# *The existence and the uniqueness of regular solution of non-stationary Navier-Stokes equation*

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(Received May 29, 1961)

**Introduction.** The mathematical investigation concerning the existence and the uniqueness of the solution of the *non-stationary equation for the motion of incompressible viscous fluid* (*Navier-Stokes equation*) was originated by J. Leray, and several remarkable results have been obtained by many authors—especially by J. Leray [10], [11], [12], E. Hopf [5], A. A. Kiselev, O. A. Ladyzhenskaia [8], [9] and J.-L. Lions [13],[14]. E. Hopf [5] proved the existence of a weak solution of the equation for any $n$-dimensional domain and infinite time interval, while A. A. Kiselev and O. A. Ladyzhenskaia [8] proved the existence and the uniqueness of the solution for 3-dimensional bounded domain and a finite time-interval, and extended the result to the case of infinite time-interval under the assumption that the exterior force has a potential and the initial Reynolds number is small. J.-L. Lions [13], [14] obtained more general results on the existence of the solution by an entirely different approach. O. A. Ladyzhenskaia [9] proved also the existence and the uniqueness of the solution for any 2-dimensional domain and infinite time interval. In any of these papers, the differentiability (in usual sense) of the solution is not proved, and the solution takes the boundary value zero in certain generalized sense.

Recently P. E. Sobolevskiĭ [21] has reported the outline of his construction of a classical solution of the non-stationary Navier-Stokes equation for a bounded domain, taking the boundary value zero, in a finite time-interval. The most essential part of his method is based on properties of fractional powers of differential operators established in [18], [19], [20] and on properties of *Green tensor* constructed by F. K. G. Odqvist [15]. But, as far as we know, detailed argument for Sobolevskiĭ's method in [21] has not yet been published.

In the present paper, we shall prove the existence and the uniqueness of classical solution of the non-stationary Navier-Stokes equation for any bounded 3-dimensional domain with a sufficiently smooth boundary, in a suitable finite time-interval; the solution has as many continuous derivatives as enter in the equation and takes given initial value and boundary value in usual sense, where the boundary value is any function satisfying a certain smoothness (THEOREMS 1 and 2). Our method is also based on some properties of the Green tensor given in [15], but entirely different from Sobolevskiĭ's. We shall also show under the assumption of Theorem 1 that our solution coincides with Kiselev-Ladyzhenskaia's if the boundary value is identically zero.

The author wishes to express his hearty thanks to Professor K. Yosida and Professor T. Kato who have encouraged him with kind discussions, and to Dr. H. Fujita whose suggestion on the result of F.K.G. Odqvist [15] has meant so much to him throughout the present paper (especially the proof of Lemma 3.2 is due to Dr. H. Fujita; the author's original proof was incomplete).

§1. **Fundamental notions and main results.** Let $D$ be a bounded domain in the Euclidean 3-space $\boldsymbol{R}^3$ whose boundary $S=\bar{D}-D$[1] consists of a finite number of components each of which is a simple closed surface of class $C^4$. For any two points $x$ and $y$ in $\boldsymbol{R}^3$, $r_{xy}$ denotes the distance between $x$ and $y$; for any point $x$ on the boundary $S$ of the domain $D$, $\boldsymbol{n}=\boldsymbol{n}(x)$ denotes the unit vector with outer normal direction at $x$. Throughout this paper, we consider spaces of *real valued* or *real-vector valued* functions defined in subdomains of $\boldsymbol{R}^3$. For any $\gamma\geqq 0$, we shall use notations $C_0^\gamma(D)$ and $C^\gamma(D)$ as usual, and denote by $C^\gamma(\bar{D})$ the union of $C^\gamma(D')$'s for all domains $D'$ containing $\bar{D}$. We shall denote by $\boldsymbol{C}^\gamma(D)$ the totality of $\boldsymbol{R}^3$-valued functions $\boldsymbol{u}(x)=\langle u^1(x), u^2(x), u^3(x)\rangle$ each component $u^j(x)$ of which belongs to $C^\gamma(D)$. $\boldsymbol{C}_0^\gamma(D)$ and $\boldsymbol{C}^\gamma(\bar{D})$ shall be defined analogously.

For any $\boldsymbol{R}^3$-valued functions $\boldsymbol{u}(x)=\langle u^1(x), u^2(x), u^3(x)\rangle$ and $\boldsymbol{v}(x)=\langle v^1(x), v^2(x), v^3(x)\rangle$, we shall use the following notations (whenever the right-hand side of each formula makes sense);—

$$\nabla\boldsymbol{u}(x)=\left\langle \frac{\partial u^j(x)}{\partial x^k}\ ;\ j,k=1,2,3\right\rangle, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u}(x)=\sum_{j=1}^{3}\frac{\partial u^j(x)}{\partial x^j}, \tag{1.2}$$

$$\Delta\boldsymbol{u}(x)=\left\langle \sum_{k=1}^{3}\frac{\partial^2 u^j(x)}{(\partial x^k)^2}\ ;\ j=1,2,3\right\rangle, \tag{1.3}$$

$$(\boldsymbol{u}\cdot\boldsymbol{v})(x)=\sum_{j=1}^{3}u^j(x)v^j(x), \tag{1.4}$$

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}(x)=\left\langle \sum_{k=1}^{3}u^k(x)\frac{\partial v^j(x)}{\partial x^k};\ j=1,2,3\right\rangle, \tag{1.5}$$

$$(\nabla\boldsymbol{u}:\nabla\boldsymbol{v})(x)=\sum_{j,k=1}^{3}\frac{\partial u^j(x)}{\partial x^k}\cdot\frac{\partial v^k(x)}{\partial x^j}, \tag{1.6}$$

1) $\bar{D}$ denotes the closure of $D$ as a subset of $\boldsymbol{R}^3$.

$$(1.7)\qquad (\boldsymbol{u}, \boldsymbol{v})=\sum_{j=1}^{3}\int_{D} u^j(x)v^j(x)dx,$$

$$(1.8)\qquad \|\boldsymbol{u}\|=(\boldsymbol{u}, \boldsymbol{u})^{1/2}=\left\{\sum_{j=1}^{3}\int_{D}|u^j(x)|^2dx\right\}^{1/2}$$

and

$$(1.9)\qquad \begin{cases} |\boldsymbol{u}(x)|=\max\limits_{1\leq j\leq 3}|u^j(x)|, \\ \|\!|\boldsymbol{u}|\!\|=\sup\limits_{x\in D}|\boldsymbol{u}(x)|. \end{cases}$$

Even if $\boldsymbol{u}$ is a function of both the time $t$ and the point $x\in D$, we operate $\nabla$, $div$ and $\Delta$ to $\boldsymbol{u}$ as a function of $x$ for each fixed $t$. In the case where $\boldsymbol{u}(x)$ and $\boldsymbol{v}(x)$ are $\boldsymbol{R}^m$-valued functions, for any given positive integer $m$, the notations $(\boldsymbol{u}, \boldsymbol{v})$, $\|\boldsymbol{u}\|$ and $\|\!|\boldsymbol{u}|\!\|$ shall be defined analogously. For example, if $\boldsymbol{u}(x)$ and $\boldsymbol{v}(x)$ are $\boldsymbol{R}^3$-valued functions, then we have

$$(1.10)\qquad (\nabla\boldsymbol{u}, \nabla\boldsymbol{v})=\sum_{j,k=1}^{3}\int_{D}\frac{\partial u^j(x)}{\partial x^k}\cdot\frac{\partial v^j(x)}{\partial x^k}\,dx$$

whenever the right-hand side makes sense.

We denote by $\mathfrak{H}$ the totality of $\boldsymbol{R}^3$-valued measurable functions $\boldsymbol{u}(x)$ on $D$ satisfying $\|\boldsymbol{u}\|<\infty$ (see (1.8)). Then $\mathfrak{H}$ is a real Hilbert space where the inner product is defined by (1.7).

We extend the definition of $\nabla$ and $div$. Let $\mathfrak{D}_\nabla$ be the totality of $\boldsymbol{v}\in\mathfrak{H}$ such that there exists a sequence $\{\boldsymbol{v}_n\}\subset\mathfrak{H}\cap \boldsymbol{C}^1(D)$ satisfying

$$(1.11)\qquad \|\nabla\boldsymbol{v}_n\|<\infty\ (n=1, 2, \cdots),\quad \lim_{n\to\infty}\|\boldsymbol{v}_n-\boldsymbol{v}\|=0$$

and

$$(1.12)\qquad \lim_{n\to\infty}\|\nabla\boldsymbol{v}_n-\hat{\boldsymbol{v}}\|=0$$

for a certain $\boldsymbol{R}^9$-valued function $\hat{\boldsymbol{v}}$ defined almost everywhere in $D$ with finite norm $\|\hat{\boldsymbol{v}}\|$. Then we may easily show that $\hat{\boldsymbol{v}}$ is uniquely determined by $\boldsymbol{v}\in\mathfrak{D}_\nabla$ and independent of the choice of sequence $\{\boldsymbol{v}_n\}$. Therefore we may denote such $\hat{\boldsymbol{v}}$ by $\nabla\boldsymbol{v}$; thus $\nabla$ is extended to any $\boldsymbol{v}\in\mathfrak{D}_\nabla$. Accordingly we may define $\mathrm{div}\,\boldsymbol{v}$ for any $\boldsymbol{v}\in\mathfrak{D}_\nabla$.

For any vector function $\boldsymbol{u}=\boldsymbol{u}(t, x)$ $(t\geq 0, x\in D)$, we define the following notations: $(\boldsymbol{u}(t)\equiv\boldsymbol{u}(t, \cdot))$

$$\begin{cases} \|\boldsymbol{u}\|_t=\sup\limits_{t\geq\tau\geq 0}\|\boldsymbol{u}(\tau)\|, \\ \|\boldsymbol{u}\|_t'=\sup\limits_{t>t_1>t_2>0}\|\boldsymbol{u}(t_1)-\boldsymbol{u}(t_2)\|\cdot|t_1-t_2|^{-1/2}t_2^{1/2}, \\ \|\!|\boldsymbol{u}|\!\|_t=\sup\limits_{t\geq\tau\geq 0}\|\!|\boldsymbol{u}(\tau)|\!\|, \\ \|\!|\boldsymbol{u}|\!\|_t'=\sup\limits_{t>t_1>t_2>0}\|\!|\boldsymbol{u}(t_1)-\boldsymbol{u}(t_2)|\!\|\cdot|t_1-t_2|^{-1/2}t_2^{1/2}. \end{cases}$$

For any vector function $\boldsymbol{b}=\boldsymbol{b}(t, x)$ $(t\geqq 0, x\in S)$ and any relatively open subset $\Omega$ of $S$, we define $\|\boldsymbol{b}\|_{t,\Omega}$ and $\|\boldsymbol{b}\|'_{t,\Omega}$ as follows:

$$\begin{cases} \|\boldsymbol{b}\|_{t,\Omega}=\sup\limits_{t\geqq\tau\geqq 0,\, x\in\Omega}|\boldsymbol{b}(t, x)|, \\ \|\boldsymbol{b}\|'_{t,\Omega}=\sup\limits_{t>t_1>t_2>0,\, x\in\Omega}|\boldsymbol{b}(t_1, x)-\boldsymbol{b}(t_2, x)|\cdot|t_1-t_2|^{-1/2}t_2^{1/2}. \end{cases}$$

It is well known that

$$\int_D \operatorname{div}\boldsymbol{u}dx=\int_S(\boldsymbol{u}\cdot\boldsymbol{n})dS \tag{1.13}$$

for any $\boldsymbol{u}\in \boldsymbol{C}^1(D)\cap \boldsymbol{C}^0(\bar{D})$ such that $\operatorname{div}\boldsymbol{u}\in L^1(D)$. Hence we have the following

LEMMA 1.1. *If* $\boldsymbol{u}\in \boldsymbol{C}^1(D)\cap \boldsymbol{C}^0(\bar{D})$ *and* $\operatorname{div}\boldsymbol{u}=0$ *in* $D$, then

$$\int_S(\boldsymbol{u}\cdot\boldsymbol{n})dS=0.$$

Conversely,

LEMMA 1.2. *If* $\boldsymbol{b}$ *is an* $R^3$*-valued function of class* $C^1$ *on* $S$ *and satisfies* $\int_S(\boldsymbol{b}\cdot\boldsymbol{n})dS=0$, *then* $\boldsymbol{b}$ *is the boundary value of a vector function* $\boldsymbol{u}\in \boldsymbol{C}^3(D)\cap \boldsymbol{C}^0(\bar{D})$ *satisfying* $\operatorname{div}\boldsymbol{u}=0$ *in* $D$.

Proof of this lemma is essentially contained in the proof of Lemma 2.9 stated later; so is omitted.

LEMMA 1.3. *Assume that* $\boldsymbol{u}\in\mathfrak{H}$ *and* $\boldsymbol{v}\in \boldsymbol{C}^0(D)$, *and that*

$$(\boldsymbol{u}, \Delta\Psi)=(\boldsymbol{v}, \Psi) \qquad \textit{for any} \qquad \Psi\in \boldsymbol{C}_0^\infty(D).$$

*Then* i) $\boldsymbol{u}\in \boldsymbol{C}^1(D)$; ii) *if especially* $\boldsymbol{u}\in \boldsymbol{C}^0(\bar{D})$, $\boldsymbol{u}|_S\in \boldsymbol{C}^2(S)$ *and* $\boldsymbol{v}\in\mathfrak{H}$, *then* $\|\nabla\boldsymbol{u}\|<\infty$.

This fact may be proved by means of properties of the Green function of the first boundary value problem (for instance, see [6; §10]).

Now we consider the non-stationary Navier-Stokes equation:

$$\frac{\partial\boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=\nu\cdot\Delta\boldsymbol{u}-\nabla p+\boldsymbol{f} \qquad (t>0,\ x\in D) \tag{1.14}$$

with the equation of continuity:

$$\operatorname{div}\boldsymbol{u}=0, \tag{1.15}$$

initial condition:

$$\boldsymbol{u}|_{t=0}=\boldsymbol{a} \qquad (\boldsymbol{a}=\boldsymbol{a}(x); x\in D), \tag{1.16}$$

and boundary condition:

$$\boldsymbol{u}|_S=\boldsymbol{b} \qquad (\boldsymbol{b}=\boldsymbol{b}(t, x);\ t>0,\ x\in S) \tag{1.17}$$

where $\boldsymbol{u}=\langle u^1(t, x),\ u^2(t, x),\ u^3(t, x)\rangle$ and $p=p(t, x)$ are unknown functions representing the velocity of the fluid and the presure respectively ; $\nu$ (positive constant) is the kinematic viscosity and $\boldsymbol{f}=\langle f^1(t, x),\ f^2(t, x),\ f^3(t, x)\rangle$ is the exterior force. (The density of the fluid is assumed to be one.) It is natural to assume that

$$(1.18)\qquad \operatorname{div} \boldsymbol{a}=0 \text{ in } D \text{ and } \boldsymbol{a}|_S=\boldsymbol{b}(0, \bullet)$$

and, in view of Lemma 1.1,

$$(1.19)\qquad \int_S (\boldsymbol{b}\cdot\boldsymbol{n})dS=0 \qquad \text{for any } t>0.$$

We also assume that[2]

(1.20) $\begin{cases} \boldsymbol{f} \text{ is expressible in the form} \\ \qquad \boldsymbol{f}=\boldsymbol{f}_0(t, x)+\nabla\varphi_0(t, x) \\ \text{where } \boldsymbol{f}_0(t, \bullet)\in\mathfrak{D}_F\cap \boldsymbol{C}^\gamma(\bar{D})\ (\gamma>0) \text{ and } \varphi_0(t, \bullet)\subset C^1(D) \text{ for any } t\geqq 0 \text{ and } \|\boldsymbol{f}_0\|_t, \\ \|\boldsymbol{f}_0\|_t', \ \|\operatorname{div}\boldsymbol{f}_0\|_t \text{ and } \|\operatorname{div}\boldsymbol{f}_0\|_t' \text{ are finite for any } t>0, \end{cases}$

$$(1.21)\qquad \boldsymbol{a}\in\boldsymbol{C}^0(\bar{D})\cap\boldsymbol{C}^2(D),\ \ \Delta\boldsymbol{a}\in\mathfrak{H}\cap\boldsymbol{C}^1(D)^{3)},$$

$$(1.22)\qquad \boldsymbol{b}\in\boldsymbol{C}^2([0, \infty]\times S)$$

and

(1.23) $\begin{cases} \text{for any } x\in S, \text{ there exists a coordinate neighborhood } \Omega \text{ of } x \text{ in the surface} \\ S \text{ for which } \left\|\dfrac{\partial^2\boldsymbol{b}}{\partial\xi^j\partial\xi^k}\right\|_{t,\Omega}, \ \left\|\dfrac{\partial^2\boldsymbol{b}}{\partial\xi^j\partial\xi^k}\right\|_{t,\Omega}', \ \left\|\dfrac{\partial^2\boldsymbol{b}}{\partial t\partial\xi^j}\right\|_{t,\Omega} \text{ and } \left\|\dfrac{\partial^2\boldsymbol{b}}{\partial t\partial\xi^j}\right\|_{t,\Omega}' \ (j, k \\ =1, 2) \text{ are finite for any } t>0 \text{ where } (\xi^1, \xi^2) \text{ denotes a local coordinate} \\ \text{around } x. \end{cases}$

Our main results in the present paper are stated as follows.

THEOREM 1 (Existence theorem). *Under the assumptions* (1.18)-(1.23), *there exist a positive number $T$ and a solution* $\{\boldsymbol{u}(t, x), p(t, x)\}$ of (1.14) (1.17), *in the region: $0<t<T$ and $x\in D$, with following properties:* i) $\dfrac{\partial\boldsymbol{u}(t, x)}{\partial t}$, $\dfrac{\partial^2\boldsymbol{u}(t, x)}{\partial x^j\partial x^k}$ *and* $\dfrac{\partial p(t, x)}{\partial x^j}$ $(j, k=1, 2, 3)$ *in usual sense exist and are continuous in* $(0, T)\times D$, *and they satisfy* (1.14) *and* (1.15) *in the region;* ii) $\boldsymbol{u}(t, x)$ *is continuous in the closed region* $[0, T]\times\bar{D}$ *and satisfies* (1.16) *and* (1.17) *in usual sense;* iii) $\boldsymbol{u}=\boldsymbol{u}(t, \bullet)$ *has its strong derivative* $\dfrac{d\boldsymbol{u}(t)}{dt}$ *in* $\mathfrak{H}$ *and* $\dfrac{d\boldsymbol{u}(t)}{dt}=\dfrac{\partial\boldsymbol{u}(t, \bullet)}{\partial t}$; iv) $\left\|\dfrac{d\boldsymbol{u}}{dt}\right\|_t<\infty$, $\|\nabla\boldsymbol{u}\|_t<\infty$ *and* $p(t, \bullet)-\varphi_0(t, \bullet)\in L^2(D)$ (see (1.20)) *for any* $t\in(0, T)$.

2) Assumptions (1.20)-(1.23) may be replaced by slightly weaker ones. But these are complicated; so we assume (1.20)-(1.23).

3) (1.18), (1.21) and (1.22) imply $\|\nabla\boldsymbol{a}\|<\infty$ by Lemma 1.3.

Theorem 2 (Uniqueness theorem). *If* $\{u, p\}$ *and* $\{v, q\}$ *are solutions of* (1.14-17) *satisfying all conditions stated in Theorem* 1, *then* $u(t, x)\equiv v(t, x)$ *and* $\nabla p(t, x)\equiv\nabla q(t, x)$ *in* $(0, T)\times D$.

Proofs of these theorems will be given in §§ 6 and 7. We shall also remark in § 7 that our proof of Theorem 2 is available in case $\{u, p\}$ is our solution stated in Theorem 1 and $\{v, q\}$ is Kiselev-Ladyzhenskaia's solution in [8] if (1.18), (1.20) and (1.21) are satisfied and $b\equiv 0$ in (1.18). Hence, under these conditions, our solution coincides with Kiselev-Ladyzhenskaia's. Accordingly, combining our results with those in [8], we may conclude, for example, that

Theorem 3. *Assume that* (1.18), (1.20) *and* (1.21) *are satisfied and* $b\equiv 0$, *that* $f$ *has a potential* (*namely we may put* $f_0\equiv 0$ *in* (1.20)) *and that the Reynolds number at the initial moment* ($t=0$) *is sufficiently small in the sense of Kiselev-Ladyzhenskaia's paper* [8]. *Then the solution* $\{u, p\}$ *stated in Theorem* 1 (*with* $b\equiv 0$) *exists in the infinite time-interval*: $0<t<\infty$.

§ 2. **Preliminaries.** Let $\mathfrak{H}_1$ and $\mathfrak{H}_0$ be subspaces of the Hilbert space $\mathfrak{H}$ (see § 1) defined as follows:

(2.1) $$\mathfrak{H}_1=\{\nabla\varphi\,;\ \varphi\in C^1(D),\ \nabla\varphi\in\mathfrak{H}\}^a$$

(the superscript $a$ denotes the closure operation in $\mathfrak{H}$), and

(2.2) $$\mathfrak{H}_0=\mathfrak{H}\ominus\mathfrak{H}_1 \text{ (ortho-complement of } \mathfrak{H}_1).$$

Then;

Lemma 2.1. *If* $u\in\mathfrak{H}\cap C^1(D)$ *and* $(u, v)=0$ *for any* $v\in C_0^2(D)$ *satisfying* $\operatorname{div} v=0$ *in* $D$, *then there exists a single-valued function* $\varphi\in C^2(D)$ *such that* $u=\nabla\varphi\in\mathfrak{H}_1$. (See [5; pp. 214-215])

Lemma 2.2. *Let* $\{D_m\}$ *be a monotone increasing sequence of subdomains of* $D$ *such that* $\bigcup_{m=1}^{\infty} D_m=D$, *that the boundary* $S_m$ *of each* $D_m$ *consists of a finite number of components each of which is a simple closed surface of class* $C^4$ *and that the sequence* $\{S_m\}$ *approaches to* $S$ *smoothly, and assume that* $v\in C^1(D)\cap C^0(\bar{D})$, $v|_S=0$ *and* $\operatorname{div} v=0$ *in* $D$. *Assume also that* $\varphi\in C^1(D)$ *and that at least one of*

$$\|\nabla\varphi\|<\infty \text{ and } \|\varphi\|<\infty \qquad \text{(see (1.8) and (1.9))}$$

*holds. Then*

(2.3) $$\lim_{m\to\infty}\int_{D_m}(v\cdot\nabla\varphi)dx=0.$$

Proof. For any $m$ and any $x\in S_m$, we denote by $n_m=n_m(x)$ the unit vector with outer normal direction at $x$. Then the assumption of this lemma implies that

$$(2.4)\qquad \int_{D_m}(\boldsymbol{v}\cdot\nabla\varphi)dx=\int_{S_m}(\boldsymbol{v}\cdot\boldsymbol{n}_m)\varphi dS$$

and that

$$(2.5)\qquad \lim_{m\to\infty}\sup_{x\in S_m}|(\boldsymbol{v}\cdot\boldsymbol{n}_m)(x)|=0.$$

Hence we get (2.3) if $\|\varphi\|<\infty$. In case: $\|\nabla\varphi\|<\infty$, we may prove that there exists a constant $C_N$ for sufficiently large $N$ such that

$$\int_{S_m}|\varphi(x)|dS\leqq\int_{S_N}|\varphi(x)|dS+C_N\|\nabla\varphi\| \qquad \text{for any } m>N.$$

Combining this relation with (2.4) and (2.5), we obtained (2.3), q.e.d.

LEMMA 2.3. *Under the assumption: $\boldsymbol{u}\in\boldsymbol{C}^1(D)\cap\boldsymbol{C}^0(\bar{D})$, $\boldsymbol{u}(x)$ belongs to $\mathfrak{H}_0$ if and only if*

$$(2.6)\qquad \operatorname{div}\boldsymbol{u}=0 \text{ in } D \text{ and } (\boldsymbol{u}\cdot\boldsymbol{n})=0 \text{ on } S.$$

PROOF. If $\boldsymbol{u}\in\mathfrak{H}_0$, then $(\operatorname{div}\boldsymbol{u},\varphi)=-(\boldsymbol{u},\nabla\varphi)=0$ for any $\varphi\in C_0^1(D)$ and hence $\operatorname{div}\boldsymbol{u}=0$ in $D$. On the other hand, any function $\varphi_0(x)$ of class $C^1$ on $S$ may be extended to a function $\varphi(x)\in C^1(\bar{D})$. Hence we have

$$\int_S(\boldsymbol{u}\cdot\boldsymbol{n})\varphi_0 dS=\int_S(\boldsymbol{u}\cdot\boldsymbol{n})\varphi dS-(\operatorname{div}\boldsymbol{u},\varphi)=(\boldsymbol{u},\nabla\varphi)=0\,;$$

this implies that $(\boldsymbol{u}\cdot\boldsymbol{n})=0$ on $S$. The converse follows immediately from Lemma 2.2.

LEMMA 2.4. *Assume that $\boldsymbol{v}\in\boldsymbol{C}^1(D)\cap\boldsymbol{C}^0(\bar{D})\cap\mathfrak{H}_0\cap\mathfrak{D}_\Gamma$, $\boldsymbol{v}|_S=0$ and $\boldsymbol{w}\in\boldsymbol{C}^2(D)\cap\boldsymbol{C}^1(\bar{D})\cap\mathfrak{H}_0$ and that $\varphi\in C^1(D)$, $\|\varphi\|<\infty$ and $\boldsymbol{u}=\nabla\varphi-\Delta\boldsymbol{w}$ belongs to $\mathfrak{H}$. Then*

$$(2.7)\qquad (\nabla\boldsymbol{v},\nabla\boldsymbol{w})=(\boldsymbol{v},\boldsymbol{u}).$$

PROOF. Let $\{D_m\}$ be such a sequence of subdomains of $D$ as stated in Lemma 2.2. Then we have

$$(2.8)\qquad \int_{D_m}(\nabla\boldsymbol{w}\cdot\nabla\boldsymbol{v})dx=\int_{S_m}\left(\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}_m}\cdot\boldsymbol{v}\right)dS-\int_{D_m}(\{\nabla\varphi-\boldsymbol{u}\}\cdot\boldsymbol{v})dx.$$

On the other hand, it follows from the assumption of this lemma that $\operatorname{div}\boldsymbol{v}=0$ in $D$ by Lemma 2.3. Hence letting $m\to\infty$ in (2.8) and applying Lemma 2.2, we obtain (2.7), q. e. d.

Assume that $\boldsymbol{u}\in\boldsymbol{C}^1(D)\cap\boldsymbol{C}^0(\bar{D})$, that $\operatorname{div}\boldsymbol{u}$ and $(\boldsymbol{u}\cdot\boldsymbol{n})$ are Hölder-continuous in $D$ and on $S$ respectively and that $\|\operatorname{div}\boldsymbol{u}\|<\infty$. Then it follows from (1.13) that the solution $\varphi=\varphi_{\boldsymbol{u}}$ of the second boundary value problem:

$$(2.9)\qquad \Delta\varphi=\operatorname{div}\boldsymbol{u},\qquad \left.\frac{\partial\varphi}{\partial\boldsymbol{n}}\right|_S=(\boldsymbol{u}\cdot\boldsymbol{n})$$

exists and is unique up to additive constant, and accordingly $\nabla\varphi_u$ is uniquely determined by $\boldsymbol{u}$. In fact, if we denote by $K(x, y)$ the kernel function of the second boundary value problem (2.9) (see [7]), we have

$$\varphi_u(x) = -\int_D K(x, y)\operatorname{div}\boldsymbol{u}(y)dy + \int_S K(x, y)(\boldsymbol{u}\cdot\boldsymbol{n})(y)dS_y + \text{const}. \tag{2.10}$$

For any $\boldsymbol{u}\in\mathfrak{D}_\Gamma\cap\boldsymbol{C}^0(\bar{D})$, (1.13) still holds and $\varphi_u(x)$ defined by (2.10) satisfies $\nabla\varphi_u \in\mathfrak{H}_1$; these facts may be seen from the definition of *div* in generalized sense (see §1) and properties of $K(x, y)$ given in [7].

We denote the projection of $\mathfrak{H}$ onto $\mathfrak{H}_0$ (resp. $\mathfrak{H}_1$) by $P_0$ (resp. $P_1$). Then we have the following

LEMMA 2.5. *If we define* $\varphi_u$ *by* (2.10) *for any* $\boldsymbol{u}\in\mathfrak{D}_\Gamma\cap\boldsymbol{C}^0(\bar{D})$, *then* $P_0\boldsymbol{u}=\boldsymbol{u}-\nabla\varphi_u$ *and* $P_1\boldsymbol{u}=\nabla\varphi_u$.

PROOF. We prove this lemma only in the case where $\operatorname{div}\boldsymbol{u}$ and $(\boldsymbol{u}\cdot\boldsymbol{n})$ are Hölder-continuous; general case may easily be reduced to the case. If we put $\boldsymbol{v}=\boldsymbol{u}-\nabla\varphi_u$, we have $\operatorname{div}\boldsymbol{v}=\operatorname{div}\boldsymbol{u}-\Delta\varphi_u=0$ and $(\boldsymbol{v}\cdot\boldsymbol{n})=(\boldsymbol{u}\cdot\boldsymbol{n})-\frac{\partial\varphi_u}{\partial\boldsymbol{n}}=0$ by (2.9). Hence $\boldsymbol{v}\in\mathfrak{H}_0$ by Lemma 2.3. Since $\boldsymbol{u}=\boldsymbol{v}+\nabla\varphi_u$ and $\nabla\varphi_u\in\mathfrak{H}_1$, it follows that $P_0\boldsymbol{u}=\boldsymbol{v}=\boldsymbol{u}-\nabla\varphi_u$ and $P_1\boldsymbol{u}=\nabla\varphi_u$, q. e. d.

LEMMA 2.6. *If* $\boldsymbol{u}\in\mathfrak{H}\cap\boldsymbol{C}^1(D)$ *and if* $\operatorname{div}\boldsymbol{u}\in\boldsymbol{C}^1(D)$, *then* $P_1\boldsymbol{u}=\nabla\varphi$ *for a suitable* $\varphi\in C^2(D)$ *satisfying* $\Delta\varphi=\operatorname{div}\boldsymbol{u}$ in $D$.

PROOF. For any $\Psi\in\boldsymbol{C}_0^3(D)$, we have $\nabla\operatorname{div}\Psi\in\mathfrak{H}_1$ and $\operatorname{rot}\operatorname{rot}\Psi\in\mathfrak{H}_0$ by Lemma 2.3. Hence

$$\begin{aligned}(P_1\boldsymbol{u}, \Delta\Psi) &= (P_1\boldsymbol{u}, \nabla\operatorname{div}\Psi-\operatorname{rot}\operatorname{rot}\Psi)\\ &= (\boldsymbol{u}, \nabla\operatorname{div}\Psi)=(\nabla\operatorname{div}\boldsymbol{u}, \Psi),\end{aligned} \tag{2.11}$$

here $\nabla\operatorname{div}\boldsymbol{u}\in\boldsymbol{C}^0(D)$. Hence $P_1\boldsymbol{u}\in\boldsymbol{C}^1(D)$ by Lemma 1.3. On the other hand, we have $(P_1\boldsymbol{u}, \boldsymbol{v})=0$ for any $\boldsymbol{v}\in\boldsymbol{C}_0^1(D)$ satisfying $\operatorname{div}\boldsymbol{v}=0$ in $D$ (since any such $\boldsymbol{v}$ belongs to $\mathfrak{H}_0$ by Lemma 2.3). Hence, by Lemma 2.1, we obtain that $P_1\boldsymbol{u}=\nabla\varphi$ for a suitable $\varphi\in C^2(D)$ and accordingly that $(\Delta\varphi, \psi)=(\nabla\varphi, \nabla\psi)=(\boldsymbol{u}-P_0\boldsymbol{u}, \nabla\psi)=(\operatorname{div}\boldsymbol{u}, \psi)$ for any $\psi\in C_0^2(D)$. Hence $\Delta\varphi=\operatorname{div}\boldsymbol{u}$ in $D$, q. e. d.

Let $K_0(x, y)$ be the kernel function of the Neumann problem in the interior of the solid sphere $Q_0=\{x\,;\,r_{xO}\leqq 1\}$ where $O$ denotes the origin of $\boldsymbol{R}^3$. Then, for any $z\in\boldsymbol{R}^3$ and any $\rho>0$, the function

$$K_z(x, y\,;\rho)=\frac{1}{\rho}K_0\left(\frac{x-z}{\rho}, \frac{y-z}{\rho}\right) \tag{2.12}$$

is the kernel function of the Neumann problem in $Q_z(\rho)=\{x\,;\,r_{xz}<\rho\}$. Using this fact, we shall prove the following

LEMMA 2.7. *Assume that* $\varphi(t, \cdot)\in C^2(D)$, $\|\nabla\varphi\|_t<\infty$ *and* $\|\nabla\varphi\|_t'<\infty$ *for any* $t>0$. *Then—*

i) *if* $\Delta\varphi(t,x)\equiv 0$, *then* $\nabla\varphi(t,x)$, $\nabla\nabla\varphi(t,x)$ *and* $\nabla\nabla\nabla\varphi(t,x)$ *are Hölder-continuous in the interior of* $(0, \infty)\times D$;

ii) *if* $\Delta\varphi(t, x)$ *is Hölder-continuous in the interior of* $(0, \infty)\times D$, *then so is* $\nabla\varphi(t, x)$.

PROOF. i) Let $z$ be an arbitrary point in $D$, and $R$ be a positive number such that $Q_z(3R)\subset D$. Then it is sufficient to prove the Hölder-continuity of $\nabla\varphi$, $\nabla\nabla\varphi$ and $\nabla\nabla\nabla\varphi$ in $(0, \infty)\times Q_z(R)$. Since $\Delta\varphi(t, x)\equiv 0$, we have

$$\varphi(t, x)=\int_{S(\rho)} K_z(x, y\,;\rho)\frac{\partial\varphi(t, y)}{\partial n_\rho}\,dS_y$$

for any $t>0$, $x\in Q_z(R)$ and $\rho>R$, where $S(\rho)=\partial Q_z(\rho)\equiv\{y\,;r_{yz}=\rho\}$ and $\frac{\partial}{\partial n_\rho}$ denotes the normal derivative on $S(\rho)$. Hence, if $t>t'>0$ and $x, x'\in Q_z(R)$, it holds that

$$\begin{aligned}(2.13)\qquad &|\nabla\varphi(t, x)-\nabla\varphi(t', x')|\\ &\leqq\frac{1}{R}\int_{2R}^{3R}d\rho\int_{S(\rho)}|\nabla_x K_z(x, y\,;\rho)|\cdot|\nabla\varphi(t, y)-\nabla\varphi(t', y)|dS_y{}^{4)}\\ &+\frac{1}{R}\int_{2R}^{3R}d\rho\int_{S(\rho)}|\nabla_x K_z(x, y\,;\rho)-\nabla_x K_z(x', y\,;\rho)|\cdot|\nabla\varphi(t', y)|dS_y\,.\end{aligned}$$

On the other hand, it follows from (2.12) that

$$\sup_{\substack{r_{xz}<R,\ r_{yz}=\rho\\ 2R<\rho<3R}}\{|\nabla_x K_z(x, y\,;\rho)|+|\nabla_x\nabla_x K_z(x, y\,;\rho)|\}$$

is finite. Applying this fact and the assumption ($\|\nabla\varphi\|_t<\infty$ and $\|\nabla\varphi\|_t'<\infty$) to (2.13), we may easily see that $\nabla\varphi$ is Hölder-continuous in $(0, t)\times Q_z(R)$. Similarly we may show the Hölder-continuity of $\nabla\nabla\varphi$ and that of $\nabla\nabla\nabla\varphi$.

ii) Let $z$ and $R$ be as stated above. Then, for any $t>0$, $x\in Q_z(R)$ and $\rho>R$, we have $\varphi(t, x)=\varphi_1(t, x)+\varphi_2(t, x)$ where

$$\varphi_1(t, x)=-\int_{Q_z(\rho)} K_z(x, y\,;\rho)\Delta\varphi(t, y)dy$$

and

$$\varphi_2(t, x)=\int_{S(\rho)} K_z(x, y\,;\rho)\frac{\partial\varphi(t, y)}{\partial n_\rho}\,dS_y.$$

The Hölder-continuity of $\nabla\varphi_1(t, x)$ in $(0, \infty)\times Q_z(R)$ may be shown from that of $\Delta\varphi(t, y)$ and properties of the kernel function $K_0(x, y)$, while the Hölder-continuity of $\nabla\varphi_2(t, x)$ is already proved above. Hence $\nabla\varphi(t, x)$ is Hölder-continuous in $(0, \infty)$

4) The subscript $x$ to $\nabla$ means to operate $\nabla$ to $K_z(x, y;\rho)$ as a function of $x$.

$\times Q_z(R)$, q. e. d.

If $\Delta\varphi(t, x)\equiv 0$ in $(0, \infty)\times D$, we have also that

$$\varphi(t, x)=\int_{S(\rho)} P_z(x, y; \rho)\varphi(t, y)dS_y \qquad \text{for any } x\in Q_z(\rho)$$

where

$$P_z(x, y; \rho)=\frac{\rho^2-r_{xz}^2}{4\pi\rho r_{xy}^3} \qquad (x\in Q_z(\rho),\ y\in S(\rho)).$$

Hence we may prove by the similar argument to the proof of Lemma 2.7 that

LEMMA 2.8. *If* $\Delta\varphi(t, x)\equiv 0$, $\|\varphi\|_t<\infty$ *and* $\|\varphi\|_t'<\infty$ *for any* $t>0$, *then* $\nabla\varphi$ *is Hölder-continuous in the interior of* $(0, \infty)\times D$.

Finally we modify the equations (1.14) and (1.15) to the following form:

$$\frac{\partial \boldsymbol{v}}{\partial t}=\nu\cdot\Delta\boldsymbol{v}+\boldsymbol{F}_t(\boldsymbol{v})-\nabla q \tag{2.14}$$

($\boldsymbol{F}_t(\cdot)$ will be defined later) and

$$\operatorname{div}\boldsymbol{v}=0, \tag{2.15}$$

and reduce the initial-boundary value problem (1.14)-(1.17) to the case: $\boldsymbol{a}\equiv 0$ and $\boldsymbol{b}\equiv 0$.

LEMMA 2.9. *Under the assumptions* (1.19), (1.22) *and* (1.23), *there exists a vector function* $\boldsymbol{U}_0=\boldsymbol{U}_0(t, x)\in\boldsymbol{C}^0([0, \infty)\times\bar{D})$ *with following properties:* i) $\boldsymbol{U}_0(t, \cdot)\in\boldsymbol{C}^3(D)$, $\operatorname{div}\boldsymbol{U}_0=0$ *in* $D$ *and* $\boldsymbol{U}_0|_S=\boldsymbol{b}$ *for any* $t\geqq 0$; ii) *the following quantities are finite for any* $t>0$:

$$\|\boldsymbol{U}_0\|_t,\ \|\boldsymbol{U}_0\|_t',\ \|\nabla\boldsymbol{U}_0\|_t,\ \|\nabla\boldsymbol{U}_0\|_t',\ \|\Delta\boldsymbol{U}_0\|_t,\ \|\Delta\boldsymbol{U}_0\|_t',\ \left\|\frac{\partial\boldsymbol{U}_0}{\partial t}\right\|_t \ \textit{and}\ \left\|\frac{\partial\boldsymbol{U}_0}{\partial t}\right\|_t';$$

*and* iii) $\nabla\boldsymbol{U}_0$ *and* $\Delta\boldsymbol{U}_0$ *are Hölder-continuous in the interior of* $(0, \infty)\times D$.

PROOF. It follows from the assumptions (1.19), (1.22) and (1.23) that there exists a function $\varphi\equiv\varphi(t, x)\in C^1([0, \infty)\times\bar{D})$ such that

$$\varphi(t, \cdot)\in C^2(\bar{D})\cap C^\infty(D),\ \Delta\varphi=0 \text{ in } D \text{ and } \left.\frac{\partial\varphi}{\partial\boldsymbol{n}}\right|_S=(\boldsymbol{b}\cdot\boldsymbol{n}) \text{ for any } t>0 \tag{2.16}$$

and

$$\|\nabla\nabla\varphi\|_t,\ \|\nabla\nabla\varphi\|_t',\ \left\|\frac{\partial}{\partial t}\nabla\varphi\right\|_t \text{ and } \left\|\frac{\partial}{\partial t}\nabla\varphi\right\|_t' \text{ are finite for any } t>0; \tag{2.17}$$

this fact may easily be seen if

$$\boldsymbol{b}\in\boldsymbol{C}^3([0, \infty)\times S), \tag{$*$}$$

and this condition ($*$) may be replaced by (1.22) and (1.23) by virtue of Schauder estimates (interior and near the boundary) for solutions of linear elliptic equations (see [1] and [2] for example). Similarly we may show that there exist vector functions $\boldsymbol{u}\equiv\boldsymbol{u}(t, x)\in \boldsymbol{C}^0([0, \infty)\times\bar{D})$ and $\boldsymbol{v}\equiv\boldsymbol{v}(x)\in\boldsymbol{C}^2(\bar{D})\cap\boldsymbol{C}^\infty(D)$ such that

(2.18) $\boldsymbol{u}(t, \cdot)\in\boldsymbol{C}^2(\bar{D})\cap\boldsymbol{C}^\infty(D)$, $\Delta\boldsymbol{u}=0$ in $D$ and $\boldsymbol{u}|_S=\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{n})\boldsymbol{n}$ for any $t\geqq 0$,

(2.19) $\|\nabla\nabla\boldsymbol{u}\|_t$, $\|\nabla\nabla\boldsymbol{u}\|'_t$, $\left\|\frac{\partial}{\partial t}\nabla\boldsymbol{u}\right\|_t$ and $\left\|\frac{\partial}{\partial t}\nabla\boldsymbol{u}\right\|'_t$ are finite for any $t>0$

and

(2.20) $$\Delta\boldsymbol{v}=0 \text{ in } D \text{ and } \boldsymbol{v}|_S=\boldsymbol{n}.$$

Furthermore, we may easily construct a function $\psi\in C^3(\bar{D})\cap C^4(D)$ satisfying

(2.21) $$\psi|_S\equiv 0 \text{ and } \left.\frac{\partial\psi}{\partial\boldsymbol{n}}\right|_S\equiv 1.$$

We put

(2.22) $$\boldsymbol{w}=\text{rot}\,[\boldsymbol{u}\wedge(\psi\boldsymbol{v})]$$

(where $[\boldsymbol{u}\wedge\boldsymbol{v}]$ denotes the 'vector product' of $\boldsymbol{u}$ with $\boldsymbol{v}$). Then

(2.23) $$\boldsymbol{w}\in\boldsymbol{C}^0([0, \infty)\times\bar{D}),$$

(2.24) $$\boldsymbol{w}(t, \cdot)\in\boldsymbol{C}^0(\bar{D})\cap\boldsymbol{C}^3(D) \quad \text{for any } t>0,$$

(2.25) $$\begin{cases} \|\boldsymbol{w}\|_t,\ \|\boldsymbol{w}\|'_t,\ \|\nabla\boldsymbol{w}\|_t,\ \|\nabla\boldsymbol{w}\|'_t,\ \|\Delta\boldsymbol{w}\|_t,\ \|\Delta\boldsymbol{w}\|'_t,\ \left\|\frac{\partial\boldsymbol{w}}{\partial t}\right\|_t \text{ and } \left\|\frac{\partial\boldsymbol{w}}{\partial t}\right\|'_t \\ \text{are finite for any } t>0 \end{cases}$$

and

(2.26) $\nabla\boldsymbol{w}$ is Hölder-continuous in the interior of $(0, \infty)\times D$

(since $\nabla\nabla\varphi$ and $\nabla\nabla\boldsymbol{u}$ are Hölder-continuous in $(0, \infty)\times D$ by means of Lemma 2.7). Futhermore (2.22) is written as follows:

(2.22′) $$\boldsymbol{w}=(\boldsymbol{u}\cdot\nabla)[\psi\boldsymbol{v}]-(\psi\boldsymbol{v}\cdot\nabla)\boldsymbol{u}+\text{div}(\psi\boldsymbol{v})\cdot\boldsymbol{u}-\text{div}\,\boldsymbol{u}\cdot(\psi\boldsymbol{v}).$$

Applying (2.16), (2.18), (2.20) and (2.21) to the above formula, we may show that

$$\boldsymbol{w}|_S=\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{n})\boldsymbol{n}=\boldsymbol{b}-(\nabla\varphi)|_S.$$

Hence, if we define $\boldsymbol{U}_0\equiv\boldsymbol{U}_0(t, x)$ by

$$\boldsymbol{U}_0=\boldsymbol{w}+\nabla\varphi,$$

then $\boldsymbol{U}_0$ satisfies i), ii) and iii) required in Lemma 2.9 ($\text{div}\,\boldsymbol{U}_0=\text{div}\,\boldsymbol{w}+\text{div}\nabla\varphi=\text{div rot}[\boldsymbol{u}\wedge(\psi\boldsymbol{v})]+\Delta\varphi=0$; the other properties are obvious).

Now we put

(2.27) $$\boldsymbol{U}(t, x)=\boldsymbol{a}(x)+\boldsymbol{U}_0(t, x)-\boldsymbol{U}_0(0, x).$$

Then it follows from (1.18), (1.21) and Lemma 2.9 that

(2.28) $$U \in C^0([0, \infty) \times \bar{D}) \quad \text{and} \quad U(0, x) \equiv a(x),$$

(2.29) $$U(t, \cdot) \in \mathfrak{D}_\Gamma \cap C^2(D), \ \operatorname{div} U = 0 \text{ in } D \text{ and } U|_S = b \text{ for any } t \geqq 0,$$

(2.30) $$\begin{cases} \|U\|_t, \ \|U\|_t', \ \|\nabla U\|_t, \ \|\nabla U\|_t', \ \|\Delta U\|_t, \ \|\Delta U\|_t', \\ \left\|\dfrac{\partial U}{\partial t}\right\|_t \text{ and } \left\|\dfrac{\partial U}{\partial t}\right\|_t' \text{ are finite for any } t > 0 \end{cases}$$

and

(2.31) $$U \text{ and } \Delta U \text{ are Hölder-continuous in } (0, \infty) \times D.$$

Hence

(2.32) $$\begin{cases} \nu \cdot \Delta U - \dfrac{\partial U}{\partial t} - (U \cdot \nabla) U \in \mathfrak{H}, \ \operatorname{div}(U \cdot \nabla) U = (\nabla U : \nabla U) \\ \text{and} \quad \operatorname{div} \Delta U = \operatorname{div} \dfrac{\partial U}{\partial t} = 0 \quad \text{for any } t \geqq 0, \end{cases}$$

and hence, applying Lemma 2.5 to $f_0$ (see (1.20)) and Lemmas 2.6 and 2.7 to $\Delta U$, $\frac{\partial U}{\partial t}$ and $(U \cdot \nabla) U$, we may see that there exists a function $\varphi_1(t, x)$ on $(0, \infty) \times D$ such that $\varphi_1(t, \cdot) \in C^1(D)$ for any $t \geqq 0$ and $\nabla \varphi_1(t, x)$ is Hölder-continuous in $(0, \infty) \times D$ and that

(2.33) $$P_1\left\{f_0 + \nu \cdot \Delta U - \frac{\partial U}{\partial t} - (U \cdot \nabla) U\right\} = \nabla \varphi_1.$$

Accordingly, if we put

(2.34) $$W = P_0\left\{f_0 + \nu \cdot \Delta U - \frac{\partial U}{\partial t} - (U \cdot \nabla) U\right\},$$

then

(2.35) $$\begin{cases} \|W\|_t \text{ and } \|W\|_t' < \infty \text{ for any } t > 0, \text{ and} \\ W(t, x) \text{ is Hölder-continuous in } (0, \infty) \times D. \end{cases}$$

Now we put $u = U + v$ and $p = \varphi_0 + \varphi_1 + q$ in (1.14)-(1.17). Then we have

(2.36) $$\begin{cases} \dfrac{\partial v}{\partial t} = \nu \cdot \Delta v + F_t(v) - \nabla q & \text{(cf. (2.14))}, \\ \operatorname{div} v = 0 & \text{(cf. (2.15))}, \\ v|_{t=0} = 0 \text{ and } v|_S = 0 \end{cases}$$

where $F_t(\cdot)$ is a nonlinear mapping defined by

(2.37) $$F_t(v) = W(t) - (v \cdot \nabla) U(t) - (U(t) \cdot \nabla) v - (v \cdot \nabla) v.$$

Hence, in order to prove Theorem 1, it is sufficient to show the existence of $v = v(t, x)$ and $q = q(t, x)$ which satisfy (2.36) and have properties corresponding to those of $u$ and $p$ in Theorem 1.

§3. **Green tensor and its eigenfunctions.** F. K. G. Odqvist [15] constructed the *Green tensor* $\{G^{jk}(x, y), h^k(x, y);\ j, k=1, 2, 3\}$ *for Stokes's boundary value problem*, with properties as follows;

1°) $G^{jk}(x, y)=G^{kj}(y, x) \qquad (x, y\in\bar{D};\ x\neq y)$;

2°) each $G^{jk}(x, y)$ is of class $C^2$ in $x\in D-\{y\}$;

3°) each $h^k(x, y)$ is of class $C^1$ in $x\in D-\{y\}$;

4°) $\sum_{j=1}^{3}\frac{\partial G^{jk}(x, y)}{\partial x^j}=0$ for $x\in D-\{y\}$ and $G^{jk}(x, y)=0$ for $x\in S$ (for each $k$ and any fixed $y\in\bar{D}$);

5°) there exists a constant $C>0$ such that

$$|G^{jk}(x, y)|<\frac{C}{r_{xy}},\quad \left|\frac{\partial G^{jk}(x,y)}{\partial x^l}\right|<\frac{C}{r_{xy}^2}\quad\text{and}\quad |h^k(x, y)|<\frac{C}{r_{xy}^2}.$$

By virtue of 5°), the formulae

$$(Gu)^j(x)=\sum_k\int_D G^{jk}(x, y)u^k(y)dy \tag{3.1}$$

and

$$(Hu)(x)=\sum_k\int_D h^k(x, y)u^k(y)dy \tag{3.2}$$

define respectively a bounded linear operator $G$ in $\mathfrak{H}$ and a bounded linear mapping $H$ of $\mathfrak{H}$ into $L^2(D)$, and we have, for a suitable constant $C'$,

$$\|\nabla Gu\|\leqq C'\|u\| \qquad \text{for any } u\in\mathfrak{H}. \tag{3.3}$$

The following property [15] is most important:

6°) if $u$ is Hölder-continuous in $D$, then $Gu\in C^2(D)\cap C^{1+\gamma}(\bar{D})$ and $Hu\in C^1(D)\cap C^\gamma(\bar{D})$ for some $\gamma>0$, and

$$\operatorname{div} Gu=0,\ [Gu]|_S=0\ \text{and}\ \nu\cdot\Delta Gu-\nabla Hu=-u \tag{3.4}$$

where $\nu$ is the constant which appears in the Navier-Stokes equation (1.14) as kinematic viscosity.

We shall investigate some properties of eigenfunctions of the operator $G$.

LEMMA 3.1. *$G$ is symmetric, positive definite and completely continuous.*

PROOF. The symmetricity of $G$ is obvious from 1°). For any $u\in C_0^1(D)$, we have $Gu\in C^2(D)\cap C^1(\bar{D})\cap\mathfrak{H}_0$, $Hu\in C^1(D)\cap C^0(\bar{D})$ and $u=\nabla Hu-\nu\cdot\Delta Gu$ by 6°) and Lemma 2.3. Hence, by Lemma 2.4, we have

$$(Gu, u)=\nu(\nabla Gu, \nabla Gu)\geqq 0,$$

which implies that $G$ is positive definite since $C_0^1(D)$ is dense in $\mathfrak{H}$. The complete continuity may be proved in the same way as the case of usual integral operators in the $L^2$-space of scalar functions on $D$.

LEMMA 3.2. *The closure of the range $\mathfrak{R}_G$ of $G$ coincides with $\mathfrak{H}_0$.*

PROOF. The relation $\mathfrak{R}_G^a \subset \mathfrak{H}_0$ is obvious from (3.4) and Lemma 2.3. In order to prove that $\mathfrak{R}_G^a = \mathfrak{H}_0$, we first show that any $\boldsymbol{u} \in \boldsymbol{C}_0^\infty(D)$ satisfying $\operatorname{div} \boldsymbol{u} = 0$ belongs to $\mathfrak{R}_G$ (such $\boldsymbol{u}$ clearly belongs to $\mathfrak{H}_0$). If we put $\boldsymbol{w} = -\Delta \boldsymbol{u}$, then $\boldsymbol{w} \in \boldsymbol{C}_0^\infty(D)$ and

$$\Delta G\boldsymbol{w} - \nabla H\boldsymbol{w} = -\boldsymbol{w} = \Delta \boldsymbol{u} \qquad \text{(by (3.4))},$$

and hence, we have

$$-\|\nabla(G\boldsymbol{w} - \boldsymbol{u})\|^2 = (\Delta(G\boldsymbol{w} - \boldsymbol{u}), G\boldsymbol{w} - \boldsymbol{u}) = (\nabla H\boldsymbol{w}, G\boldsymbol{w} - \boldsymbol{u}) = 0$$

(the last equality may be shown by means of Lemma 2.2). Therefore $G\boldsymbol{w} - \boldsymbol{u}$ is constant. Since $(G\boldsymbol{w})|_S = \boldsymbol{u}|_S = 0$, we get $\boldsymbol{u} = G\boldsymbol{w} \in \mathfrak{R}_G$. Now suppose that $\boldsymbol{v} \in \mathfrak{H}_0 \ominus \mathfrak{R}_G^a$, and take arbitrary $\boldsymbol{w} \in \boldsymbol{C}_0^\infty(D)$. Then the above result can be applied to $\boldsymbol{u} = \operatorname{rot}\operatorname{rot}\boldsymbol{w}$, and hence $(\boldsymbol{v}, \operatorname{rot}\operatorname{rot}\boldsymbol{w}) = 0$. Therefore

$$(\boldsymbol{v}, \Delta \boldsymbol{w}) = (\boldsymbol{v}, \nabla \operatorname{div} \boldsymbol{w} - \operatorname{rot}\operatorname{rot}\boldsymbol{w}) = 0.$$

Hence $\boldsymbol{v} \in \boldsymbol{C}^\infty(D)$ by virtue of Weyl's lemma [16] [17]. Furthermore $(\boldsymbol{u}, \boldsymbol{v}) = 0$ for any $\boldsymbol{u} \in \boldsymbol{C}_0^\infty(D)$ satisfying $\operatorname{div} \boldsymbol{u} = 0$ (since such $\boldsymbol{u}$ belongs to $\mathfrak{R}_G$ as proved above). Hence $\boldsymbol{v} \in \mathfrak{H}_1$ by Lemma 2.1. Thus we get $\boldsymbol{v} \in \mathfrak{H}_0 \cap \mathfrak{H}_1$ which implies $\boldsymbol{v} = 0$, q. e. d.

It follows from 5°), 6°) and above two lemmas that there exists a system of eigenvalues and eigen(vector)functions $\{\lambda_n, \boldsymbol{g}_n\}_{n=1,2,\cdots}$ of $G$ with following properties:

(3.5) $$\boldsymbol{g}_n = \lambda_n G \boldsymbol{g}_n \in \boldsymbol{C}^2(D) \cap \boldsymbol{C}^1(\bar{D}),$$

(3.6) $$\operatorname{div} \boldsymbol{g}_n = 0, \qquad \boldsymbol{g}_n|_S = 0,$$

(3.7) $$0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n \leqq \cdots \text{ and } \lim_{n \to \infty} \lambda_n = \infty,$$

(3.8) $\{\boldsymbol{g}_n\}$ is a complete orthonormal system in $\mathfrak{H}_0$,

(3.9) $$\nu \cdot \Delta \boldsymbol{g}_n - \nabla p_n = -\lambda_n \boldsymbol{g}_n \qquad \text{where} \qquad p_n = \lambda_n H \boldsymbol{g}_n.$$

We shall denote the components of $\boldsymbol{g}_n(x)$ by $g_n^j(x)$; $j = 1, 2, 3$.

LEMMA 3.3. *If we fix a point $y \in \bar{D}$ and $k$, and put $u^j(x) = G^{jk}(x, y)$ $(j = 1, 2, 3)$, then $\boldsymbol{u} \equiv \langle u^1(x), u^2(x), u^3(x) \rangle \in \mathfrak{H}_0$.*

PROOF. For any $\rho > 0$, we put $Q_\rho = \{z; r_{yz} < \rho\}$ and define $\boldsymbol{v}_\rho = \langle v_\rho^1(x), v_\rho^2(x), v_\rho^3(x) \rangle$ by

$$v_\rho^j(x) = \begin{cases} \delta^{jk} c_\rho & \text{for } x \in D \cap Q_\rho \\ 0 & \text{for } x \in D - Q_\rho \end{cases}$$

where $\delta^{jk}$ denotes the Kronecker's delta and $c_\rho = \left( \int_{D \cap Q_\rho} dz \right)^{-1}$. We further put

$\boldsymbol{u}_\rho = G\boldsymbol{v}_\rho$, namely

$$u_\rho^j(x) = \int_{D \cap Q_\rho} G^{jk}(x, z) c_\rho dz \qquad (j=1, 2, 3).$$

Then, by Schwarz's inequality,

$$\sum_j |u_\rho^j(x)|^2 \leqq \sum_j \int_{D \cap Q_\rho} |G^{jk}(x, z)|^2 c_\rho dz \cdot \int_{D \cap Q_\rho} c_\rho dz.$$

Integrating both sides of the above inequality over $D$, we obtain by 5°) that

$$\|\boldsymbol{u}_\rho\|^2 \leqq 3C^2 \int_{Q_R} r_{yx}^{-2} dx \ (<\infty) \qquad \text{for any } \rho>0 \tag{3.10}$$

where $R$ is a positive number such that $Q_{R/2} \supset D$. For any $\boldsymbol{w} \in \boldsymbol{C}_0^0(D-\{y\})$, we have

$$(\boldsymbol{w}, \boldsymbol{u}_\rho - \boldsymbol{u}) = \int_D dx \int_{D \cap Q_\rho} \sum_j w^j(x) \{G^{jk}(x, z) - G^{jk}(x, y)\} c_\rho dz,$$

and the distance between $Q$ and the carrier of $\boldsymbol{w}$ is positive for sufficiently small $\rho$. Hence, applying 5°) to the above equality, we get

$$\lim_{\rho \to 0} (\boldsymbol{w}, \boldsymbol{u}_\rho - \boldsymbol{u}) = 0. \tag{3.11}$$

Since $\boldsymbol{C}_0^0(D-\{y\})$ is dense in $\mathfrak{H}$, it follows from (3.10) and (3.11) that

$$\lim_{\rho \to 0} (\boldsymbol{w}, \boldsymbol{u}_\rho - \boldsymbol{u}) = 0 \qquad \text{for any } \boldsymbol{w} \in \mathfrak{H}. \tag{3.12}$$

On the other hand, $\boldsymbol{u}_\rho = G\boldsymbol{v}_\rho \in \mathfrak{H}_0$ by Lemma 3.2. Hence we may see by (3.12) that $\boldsymbol{u} \in \mathfrak{H}_0$.

LEMMA 3.4.

$$\sum_{n=1}^{\infty} \left| \frac{g_n^j(x)}{\lambda_n} \right|^2 = \sum_{k=1}^{3} \int_D |G^{jk}(x, y)|^2 dy \qquad (j=1, 2, 3); \tag{3.13}$$

*the series in the left-hand side converges uniformly in* $x \in \bar{D}$.

PROOF. We fix a point $x \in \bar{D}$ and $j$, and put $\boldsymbol{u} = \langle G^{jk}(x, \cdot)\,;\, k=1, 2, 3\rangle$. Then $\boldsymbol{u} \in \mathfrak{H}_0$ by 1°) and Lemma 3.3, and accordingly

$$\sum_{n=1}^{\infty} (\boldsymbol{u}, \boldsymbol{g}_n)^2 = \|\boldsymbol{u}\|^2 = \sum_k \int_D |G^{jk}(x, y)|^2 dy \qquad \text{by (3.8).}$$

On the other hand, (3.5) implies that $g_n^j(x) = \lambda_n(\boldsymbol{u}, \boldsymbol{g}_n)$. Hence we obtain (3.13). Furthermore $\sum_{n=1}^{N} |g_n^j(x)/\lambda_n|^2$ $(N=1, 2, \cdots)$ and the right-hand side of (3.13) are continuous in $x \in \bar{D}$. Hence the series in the left-hand side of (3.13) converges uniformly in $x \in \bar{D}$.

LEMMA 3.5. *$\|\boldsymbol{g}_n\| \leqq C'\lambda_n$, $\|\nabla \boldsymbol{g}_n\| \leqq C'\lambda_n^2$ and $\|p_n\| \leqq C'\lambda_n^2$ $(n=1,2,\cdots)$ for some constant $C'$.*

PROOF. We may see from (3.5) and 5°) that

$$\|\boldsymbol{g}_n\| \leqq \lambda_n \left(3\int_D \left|\frac{C}{r_{xy}}\right|^2 dx\right)^{1/2} \|\boldsymbol{g}_n\| \leqq C_1\lambda_n \tag{3.14}$$

(by Schwarz's inequality) for some constant $C_1$, and accordingly that

$$\|\nabla \boldsymbol{g}_n\| \leqq \lambda_n \int_D \frac{3C}{r_{xy}^2} dx \cdot \|\boldsymbol{g}_n\| \leqq C_2\lambda_n^2$$

for some constant $C_2$. Since $p_n = \lambda_n H\boldsymbol{g}_n$, it follows from (3.14) and 5°) that $\|p_n\| \leqq C_3\lambda_n^2$ for some constant $C_3$. Hence we obtain Lemma 3.5 by putting $C' = \max\{C_1, C_2, C_3\}$.

LEMMA 3.6.

$$\nu\left(\frac{\nabla \boldsymbol{g}_n}{\lambda_n^{1/2}}, \frac{\nabla \boldsymbol{g}_m}{\lambda_m^{1/2}}\right) = \delta_{nm} \qquad \text{(Kronecker's delta)}. \tag{3.15}$$

PROOF. By virtue of 6°) and (3.5), we may put in Lemma 2.4

$$\boldsymbol{v} = \boldsymbol{g}_n, \quad \boldsymbol{w} = \nu \boldsymbol{g}_m, \quad \varphi = \lambda_m H\boldsymbol{g}_m$$

and accordingly $\boldsymbol{u} = \nabla\varphi - \Delta\boldsymbol{w} = \lambda_m \boldsymbol{g}_m$. Hence we get

$$\nu(\nabla \boldsymbol{g}_n, \nabla \boldsymbol{g}_m) = \lambda_m(\boldsymbol{g}_n, \boldsymbol{g}_m) = \lambda_m \delta_{nm} \qquad \text{(by (3.8))}$$

which implies (3.15).

LEMMA 3.7. *$p_n \in C^\infty(D)$ and $\Delta p_n = 0$ in $D$ $(n=1, 2, \cdots)$.*

PROOF. For any $\varphi \in C_0^\infty(D)$, we have by (3.9)

$$-(p_n, \Delta\varphi) = (\nabla p_n, \nabla\varphi) = (\nu\Delta \boldsymbol{g}_n + \lambda_n \boldsymbol{g}_n, \nabla\varphi) = (\boldsymbol{g}_n, \nabla(\nu\Delta\varphi + \lambda_n\varphi)) = 0.$$

Hence $p_n \in C^\infty(D)$ and $\Delta p_n = 0$ by Weyl's lemma [16] [17].

It follows from Lemma 3.4 and 5°) that there exists a constant $M$ such that

$$\sum_{n=1}^\infty \left|\frac{\boldsymbol{g}_n(x)}{\lambda_n}\right|^2 \leqq M^2 \qquad \text{for any } x \in \bar{D}. \tag{3.16}$$

Integrating both sides of (3.16) over the domain $D$, we obtain that

$$\sum_{n=1}^\infty \frac{1}{\lambda_n^2} < \infty. \tag{3.17}$$

Furthermore, since

$$\lambda^\beta e^{-\lambda t} \leqq (\beta/e)^\beta t^{-\beta} \qquad \text{for any positive } \beta, \lambda \text{ and } t, \tag{3.18}$$

and since

$$\sum_{n=N}^{\infty}|\lambda_n^{\beta-1}e^{-\lambda_n t}\alpha_n \boldsymbol{g}_n(x)|\leq\left\{\sum_{n=N}^{\infty}\lambda_n^{2\beta}e^{-2\lambda_n t}\left|\frac{\boldsymbol{g}_n(x)}{\lambda_n}\right|^2\right\}^{1/2}\left\{\sum_{n=N}^{\infty}\alpha_n^2\right\}^{1/2} \tag{3.19}$$

(by Schwarz's inequality) for any sequence $\{\alpha_n\}$, we have

$$\sum_{n=1}^{\infty}|\lambda_n^{\beta-1}e^{-\lambda_n t}\alpha_n \boldsymbol{g}_n(x)|\leq\begin{cases} M(\beta/e)^\beta t^{-\beta}(\sum_{n=1}^{\infty}\alpha_n^2)^{1/2} & \text{if } \beta>0 \\ M(\sum_{n=1}^{\infty}\alpha_n^2)^{1/2} & \text{if } \beta=0,\end{cases} \tag{3.20}$$

where the series in the left-hand side converges uniformly in $\langle t, x\rangle\in[t_0, \infty)\times\bar{D}$ for any $t_0>0$ by virtue of Lemma 3.4, if $\sum_{n=1}^{\infty}\alpha_n^2<\infty$. Similarly we may prove that

$$\sum_{n=1}^{\infty}e^{-\lambda_n t}|\boldsymbol{g}_n(x)|\cdot|\boldsymbol{g}_n(y)|\leq M't^{-2} \tag{3.21}$$

and

$$\sum_{n=1}^{\infty}e^{-\lambda_n t}|p_n(x)|\cdot|\boldsymbol{g}_n(y)|\leq M't^{-4} \qquad \text{(see Lemma 3.5 and (3.18))} \tag{3.22}$$

for some constant $M'$; the series in the left-hand side of each of (3.21) and (3.22) converges uniformly in $\langle t, x, y\rangle\in[t_0, \infty)\times\bar{D}\times\bar{D}$ for any $t_0>0$. Hence we can define

$$G^{jk}(t, x, y)=\sum_{n=1}^{\infty}e^{-\lambda_n t}g_n^j(x)g_n^k(y) \qquad (j, k=1, 2, 3) \tag{3.23}$$

and

$$H^k(t, x, y)=\sum_{n=1}^{\infty}e^{-\lambda_n t}p_n(x)g_n^k(y) \qquad (k=1, 2, 3) \tag{3.24}$$

for $\langle t, x, y\rangle\in(0, \infty)\times\bar{D}\times\bar{D}$, and, for any fixed $t>0$, the formulae

$$[G(t)\boldsymbol{u}]^j(x)=\sum_{k=1}^{3}\int_D G^{jk}(t, x, y)u^k(y)dy \tag{3.25}$$

and

$$[H(t)\boldsymbol{u}](x)=\sum_{k=1}^{3}\int_D H^k(t, x, y)u^k(y)dy \tag{3.26}$$

define respectively a bounded linear operator $G(t)$ in $\mathfrak{H}$ and a bounded linear mapping $H(t)$ of $\mathfrak{H}$ into $L^2(D)$. We call the system $\{G(t), H(t)\}$ a *fundamental solution* (*of the Stokes's initial value problem*) in view of the following

LEMMA 3.8. *If we define $\boldsymbol{u}(t)=G(t)\boldsymbol{u}_0$ and $p(t)=H(t)\boldsymbol{u}_0$ for any $\boldsymbol{u}_0\in\mathfrak{H}_0$, then $\frac{\partial \boldsymbol{u}}{\partial t}=\nu\Delta\boldsymbol{u}-\nabla p$, $\boldsymbol{u}|_S=0$ and $\lim_{t\downarrow 0}\|\boldsymbol{u}(t)-\boldsymbol{u}_0\|=0$.*

We shall not use this lemma in the sequel, so omit the proof.

§4. **Some estimations concerning the fundamental solution.** Let $\boldsymbol{w}(t)$ be a

$\mathfrak{H}_0$-valued function of $t\geqq 0$ such that $\|\boldsymbol{w}\|_t$ and $\|\boldsymbol{w}\|_t'$ are finite for any $t>0$, and put

$$\boldsymbol{v}(t)=\int_0^t G(t-\tau)\boldsymbol{w}(\tau)d\tau\,; \tag{4.1}$$

the right-hand side is understood as Bochner integral [3] [4], and it may be seen by Fubini's theorem that each component $v^j(t, x)$ of $\boldsymbol{v}(t)$ is equal to the following function defined by means of usual integral of real-valued functions:

$$\int_0^t\left\{\sum_{k=1}^3\int_D G^{jk}(t-\tau, x, y)w^k(\tau, y)dy\right\}d\tau. \tag{4.2}$$

It follows from (4.1) that

$$\boldsymbol{v}(t)=\boldsymbol{v}_1(t)-\boldsymbol{v}_2(t) \tag{4.3}$$

where

$$\begin{aligned}\boldsymbol{v}_1(t)&=\int_0^t G(t-\tau)\boldsymbol{w}(t)d\tau=\int_0^t G(\tau)\boldsymbol{w}(t)d\tau \quad \text{and}\\ \boldsymbol{v}_2(t)&=\int_0^t G(t-\tau)\{\boldsymbol{w}(t)-\boldsymbol{w}(\tau)\}d\tau.\end{aligned} \tag{4.4}$$

Since $\boldsymbol{w}(t)$ is expressible in the form:

$$\boldsymbol{w}(t)=\sum_{n=1}^\infty \alpha_n(t)\boldsymbol{g}_n \qquad \text{where} \qquad \sum_{n=1}^\infty \alpha_n(t)^2=\|\boldsymbol{w}(t)\|^2, \tag{4.5}$$

we have (see (3.23) and (3.25))

$$\begin{cases}\boldsymbol{v}_1(t)\equiv\boldsymbol{v}_1(t, x)=\displaystyle\sum_{n=1}^\infty \frac{1-e^{-\lambda_n t}}{\lambda_n}\alpha_n(t)\boldsymbol{g}_n(x) \qquad \text{and}\\ \boldsymbol{v}_2(t)\equiv\boldsymbol{v}_2(t, x)=\displaystyle\sum_{n=1}^\infty\int_0^t e^{-\lambda_n(t-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}\boldsymbol{g}_n(x)d\tau;\end{cases} \tag{4.6}$$

these formulae make sense in virtue of the following

LEMMA 4.1. *The series in the right-hand of each equality in* (4.6) *converges uniformly in* $[0, t_0)\times\bar{D}$ *for any* $t_0>0$.

PROOF. It follows from (3.19) and (3.16) that, for any $x\in\bar{D}$,

$$\left|\sum_{n=N}^\infty \frac{1-e^{-\lambda_n t}}{\lambda_n}\alpha_n(t)\boldsymbol{g}_n(x)\right|\leqq\|\boldsymbol{w}\|_t\left\{\sum_{n=N}^\infty\left|\frac{\boldsymbol{g}_n(x)}{\lambda_n}\right|^2\right\}^{1/2}\leqq M\cdot\|\boldsymbol{w}\|_t \tag{4.7}$$

and

$$\begin{aligned}&\sum_{n=N}^\infty\int_0^t\left|e^{-\lambda_n(t-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}\,\boldsymbol{g}_n(x)\right|d\tau\\ &\quad\leqq\int_0^t e^{-1}(t-\tau)^{-1}\|\boldsymbol{w}(t)-\boldsymbol{w}(\tau)\|\left\{\sum_{n=N}^\infty\left|\frac{\boldsymbol{g}_n(x)}{\lambda_n}\right|^2\right\}^{1/2}d\tau\end{aligned} \tag{4.8}$$

$$\leqq \int_0^t e^{-1}\|w\|_t'(t-\tau)^{-1/2}\tau^{-1/2}d\tau \cdot \left\{\sum_{n=N}^{\infty}\left|\frac{g_n(x)}{\lambda_n}\right|^2\right\}^{1/2}$$

$$=\pi e^{-1}\|w\|_t'\left\{\sum_{n=N}^{\infty}\left|\frac{g_n(x)}{\lambda_n}\right|^2\right\}^{1/2}\leqq M\pi e^{-1}\cdot\|w\|_t'.$$

Hence we obtain Lemma 4.1 by means of Lemma 3.4.

From (3.5), (3.6), (4.3) and the above lemma immediately follows that

COROLLARY. $v(t)\in C^0(\bar{D})$ and $v(t)|_S=0$ for any $t\geqq 0$.

LEMMA 4.2. *There exists a monotone function $\varepsilon_1(t)$ of $t>0$ independent of $w$ and such that*

$$\|v\|_t\leqq\varepsilon_1(t)(\|w\|_t+\|w\|_t') \quad \textit{and} \quad \lim_{t\downarrow 0}\varepsilon_1(t)=0.$$

PROOF. From (4.3), (4.6), (4.7) and (4.8) follows that

$$\|v\|_t\leqq M(\|w\|_t+\pi e^{-1}\|w\|_t') \quad \text{for any } t>0.$$

Hence it suffices to prove that, for any $\varepsilon>0$, there exists $\eta=\eta(\varepsilon)>0$ such that $\|v\|_t\leqq\varepsilon(\|w\|_t+\|w\|_t')$ whenever $0<t<\eta$. For any $\varepsilon>0$, there exists $N=N_\varepsilon$ such that

$$\left\{\sum_{n=N}^{\infty}\left|\frac{g_n(x)}{\lambda_n}\right|^2\right\}^{1/2}<\frac{e\varepsilon}{2\pi} \quad \text{for any } x\in\bar{D} \tag{4.9}$$

(by means of Lemma 3.4). We put $\eta=(2M\pi\lambda_N)^{-1}\varepsilon$. Then, if $0<t<\eta$, we have $1-e^{-\lambda_N t}\leqq\lambda_N t<\varepsilon/2M$, and hence

$$\begin{aligned}
&\sum_{n=1}^{N-1}\left|\frac{1-e^{-\lambda_n t}}{\lambda_n}\alpha_n(t)g_n(x)\right| \\
&\quad\leqq\left\{\sum_{n=1}^{N-1}(1-e^{-\lambda_N t})^2\alpha_n(t)^2\right\}^{1/2}\left\{\sum_{n=1}^{N-1}\left|\frac{g_n(x)}{\lambda_n}\right|^2\right\}^{1/2} \\
&\quad\leqq(1-e^{-\lambda_N t})\|w\|_t\cdot M\leqq\frac{\varepsilon}{2}\|w\|_t
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
&\sum_{n=1}^{N-1}\int_0^t\left|e^{-\lambda_n(t-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}g_n(x)\right|d\tau \\
&\leqq\lambda_N\int_0^t\left\{\sum_{n=1}^{N-1}|\alpha_n(t)-\alpha_n(\tau)|^2\right\}^{1/2}\left\{\sum_{n=1}^{N-1}\left|\frac{g_n(x)}{\lambda_n}\right|^2\right\}^{1/2}d\tau \\
&\leqq M\lambda_N\|w\|_t'\int_0^t(t-\tau)^{1/2}\tau^{-1/2}d\tau\leqq M\pi\lambda_N t\|w\|_t'\leqq\frac{\varepsilon}{2}\|w\|_t' \quad \text{(by (3.16))}.
\end{aligned}\tag{4.11}$$

It follows from (4.6–4.11) that

$$\begin{aligned}
\|v\|_t&\leqq\|v_1\|_t+\|v_2\|_t \\
&\leqq\left(\frac{\varepsilon}{2}\|w\|_t+\frac{e\varepsilon}{2\pi}\|w\|_t\right)+\left(\frac{\varepsilon}{2}\|w\|_t'+\frac{\varepsilon}{2}\|w\|_t'\right)
\end{aligned}$$

$$\leqq \varepsilon(\|\boldsymbol{w}\|_t+\|\boldsymbol{w}\|_t') \qquad \text{whenever } 0<t<\eta.$$

Lemma 4.2 is thus proved.

LEMMA 4.3. *There exists a monotone function $\varepsilon_2(t)$ of $t>0$ independent of $\boldsymbol{w}$ and such that*

$$\|\boldsymbol{v}\|_t'\leqq \varepsilon_2(t)(\|\boldsymbol{w}\|_t+\|\boldsymbol{w}\|_t') \qquad \textit{and} \qquad \lim_{t\downarrow 0}\varepsilon_2(t)=0.$$

PROOF. We first prove that

$$\|\boldsymbol{v}\|_t'\leqq M(e^{-1}\|\boldsymbol{w}\|_t+8\|\boldsymbol{w}\|_t') \qquad \text{for any } t>0. \tag{4.12}$$

By means of (3.20), we have following estimations for any $x\in\bar{D}$ whenever $t_1>t_2>0$ and $\sigma>0$:

$$\left\{\begin{aligned}&\sum_{n=1}^{\infty}\left|\frac{1-e^{-\lambda_n(t_2+\sigma)}}{\lambda_n}\alpha_n(t_1)\boldsymbol{g}_n(x)\right|\leqq M\cdot\|\boldsymbol{w}\|_{t_1},\\&\sum_{n=1}^{\infty}|e^{-\lambda_n(t_2+\sigma)}\alpha_n(t_1)\boldsymbol{g}_n(x)|\leqq Me^{-1}\|\boldsymbol{w}\|_{t_1}(t_2+\sigma)^{-1}\leqq(2e)^{-1}M\|\boldsymbol{w}\|_{t_1}t_2^{-1/2}\sigma^{-1/2},\end{aligned}\right. \tag{4.13}$$

$$\sum_{n=1}^{\infty}\frac{1-e^{-\lambda_n t_2}}{\lambda_n}|\alpha_n(t_1)-\alpha_n(t_2)|\cdot|\boldsymbol{g}_n(x)|\leqq M\|\boldsymbol{w}\|_{t_1}'|t_1-t_2|^{1/2}t_2^{-1/2}, \tag{4.14}$$

$$\left\{\begin{aligned}&\sum_{n=1}^{\infty}\int_0^t|e^{-\lambda_n(t_2+\sigma-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}\boldsymbol{g}_n(x)|d\tau\\&\leqq\int_0^t Me^{-1}(t_2+\sigma-\tau)^{-1}\|\boldsymbol{w}\|_t'(t-\tau)^{1/2}\tau^{-1/2}d\tau\\&\leqq Me^{-1}\|\boldsymbol{w}\|_t'\int_0^t(t-\tau)^{-1/2}\tau^{-1/2}d\tau\leqq M\pi e^{-1}\|\boldsymbol{w}\|_t',\\&\sum_{n=1}^{\infty}\int_0^t|\lambda_n e^{-\lambda_n(t+\sigma-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}\,\boldsymbol{g}_n(x)|d\tau\\&\leqq\int_0^t 4e^{-2}M(t+\sigma-\tau)^{-2}\|\boldsymbol{w}\|_t'(t-\tau)^{1/2}\tau^{-1/2}d\tau\\&\leqq 4e^{-2}M\|\boldsymbol{w}\|_t'\left[\int_0^{t/3}+\int_{t/3}^{t}\right](t+\sigma-\tau)^{-3/2}\tau^{-1/2}d\tau\\&\leqq 4e^{-2}M\|\boldsymbol{w}\|_t'\left\{\int_0^{t/3}\left(\frac{2t}{3}+\sigma\right)^{-3/2}\tau^{-1/2}d\tau+\int_{t/3}^{t}(t+\sigma-\tau)^{-3/2}\left(\frac{t}{3}\right)^{-1/2}d\tau\right\}\\&\leqq 4e^{-2}M\|\boldsymbol{w}\|_t'\left\{2t^{-1}(3\sigma)^{-1/2}\left(\frac{t}{3}\right)^{1/2}+2\sigma^{-1/2}\left(\frac{t}{3}\right)^{-1/2}\right\}^{5)}\\&\leqq 17e^{-2}M\|\boldsymbol{w}\|_t'\sigma^{-1/2}t^{-1/2}\qquad(t>0),\end{aligned}\right. \tag{4.15}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\int_0^{t_2}e^{-\lambda_n(t_1-\tau)}|\{\alpha_n(t_1)-\alpha_n(\tau)\}-\{\alpha_n(t_2)-\alpha_n(\tau)\}|\cdot|\boldsymbol{g}_n(x)|d\tau\\&\leqq\sum_{n=1}^{\infty}\frac{e^{-\lambda_n(t_1-t_2)}-e^{-\lambda_n t_1}}{\lambda_n}|\alpha_n(t_1)-\alpha_n(t_2)|\cdot|\boldsymbol{g}_n(x)|\end{aligned} \tag{4.16}$$

5) Here we use the fact: $\frac{2}{3}t+\sigma=\frac{1}{3}(t+t+3\sigma)\geqq(3\sigma t^2)^{1/3}$.

$$\leqq M\cdot\|\boldsymbol{w}(t_1)-\boldsymbol{w}(t_2)\|\leqq M\cdot\|\boldsymbol{w}\|'_{t_1}|t_1-t_2|^{1/2}t_2^{-1/2}$$

and

$$\sum_{n=1}^{\infty}\int_{t_2}^{t_1}e^{-\lambda_n(t_1-\tau)}|\alpha_n(t_1)-\alpha_n(\tau)|\cdot|\boldsymbol{g}_n(x)|d\tau \tag{4.17}$$

$$\leqq\int_{t_2}^{t_1}Me^{-1}(t_1-\tau)^{-1}\|\boldsymbol{w}\|'_{t_1}(t_1-\tau)^{1/2}\tau^{-1/2}d\tau$$

$$\leqq Me^{-1}\|\boldsymbol{w}\|'_{t_1}\int_{t_2}^{t_1}(t_1-\tau)^{-1/2}t_2^{-1/2}d\tau\leqq 2e^{-1}M\|\boldsymbol{w}\|'_{t_1}(t_1-t_2)^{1/2}t_2^{-1/2}.$$

It follows from (4.13) that the function

$$\boldsymbol{v}_1(t_1, t_2\,;\sigma, x)=\sum_{n=1}^{\infty}\frac{1-e^{-\lambda_n(t_2+\sigma)}}{\lambda_n}\alpha_n(t_1)\boldsymbol{g}_n(x) \tag{4.18}$$

is differentiable in $\sigma>0$ and satisfies

$$\left|\frac{\partial\boldsymbol{v}_1(t_1, t_2\,;\sigma, x)}{\partial\sigma}\right|\leqq(2e)^{-1}M\|\boldsymbol{w}\|_{t_1}t_2^{-1/2}\sigma^{-1/2}.$$

Hence

$$|\boldsymbol{v}_1(t_1, t_2, t_1-t_2, x)-\boldsymbol{v}_1(t_1, t_2, 0, x)|$$

$$\leqq\int_0^{t_1-t_2}\left|\frac{\partial\boldsymbol{v}_1(t_1, t_2, \sigma, x)}{\partial\sigma}\right|d\sigma\leqq Me^{-1}\|\boldsymbol{w}\|_{t_1}|t_1-t_2|^{1/2}t_2^{-1/2}.$$

Combining this inequality with (4.14), we obtain (see (4.6))

$$|\boldsymbol{v}_1(t_1, x)-\boldsymbol{v}_1(t_2, x)|\leqq M(e^{-1}\|\boldsymbol{w}\|_{t_1}+\|\boldsymbol{w}\|'_{t_1})|t_1-t_2|^{1/2}t_2^{-1/2}\qquad(t_1>t_2>0),$$

which implies that

$$\|\boldsymbol{v}_1\|'_t\leqq M(e^{-1}\|\boldsymbol{w}\|_t+\|\boldsymbol{w}\|'_t). \tag{4.19}$$

Similarly it follows from (4.15) that the function

$$\boldsymbol{v}_2(t\,;\sigma, x)=\sum_{n=1}^{\infty}\int_0^t e^{-\lambda_n(t+\sigma-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}\boldsymbol{g}_n(x)d\tau \tag{4.20}$$

satisfies

$$|\boldsymbol{v}_2(t_2\,;t_1-t_2, x)-\boldsymbol{v}_2(t_2\,;0, x)|\leqq 34e^{-2}M\|\boldsymbol{w}\|'_{t_2}|t_1-t_2|^{1/2}t_2^{-1/2}.$$

Combining this inequality with (4.16) and (4.17), we obtain (see (4.6))

$$|\boldsymbol{v}_2(t_1, x)-\boldsymbol{v}_2(t_2, x)|\leqq(34e^{-2}+1+2e^{-1})M\|\boldsymbol{w}\|'_{t_1}|t_1-t_2|^{1/2}t_2^{-1/2},$$

which implies that

$$\|\boldsymbol{v}_2\|'_t\leqq 7M\|\boldsymbol{w}\|'_t. \tag{4.21}$$

From (4.19) and (4.21), we obtain (4.12). Accordingly similar argument to the proof of Lemma 4.2 may be applied to estimations (4.13–17), and it may be proved that, for any $\varepsilon>0$, there exists $\eta=\eta(\varepsilon)>0$ such that $\|v\|'_t\leqq\varepsilon(\|w\|_t+\|w\|'_t)$ whenever $0<t<\eta$. This fact and (4.12) imply Lemma 4.3.

LEMMA 4.4. *$v\in\mathfrak{D}_\Gamma$ and* $\|\nabla v\|_t\leqq\nu^{-1/2}t^{1/2}\|w\|_t$.

PROOF. By virtue of the orthogonality relation of the system $\{g_n\}$ and that of $\{\nabla g_n\}$ (see (3.8) and (3.15)), we have

$$\left\|\sum_{n=N}^{\infty}e^{-\lambda_n(t-\tau)}\alpha_n(\tau)g_n\right\|^2\leqq\sum_{n=N}^{\infty}|\alpha_n(\tau)|^2$$

and

$$\left\|\sum_{n=N}^{\infty}\lambda_n^{1/2}e^{-\lambda_n(t-\tau)}\alpha_n(\tau)\frac{\nabla g_n}{\lambda_n^{1/2}}\right\|^2\leqq(2e\nu)^{-1}(t-\tau)^{-1}\sum_{n=N}^{\infty}|\alpha_n(\tau)|^2 \qquad \text{(by (3.18))}.$$

Hence $\nabla v(t)=\int_0^t\sum_{n=1}^{\infty}e^{-\lambda_n(t-\tau)}\alpha_n(\tau)\nabla g_n d\tau$ is well defined ($\nabla$ is understood in the generalized sense defined in §1) and

$$\|\nabla v(t)\|\leqq(2e\nu)^{-1/2}\|w\|_t\int_0^t(t-\tau)^{-1/2}d\tau\leqq\nu^{-1/2}t^{1/2}\|w\|_t,$$

which implies $\|\nabla v\|_t\leqq\nu^{-1/2}t^{1/2}\|w\|_t$, q. e. d.

LEMMA 4.5. $\|\nabla v\|'_t\leqq\nu^{-1/2}t^{1/2}(\|w\|_t+5\|w\|'_t)$.

PROOF. By similar argument to the proof of Lemma 4.4, we obtain from (4.6) that

$$\nabla v_1(t)=\sum_{n=1}^{\infty}\frac{1-e^{-\lambda_n t}}{\lambda_n^{1/2}}\alpha_n(t)\frac{\nabla g_n}{\lambda_n^{1/2}}=\sum_{n=1}^{\infty}\alpha_n(t)\frac{\nabla g_n}{\lambda_n^{1/2}}\int_0^t\lambda_n^{1/2}e^{-\lambda_n(t-\tau)}d\tau$$

and

$$\nabla v_2(t)=\sum_{n=1}^{\infty}\int_0^t\lambda_n^{1/2}e^{-\lambda_n(t-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}\frac{\nabla g_n}{\lambda_n^{1/2}}d\tau.$$

The following estimations are obtained by virtue of the orthogonality relation (3.15) of $\{\nabla g_n\}$ and by similar computations to those in the proof of Lemma 4.3:

$$\left\|\sum_{n=1}^{\infty}\lambda_n^{1/2}e^{-\lambda_n(t_2+\sigma)}\alpha_n(t_1)\frac{\nabla g_n}{\lambda_n^{1/2}}\right\|\leqq(2e\nu)^{-1/2}\|w\|_{t_1}\sigma^{-1/2}, \tag{4.22}$$

$$\left\|\sum_{n=1}^{\infty}\{\alpha_n(t_1)-\alpha_n(t_2)\}\frac{\nabla g_n}{\lambda_n^{1/2}}\int_0^{t_2}\lambda_n^{1/2}e^{-\lambda_n(t_2-\tau)}d\tau\right\|\leqq2^{1/2}(e\nu)^{-1/2}\|w\|'_{t_1}|t_1-t_2|^{1/2}, \tag{4.23}$$

$$\begin{aligned}\left\|\sum_{n=1}^{\infty}\int_0^t\lambda_n^{3/2}e^{-\lambda_n(t+\sigma-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}\frac{\nabla g_n}{\lambda_n^{1/2}}d\tau\right\|&\\ \leqq\int_0^t\left(\frac{3}{2e}\right)^{3/2}\nu^{-1/2}(t+\sigma-\tau)^{-3/2}\|w\|'_t(t-\tau)^{1/2}\tau^{-1/2}d\tau&\\ \leqq e^{-3/2}\nu^{-1/2}\|w\|'_t\int_0^t\sigma^{-1/2}(t-\tau)^{-1/2}\tau^{-1/2}d\tau&\\ =2e^{-3/2}\pi\nu^{-1/2}\|w\|'_t\sigma^{-1/2},&\end{aligned} \tag{4.24}$$

$$(4.25)\qquad \left\|\sum_{n=1}^{\infty}\int_0^{t_2}\lambda_n^{1/2}e^{-\lambda_n(t_1-\tau)}\{\alpha_n(t_1)-\alpha_n(t_2)\}\frac{\nabla g_n}{\lambda_n^{1/2}}d\tau\right\|$$
$$\leqq\int_0^{t_2}(2e\nu)^{-1/2}(t_2-\tau)^{-1/2}\|w\|'_{t_2}|t_1-t_2|^{1/2}t_2^{-1/2}d\tau$$
$$\leqq 2^{1/2}(e\nu)^{-1/2}\|w\|'_{t_1}|t_1-t_2|^{1/2}$$

and

$$(4.26)\qquad \left\|\sum_{n=1}^{\infty}\int_{t_2}^{t_1}\lambda_n^{1/2}e^{-\lambda_n(t_1-\tau)}\{\alpha_n(t_1)-\alpha_n(\tau)\}\frac{\nabla g_n}{\lambda_n^{1/2}}d\tau\right\|$$
$$\leqq 2^{1/2}(e\nu)^{-1/2}\|w\|'_{t_1}(t_1^{1/2}-t_2^{1/2})\leqq 2^{1/2}(e\nu)^{-1/2}\|w\|'_{t_1}|t_1-t_2|^{1/2}.$$

It follows from (4.22) and (4.24) that $v_1(t_1, t_2; \sigma, \cdot)$ defined by (4.18) satisfies

$$\left\|\frac{\partial}{\partial\sigma}\nabla v_1(t_1, t_2; \sigma, \cdot)\right\|\leqq(2e\nu)^{-1/2}\|w\|_{t_1}\sigma^{-1/2}$$

and that $v_2(t, \sigma, \cdot)$ defined by (4.20) satisfies

$$\left\|\frac{\partial}{\partial\sigma}\nabla v_2(t_2; \sigma, \cdot)\right\|\leqq e^{-3/2}\pi\nu^{-1/2}\|w\|'_{t_2}\sigma^{-1/2},$$

Hence

$$\|\nabla v_1(t_1, t_2; t_1-t_2, \cdot)-\nabla v_1(t_1, t_2; 0, \cdot)\|\leqq(2/e)^{1/2}\nu^{-1/2}\|w\|_{t_1}|t_1-t_2|^{1/2}$$

and

$$\|\nabla v_2(t_2; t_1-t_2, \cdot)-\nabla v_2(t_2, 0, \cdot)\|\leqq 2e^{-3/2}\pi\nu^{-1/2}\|w\|'_{t_2}|t_1-t_2|^{1/2}.$$

Combining these inequalities with (4.23), (4.25) and (4.26), we get

$$\|\nabla v_1\|'_t\leqq\nu^{-1/2}t^{1/2}(\|w\|_t+\|w\|'_t)$$

and

$$\|\nabla v_2\|'_t\leqq 4\nu^{-1/2}t^{1/2}\|w\|'_t.$$

Hence we obtain

$$\|\nabla v\|'_t\leqq\nu^{-1/2}t^{1/2}(\|w\|_t+5\|w\|'_t),\qquad \text{q. e. d.}$$

Next we put

$$(4.27)\qquad q(t)=\int_0^t H(t-\tau)w(\tau)d\tau$$

and

$$(4.28)\qquad \begin{cases} q_1(t)=\int_0^t H(t-\tau)w(t)d\tau=\int_0^t H(\tau)w(t)d\tau \\ q_2(t)=\int_0^t H(t-\tau)\{w(t)-w(\tau)\}d\tau. \end{cases}$$

Then

$$q(t)=q_1(t)-q_2(t) \tag{4.29}$$

(cf. (4.1), (4.3) and (4.4)), and we have (see (3.24), (3.26) and (3.2))

$$\left\{\begin{aligned} q_1(t)&\equiv q_1(t,x)=\sum_{n=1}^{\infty}\frac{1-e^{-\lambda_n t}}{\lambda_n}\alpha_n(t)p_n(x)=H[\sum_{n=1}^{\infty}(1-e^{-\lambda_n t})\alpha_n(t)\boldsymbol{g}_n](x),\\ q_2(t)&\equiv q_2(t,x)=\sum_{n=1}^{\infty}\int_0^t e^{-\lambda_n(t-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}p_n(x)d\tau\\ &=H[\sum_{n=1}^{\infty}\int_0^t\lambda_n e^{-\lambda_n(t-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}\boldsymbol{g}_n](x)d\tau\ ; \end{aligned}\right. \tag{4.30}$$

these equalities make sense by virtue of the following relations:

$$\left\{\begin{aligned} &\left\{\sum_{n=1}^{\infty}\alpha_n(t)^2\right\}^{1/2}\leqq\|\boldsymbol{w}\|_t<\infty\\ &\int_0^t\left(\sum_{n=1}^{\infty}[\lambda_n e^{-\lambda_n(t-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}]^2\right)^{1/2}d\tau\leqq e^{-1}\|\boldsymbol{w}\|_t'<\infty \end{aligned}\right. \tag{4.31}$$

(cf. (4.7) and (4.8)). Hence we have

$$\|q_1\|_t\leqq\|H\|\cdot\|\boldsymbol{w}\|_t \quad\text{and}\quad \|q_2\|_t\leqq\pi e^{-1}\|H\|\cdot\|\boldsymbol{w}\|_t'$$

where $\|H\|$ denotes the norm of $H$ as a bounded linear mapping of $\mathfrak{H}$ into $L^2(D)$. Similarly we may show by the same argument as the proof of Lemma 4.3 that

$$\|q_1\|_t'\leqq\|H\|(e^{-1}\|\boldsymbol{w}\|_t+\|\boldsymbol{w}\|_t') \quad\text{and}\quad \|q_2\|_t'\leqq 7\|H\|\cdot\|\boldsymbol{w}\|_t'.$$

Thus we obtain that

LEMMA 4.6.

$$\|q\|_t\leqq\|H\|\cdot(\|\boldsymbol{w}\|_t+\pi e^{-1}\|\boldsymbol{w}\|_t') \quad\text{and}\quad \|q\|_t'\leqq\|H\|(e^{-1}\|\boldsymbol{w}\|_t+8\|\boldsymbol{w}\|_t').$$

**§5. A solution of the equation $\partial v/\partial t=\nu\Delta v+w-\nabla q$ and some properties of $F_t(v)$.** Let $\boldsymbol{w}(t)$ be a given $\mathfrak{H}_0$-valued function of $t\in[0,T)$ where $0<T\leqq\infty$, and assume that $\|\boldsymbol{w}\|_t$ and $\|\boldsymbol{w}\|_t'$ are finite for any $t\in(0,T)$. We consider the linear equation

$$\frac{\partial \boldsymbol{v}}{\partial t}=\nu\cdot\Delta\boldsymbol{v}+\boldsymbol{w}-\nabla q \qquad (0<t<T), \tag{5.1}$$

and we shall show that a 'weak solution' $\{\boldsymbol{v},q\}$ of (5.1) is given by

$$\boldsymbol{v}(t)=\int_0^t G(t-\tau)\boldsymbol{w}(\tau)d\tau \tag{5.2}$$

and

$$q(t)=\int_0^t H(t-\tau)\boldsymbol{w}(\tau)d\tau \tag{5.3}$$

and that, if $w(t)\equiv w(t, x)$ is Hölder-continuous in $(0, T)\times D$, the system $\{v, q\}$ is a classical solution of (5.1).

LEMMA 5.1. *Assume that* $u=u(t)$ *be a* $\mathfrak{H}_0$*-valued function continuous in* $t\in[t_1, t_2]$ *with respect to the norm in* $\mathfrak{H}_0$, *and put*

$$u_N(t)=\sum_{n=1}^{N}\beta_n(t)g_n \qquad \textit{where} \qquad \beta_n(t)=(u(t), g_n).$$

*Then*

$$\lim_{N\to\infty}\sup_{t_1\leq t\leq t_2}\|u_N(t)-u(t)\|=0$$

(namely $\{u_N(t)\,;\,N=1, 2,\cdots\}$ converges to $u(t)$ as $N\to\infty$ uniformly in $t$ with respect to the norm in $\mathfrak{H}_0$).

PROOF. The function $\|u_N(t)\|^2=\sum_{n=1}^{N}\beta_n(t)^2$ is continuous in $t\in[t_1, t_2]$ for each $N$, the sequence $\{\|u_N(t)\|^2; N=1, 2,\cdots\}$ is monotone increasing and converges to $\|u(t)\|^2$ for any fixed $t$, and $\|u(t)\|^2$ is also continuous in $t\in[t_1, t_2]$. Hence $\|u_N(t)\|^2$ converges to $\|u(t)\|^2$ uniformly in $t\in[t_1, t_2]$. From this fact and the orthogonality relation of $\{g_n\}$, it follows that

$$\lim_{N\to\infty}\|u_N(t)-u(t)\|^2=\lim_{N\to\infty}\{\|u(t)\|^2-\|u_N(t)\|^2\}=0$$

and the convergence is uniform in $t\in[t_1, t_2]$, q. e. d.

We put, for $N=1, 2,\cdots$,

$$w_N(t)=\sum_{n=1}^{N}\alpha_n(t)g_n \qquad \text{where} \qquad \alpha_n(t)=(w(t), g_n), \tag{5.4}$$

$$v_N(t)=\int_0^t G(t-\tau)w_N(\tau)d\tau \tag{5.5}$$

and

$$q_N(t)=\int_0^t H(t-\tau)w_N(\tau)d\tau. \tag{5.6}$$

Then, by Lemma 5.1, it holds that

$$\lim_{N\to\infty}\|w_N-w\|_t=0 \tag{5.7}$$

and accordingly

$$\lim_{N\to\infty}\|v_N-v\|_t=0 \qquad \text{(by Lemma 4.2)} \tag{5.8}$$

and

$$\lim_{N\to\infty}\|\nabla v_N-\nabla v\|_t=0 \quad \text{(by Lemma 4.4)} \tag{5.9}$$

for any $t$. Further we have

$$v(t)=\sum_{n=1}^{\infty}\int_0^t e^{-\lambda_n(t-\tau)}\alpha_n(\tau)g_n d\tau, \tag{5.10}$$

$$v_N(t)=\sum_{n=1}^{N}\int_0^t e^{-\lambda_n(t-\tau)}\alpha_n(\tau)g_n d\tau \tag{5.11}$$

and

$$q_N(t)=\sum_{n=1}^{N}\int_0^t e^{-\lambda_n(t-\tau)}\alpha_n(\tau)p_n d\tau. \tag{5.12}$$

Hence each $v_N(t)$ has its strong derivative $dv_N(t)/dt$ in $\mathfrak{H}_0$ and

$$\begin{aligned}\frac{dv_N(t)}{dt}&=\sum_{n=1}^{N}\int_0^t(-\lambda_n)e^{-\lambda_n(t-\tau)}\alpha_n(\tau)g_n d\tau+\sum_{n=1}^{N}\alpha_n(t)g_n\\&=\sum_{n=1}^{N}\int_0^t e^{-\lambda_n(t-\tau)}\alpha_n(\tau)(\nu\cdot\Delta g_n-\nabla p_n)d\tau+\sum_{n=1}^{N}\alpha_n(t)g_n \qquad \text{(by (3.9))}\\&=\nu\Delta v_N+w_N-\nabla q_N.\end{aligned} \tag{5.13}$$

We have also that

$$\frac{dv_N(t)}{dt}=\sum_{n=1}^{N}e^{-\lambda_n t}\alpha_n(t)g_n+\sum_{n=1}^{N}\int_0^t\lambda_n e^{-\lambda_n(t-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}g_n d\tau. \tag{5.14}$$

On the other hand, we may see by means of (4.13) that

$$\hat{v}(t)=\sum_{n=1}^{\infty}e^{-\lambda_n t}\alpha_n(t)g_n+\sum_{n=1}^{\infty}\int_0^t\lambda_n e^{-\lambda_n(t-\tau)}\{\alpha_n(t)-\alpha_n(\tau)\}g_n d\tau \tag{5.15}$$

is well defined for any $t\in[0, T)$ and satisfies

$$\|\hat{v}\|_t\leqq\|w\|_t+\pi e^{-1}\|w\|_t' \qquad \text{for any } t\in(0, T). \tag{5.16}$$

Furthermore we may prove from (5.15) and by similar computations to those in the proof of Lemma 4.3 that

$$\|\hat{v}\|_t'\leqq e^{-1}\|w\|_t+8\|w\|_t' \qquad \text{for any } t\in(0, T). \tag{5.17}$$

Here we show the following

LEMMA 5.2. *The function $v(t)$ has its strong derivative $dv(t)/dt$ in $\mathfrak{H}_0$ whenever $0<t<T$, and satisfies that*

$$\left\|\frac{dv}{dt}\right\|_t<\infty \textit{ and } \left\|\frac{dv}{dt}\right\|_t'<\infty \qquad \textit{for any } t\in(0, T) \tag{5.18}$$

*and that*

$$\lim_{N\to\infty}\sup_{t_1\leqq t\leqq t_2}\left\|\frac{dv_N(t)}{dt}-\frac{dv(t)}{dt}\right\|=0 \quad \textit{whenever} \quad 0<t_1<t_2<T. \tag{5.19}$$

PROOF. The function $\hat{v}(t)$ defined by (5.15) is strongly continuous in $[t_1, t_2]$ by means of (5.16) and (5.17). Hence, by Lemma 5.1 (see also (5.14)), we have

$$\lim_{N\to\infty} \sup_{t_1\leq t\leq t_2} \left\| \frac{dv_N(t)}{dt} - \hat{v}(t) \right\| = 0. \tag{5.20}$$

It follows from (5.8) and (5.20) that $v(t)$ has its strong derivative $dv(t)/dt$ in $\mathfrak{H}_0$ whenever $0<t<T$ and $dv(t)/dt=\hat{v}(t)$; this fact implies (5.18) and (5.19), q. e. d.

LEMMA 5.3. i) *$q(t)\in C^\infty(D)$ and $\Delta q(t, \cdot)=0$ in $D$;* ii) *$\nabla q(t, x)$ is Hölder-continuous in $(0, T)\times D$;* iii) *$\lim_{N\to\infty} \sup_{t_1<t<t_2} \|q_N(t)-q(t)\|=0$ whenever $0<t_1<t_2<T$.*

PROOF. By virtue of (4.31), Lemma 4.6 and Lemma 5.1, we may show part iii) of this lemma by the same argument to the proof of (5.20). On the other hand, each $q_N(t)$ belongs to $C^\infty(D)$ and satisfies $\Delta q_N(t, \cdot)=0$ in $D$ by means of (5.12) and Lemma 3.7. Hence

$$(q(t), \Delta\varphi)=\lim_{N\to\infty}(q_N(t), \Delta\varphi)=0 \qquad \text{for any} \qquad \varphi\in C_0^\infty(D),$$

which implies part i) by Weyl's lemma [17]. Accordingly we obtain part ii) by means of Lemma 2.8 and Lemma 4.6.

LEMMA 5.4 (Fundamental lemma I).

$$\frac{d}{dt}(v, \Psi)=\left(\frac{dv}{dt}, \Psi\right)=\nu(v, \Delta\Psi)+(w, \Psi)-(\nabla q, \Psi) \tag{5.21}$$

*for any $\Psi\in C_0^2(D)$ whenever $0<t<T$.*

PROOF. The first equality in (5.21) is obvious from Lemma 5.2. The second equality is proved as follows. We obtain from (5.13) that

$$\left(\frac{dv_N}{dt}, \Psi\right)=\nu(v_N, \Delta\Psi)+(w_N, \Psi)+(q_N, \operatorname{div}\Psi) \qquad (N=1, 2, \cdots).$$

Letting $N\to\infty$ and using (5.7), (5.8), (5.19) and Lemma 5.3, wet get

$$\left(\frac{dv}{dt}, \Psi\right)=\nu(v, \Delta\Psi)+(w, \Psi)+(q, \operatorname{div}\Psi)=\nu(v, \Delta\Psi)+(w, \Psi)-(\nabla q, \Psi), \qquad \text{q. e. d.}$$

LEMMA 5.5. (Fundamental lemma II). i) *$v(t)\in C^0(\bar{D})\cap C^1(D)$, $v(t)|_S=0$ and $\operatorname{div} v(t)=0$ in $D$ for any $t\in(0, T)$;* ii) *$v(t, x)$ and $\nabla v(t, x)$ are Hölder-continuous in $(0, T)\times D$;* iii) *if especially $w(t)\equiv w(t, x)$ is Hölder-continuous in $(0, T)\times D$, then $\dfrac{dv}{dt}$ and $\dfrac{\partial^2 v}{\partial x^j \partial x^k}$ $(j, k=1, 2, 3)$ exist and are continuous in $(0, T)\times D$.*

PROOF. $v(t)\in C^0(\bar{D})$ and $v(t)|_S=0$ are already proved (see Corollary to Lemma 4.1). Let $D_0$, $D_1$, $D_2$ and $D_3$ be arbitrary subdomains of $D$ such that

$$\bar{D}_0\subset D_1\subset \bar{D}_1\subset D_2\subset \bar{D}_2\subset D_3\subset \bar{D}_3\subset D,$$

and let $\omega_1(x)$ and $\omega_2(x)$ be functions of class $C^\infty$ in $D$ satisfying

$$\omega_1(x)=1 \text{ in } \bar{D}_0, \qquad \omega_1(x)=0 \text{ in } D-D_1$$

and

$$\omega_2(x)=1 \text{ in } \bar{D}_2, \qquad \omega_2(x)=0 \text{ in } D-D_3 .$$

We put

$$K(t, x, y)=(4\pi\nu t)^{-3/2}\omega_1(x)\omega_2(y)\exp(-r_{xy}^2/4\nu t) \qquad (t>0;\, x, y\in D) \tag{5.22}$$

and

$$J(t, x, y)=\begin{cases} \left[\nu\Delta_y-\dfrac{\partial}{\partial t}\right]K(t, x, y)^{6)} & (t>0) \\ 0 & (t\leqq 0). \end{cases} \tag{5.23}$$

Then $J(t, x, y)$ is of class $C^\infty$ in the region $\{-\infty<t<\infty;\, x, y\in D\}$. We may see from (5.21) that each component $v^j(t, x)$ of the vector function $\boldsymbol{v}(t, x)$ satisfies

$$\frac{\partial}{\partial\tau}\int_D K(t-\tau, x, y)v^j(\tau, y)dy$$
$$=\int_D J(t-\tau, x, y)v^j(\tau, y)dy+\int_D K(t-\tau, x, y)\left\{w^j(\tau, y)+\frac{\partial q(\tau, y)}{\partial y^j}\right\}dy.$$

We fix arbitrary $t_0\in(0, T)$ and assume that $t>t_0$. Then, integrating both sides of the above equality in the interval: $t_0<\tau<t$, we get

$$\begin{aligned} v^j(t, x)=&\int_{D_3} K(t-t_0, x, y)v^j(t_0, y)dy+\int_{t_0}^t d\tau\int_{D_3-D_2} J(t-\tau, x, y)w^j(\tau, y)dy \\ &+\int_{t_0}^t d\tau\int_{D_3} K(t-\tau, x, y)\left\{w^j(\tau, y)+\frac{\partial q(\tau, y)}{\partial y^j}\right\}dy \qquad \text{for } x\in D_0 \end{aligned} \tag{5.24}$$

by virtue of (5.18) and part ii) of Lemma 5.3. Hence by virtue of the fact that $\|\boldsymbol{w}\|_t$ and $\|\boldsymbol{w}\|_t'$ are finite and by the arbitrariness of $D_0$ and $t_0$, we may see that $v^j(t, \cdot)\in C^1(D)$ for any $t\in(0, T)$. On the other hand, $\boldsymbol{v}(t)$ clearly belongs to $\mathfrak{H}_0$ by (5.10). Hence $\operatorname{div}\boldsymbol{v}(t)=0$ in $D$ by Lemma 2.3. The Hölder-continuity of $\boldsymbol{v}(t, x)$ and that of $\nabla\boldsymbol{v}(t, x)$ in $(0, T)\times D$ also follow from (5.22–24) and the arbitrariness of $D_0$ and $t_0$. Finally assume that $\boldsymbol{w}(t, x)$ is Hölder-continuous in $(0, T)\times D$. Then $w^j(\tau, y)+\partial q(\tau, y)/\partial y^j$ is Hölder-continuous in $(0, T)\times D$ for each $j$. Hence it follows from (5.22–24) that $\partial\boldsymbol{v}/\partial t$ and $\partial^2\boldsymbol{v}/\partial x^j\partial x^k$ $(j, k=1, 2, 3)$ exist and are continuous in $(t_0, T)\times D_0$ and accordingly in $(0, T)\times D$, q. e. d.

We next investigate the nonlinear mapping $\boldsymbol{F}_t(\boldsymbol{v})$ defined by (2.37). Let $\boldsymbol{w}(t)$ and $\boldsymbol{v}(t)$ be $\mathfrak{H}_0$-valued functions stated above (the Hölder-continuity of $\boldsymbol{w}(t)$ assumed in part iii) of Lemma 5.5 will not be used), and put

$$\tilde{\boldsymbol{w}}(t)=\boldsymbol{F}_t(\boldsymbol{v}(t)). \tag{5.25}$$

6) The subscript $y$ to $\Delta$ means to operate $\Delta$ to $K(t, x, y)$ as a function of $y$.

Then

LEMMA 5.6. *$\tilde{\boldsymbol{w}}(t)\in\mathfrak{H}$ for any $t\in[0, T]$, and we have*

$$\|\tilde{\boldsymbol{w}}\|_t\leqq\|\boldsymbol{W}\|_t+\|\nabla\boldsymbol{U}\|_t\cdot|\boldsymbol{v}|_t+|\boldsymbol{U}|_t\cdot\|\nabla\boldsymbol{v}\|_t+|\boldsymbol{v}|_t\cdot\|\nabla\boldsymbol{v}\|_t \tag{5.26}$$

*and*

$$\begin{aligned}\|\tilde{\boldsymbol{w}}\|_t'\leqq\|\boldsymbol{W}\|_t'+\|\nabla\boldsymbol{U}\|_t'\cdot|\boldsymbol{v}|_t+\|\nabla\boldsymbol{U}\|_t\cdot|\boldsymbol{v}|_t'+|\boldsymbol{U}|_t'\cdot\|\nabla\boldsymbol{v}\|_t+|\boldsymbol{U}|_t\cdot\|\nabla\boldsymbol{v}\|_t'\\+|\boldsymbol{v}|_t'\cdot\|\nabla\boldsymbol{v}\|_t+|\boldsymbol{v}|_t\cdot\|\nabla\boldsymbol{v}\|_t'.\end{aligned} \tag{5.27}$$

This lemma immediately follows from (2.30) and (2.35).

LEMMA 5.7. *There exists a function $\varphi=\varphi(t, x)$ on $(0, T)\times D$ such that $\varphi(t, \cdot)\in C^2(D)$, $P_1\tilde{\boldsymbol{w}}=\nabla\varphi$ and $\Delta\varphi=2(\nabla\boldsymbol{U}:\nabla\boldsymbol{v})+(\nabla\boldsymbol{v}:\nabla\boldsymbol{v})$ for any $t$.*

PROOF. We define $\boldsymbol{w}_N(t)$ and $\boldsymbol{v}_N(t)$ by (5.4) and (5.5), and put $\tilde{\boldsymbol{w}}_N(t)=\boldsymbol{F}_t(\boldsymbol{v}_N(t))$ $(N=1, 2, \cdots)$. Then $\operatorname{div}\boldsymbol{v}_N=0$ and $\tilde{\boldsymbol{w}}_N(t)-\boldsymbol{W}(t)\in\mathfrak{H}\cap C^1(D)$ by (2.29), (3.5), (3.6) and (5.11), and accordingly

$$\operatorname{div}(\tilde{\boldsymbol{w}}_N-\boldsymbol{W})=2(\nabla\boldsymbol{U}:\nabla\boldsymbol{v}_N)+(\nabla\boldsymbol{v}_N:\nabla\boldsymbol{v}_N)\in C^1(D). \tag{5.28}$$

Hence, by (5.8) and (5.9), we have

$$\lim_{N\to\infty}\|\tilde{\boldsymbol{w}}_N-\tilde{\boldsymbol{w}}\|_t=0 \tag{5.29}$$

and

$$\lim_{N\to\infty}\int_D|\operatorname{div}(\tilde{\boldsymbol{w}}_N-\boldsymbol{W})-\{2(\nabla\boldsymbol{U}:\nabla\boldsymbol{v})+(\nabla\boldsymbol{v}:\nabla\boldsymbol{v})\}|dx=0 \qquad \text{for any fixed } t. \tag{5.30}$$

It follows from (2.34), (5.28) and by Lemma 2.6 that

$$P_1\tilde{\boldsymbol{w}}_N=P_1(\tilde{\boldsymbol{w}}_N-\boldsymbol{W})=\nabla\varphi_N \tag{5.31}$$

for a suitable function $\varphi_N=\varphi_N(t, x)$ such that $\Delta\varphi_N=\operatorname{div}(\tilde{\boldsymbol{w}}_N-\boldsymbol{W})$ for any $t$. Let $Q_z(\rho)$ and $K_z(x, y; \rho)$ be as defined in § 2 for any $z\in D$ and $\rho>0$. Then, if $Q_z(3R)\subset D$, we have

$$\varphi_N(t, x)=-\int_{D_z(\rho)}K_z(x, y; \rho)\operatorname{div}\{\tilde{\boldsymbol{w}}_N(t, y)-\boldsymbol{W}(t, y)\}dy+\int_{S(\rho)}K_z(x, y; \rho)\frac{\partial\varphi_N(t, y)}{\partial\boldsymbol{n}_\rho}dS_y$$

for any $x\in Q_z(R)$ and $\rho>R$ where $S(\rho)=\partial Q_z(\rho)$ (see the proof of Lemma 2.7), and accordingly

$$\begin{aligned}\varphi_N(t, x)=&-\frac{1}{R}\int_{2R}^{3R}d\rho\int_{Q_z(\rho)}K_z(x, y; \rho)\operatorname{div}\{\tilde{\boldsymbol{w}}_N(t, y)-\boldsymbol{W}(t, y)\}dy\\&+\frac{1}{R}\int_{2R}^{3R}d\rho\int_{S(\rho)}K_z(x, y; \rho)(P_1\tilde{\boldsymbol{w}}_N(t)\cdot\boldsymbol{n}_\rho)(y)dS_y\end{aligned}$$

since $\dfrac{\partial\varphi_N}{\partial\boldsymbol{n}_\rho}=(\nabla\varphi_N\cdot\boldsymbol{n}_\rho)=(P_1\tilde{\boldsymbol{w}}_N\cdot\boldsymbol{n}_\rho)$ by (5.31). Therefore, if we put

$$\varphi_{zR}(t, x) = -\frac{1}{R}\int_{2R}^{3R} d\rho \int_{Q_z(\rho)} K_z(x, y; \rho)\{2(\nabla U(t) : \nabla v(t))(y) + (\nabla v(t) : \nabla v(t))(y)\}\, dy$$
$$+\frac{1}{R}\int_{2R}^{3R} d\rho \int_{S(\rho)} K_z(x, y; \rho)(P_1\tilde{w}(t)\cdot n_\rho)(y)\, dS_y,$$

we may see that

(5.32) $\qquad \varphi_{zR} \in C^2(Q_z(R))$ and $\Delta\varphi_{zR} = 2(\nabla U : \nabla v) + (\nabla v : \nabla v)$ in $Q_z(R)$

for any $t$ (since $\nabla U$ and $\nabla v$ are Hölder-continuous in $x$ by (2.31) and part ii) of Lemma 5.5), and also that

$$\lim_{N\to\infty} |\nabla\varphi_N(t, x) - \nabla\varphi_{zR}(t, x)| = 0 \text{ for any } \langle t, x\rangle \in (0, T)\times Q_z(R)$$

by means of (5.29), (5.30) and by similar estimation to (2.13). On the other hand

$$\lim_{N\to\infty} \|\nabla\varphi_N - P_1\tilde{w}\|_t = \lim_{N\to\infty} \|P_1(\tilde{w}_N - \tilde{w})\|_t = 0$$

by (5.31) and (5.29). Hence we get

$$P_1\tilde{w} = \nabla\varphi_{zR} \text{ whenever } x \in Q_z(R).$$

Since $z$ is an arbitrary point in $D$, we have

$$P_1\tilde{w} \in C^1(D) \text{ and } \operatorname{div} P_1\tilde{w} = 2(\nabla U : \nabla v) + (\nabla v : \nabla v) \text{ for any } t$$

by (5.32). Hence, by Lemma 2.1, there exists a function $\varphi \equiv \varphi(t, x)$ on $(0, T)\times D$ such that

$$\varphi(t, \cdot) \in C^2(D) \text{ and } P_1\tilde{w} = \nabla\varphi \qquad \text{for any } t$$

and accordingly that

$$\Delta\varphi = \operatorname{div} P_1\tilde{w} = 2(\nabla U : \nabla v) + (\nabla v : \nabla v), \qquad \text{q. e. d.}$$

COROLLARY. *$P_0\tilde{w}$ and $P_1\tilde{w}$ are Hölder-continuous in* $(0, T)\times D$.

PROOF. It follows from (2.30), (2.31) and part ii) of Lemma 5.5 that $\tilde{w}$ and $\Delta\varphi$ are Hölder-continuous in $(0, T)\times D$. Hence, by Lemma 2.7, $P_1\tilde{w} \equiv \nabla\varphi$ is Hölder-continuous in $(0, T)\times D$, and accordingly so is $P_0\tilde{w} \equiv \tilde{w} - \nabla\varphi$.

**§6. Construction of a solution of the Navier-Stokes equation.** Let $w(t)$ be a $\mathfrak{H}_0$-valued function of $t \in [0, \infty)$ satisfying $\|w\|_t < \infty$ and $\|w\|_t' < \infty$ for any $t > 0$, and put

$$v(t) = \int_0^t G(t-\tau)w(\tau)d\tau \quad \text{and} \quad \tilde{w}(t) = F_t(v(t)).$$

Then, by means of Lemmas 4.2–4.5, we have

$$\left.\begin{array}{l}\|\boldsymbol{v}\|_t\\ \|\boldsymbol{v}\|'_t\\ \|\nabla\boldsymbol{v}\|_t\\ \|\nabla\boldsymbol{v}\|'_t\end{array}\right\}\leqq\varepsilon(t)(\|\boldsymbol{w}\|_t+\|\boldsymbol{w}\|'_t) \tag{6.1}$$

where $\varepsilon(t)$ is a monotone function of $t>0$, independent of $\boldsymbol{w}$ and satisfying

$$\lim_{t\searrow 0}\varepsilon(t)=0. \tag{6.2}$$

It follows from (6.1) and Lemma 5.6 that

$$\|\tilde{\boldsymbol{w}}\|_t\leqq\|\boldsymbol{W}\|_t+\varepsilon(t)(\|\boldsymbol{w}\|_t+\|\boldsymbol{w}\|'_t)(\|\nabla\boldsymbol{U}\|_t+\|\boldsymbol{U}\|_t)+\varepsilon(t)^2(\|\boldsymbol{w}\|_t+\|\boldsymbol{w}\|'_t)^2 \tag{6.3}$$

and

$$\|\tilde{\boldsymbol{w}}\|'_t\leqq\|\boldsymbol{W}\|'_t+\varepsilon(t)(\|\boldsymbol{w}\|_t+\|\boldsymbol{w}\|'_t)(\|\nabla\boldsymbol{U}\|_t+\|\nabla\boldsymbol{U}\|'_t+\|\boldsymbol{U}\|_t+\|\boldsymbol{U}\|'_t)+2\varepsilon(t)^2(\|\boldsymbol{w}\|_t+\|\boldsymbol{w}\|'_t)^2 \tag{6.4}$$

for any $t>0$. We put

$$A_t=\|\boldsymbol{W}\|_t+\|\nabla\boldsymbol{U}\|_t+\|\boldsymbol{U}\|_t+1 \tag{6.5}$$

and

$$A'_t=\|\boldsymbol{W}\|'_t+\|\nabla\boldsymbol{U}\|_t+\|\nabla\boldsymbol{U}\|'_t+\|\boldsymbol{U}\|_t+\|\boldsymbol{U}\|'_t+2. \tag{6.6}$$

Then $A_T$ and $A'_t$ are finite for any $t>0$ by (2.30) and (2.35), and increase when $t$ increases. Hence, by virtue of (6.2), there exists a positive number $T$ for which $\varepsilon(T)\leqq(A_T+3A'_T)^{-1}$ holds. (We choose such $T$ as large as possible.) Accordingly, from (6.3)-(6.6) immediately follows that

LEMMA 6.1. *If* $\|\boldsymbol{w}\|_T<A_T$ *and* $\|\boldsymbol{w}\|'_T<A'_T$, *then* $\|\tilde{\boldsymbol{w}}\|_T<A_T$ *and* $\|\tilde{\boldsymbol{w}}\|'_T<A'_T$.

Next, let $\boldsymbol{w}_n(t)$ $(n=1, 2)$ be $\mathfrak{H}_0$-valued functions of $t\in[0, \infty)$ satisfying $\|\boldsymbol{w}_n\|_t<\infty$ $\|\boldsymbol{w}_n\|'_t<\infty$ for any $t>0$, and put

$$\boldsymbol{v}_n(t)=\int_0^t G(t-\tau)\boldsymbol{w}_n(\tau)d\tau \text{ and } \tilde{\boldsymbol{w}}_n(t)=\boldsymbol{F}_t(\boldsymbol{v}_n(t)) \qquad (n=1, 2).$$

Then it follows from (6.1), (6.3) and (6.4) that

$$\begin{aligned}\|\tilde{\boldsymbol{w}}_1-\tilde{\boldsymbol{w}}_2\|_t\leqq&\,\varepsilon(t)(\|\boldsymbol{w}_2-\boldsymbol{w}_1\|_t+\|\boldsymbol{w}_2-\boldsymbol{w}_1\|'_t)(\|\nabla\boldsymbol{U}\|_t+\|\boldsymbol{U}\|_t)\\ &+\varepsilon(t)^2(\|\boldsymbol{w}_2-\boldsymbol{w}_1\|_t+\|\boldsymbol{w}_2-\boldsymbol{w}_1\|'_t)(\|\boldsymbol{w}_2\|_t+\|\boldsymbol{w}_2\|'_t+\|\boldsymbol{w}_1\|_t+\|\boldsymbol{w}_1\|'_t)\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}\|\tilde{\boldsymbol{w}}_1-\tilde{\boldsymbol{w}}_2\|_t\leqq&\,\varepsilon(t)(\|\boldsymbol{w}_2-\boldsymbol{w}_1\|_t+\|\boldsymbol{w}_2-\boldsymbol{w}_1\|'_t)(\|\nabla\boldsymbol{U}\|_t+\|\nabla\boldsymbol{U}\|'_t+\|\boldsymbol{U}\|_t+\|\boldsymbol{U}\|'_t)\\ &+2\varepsilon(t)^2(\|\boldsymbol{w}_2-\boldsymbol{w}_1\|_t+\|\boldsymbol{w}_2-\boldsymbol{w}_1\|'_t)(\|\boldsymbol{w}_2\|_t+\|\boldsymbol{w}_2\|'_t+\|\boldsymbol{w}_1\|_t+\|\boldsymbol{w}_1\|'_t)\end{aligned} \tag{6.8}$$

for any $t>0$. Let $A_t$, $A'_t$ and $T$ be as mentioned above. Then

LEMMA 6.2. *If* $\|\boldsymbol{w}_n\|_T<A_T$ *and* $\|\boldsymbol{w}_n\|'_T<A'_T$ $(n=1, 2)$, *then*

$$\|\tilde{\boldsymbol{w}}_2-\tilde{\boldsymbol{w}}_1\|_T+\|\tilde{\boldsymbol{w}}_2-\tilde{\boldsymbol{w}}_1\|'_T<\frac{6}{7}(\|\boldsymbol{w}_2-\boldsymbol{w}_1\|_T+\|\boldsymbol{w}_2-\boldsymbol{w}_1\|'_T). \tag{6.9}$$

PROOF. Since $\|\boldsymbol{w}_2\|_T+\|\boldsymbol{w}_2\|'_T+\|\boldsymbol{w}_1\|_T+\|\boldsymbol{w}_1\|'_T<2(A_T+A'_T)$ (by the assumption), we obtain from (6.7) and (6.8) that

$$\|\tilde{\boldsymbol{w}}_1-\tilde{\boldsymbol{w}}_2\|_T\leqq\varepsilon(T)(\|\nabla\boldsymbol{U}\|_T+\|\boldsymbol{U}\|_T+2)(\|\boldsymbol{w}_2-\boldsymbol{w}_1\|_T+\|\boldsymbol{w}_2-\boldsymbol{w}_1\|'_T)$$

and

$$\|\tilde{\boldsymbol{w}}_1-\tilde{\boldsymbol{w}}_2\|'_T\leqq\varepsilon(T)(\|\nabla\boldsymbol{U}\|_T+\|\nabla\boldsymbol{U}\|'_T+\|\boldsymbol{U}\|_T+\|\boldsymbol{U}\|'_T+4)(\|\boldsymbol{w}_2-\boldsymbol{w}_1\|_T+\|\boldsymbol{w}_2-\boldsymbol{w}_1\|'_T).$$

Hence we have

$$\begin{aligned}&\|\tilde{\boldsymbol{w}}_2-\tilde{\boldsymbol{w}}_1\|_T+\|\tilde{\boldsymbol{w}}_2-\tilde{\boldsymbol{w}}_1\|'_T\\ &\leqq(A_T+3A'_T)^{-1}(2\|\nabla\boldsymbol{U}\|_T+2\|\boldsymbol{U}\|_T+\|\nabla\boldsymbol{U}\|'_T+\|\boldsymbol{U}\|'_T+6)(\|\boldsymbol{w}_2-\boldsymbol{w}_1\|_T+\|\boldsymbol{w}_2-\boldsymbol{w}_1\|'_T).\end{aligned}$$

Since $A_T+3A'_T=\|\boldsymbol{W}\|_T+3\|\boldsymbol{W}\|'_T+4\|\boldsymbol{U}\|_T+4\|\nabla\boldsymbol{U}\|_T+3\|\nabla\boldsymbol{U}\|'_T+3\|\boldsymbol{U}\|'_T+7$, the above inequality implies (6.9), q. e. d.

Using the above results, we shall construct a solution $\{\boldsymbol{v}, q\}$ of (2.36) and prove Theorem 1. We first define two sequences $\{\boldsymbol{v}_n(t)\}$ and $\{\boldsymbol{w}_n(t)\}$ of $\mathfrak{H}_0$-valued functions on $[0, T]$ by means of successive substitution as follows:

$$\boldsymbol{v}_0(t)\equiv 0 \tag{6.10}$$

and

$$\left.\begin{aligned}\boldsymbol{w}_n(t)&=P_0\boldsymbol{F}_t(\boldsymbol{v}_{n-1}(t))\\ \boldsymbol{v}_n(t)&=\int_0^t G(t-\tau)\boldsymbol{w}_n(\tau)d\tau\end{aligned}\right\}\quad(n=1, 2, \cdots). \tag{6.11}$$

Then we may see by means of Lemma 6.1 and by mathematical induction that

$$\|\boldsymbol{w}_n\|_T<A_T \text{ and } \|\boldsymbol{w}_n\|'_T<A'_T \qquad (n=1, 2, \cdots), \tag{6.12}$$

and accordingly, by Lemma 6.2, that

$$\|\boldsymbol{w}_{n+1}-\boldsymbol{w}_n\|_T+\|\boldsymbol{w}_{n+1}-\boldsymbol{w}_n\|'_T\leqq\frac{6}{7}(\|\boldsymbol{w}_n-\boldsymbol{w}_{n-1}\|_T+\|\boldsymbol{w}_n-\boldsymbol{w}_{n-1}\|'_T). \tag{6.13}$$

Hence there exists a $\mathfrak{H}_0$-valued function $\boldsymbol{w}(t)$ on $[0, T]$ such that

$$\|\boldsymbol{w}\|_T\leqq A_T, \qquad \|\boldsymbol{w}\|'_T\leqq A'_T \tag{6.14}$$

and

$$\lim_{n\to\infty}\|\boldsymbol{w}_n-\boldsymbol{w}\|_T=\lim_{n\to\infty}\|\boldsymbol{w}_n-\boldsymbol{w}\|'_T=0. \tag{6.15}$$

Therefore, if we put

$$\boldsymbol{v}(t)=\int_0^t G(t-\tau)\boldsymbol{w}(\tau)d\tau \quad\text{and}\quad \tilde{\boldsymbol{w}}(t)=\boldsymbol{F}_t(\boldsymbol{v}(t)), \tag{6.16}$$

we have

$$\lim_{n\to\infty} \|v_n - v\|_T = \lim_{n\to\infty} \|v_n - v\|'_T = 0$$

and

$$\lim_{n\to\infty} \|\Gamma v_n - \Gamma v\|_T = \lim_{n\to\infty} \|\Gamma v_n - \Gamma v\|'_T = 0$$

by Lemmas 4.2–4.5, and accordingly

$$\lim_{n\to\infty} \|w_n - P_0\tilde{w}\|_T = \lim_{n\to\infty} \|w_n - P_0\tilde{w}\|'_T = 0$$

by Lemma 5.6 (The difference $w_n - P_0\tilde{w}$ does not contain the term $W$). Hence we have, by (6.15),

$$w(t) = P_0\tilde{w}(t) \qquad \text{for any } t \in [0, T]. \tag{6.17}$$

Furthermore, by Lemma 5.7, there exists a function $\varphi_2 = \varphi_2(t, x)$ on $(0, T)\times D$ such that

$$\varphi_2(t, \cdot) \in C^2(D) \quad \text{and} \quad P_1\tilde{w} = \Gamma\varphi_2 \qquad \text{for any } t. \tag{6.18}$$

We define $\tilde{q}(t) \equiv \tilde{q}(t, x)$ and $q(t, x)$ as follows:

$$\tilde{q}(t) = \int_0^t H(t-\tau) w(\tau) d\tau \qquad \text{(see (5.3) and Lemma 5.3)}, \tag{6.19}$$

$$q(t, x) = \tilde{q}(t, x) + \varphi_2(t, x). \tag{6.20}$$

We shall prove that $\{v, q\}$ is a solution of (2.36) in usual sense; this fact essentially implies i) and ii) in Theorem 1 as is shown in §2. By means of Lemma 5.4 and part i) of Lemma 5.5, we have that

$$\frac{d}{dt}(v, \Psi) = \left(\frac{dv}{dt}, \Psi\right) = \nu(v, \Delta\Psi) + (w, \Psi) - (\Gamma\tilde{q}, \Psi) \tag{6.21}$$

for any $\Psi \in C_0^2(D)$ whenever $0 < t < T$ and that

$$v(t) \in C^0(\bar{D}) \cap C^1(D), \ v|_S = 0 \text{ and } \operatorname{div} v = 0 \text{ in } D \tag{6.22}$$

for any $t \in (0, T)$. On the other hand, it follows from (6.17) and Corollary to Lemma 5.7 that $w = w(t, x)$ is Hölder-continuous in $(0, T)\times D$. Hence, by part iii) of Lemma 5.5, $\frac{\partial v}{\partial t}$ and $\frac{\partial^2 v}{\partial x^j \partial x^k}$ $(j, k = 1, 2, 3)$ exist and are continuous in $(0, T)\times D$. Therefore

$$\frac{d}{dt}(v, \Psi) = \left(\frac{dv}{dt}, \Psi\right) = \left(\frac{\partial v}{\partial t}, \Psi\right)^{7)} \qquad \text{for any } \Psi \in C_0^2(D),$$

7) $\frac{dv}{dt}$ denotes the strong derivative of $v(t) = v(t, \cdot)$ as a $\mathfrak{H}_0$-valued function of $t \in (0, T)$, while $\frac{\partial v}{\partial t}$ denotes the usual partial derivative of $v(t, x)$ as an $R^3$-valued function of $\langle t, x\rangle \in (0, T)\times D$.

while it follows from (6.17), (6.18) and (6.20) that

$$\boldsymbol{w}-\nabla\tilde{q}=\tilde{\boldsymbol{w}}-\nabla\varphi_2-\nabla\tilde{q}=\boldsymbol{F}_t(\boldsymbol{v})-\nabla q$$

Hence (6.21) implies that

$$\frac{d\boldsymbol{v}(t)}{dt}=\frac{\partial\boldsymbol{v}(t,\cdot)}{\partial t} \quad\text{and}\quad \frac{d\boldsymbol{v}}{dt}=\nu\cdot\Delta\boldsymbol{v}+\boldsymbol{F}_t(\boldsymbol{v})-\nabla q.$$

Thus we have proved (2.36).

If we put

$$\text{(6.23)}\qquad \boldsymbol{u}=\boldsymbol{U}+\boldsymbol{v} \quad\text{and}\quad p=\varphi_0+\varphi_1+q,$$

then it is clear that $\boldsymbol{u}(t,x)$ and $p(t,x)$ satisfy i) and ii) in Theorem 1. Since $\left\|\frac{d\boldsymbol{v}}{dt}\right\|_t<\infty$ (see Lemma 5.2) and $\|\nabla\boldsymbol{v}\|_t<\infty$ for any $t\in(0,T)$, we obtain that $\frac{d\boldsymbol{u}(t)}{dt}=\frac{\partial\boldsymbol{u}(t,\cdot)}{\partial t}\in\mathfrak{H}$, $\left\|\frac{d\boldsymbol{u}}{dt}\right\|_t<\infty$ and $\|\nabla\boldsymbol{u}\|_t<\infty$ for any $t\in(0,T)$. Hence it remains only to prove that $p-\varphi_0\in L^2(D)$, namely

$$\text{(6.24)}\qquad \varphi_1(t)+\varphi_2(t)+\tilde{q}(t)\in L^2(D) \qquad \text{(see (6.20) and (6.23))}$$

for any $t\in(0,T)$. However $\tilde{q}(t)\in L^2(D)$ by Lemma 4.6, while it follows from (2.33) and (6.18) that

$$\nabla\{\varphi_1(t)+\varphi_2(t)\}\in\mathfrak{H}$$

which implies $\varphi_1(t)+\varphi_2(t)\in L^2(D)$. Hence we obtain (6.24). Proof of Theorem 1 is thus complete.

## §7. The uniqueness of the solution.

LEMMA 7.1. *For any* $\boldsymbol{w}\in\boldsymbol{C}^1(\bar{D})$ *satisfying* $\boldsymbol{w}|_S=0$, *there exists a sequence* $\{\boldsymbol{w}_m\}\subset\boldsymbol{C}_0^1(D)$ *such that*

$$\text{(7.1)}\qquad \lim_{m\to\infty}\|\boldsymbol{w}_m-\boldsymbol{w}\|=0 \quad\textit{and}\quad \lim_{m\to\infty}\|\nabla\boldsymbol{w}_m-\nabla\boldsymbol{w}\|=0.$$

PROOF. Let $\{D_m\}$ be such a sequence of subdomains of $D$ as stated in Lemma 2.2, where we may assume that the distance between $S_m$ $(=\partial D_m)$ and $S$ is between $\frac{1}{2m}$ and $\frac{1}{m}$ for any $m$. Then, for each $m$, there exists a function $\varphi_m\in C_0^1(D)$ such that $\varphi_m(x)\equiv1$ in $D_m$ and $|\nabla\varphi_m|\leqq4m$. We put $\boldsymbol{w}_m(x)=\varphi_m(x)\boldsymbol{w}(x)$. Then, by virtue of the assumption: $\boldsymbol{w}\in\boldsymbol{C}^1(\bar{D})$, we may easily show (7.1).

LEMMA 7.2. *Assume that* $\boldsymbol{w}\in\boldsymbol{C}^1(D)\cap\boldsymbol{C}^0(\bar{D})\cap\mathfrak{H}_0\cap\mathfrak{D}_\Gamma$ *and* $\boldsymbol{w}|_S=0$, *and put*

$$\text{(7.2)}\qquad \boldsymbol{w}_N=\sum_{n=1}^{N}\alpha_n\boldsymbol{g}_n \qquad \textit{where}\ \alpha_n=(\boldsymbol{w},\boldsymbol{g}_n).$$

*Then*

$$\lim_{N\to\infty}\|\boldsymbol{w}_N-\boldsymbol{w}\|=0 \quad \textit{and} \quad \lim_{N\to\infty}\|\nabla\boldsymbol{w}_N-\nabla\boldsymbol{w}\|=0. \tag{7.3}$$

PROOF. From (7.2) immediately follows that $\lim_{n\to\infty}\|\boldsymbol{w}_N-\boldsymbol{w}\|=0$. Since $\nabla p_n-\nu\Delta\boldsymbol{g}_n=\lambda_n\boldsymbol{g}_n\in\boldsymbol{C}^2(D)\cap\boldsymbol{C}^1(\bar{D})\cap\mathfrak{H}_0$ and $\|p_n\|<\infty$ by (3.9) and Lemma 3.5, we obtain by Lemma 2.4 that

$$\nu(\nabla\boldsymbol{w},\nabla\boldsymbol{g}_n)=(\boldsymbol{w},\lambda_n\boldsymbol{g}_n)=\lambda_n\alpha_n.$$

Hence

$$\sum_{n=1}^{\infty}\lambda_n\alpha_n^2\leqq\nu\|\nabla\boldsymbol{w}\|^2 \qquad \text{(by Lemma 3.6)},$$

accordingly

$$\lim_{N,N'\to\infty}\|\nabla\boldsymbol{w}_N-\nabla\boldsymbol{w}_{N'}\|=0.$$

Hence it follows from the definition of generalized $\nabla$ (see § 1) that

$$\lim_{N\to\infty}\|\nabla\boldsymbol{w}_N-\nabla\boldsymbol{w}\|=0.$$

LEMMA 7.3. *For any $\boldsymbol{w}\in\boldsymbol{C}^1(D)\cap\boldsymbol{C}^0(\bar{D})\cap\mathfrak{H}_0\cap\mathfrak{D}_\Gamma$ satisfying $\boldsymbol{w}|_S=0$, there exists a sequence $\{\tilde{\boldsymbol{w}}_N\}\subset\boldsymbol{C}_0^1(D)$ such that*

$$\lim_{N\to\infty}\|\tilde{\boldsymbol{w}}_N-\boldsymbol{w}\|=0 \quad \textit{and} \quad \lim_{N\to\infty}\|\nabla\tilde{\boldsymbol{w}}_N-\nabla\boldsymbol{w}\|=0. \tag{7.4}$$

PROOF. We define $\boldsymbol{w}_N$ $(N=1, 2, \cdots)$ by (7.2). Then $\boldsymbol{w}_N$'s are in $\boldsymbol{C}^1(\bar{D})$ as so are $\boldsymbol{g}_n$'s. Hence, by Lemma 7.1, there exists $\tilde{\boldsymbol{w}}_N\in\boldsymbol{C}_0^1(D)$ for each $N$ such that $\|\tilde{\boldsymbol{w}}_N-\boldsymbol{w}_N\|<1/N$ and $\|\nabla\tilde{\boldsymbol{w}}_N-\nabla\boldsymbol{w}_N\|<1/N$. Hence we obtain (7.4) by Lemma 7.2.

LEMMA 7.4. *Assume that $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ belong to $\boldsymbol{C}^1(D)\cap\boldsymbol{C}^0(\bar{D})$, that $\operatorname{div}\boldsymbol{u}=0$ in $D$ and that at least one of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ vanishes on $S$. Then* i) $((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{w})=-((\boldsymbol{u}\cdot\nabla)\boldsymbol{w}, \boldsymbol{v})$; ii) *if especially $\boldsymbol{v}=\boldsymbol{w}$ then both sides of the above equality are equal to zero.*

The assertion i) may be proved by means of partial integration. The assertion ii) immediately follows from i).

LEMMA 7.5. *If $\{\boldsymbol{u}, p\}$ satisfies all conditions stated in Theorem* 1 *and if $\boldsymbol{w}\in\boldsymbol{C}^1(D)\cap\boldsymbol{C}^0(\bar{D})\cap\mathfrak{H}_0\cap\mathfrak{D}_\Gamma$ and $\boldsymbol{w}|_S=0$, then*

$$\left(\frac{d\boldsymbol{u}}{dt},\boldsymbol{w}\right)+\nu(\nabla\boldsymbol{u},\nabla\boldsymbol{w})=((\boldsymbol{u}\cdot\nabla)\boldsymbol{w},\boldsymbol{u})+(\boldsymbol{f}_0,\boldsymbol{w}). \tag{7.5}$$

PROOF. By Lemma 7.3, there exists a sequence $\{\tilde{\boldsymbol{w}}_N\}\subset\boldsymbol{C}_0^1(D)$ satisfying (7.4). Since $\{\boldsymbol{u}, p\}$ satisfies the equation (1.14) and since $\dfrac{d\boldsymbol{u}(t)}{dt}=\dfrac{\partial\boldsymbol{u}(t,\cdot)}{\partial t}$, we have

$$(7.6)\qquad \nu\cdot\Delta u-\nabla(p-\varphi_0)=\frac{du}{dt}+(u\cdot\nabla)u-f_0\in\mathfrak{H}\qquad \text{(see (1.20))}.$$

Hence, letting $N\to\infty$ in the identity:

$$(\nu\cdot\Delta u-\nabla(p-\varphi_0),\ \tilde{w}_N)=-\nu(\nabla u,\ \nabla\tilde{w}_N)+(p-\varphi_0,\ \operatorname{div}\tilde{w}_N),$$

and remembering iv) in Theorem 1, we obtain

$$(\nu\cdot\Delta u-\nabla(p-\varphi_0),\ w)=-\nu(\nabla u,\ \nabla w).$$

On the other hand, since $((u\cdot\nabla)u,\ w_N)=-((u\cdot\nabla)w_N,\ u)$ $(N=1, 2,\cdots)$ by Lemma 7.4, we have $((u\cdot\nabla)u,\ w)=-((u\cdot\nabla)w,\ u)$. Hence it follows from (7.6) that

$$-\nu(\nabla u,\ \nabla w)=\left(\frac{du}{dt},\ w\right)-((u\cdot\nabla)w,\ u)-(f_0,\ w),$$

which implies (7.5).

Now we prove Theorem 2 (uniqueness theorem) as follows. Assume that $\{u,\ p\}$ and $\{v,\ q\}$ satisfy all conditions stated in Theorem 1, and put $w=u-v$. Then, for any fixed $t\in(0,\ T)$, $w\in C^1(D)\cap C^0(\bar{D})\cap\mathfrak{D}_\Gamma$, $\operatorname{div}w=0$ in $D$ and $w|_S=0$, and accordingly $w\in\mathfrak{H}_0$. Hence, by Lemma 7.5, we have

$$(7.5')\qquad \left(\frac{du}{dt},\ w\right)+\nu(\nabla u,\ \nabla w)=((u\cdot\nabla)w,\ u)+(f_0,\ w)$$

and

$$(7.5'')\qquad \left(\frac{dv}{dt},\ w\right)+\nu(\nabla v,\ \nabla w)=((v\cdot\nabla)w,\ v)+(f_0,\ w)\,.$$

Subtracting (7.5″) from (7.5′) term by term, we get

$$(7.7)\qquad \left(\frac{dw}{dt},\ w\right)+\nu\|\nabla w\|^2=((w\cdot\nabla)w,\ u)+((v\cdot\nabla)w,\ w)\,;$$

here the last term vanishes by part ii) of Lemma 7.4. By virtue of condition ii) in Theorem 1, $C=\|u\|_T$ is a finite constant. Hence

$$((w\cdot\nabla)w,\ u)\leq C\|w\|\cdot\|\nabla w\|\leq\nu\|\nabla w\|^2+\frac{C}{4\nu}\|w\|^2$$

for any $t\in(0,\ T)$. On the other hand, it is clear that

$$\left(\frac{dw}{dt},\ w\right)=\frac{1}{2}\cdot\frac{d}{dt}\|w\|^2.$$

Hence it follows from (7.7) that

$$\frac{d}{dt}\|w(t)\|^2\leq\frac{C}{2\nu}\|w(t)\|^2\qquad (0<t<T).$$

Since $\lim_{t\downarrow 0}\|w(t)\|^2=0$ (see ii) in Theorem 1), the above differential inequality implies $\|w(t)\|=0$ for any $t\in(0, T)$. Hence we get $u(t, x)\equiv v(t, x)$ and accordingly $\nabla p(t, x)\equiv\nabla q(t, x)$ in $(0, T)\times D$. Theorem 2 is thus proved.

Finally we show that our solution coincides with Kiselev-Ladyzhenskaia's solution [8] under our assumptions (1.18-21) with $b\equiv 0$. Let $\{u, p\}$ be the solution constructed in §6 and $\{v, q\}$ be Kiselev-Ladyzhenskaia's solution. Then, for any fixed $t$, $u(t)\in C^1(D)\cap C^0(\bar{D})\cap\mathfrak{H}_0\cap\mathfrak{D}_\Gamma$ (since $u|_S=0$) and accordingly Lemma 7.3 may be applied to $u(t)$, while it may be seen from the argument in [8] that there exists a sequence $\{v_N(t)\}\subset C_0^1(D)$, for almost every $t\in(0, T)$, such that

$$\lim_{N\to\infty}\|v_N(t)-v(t)\|=0 \quad\text{and}\quad \lim_{N\to\infty}\|\nabla v_N(t)-\nabla v(t)\|=0.$$

Hence, if we put $w(t)=u(t)-v(t)$, there exists a sequence $\{\tilde{w}_N(t)\}\subset C_0^1(\bar{D})$ satisfying (7.4) for almost every $t$. Hence we obtain (7.5′) for almost all $t$, while the results of [8] implies (7.5)″ for almost all $t$. Hence, as we have shown above, it holds that

$$\left(\frac{dw(t)}{dt}, w(t)\right)\leqq\frac{C}{4\nu}\|w(t)\|^2 \qquad \text{for almost all } t$$

($C\equiv\|u\|_T<\infty$ since $u$ is *our* solution), and accordingly

$$\|w(t)\|^2=\int_0^t 2\left(\frac{dw(\tau)}{d\tau}, w(\tau)\right)d\tau\leqq\frac{C}{2\nu}\int_0^t\|w(\tau)\|^2 d\tau \qquad (0<t<T).$$

Since $\lim_{t\downarrow 0}\|w(t)\|^2=0$, the above relation implies $\|w(t)\|\equiv 0$.

Combining this result with Theorem 1 and Kiselev-Ladyzhenskaia's result [8], we may easily derive Theorem 3.

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