

On the Hilbert-Siegel modular group and abelian varieties

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1. The Hilbert-Siegel modular group was first introduced in C. L. Siegel [6] and treated systematically by Pyatetzki-Shapiro [2]. In his paper [1], W. L. Baily gave the compactification of the Hilbert-Siegel modular space which is the generalization of Satake's compactification [3] to the case and proved that every meromorphic function on the Hilbert-Siegel upper-half plane invariant under the Hilbert-Siegel group can be represented as a quotient of two automorphic forms with Fourier developments, except the case of elliptic modular functions. On the other hand, G. Shimura has developed the general theory on automorphic functions and abelian varieties, in [5], and showed that if the system of polarized abelian varieties parametrized by suitable meromorphic functions (f_1, \dots, f_d) is complete⁰⁾ with respect to a subfield K , (with some properties) of \mathbb{C} , then the field $K(f_1, \dots, f_d)$ and \mathbb{C} are linearly disjoint over K ([5] Theorem 2). In [4], he constructed the system of abelian varieties attached to the Siegel's (para-) modular group and proved that the system is complete with respect to \mathbb{Q} . Therefore, by Baily's result mentioned above, (in particular, in the case of Siegel), it has been shown that the field of Siegel's (para-) modular functions is defined over \mathbb{Q} . Also, in [5], he has constructed the system of polarized abelian varieties whose endomorphism rings are isomorphic to an order of a quaternion algebra over \mathbb{Q} and proved that the system is complete with respect to \mathbb{Q} .

In the present paper, we shall construct systems of polarized abelian varieties whose endomorphism rings are isomorphic to an order in a totally real algebraic number field and prove the completeness of the systems with respect to \mathbb{Q} , in the sense of Shimura, when the order is the ring of integers in that field. This implies also, similar to the above result of Shimura, that the field of Hilbert-Siegel (para-) modular functions is defined over \mathbb{Q} , by the above mentioned Baily's result and Shimura's Theorem 2 in [4].

In section 2, we first recall the theory of Shimura [4], [5], which is fundamental for the present work. In section 3, we shall learn the back-ground from the theory of lattices and give the key Lemma 1 due to Shimura. In section 4, we shall define precisely the Hilbert-Siegel para-modular groups. In section 6, we give the characterization of abelian varieties of our present purpose and construct the system of abelian varieties from the data obtained (Theorem 1, 2). In section 8, it will be given that isomorphic classes of our abelian varieties and points of the quotient space of the generalized Hilbert-Siegel upper-half

0) For the definition of completeness cf. G. Shimura [5], p. 124, and also Theorem 4 in this paper.

plane by our group are in one-to-one correspondence (Theorem 3). In section 9, we shall obtain the completeness of our system with respect to \mathbf{Q} (Theorem 4). In section 10, we shall show that the field of Hilbert-Siegel (para-) modular functions is defined over \mathbf{Q} (Theorems 5, 6). Their proofs are quite analogous to that of Theorems in Shimura [4], [5] corresponding to our Theorems.

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Notation:

We denote by \mathbf{Z} , \mathbf{Q} , \mathbf{R} and \mathbf{C} , respectively, the ring of rational integers, the fields of rational numbers, real numbers and complex numbers. \mathbf{R}^n and \mathbf{C}^n denote the row vector spaces composed of all n -tuples with real and complex coefficients, respectively. We denote by \bar{u} the complex conjugate of u , where u may be complex number or a matrix with complex entries. A being an abelian variety, we denote, as usual, by $\mathcal{V}(A)$ and $\mathcal{M}(A)$ the ring of endomorphisms of A and the algebra $\mathcal{V}(A) \otimes \mathbf{Q}$. For every algebraic variety V in projective space, $c(V)$ denotes the Chow point of V . X and Y being divisors on a variety, $X \equiv Y$ means that X is equivalent algebraically to Y .

2. First, we recall the theory of G. Shimura [4], [5].

Let V be a complete algebraic variety, non-singular in codimension 1, and X a positive divisor on V . We denote by $\mathcal{E}(X)$ the set of all positive divisors X' on V for which there exist two positive integers m, m' such that $mX \equiv m'X'$. If an ample divisor Y is contained in $\mathcal{E}(X)$, we call $\mathcal{E}(X)$ a polarization of V ; a positive divisor Y is called ample if the linear system determined by Y gives a birational biregular imbedding of V into a projective space.

\mathcal{E} being a polarization of V , a couple (V, \mathcal{E}) is called a polarized variety. If V is defined over a field K and \mathcal{E} contains a divisor which is rational over K , then we say that (V, \mathcal{E}) is defined over K .

Let σ be an isomorphism of K into a field K' . (V, \mathcal{E}) being defined over K , we denote by \mathcal{E}^σ the polarization $\mathcal{E}(X^\sigma)$ of V^σ where X is a rational divisor in \mathcal{E} over K . Let (A, \mathcal{E}) and (A', \mathcal{E}') be two polarized abelian varieties and A a homomorphism (an isomorphism) of A onto A' . We say that A is a homomorphism (an isomorphism) of (A, \mathcal{E}) onto (A', \mathcal{E}') if there exists a divisor X' in \mathcal{E}' such that $A^{-1}(X')$ is contained in \mathcal{E} .

Let r be a ring having a finite basis over \mathbf{Z} and A be an abelian variety. We understand by a polarized abelian variety of type r a triplet (A, \mathcal{E}, ι) formed by a polarized abelian variety (A, \mathcal{E}) and an isomorphism ι of r into $\mathcal{V}(A)$. We say that (A, \mathcal{E}, ι) is defined over a field K if (A, \mathcal{E}) and all elements of $\iota(r)$ are defined over K . (A, \mathcal{E}, ι) being defined over K , let σ be an isomorphism of K into a field K' ; then we obtain a polarized abelian variety $(A^\sigma, \mathcal{E}^\sigma, \iota^\sigma)$ of type r by putting $\iota^\sigma(f) = \iota(f)^\sigma$. If σ is the identity on a subfield K_1 of K , we call $(A^\sigma, \mathcal{E}^\sigma, \iota^\sigma)$ a generic specialization of (A, \mathcal{E}, ι) over K_1 .

Let (A, \mathcal{E}, ι) and $(A', \mathcal{E}', \iota')$ be two polarized abelian varieties of type r .

A homomorphism (an isomorphism) A of (A, \mathcal{C}) onto (A', \mathcal{C}') is called a homomorphism (an isomorphism) of (A, \mathcal{C}, ι) on to $(A', \mathcal{C}', \iota')$ if $A \cdot \iota(f) = \iota'(f) \cdot A$ holds for any $f \in \mathfrak{r}$.

Let A be an abelian variety defined over K in a projective space \mathcal{P}^N and ι an isomorphism of \mathfrak{r} into $\mathcal{N}(A)$ such that $\iota(r_i)$ is defined over K for every i , where $\{r_i\}_{i=1}^h$ is a basis of \mathfrak{r} over \mathbb{Z} . Take a $(N+1, N+1)$ matrix (t_{ij}) with independent variables t_{ij} over K and h independent generic points u_1, \dots, u_h of A over $K(t_{ij})$. Let A_0 be the transform of A by the projective transformation $\varphi; (x_i) \rightarrow (\sum t_{ij} x_j)$ of \mathcal{P}^N ; let a be the Chow point of A_0 ; then $K(a)$ is regular over K . Let W_i be the graph of the rational mapping $x \rightarrow \varphi[\iota(r_i)\varphi^{-1}(x) + u_i]$ of A_0 onto itself and z_i the Chow point of W_i ; then $K(a, z_1, \dots, z_h)$ is regular over K . Denote by $\mathcal{F}(A, \iota)$ the locus of $a \times z_1 \times z_2 \times \dots \times z_h$ over K . The variety $\mathcal{F}(A, \iota)$ does not depend upon the choice of K , (t_{ij}) and u_i . A being an abelian variety in a projective space, and \mathcal{C} being the polarization of A defined by hyperplane sections, let (A, \mathcal{C}, ι) be a polarized abelian variety of type \mathfrak{r} . Let F be the smallest field of definition for $\mathcal{F}(A, \iota)$ and K a field of definition for A, \mathcal{C}' and ι containing F . Then, F has the following property.

(SS1) σ being an isomorphism of K into a field K' , (A, \mathcal{C}, ι) is isomorphic to $(A^\sigma, \mathcal{C}^\sigma, \iota^\sigma)$ if and only if σ is the identity on F .

F does not depend on the choice of a basis of \mathfrak{r} . Characteristic being 0, we call F the field of moduli of (A, \mathcal{C}, ι) .

Let A and A' be abelian varieties in \mathcal{P}^N , \mathcal{C} and \mathcal{C}' the polarizations of A and A' , respectively, defined by hyperplane sections, and ι and ι' be isomorphisms of \mathfrak{r} into $\mathcal{N}(A)$ and $\mathcal{N}(A')$, respectively.

(SS2) Suppose that neither of A, A' is contained in any hyperplane and that the linear systems on A, A' defined by the hyperplane sections are complete. Then, (A, \mathcal{C}, ι) is isomorphic to $(A', \mathcal{C}', \iota')$ if and only if $\mathcal{F}(A, \iota) = \mathcal{F}(A', \iota')$.

Let \mathcal{A} be a discrete subgroup of C^m of rank $2m$. In order that the complex torus C^m/\mathcal{A} has a structure of abelian variety, it is necessary and sufficient that there exists a non-degenerate Riemann form $\mathcal{E}(x, y)$ on C^m/\mathcal{A} ; by a non-degenerate Riemann form $\mathcal{E}(x, y)$ we mean as usual an R -valued R -bilinear form on $C^m \times C^m$ with the following properties:

- (R 1) $\mathcal{E}(x, y) \in \mathbb{Z}$ for every $x, y \in \mathcal{A}$;
- (R 2) $\mathcal{E}(x, y) = -\mathcal{E}(y, x)$;
- (R 3) $\mathcal{E}(x, \sqrt{-1}y)$ is symmetric and positive definite.

Let X be a positive analytic divisor on C^m/\mathcal{A} . Then, there exists a holomorphic function θ (so-called theta-function attached to X) on C^m such that $(\theta) = X$ and

$$\begin{aligned} \theta(x+d) &= \theta(x) \exp [2\pi\sqrt{-1} \{H(d, x) + 1/2H_0(d, d) + b(d)\}] \text{ for } d \in \mathcal{A}, \\ (1) \quad H_0(u, v) &= H_0(v, u), \\ H(d_1, d_2) &\equiv H_0(d_1, d_2) \pmod{\mathbb{Z}} \text{ for } d_1, d_2 \in \mathcal{A}. \end{aligned}$$

Putting $\mathcal{E}(x, y) = H(x, y) - H(y, x)$, we obtain a Riemann form $\mathcal{E}(x, y)$ (Riemann

form defined by θ) on C^m/J . $\mathcal{E}(x, y)$ depends only on X , so we denote $\mathcal{E}(x, y) = \mathcal{E}(X)(x, y)$. If H is skew-hermitian and b is R -valued, θ is said to be normalized. Every non-degenerate Riemann form on C^m/J is obtained from a positive non-degenerate analytic divisor on C^m/J and every positive non-degenerate analytic divisor on C^m/J defines a non-degenerate Riemann form on C^m/J ; hence if A is an abelian variety isomorphic to C^m/J , every non-degenerate Riemann form on C^m/J defines a polarization of A .

Let u_1, \dots, u_{2m} be a basis of J over Z . Put

$${}^t\Omega = \begin{pmatrix} u_1 \\ \vdots \\ u_{2m} \end{pmatrix}.$$

${}^t\Omega$ is a $(2m, m)$ matrix with complex entries. Then there exists a matrix P of degree $2m$ such that

$$\mathcal{E}(x {}^t\Omega, y {}^t\Omega) = x P {}^t y \quad \text{for } x, y \in R^{2m}.$$

Then, (R 1), (R 2) and (R 3) are equivalent to the following properties:

(G₀ 1) The entries of P are integers:

(G₀ 2) ${}^tP = -P$:

(G₀ 3) $\Omega {}^tP {}^{-t}\Omega = 0$:

(G₀ 4) $\sqrt{-1} \Omega {}^tP {}^{-t}\overline{\Omega} > 0$ (positive definite hermitian matrix).¹⁾

Let $\delta_1, \dots, \delta_m$ be m positive integers such that

$$\delta_1 = 1, \quad \delta_i \mid \delta_{i+1} \quad 1 \leq i \leq m-1;$$

put

$$P_{\delta} = \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} \quad \text{with a diagonal matrix } \delta = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_m \end{pmatrix}.$$

Put $\mathcal{H}(m) = \{Z \in GL(m, C); {}^tZ = Z \text{ and } \text{Im } Z > 0\}$. Denote by $J_{\delta}(Z)$ the discrete subgroup of C^m generated over Z by the rows of the matrix $\begin{pmatrix} Z \\ \delta \end{pmatrix}$. Then, we see that

$$\mathcal{E}\left(x \begin{pmatrix} Z \\ \delta \end{pmatrix}, y \begin{pmatrix} Z \\ \delta \end{pmatrix}\right) = x P_{\delta} {}^t y \quad \text{for } x, y \in R^{2m}$$

is a non-degenerate Riemann form on $C^m/J_{\delta}(z)$; so the complex torus, with the form \mathcal{E} , has a structure of abelian variety. Then, for each δ , there exists a

1) In our case, the equivalence between (R 1), (R 2), (R 3) and (G 1), (G 2), (G 3), (G 4) will be shown. See section 6.

system of abelian varieties $A_s(Z)$ parametrized by holomorphic functions

$$\theta_0(u, Z), \dots, \theta_N(u, Z)$$

on $\mathbb{C}^n \times \mathcal{H}^o(m)$ satisfying the conditions (S 1), (S 2) and (S 3) in Shimura [5], p. 115. We denote by $\Theta_s(u, Z)$ the point

$$(\theta_0(u, Z), \dots, \theta_N(u, Z))$$

in \mathcal{P}^N . Then, the system $\{A_s(Z); Z \in \mathcal{H}^o(m)\}$ has the following properties:

(SS 3) $A_s(Z)$ is an abelian variety in \mathcal{P}^N whose hyperplane sections define a complete linear system and whose degree is independent of Z . There is no hyperplane containing $A_s(Z)$.

(SS 4) For every $Z \in \mathcal{H}^o(m)$, the mapping $u \rightarrow \Theta_s(u, Z)$ gives an analytic isomorphism of $\mathbb{C}^m/J_s(Z)$ onto $A_s(Z)$.

(SS 5) The polarization of $A_s(Z)$ determined by the hyperplane sections corresponds to the Riemann form given by P_s .

Let X be a non-degenerate positive divisor on A and θ an analytic isomorphism of A onto \mathbb{C}^m/J . Then, $\theta(X)$ is a positive analytic divisor on \mathbb{C}^m/J . Put $\mathcal{E}_0(X) = \mathcal{E}(\theta(X))$. Let Y be a non-degenerate positive divisor on A ; then $\mathcal{E}_0(X) = \mathcal{E}_0(Y)$ holds if and only if $X = Y$.

Moreover, we have

(SS 6) The mapping $Z \rightarrow c(A_s(Z))$ is everywhere holomorphic on $\mathcal{H}^o(m)$.

(Theorem S in [5])

3. Let k be an algebraic number field of finite degree and \mathfrak{o} the ring of integers in k . We denote by $\mathcal{V}^o(m, k)$ the row vector space of dimension m over k . By a lattice in $\mathcal{V}^o(m, k)$, we shall understand a free \mathbb{Z} -submodule \mathfrak{M} of $\mathcal{V}^o(m, k)$ of rank ms . If $\mathfrak{o}\mathfrak{M} \subset \mathfrak{M}$, we call \mathfrak{M} an \mathfrak{o} -lattice. We know that every \mathfrak{o} -lattice \mathfrak{M} in $\mathcal{V}^o(m, k)$ has a basis $\{x_1, \dots, x_m\}$ such that

$$\mathfrak{M} = \alpha_1 x_1 + \dots + \alpha_m x_m,$$

where $\alpha_1, \dots, \alpha_m$ are ideals in k and moreover \mathfrak{M} has a basis $\{y_1, \dots, y_m\}$ such that

$$(2) \quad \mathfrak{M} = \mathfrak{o} y_1 + \dots + \mathfrak{o} y_{m-1} + \mathfrak{a} y_m,$$

with an ideal \mathfrak{a} of k . We say that two \mathfrak{o} -lattices $\mathfrak{M}, \mathfrak{M}'$ are equivalent if there exists an element M of $GL(m, k)$, the general linear group of degree m over k , such that $\mathfrak{M}M = \mathfrak{M}'$. Suppose that \mathfrak{M} has the form (2) and another lattice \mathfrak{M}' has a basis $\{y_1', \dots, y_m'\}$ such that $\mathfrak{M}' = \mathfrak{o} y_1' + \dots + \mathfrak{o} y_{m-1}' + \mathfrak{a}' y_m'$. Then, \mathfrak{M} is equivalent to \mathfrak{M}' if and only if \mathfrak{a} and \mathfrak{a}' belong to the same ideal class in k . For m ideals $\alpha_1, \dots, \alpha_m$ of k , we put

$$[\alpha_1, \dots, \alpha_m] = \{(\alpha_1, \dots, \alpha_m) \in \mathcal{V}^o(m, k) \mid \alpha_i \in \alpha_i\}.$$

We denote by $\text{End } \mathcal{V}^o(m, k)$ the endomorphism ring of $\mathcal{V}^o(m, k)$. Put

$$\mathfrak{O} = \{M \in \text{End } \mathcal{V}(m, k); \mathfrak{M}M \subset \mathfrak{M}\}.$$

Then, \mathfrak{O} is a maximal order in $\text{End } \mathcal{V}(m, k)$. If $\mathfrak{M} = [a_1, \dots, a_m]$, \mathfrak{O} can be written in the form

$$\mathfrak{O} = \begin{pmatrix} a_1 & b_{12} & \dots & b_{1m} \\ b_{21} & 0 & & \\ \vdots & & \ddots & \\ b_{m1} & \dots & \dots & 0 \end{pmatrix}$$

where $b_{ij} = a_i^{-1}a_j$, and $b_{ij} = b_{ji}^{-1}$. Then \mathfrak{O} will be denoted by $\mathfrak{O}(a_1, \dots, a_m)$.

LEMMA 1. *Let \mathfrak{M} be an \mathfrak{o} -lattice in $\mathcal{V}(2n, k)$ and $P(x, y)$ a k -bilinear non-degenerate alternating form which is \mathfrak{o} -valued on \mathfrak{M} . Then, there exist n ideals \mathfrak{a}_i in k and a basis $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ of $\mathcal{V}(2n, k)$ such that $\mathfrak{M} = \mathfrak{o}x_1 + \mathfrak{a}_1y_1 + \dots + \mathfrak{o}x_n + \mathfrak{a}_ny_n$, $\mathfrak{o} \supset \mathfrak{a}_1 \supset \dots \supset \mathfrak{a}_n$, $P(x_i, x_j) = P(y_i, y_j) = 0$, $P(x_i, y_j) = \delta_{ij}$ (Kronecker's delta). Moreover the ideals \mathfrak{a}_i are uniquely determined by P and \mathfrak{M} .*

PROOF. For x in \mathfrak{M} , $\mathfrak{a}_x = P(x, \mathfrak{M})$ are integral ideals in k . Among these ideals \mathfrak{a}_x , take a maximal one and denote it by \mathfrak{a}_1 . Then, $\mathfrak{a}_1 = P(x_1, \mathfrak{M})$ for an element x_1 in \mathfrak{M} , and $\mathfrak{o} = P(x_1, \mathfrak{a}_1^{-1}\mathfrak{M})$. Therefore, there is an element y_1 in $\mathfrak{a}_1^{-1}\mathfrak{M}$ such that $P(x_1, y_1) = 1$ and $\mathfrak{a}_1 y_1 \subset \mathfrak{M}$. Put $\mathfrak{b} = P(\mathfrak{M}, y_1)$; then $\mathfrak{a}_1 \mathfrak{b} = P(\mathfrak{M}, \mathfrak{a}_1 y_1) \supset \mathfrak{a}_1$. If $\mathfrak{b} \neq \mathfrak{o}$, then $P(\mathfrak{M}, \mathfrak{a}_1 y_1) \neq \mathfrak{a}_1$. Hence, there would exist an element c in \mathfrak{o} which is not contained in \mathfrak{a}_1 , an element z in \mathfrak{M} and an element a in \mathfrak{a}_1 such that $P(z, \mathfrak{a}_1 y_1) = c$. Put $d = P(x_1, z)$, which is in \mathfrak{a}_1 . Now, we would have $P(x_1 + ay_1, z - dy_1) = c$ and $P(x_1 + ay_1, \mathfrak{a}_1 y_1) = \mathfrak{a}_1$. The elements $x_1 + ay_1$ and $z - dy_1$ would be contained in \mathfrak{M} . Hence, $P(x_1 + ay_1, \mathfrak{M})$ would contain an ideal (\mathfrak{a}_1, c) which contains strictly \mathfrak{a}_1 , and so it contradicts the maximality of \mathfrak{a}_1 . Therefore, we must have $\mathfrak{b} = \mathfrak{o}$ and $P(\mathfrak{M}, y_1) = \mathfrak{o}$. Put $\mathfrak{M}' = \{x; x \in \mathfrak{M}, P(x_1, x) = P(y_1, x) = 0\}$. Take an arbitrary element z of \mathfrak{M} and put $P(x_1, z) = f$, $P(y_1, z) = g$. Then, $f \in \mathfrak{a}_1$ and $g \in \mathfrak{o}$. We have $P(x_1, z + gx_1 - fy_1) = 0$, $P(y_1, z + gx_1 - fy_1) = 0$. Hence, $w \in \mathfrak{M}'$ and $z = w + gx_1 + fy_1$ is contained in $\mathfrak{M}' + \mathfrak{o}x_1 + \mathfrak{a}_1y_1$. Thus, as it follows easily that the last sum is direct, we have

$$\mathfrak{M} = \mathfrak{o}x_1 + \mathfrak{a}_1y_1 + \mathfrak{M}'. \quad (\text{direct})$$

We shall prove the lemma by induction on n . In case $n=1$ we have $\mathfrak{M}'=0$ and the lemma holds. For arbitrary n , using the induction assumption on \mathfrak{M}' , we have

$$\mathfrak{M}' = \mathfrak{o}x_2 + \mathfrak{a}_2y_2 + \dots + \mathfrak{o}x_n + \mathfrak{a}_ny_n,$$

where $\mathfrak{o} \supset \mathfrak{a}_2 \supset \dots \supset \mathfrak{a}_n$, $P(x_i, x_j) = P(y_i, y_j) = 0$ and $P(x_i, y_j) = \delta_{ij}$ for $i, j=2, \dots, n$. Thus it suffices to show that $\mathfrak{a}_1 \supset \mathfrak{a}_2$. For elements $z, w \in \mathfrak{M}'$, $a \in \mathfrak{a}_1$ and $b \in \mathfrak{o}$, we have $P(x_1 + z, ay_1 + bw) = a + bP(z, w)$. Hence, $P(x_1 + z, \mathfrak{M})$ contains $\mathfrak{a}_1 + P(z, \mathfrak{M}')$

which contains a_1 . By the maximality of a_1 , we have $P(z, \mathcal{M}) \subset a_1$. From this, it follows easily that $a_1 \supset a_2$.

Let \mathfrak{p} be a prime ideal in k , k^* and \mathfrak{o}^* the \mathfrak{p} -completion of k and \mathfrak{o} , $a_i^* = \mathfrak{o}^* \cdot a_i$ and $\mathcal{M}^* = \mathfrak{o}^* \cdot \mathcal{M}$. Let $P^*(x, y)$ be the alternating form on $\mathcal{V}(2n, k^*)$ obtained by the extension of $P(x, y)$. Then, by the first part of the lemma, we can write $\mathcal{M}^* = \mathfrak{o}^* x_{1,\mathfrak{p}} + \alpha_1 \mathfrak{o}^* y_{1,\mathfrak{p}} + \dots + \mathfrak{o}^* x_{n,\mathfrak{p}} + \alpha_n \mathfrak{o}^* y_{n,\mathfrak{p}}$, where $P^*(x_{i,\mathfrak{p}}, y_{j,\mathfrak{p}}) = \delta_{i,j}$, $P^*(x_{i,\mathfrak{p}}, x_{j,\mathfrak{p}}) = P^*(y_{i,\mathfrak{p}}, y_{j,\mathfrak{p}}) = 0$ and $\alpha_i \in \mathfrak{o}^*$ for every i and by the well-known theory of elementary divisors for a principal ideal domain, $\alpha_i \mathfrak{o}^*$ are uniquely determined by \mathcal{M}^* and P^* and $\alpha_i \mathfrak{o}^* = a_i^*$. Since this holds for every \mathfrak{p} , we see that a_i are uniquely determined by \mathcal{M} and P . q.e.d.

We call the basis $\{x_i, y_i\}$ in the above lemma a *canonical basis* of the couple (\mathcal{M}, P) .

COROLLARY. Let \mathcal{M} be an \mathfrak{o} -lattice in $\mathcal{V}(2n, k)$ and P a non-degenerate skew-symmetric matrix such that $xP'y \in \mathfrak{c}$ for $x, y \in \mathcal{M}$, where \mathfrak{c} is an ideal in k . Then, there exists an element T in $GL(2n, k)$ such that

$$\mathcal{M} = [\mathfrak{o}, \dots, \mathfrak{o}, ca_1, \dots, ca_n]T$$

and
$$TP'T = J, \quad \text{where} \quad J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

This follows easily from the above lemma.

Let \mathfrak{a} be an integral ideal in k . We denote by $\mathfrak{o}/\mathfrak{a}$ the quotient module of \mathfrak{o} by \mathfrak{a} . The symbol \oplus will mean the direct sum of modules.

LEMMA 2. Let $a_1, \dots, a_n, b_1, \dots, b_m$ be ideals in k such that

- (i) $a_1 \subset a_2 \subset \dots \subset a_n \subset \mathfrak{o}$,
- (ii) $b_1 \subset b_2 \subset \dots \subset b_m \subset \mathfrak{o}$,
- (iii) $\mathfrak{o}/a_1 \oplus \dots \oplus \mathfrak{o}/a_n$ and $\mathfrak{o}/b_1 \oplus \dots \oplus \mathfrak{o}/b_m$ are isomorphic as \mathfrak{o} -modules.

Then, there exists an integer t such that $a_i = b_i$ for $1 \leq i \leq t$ and $a_k \subset a$, $b_j \subset \mathfrak{o}$ for $j > t$.

PROOF. \mathfrak{p} being a prime ideal, we consider the set

$$A_{\mathfrak{p}} = \{x \in \mathfrak{o}/a_1 \oplus \dots \oplus \mathfrak{o}/a_n; \mathfrak{p}^{\nu}x = 0 \text{ for some } \nu > 0\}.$$

Then, $A_{\mathfrak{p}} = \mathfrak{o}/\mathfrak{p}^{\nu_1} \oplus \dots \oplus \mathfrak{o}/\mathfrak{p}^{\nu_n}$, where \mathfrak{p}^{ν_i} is the \mathfrak{p} -part of a_i and

$$(3) \quad \nu_1 \geq \nu_2 \geq \dots \geq \nu_n.$$

In the same way, putting

$$B_{\mathfrak{p}} = \{x \in \mathfrak{o}/b_1 \oplus \dots \oplus \mathfrak{o}/b_m; \mathfrak{p}^{\nu}x = 0 \text{ for some } \nu > 0\},$$

we have $B_{\mathfrak{p}} = \mathfrak{o}/\mathfrak{p}^{\mu_1} \oplus \dots \oplus \mathfrak{o}/\mathfrak{p}^{\mu_m}$, where \mathfrak{p}^{μ_i} is the \mathfrak{p} -part of b_i and

$$(4) \quad \mu_1 \geq \mu_2 \geq \dots \geq \mu_m.$$

By (iii), we have

$$(5) \quad \mathfrak{o}/\mathfrak{p}^{\nu_1} \oplus \cdots \oplus \mathfrak{o}/\mathfrak{p}^{\nu_n} \cong \mathfrak{o}/\mathfrak{p}^{\mu_1} \oplus \cdots \oplus \mathfrak{o}/\mathfrak{p}^{\mu_m}.$$

Let \mathfrak{o}^* be the \mathfrak{p} -completion of \mathfrak{o} and $\mathfrak{p}^* = \mathfrak{o}^* \mathfrak{p}$. Then, $\mathfrak{o}/\mathfrak{p}^\lambda \cong \mathfrak{o}^*/\mathfrak{p}^{*\lambda}$; and $\mathfrak{p}^{*\lambda}$ is a principal ideal for every λ . Therefore, by the theory of elementary divisors, ordered sets $\{\nu_i\}$ and $\{\mu_i\}$ coincide, after removing the ν_i and μ_i which are equal to 0. As this result holds for any \mathfrak{p} , we obtain our lemma.

For a later use, we write the following

COROLLARY. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_m, \mathfrak{b}_1, \dots, \mathfrak{b}_n$ be ideals in k such that $\mathfrak{o} \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_m$, $\mathfrak{o} \supset \mathfrak{b}_1 \supset \cdots \supset \mathfrak{b}_n$, and

$$\bigoplus_{i=1}^m (\mathfrak{o}/\mathfrak{a}_i + \mathfrak{o}/\mathfrak{a}_i) \cong \bigoplus_{i=1}^n (\mathfrak{o}/\mathfrak{b}_i + \mathfrak{o}/\mathfrak{b}_i).$$

Then, we have $\mathfrak{a}_i = \mathfrak{b}_i$ for every i .

This is an immediate consequence of Lemma 2.

4. Let k be a totally real algebraic number field of finite degree s over \mathbb{Q} , \mathfrak{o} the ring of integers of k , and $k^{(1)} = k, \dots, k^{(s)}$ the conjugate fields of k over \mathbb{Q} . Let $Z^{(\lambda)}$ be complex $n \times n$ matrices for $\lambda = 1, \dots, s$; and let us denote $Z = (Z^{(1)}, \dots, Z^{(s)})$. We write $'Z = ('Z^{(1)}, \dots, 'Z^{(s)})$ and put $\mathcal{H}(n, s) = \{Z; Z = 'Z \text{ and } \text{Im } Z = Y > 0\}$, where $\text{Im } Z = Y > 0$ means that $\text{Im } Z^{(\lambda)} = Y^{(\lambda)}$ is positive definite for every λ . For $M \in GL(2n, k)$, we denote $M^* = (M^{(1)}, \dots, M^{(s)})$, where $M^{(\lambda)}$ is the λ -th conjugate of M . Put

$$Sp(n, k) = \{M; M \in GL(2n, k) \text{ } MJ'M = J\}, \text{ where } J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

For every $M \in Sp(n, k)$, we define an operator on $\mathcal{H}(n, s)$ as follows:

$$(6) \quad Z \mapsto M^*[Z] = (M^{(1)}[Z]^{(1)}, \dots, M^{(s)}[Z]^{(s)}), \quad Z \in \mathcal{H}(n, s) \\ M^{(\lambda)}[Z]^{(\lambda)} = (A^{(\lambda)}Z^{(\lambda)} + B^{(\lambda)})(C^{(\lambda)}Z^{(\lambda)} + D^{(\lambda)})^{-1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A, B, C, D in $M_n(k)$, the total matrix ring of degree n over k . Let \mathfrak{M} be an \mathfrak{o} -lattice in $\mathcal{H}(2n, k)$ and $P(x, y)$ a k -bilinear non-degenerate alternating form which is \mathfrak{o} -valued on \mathfrak{M} with an ideal \mathfrak{c} in k . Define a skew symmetric matrix P by $P(x, y) = xP'y$. Put

$$I'(P, \mathfrak{M}) = \{M \in GL(2n, k); MP'M = P \text{ and } \mathfrak{M}M = \mathfrak{M}\}.$$

By Corollary to Lemma 1, there exists an element T of $GL(2n, k)$ such that $\mathfrak{M} = [\mathfrak{o}, \dots, \mathfrak{o}, \mathfrak{ca}_1, \dots, \mathfrak{ca}_n]T$ and $TP'T = J$. Put

$$I'_\mathfrak{o}(\mathfrak{ca}_1, \dots, \mathfrak{ca}_n) = \{TMT^{-1}; M \in I'(P, \mathfrak{M})\}.$$

Then, $I'_\mathfrak{o}(\mathfrak{ca}_1, \dots, \mathfrak{ca}_n) = Sp(n, k) \cap \mathfrak{D}(\mathfrak{o}, \dots, \mathfrak{o}, \mathfrak{ca}_1, \dots, \mathfrak{ca}_n)$. It is easily seen that

$\Gamma_0(a_1, \dots, a_n)$ and $\Gamma_0(b_1, \dots, b_n)$ are commensurable. $\Gamma_0(ca_1, \dots, ca_n)$ operates discontinuously on $\mathcal{H}(n, s)$ with the operation defined by (6). If $M_1, M_2 \in \Gamma_0(ca_1, \dots, ca_n)$ define one and the same operator, then we have $M_1 = \pm M_2$ (cf. C. L. Siegel [7]).

We put $\Gamma'(ca_1, \dots, ca_n) = \Gamma_0(ca_1, \dots, ca_n) / \{ \pm (1_{2n}, \dots, 1_n) \}$ and call it the *Hilbert-Siegel para-modular group of type* (ca_1, \dots, ca_n) . In particular, $\Gamma'(0, \dots, 0)$ is called the Hilbert-Siegel modular group.

Being $s=1$ (the classical case), let \mathfrak{M} be a lattice in $\mathcal{V}(2n, \mathbf{Q})$, and P an integral non-singular skew-symmetric matrix, such that $xP'y$ is contained in \mathbf{Z} for $x, y \in \mathfrak{M}$. Then, there exists a unimodular matrix T such that

$$\mathfrak{M} = [Z, \dots, Z, \delta_1 Z, \dots, \delta_n Z] T$$

and

$$TP^tT = J,$$

where the δ_i are integers and $\delta_1 | \delta_2 | \delta_3 | \dots | \delta_n$. Put

$$\delta = \begin{pmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_n \end{pmatrix}.$$

Then, we have

$$\Gamma(P, \mathfrak{M}) = \{ M \in GL(2n, \mathbf{Q}) ; MP^tM = P \text{ and } \mathfrak{M}M = \mathfrak{M} \},$$

and

$$\Gamma'((\delta_1), \dots, (\delta_n)) = \{ TMT^{-1} ; M \in \Gamma(P, \mathfrak{M}) \}.$$

This group was defined as a kind of para-modular groups in Seminaire H. Cartain 1957/58 and denoted by $\Gamma(\delta)$. In particular, $\Gamma'((1), \dots, (1))$ is the Siegel modular group.

5. Let A be an abelian variety. We consider the relation between polarizations of A and involutions of $\mathcal{N}_0(A)$.

Let B be the Picard variety of A . Every divisor X on A which is algebraically equivalent to 0 defines a point $\text{Cl}(X)$ of B . For every element α of $\mathcal{N}(A)$, there exists an element β of $\mathcal{N}(B)$ defined by $\beta(\text{Cl}(X)) = \text{Cl}(\alpha^{-1}(X))$. β is called the transpose of α and denoted by ${}^t\alpha$. The mapping $\alpha \rightarrow {}^t\alpha$ is an anti-isomorphism of $\mathcal{N}(A)$ into $\mathcal{N}(B)$. We extend this to an anti-isomorphism of $\mathcal{N}_0(A)$ into $\mathcal{N}_0(B)$ and denote it also by the same letter ${}^t\alpha$.

The mapping φ_X of A into B defined by $\varphi_X(t) = \text{Cl}(X_t - X)$ for $t \in A$ gives a homomorphism of A into B . X is non-degenerate if and only if $\varphi_X(A) = B$. X being non-degenerate, for every $\alpha \in \mathcal{N}_0(A)$, define an element α^* of $\mathcal{N}_0(A)$ by

$$\varphi_X^{-1} \cdot {}^t\alpha \cdot \varphi_X = \alpha^*.$$

The mapping $\alpha \rightarrow \alpha^*$ of $\mathcal{N}_0(A)$ onto itself gives an involution of $\mathcal{N}_0(A)$; and if a suitable positive multiple of X is ample, we have

$$\mathrm{tr}(\alpha\alpha^*) > 0$$

for every $\alpha \neq 0$ of $\mathcal{N}_0(A)$, where $\mathrm{tr}(\alpha)$ means the trace of l -adic representation of α . Two non-degenerate divisors give the same involution if they are contained in the same polarization of A ; so every polarization of A defines an involution of $\mathcal{N}_0(A)$. Characteristic being 0, let a complex torus C^n/\mathcal{A} be an analytic model of A of dimension n . Fixing an isomorphism of A onto C^n/\mathcal{A} , every positive non-degenerate divisor X on A corresponds to a non-degenerate Riemann form $\mathcal{E}(x, y)$ on C^n/\mathcal{A} and every element $\alpha \in \mathcal{N}_0(A)$ corresponds to a matrix $M(\alpha)$ of degree n , then we have

$$(7) \quad \mathcal{E}(xM(\alpha), y) = \mathcal{E}(x, yM(\alpha^*))$$

for the Riemann form corresponding to the non-degenerate X .

6. k being a totally real algebraic number field of finite degree s over \mathbb{Q} , let (A, \mathcal{E}) be a polarized abelian variety of dimension m defined over C such that there exists an isomorphism ι of k into $\mathcal{N}_0(A)$ and the following conditions are satisfied:

- a) $\iota(1)$ is the identity of $\mathcal{N}_0(A)$;
- b) The involution $*$ which is determined by \mathcal{E} is the identity on $\iota(k)$, i.e., $\iota(f)^* = \iota(f)$ for any $f \in k$.²⁾

Take and fix an analytic isomorphism of A onto a complex torus C^m/\mathcal{A} of dimension m , where \mathcal{A} denotes a discrete subgroup of C^m of rank $2m$. Every element of $\mathcal{N}_0(A)$ corresponds to a matrix of degree m with complex entries. We denote by the same letter $\iota(f)$ the complex matrix which corresponds to $\iota(f) \in \mathcal{N}_0(A)$.³⁾ Then, $\iota(f)$, considered as a representation of k , is a sum of representations $f \mapsto f^{(\lambda)}$ for $1 \leq \lambda \leq s$ with suitable multiplicities. In view of a), no 0-representation can be included. As is well-known, the sum of ι and its complex conjugate is equivalent to a rational representation. As k is totally real, we observe that the multiplicities of $f \mapsto f^{(\lambda)}$ in ι are the same for all λ . It follows, in particular, that m is divisible by s ; put $m = ns$.

Thus, after choosing a suitable basis of $C^m = C^{ns}$, we may assume that $\iota(f)$ is of the form

$$(8) \quad \iota(f) = \begin{pmatrix} f^{(1)} \cdot \mathbf{1}_n & & 0 \\ & \ddots & \\ 0 & & f^{(s)} \cdot \mathbf{1}_n \end{pmatrix}$$

2) This follows from the condition that $*$ fixes $\iota(k)$ as a whole.

3) Namely, $\Theta(x, Z) \cdot \iota(f) = \Theta(x \cdot \iota(f), Z)$ for $\Theta(x, Z)$ given in (SS4).

for every $f \in k$. Since \mathcal{A} has rank $2ns$ over Z , $Q\mathcal{A}$ is a vector space over $\iota(k)$ of dimension $2n$. We can write

$$(9) \quad Q\mathcal{A} = \sum_{i=1}^{2n} u_i \iota(k)$$

with $2n$ vectors u_i of C^{ns} .

PROPOSITION 1. Let u_i for $i=1, \dots, 2n$, be $2n$ vectors of C^{ns} . For every i , put $u_i = (x_i^{(1)}, \dots, x_i^{(s)})$ with vectors $x_i^{(\lambda)}$ in C^n . Let ω_j , $j=1, \dots, s$, be a basis of k over Q . Then, the vectors

$$(u_i, \bar{u}_i) \begin{pmatrix} \iota(\omega_j) & 0 \\ 0 & \iota(\omega_j) \end{pmatrix}, \quad 1 \leq i \leq 2n, \quad 1 \leq j \leq s,$$

are linearly independent over R if and only if

$$(10) \quad \det \begin{pmatrix} x_1^{(\lambda)} & x_1^{(\lambda)} \\ \vdots & \vdots \\ \vdots & \vdots \\ x_{2n}^{(\lambda)} & x_{2n}^{(\lambda)} \end{pmatrix} \neq 0 \quad \text{for every } \lambda.$$

PROOF. We have

$$\begin{aligned} \sum_j \iota(\omega_j) a_{ij} &= \begin{pmatrix} \sum_j \omega_j^{(1)} a_{ij} \cdot 1_n & & \\ & \ddots & \\ & & \sum_j \omega_j^{(s)} a_{ij} \cdot 1_n \end{pmatrix} \\ &= \begin{pmatrix} b_i^{(1)} \cdot 1_n & & \\ & \ddots & \\ & & b_i^{(s)} \cdot 1_n \end{pmatrix}, \end{aligned}$$

where a_{ij} , $b_i^{(\lambda)} = \sum_j \omega_j^{(\lambda)} a_{ij} \in R$.

Therefore, $\sum_{i,j} (u_i, \bar{u}_i) \begin{pmatrix} \iota(\omega_j) & 0 \\ 0 & \iota(\omega_j) \end{pmatrix} a_{ij} = 0$ holds if and only if $\sum_i (x_i^{(\lambda)}, x_i^{(\lambda)}) \cdot b_i^{(\lambda)} \cdot 1_n = 0$ holds for every λ . From this, our proposition easily follows.

Coming back to (9), as \mathcal{A} is discrete in C^{ns} , we have

$$\det \begin{pmatrix} x_1^{(\lambda)} & x_1^{(\lambda)} \\ \vdots & \vdots \\ \vdots & \vdots \\ x_{2n}^{(\lambda)} & x_{2n}^{(\lambda)} \end{pmatrix} \neq 0 \quad \text{for every } \lambda.$$

by Proposition 1. Hence, putting

$$(11) \quad \begin{pmatrix} u_1 \\ \vdots \\ \vdots \\ u_{2n} \end{pmatrix} = \begin{pmatrix} {}^t U^{(1)} & {}^t U^{(2)} & \dots & {}^t U^{(s)} \\ {}^t V^{(1)} & {}^t V^{(2)} & \dots & {}^t V^{(s)} \end{pmatrix}$$

with $U^{(\lambda)}, V^{(\lambda)} \in \mathcal{M}_n(C)$, we have

$$(12) \quad \det \begin{pmatrix} {}^t U^{(\lambda)} & {}^t \overline{U}^{(\lambda)} \\ {}^t V^{(\lambda)} & {}^t \overline{V}^{(\lambda)} \end{pmatrix} \neq 0 \quad \text{for every } \lambda.$$

Let $\mathcal{S}(x, y)$ be a non-degenerate Riemann form on C^{ns}/\mathcal{A} .

Consider a polarization \mathcal{S} of A ; take a divisor X which corresponds to a Riemann form $\mathcal{S}(x, y)$ on C^{ns}/\mathcal{A} . By the property b) of \mathcal{S} and (7),

$$(13) \quad \mathcal{S}(x \cdot \iota(f), y) = \mathcal{S}(x, y \cdot \iota(f)) \quad \text{for } x, y \in C^{ns}.$$

Put

$$(14) \quad \mathfrak{M} = \{(f_1, \dots, f_{2n}) \in \mathcal{V}(2n, k); (f_1, \dots, f_{2n}) \begin{pmatrix} {}^t U \\ {}^t V \end{pmatrix} \in \mathcal{A}\},$$

where $(f_1, \dots, f_{2n}) \begin{pmatrix} {}^t U \\ {}^t V \end{pmatrix}$ means a vector

$$\left((f_1^{(1)}, \dots, f_{2n}^{(1)}) \begin{pmatrix} {}^t U^{(1)} \\ {}^t V^{(1)} \end{pmatrix}, \dots, (f_1^{(s)}, \dots, f_{2n}^{(s)}) \begin{pmatrix} {}^t U^{(s)} \\ {}^t V^{(s)} \end{pmatrix} \right)$$

in C^{ns} . Then, \mathfrak{M} is a lattice of $\mathcal{V}(2n, k)$. Put

$$c) \quad \mathfrak{r} = \iota^{-1}[\iota(k) \cap \mathcal{S}(A)];$$

then, \mathfrak{r} is an order in k and $\mathfrak{r} = \{f \in k; f\mathfrak{M} \subset \mathfrak{M}\}$ by (14).

We write $x, y \in Q\mathcal{A}$ in the form

$$(15) \quad x = (f_1, \dots, f_{2n}) \begin{pmatrix} {}^t U \\ {}^t V \end{pmatrix}, \quad y = (g_1, \dots, g_{2n}) \begin{pmatrix} {}^t U \\ {}^t V \end{pmatrix}$$

with $f_i, g_i \in k$. For $(f_1, \dots, f_{2n}), (g_1, \dots, g_{2n}) \in \mathcal{V}(2n, k)$,

$$\begin{aligned} \text{put} \quad & A((f_1, \dots, f_{2n}), (g_1, \dots, g_{2n})) \\ &= \mathcal{S} \left((f_1, \dots, f_{2n}) \begin{pmatrix} {}^t U \\ {}^t V \end{pmatrix}, (g_1, \dots, g_{2n}) \begin{pmatrix} {}^t U \\ {}^t V \end{pmatrix} \right); \end{aligned}$$

then, A is a \mathbf{Q} -bilinear mapping of $\mathcal{V}(2n, k)$ into \mathbf{Q} by (R1). By (13) and by using the fact that any \mathbf{Q} -bilinear mapping λ of k into \mathbf{Q} can be written as

$$\lambda(f) = \text{tr}_{k/\mathbf{Q}}(\zeta f)$$

with an element ζ in k , we have

$$(16) \quad A((f_1, \dots, f_{2n}), (g_1, \dots, g_{2n})) = \text{tr}_{k/\mathbf{Q}} \left((f_1, \dots, f_{2n}) P \begin{pmatrix} g_1 \\ \vdots \\ g_{2n} \end{pmatrix} \right)$$

with $P=(p_{ij}) \in GL(2n, k)$.

Then, we have, by (R 2),

$$(G\ 2) \quad P = {}^t P.$$

By (16), we have

$$\mathcal{E}(x, y) = \sum_{\lambda=1}^s (f_1^{(\lambda)}, \dots, f_{2n}^{(\lambda)}) P^{(\lambda)} \begin{pmatrix} g_1^{(\lambda)} \\ \vdots \\ g_{2n}^{(\lambda)} \end{pmatrix}$$

for the vectors x, y given by (15). Note that the set of vectors $(f^{(1)}, \dots, f^{(s)})$ for $f \in k$ is dense in K^s . Therefore, we have

$$\mathcal{E}(x, y) = \sum_{\lambda=1}^s r^{(\lambda)} P^{(\lambda)} t^{(\lambda)}$$

for $r^{(\lambda)}, t^{(\lambda)} \in K^{2n}$, and

$$x = \left(r^{(1)} \begin{pmatrix} {}^t U^{(1)} \\ {}^t V^{(1)} \end{pmatrix}, \dots, r^{(s)} \begin{pmatrix} {}^t U^{(s)} \\ {}^t V^{(s)} \end{pmatrix} \right), \quad y = \left(t^{(1)} \begin{pmatrix} {}^t U^{(1)} \\ {}^t V^{(1)} \end{pmatrix}, \dots, t^{(s)} \begin{pmatrix} {}^t U^{(s)} \\ {}^t V^{(s)} \end{pmatrix} \right).$$

Putting $x^{(\lambda)} = r^{(\lambda)} \begin{pmatrix} {}^t U^{(\lambda)} \\ {}^t V^{(\lambda)} \end{pmatrix}$ and $y^{(\lambda)} = t^{(\lambda)} \begin{pmatrix} {}^t U^{(\lambda)} \\ {}^t V^{(\lambda)} \end{pmatrix}$, we get

$$(x^{(\lambda)}, x^{(\lambda)}) = r^{(\lambda)} \begin{pmatrix} {}^t U^{(\lambda)} & {}^t \bar{V}^{(\lambda)} \\ {}^t V^{(\lambda)} & {}^t \bar{V}^{(\lambda)} \end{pmatrix}, \quad (y^{(\lambda)}, y^{(\lambda)}) = t^{(\lambda)} \begin{pmatrix} {}^t U^{(\lambda)} & {}^t \bar{U}^{(\lambda)} \\ {}^t V^{(\lambda)} & {}^t \bar{V}^{(\lambda)} \end{pmatrix}.$$

By (12), the matrix $\begin{pmatrix} {}^t U^{(\lambda)} & {}^t \bar{U}^{(\lambda)} \\ {}^t V^{(\lambda)} & {}^t \bar{V}^{(\lambda)} \end{pmatrix}$ has the inverse; denote it by $\begin{pmatrix} F_1^{(\lambda)} & F_3^{(\lambda)} \\ F_2^{(\lambda)} & F_4^{(\lambda)} \end{pmatrix}$.

Then

$$\mathcal{E}(x, y) = \sum_{\lambda=1}^s (x^{(\lambda)}, x^{(\lambda)}) \begin{pmatrix} F_1^{(\lambda)} & F_3^{(\lambda)} \\ F_2^{(\lambda)} & F_4^{(\lambda)} \end{pmatrix} P^{(\lambda)} \begin{pmatrix} F_1^{(\lambda)} & F_3^{(\lambda)} \\ F_2^{(\lambda)} & F_4^{(\lambda)} \end{pmatrix} \cdot \begin{pmatrix} y^{(\lambda)} \\ y^{(\lambda)} \end{pmatrix}.$$

Fix a λ and consider the vectors x and y such that $x^{(\nu)} = y^{(\nu)} = 0$ for $\nu \neq \lambda$. Then, the properties (R 2, 3) of \mathcal{E} imply that, for every λ ,

$$E^{(\lambda)}(x^{(\lambda)}, y^{(\lambda)}) = (x^{(\lambda)}, x^{(\lambda)}) \begin{pmatrix} F_1^{(\lambda)} & F_3^{(\lambda)} \\ F_2^{(\lambda)} & F_4^{(\lambda)} \end{pmatrix} P^{(\lambda)} \begin{pmatrix} F_1^{(\lambda)} & F_3^{(\lambda)} \\ F_2^{(\lambda)} & F_4^{(\lambda)} \end{pmatrix} \cdot \begin{pmatrix} y^{(\lambda)} \\ y^{(\lambda)} \end{pmatrix}$$

has the same properties; namely,

(*) $E^{(\lambda)}(x^{(\lambda)}, \sqrt{-1}y^{(\lambda)})$ is symmetric and positive definite.

As the following treatment holds for every λ , we shall omit λ .

We have then

$$UF_1 + VF_2 = 1, \quad UF_3 + VF_4 = 0, \quad \bar{U}F_1 + \bar{V}F_2 = 0 \quad \text{and} \quad \bar{U}F_3 + \bar{V}F_4 = 1$$

and taking the conjugates of the above first two formulae, we have

$$\overline{UF_1} + \overline{VF_2} = 1 \quad \text{and} \quad \overline{UF_3} + \overline{VF_4} = 0.$$

From these, it follows that $\overline{F_1} = F_3$ and $\overline{F_2} = F_4$, since F_1, F_2, F_3 and F_4 are determined uniquely by the above relations. Thus, we have

$$\begin{pmatrix} U & V \\ U & V \end{pmatrix}^{-1} = \begin{pmatrix} F_1 & \overline{F_1} \\ F_2 & \overline{F_2} \end{pmatrix}.$$

Putting $S = \begin{pmatrix} F_1 & \overline{F_1} \\ F_2 & \overline{F_2} \end{pmatrix} P \begin{pmatrix} F_1 & \overline{F_1} \\ F_2 & \overline{F_2} \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$, we get

$$S_1 = ({}^t F_1, {}^t F_2) P \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad S_2 = ({}^t F_1, {}^t F_2) P \begin{pmatrix} \overline{F_1} \\ \overline{F_2} \end{pmatrix},$$

$$S_3 = ({}^t \overline{F_1}, {}^t \overline{F_2}) P \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \quad \text{and} \quad S_4 = ({}^t \overline{F_1}, {}^t \overline{F_2}) P \begin{pmatrix} \overline{F_1} \\ \overline{F_2} \end{pmatrix}.$$

As P is real, we have

$$S_1 = \overline{S_4} \quad \text{and} \quad S_2 = \overline{S_3}$$

By (*), $S \begin{pmatrix} \sqrt{-1} \cdot 1_n & 0 \\ 0 & -\sqrt{-1} \cdot 1_n \end{pmatrix}$ is symmetric, so that

$$(\#) \quad S_1 = {}^t S_1, \quad -{}^t S_2 = \overline{S_2}$$

Now as S is skew-symmetric,

$$S_1 = -{}^t S_1.$$

Therefore, in view of (#), we have

$$S_1 = S_4 = 0.$$

Put $S_2 = H$; then, we have $S = \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix}$ and

$$\begin{aligned} E(x, \sqrt{-1}y) &= (x, \bar{x}) \begin{pmatrix} 0 & -\sqrt{-1}H \\ \sqrt{-1}\overline{H} & 0 \end{pmatrix} \begin{pmatrix} {}^t y \\ {}^t \overline{y} \end{pmatrix} = \sqrt{-1} (xH{}^t \overline{y} - xH{}^t y) \\ &= -2 \operatorname{Re} (x\sqrt{-1}H{}^t y). \end{aligned}$$

Therefore, by (*), it follows that

$$(\#\#) \quad -\sqrt{-1}H \text{ is hermitian and positive definite.}$$

We have

$$\begin{pmatrix} {}^t U & {}^t \overline{U} \\ {}^t V & {}^t \overline{V} \end{pmatrix} \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix} \begin{pmatrix} U & V \\ \overline{U} & \overline{V} \end{pmatrix} = P$$

and

$$\begin{pmatrix} U & {}^tV \\ \bar{U} & \bar{V} \end{pmatrix} {}^tP^{-1} \begin{pmatrix} {}^tU & {}^tU \\ {}^tV & {}^tV \end{pmatrix} = \begin{pmatrix} 0 & {}^tH^{-1} \\ {}^tH^{-1} & 0 \end{pmatrix}.$$

Thus,

$$(G\ 3) \quad (U^{(\lambda)} V^{(\lambda)}) {}^tP^{(\lambda)-1} \begin{pmatrix} {}^tU^{(\lambda)} \\ {}^tV^{(\lambda)} \end{pmatrix} = 0 \quad \text{for every } \lambda$$

and by (##),

$$(G\ 4) \quad \sqrt{-1} (U^{(\lambda)} V^{(\lambda)}) {}^tP^{(\lambda)-1} \begin{pmatrix} {}^t\bar{U}^{(\lambda)} \\ {}^t\bar{V}^{(\lambda)} \end{pmatrix} = \sqrt{-1} {}^tH^{(\lambda)-1}$$

is hermitian and positive definite, for every λ .

By (R 1), we have

(G 1) $\text{tr}_{k/Q}(\mathfrak{M} P {}^t\mathfrak{M}) \subset Z$; if $v=0$, \mathfrak{M} is an v -lattice and $\mathfrak{M} P {}^t\mathfrak{M} \subset \delta^{-1}$, where δ is the different of k with respect to Q .

We have thus shown that a polarized abelian variety (A, \mathcal{E}, ι) which satisfies a), b) and c), determines a lattice \mathfrak{M} in $\mathscr{V}(2n, k)$, a matrix $\begin{pmatrix} {}^tU^{(1)} & \dots & {}^tU^{(s)} \\ {}^tV^{(1)} & \dots & {}^tV^{(s)} \end{pmatrix}$ and element P of $GL(2n, k)$. Now, we shall construct a triplet (A, \mathcal{E}, ι) which satisfies a), b) and c) from these data: a lattice \mathfrak{M} of $\mathscr{V}(2n, k)$ and matrices $\begin{pmatrix} {}^tU \\ {}^tV \end{pmatrix} = \begin{pmatrix} {}^tU^{(1)} & \dots & {}^tU^{(s)} \\ {}^tV^{(1)} & \dots & {}^tV^{(s)} \end{pmatrix}$ and P such that (G 1), (G 2), (G 3) and (G 4) are satisfied. Let the real representation $\iota(f)$ of k of degree ns be determined by (8). Take a lattice \mathfrak{M} and $2ns$ vectors u_1, \dots, u_{2n} of C^{ns} such that

$$\det \begin{pmatrix} {}^tU^{(\lambda)} & {}^t\bar{U}^{(\lambda)} \\ {}^tV^{(\lambda)} & {}^t\bar{V}^{(\lambda)} \end{pmatrix} \neq 0 \quad \text{for every } \lambda,$$

writing $\begin{pmatrix} u_1 \\ \vdots \\ u_{2n} \end{pmatrix}$ in the form (11).

$$\text{Put} \quad \mathcal{A} = \left\{ (f_1, \dots, f_{2n}) \begin{pmatrix} {}^tU \\ {}^tV \end{pmatrix}; (f_1, \dots, f_{2n}) \in \mathfrak{M} \right\}.$$

Then, by Proposition 1, \mathcal{A} is a discrete subgroup of C^{ns} of rank $2ns$, so C^{ns}/\mathcal{A} is a complex torus. Put

$$\begin{aligned} \mathcal{E} \left((f_1, \dots, f_{2n}) \begin{pmatrix} {}^tU \\ {}^tV \end{pmatrix}, (g_1, \dots, g_{2n}) \begin{pmatrix} {}^tU \\ {}^tV \end{pmatrix} \right) \\ = \sum_{\lambda=1}^s \left((f_1^{(\lambda)}, \dots, f_{2n}^{(\lambda)}) P^{(\lambda)} \begin{pmatrix} g_1^{(\lambda)} \\ \vdots \\ g_{2n}^{(\lambda)} \end{pmatrix} \right) \end{aligned}$$

for $f^{(\lambda)}, g^{(\lambda)} \in R$.

Then, by (G 2), $\mathcal{E}\left((f_1, \dots, f_{2n})\begin{pmatrix} {}^tU \\ {}^tV \end{pmatrix}, (g_1, \dots, g_{2n})\begin{pmatrix} {}^tU \\ {}^tV \end{pmatrix}\right)$ is an alternating R -bilinear form on C^{ns} and Z -valued on A . By (G 3) and (G 4), we have

$$\begin{pmatrix} U^{(\lambda)} & V^{(\lambda)} \\ U^{(\lambda)} & V^{(\lambda)} \end{pmatrix} P^{(\lambda)^{-1}} \begin{pmatrix} {}^tU^{(\lambda)} & {}^t\bar{U}^{(\lambda)} \\ {}^tV^{(\lambda)} & {}^t\bar{V}^{(\lambda)} \end{pmatrix} = \begin{pmatrix} 0 & {}^tH^{(\lambda)^{-1}} \\ {}^tH^{(\lambda)^{-1}} & 0 \end{pmatrix}$$

for every λ , where $\sqrt{-1} {}^tH^{(\lambda)^{-1}}$ is hermitian and positive definite; hence

$$P^{(\lambda)} = \begin{pmatrix} {}^tU^{(\lambda)} & {}^tU^{(\lambda)} \\ {}^tV^{(\lambda)} & {}^tV^{(\lambda)} \end{pmatrix} \begin{pmatrix} 0 & H^{(\lambda)} \\ \bar{H}^{(\lambda)} & 0 \end{pmatrix} \begin{pmatrix} U^{(\lambda)} & V^{(\lambda)} \\ U^{(\lambda)} & V^{(\lambda)} \end{pmatrix},$$

and

$$\begin{aligned} & \mathcal{E}\left((x_1, \dots, x_{2n})\begin{pmatrix} {}^tU \\ {}^tV \end{pmatrix}, (y_1, \dots, y_{2n})\begin{pmatrix} {}^tU \\ {}^tV \end{pmatrix}\right) \\ &= \sum_{\lambda=1}^s \left((x_1^{(\lambda)}, \dots, x_{2n}^{(\lambda)}) P^{(\lambda)} \begin{pmatrix} y_1^{(\lambda)} \\ \vdots \\ y_{2n}^{(\lambda)} \end{pmatrix} \right) \\ &= \sum_{\lambda=1}^s (x_1^{(\lambda)}, \dots, x_{2n}^{(\lambda)}) \begin{pmatrix} {}^tU^{(\lambda)} & {}^t\bar{U}^{(\lambda)} \\ {}^tV^{(\lambda)} & {}^t\bar{V}^{(\lambda)} \end{pmatrix} \begin{pmatrix} 0 & H^{(\lambda)} \\ \bar{H}^{(\lambda)} & 0 \end{pmatrix} \begin{pmatrix} U^{(\lambda)} & V^{(\lambda)} \\ U^{(\lambda)} & V^{(\lambda)} \end{pmatrix} \begin{pmatrix} y_1^{(\lambda)} \\ \vdots \\ y_{2n}^{(\lambda)} \end{pmatrix}. \end{aligned}$$

Thus, putting

$$x^{(\lambda)} = (x_1^{(\lambda)}, \dots, x_{2n}^{(\lambda)}) \begin{pmatrix} {}^tU^{(\lambda)} \\ {}^tV^{(\lambda)} \end{pmatrix}, \quad y^{(\lambda)} = (y_1^{(\lambda)}, \dots, y_{2n}^{(\lambda)}) \begin{pmatrix} {}^tU^{(\lambda)} \\ {}^tV^{(\lambda)} \end{pmatrix},$$

$x = (x^{(1)}, \dots, x^{(s)})$ and $y = (y^{(1)}, \dots, y^{(s)})$, we have

$$\mathcal{E}(x, y) = \sum_{\lambda=1}^s (x^{(\lambda)}, \bar{x}^{(\lambda)}) \begin{pmatrix} 0 & H^{(\lambda)} \\ \bar{H}^{(\lambda)} & 0 \end{pmatrix} \begin{pmatrix} {}^ty^{(\lambda)} \\ {}^t\bar{y}^{(\lambda)} \end{pmatrix}.$$

Since

$$\begin{aligned} \mathcal{E}(x, \sqrt{-1}y) &= \sum_{\lambda=1}^s (x^{(\lambda)}, \bar{x}^{(\lambda)}) \begin{pmatrix} 0 & H^{(\lambda)} \\ \bar{H}^{(\lambda)} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{-1} {}^ty^{(\lambda)} \\ -\sqrt{-1} {}^t\bar{y}^{(\lambda)} \end{pmatrix} \\ &= \sum_{\lambda=1}^s (x^{(\lambda)}, \bar{x}^{(\lambda)}) \begin{pmatrix} 0 & \sqrt{-1} H^{(\lambda)} \\ \sqrt{-1} \bar{H}^{(\lambda)} & 0 \end{pmatrix} \begin{pmatrix} {}^ty^{(\lambda)} \\ {}^t\bar{y}^{(\lambda)} \end{pmatrix} \\ &= \sum_{\lambda=1}^s (\sqrt{-1} \bar{x}^{(\lambda)} \bar{H}^{(\lambda)} {}^ty - \sqrt{-1} x^{(\lambda)} H^{(\lambda)} {}^t\bar{y}^{(\lambda)}) \\ &= 2 \sum_{\lambda=1}^s \operatorname{Re} (x^{(\lambda)} \sqrt{-1} H {}^ty^{(\lambda)}), \end{aligned}$$

we can easily show that $\mathcal{E}(x, y)$ has the property (R 3), on account of (G 3) and

(G 4). Thus, we obtain a Riemann form $\mathcal{E}(x, y)$ on C^{ns}/J , hence there exists an analytic isomorphism of C^{ns}/J onto an abelian variety defined over C . $\mathcal{E}(x, y)$ determines a polarization \mathcal{E} on A . Now, let r be the set of all elements f of k such that $f\mathfrak{M} \subset \mathfrak{M}$; then, r is an order in k . For every $f \in k$, we have $J \cdot \iota(f) \subset J$ if and only if $f \in r$; hence, for every $f \in r$, $\iota(f)$ gives an endomorphism of C^{ns}/J ; denote by the same letter $\iota(f)$ the corresponding endomorphism of A ; ι is extended to an isomorphism of k into $\mathcal{A}_0(A)$ which is denoted by ι again. We have by the form of $\mathcal{E}(x, y)$,

$$\mathcal{E}(x \cdot \iota(f), x) = \mathcal{E}(x, y \cdot \iota(f));$$

so, \mathcal{E} satisfies b) and c). Thus, we have obtained the following theorems.

THEOREM 1. *Let k be a totally real algebraic number field of finite degree s over \mathbb{Q} , \mathfrak{M} be a free submodule of $\mathcal{V}(2n, k)$ of rank $2ns$ over \mathbb{Z} , u_1, \dots, u_{2n} be $2n$ vectors of C^{ns} such that*

$$\det \begin{pmatrix} U^{(\lambda)} & \overline{U}^{(\lambda)} \\ V^{(\lambda)} & V^{(\lambda)} \end{pmatrix} \neq 0 \quad \text{for every } \lambda,$$

in writing as

$$\begin{pmatrix} u_1 \\ \vdots \\ u_{2n} \end{pmatrix} = \begin{pmatrix} {}^t U^{(1)} & {}^t U^{(2)} & \dots & {}^t U^{(s)} \\ {}^t V^{(1)} & {}^t V^{(2)} & \dots & {}^t V^{(s)} \end{pmatrix}$$

with $U^{(\lambda)}, V^{(\lambda)} \in \mathcal{M}_n(R)$, and P be an element of $GL(2n, k)$. Put

$$J = \left\{ (f_1, \dots, f_{2n}) \begin{pmatrix} {}^t U \\ {}^t V \end{pmatrix}; \quad (f_1, \dots, f_{2n}) \in \mathfrak{M} \right\}.$$

Then, it is necessary and sufficient for

$$\mathcal{E}(x, y) = \sum_{\lambda=1}^s (x_1^{(\lambda)}, \dots, x_{2n}^{(\lambda)}) P^{(\lambda)} \begin{pmatrix} y_1^{(\lambda)} \\ \vdots \\ y_{2n}^{(\lambda)} \end{pmatrix}$$

with

$$x = \left((x_1^{(1)}, \dots, x_{2n}^{(1)}) \begin{pmatrix} {}^t U^{(1)} \\ {}^t V^{(1)} \end{pmatrix}, \dots, (x_1^{(s)}, \dots, x_{2n}^{(s)}) \begin{pmatrix} {}^t U^{(s)} \\ {}^t V^{(s)} \end{pmatrix} \right) \quad \text{and}$$

$$y = \left((y_1^{(1)}, \dots, y_{2n}^{(1)}) \begin{pmatrix} {}^t U^{(1)} \\ {}^t V^{(1)} \end{pmatrix}, \dots, (y_1^{(s)}, \dots, y_{2n}^{(s)}) \begin{pmatrix} {}^t U^{(s)} \\ {}^t V^{(s)} \end{pmatrix} \right),$$

to be a Riemann form on C^{ns}/J that the following conditions are satisfied;

$$(G 1) \quad \sum_{\lambda=1}^s (f_1^{(\lambda)}, \dots, f_{2n}^{(\lambda)}) P^{(\lambda)} \begin{pmatrix} g_1^{(\lambda)} \\ \vdots \\ g_{2n}^{(\lambda)} \end{pmatrix} \in \mathbb{Z} \quad \text{for every } (f_1, \dots, f_{2n}), (g_1, \dots, g_{2n}) \text{ in } \mathfrak{M}$$

$$(G2) \quad P = \dots = {}^tP.$$

$$(G3) \quad (U^{(\lambda)} V^{(\lambda)})({}^tP^{(\lambda)})^{-1} \begin{pmatrix} {}^tU^{(\lambda)} \\ {}^tV^{(\lambda)} \end{pmatrix} = 0 \quad \text{for every } \lambda.$$

$$(G4) \quad \sqrt{-1} (U^{(\lambda)} V^{(\lambda)})({}^tP^{(\lambda)})^{-1} \begin{pmatrix} {}^t\bar{U}^{(\lambda)} \\ {}^t\bar{V}^{(\lambda)} \end{pmatrix} > 0 \quad \text{for every } \lambda.$$

THEOREM 2. Notations being as above, let ι be the representation of k of degree ns defined by

$$\iota(f) = \begin{pmatrix} f^{(1)} \cdot 1_n & & \\ & \ddots & \\ & & f^{(s)} \cdot 1_n \end{pmatrix}$$

for $f \in k$.

Assume that \mathfrak{M} , $\begin{pmatrix} {}^tU^{(1)} & {}^tU^{(2)} & \dots & {}^tU^{(s)} \\ {}^tV^{(1)} & {}^tV^{(2)} & \dots & {}^tV^{(s)} \end{pmatrix}$ and P satisfy the conditions (G1), (G2), (G3) and (G4). Put $\mathfrak{r} = \{f \in k; f\mathfrak{M} \subset \mathfrak{M}\}$ and let $\mathcal{E}(x, y)$ be the Riemann form defined in the above theorem. Then, there exists an analytic isomorphism of the complex torus C^{ns}/Δ onto an abelian variety A defined over C ; $f \mapsto \iota(f)$ gives an isomorphism of k into $\mathcal{A}_v(A)$, which is denoted by ι again, such that

a) $\iota(1)$ is the identity of $\mathcal{A}_v(A)$;

$\mathcal{E}(x, y)$ determines a polarization \mathcal{E} of A such that

b) the involution given by \mathcal{E} is the identity on $\iota(k)$;

and we have

c) $\iota(\mathfrak{r}) = \mathcal{A}_v(A) \cap \iota(k)$.

Conversely, A being an abelian variety of dimension ns defined over C , if there exist an isomorphism ι of k into $\mathcal{A}_v(A)$ and a polarization \mathcal{E} of A such that the conditions a), b) and c) hold for them, then ι , considered as the representation of k , reduces to the above form and A is obtained from a lattice \mathfrak{M} , $2n$ vectors u_1, \dots, u_{2n} and an element P of $GL(2n, k)$ in this manner and the polarization \mathcal{E} is determined by the Riemann form $\mathcal{E}(x, y)$ given in Theorem 1.

P is called the principal matrix for A or $\mathcal{E}(x, y)$. Thus, the data \mathfrak{M} , U , V , P determine a polarized abelian variety (A, \mathcal{E}, ι) of type \mathfrak{r} , the ring \mathfrak{r} being an order in k ; we denote $(A, \mathcal{E}, \iota) = \mathfrak{P}(\mathfrak{M}, U, V, P)$.

7. Notations being as in the last section, we consider a triplet $(A, \mathcal{E}, \iota) = \mathfrak{P}(\mathfrak{M}, U, V, J)$; note that it is assumed that the principal matrix for A is J . Put $V^{(\lambda)-1}U^{(\lambda)} = Z^{(\lambda)}$ for every λ . This is possible since $V^{(\lambda)}$ is non-singular by (G4) for $J = P$. Put $Z = (Z^{(1)}, \dots, Z^{(s)})$ and

$$\mathcal{J}' = \left\{ (f_1, \dots, f_{2n}) \begin{pmatrix} Z \\ 1 \end{pmatrix} : (f_1, \dots, f_{2n}) \in \mathfrak{M} \right\},$$

$$\text{where } (f_1, \dots, f_{2n}) \begin{pmatrix} Z \\ 1 \end{pmatrix} = \left((f_1^{(1)}, \dots, f_{2n}^{(1)}) \begin{pmatrix} Z^{(1)} \\ 1 \end{pmatrix}, \dots, (f_1^{(s)}, \dots, f_{2n}^{(s)}) \begin{pmatrix} Z^{(s)} \\ 1 \end{pmatrix} \right)$$

belongs to C^{ns} .

By (G 3), $Z = V^{-1}U = (V^{(s)-1}U^{(1)}, \dots, V^{(s)-1}U^{(s)})$ is symmetric and by (G 4), $\text{Im } Z$ is positive definite; hence Z belongs to $\mathcal{H}^s(n, s)$.

Then, the linear mapping

$$(x_1, \dots, x_{2n}) \begin{pmatrix} {}^tU \\ {}^tV \end{pmatrix} \longrightarrow (x_1, \dots, x_{2n}) \begin{pmatrix} {}^tU \\ {}^tV \end{pmatrix} {}^tV^{-1} = (x_1, \dots, x_{2n}) \begin{pmatrix} Z \\ 1 \end{pmatrix}$$

of C^{ns} onto itself gives an isomorphism A of C^{ns}/\mathcal{J} onto C^{ns}/\mathcal{J}' and for every $f \in \mathfrak{r}$,

$$A \cdot \iota(f) = \iota(f) \cdot A$$

This is easily seen by the forms of A and $\iota(f)$. Thus the discrete group \mathcal{J} is always reduced to the form

$$\mathcal{J} = \left\{ (f_1, \dots, f_{2n}) \begin{pmatrix} Z \\ 1 \end{pmatrix} : (f_1, \dots, f_{2n}) \in \mathfrak{M} \right\}$$

with $Z \in \mathcal{H}^s(n, s)$. Then, we denote $\mathfrak{P}(\mathfrak{M}, U, V, J) = \mathfrak{P}(\mathfrak{M}, Z)$.

Suppose that $r=0$, and consider $\mathfrak{P}(\mathfrak{M}, U, V, P)$. Then, \mathfrak{M} is an \mathfrak{o} -lattice in $\mathcal{V}(2n, k)$. Put

$$E(x, y) = xP^ty \quad \text{for } x, y \in \mathcal{V}(2n, k).$$

Then, E is \mathfrak{d}^{-1} -valued on \mathfrak{M} . (\mathfrak{d}^{-1} is the different of k over \mathbf{Q}). Hence by Corollary to Lemma 1, there exists an element T of $GL(2n, k)$ such that $\mathfrak{M} = [0, \dots, 0, \mathfrak{d}^{-1}a_1, \dots, \mathfrak{d}^{-1}a_n]T$, and $TP^tT = J$. Therefore, every $\mathfrak{P}(\mathfrak{M}, U, V, P)$ is isomorphic to $\mathfrak{P}([0, \dots, 0, \mathfrak{d}^{-1}a_1, \mathfrak{d}^{-1}a_n], Z)$ for some integral ideal a_1, \dots, a_n .

We denote

$$\mathfrak{P}(a_1, \dots, a_n, Z) = \mathfrak{P}([0, \dots, 0, \mathfrak{d}^{-1}a_1, \dots, \mathfrak{d}^{-1}a_n], Z).$$

8. Let (A, \mathcal{C}, ι) and $(A', \mathcal{C}', \iota')$ be two polarized abelian varieties of type r of the same dimension. As was stated in section 2, a homomorphism A of A onto A' is called a homomorphism of (A, \mathcal{C}, ι) onto $(A', \mathcal{C}', \iota')$ if there exists a divisor X' in \mathcal{C}' such that $A^{-1}(X')$ is contained in \mathcal{C} and $A \cdot \iota'(f) = \iota(f) \cdot A$ holds for any $f \in \mathfrak{r}$. Isomorphisms are used in the similar meaning. Let $(A, \mathcal{C}, \iota) = \mathfrak{P}(\mathfrak{M}, U, V, P)$ and $(A', \mathcal{C}', \iota') = \mathfrak{P}(\mathfrak{M}', U', V', P')$. Put

$$\mathcal{A} = \left\{ (f_1, \dots, f_{2n}) \begin{pmatrix} {}^tU \\ {}^tV \end{pmatrix} : (f_1, \dots, f_{2n}) \in \mathfrak{M} \right\}$$

and

$$\mathcal{A}' = \left\{ (f_1, \dots, f_{2n}) \begin{pmatrix} {}^t U' \\ {}^t V' \end{pmatrix}; (f_1, \dots, f_{2n}) \in \mathfrak{M}' \right\}.$$

If (A, \mathcal{E}, ι) and $(A', \mathcal{E}', \iota')$ are isomorphic, then analytic models C^{ns}/\mathcal{A} and C^{ns}/\mathcal{A}' of A and A' , respectively, satisfy the following conditions:

(i) There is a C -linear mapping A of C^{ns} onto itself such that

$$\mathcal{A} \cdot \mathcal{A}' = \mathcal{A}'.$$

(ii) For any f in k , $\mathcal{A} \cdot \iota'(f) = \iota(f) \cdot \mathcal{A}$.

(iii) There is a positive $r \in Q$ such that

$$\mathcal{E}'(xA, yA) = r \mathcal{E}(x, y)$$

where \mathcal{E} and \mathcal{E}' are Riemann forms, respectively, on C^{ns}/\mathcal{A} and C^{ns}/\mathcal{A}' with the principal matrices P and P' . We denote also the matrix corresponding to the mapping A by the same A . By (ii), A is of the form

$$A = \begin{pmatrix} A^{(1)} & & \\ & \ddots & \\ & & A^{(s)} \end{pmatrix}$$

where $A^{(\lambda)}$ is a complex (n, n) matrix for every λ . By definitions of \mathcal{A} and \mathcal{A}' , there is an element $G \in GL(2n, k)$ such that

$$\begin{pmatrix} {}^t U \\ {}^t V \end{pmatrix} \cdot A = G \begin{pmatrix} {}^t U' \\ {}^t V' \end{pmatrix}$$

and we have by (i),

$$\mathfrak{M}G = \mathfrak{M}'.$$

Now, we can find an element G_0 of $GL(2n, k)$ such that $G_0 P' G_0 = J$. Then, obviously, $\mathfrak{P}(\mathfrak{M}, U, V, P)$ is isomorphic to $\mathfrak{P}(\mathfrak{M}G_0^{-1}, U, V, J)$. Hence, as mentioned in section 7, V is non-singular. If we write $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then

$$(18) \quad {}^t U {}^t V^{-1} = (A {}^t U' + B {}^t V')(C {}^t U' + D {}^t V')^{-1}.$$

We have, writing $x = \left(x^{(1)} \begin{pmatrix} {}^t U^{(1)} \\ {}^t V^{(1)} \end{pmatrix}, \dots, x^{(s)} \begin{pmatrix} {}^t U^{(s)} \\ {}^t V^{(s)} \end{pmatrix} \right)$ and

$$y = \left(y^{(1)} \begin{pmatrix} {}^t U^{(1)} \\ {}^t V^{(1)} \end{pmatrix}, \dots, y^{(s)} \begin{pmatrix} {}^t U^{(s)} \\ {}^t V^{(s)} \end{pmatrix} \right) \quad \text{with} \quad x^{(\lambda)}, y^{(\lambda)} \in R^{2n},$$

$$r \mathcal{E}(x, y) = \mathcal{E}'(xA, yA) = \sum_{\lambda=1}^s x^{(\lambda)} G^{(\lambda)} P'^{(\lambda)} G^{(\lambda)} (y^{(\lambda)})$$

$$= r \sum_{\lambda=1}^s r^{(\lambda)} P^{(\lambda)} \eta^{(\lambda)},$$

by (iii). Hence, we have

$$(19) \quad GP'G = rP.$$

Conversely, if there is an element $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $GL(2n, k)$ such that (17), (18) and (19) for some positive rational r hold, then there exists an element A of $GL(n, C)$ such that

$$\begin{pmatrix} {}^tU \\ {}^tV \end{pmatrix} \cdot A = G \begin{pmatrix} {}^tU' \\ {}^tV' \end{pmatrix}$$

$$\mathcal{E}'(xA, yA) = r \mathcal{E}(x, y)$$

and

$$A \cdot A = A'.$$

THEOREM 3. (1°) $\mathfrak{P}(\mathfrak{M}, U, V, P)$ and $\mathfrak{P}(\mathfrak{M}', U', V', P)$ are isomorphic if and only if there is an element $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $GL(2n, k)$ such that

$$\mathfrak{M}G = \mathfrak{M}', \quad {}^tU'V^{-1} = (A{}^tU' + B{}^tV')(C{}^tU' + D{}^tV')^{-1},$$

and $GP'G = rP$ for some positive $r \in \mathbb{Q}$.

(2°) In particular, when $r=0$, $\mathfrak{P}(a_1, \dots, a_n, Z)$ and $\mathfrak{P}(a_1, \dots, a_n, Z')$ are isomorphic if and only if there exists an element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $I'(\mathfrak{d}^{-1}a_1, \dots, \mathfrak{d}^{-1}a_n)$ such that

$$Z = (AZ' + B)(CZ' + D)^{-1},$$

where $Z = V^{-1}U$ and $Z' = V'^{-1}U'$.

Proof of (2°). As $[0, \dots, 0, \mathfrak{d}^{-1}a_1, \dots, \mathfrak{d}^{-1}a_n]G = [0, \dots, 0, \mathfrak{d}^{-1}a_1, \dots, \mathfrak{d}^{-1}a_n]$, we see that $\det G$ is a unit of k , so that $r=1$ in (1), so putting $M = {}^tG$, with G in (1°), M belongs to $I'(\mathfrak{d}^{-1}a_1, \dots, \mathfrak{d}^{-1}a_n)$. Hence, we can easily obtain (2°) by (1°).

Therefore, isomorphic classes of $\mathfrak{P}(a_1, \dots, a_n, Z)$ and points of $I'(\mathfrak{d}^{-1}a_1, \dots, \mathfrak{d}^{-1}a_n) \setminus \mathcal{H}(n, s)$ are in one-to-one correspondence.

9. First, we shall consider an analytic model of the Picard variety of an abelian variety A of dimension m and an analytic representation of φ_X of A onto B , defined in section 5 with non-degenerate divisor X on A . c.f. [6]

Let C^m/A be an analytic model of A and θ an analytic isomorphism of A onto C^m/A . Let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ be two elements of C^m . Let $\langle x, y \rangle$ be a non-degenerate not necessarily symmetric R -valued R -bilinear form on C^m such that

$$(19') \quad \langle \sqrt{-1}x, y \rangle = \langle x, -\sqrt{-1}y \rangle$$

Hence, C^m is considered as the dual vector space over R of itself, with respect to $\langle x, y \rangle$.

Denote by \mathcal{A}^* the set of all vectors $y \in C^m$ such that $\langle x, y \rangle$ is contained in Z for every $x \in \mathcal{A}$. Then, \mathcal{A}^* is a discrete subgroup of C^m , so we obtain a complex torus C^m/\mathcal{A}^* of dimension m . Let T be the set of all characters of \mathcal{A} . We identify T with C^m/\mathcal{A}^* by the isomorphism given by

$$C^m \ni y \longrightarrow \exp [2\pi\sqrt{-1} \langle y, \cdot \rangle].$$

Let X be a divisor on A and ϑ a normalized theta-function on C^m/\mathcal{A} such that $(\vartheta) = \theta(X)$. Suppose that X is algebraically equivalent to 0. Then, we have $\mathcal{E}_\vartheta(X) = 0$. ϑ satisfies the following formula:

$$\vartheta(x+d) = \nu(d)\vartheta(x)$$

for $d \in \mathcal{A}$, with $\nu \in T$. We can see that ν is determined only by X . Let $\phi(X)$ denote the point of C^m/\mathcal{A}^* corresponding to ν . Then, we obtain an isomorphism θ^* of B onto C^m/\mathcal{A}^* by putting

$$\theta^*(\text{Cl}(X)) = \phi(X).$$

Thus, C^m/\mathcal{A}^* can be considered as an analytic model of B . X, ϑ being as above ϑ satisfies the formula (1) in section 2. Put

$$\Theta_u(x) = \vartheta(x)^{-1} \vartheta(x+u) \exp [2\pi\sqrt{-1} H(u, x)]$$

for every $u \in C^m$. t being an element of A such that $\theta(t) = u$, we see that

$$(19\#) \quad (\Theta_u) = \theta(X_t - X)$$

and

$$(19\#\#) \quad \Theta_u(x+d) = \Theta_u(x) \exp [2\pi\sqrt{-1} \mathcal{E}_\vartheta(X)(u, d)]$$

for $d \in \mathcal{A}$.

On the other hand, we have an R -linear mapping L of C^m into itself by putting

$$(19'') \quad \langle L(x), y \rangle = \mathcal{E}_\vartheta(X)(x, y).$$

By (R1), (R2), (R3) and (19'), L is C -linear and maps \mathcal{A} into \mathcal{A}^* , so L gives a homomorphism of C^m/\mathcal{A} into C^m/\mathcal{A}^* . Then, by (19\#) (19\#\#) and the definition of ν , we have

$$L(\theta(t)) = \theta^*(\text{Cl}(X_t - X)).$$

Thus, L is the analytic representation of φ_X with respect to θ and θ^* .

Let (A, \mathcal{E}, ι) be a polarized abelian variety of type \mathfrak{o} defined over a field K . Namely, $\iota(\mathfrak{o}) = \iota(k) \cap \bigcap \mathcal{V}(A)$ for $(A, \mathcal{E}, \iota) = \mathfrak{P}(\mathfrak{M}, Z)$. Then, \mathfrak{M} is an \mathfrak{o} -lattice.

THEOREM 4. *The family of polarized abelian varieties*

$$\{\mathfrak{P}(a_1, \dots, a_n, Z); \quad Z \in \mathcal{H}(n, s)\}$$

is complete with respect to \mathbf{Q} in the sense of Shimura; namely let $(A', \mathcal{E}', \iota')$ be a generic specialization of (A, \mathcal{E}, ι) over \mathbf{Q} ; then there exists an element Z' of $\mathcal{H}(n, s)$ such that

$$(A', \mathcal{E}', \iota') = \mathfrak{P}(a_1, \dots, a_n, Z').$$

PROOF. Let K be a field of definition for (A, \mathcal{E}, ι) . By definition, there exists an isomorphism σ of the universal domain over \mathbf{Q} such that

$$(A', \mathcal{E}', \iota') = (A^\sigma, \mathcal{E}^\sigma, \iota^\sigma).$$

We denote by the same ι' the extension of ι' to k . We see that $\iota'(0) \in \iota'(k) \cap \mathcal{N}(A')$ holds. Let X be a non-degenerate divisor of \mathcal{E} on A rational over K . Let B be the Picard variety of A and φ_X the homomorphism of A onto B defined by

$$\varphi_X(t) = \text{Cl}(X_t - X)$$

for $t \in A$. Then, we have

$$\iota(f) = \varphi_X^{-1} \cdot \iota'(f) \cdot \varphi_X$$

for $f \in k$. We see that B^σ is the Picard variety of A^σ and we have

$$\iota'(f) = (\varphi_X^\sigma)^{-1} \cdot \iota'(\iota'(f)) \cdot \varphi_X^\sigma,$$

and

$$\varphi_X^\sigma(u) = \text{Cl}(X_u^\sigma - X^\sigma)$$

for $u \in A'$. Hence, we see that the involution of $\mathcal{N}_0(A')$ determined by the polarization $\mathcal{E}' = \mathcal{E}^\sigma$ is the identity on $\iota'(k)$; Thus, we see that $(A^\sigma, \mathcal{E}^\sigma, \iota^\sigma)$ satisfies the conditions a), b) and c) (now, $r=0$).

We have

$$(20) \quad (\text{kernel of } \varphi_X)^\sigma = (\text{kernel of } \varphi_X^\sigma).$$

Let $\mathcal{E}(x, y)$ be the Riemann form determined by

$$\begin{aligned} & \mathcal{E}\left((f_1, \dots, f_{2n}) \begin{pmatrix} Z \\ 1 \end{pmatrix}, (g_1, \dots, g_{2n}) \begin{pmatrix} Z \\ 1 \end{pmatrix}\right) \\ &= \text{tr}_{k/\mathbf{Q}}\left((f_1, \dots, f_{2n}) J \begin{pmatrix} g_1 \\ \vdots \\ g_{2n} \end{pmatrix}\right) \end{aligned}$$

for $(f_1, \dots, f_{2n}), (g_1, \dots, g_{2n}) \in [0, \mathfrak{d}^{-1}a_1, 0, \mathfrak{d}^{-1}a_2, \dots, 0, \mathfrak{d}^{-1}a_n]$ where J is considered

as $\begin{pmatrix} 01 \\ -10 & \cdot \\ & \cdot & 01 \\ & & -10 \end{pmatrix}$. Let X be the divisor on A corresponding to $\mathcal{E}(x, y)$; and

$\mathcal{E}'(x, y)$ be the Riemann form determined by X^σ . Then, by the result of section 7, there exist n ideals b_1, \dots, b_n and a point Z' of $\mathcal{H}(n, s)$ such that

$$(A', \mathcal{E}', \iota') = \mathfrak{P}(b_1, \dots, b_n, Z'),$$

and

$$\begin{aligned} & \mathcal{E}'\left((f_1, \dots, f_{2n})\begin{pmatrix} Z \\ 1 \end{pmatrix}, (g_1, \dots, g_{2n})\begin{pmatrix} Z \\ 1 \end{pmatrix}\right) \\ &= \text{tr}_{k/Q}\left((f_1, \dots, f_{2n})J\begin{pmatrix} g_1 \\ \vdots \\ g_{2n} \end{pmatrix}\right) \end{aligned}$$

for $(f_1, \dots, f_{2n}), (g_1, \dots, g_{2n}) \in [0, \mathfrak{d}^{-1}b_1, 0, \mathfrak{d}^{-1}b_2, \dots, 0, \mathfrak{d}^{-1}b_n]$. Put

$$\mathfrak{M} = [0, \mathfrak{d}^{-1}a_1, 0, \mathfrak{d}^{-1}a_2, \dots, 0, \mathfrak{d}^{-1}a_n],$$

$$\mathfrak{M}' = [0, \mathfrak{d}^{-1}b_1, 0, \mathfrak{d}^{-1}b_2, \dots, 0, \mathfrak{d}^{-1}b_n],$$

$$\begin{aligned} \mathfrak{M} = \left\{ (f_1, \dots, f_{2n}) \in \mathcal{V}(2n, k); \quad (f_1, \dots, f_{2n})J\begin{pmatrix} g_1 \\ \vdots \\ g_{2n} \end{pmatrix} \in \mathfrak{d}^{-1} \text{ for any} \right. \\ \left. (g_1, \dots, g_{2n}) \in \mathfrak{M} \right\}, \end{aligned}$$

and

$$\begin{aligned} \mathfrak{M} = \left\{ (f_1, \dots, f_{2n}) \in \mathcal{V}(2n, k); \quad (f_1, \dots, f_{2n})J\begin{pmatrix} g_1 \\ \vdots \\ g_{2n} \end{pmatrix} \in \mathfrak{d}^{-1} \text{ for any} \right. \\ \left. (g_1, \dots, g_{2n}) \in \mathfrak{M}' \right\}. \end{aligned}$$

Since

$$\mathcal{E}(\sqrt{-1}x, y)$$

$$= -\mathcal{E}(y, \sqrt{-1}x) \quad (\text{skew-symmetry})$$

$$= -\mathcal{E}(x, \sqrt{-1}y) \quad (\text{symmetry of } \mathcal{E}(x, \sqrt{-1}y))$$

$$= \mathcal{E}(x, -\sqrt{-1}y),$$

we can take $\mathcal{E}(x, y)$ as the form $\langle x, y \rangle$ considered in the first part of this section. Then, we get the identity mapping as the analytic representation L

of φ_X , which is given in (19''). Therefore, we see easily that

$$(21) \quad (\text{kernel of } \varphi_X) \cong \tilde{\mathfrak{M}} \left(\begin{smallmatrix} Z \\ 1 \end{smallmatrix} \right) / \mathfrak{M} \left(\begin{smallmatrix} Z \\ 1 \end{smallmatrix} \right)$$

and

$$(22) \quad (\text{kernel of } \varphi_X^\sigma) \cong \tilde{\mathfrak{M}}' \left(\begin{smallmatrix} Z' \\ 1 \end{smallmatrix} \right) / \mathfrak{M}' \left(\begin{smallmatrix} Z' \\ 1 \end{smallmatrix} \right).$$

We shall determine explicitly $\tilde{\mathfrak{M}}$ and $\tilde{\mathfrak{M}}'$. For any element (g_1, \dots, g_{2n}) of \mathfrak{M} , we have

$$(f_1, \dots, f_{2n}) J \begin{pmatrix} g_1 \\ \vdots \\ g_{2n} \end{pmatrix} \in \mathfrak{d}^{-1},$$

for $(f_1, \dots, f_{2n}) \in \mathfrak{M}$; therefore, in particular for $(g_1, 0, \dots, 0)$ with $g_1 \in \mathfrak{o}$, we have

$$(f_1, \dots, f_{2n}) J \begin{pmatrix} g_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathfrak{d}^{-1};$$

so f_2 is contained in \mathfrak{d}^{-1} ; using $(0, g_2, 0, \dots, 0)$ with $g_2 \in \mathfrak{d}^{-1}a_1$, we see that f_1 is contained in a_1^{-1} .

Continuing this process, we see that (f_1, \dots, f_{2n}) is contained in $[a_1^{-1}, \mathfrak{d}^{-1}, \dots, a_n^{-1}, \mathfrak{d}^{-1}]$. Conversely, if (f_1, \dots, f_{2n}) is contained in $[a_1^{-1}, \mathfrak{d}^{-1}, \dots, a_n^{-1}, \mathfrak{d}^{-1}]$, then for any (g_1, \dots, g_{2n}) in \mathfrak{M} , we have

$$(f_1, \dots, f_{2n}) J \begin{pmatrix} g_1 \\ \vdots \\ g_{2n} \end{pmatrix} \in \mathfrak{d}^{-1}.$$

Hence,

$$(23) \quad \tilde{\mathfrak{M}} = [a_1^{-1}, \mathfrak{d}^{-1}, \dots, a_n^{-1}, \mathfrak{d}^{-1}].$$

In the same way,

$$(24) \quad \tilde{\mathfrak{M}}' = [b_1^{-1}, \mathfrak{d}^{-1}, \dots, b_n^{-1}, \mathfrak{d}^{-1}].$$

In view of (21) and (22), we see easily

$$\tilde{\mathfrak{M}}/\mathfrak{M} \cong \bigoplus_{i=1}^n (\mathfrak{o}/a_i \oplus \mathfrak{o}/a_i)$$

and

$$\tilde{\mathcal{M}}'/\mathcal{M}' \cong \bigoplus_{i=1}^n (\mathfrak{o}/\mathfrak{b}_i \oplus \mathfrak{o}/\mathfrak{b}_i).$$

Since

$$\tilde{\mathcal{M}}/\mathcal{M} \cong \tilde{\mathcal{M}}'/\mathcal{M}'$$

holds by (20), we have $\mathfrak{a}_i = \mathfrak{b}_i$ for every i , by Corollary to Lemma 3. Thus

$$(A', \mathcal{C}', \epsilon) = \mathfrak{P}(a_n, \dots, a_n, Z')$$

as desired.

10. Let $(A, \mathcal{C}, \epsilon) = \mathfrak{P}(a_1, \dots, a_n, Z)$ be a polarized abelian variety of type \mathfrak{o} constructed in the above sections. Put $\mathcal{M} := [\mathfrak{o}, \dots, \mathfrak{o}, \delta^{-1}a_1, \dots, \delta^{-1}a_n] = \sum_{i=1}^{2ns} Z \cdot \mathfrak{x}_i$ with $2ns$ vectors \mathfrak{x}_i in $\mathcal{V}(2n, k)$ and

$${}^t W(Z) = \begin{pmatrix} \mathfrak{x}_1 \cdot \begin{pmatrix} Z \\ 1 \end{pmatrix} \\ \vdots \\ \mathfrak{x}_{2ns} \cdot \begin{pmatrix} Z \\ 1 \end{pmatrix} \end{pmatrix}.$$

Then $W(Z)$ is a complex $(ns, 2ns)$ matrix. The group $\mathcal{A}(Z)$ generated by the rows of $W(Z)$ over Z is a discrete subgroup of C^{ns} , of rank $2ns$. Denote by P the matrix of degree $2ns$ whose (i, j) entry is $c \operatorname{tr}(\mathfrak{x}_i J' \mathfrak{x}_j)$, where c is a constant integer such that $c \operatorname{tr}(\mathfrak{x}_i J' \mathfrak{x}_j)$ are integers for all i, j . Then P defines a Riemann form on $C^{ns}/\mathcal{A}(Z)$ for every $Z \in \mathcal{H}(n, s)$. There exists a unimodular matrix G of degree $2ns$ such that

$$G P {}^t G = \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix}$$

where

$$\delta = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_{ns} \end{pmatrix},$$

the δ_i are integers and $\delta_1 = 1$, $\delta_i | \delta_{i+1}$, $1 \leq i \leq ns-1$. Put

$$W(Z) {}^t G = (U(Z) \ V(Z))$$

with complex matrices $U(Z)$, $V(Z)$ of degree ns . Then, by (G3) (G4) for $W(Z)$ and P , we have

$$V(Z) \delta^{-1} {}^t U(Z) = U(Z) \delta^{-1} {}^t V(Z)$$

and

$$\sqrt{-1} [V(Z) \delta^{-1} {}^t U(Z) - U(Z) \delta^{-1} {}^t V(Z)]$$

is a positive definite hermitian matrix; from this, $V(Z)$ has the inverse; put

$$S(Z) = \delta V(Z)^{-1} U(Z) \quad \text{for} \quad Z \in \mathcal{H}^p(n, s).$$

Then, the linear mapping $x \rightarrow x \cdot {}^t V(Z)^{-1}$ of C^{ns} onto itself gives an isomorphism of $C^{ns}/J(Z)$ onto $C^{ns}/J_\delta(S(Z))$ for each $Z \in \mathcal{H}^p(n, s)$. Put $A(Z) = A_\delta(S(Z))$ and $\theta(x, Z) = \theta_\delta(x \cdot {}^t V(Z)^{-1}, S(Z))$ for $x \in C^{ns}$, $Z \in \mathcal{H}^p(n, s)$. $\theta(x, Z)$ is an analytic mapping of $C^{ns} \times \mathcal{H}^p(n, s)$ into \mathcal{P}^N ; the mapping $x \rightarrow \theta(x, Z)$ gives an analytic isomorphism of $C^{ns}/J(Z)$ onto $A(Z)$; the polarization \mathcal{C}_δ of $A(Z)$ determined by the hyperplane sections corresponds to the Riemann form on $C^{ns}/J(Z)$ given by P . Since $\iota(f)$ is a faithful representation of k and the linear mapping $x \rightarrow x \cdot \iota(f)$ of C^{ns} into itself gives an endomorphism of $C^{ns}/J(Z)$, there is an endomorphism $\iota_Z(f)$ of $A(Z)$ such that

$$\theta(x, Z) \cdot \iota_Z(f) = \theta(x \cdot \iota(f), Z)$$

and $f \rightarrow \iota_Z(f)$ defines an isomorphism of \mathfrak{o} into $\mathcal{N}(A(Z))$ for each $Z \in \mathcal{H}^p(n, s)$. Thus, we have obtained a system of polarized abelian varieties $(A(Z), \mathcal{C}_\delta, \iota_Z)$ of type \mathfrak{o} , which satisfies the conditions (SS3), (SS4) and (SS5). Then, we can construct the variety $\mathcal{F}(A(Z), \iota_Z)$ with respect to a basis of \mathfrak{o} , for each $Z \in \mathcal{H}^p(n, s)$. Put $\mathcal{F}(Z) = \mathcal{F}(A(Z), \iota_Z)$. Then, by (SS2) we have the following

(SS2') for two elements $Z, Z' \in \mathcal{H}^p(n, s)$, $(A(Z), \mathcal{C}_\delta, \iota_Z)$ is isomorphic to $(A(Z'), \mathcal{C}_{\delta'}, \iota_{Z'})$ if and only if $\mathcal{F}(Z) = \mathcal{F}(Z')$.

We restate Lemma 8 and Theorem 1 in Shimura [5].

(SS7) The variety $\mathcal{F}(Z)$ is of the same dimension for every $Z \in \mathcal{H}^p(n, s)$.

(SS8) There exists an analytic subset \mathfrak{G} of $\mathcal{H}^p(n, s)$ of codimension 1 and a set of meromorphic functions $\{f_1, \dots, f_d\}$ on $\mathcal{H}^p(n, s)$ such that

1° the varieties $\mathcal{F}(Z)$ for $Z \in \mathcal{H}^p(n, s) - \mathfrak{G}$ are of the same dimension and the same degree;

2° the f_i are holomorphic on $\mathcal{H}^p(n, s) - \mathfrak{G}$ and $c(\mathcal{F}(Z)) = (1, f_1(Z), \dots, f_d(Z))$ for every $Z \in \mathcal{H}^p(n, s) - \mathfrak{G}$.

(SS8') There exist an analytic subset \mathfrak{G} of $\mathcal{H}^p(n, s)$ of codimension 1 and a set of meromorphic functions $\{f_1, \dots, f_d\}$ on $\mathcal{H}^p(n, s)$ such that $Q(f_1(Z), \dots, f_d(Z))$ is the field of moduli of $(A(Z), \mathcal{C}_\delta, \iota_Z)$ for every $Z \in \mathcal{H}^p(n, s) - \mathfrak{G}$.

Since our system $\{(A(Z), \mathcal{C}_\delta, \iota_Z); Z \in \mathcal{H}^p(n, s)\}$ is complete with respect to Q , we obtain the following theorem by Theorem 2 in Shimura [5];

THEOREM 5. Let (f_1, \dots, f_d) be the set of meromorphic functions in (SS8). Then, the field $Q(f_1, \dots, f_d)$ is a regular extension of Q and

$$\dim_Q Q(f_1, \dots, f_d) = \dim_C C(f_1, \dots, f_d).$$

From Theorem 3 and (SS2) we have the following

PROPOSITION 2. For $Z, Z' \in \mathcal{H}^p(n, s)$, $\mathcal{F}(Z) = \mathcal{F}(Z')$ holds if and only if there exists an element M of $I(d^{-1}a_1, \dots, d^{-1}a_n)$ such that $Z' = M[Z]$.

PROPOSITION 3. *The functions f_i in (SS8) are invariant with respect to the group $\Gamma(\delta^{-1}a_1, \dots, \delta^{-1}a_n)$.*

PROOF. \mathfrak{G} being as in (SS8), we have

$$c(\mathcal{F}(Z)) = (1, f_1(Z), \dots, f_d(Z)) \text{ for every } Z \in \mathcal{H}(n, s) - \mathfrak{G}.$$

Let M be an element of $\Gamma(\delta^{-1}a_1, \dots, \delta^{-1}a_n)$. Take a generic point Z_0 of $\mathcal{H}(n, s)$ over \mathbb{Q} such that, for every $Z_1 \in \mathcal{H}(n, s)$, $(f_1(Z_1), \dots, f_d(Z_1))$ and $(f_1(M[Z_1]), \dots, f_d(M[Z_1]))$ is a specialization of $(f_1(Z_0), \dots, f_d(Z_0))$ and $(f_1(M[Z_0]), \dots, f_d(M[Z_0]))$, over \mathbb{Q} respectively. (Such an element Z_0 is called a generic point of $\mathcal{H}(n, s)$ for $f_i(Z)$ and $f_i(M[Z])$ over \mathbb{Q} , respectively). We can assume that Z_0 does not belong to $\mathfrak{G} \cup M^{-1}[\mathfrak{G}]$. Then, we have

$$c(\mathcal{F}(Z_0)) = (1, f_1(Z_0), \dots, f_d(Z_0))$$

and

$$c(\mathcal{F}(M[Z_0])) = (1, f_1(M[Z_0]), \dots, f_d(M[Z_0])).$$

By Proposition 2, we have

$$c(\mathcal{F}(Z_0)) = c(\mathcal{F}(M[Z_0]))$$

and

$$f_i(Z_0) = f_i(M[Z_0]).$$

Therefore, by the choice of Z_0 , we have

$$f_i(Z) = f_i(M[Z])$$

for every i , for $Z \in \mathcal{H}(n, s)$. q.e.d.

We denote the field $\mathbb{Q}(f_1, \dots, f_{2n})$ by $\mathbb{R}(a_1, \dots, a_n)$, where $(A, \mathcal{E}, \iota) = \mathbb{P}(a_1, \dots, a_n, Z)$.

Pyateczki-Shapiro first treated the functions on $\mathcal{H}(n, s)$ invariant with respect to the group commensurable with the Hilbert-Siegel modular group (para-modular functions on $\mathcal{H}(n, s)$) and proved that all the para-modular functions on $\mathcal{H}(n, s)$ form the algebraic function field of dimension $(1/2)ns(ns+1)$ over \mathbb{C} . Baily proved, using the generalization of the compactification of Satake to the case of $\Gamma(\delta^{-1}a_1, \dots, \delta^{-1}a_n)/\mathcal{H}(n, s)$, that every meromorphic function on $\mathcal{H}(n, s)$ invariant by $\Gamma(\delta^{-1}a_1, \dots, \delta^{-1}a_n)$ can be represented as quotient of two automorphic forms with Fourier developments (except the case of elliptic modular functions) and also proved that

(B) there exist automorphic forms $h_0(Z), \dots, h_M(Z)$ on $\mathcal{H}(n, s)$ with respect to $\Gamma(\delta^{-1}a_1, \dots, \delta^{-1}a_n)$ such that the mapping

$$Z \mapsto h(Z) = (h_0(Z), \dots, h_M(Z))$$

induces an isomorphism of the quotient space $\Gamma(\delta^{-1}a_1, \dots, \delta^{-1}a_n) \backslash \mathcal{H}(n, s)$ onto a Zariski open subset V_0 of an algebraic variety V in a projective space \mathcal{P}^M .

Suppose that $n > 1$ or $s > 1$. Denote by $\mathfrak{H}(a_1, \dots, a_n)$ the field of meromorphic functions on $\mathcal{H}(n, s)$ invariant by $I'(\mathfrak{d}^{-1}a_1, \dots, \mathfrak{d}^{-1}a_n)$. By the above mentioned result of Pyatezki-Shapiro, $\mathfrak{H}(a_1, \dots, a_n)$ is the function field of dimension $(1/2)ns(ns+1)$ over \mathbf{C} and we have $\mathfrak{H}(a_1, \dots, a_n) \supset \mathfrak{K}(a_1, \dots, a_n)$.

THEOREM 6. $\mathfrak{H}(a_1, \dots, a_n) \supset \mathbf{C} \cdot \mathfrak{K}(a_1, \dots, a_n)$ and $\mathfrak{K}(a_1, \dots, a_n)$ is regular over \mathbf{Q} .

PROOF. $(A(Z), \mathcal{C}_Z, \iota_Z) = \mathfrak{P}(a_1, \dots, a_n, Z)$ being a polarized abelian variety of type \mathfrak{o} , let f_i be meromorphic functions in (SS8). Then, there exist polynomials P_0, P_1, \dots, P_d with complex coefficients and the same degree such that

$$f_i(Z) = P_i(h_0(Z), \dots, h_M(Z)) / P_0(h_0(Z), \dots, h_M(Z))$$

where the h_i are defined in (B).

V, V_0 being as in (B), let K be a countable subfield of \mathbf{C} which contains coefficients of $P_i, i=0, 1, \dots, d$, and over which V and $V-V_0$ are rational. We shall show that the field $K(f_1, \dots, f_d)$ contains the functions h_j/h_j' .

Suppose that the contrary be the case. Then there exists h_p/h_q which is not contained in $K(f_1, \dots, f_d)$. \mathfrak{G} being as in (SS8), let Z_0 be a generic point of $\mathcal{H}(n, s) - \mathfrak{G}$ for the h_i over K ; then $K(f_i(Z_0))$ does not contain $h_p(Z_0)/h_q(Z_0)$; hence there is an isomorphism σ of $K(h_j(Z_0))$ into \mathbf{C} such that

$$f_i(Z_0)^\sigma = f_i(Z_0) \quad \text{for every } i$$

and

$$(24) \quad h_p(Z_0)^\sigma / h_q(Z_0)^\sigma \neq h_p(Z_0) / h_q(Z_0).$$

Since V and $V-V_0$ are rational over K , there is an element Z' of $\mathcal{H}(n, s)$ such that

$$h_j(Z') = h_j(Z_0)^\sigma \quad (0 \leq j \leq M).$$

Then, We have

$$f_i(Z') = f_i(Z_0)^\sigma = f_i(Z_0) \quad \text{for every } i.$$

Let $T(Z)$ be the graph of the law of composition of $A(Z)$ and $V(f, Z)$ the graph of $\iota_Z(f)$ for each $f \in \mathfrak{o}$, $Z \in \mathcal{H}(n, s)$. $\{\omega_i\}$ being a basis of \mathfrak{o} , let $\Psi(Z)$ be the gathering of the mapping $c(A(Z)), c(0, Z), c(T(Z))$ and $c(V(\omega_i, Z))$ for $i=1, \dots, s$. Then, we know that the variety $\mathcal{F}(Z)$ is defined over $\mathbf{Q}(\Psi(Z))$ for every $Z \in \mathcal{H}(n, s)$. We recall the determination of \mathfrak{G} in (SS8). By a place \mathfrak{p} of a field K' , we mean as usual a homomorphism of K' into a field, which maps the identity on the identity, where we admit that \mathfrak{p} takes the value ∞ . Take and fix a generic point Z^* of $\mathcal{H}(n, s)$ for Ψ ; by Lemma 6 of Appendix of [5], there exist a finite number of elements α_i in $\mathbf{Q}(\Psi(Z^*))$ such that, for every place \mathfrak{q} of $\mathbf{Q}(\Psi(Z^*))$, the cycle $\mathfrak{q}(\mathcal{F}(Z^*))$ is a variety if $\mathfrak{q}(\alpha_i) \neq 0$ for every i . Put $\alpha_i = a_i(Z^*)$ with the functions a_i in $\mathbf{Q}(\Psi)$. Let \mathfrak{G}' be the set of points Y where the a_i are holomorphic and $a_i(Y) \neq 0$ for every i . Let Y be a point of $\mathcal{H}(n, s)$ and \mathfrak{a} the set of all the functions $F(Z)$ in $\mathbf{Q}(\Psi)$ which are holomorphic

at Y ; we extend the homomorphism $F \rightarrow F(Y)$ of \mathfrak{a} into C to a place \mathfrak{p}_0 of $Q(\Psi)$ into C , and denote by \mathfrak{p} the place of $Q(\Psi(Z^*))$ corresponding to \mathfrak{p}_0 by the canonical isomorphism of $Q(\Psi)$ onto $Q(\Psi(Z^*))$. Then, we have $\mathfrak{p}(\Psi(Z^*)) = \Psi(Y)$ and if Y is contained in \mathfrak{G}' , we have $a_i(Y) \neq 0$ for every i and so $\mathfrak{p}(\alpha_i) \neq 0$ for every i , so that $\mathfrak{p}(\mathcal{F}(Z^*))$ is a variety. Let ξ be the generic projective transformation $(x_j) \rightarrow (\sum_j t_{ij} x_j)$ of \mathcal{P}^N defined by a generic matrix (t_{ij}) over $Q(\Psi(Z^*))$ and ξ' the generic projective transformation $(x_j) \rightarrow (\sum_j t'_{ij} x_j)$ of \mathcal{P}^N defined by generic matrix (t'_{ij}) over $\mathfrak{p}(Q(\Psi(Z^*)))$. Then there exists an extension \mathfrak{p}_1 of \mathfrak{p} on $Q(\Psi(Z^*), t_{ij})$ such that $\mathfrak{p}_1(t_{ij}) = t'_{ij}$. Let u_1, \dots, u_h (u'_1, \dots, u'_h) be h independent generic points of $A(Z^*)$ (resp., $A(Y)$) over $Q(\Psi(Z^*), t_{ij})$ (resp., $\mathfrak{p}(Q(\Psi(Z^*)), t_{ij})$). Then there exists an extension \mathfrak{P} of \mathfrak{p}_1 such that $\mathfrak{P}(u_i) = u'_i$ for every i . Put $B = \xi(A(Z^*))$ and $B' = \xi'(A(Y))$; then, $\mathfrak{p}_1(B) = B'$. Let λ_i (resp., λ'_i) be the rational mapping of B (resp., B') into itself defined by

$$\begin{aligned} \lambda_i(x) &= \xi[\iota_{Z^*}(\omega_i)\xi^{-1}(x) + u_i] \\ (\text{resp., } \lambda'_i(x') &= \xi'[\iota_Y(\omega_i)\xi'^{-1}(x') + u'_i]). \end{aligned}$$

Then we see that if (x', x'_1) is a specialization of $(x, \lambda_i(x))$ over \mathfrak{P} , we have $x'_1 = \lambda'_i(x')$. Let W_i (resp. W'_i) be the graph of λ_i (resp. λ'_i). Then $\mathfrak{P}(W_i) = W'_i$ and

$$\mathfrak{P}[c(B) \times c(W_1) \times \dots \times c(W_h)] = c(B') \times c(W'_1) \times \dots \times c(W'_h).$$

From this formula and by (SS7), we have $\mathfrak{p}(\mathcal{F}(Z^*)) = \mathcal{F}(Y)$. Put

$$c(\mathcal{F}(Z^*)) = (1, \beta_1, \dots, \beta_d)$$

and $f_i(Z^*) = \beta_i$ for each i with the functions f_i in $Q(\Psi)$. Let \mathfrak{G}'' be the set of all points of \mathfrak{G}' where the f_i are holomorphic.

For $Y \in \mathfrak{G}''$, we have $\mathfrak{p}(\beta_i) = f_i(Y)$ and

$$c(\mathcal{F}(Y)) = \mathfrak{p}(c(\mathcal{F}(Z^*))) = (1, f_1(Y), \dots, f_d(Y)).$$

Now, we put $\mathfrak{G} = \mathcal{H}(n, s) = \mathfrak{G}''$.

Coming back to our case, let Z' and Z_0 be as above. By (SS6) in section 2, there exists a set of holomorphic functions (ϕ_0, \dots, ϕ_k) such that there exists a neighborhood Π of Z' for whose element $Z, (\phi_0(Z), \dots, \phi_k(Z))$ is the Chow point of $A(Z)$. Let Z_1 be a generic point of $(\mathcal{H}(n, s) = \mathfrak{G}) \cap \Pi$ for the ϕ_i and the f_i over Q . We take Z_1 as Z^* and Z' as Y , where Z and Y are mentioned in the above recalling. Now, as the place \mathfrak{p} , we take the place of $Q(\Psi(Z_1))$ corresponding to the place of $Q(\Psi)$ which is the extension of the mapping $F \rightarrow F(Z')$.

Let (A', \mathcal{F}') be a specialization of $(A(Z_1), \mathcal{F}(Z_1))$ over the specialization $(\phi_i(Z_1), f_i(Z_1)) \rightarrow (\phi_i(Z'), f_i(Z'))$ with respect to \mathfrak{p} . Then, $A' = A(Z')$ and $\mathcal{F}' \supset \mathcal{F}(Z')$. On the other hand, we have

$$c(\mathcal{F}') = (1, f_i(Z')) = (1, f_i(Z_0)) = c(\mathcal{F}(Z_0));$$

hence $\mathcal{F}' = \mathcal{F}(Z_0)$. In particular, \mathcal{F}' is irreducible. Since, by (SS7), \mathcal{F}'

and $\mathcal{F}(Z')$ have the same dimension, we have $\mathcal{F}' = \mathcal{F}(Z')$. Thus we obtain $\mathcal{F}(Z') = \mathcal{F}(Z_0)$. By Proposition 2, there exists an element M of $I'(\mathfrak{d}^{-1}a_1, \dots, \mathfrak{d}^{-1}a_n)$ such that $Z' = M[Z_0]$. But this contradicts (25). Hence $K(f_1, \dots, f_d)$ contains the functions h_j'/h_j' . This completes our proof of the first part. The second part easily follows from Theorem 5.

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