal{H}(n,r)$ as follows:
Let $Z^{(i)}$ be a symmetric n-n matrix for every $i=1,\cdots,r$ and

$$Z=(Z^{(1)},\cdots,Z^{(r)}).$$

Define

$$\mathcal{H}(n,r)=\{Z=(Z^{(1)},\cdots,Z^{(r)})|\operatorname{Im}Z^{(i)}>0 \text{ or } <0\}.$$

$\mathcal{H}(n,r)$ is non-connected and has 2^r connected components.

We define an operator on $\mathcal{H}(n,r)$ for every element σ of $\boldsymbol{J}_{G,\infty}$ as follows:

$$Z\to\sigma[Z]=(\sigma^{(1)}[Z^{(1)}],\cdots,\sigma^{(r)}[Z^{(r)}]),$$
$$\sigma^{(i)}[Z^{(i)}]=(a^{(i)}Z^{(i)}+b^{(i)})(c^{(i)}Z^{(i)}+d^{(i)})^{-1} \qquad \text{for every } i,$$
$$\sigma^{(i)}=\begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)}\end{pmatrix} a^{(i)},b^{(i)},c^{(i)},d^{(i)}\in\mathcal{M}(n,\boldsymbol{R}).$$

For a $\mathfrak{g}$-lattice $\mathfrak{L}$ in $\mathfrak{V}$, there exists an element τ of G such that

$$\mathfrak{L}=[\mathfrak{g},\cdots,\mathfrak{g},\mathfrak{a}_1,\cdots,\mathfrak{a}_n]\tau,$$

by Lemma 1.1. Put

$$\Gamma^0(\mathfrak{a}_1,\cdots,\mathfrak{a}_n)=\{\tau\sigma\tau^{-1}|\sigma\in\Gamma^0(\mathfrak{L},J)\}.$$

This group is called the Hilbert-Siegel para-modular group of type $(\mathfrak{a}_1,\cdots,\mathfrak{a}_n)$.

Then the following facts are well-known. $\Gamma^0(\mathfrak{a}_1,\cdots,\mathfrak{a}_n)$ operates discontinuously on $\mathcal{H}(n,r)$ and $\Gamma^0(\mathfrak{a}_1,\cdots,\mathfrak{a}_n)\backslash\mathcal{H}(n,r)$ is of finite measure, where the volume element dv is given by

$$dv=\prod_{i=1}^{r}\frac{d(\operatorname{Re}Z^{(i)})\,d(\operatorname{Im}Z^{(i)})}{\det|\operatorname{Im}Z^{(i)}|^2},$$
$$\operatorname{Re}Z^{(i)}=(x_{p,q}^{(i)}),\qquad d(\operatorname{Re}Z^{(i)})=\prod_{p\le q}dx_{p,q}^{(i)},$$

$$\operatorname{Im} Z^{(i)}=(y_{p,q}^{(i)})\,,\quad \text{and}\quad d(\operatorname{Im} Z^{(i)})=\prod_{p\leq q} dy_{p,q}^{(i)}\,.$$

2. Let A be an abelian variety and $\mathcal{C}$ a polarization of A; namely for a positive divisor X on A, $\mathcal{C}=\{X' \mid$ positive divisor and $mX\equiv m'X'$ with positive integers $m, m'\}$ contains an ample divisor on A. A couple $(A,\mathcal{C})$ is called a polarized abelian variety. Let $(A,\mathcal{C})$ and $(A',\mathcal{C}')$ be two polarized abelian varieties and λ a homomorphism (an isomorphism) of A onto A'. Then λ is called a homomorphism (an isomorphism) of $(A,\mathcal{C})$ onto $(A',\mathcal{C}')$ if there exists a divisor X' in $\mathcal{C}'$ such that $\lambda^{-1}(X')$ is contained in $\mathcal{C}$. $\mathfrak{r}$ being a ring having a finite basis over $\boldsymbol{Z}$, we understand by a polarized abelian variety of type $\mathfrak{r}$ a triplet $(A,\mathcal{C},\iota)$ formed by a couple $(A,\mathcal{C})$ and an isomorphism ι of $\mathfrak{r}$ into $\mathcal{A}(A)$. A homomorphism (an isomorphism) λ of $(A,\mathcal{C})$ onto $(A',\mathcal{C}')$ is called a homomorphism (an isomorphism) of $(A,\mathcal{C},\iota)$ onto $(A',\mathcal{C}',\iota')$ if λ is compatible with ι and ι', that is,

$$\lambda\cdot\iota'(\alpha)=\iota(\alpha)\cdot\lambda \qquad \text{for any } \alpha\in\mathcal{A}(A)\,.$$

Every divisor X on A such that $X\equiv 0$ defines a point $\mathrm{Cl}\,(X)$ of B, the Picard variety of A. For every element α of $\mathcal{A}(A)$, an element β of $\mathcal{A}(B)$ is defined by $\beta(\mathrm{Cl}\,(X))=\mathrm{Cl}\,(\alpha^{-1}(X))$. The mapping $\alpha\to\beta$ is an anti-isomorphism of $\mathcal{A}(A)$ into $\mathcal{A}(B)$ and extended to an anti-isomorphism of $\mathcal{A}_0(A)$ into $\mathcal{A}_0(B)$. The mapping φ_X of A into B defined by $\varphi_X(t)=\mathrm{Cl}\,(X_t-X)$, for $t\in A$, is a homomorphism of A into B. φ_X is onto if and only if X is non-degenerate. If so, every $\alpha\in\mathcal{A}_0(A)$ defines an element α^* of $\mathcal{A}_0(A)$ by $\varphi_X^{-1}\cdot\beta\cdot\varphi_X=\alpha^*$.

Now two non-degenerate divisors on A define the same involution on $\mathcal{A}_0(A)$ if they belong to one and the same polarization of A; so every polarization of A defines an involution of $\mathcal{A}_0(A)$.

Characteristic being 0, let $\boldsymbol{C}^m/\Delta$ be an analytic model of A of dimension m. Fixing an analytic isomorphism of A onto $\boldsymbol{C}^m/\Delta$, every non-degenerate positive divisor X on A corresponds to a non-degenerate Riemann form on $\boldsymbol{C}^m/\Delta$; by a non-degenerate Riemann form on $\boldsymbol{C}^m/\Delta$, we mean as usual an $\boldsymbol{R}$-valued $\boldsymbol{R}$-bilinear form on $\boldsymbol{C}^m\times\boldsymbol{C}^m$ satisfying the following conditions:

(RI) $E(x,y)\in\boldsymbol{Z}$ for every $x,y\in\Delta$,

(RII) $E(x,y)=-E(y,x)$,

(RIII) $E(x,\sqrt{-1}\,y)$ is symmetric and positive definite.

Let F be a totally real algebraic number field of degree r, as above. $\mathfrak{r}$ being an order in F, we shall consider a polarized abelian variety $(A,\mathcal{C},\iota)$ of type $\mathfrak{r}$ with the following properties:

a) $\iota(1)$ is the identity of $\mathcal{A}_0(A)$,

b) The involution * on $\mathcal{A}_0(A)$ which is determined by C is the identity of $\iota(F)$; namely $\iota(\alpha)^*=\iota(\alpha)$ for any $\alpha\in F$,

c) $\mathfrak{r}=\iota^{-1}[\iota(F)\cap\mathcal{A}(A)]$.

Every element $\iota(\alpha)\in\mathcal{A}_0(A)$, $\alpha\in F$, corresponds to a complex matrix of degree m, which is denoted by the same letter $\iota(\alpha)$. As F is totally real, it follows from (a), (b) that $\iota(\alpha)$, considered as a representation of F, is reduced to the form

$$\iota(\alpha)=\begin{pmatrix}\alpha^{(1)}\cdot 1_n & & \\ & \ddots & \\ & & \alpha^{(r)}\cdot 1_n\end{pmatrix}\qquad \text{for every } \alpha\in F,$$

where, in particular, it can be seen that m is divisible by r and so we put $m=nr$.

LEMMA 2.1. *Let x_i for $i=1,\cdots,2n$ be $2n$ vectors of C^{nr}. For every i, put $x_i=(x_i^{(1)},\cdots,x_i^{(r)})$ with vectors $x_i^{(j)}$ in C^n. Let ω_j, $j=1,\cdots,r$ be a basis of F over Q. Then the vectors*

$$(x_i,\bar{x}_i)\begin{pmatrix}\iota(\omega_j) & 0\\ 0 & \iota(\omega_j)\end{pmatrix},\qquad 1\leqslant i\leqslant 2n\,,\quad 1\leqslant j\leqslant r$$

are linearly independent over R if and only if

$$\det\begin{pmatrix}x_1^{(j)} & \bar{x}_1^{(j)}\\ \vdots & \vdots\\ x_{2n}^{(j)} & \bar{x}_{2n}^{(j)}\end{pmatrix}\neq 0\qquad \textit{for every } j.$$

PROOF. See [4].

We have proved the following Theorems in [4], with some change of notation.

THEOREM 2. *Let F be a totally real algebraic number field of degree r, $\mathfrak{M}$ a free submodule of $\mathfrak{V}(2n,F)$ of rank $2nr$ over Z. Let $\mathfrak{u}_1,\cdots,\mathfrak{u}_{2n}$ be $2n$ vectors of C^{nr} such that*

$$\det\begin{pmatrix}U^{(i)} & \bar{U}^{(i)}\\ V^{(i)} & \bar{V}^{(i)}\end{pmatrix}\neq 0\qquad \textit{for every } i, \textit{ where } U^{(i)},V^{(i)}\in\mathcal{M}(n,C)$$

are defined by

$$\begin{pmatrix}\mathfrak{u}_1\\ \vdots\\ \mathfrak{u}_{2n}\end{pmatrix}=\begin{pmatrix}U^{(1)},\cdots,U^{(r)}\\ \\ V^{(1)},\cdots,V^{(r)}\end{pmatrix}.$$

Let P be an element of $GL(2n,F)$ and put

$$\Delta=\left\{(\alpha_1,\cdots,\alpha_{2n})\begin{pmatrix}U\\ V\end{pmatrix}\,\middle|\,(\alpha_1,\cdots,\alpha_{2n})\in\mathfrak{M}\right\},$$

where $(\alpha_1,\cdots,\alpha_{2n})\begin{pmatrix}U\\ V\end{pmatrix}$ means $\left((\alpha_1^{(1)},\cdots,\alpha_{2n}^{(1)})\begin{pmatrix}U^{(1)}\\ V^{(1)}\end{pmatrix},\cdots,(\alpha_1^{(r)},\cdots,\alpha_{2n}^{(r)})\begin{pmatrix}U^{(r)}\\ V^{(r)}\end{pmatrix}\right)$ which *is a vector in* C^{nr}. *Then* C^{nr}/Δ *is a complex torus.*

Furthermore, define a bilinear form $E(x, y)$ *on* $C^{nr}\times C^{nr}$ *by putting*

$$E(x, y)=\sum_{i=1}^{r}(x_1^{(i)},\cdots,x_{2n}^{(i)})P^{(i)\,t}(y_1^{(i)},\cdots,y_{2n}^{(i)})$$

with

$$x=\left((x_1^{(1)},\cdots,x_{2n}^{(1)})\begin{pmatrix}U^{(1)}\\V^{(1)}\end{pmatrix},\cdots,(x_1^{(r)},\cdots,x_{2n}^{(r)})\begin{pmatrix}U^{(r)}\\V^{(r)}\end{pmatrix}\right)$$

and

$$y=\left((y_1^{(1)},\cdots,y_{2n}^{(1)})\begin{pmatrix}U^{(1)}\\V^{(1)}\end{pmatrix},\cdots,(y_1^{(r)},\cdots,y_{2n}^{(r)})\begin{pmatrix}U^{(r)}\\V^{(r)}\end{pmatrix}\right).$$

Then $E(x, y)$ *is a Riemann form on* C^{nr}/Δ *if and only if the following conditions are satisfied:*

(GI) $\sum_{i=1}^{r}(\alpha_1^{(i)},\cdots,\alpha_{2n}^{(i)})P^{(i)\,t}(\beta_1^{(i)},\cdots,\beta_{2n}^{(i)})\in Z$ *for every* $(\alpha_1,\cdots,\alpha_{2n})$, $(\beta_1,\cdots,\beta_{2n})$ *in* $\mathfrak{M}$,

(GII) $P=-{}^tP$,

(GIII) $({}^tU^{(i)\,t}V^{(i)})\,{}^tP^{(i)-1}\begin{pmatrix}U^{(i)}\\V^{(i)}\end{pmatrix}=0$ *for every* i,

(GIV) $-\sqrt{-1}\,({}^tU^{(i)\,t}V^{(i)})\,{}^tP^{(i)-1}\begin{pmatrix}\overline{U}^{(i)}\\\overline{V}^{(i)}\end{pmatrix}>0$ *for every* i.

THEOREM 3. *Notation being as above, let* ι *be the representation of* F *of degree* nr *defined by*

$$\iota(\alpha)=\begin{pmatrix}\alpha^{(1)}\cdot 1_n & & \\ & \ddots & \\ & & \alpha^{(r)}\cdot 1_n\end{pmatrix}\qquad \text{for } \alpha\in F.$$

Take $\mathfrak{M}$, $\begin{pmatrix}U^{(1)},\cdots,U^{(r)}\\V^{(1)},\cdots,V^{(r)}\end{pmatrix}$ *and* P *satisfying the conditions* (GI, II, III, IV). *Put* $\mathfrak{r}=\{\alpha\in F|\alpha\mathfrak{M}\subset\mathfrak{M}\}$. *Let* $E(x, y)$ *be the Riemann form defined in* Theorem 1. *Then there exists an analytic isomorphism of the complex torus* C^{nr}/Δ *onto an abelian variety* A *defined over* C. $\alpha\rightarrow\iota(\alpha)$ *gives an isomorphism of* F *into* $\mathcal{A}_0(A)$ *such that the properties* a), c) *hold.* $E(x, y)$ *determines a polarization* $\mathcal{C}$ *of* A *such that the property* b) *holds.*

Conversely, $(A, \mathcal{C}, \iota)$ *being a polarized abelian variety of type* $\mathfrak{r}$ *such that* a), b) *and* c) *hold, then* ι *reduces to the above form and* A *is obtained in the above manner from a lattice* $\mathfrak{M}$, $2n$ *vectors* $\mathfrak{u}_1,\cdots,\mathfrak{u}_{2n}$ *and an element* P *of* $GL(2n, F)$ *and* $\mathcal{C}$ *is determined by the Riemann form* $E(x, y)$ *given in* Theorem 2.

Thus the data $\mathfrak{M}$, U, V, P determine a triplet $(A, \mathcal{C}, \iota)$ of type $\mathfrak{r}$, $\mathfrak{r}$ being an order in F. We denote $(A, \mathcal{C}, \iota)=\mathfrak{P}(\mathfrak{M}, U, V, P)$.

3. Let $(A, \mathcal{C}, \iota)=\mathfrak{P}(\mathfrak{M}, U, V, P)$ and $(A', \mathcal{C}', \iota')=\mathfrak{P}(\mathfrak{M}', U', V', P')$ be two polarized abelian varieties of type $\mathfrak{r}$ of the same dimension.

Let $\boldsymbol{C}^{nr}/\Delta$ and $\boldsymbol{C}^{nr}/\Delta'$ be analytic models of A and A', respectively. If $(A, \mathcal{C}, \iota)$ and $(A', \mathcal{C}', \iota')$ are isomorphic, then the following conditions are satisfied:

(i) There is a $\boldsymbol{C}$-linear mapping λ of $\boldsymbol{C}^{nr}$ into $\boldsymbol{C}^{nr}$ such that $\Delta\cdot\lambda=\Delta'$,

(ii) For any α in F, $\lambda\cdot\iota'(\alpha)=\iota(\alpha)\lambda$,

(iii) There is a positive $c\in\boldsymbol{Q}$ such that $E'(x\lambda, y\lambda)=cE(x, y)$,

where E and E' are Riemann forms on $\boldsymbol{C}^{nr}/\Delta$ and $\boldsymbol{C}^{nr}/\Delta'$, respectively, having the principal matrices P and P'.

λ, as considered the matrix corresponding to λ, reduces to the form

$$\lambda=\begin{pmatrix} \lambda^{(1)} & & \\ & \ddots & \\ & & \lambda^{(r)} \end{pmatrix}$$

with complex n-n matrix $\lambda^{(i)}$.

We have proved the following Theorem in [4].

THEOREM 4. (i) *$\mathfrak{P}(\mathfrak{M}, U, V, P)$ and $\mathfrak{P}(\mathfrak{M}', U', V', P')$ are isomorphic if and only if there exists an element σ in $GL(2n, F)$ such that*

$$\mathfrak{M}\sigma=\mathfrak{M}', \qquad UV^{-1}=(aU'+bV')(cU'+dV')^{-1}$$

and

$$\sigma P'\,{}^t\sigma=cP \qquad \text{with some positive } c\in\boldsymbol{Q},$$

where

$$\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(ii) *If $\mathfrak{r}=\mathfrak{g}$, then $\mathfrak{P}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n; Z)$ and $\mathfrak{P}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n; Z')$, for $Z, Z'\in\mathcal{H}(n, r)$ with $\operatorname{Im} Z>0$ and $\operatorname{Im} Z'>0$, are isomorphic if and only if there exists an element $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\Gamma^0(\mathfrak{d}^{-1}\mathfrak{a}_1, \cdots, \mathfrak{d}^{-1}\mathfrak{a}_n)/\{\pm 1\}$ such that*

$$Z=(aZ'+b)(cZ'+d)^{-1}.$$

where $\mathfrak{d}$ denotes the differente of F.

$\mathfrak{P}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n; Z)$ is defined as follows:

Consider $\mathfrak{P}(\mathfrak{M}, U, V, P)$. By Lemma 1.1, $(\mathfrak{M}, P)$ is reduced to the canonical form

$$\begin{aligned} &\mathfrak{M}=\mathfrak{g}x_1+\cdots+\mathfrak{g}x_n+\mathfrak{a}_1 y_1+\cdots+\mathfrak{a}_n y_n, \\ &P(x_i, y_j)=\delta_{i,j}, \\ &P(x_i, x_j)=P(y_i, y_j)=0. \end{aligned}$$

Hence $\mathfrak{P}(\mathfrak{M}, U, V, P)$ is isomorphic to $\mathfrak{P}(\mathfrak{M}, U', V', J)$ with some U', V'. From Riemann's conditions (GI, II, III, IV) for U', V', we see that V' is non-singular. Put $Z = U'V'^{-1}$. Then we have $\begin{pmatrix} Z \\ 1 \end{pmatrix}$ instead of $\begin{pmatrix} U' \\ V' \end{pmatrix} \cdot V'^{-1}$. Thus we obtain $\mathfrak{P}(\mathfrak{M}, Z, 1, J)$, which we shall denote by $\mathfrak{P}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n; Z)$, isomorphic to $\mathfrak{P}(\mathfrak{M}, U', V', J)$ by Theorem 4, (i).

4. Let G be the subgroup of $GL(2n, F)$ which consists of elements σ such that $\sigma J {}^t\sigma = m(\sigma) J$ with elements $m(\sigma) \in F$; namely $G = G(\mathfrak{B}(2n, F), J)$.

Let $\boldsymbol{J}_G$ be the idelization of G. An element $\boldsymbol{D} = \boldsymbol{D}_\infty \times \boldsymbol{D}_0$ can be written as the following form;

$$\boldsymbol{D}_\infty = \left(\begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & D^{(1)} \end{pmatrix}, \cdots, \begin{pmatrix} A^{(r)} & B^{(r)} \\ C^{(r)} & D^{(r)} \end{pmatrix} \right)$$

with elements $A^{(i)}$, $B^{(i)}$, $C^{(i)}$, $D^{(i)}$ of $\mathcal{M}(n, \boldsymbol{R})$ for every i and considering J as

$$\left(\cdots\cdots, \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \cdots\cdots \right),$$

we have

$$\boldsymbol{J}_G = \{ \boldsymbol{D} \in \boldsymbol{A} \mid \boldsymbol{D} J {}^t\boldsymbol{D} = xJ,\ x \in \boldsymbol{J}_F \},$$

where $\boldsymbol{A}$ is the adelization of $\mathcal{M}(2n, F)$. x is denoted by $m(\boldsymbol{D})$.

For $\boldsymbol{D} \in \boldsymbol{J}_G$, we define

$$U_D = (\sqrt{-1}\, A^{(1)} + B^{(1)}, \cdots, \sqrt{-1}\, A^{(r)} + B^{(r)})$$

and

$$V_D = (\sqrt{-1}\, C^{(1)} + D^{(1)}, \cdots, \sqrt{-1}\, C^{(r)} + D^{(r)}),$$

so they are both complex n-nr matrices. Put $\mathfrak{B}(\mathfrak{g}) = \mathfrak{B}(2n, \mathfrak{g})$ and

$$\mathfrak{M}_D = \bigcap_{\mathfrak{p}} \mathfrak{B}(\mathfrak{g})_\mathfrak{p} \cdot \boldsymbol{D}_\mathfrak{p}^{-1} \cap \mathfrak{B} = \{ (\alpha_1, \cdots, \alpha_{2n}) \in \mathfrak{B} \mid (\alpha_1, \cdots, \alpha_{2n}) \cdot \boldsymbol{D}_0 \in \mathfrak{B}(\mathfrak{g}) \}.$$

Then, $\mathfrak{M}_D$ is a $\mathfrak{g}$-lattice in $\mathfrak{B}$ by Lemma 2.2, and has rank $2nr$ over $\boldsymbol{Z}$. Put

$$\Delta_D = \left\{ (\alpha_1, \cdots, \alpha_{2n}) \begin{pmatrix} U_D \\ V_D \end{pmatrix} \middle| (\alpha_1, \cdots, \alpha_{2n}) \in \mathfrak{M}_D \right\}.$$

As above, for an element α of F, put

$$\iota(\alpha) = \begin{pmatrix} \alpha^{(1)} \cdot 1_n & & \\ & \ddots & \\ & & \alpha^{(r)} \cdot 1_n \end{pmatrix}.$$

Then $\Delta_D \cdot \iota(\alpha) \subset \Delta_D$ for every $\alpha \in \mathfrak{g}$. By Lemma 1.1 of § II and by definition of $\boldsymbol{D}$,

Δ_D is a discrete subgroup of C^{nr} and so C^{nr}/Δ_D is a complex torus. Put $a^{(i)}=m(D^{(i)})$, where $D^{(i)}$ means the i-th part of D_∞, and let η be an element of F such that $a^{(i)}\eta^{(i)}$ is positive for every i, where $\eta^{(i)}$ is the i-th conjugate of η.

Since

$$\begin{pmatrix} A^{(i)} & B^{(i)} \\ C^{(i)} & D^{(i)} \end{pmatrix}\begin{pmatrix} \sqrt{-1}\cdot 1_n & -\sqrt{-1}\cdot 1_n \\ 1_n & 1_n \end{pmatrix}\begin{pmatrix} \sqrt{-1}\cdot 1_n & -\sqrt{-1}\cdot 1_n \\ 1_n & 1_n \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$
$$\times {}^t\begin{pmatrix} \sqrt{-1}\cdot 1_n & -\sqrt{-1}\cdot 1_n \\ 1_n & 1_n \end{pmatrix}^{-1} {}^t\begin{pmatrix} \sqrt{-1}\cdot 1_n & -\sqrt{-1}\cdot 1_n \\ 1_n & 1_n \end{pmatrix} {}^t\begin{pmatrix} A^{(i)} & B^{(i)} \\ C^{(i)} & D^{(i)} \end{pmatrix}$$
$$=\begin{pmatrix} 0 & -a^{(i)}1_n \\ -a^{(i)}1_n & 0 \end{pmatrix}$$

holds for every i, we have

$$\begin{pmatrix} U_D^{(i)} & \bar{U}_D^{(i)} \\ V_D^{(i)} & \bar{V}_D^{(i)} \end{pmatrix}\begin{pmatrix} 0 & -\frac{\sqrt{-1}}{2}1_n \\ \frac{\sqrt{-1}}{2}1_n & 0 \end{pmatrix}\begin{pmatrix} {}^tU_D^{(i)} & {}^tV_D^{(i)} \\ {}^t\bar{U}_D^{(i)} & {}^t\bar{V}_D^{(i)} \end{pmatrix}=\begin{pmatrix} 0 & a^{(i)}1_n \\ -a^{(i)}1_n & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} U_D^{(i)} & \bar{U}_D^{(i)} \\ V_D^{(i)} & \bar{V}_D^{(i)} \end{pmatrix}\begin{pmatrix} 0 & -\frac{\sqrt{-1}}{2}a^{(i)-1}\eta^{(i)}1_n \\ \frac{\sqrt{-1}}{2}a^{(i)-1}\eta^{(i)}1_n & 0 \end{pmatrix}\begin{pmatrix} {}^tU_D^{(i)} & {}^tV_D^{(i)} \\ {}^t\bar{U}_D^{(i)} & {}^t\bar{V}_D^{(i)} \end{pmatrix}=\begin{pmatrix} 0 & \eta^{(i)}1_n \\ -\eta^{(i)}1_n & 0 \end{pmatrix}.$$

Taking the inverse and transpose of both sides, we obtain

(GIII′) $\eta^{(i)-1}({}^tU_D^{(i)}\,{}^tV_D^{(i)})\,{}^tJ^{-1}\begin{pmatrix} U_D^{(i)} \\ V_D^{(i)} \end{pmatrix}=0$ for every i,

(GIV′) $-\sqrt{-1}\,\eta^{(i)-1}({}^tU_D^{(i)}\,{}^tV_D^{(i)})\,{}^tJ^{-1}\begin{pmatrix} \bar{U}_D^{(i)} \\ \bar{V}_D^{(i)} \end{pmatrix}>0$ for every i.

Now we see that, with a suitably chosen positive integer c,

(GI′) $c\mathfrak{M}_D\eta J\,{}^t\mathfrak{M}_D\subset\mathfrak{d}$ holds.

Taking $c\eta J$ to be a principal matrix, this defines a Riemann form on C^{nr}/Δ_D, so we obtain $\wp(\mathfrak{M}_D, U_D, V_D, c\eta J)$. By Lemma 1.2, there exists an element T of G such that $\mathfrak{M}_D=[\mathfrak{g},\cdots,\mathfrak{g},\mathfrak{a}_1,\cdots,\mathfrak{a}_n]T$ and $Tc\eta J\,{}^tT=J$.

Now we define an abelian variety $\wp(\mathfrak{M}_D, U_D, V_D, \eta, J)$ by the condition (GIII′), (GIV′) and $\mathfrak{M}_D J\,{}^t\mathfrak{M}_D\subset\mathfrak{d}$. Thus we obtain $\wp([\mathfrak{g},\cdots,\mathfrak{g},\mathfrak{a}_1,\cdots,\mathfrak{a}_n], U_D, V_D, \eta, J)$ which is isomorphic to $\wp(\mathfrak{M}_D, U_D, V_D, c\eta J)$. Furthermore, by (GIV′), V_D is non-singular. Putting $Z_D=U_DV_D^{-1}$, we see that $\eta\,\mathrm{Im}\,Z_D$ is totally positive, hence $m(D_\infty)\,\mathrm{Im}\,Z_D$ is totally positive.

Thus we obtain $\wp(\mathfrak{a}_1,\cdots,\mathfrak{a}_n; m(D_\infty)Z_D)$ which satisfies the following conditions:

(I) $[\mathfrak{g},\cdots,\mathfrak{g},\mathfrak{a}_1,\cdots,\mathfrak{a}_n]J^t[\mathfrak{g},\cdots,\mathfrak{g},\mathfrak{a}_1,\cdots,\mathfrak{a}_n]\subset\mathfrak{d}$,

(II) $Z_D\in\mathfrak{H}(n,r)$ and $m(\boldsymbol{D}_\infty)\operatorname{Im} Z_D$ is totally positive.

Conversely, we shall show that for a given polarized abelian variety $\mathfrak{P}(\mathfrak{M},U,V,P)$ of type $\mathfrak{g}$, there exists an element $\boldsymbol{D}$ of $\boldsymbol{J}_G$ such that $\mathfrak{P}(\mathfrak{a}_1,\cdots,\mathfrak{a}_n;m(\boldsymbol{D}_\infty)Z_D)$ is isomorphic to $\mathfrak{P}(\mathfrak{M},U,V,P)$.

Firstly, we shall reformulate Theorm 4 in the language of $\boldsymbol{J}_G$.

For two elements $\boldsymbol{D}$, $\boldsymbol{D}'$ of $\boldsymbol{J}_G$, $\boldsymbol{D}_\infty=\boldsymbol{D}'_\infty$ holds if and only if $U_D=U_{D'}$ and $V_D=V_{D'}$. Furthermore we can see that

(*) $\boldsymbol{D}_\infty$ and $\boldsymbol{D}'_\infty$ determine one and the same $Z\in\mathfrak{H}(n,r)$ if and only if there exists an element K of $GL(n,C)$ such that $U_D=U_{D'}\cdot K$, $V_D=V_{D'}\cdot K$ hold.

If such a K exists for U_D, V_D, $U_{D'}$, $V_{D'}$, it follows that

$$\begin{pmatrix} A_D^{(i)} & B_D^{(i)} \\ C_D^{(i)} & D_D^{(i)} \end{pmatrix}=\begin{pmatrix} A_{D'}^{(i)} & B_{D'}^{(i)} \\ C_{D'}^{(i)} & D_{D'}^{(i)} \end{pmatrix}\cdot\begin{pmatrix} \dfrac{K^{(i)}+\bar{K}^{(i)}}{2} & \dfrac{\sqrt{-1}(K^{(i)}-\bar{K}^{(i)})}{2} \\ \dfrac{-\sqrt{-1}(K^{(i)}-\bar{K}^{(i)})}{2} & \dfrac{K^{(i)}+\bar{K}^{(i)}}{2} \end{pmatrix}$$

holds for every i. Hence putting $K=S-\sqrt{-1}\,T$ with real S, T, we see that

$$(**)\qquad \boldsymbol{D}_\infty=\boldsymbol{D}'_\infty\cdot\begin{pmatrix} S & T \\ -T & S \end{pmatrix},\qquad \begin{pmatrix} S & T \\ -T & S \end{pmatrix}\in\boldsymbol{J}_{G,\infty}.$$

Conversely, if there exists an element $\begin{pmatrix} S & T \\ -T & S \end{pmatrix}$ of $\boldsymbol{J}_{G,\infty}$ such that $\boldsymbol{D}_\infty=\boldsymbol{D}'_\infty\begin{pmatrix} S & T \\ -T & S \end{pmatrix}$, then we see that $U_D=U_{D'}K$, $V_D=V_{D'}K$ hold with $K=S-\sqrt{-1}\,T$. Therefore by (*)

$\boldsymbol{D}_\infty$ and $\boldsymbol{D}'_\infty$ determine one and the same $Z\in\mathfrak{H}(n,r)$ if and only if (**) holds with $\begin{pmatrix} S & T \\ -T & S \end{pmatrix}\in\boldsymbol{J}_{G,\infty}$.

Furthermore we see that $m\left(\begin{pmatrix} S & T \\ -T & S \end{pmatrix}\right)$ is totally positive by (*) and so $m(\boldsymbol{D}_\infty)\cdot m(\boldsymbol{D}'_\infty)$ is totally positive.

Now $\begin{pmatrix} S & T \\ -T & S \end{pmatrix}$ does not move $\sqrt{-1}\cdot 1_n$ under the operation defined in 1, § II, and we have operated $\boldsymbol{D}_\infty$ to $\sqrt{-1}\cdot 1_n$ to obtain the space $\mathfrak{H}(n,r)$. Let ε_β be any one of 2^r elements $(\pm1,\cdots,\pm1)$. Then we can see that the element $\boldsymbol{D}_\infty\in\boldsymbol{J}_{G,\infty}$ which move $\varepsilon_\beta\sqrt{-1}\cdot 1_n$ to $\sqrt{-1}\cdot 1_n$ is of the form $\boldsymbol{D}_\infty=\begin{pmatrix} \varepsilon_\beta S & T \\ \varepsilon_\beta(-T) & S \end{pmatrix}$ with $\begin{pmatrix} S & T \\ -T & S \end{pmatrix}\in\boldsymbol{J}_{G,\infty}$, and it follows that

$$m\left(\begin{pmatrix} \varepsilon_\beta S & T \\ \varepsilon_\beta(-T) & S \end{pmatrix}\right)=\varepsilon_\beta m\left(\begin{pmatrix} S & T \\ -T & S \end{pmatrix}\right)\quad \text{hold for every } \varepsilon_\beta.$$

Hence we can say that $Z_D=Z_{D'}$ holds if and only if there exists an element $K=\begin{pmatrix} \varepsilon_\beta S & T \\ \varepsilon_\beta(-T) & S \end{pmatrix}$ of $\boldsymbol{J}_{G,\infty}$ with totally positive $m(K)$ such that $\boldsymbol{D}_\infty=\boldsymbol{D}'_\infty\cdot K$.

Let $\{x_i, y_i\}$ be a basis of $\mathfrak{M}_D$ for a given $\mathfrak{P}(\mathfrak{M}_D, U_D, V_D, c\eta J)$. If we take the basis $\{x_i/c\eta, y_i\}$ instead of $\{x_i, y_i\}$, the principal matrix becomes J; namely we get $\mathfrak{P}(\mathfrak{M}_D, U_D, V_D, \eta, J)$ which is isomorphic to $\mathfrak{P}(\mathfrak{M}_D, U_D, V_D, c\eta J)$. Putting $Z_D = U_D V_D^{-1}$, we get $\mathfrak{P}(\mathfrak{M}_D, Z_D, 1, \eta, J)$ which is isomorphic to $\mathfrak{P}(\mathfrak{M}_D, U_D, V_D, \eta, J)$.

In the following, we shall consider abelian varieties of this form $\mathfrak{P}(\mathfrak{M}_D, Z_D, 1, \eta, J)$. If $\mathfrak{M}_D = \mathfrak{M}_{D'}$ holds, then for every $\mathfrak{p}$, $\mathfrak{M}_{D,\mathfrak{p}} = \mathfrak{M}_{D',\mathfrak{p}}$ holds. Hence we have $\mathfrak{M}_{D,\mathfrak{p}} = \mathfrak{B}(\mathfrak{g})_{\mathfrak{p}} D_{\mathfrak{p}}^{-1} = \mathfrak{M}_{D',\mathfrak{p}} = \mathfrak{B}(\mathfrak{g})_{\mathfrak{p}} D_{\mathfrak{p}}'^{-1}$ and there exists an element $\sigma_{\mathfrak{p}}$ of $\Gamma^0(\mathfrak{B}(\mathfrak{g}), J)$ for every $\mathfrak{p}$ such that $\sigma_{\mathfrak{p}} D_{\mathfrak{p}}^{-1} = D_{\mathfrak{p}}'^{-1}$. Conversely, if there exists such $\sigma_{\mathfrak{p}}$ for every $\mathfrak{p}$, then $\mathfrak{M}_{D,\mathfrak{p}} = \mathfrak{M}_{D',\mathfrak{p}}$ and $\mathfrak{M}_D = \mathfrak{M}_{D'}$ by Lemma 2.2. Therefore, $\mathfrak{M}_D = \mathfrak{M}_{D'}$ holds if and only if there exists an element σ_0 of $\Gamma^0(\mathfrak{B}(\mathfrak{g})_0, J)$ such that $D_0 \sigma_0 = D_0'$.

Now let $\mathfrak{P}(\mathfrak{M}_D, U_D, V_D, \eta P_D)$ and $\mathfrak{P}(\mathfrak{M}_{D'}, U_{D'}, V_{D'}, P_{D'})$ be two isomorphic polarized abelian varieties of type $\mathfrak{g}$ obtained from two elements D, D' of J_G. As is already seen, $\mathfrak{P}(\mathfrak{M}_D, U_D, V_D, P_D)$ is isomorphic to $\mathfrak{P}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n; m(D_\infty) \cdot Z_D)$ where $[\mathfrak{g}, \cdots, \mathfrak{g}, \mathfrak{a}_1, \cdots, \mathfrak{a}_n]$ is the canonical form of $\mathfrak{M}_D$. Since $\mathfrak{M}_D$ and $\mathfrak{M}_{D'}$ are equivalent, the canonical forms of them are the same, by the uniqueness part of Lemma 1.1. Hence $\mathfrak{P}(\mathfrak{M}_{D'}, U_{D'}, V_{D'}, P_{D'})$ is isomorphic to $\mathfrak{P}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n; m(D'_\infty) \cdot Z_{D'})$. By Theorem 4 there exists an element σ of $\Gamma^0([\mathfrak{g}, \cdots, \mathfrak{g}, \mathfrak{a}_1, \cdots, \mathfrak{a}_n], J)$ such that

$$Z_D = (aZ_{D'} + b)(cZ_D + d)^{-1}$$

where

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $\mathfrak{M}_D$ and $\mathfrak{M}_{L'}$ are equivalent, the ideal determined by $m(D_0)$ is equivalent to that determined by $m(D_0')$. Hence there exists an element η of F such that $m(D_0) = \eta \cdot m(D_0')$. Then, taking elements T, T' of G such that

$$\mathfrak{M}_D = [\mathfrak{g}, \cdots, \mathfrak{g}, \mathfrak{a}_1, \cdots, \mathfrak{a}_n] T, \qquad \mathfrak{M}_{D'} = [\mathfrak{g}, \cdots, \mathfrak{g}, \mathfrak{a}_1, \cdots, \mathfrak{a}_n] T'$$

and

$$T P_D {}^t T = J, \qquad T' P_{D'} {}^t T' = J,$$

we have

$$T^{-1} \sigma T'^{-1} J^t (T^{-1} \sigma T'^{-1}) = \eta J.$$

Hence

$$T^{-1} \sigma T'^{-1} \in G.$$

We define

$$\mathfrak{R}_\infty = \left\{ D_\infty \in J_{G,\infty} \,\middle|\, D_\infty = \begin{pmatrix} \varepsilon_\beta S & T \\ \varepsilon_\beta(-T) & S \end{pmatrix} \text{ and } m(D_\infty) \text{ is totally positive} \right\},$$

and put $\mathfrak{K}=\mathfrak{K}_\infty\times\mathfrak{U}_0$.

Therefore, if $\mathfrak{p}(\mathfrak{M}_D, U_D, V_D, P_D)$ is isomorphic to $\mathfrak{p}(\mathfrak{M}_{D'}, U_{D'}, V_{D'}, P_{D'})$ then $\boldsymbol{D}$, $\boldsymbol{D}'$ belong to the same double coset of $G\backslash \boldsymbol{J}_G/\mathfrak{K}$.

Conversely, we can show easily that if $\boldsymbol{D}$, $\boldsymbol{D}'$ belong to one and the same double coset of $G\backslash \boldsymbol{J}_G/\mathfrak{K}$, then $\mathfrak{p}(\mathfrak{M}_D, U_D, V_D, P_D)$ is isomorphic to $\mathfrak{p}(\mathfrak{M}_{D'}, U_{D'}, V_{D'}, P_{D'})$.

Lastly, we shall show that $\mathfrak{p}(\mathfrak{a}_1,\cdots,\mathfrak{a}_n;\eta Z)$, where $\eta\,\mathrm{Im}\,Z>0$, can be obtained as $\mathfrak{p}(\mathfrak{M}_D; m(\boldsymbol{D}_\infty)Z_D)$ with $\boldsymbol{D}$ of $\boldsymbol{J}_G$. Since $\eta\,\mathrm{Im}\,Z$ is totally positive, $\frac{\sqrt{-1}}{2}\eta^{-1}(\bar{Z}-Z)=\frac{\sqrt{-1}}{2}\eta^{-1}H>0$, hence we can write $\frac{\sqrt{-1}}{2}\eta^{-1}H={}^t\bar{V}^{-1}a1_nV^{-1}$ with $V\in GL(n,\boldsymbol{C})$ and $a\in\boldsymbol{J}_{F,\infty}$, $a>0$. Put $ZV=U$, then $Z=UV^{-1}$. We have $-\frac{\sqrt{-1}}{2}\eta^{-1}({}^tU\,{}^tV){}^tJ^{-1}\begin{pmatrix}\bar{U}\\ \bar{V}\end{pmatrix}=a1_n$ and furthermore by symmetricity of Z, we have $\eta^{-1}({}^tU\,{}^tV){}^tJ^{-1}\begin{pmatrix}U\\ V\end{pmatrix}=0$. From this it follows that, with $U=\sqrt{-1}\,A+B$, $V=\sqrt{-1}\,C+D$,

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}0 & \eta 1_n\\ -\eta 1_n & 0\end{pmatrix}\begin{pmatrix}{}^tA & {}^tC\\ {}^tB & {}^tD\end{pmatrix}=\begin{pmatrix}0 & a1_n\\ -a1_n & 0\end{pmatrix},$$

also we can see $m(\boldsymbol{D}_\infty)=\eta\cdot a$ has the same signature as that of η and $m(\boldsymbol{D}_\infty)\,\mathrm{Im}\,Z>0$. Hence putting $\boldsymbol{D}_\infty=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$, we see that $\boldsymbol{D}_\infty$ belongs to $\boldsymbol{J}_{G,\infty}$.

Now we can see that with a suitable $\boldsymbol{D}_\mathfrak{p}$,

$$[\mathfrak{g},\cdots,\mathfrak{g},\mathfrak{a}_1,\cdots,\mathfrak{a}_n]=\bigcap_{\mathfrak{p}}\mathfrak{B}(\mathfrak{g})_\mathfrak{p}\cdot\boldsymbol{D}_\mathfrak{p}^{-1}\cap\mathfrak{B}\qquad\text{holds.}$$

Put $\boldsymbol{D}=\boldsymbol{D}_\infty\times\boldsymbol{D}_0$ with $\boldsymbol{D}_\infty$, $\boldsymbol{D}_0$ chosen above. Then $\boldsymbol{D}$ belongs to $\boldsymbol{J}_G$ and we have $\mathfrak{p}(\mathfrak{a}_1,\cdots,\mathfrak{a}_n;\eta Z)=\mathfrak{p}(\mathfrak{a}_1,\cdots,\mathfrak{a}_n;m(\boldsymbol{D}_\infty)Z_D)$.

Thus we have the following

THEOREM 5. *Let $\boldsymbol{D}=\boldsymbol{D}_\infty\times\boldsymbol{D}_0$ be an element of $\boldsymbol{J}_G$. Put*

$$\boldsymbol{D}_\infty=\left(\begin{pmatrix}A^{(1)} & B^{(1)}\\ C^{(1)} & D^{(1)}\end{pmatrix},\cdots,\begin{pmatrix}A^{(r)} & B^{(r)}\\ C^{(r)} & D^{(r)}\end{pmatrix}\right),$$

$$\begin{pmatrix}U_D\\ V_D\end{pmatrix}=\left(\begin{pmatrix}\sqrt{-1}\,A^{(1)}+B^{(1)}\\ \sqrt{-1}\,C^{(1)}+D^{(1)}\end{pmatrix},\cdots,\begin{pmatrix}\sqrt{-1}\,A^{(r)}+B^{(r)}\\ \sqrt{-1}\,C^{(r)}+D^{(r)}\end{pmatrix}\right),$$

and

$$\mathfrak{M}_D=\bigcap_{\mathfrak{p}}\mathfrak{B}(\mathfrak{g})_\mathfrak{p}\cdot\boldsymbol{D}_\mathfrak{p}^{-1}\cap\mathfrak{B}.$$

Let $Z_D=U_DV_D^{-1}$ and $\mathfrak{a}_1,\cdots,\mathfrak{a}_n$ be the elementary divisors of $\mathfrak{M}_D$. Put

$$\Delta_D=\left\{(\alpha_1,\cdots,\alpha_{2n})\cdot\begin{pmatrix}Z_D\\ 1_n\end{pmatrix}\,\middle|\,(\alpha_1,\cdots,\alpha_{2n})\in[\mathfrak{g},\cdots,\mathfrak{g},\mathfrak{a}_1,\cdots,\mathfrak{a}_n]\right\}.$$

Then, $\boldsymbol{C}^{nr}/\Delta_D$ is a complex torus and $[\mathfrak{g},\cdots,\mathfrak{g},\mathfrak{a}_1,\cdots,\mathfrak{a}_n]$, Z_D, J determines a *polarized abelian variety* $\mathfrak{P}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n; m(\boldsymbol{D}_\infty)\cdot Z_D)$ *of type* $\mathfrak{g}$, *where*

$$[\mathfrak{g}, \cdots, \mathfrak{g}, \mathfrak{a}_1, \cdots, \mathfrak{a}_n] J' [\mathfrak{g}, \cdots, \mathfrak{g}, \mathfrak{a}_1, \cdots, \mathfrak{a}_n] \subset \mathfrak{d}$$

and

$$m(\boldsymbol{D}_\infty)\cdot \mathrm{Im}\, Z_D \quad \textit{is totally positive.}$$

Conversely, a polarized abelian variety $\mathfrak{P}(\mathfrak{M}, U, V, P)$ *of type* $\mathfrak{g}$ *is obtained as* $\mathfrak{P}(\mathfrak{a}_1, \cdots, \mathfrak{a}_n; m(\boldsymbol{D}_\infty)\cdot Z_D)$ *with* $\boldsymbol{D} \in \boldsymbol{J}_G$ *in the above manner.*

THEOREM 6. *Let* $\boldsymbol{D}$, $\boldsymbol{D}'$ *be two elements of* $\boldsymbol{J}_G$ *and* $\mathfrak{P}_D = \mathfrak{P}(\mathfrak{M}_D, m(\boldsymbol{D}_\infty)\cdot Z_D)$, $\mathfrak{P}_{D'} = \mathfrak{P}(\mathfrak{M}_{D'}; m(\boldsymbol{D}'_\infty)\cdot Z_{D'})$ *two polarized abelian varietis of type* $\mathfrak{g}$ *obtained in* Theorem 5. *Then* $\mathfrak{P}_D$ *is isomorphic to* $\mathfrak{P}_{D'}$ *if and only if* $\boldsymbol{D}$ *and* $\boldsymbol{D}'$ *belong to one and the same double coset of* $G\backslash \boldsymbol{J}_G/\mathfrak{K}$.

By Theorem 5, 6, we have the following

COROLLARY. *The isomorphism classes of polarized abelian varieties of type* $\mathfrak{g}$ *parametrized by* $\mathfrak{H}(n, r)$ *and the double cosets of* $G\backslash \boldsymbol{J}_G/\mathfrak{K}$ *are in bijective correspondence.*

§ 3. Isogenies defined by Hecke operators

1. x_λ, $\mathfrak{L}_\lambda$, $\mathfrak{U}_\lambda$ being as in the preceding sections, let $\boldsymbol{D}_\lambda$ be an element of $\boldsymbol{J}_G$ such that

$$\boldsymbol{D}_{\lambda,0} = x_\lambda\,,$$
$$\boldsymbol{D}_{1,\infty} = \boldsymbol{D}_{\lambda,\infty} \qquad \text{for every } \lambda$$

and

$$m(\boldsymbol{D}_{1,\infty}) = a \qquad \text{has the same signature as } \eta \in F.$$

Now we take x_λ so that $\mathfrak{L}_\lambda$ is of canonical form. We obtain h polarized abelian varieties of type $\mathfrak{g}$:

$$A_\lambda = \mathfrak{P}(\mathfrak{L}_\lambda; aZ)$$

with $Z = UV^{-1}$ defined by D_1, and $\mathfrak{L}_\lambda = [\mathfrak{g}, \cdots, \mathfrak{g}, \mathfrak{a}_1, \cdots, \mathfrak{a}_n]$.

We define an abelian variety A as

$$A = A_1 \times \cdots \times A_h\,.$$

In the following, we shall show that every element of the Hecke ring defines an isogeny of A.

2. Let $\mathfrak{a}$ be an ideal of F. We denote by $\mathfrak{h}_\lambda(\mathfrak{a})$ the set of $\mathfrak{a}$-section points on A_λ; namely

$$\mathfrak{h}_\lambda(\mathfrak{a}) = \{t \in A_\lambda \mid t\cdot \iota(\alpha) = 0 \quad \alpha \in \mathfrak{a}\}\,.$$

Let X_λ be a divisor on A_λ corresponding to J. For $\alpha\in F$ with $\alpha^{(i)}>0$ for every i, we use the symbol $e_{X_\lambda,\alpha}(s,t)$ for s, t such that $s\cdot\iota(\alpha)=0$, $s\cdot\iota(\alpha)=0$ defined by

$$e_{X_\lambda,\alpha}(s,t)=\exp.\left(2\pi\sqrt{-1}\sum_{i=1}^{r}x^{(i)}\alpha^{(i)}J^{t}y^{(i)}\right)$$

where $x=(x^{(i)})$, $y=(y^{(i)})$ are vectors in C^{nr} corresponding to $s,t\in A_\lambda$ (Weil [5]). Let H be a homomorphism of A_λ onto another polarized abelian variety $A'=\wp(\mathfrak{L}',Z')$ of type $\mathfrak{g}$ of the same dimension as A_λ. (α) being a principal ideal in F, let $\mathfrak{h}(H)$, the kernel of H, be contained in $\mathfrak{h}_\lambda((\alpha))$.

PROPOSITION 2.1. *Notation being as above, for every* $s,t\in\mathfrak{h}(H)$,

$$e_{X_\lambda,\alpha}(s,t)=1$$

holds.

PROOF. Denote by the same letter H the matrix of linear mapping of C^{nr} into C^{nr} corresponding to the homomorphism H. We see that there exists an element σ of $GL(2n,F)$ such that

$$\begin{pmatrix}Z\\1\end{pmatrix}\cdot H=\sigma\begin{pmatrix}Z'\\1\end{pmatrix},$$

observing that $\Delta_\lambda\cdot H\subset\Delta'$. Moreover, since H is compatible with polarization of A_λ, we have $\sigma J^{t}\sigma=\beta J$. But we may take $\alpha=\beta$. Because, for every $\mathfrak{p}$, we have

$$\begin{aligned}\mathfrak{h}(H)_\mathfrak{p}&\cong\mathfrak{L}'_\mathfrak{p}/\mathfrak{L}_{\lambda,\mathfrak{p}}\\&\cong\mathfrak{g}_\mathfrak{p}/(\alpha_{1,\mathfrak{p}})\mathfrak{g}_\mathfrak{p}\oplus\cdots\oplus\mathfrak{g}_\mathfrak{p}/(\alpha_{n,\mathfrak{p}})\mathfrak{g}_\mathfrak{p}\oplus\mathfrak{g}_\mathfrak{p}/(\beta_{1,\mathfrak{p}})\mathfrak{g}_\mathfrak{p}\oplus\cdots\oplus\mathfrak{g}_\mathfrak{p}/(\beta_{n,\mathfrak{p}})\mathfrak{g}_\mathfrak{p},\end{aligned}$$

where $\mathrm{diag}(\alpha_{1,\mathfrak{p}},\cdots,\alpha_{n,\mathfrak{p}},\beta_{1,\mathfrak{p}},\cdots,\beta_{n,\mathfrak{p}})$ is the diagonal form of σ for every $\mathfrak{p}$, σ being considered as an element of $G_\mathfrak{p}$. By assumption $\mathfrak{h}(H)\subset\mathfrak{h}_\lambda((\alpha))$, we see $\alpha_{i,\mathfrak{p}}\beta_{i,\mathfrak{p}}=\alpha$ for every i, $\mathfrak{p}$. Hence $\alpha=\beta$ and

$$\sigma J^{t}\sigma=\alpha J.$$

Then for $s,t\in\mathfrak{h}(H)$, we have, observing that J is principal matrix for $A'=\wp(\mathfrak{L}',Z')$,

$$\begin{aligned}e_{X_\lambda,\alpha}(s,t)&=\exp\left(2\pi\sqrt{-1}\sum_{i=1}^{r}x^{(i)}\alpha^{(i)}J^{t}y^{(i)}\right)\\&=\exp\left(2\pi\sqrt{-1}\sum_{i=1}^{r}x^{(i)}\sigma^{(i)}J^{t}\sigma^{(i)\,t}y^{(i)}\right)\\&=\exp\left(2\pi\sqrt{-1}\sum_{i=1}^{r}x'^{(i)}J^{t}y'^{(i)}\right)=1,\end{aligned}$$

where x, y correspond to s, t and $x\cdot\sigma^{-1}=x'$, $y\cdot\sigma^{-1}=y'\in\mathfrak{L}'$.

PROPOSITION 2.2. *Let τ be an element of $\Gamma(\mathfrak{B}_0, J)$ such that the diagonal form of $x_\mu^{-1}\tau x_\lambda$ is* $\mathrm{diag}(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n)$ *with $\alpha_i, \beta_i \in J_{\mathfrak{g},0}$. Assume that $(m(\tau))=(\alpha)$ is a principal ideal in F.*

Then the cosets $\tau_i\mathfrak{U}_\lambda$ of $\mathfrak{U}_\mu\tau\mathfrak{U}_\lambda$ and subgroups $\mathfrak{h}_{\mu,i}^\lambda$ of $\mathfrak{h}((\alpha), A_\mu)$ such that

i) $\mathfrak{h}_{\mu,i,\mathfrak{p}}^\lambda = \mathfrak{g}_\mathfrak{p}/(\alpha_{1,\mathfrak{p}})\mathfrak{g}_\mathfrak{p} \oplus \cdots \oplus \mathfrak{g}_\mathfrak{p}/(\alpha_{n,\mathfrak{p}})\mathfrak{g}_\mathfrak{p} \oplus \mathfrak{g}_\mathfrak{p}/(\beta_{1,\mathfrak{p}})\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}_\mathfrak{p}/(\beta_{n,\mathfrak{p}})\mathfrak{g}_\mathfrak{p}$,

ii) $e_{X_{\mu,\alpha}}(s, t) = 1$ *for* $s, t \in \mathfrak{h}_{\mu,i}^\lambda$,

are in one-to-one correspondence.

PROOF. Take a coset $\tau_i\mathfrak{U}_\lambda$ of $\mathfrak{U}_\mu\tau\mathfrak{U}_\lambda$ and put $\mathfrak{L}_{\mu,i} = \bigcap_\mathfrak{p} \mathfrak{L}_{\lambda,\mathfrak{p}}\tau_{i,\mathfrak{p}}^{-1} \cap \mathfrak{B}$. Then by Lemma 2.2, $\mathfrak{L}_{\mu,i}$ is a $\mathfrak{g}$-lattice which contains $\mathfrak{L}_\mu$ and $\mathfrak{L}_{\mu,i,\mathfrak{p}} = \mathfrak{L}_{\lambda,\mathfrak{p}}\tau_{i,\mathfrak{p}}^{-1}$ for every $\mathfrak{p}$. Put $\mathfrak{h}_{\mu,i}^\lambda = \mathfrak{L}_{\mu,i}/\mathfrak{L}_\mu$, then $\mathfrak{h}_{\mu,i}^\lambda$ is a $\mathfrak{g}$-invariant finite subgroup of A_μ. By construction we have

$$\begin{aligned}\mathfrak{h}_{\mu,i,\mathfrak{p}}^\lambda &= \mathfrak{L}_{\mu,i,\mathfrak{p}}/\mathfrak{L}_{\mu,\mathfrak{p}} \\ &\cong \mathfrak{L}_{\lambda,\mathfrak{p}}/\mathfrak{L}_{\mu,\mathfrak{p}}\tau_{i,\mathfrak{p}} \\ &\cong \mathfrak{B}(\mathfrak{g})_\mathfrak{p}/\mathfrak{B}(\mathfrak{g})_\mathfrak{p} x_{\mu,\mathfrak{p}}^{-1}\tau_{i,\mathfrak{p}}x_{\lambda,\mathfrak{p}} .\end{aligned}$$

Now the diagonal form of $x_{\mu,\mathfrak{p}}^{-1}\tau_{i,\mathfrak{p}}x_{\lambda,\mathfrak{p}}$ is $\mathrm{diag}(\alpha_{1,\mathfrak{p}}, \cdots, \alpha_{n,\mathfrak{p}}, \beta_{1,\mathfrak{p}}, \cdots, \beta_{n,\mathfrak{p}})$. Hence we have

$$\mathfrak{h}_{\mu,i,\mathfrak{p}}^\lambda \cong \mathfrak{g}_\mathfrak{p}/(\alpha_{1,\mathfrak{p}})\mathfrak{g}_\mathfrak{p} \oplus \cdots \oplus \mathfrak{g}_\mathfrak{p}/(\alpha_{n,\mathfrak{p}})\mathfrak{g}_\mathfrak{p} \oplus \mathfrak{g}_\mathfrak{p}/(\beta_{1,\mathfrak{p}})\mathfrak{g}_\mathfrak{p} \oplus \cdots \oplus \mathfrak{g}_\mathfrak{p}/(\beta_{n,\mathfrak{p}})\mathfrak{g}_\mathfrak{p} .$$

Since $(m(\tau)) = (m(\tau_i)) = (\alpha)$ is principal, we see that ii) holds for $s, t \in \mathfrak{h}_{\mu,i}^\lambda$ by Proposition 2.1.

Conversely, assume that a $\mathfrak{g}$-invariant finite subgroup $\mathfrak{h}_{\mu,i}^\lambda$ of $\boldsymbol{C}^{nr}/\Delta_\mu \cong A_\mu$ satisfies i) and ii). For every $\mathfrak{p}$, there exists a $\mathfrak{g}_\mathfrak{p}$-lattice $\mathfrak{L}_{\mu,i,\mathfrak{p}}$ such that

$$\begin{aligned}\mathfrak{h}_{\mu,i,\mathfrak{p}}^\lambda &= \mathfrak{L}_{\mu,i,\mathfrak{p}}/\mathfrak{L}_{\mu,\mathfrak{p}} \\ &\cong \mathfrak{g}_\mathfrak{p}/(\alpha_{1,\mathfrak{p}})\mathfrak{g}_\mathfrak{p} \oplus \cdots \oplus \mathfrak{g}_\mathfrak{p}/(\alpha_{n,\mathfrak{p}})\mathfrak{g}_\mathfrak{p} \oplus \mathfrak{g}_\mathfrak{p}/(\beta_{1,\mathfrak{p}})\mathfrak{g}_\mathfrak{p} \oplus \cdots \oplus \mathfrak{g}_\mathfrak{p}/(\beta_{n,\mathfrak{p}})\mathfrak{g}_\mathfrak{p} .\end{aligned}$$

We take an element $\tau_\mathfrak{p}$ of $G_\mathfrak{p}$ such that the diagonal form of $x_{\mu,\mathfrak{p}}^{-1}\tau_\mathfrak{p}x_{\lambda,\mathfrak{p}}$ is $\mathrm{diag}(\alpha_{1,\mathfrak{p}}, \cdots, \alpha_{n,\mathfrak{p}}, \beta_{1,\mathfrak{p}}, \cdots, \beta_{n,\mathfrak{p}})$ and $\mathfrak{L}_{\mu,i,\mathfrak{p}}\tau_{i,\mathfrak{p}} = \mathfrak{L}_{\lambda,\mathfrak{p}}$ for every $\mathfrak{p}$.

Putting $\mathfrak{L}_{\mu,i} = \bigcap_\mathfrak{p} \mathfrak{L}_{\mu,i,\mathfrak{p}} \cap \mathfrak{B}$, we have a $\mathfrak{g}$-lattice $\mathfrak{L}_{\mu,i}$. Since we have

$$\begin{aligned}\mathfrak{L}_{\mu,i,\mathfrak{p}}/\mathfrak{L}_{\mu,\mathfrak{p}} &= \mathfrak{L}_{\lambda,\mathfrak{p}}\tau_{i,\mathfrak{p}}^{-1}/\mathfrak{L}_{\mu,\mathfrak{p}} \\ &\cong \mathfrak{B}(\mathfrak{g})_\mathfrak{p}\cdot x_{\lambda,\mathfrak{p}}^{-1}\tau_{i,\mathfrak{p}}^{-1}/\mathfrak{B}(\mathfrak{g})_\mathfrak{p}\cdot x_{\mu,\mathfrak{p}}^{-1} \cong \mathfrak{B}(\mathfrak{g})_\mathfrak{p}/\mathfrak{B}(\mathfrak{g})_\mathfrak{p}\cdot x_{\mu,\mathfrak{p}}^{-1}\tau_{i,\mathfrak{p}}x_{\lambda,\mathfrak{p}} ,\end{aligned}$$

there exist $u_\mathfrak{p}$, $v_\mathfrak{p}$ of $\mathfrak{O}(\mathfrak{B}(\mathfrak{g})_\mathfrak{p})$ such that

$$u_\mathfrak{p}x_{\mu,\mathfrak{p}}^{-1}\tau_{i,\mathfrak{p}}x_{\lambda,\mathfrak{p}}v_\mathfrak{p} = \mathrm{diag.}\,(\alpha_{1,\mathfrak{p}}, \cdots, \alpha_{n,\mathfrak{p}}, \beta_{1,\mathfrak{p}}, \cdots, \beta_{n,\mathfrak{p}})$$

for every $\mathfrak{p}$. By Lemma 4.4 without J, we can take an element σ_i of $\mathfrak{O}(\mathfrak{L}_\mu)$ such that $\mathfrak{L}_{\mu,i}\sigma_i = \mathfrak{L}_\mu$. But by condition ii), σ_i satisfies

(##) $$\sigma_i J^t\sigma_i = \alpha J .$$

Hence, it may be considered that $\tau_{i,\mathfrak{p}}$ belongs to $G_\mathfrak{p}$, and $u_\mathfrak{p}$, $v_\mathfrak{p}$ belong to $\Gamma^0(\mathfrak{B}(\mathfrak{g})_\mathfrak{p}, J)$.

Therefore, $\tau_i \mathfrak{U}_\lambda$ is a coset of $\mathfrak{U}_\mu \tau \mathfrak{U}_\lambda$. It is easily seen that the above correspondence between $\mathfrak{y}^\lambda_{\mu,i}$ and $\tau_i \mathfrak{U}_\lambda$ is one-to-one. Q. E. D.

We say that σ_i in (✲✲) corresponds to τ_i.

PROPOSITION 2.3. *Notation being as in* Proposition 2.2, *there exists a homomorphism H_i, for every i, of A_μ onto some polarized abelian variety of type* $\mathfrak{g}$ *contained in A_λ such that the kernel of H_i is $\mathfrak{y}^\lambda_{\mu,i}$.*

PROOF. Notation being as in the proof of Proposition 2.2, we take a $\mathfrak{g}$-lattice $\mathfrak{L}_{\mu,i}$ for every i. Then there exists an element $\sigma_i \in G$ such that

$$\mathfrak{L}_{\mu,i}\sigma_i = \mathfrak{L}_\mu \quad \text{and} \quad m(\sigma_i) = \alpha .$$

Putting $\sigma_i[Z] = Z_i$, we can form a polarized abelian variety $\wp(\mathfrak{L}_{\mu,i}, Z_i, 1, \alpha^{-1}J)$ for every i. Then there exists a complex matrix H_i such that $\sigma_i \begin{pmatrix} Z \\ 1 \end{pmatrix} = \begin{pmatrix} Z_i \\ 1 \end{pmatrix} \cdot H_i$. Then the mapping $x \to xH_i$ defines a homomorphism of $C^{nr}/\Delta \cong A_\mu$ onto C^{nr}/Δ_i, where $\Delta_i = \mathfrak{L}_{\mu,i} \begin{pmatrix} Z_i \\ 1 \end{pmatrix}$.

Denote by the same letter H_i the homomorphism of A_μ onto the abelian variety whose analytic model is C^{nr}/Δ_i. By ii), we see that H_i is a homomorphism of A_μ onto polarized abelian variety $\wp(\mathfrak{L}_{\mu,i}, Z_i, 1, \alpha^{-1}J)$. $\wp(\mathfrak{L}_{\mu,i}, Z_i, 1, \alpha^{-1}J)$ is isomorphic to $\wp(\mathfrak{L}_\mu; a \cdot m(\sigma_i)Z_i)$. Therefore for every i, there exists a homomorphism H_i, whose kernel is $\mathfrak{y}^\lambda_{\mu,i}$, of $\wp(\mathfrak{L}_\mu, aZ)$ onto $\wp(\mathfrak{L}_\mu, a \cdot m(\sigma_i)Z_i)$.

By Proposition 2.2 and 2.3 we have the following

PROPOSITION 2.4. *Notation being as above, put* $\mathfrak{U}_\mu \tau \mathfrak{U}_\lambda = \bigcup_{i=1}^{d} \tau_i \mathfrak{U}_\lambda$. *Then, there exist d polarized abelian varieties $\wp(\mathfrak{L}_\mu, a \cdot m(\sigma_i)Z_i)$ of type* $\mathfrak{g}$ *and a homomorphism H_i of $\wp(\mathfrak{L}_\mu, Z)$ onto $\wp(\mathfrak{L}_\mu, a \cdot m(\sigma_i)Z_i)$ for every i such that the kernel of H_i is $\mathfrak{y}^\lambda_{\mu,i}$, where $\sigma_i[Z] = Z_i$ and σ_i corresponds to τ_i as above.*

3. We defined a polarized abelian variety

$$A = A_1 \times \cdots \times A_h$$

of type $\mathfrak{g}$.

Let $\mathfrak{c}$ be an ideal of F and $T(\mathfrak{c}) = \sum T(e)$, where the sum is extended over all invariant e such that $N(e) = \mathfrak{c}$. Let e be given as $e = \operatorname{inv}(\mathfrak{L}_\lambda : \mathfrak{L}_\mu \tau(e))$ with $\tau(e) \subset G$, for every λ. (Notice that for a given λ, μ is uniquely determined.) Then $T(e) = \mathfrak{U}_0 \tau'(e) \mathfrak{U}_0$ with $\tau'(e) = x_\mu^{-1} \cdot \tau(e) \cdot x_\lambda$. Let $\mathfrak{U}_0 \tau'(e) \mathfrak{U}_0 = \bigcup_{i=1}^{d} \tau'(e)_i \mathfrak{U}_0$ be disjoint sum. Then

$$x_\mu \mathfrak{U}_0 x_\mu^{-1} \tau(e) x_\lambda \mathfrak{U}_0 x_\lambda^{-1} = \bigcup_{i=1}^{d} x_\mu \tau(e)_i x_\lambda^{-1} x_\lambda \mathfrak{U}_0 x_\lambda^{-1}$$

and

$$\mathfrak{U}_\mu \tau(e) \mathfrak{U}_\lambda = \bigcup_{i=1}^{d} \tau(e)_{\mu,i} \mathfrak{U}_\lambda \qquad \text{with } \tau(e)_{\mu,i} = x_\mu \tau(e)_i x_\lambda^{-1}$$

is disjoint. Since $\mathfrak{L}_{\mu,0}=\mathfrak{L}_{1,0}x_{\mu}^{-1}$ for every μ and $(m(\tau))=(\alpha)$ is principal, we can apply Proposition 2.2, 2.3, and 2.4 to $A_{\mu}, \mathfrak{U}_{\mu}\tau(e)\mathfrak{U}_{\lambda}$ for every μ. Therefore, putting

$$\mathfrak{U}_{\mu}\tau(e)\mathfrak{U}_{\lambda}=\bigcup_{i=1}^{d(e)}\tau(e)_{\mu,i}\mathfrak{U}_{\lambda},$$

we obtain $d(e)$ polarized abelian varieties $\wp(\mathfrak{L}_{\mu}, a\cdot m(\sigma(e)_{\mu,i})Z_{\mu,i}(e))$ of type $\mathfrak{g}$ for every μ, and a homomorphism $H_{\mu,i}(e)$ of $\wp(\mathfrak{L}_{\mu}, aZ_{\mu})$ onto $\wp(\mathfrak{L}_{\mu}, a\cdot m(\sigma(e)_{\mu,i})Z_{\mu,i}(e))$ whose kernel is $\mathfrak{h}_{\mu,i}^{\lambda}(e)$, where $Z_{\mu,i}(e)=\sigma(e)_{\mu,i}[Z_{\mu}]$ and $\sigma(e)_{\mu,i}$ corresponds to $\tau(e)_{\mu,i}$. For every μ and i, $\mathfrak{h}_{\mu,i}(e)$ is a $\mathfrak{g}$-invariant subgroup of A_{μ} satisfying i), ii) in Proposition 2.2. Put $\wp(\mathfrak{L}_{\mu}, a\cdot m(\sigma(e)_{\mu,i})Z_{\mu,i}(e))=A_{\mu,i}(e)$. Thus we have $d(e)$ polarized abelian varieties $A_{1,i}(e)\times\cdots\times A_{h,i}(e)$ of type $\mathfrak{g}$ and a homomorphism $H_i(e)=H_{1,i}(e)\times\cdots\times H_{h,i}(e)$ of A onto $A_{1,i}(e)\times\cdots\times A_{h,i}(e)$ such that the kernal of $H_i(e)$ is $\mathfrak{h}_{1,i}^{\lambda_1}(e)\times\cdots\times\mathfrak{h}_{h,i}^{\lambda_h}(e)$. Since we obtain $H_i(e)$ for every e such that $N(e)=\mathfrak{c}$, $T(\mathfrak{c})$ can be represented as a collection of isogenies $T(e)$ of A.

4. Let τ_0 be an integral element of $J_{G,0}$ and

$$\mathfrak{U}_0\tau_0\mathfrak{U}_0=\bigcup_{i=1}^{d}\tau_i\mathfrak{U}_0$$

be a disjoint sum.

We defined A_1 by $D_1=D_{1,\infty}\times D_{1,0}\in J_G$ with $D_{1,0}=x_1$. With the D, put $D_i=D_{1,\infty}\times\tau_i D_{1,0}$ for every i. We can define one-to-d corresponcence

$$(\sharp)\qquad \mathfrak{T}(e);\ D\rightarrow(D_1,\cdots,D_d),$$

where, since $\mathfrak{U}_0\tau_0\mathfrak{U}_0=\mathfrak{U}_0x_{\mu}^{-1}\alpha x_{\lambda}\mathfrak{U}_0$ with $\alpha\in G$, e is given as $e=\mathrm{inv}(\mathfrak{L}_{\lambda}:\mathfrak{L}_{\mu}\alpha)$. $\mathfrak{T}(e)$ is called a Hecke operator.

Take a D_i in ($\sharp$). Since $\mathfrak{U}_0\tau_0\mathfrak{U}_0=\bigcup_{i=1}^{d}\tau_i\mathfrak{U}_0$, we have

$$x_{\mu}\mathfrak{U}_0x_{\mu}^{-1}x_{\mu}\alpha x_{\lambda}^{-1}x_{\lambda}\mathfrak{U}_0x_{\lambda}^{-1}=\bigcup_{i=1}^{d}\alpha_i\mathfrak{U}_0x_{\lambda}^{-1},\qquad \text{with } \alpha_i=x_{\mu}\tau_ix_{\lambda}^{-1}.$$

Hence $D_i=D_{1,\infty}\times x_{\mu}^{-1}\alpha_ix_{\lambda}D_{1,0}$.

By $D_{1,\infty}\times x_{\lambda}D_{1,0}$, we constructed $A_{\lambda}=\wp(\mathfrak{L}_{\lambda};aZ)$, $a=m(D_{\infty})$.

Let $\mathfrak{c}$ be an ideal in F. Then, the number of τ_0 such that $(m(\tau_0))=\mathfrak{c}$ is finite; let $\{\tau_0^1(\mathfrak{c}),\cdots,\tau_0^k(\mathfrak{c})\}$ be the set of all such τ_0's. We define a correspondence

$$\mathfrak{T}(\mathfrak{c}):\quad D\rightarrow(D_1^1(\mathfrak{c}),\cdots,D_{d_1}^1(\mathfrak{c}),\cdots,D_1^k(\mathfrak{c}),\cdots,D_{d_k}^k(\mathfrak{c}))$$

where $(D_1^j(\mathfrak{c}),\cdots,D_{d_j}^j(\mathfrak{c}))$ is defined with respect to

$$\mathfrak{U}_0\tau_0^j(\mathfrak{c})\mathfrak{U}_0=\bigcup_{i=1}^{d_j}\tau_i^j(\mathfrak{c})\mathfrak{U}_0,\qquad D_i^j(\mathfrak{c})=D_{1,\infty}\times\tau_i^j(\mathfrak{c})\cdot D_{1,0}$$

in the above manner. $\mathfrak{T}(\mathfrak{c})$ is also called a Hecke operator for $\mathfrak{c}$.

We obtain the following

THEOREM 7. *Let* $\mathfrak{c}$ *be an ideal in* F *and* $\{\tau_0^1(\mathfrak{c}),\cdots,\tau_0^h(\mathfrak{c})\}$ *be the set of all* τ_0's *such that* $(m(\tau_0))=\mathfrak{c}$.

Let $\mathrm{diag}\,(\alpha_{1,\mathfrak{p}}^l,\cdots,\alpha_{n,\mathfrak{p}}^l,\beta_{1,\mathfrak{p}}^l,\cdots,\beta_{n,\mathfrak{p}}^l)$ *be the diagonal form of* $\tau_0^l(\mathfrak{c})$, *and* $e^l=\mathrm{inv}\,(\mathfrak{L}_\mu:\mathfrak{L}_\lambda\alpha^l)$ *where* $\tau_0^l=x_\mu\alpha^l x_\lambda^{-1}$ *with* $\alpha^l\in G$.

Let $\mathfrak{T}(e^l)$ *be a Hecke operator defined with respect to the disjoint sum decomposition* $\mathfrak{U}_0\tau_0^l(\mathfrak{c})\mathfrak{U}_0=\bigcup_{i=1}^{d_l}\tau_i^l(\mathfrak{c})\mathfrak{U}_0$ *and put* $A_1=\wp(\mathfrak{L}_1;m(\boldsymbol{D}_{1,\infty})\cdot Z\mathfrak{p}_1)$. *Then* $\mathfrak{T}(e^l)$ *acts on* $A=A_1\times\cdots\times A_h$ *as* $T(e^l)$. *Put* $\alpha_{\mu,i}^l=x_\mu^{-1}\tau_i^l(\mathfrak{c})x_\lambda$ *with* $\alpha_{\mu,i}^l\in G$. *Then there exist* d_l *polarized abelian varieties*

$$A^l=A_1^l\times\cdots\times A_h^l$$

where

$$A_\mu^l=\wp(\mathfrak{L}_\mu,m(\boldsymbol{D}_{1,\infty})\,m(\alpha_{\mu,i}^l)\,Z_{\mu,i})\quad \textit{and}\quad Z_{\mu,i}=\alpha_{\mu,i}^l[Z\mathfrak{p}_1]$$

and a homomorphism $H_l=H_{1,l}\times\cdots\times H_{h,l}$ *of* A *onto* A', *whose kernel is* $\mathfrak{h}_l=\mathfrak{h}_{1,l}^{\lambda_1}\times\cdots\times\mathfrak{h}_{\lambda,l}^{\lambda_h}$, *for every* i. $\mathfrak{h}_{\mu,l}^{\lambda}$ *satisfies* i), ii) *in Proposition* 2.2 *for* $\mathrm{diag}\,(\alpha_1^l,\cdots,\alpha_n^l,\beta_n^l,\cdots,\beta_n^l)$.

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