# *Integration of the generalized Kolmogorov-Feller backward equations*

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We consider, in the Banach space of continuous functions, the initial value problem of the integro-differential equation of the type

(0.1)
$$\frac{\partial u(t, x)}{\partial t}=Au(t, x)$$
$$Au(x)=\sum_{i,j}a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j}+\sum_i b_i(x)\frac{\partial u(x)}{\partial x_i}+c(x)u(x)$$
$$+\int\left[u(x+\xi)-u(x)-\sum_i \frac{\partial u(x)}{\partial x_i}\varphi(\xi_i)\right]g(x, \xi)G(d\xi)\ .$$

Here $a_{ij}(x)$ is symmetric and non-negative definite (not necessarily positive definite) at every point $x$, $\varphi(\xi_i)$ is a bounded function coinciding with $\xi_i$ near $\xi_i=0$, $g(x, \xi)$ is non-negative, and $G(d\xi)$ is possibly of infinite total measure. In the one-dimensional case, the equation (0.1) is the temporally homogeneous version of the backward equation considered by A. Kolmogorov [9], W. Feller [1] and K. Itô [6]. K. Itô's treatment is very general, but we use an entirely different method not confined to one dimension. In the spatially homogeneous case, that is, in the case where $a_{ij}$, $b_i$, $c$ and $g$ are constant, G. A. Hunt [5] proved the existence of the semi-group generated by $A$.

Here we sketch our method of integration. We use the Hille-Yosida theory of semi-groups [4, 12], and the problem is reduced to solving the equation

(0.2) $$(\alpha-A)u=f$$

for large $\alpha$ and dense $f$. We define an operator $A_n$ from $A$, replacing $a_{ii}(x)$ by $a_{ii}(x)+1/n$ and cutting off that part of the measure $G$ which lies on the $1/n$-neighborhood of the origin so that the total measure becomes finite. Then $A_n$ is the sum of a non-degenerated elliptic differential operator and a bounded operator. So we can solve

$$(\alpha-A_n)u_n=f$$

for smooth $f$. If we can prove $\|Au_n-A_nu_n\|\to 0$ $(n\to\infty)$, then (0.2) is solved when the right side $f$ is replaced by a function near $f$ in the sense of the norm, namely, by $f-(Au_n-A_nu_n)$. The estimation of $\|Au_n-A_nu_n\|$, however, is difficult in general, so that our treatment is restricted to two special cases. The first is the case where the coefficients in $A$ are periodic (i.e. $x$ space is a circle, a torus or the like in higher dimensions)[1] and moreover some of them are constant in some directions. The second is the case where $x$ space is a real line and the coefficients satisfy certain conditions near $+\infty$ and $-\infty$.

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## 1. Integration on a circle.

Let $C$ be the set of all real-valued continuous functions on the real line with period $2\pi$, and $C^n$ be the set of all $n$-times continuously differentiable functions belonging to $C$. $C$ is a Banach space with the norm $\|u\|=\max_x|u(x)|$. Suppose that the following integro-differential operator is given:

$$\begin{aligned} Au(x)&=a(x)u''(x)+b(x)u'(x)+c(x)u(x) \\ &\quad+\int_{R_0}[u(x+\xi)-u(x)-u'(x)\xi]g(x,\ \xi)G(d\xi)\ , \end{aligned} \tag{1.1}$$

where $R_0=[-\pi,\ 0)\cup(0,\ \pi)$. Here we assume that

(i) $a,\ b,\ c\in C^2$, $a(x)$ is non-negative and $a''(x)$ is uniformly Hölder continuous;

(ii) $g(x,\ \xi)$ is non-negative, belongs to $C^2$ as a function of $x$, and is measurable as a function of $\xi$, $G(d\xi)$ is a measure on $R_0$ and there exist functions $g_i(\xi)$ $(i=0,\ 1,\ 2)$ satisfying

$$g(x,\ \xi)\leqq g_0(\xi) \tag{1.2}$$

$$\left|\frac{\partial g(x,\ \xi)}{\partial x}\right|\leqq g_1(\xi) \tag{1.3}$$

$$\left|\frac{\partial^2 g(x,\ \xi)}{\partial x^2}\right|\leqq g_2(\xi) \tag{1.4}$$

and

$$\int_{R_0}\xi^2 g_i(\xi)G(d\xi)<+\infty \qquad (i=0,\ 2) \tag{1.5}$$

$$\int_{R_0}|\xi|g_1(\xi)G(d\xi)<+\infty\ . \tag{1.6}$$

Since $u\in C^2$ implies $Au\in C$, we take $C^2$ as the domain $\mathrm{D}(A)$. We prepare a lemma before the statement of the theorem.

LEMMA 1.1. *A has the smallest closed extension* $\bar{A}$.

PROOF.[2] We have to show that $u_n\in C^2$ $(n=1,\ 2,\ \cdots)$, $v\in C$, $\|u_n\|\to 0$, and $\|Au_n-v\|\to 0$ imply $v=0$. Define

$$A^0u(x)=Au(x)-c(x)u(x) \qquad \text{for } u\in C^2.$$

Then $\|A^0u_n-v\|\to 0$. Suppose $v\neq 0$. We can assume that $v$ is positive at some

1) Periodicity assumption was suggested by K. Yosida.

2) Suggested by the lemma of А.Д. Вентцель [15].

point $x_0$, without losing generalities. There exist positive $\varepsilon$ and $r$ such that for any sufficiently large $n$

$$A^0 u_n(x) > \varepsilon \qquad \text{when } |x-x_0| \leqq r.$$

Take $\psi \in C^2$ satisfying $\psi(x_0)=0$ and $\psi(x) \geqq \delta > 0$ when $r \leqq |x-x_0| \leqq \pi$ and put

$$\tilde{u}_n(x) = u_n(x) - \frac{\varepsilon}{K}\psi(x)\,, \qquad \text{where } K > \max_{|y-x_0|\leqq r} |A^0\psi(y)|\,.$$

Then,

$$A^0\tilde{u}_n(x) = A^0 u_n(x) - \frac{\varepsilon}{K} A^0\psi(x) > A^0 u_n(x) - \varepsilon > 0\,,$$

when $|x-x_0| \leqq r$. Thus $\tilde{u}_n(x) < \max_{|y-x_0|\geqq r} \tilde{u}_n(y)$ $(|x-x_0|<r)$, since $A^0 u(x)$ is not positive at the point where $u$ reaches its maximum. This implies

$$u_n(x_0) < \max_{|y-x_0|\geqq r} u_n(y) - \frac{\varepsilon}{K}\delta\,,$$

which contradicts that $\|u_n\|$ tends to zero.

THEOREM 1. *Under the assumptions* (i) *and* (ii), $\bar{A}$ *is the infinitesimal generator of a uniquely determined strongly continuous semi-group* $\{T_t;\ t \geqq 0\}$ *of bounded linear operators on* $C$. $T_t$ *is positivity-preserving*

$$(1.7) \qquad T_t f \geqq 0 \qquad \textit{if} \quad f \geqq 0,$$

*and satisfies*

$$(1.8) \qquad \|T_t\| \leqq e^{\gamma t}$$

*where*

$$(1.9) \qquad \gamma = \max_x c(x)\,.$$

By "*a strongly continuous semi-group*" we mean that

$$T_0 = \text{identity operator},$$

$$T_t T_s = T_{t+s} \qquad \text{for any } t,\ s \geqq 0,$$

and

$$\|T_{t+s}f - T_t f\| \to 0 \quad (s \downarrow 0) \qquad \text{for any } f \in C \text{ and } t \geqq 0.$$

Before the proof, we prepare two lemmas.

LEMMA 1.2. *Let*

$$Bu(x)=p(x)u''(x)+q(x)u'(x)+r(x)u(x) + \int_{R_0}[u(x+\xi)-u(x)-u'(x)\xi]h(x,\xi)H(d\xi)$$

*where* $p(x)$ *and* $h(x,\xi)$ *are non-negative and* $\int_{R_0}\xi^2h(x,\xi)H(d\xi)<+\infty$. *If* $u\in C^2$, $f\in C$, $\alpha>\beta=\max_x r(x)$, *and* $(\alpha-B)u=f$, *then*

$$\|u\|\leq\frac{1}{\alpha-\beta}\|f\| .$$

This lemma is easily proved by the fact that if $u$ reaches its positive maximum at $x_0$ then

$$Bu(x_0)\leq\beta u(x_0)$$

holds.

LEMMA 1.3. *Let* $\{S_t;\ t\geq 0\}$ *be a strongly continuous semi-group on* $C$ *with the infinitesimal generator* $M_1$ *and let* $M_2$ *be a bounded linear operator on* $C$. *Suppose that there exists a constant* $\beta$ *such that*

$$(M_1+M_2)u(x_0)\leq\beta u(x_0) \quad \text{if } u \text{ reaches its positive maximum at } x_0. \tag{1.10}$$

*Then* $M_1+M_2$ (*with* $\mathrm{D}(M_1+M_2)=\mathrm{D}(M_1)$) *is the infinitesimal generator of a strongly continuous semi-group* $\{\tilde{S}_t;\ t\geq 0\}$ *satisfying*

$$\|\tilde{S}_t\|\leq e^{\beta t} . \tag{1.11}$$

PROOF. According to the Hille-Yosida theorem [4, 12], we need only prove the following three facts for any sufficiently large $\alpha$. (i) $\mathrm{D}(M_1+M_2)$ is dense in $C$. (ii) For any $f\in C$

$$(\alpha-M_1-M_2)u=f \tag{1.12}$$

has a solution $u$. (iii) If (1.12) holds, then

$$\|u\|\leq\frac{1}{\alpha-\beta}\|f\| .$$

(i) is obvious. (iii) is proved by (1.10) similarly to the proof of Lemma 1.2. The proof of (ii) is as follows. Since $M_1$ is the infinitesimal generator of a semi-group, the resolvent operator $G_\alpha=(\alpha-M_1)^{-1}$ is defined everywhere on $C$ and $\|G_\alpha\|\to 0$ $(\alpha\to+\infty)$. Hence (1.12) is equivalent to

$$u-G_\alpha M_2u=G_\alpha f .$$

The solution of this equation is obtained by the so-called C. Neumann's series because $\|G_\alpha M_2\|$ $(\leq\|G_\alpha\|\cdot\|M_2\|)$ is smaller than 1 when $\alpha$ is large, which completes the proof.

Lemma 1.3 implies that (1.12) has a solution for any $\alpha$ larger than $\beta$. The fact that the value $\|M_2\|$ has no influence shall be used later.

PROOF OF THEOREM 1. The following four facts are to be proved for any sufficiently large $\alpha$. (i) The domain $D(\bar{A})$ is dense in $C$. (ii) For any $f\in C$

$$(\alpha-\bar{A})u=f \tag{1.13}$$

has a solution in $D(\bar{A})$. (iii) If (1.13) holds, then

$$\|u\|\leq \frac{1}{\alpha-\gamma}\|f\| . \tag{1.14}$$

(iv) If (1.13) holds and $f\geqq 0$, then $u\geqq 0$.

Among them (i) is obvious. In order to prove (ii)—(iv), it suffices to show the following (ii′)—(iv′). (ii′) The range $R(\alpha-A)$, namely $\{(\alpha-A)u;\ u\in C^2\}$, is dense in $C$. (iii′) If

$$(\alpha-A)u=f \tag{1.15}$$

holds, then (1.14) holds. (iv′) If (1.15) holds and $f\geqq 0$, then $u\geqq 0$.

(iii′) was proved in Lemma 1.2. If (1.15) holds, $f\geqq 0$, and $u$ reaches its negative minimum at $x_0$, then

$$f(x_0)=(\alpha-A)u(x_0)\leqq(\alpha-\gamma)u(x_0)<0 ,$$

which is absurd. Hence (iv′) is proved.

In order to prove (ii′), define

$$A_n u(x)=\left(a(x)+\frac{1}{n}\right)u''(x)+b(x)u'(x)+c(x)u(x) \tag{1.16}$$

$$+\int_{R_n}[u(x+\xi)-u(x)-u'(x)\xi]g(x,\xi)G(d\xi) \qquad \text{for } u\in C^2,$$

$$D_n u(x)=\left(a(x)+\frac{1}{n}\right)u''(x)+b_n(x)u'(x) \qquad \text{for } u\in C^2,$$

and

$$J_n u(x)=c(x)u(x)+\int_{R_n}[u(x+\xi)-u(x)]g(x,\xi)G(d\xi) \qquad \text{for } u\in C$$

where $R_n=[-\pi,\ -1/n)\cup(1/n,\ \pi)$ and $b_n(x)=b(x)-\int_{R_n}\xi g(x,\xi)G(d\xi)$ . Obviously $A_n=D_n+J_n$. $D_n$ is the infinitesimal generator of a strongly continuous semi-group. For, $D_n$ is a non-degenerated elliptic differential operator and our assumptions are sufficient to apply the result of S. Itô [7, 8] (see also W. Feller [1, 2] and E. Hille [3]). $J_n$ is a bounded operator, and (1.10) holds when $M_1+$

$M_2$ is replaced by $A_n$ and $\beta$ by $\gamma$. Accordingly, by virtue of Lemma 1.3, $A_n$ (with the domain $C^2$) is the infinitesimal generator of a strongly continuous semi-group satisfying (1.8). Take an $f$ from $C^2$ and a positive $\varepsilon$ arbitrarily. We will show that there is such a function $u \in C^2$ that

$$\|(\alpha - A)u - f\| < \varepsilon , \tag{1.17}$$

which completes the proof. Up to the present we know that for $\alpha > \gamma$ there exists a solution $u_n \in C^2$ of

$$(\alpha - A_n)u_n = f \tag{1.18}$$

and that

$$\|u_n\| \leq \frac{1}{\alpha - \gamma}\|f\| . \tag{1.19}$$

Moreover, $u_n$ belongs to $C^4$ since we have taken $f$ from $C^2$. Differentiating the both sides of (1.18) we have

$$\begin{aligned}(\alpha - A_n)u_n' - a'u_n'' - b'u_n' - c'u_n \\ - \int_{R_n} [u_n(x+\xi) - u_n(x) - u_n'(x)\xi] \frac{\partial g(x, \xi)}{\partial x} G(d\xi) = f' ,\end{aligned} \tag{1.20}$$

that is,

$$\begin{aligned}\left(\alpha - A_n - a'\frac{d}{dx}\right)u_n' = f' + b'u_n' + c'u_n \\ + \int_{R_n} [u_n(x+\xi) - u_n(x) - u_n'(x)\xi] \frac{\partial g(x, \xi)}{\partial x} G(d\xi) .\end{aligned} \tag{1.21}$$

Let a constant $K$ be larger than

$$1,\ \gamma,\ \|a''\|,\ \|b'\|,\ \|b''\|,\ \|c'\|,\ \|c''\|,\ \int_{R_0} |\xi| g_1(\xi) G(d\xi),\ \int_{R_0} \xi^2 g_2(\xi) G(d\xi) . \tag{1.22}$$

Take $\alpha$ so large that

$$0 < \frac{4K}{\alpha - 17/2 \cdot K} < 1 . \tag{1.23}$$

Since

$$\left|\int_{R_n} [u_n(x+\xi) - u_n(x)] \frac{\partial g(x, \xi)}{\partial x} G(d\xi)\right| \leq \|u_n'\| \int_{R_0} |\xi| g_1(\xi) G(d\xi) \tag{1.24}$$

and

$$\left|\int_{R_n} u_n'(x)\xi \frac{\partial g(x, \xi)}{\partial x} G(d\xi)\right| \leq \|u_n'\| \int_{R_0} |\xi| g_1(\xi) G(d\xi) , \tag{1.25}$$

the norm of the right hand side of (1.21) is not greater than $\|f'\|+3K\|u_n'\|+K\|u_n\|$. Applying Lemma 1.2 to (1.21) we get

$$\|u_n'\| \leqq \frac{1}{\alpha-\gamma}(\|f'\|+3K\|u_n'\|+K\|u_n\|),$$

so that, by (1.19) and (1.23),

$$\|u_n'\| \leqq \frac{1}{\alpha-\gamma-3K}(\|f'\|+K\|u_n\|) \leqq \|f'\|+\|f\| . \tag{1.26}$$

Differentiating the both sides of (1.20) once more, we have

$$\begin{aligned}(\alpha-A_n)u_n''&-2a'u_n'''-a''u_n''-2b'u_n''-b''u_n'-2c'u_n'-c''u_n\\ &-2\int_{R_n}[u_n'(x+\xi)-u_n'(x)-u_n''(x)\xi]\frac{\partial g(x,\xi)}{\partial x}G(d\xi)\\ &-\int_{R_n}[u_n(x+\xi)-u_n(x)-u_n'(x)\xi]\frac{\partial^2 g(x,\xi)}{\partial x^2}G(d\xi)=f'' .\end{aligned}$$

Use the estimates similar to (1.24) and (1.25) together with the estimate

$$\left|\int_{R_n}[u_n(x+\xi)-u_n(x)-u_n'(x)\xi]\frac{\partial^2 g(x,\xi)}{\partial x^2}G(d\xi)\right| \leqq \frac{1}{2}\|u_n''\|\int_{R_0}\xi^2 g_2(\xi)G(d\xi) .$$

Apply Lemma 1.2 again. Then, using (1.22) we get

$$\|u_n''\| \leqq \frac{1}{\alpha-\gamma}\left(\|f''\|+\frac{15}{2}K\|u_n''\|+3K\|u_n'\|+K\|u_n\|\right) .$$

Hence, by (1.19), (1.26) and (1.23),

$$\|u_n''\| \leqq \|f''\|+\|f'\|+\|f\| . \tag{1.27}$$

From (1.18) and (1.16) we find

$$\begin{aligned}(\alpha-A)u_n&=f+(A_n-A)u_n\\ &=f+\frac{1}{n}u_n''-\int_{R_n^c}[u_n(x+\xi)-u_n(x)-u_n'(x)\xi]g(x,\xi)G(d\xi)\end{aligned}$$

where $R_n^c=R_0\cap[-1/n,1/n]$. Hence,

$$\begin{aligned}\|(\alpha-A)u_n-f\| &\leqq \|u_n''\|\left(\frac{1}{n}+\frac{1}{2}\int_{R_n^c}\xi^2 g(x,\xi)\,G(d\xi)\right)\\ &\leqq (\|f''\|+\|f'\|+\|f\|)\left(\frac{1}{n}+\frac{1}{2}\int_{R_n^c}\xi^2 g_0(\xi)\,G(d\xi)\right),\end{aligned}$$

which is smaller than $\varepsilon$ when $n$ is taken sufficiently large. Thus the theorem is proved.

**2. Integration on a whole line.**

For a bounded continuous function on a whole line, we cannot prove Lemma 1.2, since the maximum and the minimum are not necessarily attained at finite points. To get the analogue to Lemma 1.2 (Lemma 2.2) we have to restrict ourself to a smaller class of functions. Hence, in this section, we let $C$ denote the Banach space of all continuous functions $f(x)$ on the real line converging to zero when $|x|\to\infty$. $C_0$ denotes the set of all continuous functions with compact carriers, and $C^n$ denotes the set of all $n$ times continuously differentiable functions belonging to $C$.

Suppose that $A$ is given in almost the same form as (1.1):

$$Au(x)=a(x)u''(x)+b(x)u'(x)+c(x)u(x) \tag{2.1}$$
$$+\int_{R_0}[u(x+\xi)-u(x)-u'(x)\varphi(\xi)]g(x,\xi)G(d\xi)$$

where $R_0=(-\infty, 0)\cup(0, +\infty)$ and $\varphi(\xi)$ is a fixed bounded continuous function coinciding with $\xi$ in a neighborhood of zero. We assume the following three conditions:

(i) $a$, $b$ and $c$ are twice continuously differentiable, $a(x)$ is non-negative, $a''(x)$ is uniformly Hölder continuous on every compact set, and moreover $a''$ and $b'$ are bounded from above and $b$, $b''$ and $c$ are bounded.

(ii) $g(x, \xi)$ is non-negative, twice continuously differentiable in $x$ and measurable in $\xi$. $G(d\xi)$ is a measure on $R_0$ and there exist measurable functions $g_i(\xi)$ $(i=0, 1, 2)$ satisfying (1.2)—(1.4) and

$$\int_{\{|\xi|<1\}}\xi^2 g_i(\xi)G(d\xi)<+\infty \qquad (i=0, 2) \tag{2.2}$$

$$\int_{\{|\xi|\geq 1\}}g_i(\xi)G(d\xi)<+\infty \qquad (i=0, 2) \tag{2.3}$$

$$\int_{R_0}|\xi|g_1(\xi)G(d\xi)<+\infty\ . \tag{2.4}$$

(iii) For every $n$, neither $+\infty$ nor $-\infty$ is an entrance boundary, in the sense of W. Feller [2], with respect to the differential operator

$$D_n u(x)=\left(a(x)+\frac{1}{n}\right)u''(x)+b_n(x)u'(x) \tag{2.5}$$

where $b_n(x)=b(x)-\int_{R_n}\varphi(\xi)g(x,\xi)G(d\xi)$ and $R_n=(-n, -1/n)\cup(1/n, n)$.

Concerning the assumption (iii), we will only make use of the implication that $D_n$ is the infinitesimal generator of a strongly continuous semi-group on $C$.

We take as $\mathrm{D}(A)$ the set of all $u$ in $C^2$ such that $Au$ belongs to $C$. This domain contains $C^2 \cap C_0$. The following three lemmas are proved in the same way as Lemmas 1.1—1.3.

LEMMA 2.1. *$A$ has the smallest closed extension $\bar{A}$.*

LEMMA 2.2. *After changing the meanings of $C$, $C^2$ and $R_0$,* Lemma 1.2 *still holds.*

LEMMA 2.3. *After changing the meaning of $C$,* Lemma 1.3 *still holds.*

THEOREM 2. *Under the above assumptions* (i)—(iii), *$\bar{A}$ is the infinitesimal generator of a uniquely determined strongly continuous semi-group $\{T_t;\ t\geq 0\}$ of bounded linear operators on $C$. This $T_t$ is positivity-preserving and its norm satisfies* (1.8).

Define

$$A_n u(x)=\left(a(x)+\frac{1}{n}\right)u''(x)+b(x)u'(x) \tag{2.6}$$
$$+\int_{R_n}[u(x+\xi)-u(x)-u'(x)\varphi(\xi)]g(x,\xi)G(d\xi)\,,$$

$\mathrm{D}(A_n)$ being the set of all $u$ in $C^2$ such that $A_n u$ belongs to $C$. To prove the theorem, we need the fact that the derivatives of the solution of (2.7) belong to $C$. Let $\hat{C}^n$ be the aggregate of all functions in $C^n$ the derivatives of which up to and including the $n$-th order belong to $C$.

LEMMA 2.4. *For every $\alpha$ larger than $\alpha_0$ and every $f$ in $\hat{C}^1$, there exists one and only one solution $u\in\mathrm{D}(A_n)$ of*

$$(\alpha-A_n)u=f\,. \tag{2.7}$$

*This $u$ necessarily belongs to $\hat{C}^2$. Here $\alpha_0$ is a constant independent of $n$ and $f$.*

PROOF. Define

$$J_n u(x)=\int_{R_n}[u(x+\xi)-u(x)]g(x,\xi)G(d\xi)\,.$$

Then $J_n$ is a bounded operator on $C$ and $A_n=D_n+J_n$ where $D_n$ was defined by (2.5). Since, by virtue of the assumption (iii), $D_n$ (its domain being equal to $\mathrm{D}(A_n)$) is the infinitesimal generator of a strongly continuous semi-group on $C$ with norm not exceeding one, so is $A_n$ by Lemma 2.3. Thus the first half of our lemma is clear.

To prove the latter half, we use the assumption that $f$ lies in $\hat{C}^1$. We show that there exists such a constant $\alpha_n$ that the solution in $\mathrm{D}(A_n)$ of (2.7) necessarily belongs to $\hat{C}^1$ if $\alpha>\alpha_n$. Firstly we observe that

$$\limsup_{x\to+\infty} u'(x) \geqq 0\,, \tag{2.8}$$

for, if otherwise we would have $\lim_{x\to+\infty} u(x)=-\infty$. Similarly,

$$\liminf_{x\to+\infty} u'(x) \leqq 0\,, \tag{2.9}$$

Take $\alpha_n$ so large that

$$\sup_x b'(x),\ \int_{R_n} g_0(\xi)G(d\xi),\ \int_{R_n} g_1(\xi)G(d\xi),\ \int_{R_n} |\varphi(\xi)|g_1(\xi)G(d\xi) \leqq \frac{\alpha_n}{3}\,, \tag{2.10}$$

and take $\alpha$ larger than $\alpha_n$. Suppose that $u'$ takes positive local maxima at points $x_1, x_2, \cdots$ with $x_n \uparrow +\infty$ and that $\sup_{y>N} u'(y) \leqq \sup_{x_i\in X(N)} u'(x_i)$ holds for every large $N$. Here $X(N)$ consists of all $x_i$ such that $x_{i+1}>N$. Then, we have

$$\lim_{i\to\infty} u'(x_i)=0\,. \tag{2.11}$$

In fact, observing that $u\in C^3$, differentiating the both sides of (2.7) and making use of the fact $u'(x_i)>0$, $u''(x_i)=0$ and $u'''(x_i)\leqq 0$, we get

$$\alpha u'(x_i)-b'(x_i)u'(x_i)-\int_{R_n} u'(x_i+\xi)g(x_i,\xi)G(d\xi)$$
$$-\int_{R_n}[u(x_i+\xi)-u(x_i)-u'(x_i)\varphi(\xi)]\frac{\partial g(x_i,\xi)}{\partial x}G(d\xi) \leqq f'(x_i) \qquad (i=1, 2, \cdots).$$

By (2.10),

$$\int_{R_n} u'(x_i+\xi)g(x_i,\xi)G(d\xi) \leqq \sup_{y>x_i-n} u'(y)\int_{R_n} g(x_i,\xi)G(d\xi)$$
$$\leqq \frac{\alpha_n}{3}\sup_{x_j\in X(x_i-n)} u'(x_j)\,,$$

$$\left|\int_{R_n}[u(x_i+\xi)-u(x_i)]\frac{\partial g(x_i,\xi)}{\partial x}G(d\xi)\right| \leqq \frac{2}{3}\alpha_n \sup_{y>x_i-n}|u(y)|\,,$$

and

$$\left|\int_{R_n} u'(x_i)\varphi(\xi)\frac{\partial g(x_i,\xi)}{\partial x}G(d\xi)\right| \leqq \frac{\alpha_n}{3}u'(x_i)\,,$$

so that we get

$$\alpha u'(x_i)-\frac{\alpha_n}{3}u'(x_i)-\frac{\alpha_n}{3}\sup_{x_j\in X(x_i-n)} u'(x_j)-\frac{2}{3}\alpha_n\sup_{y>x_i-n}|u(y)|-\frac{\alpha_n}{3}u'(x_i) \leqq f'(x_i)\,.$$

This implies (2.11), since $u(x)\to 0$ and $f'(x)\to 0$ $(x\to+\infty)$. Similarly, if $u'$ takes negative local minima at points $x_1, x_2, \cdots$ with $x_n \uparrow +\infty$ and if $\inf_{y>N} u'(y) \geqq \inf_{x_i \in X(N)} u'(x_i)$ holds for every large $N$, then $\lim_{i\to\infty} u'(x_i)=0$ holds. Combining these facts with (2.8) and (2.9), we see that $u$ belongs to $\hat{C}^1$ when $\alpha>\alpha_n$.[3] Now, define a new norm $\|f\|_1=\|f\|+\|f'\|$ for $f\in\hat{C}^1$. With respect to this norm, $\hat{C}^1$ is a Banach space. Let $\hat{A}_n$ be the restriction of $A_n$ to the set $D(\hat{A}_n)$ of all $u$ in $\hat{C}^1\cap C^2$ such that $A_n u$ lies in $\hat{C}^1$. Take $f$ from $\hat{C}^1$ and $\alpha>\alpha_n$. Then the solution $u$ in $D(A_n)$ of (2.7) belongs to $D(\hat{A}_n)$. Since $u\in C^3$ is obvious, we differentiate the both sides of (2.7) and get

$$(\alpha-A_n)u'-a'u''-b'u'-\int_{R_n}[u(x+\xi)-u(x)-u'(x)\varphi(\xi)]\frac{\partial g(x,\xi)}{\partial x}G(d\xi)=f' ,$$

which implies

$$\|u'\|\leqq\frac{1}{\alpha-3K}\|f'\| \qquad (\alpha>3K) \tag{2.12}$$

by Lemma 2.2. Here we have taken $K$ larger than

$$\sup_x b'(x),\ \int_{R_0}|\xi|g_1(\xi)G(d\xi),\ \int_{R_0}|\varphi(\xi)|g_1(\xi)G(d\xi) .$$

Combining (2.12) with the inequality $\|u\|\leqq 1/\alpha\cdot\|f\|$, we have

$$\|u\|_1\leqq\frac{1}{\alpha-3K}\|f\|_1 .$$

Since $D(\hat{A}_n)$ is dense in $\hat{C}^1$ with respect to the norm $\|\cdot\|_1$, $\hat{A}_n$ is the infinitesimal generator of a strongly continuous semi-group $\{T_t^{(n)};\ t\geqq 0\}$ on the Banach space $\hat{C}^1$ with norm $\|T_t^{(n)}\|_1\leqq e^{3Kt}$. This shows that (2.7) has a solution in $D(\hat{A}_n)$ for ever $\alpha$ larger than $3K$ and every $f$ in $C^1$.

Once it is proved that $u$ belongs to $\hat{C}^1$, it is easy to show that it belongs to $\hat{C}^2$. In fact (2.7) is written as

$$\left(a(x)+\frac{1}{n}\right)u''=-f+\alpha u-bu'-\int_{R_n}[u(x+\xi)-u(x)-u'(x)\varphi(\xi)]g(x,\xi)G(d\xi) ,$$

and the right hand side belongs to $C$. The proof of Lemma 2.4 is completed.

Proof of Theorem 2. Define $A^0u$ by $A^0u=Au-cu$. Since we have assumed that $c$ is bounded, it is sufficient to prove the theorem for $A^0$ instead of $A$ (cf. Lemma 2.3). By Lemma 2.4 the integro-differential equation (2.7) has a solution in $\hat{C}^2$ for every sufficiently large $\alpha$ and every $f$ in $\hat{C}^1$. Therefore we can estimate $\|u'\|$ and $\|u''\|$ through differentiating (2.7) and applying Lemma

3) We owe this part of the proof to S. Itô's suggestion.

2.2. The remaining parts of the proof are almost the same as those of Theorem 1.

**3. Integration on a torus.**

In this section $x$ denotes a point $(x_1, \cdots, x_m)$ in the Euclidean $m$-space $R^m$. Let $C$ be the Banach space of all continuous functions $f(x)$ on $R^m$ such that

$$f(x_1, \cdots, x_m)=f(x_1+2n_1\pi, \cdots, x_m+2n_m\pi)$$

for all integers $n_1, \cdots, n_m$, and $C^n$ be the set of all $n$ times continuously differentiable functions belonging to $C$.

Let $Au$ be

$$Au(x)=\sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j}+\sum_{i=1}^{m} b_i(x)\frac{\partial u(x)}{\partial x_i}+c(x)u(x)$$
$$+\int_{R_0}\left[u(x+\xi)-u(x)-\sum_{i=1}^{m}\frac{\partial u(x)}{\partial x_i}\xi_i\right]g(x, \xi)G(d\xi),$$

where $R_0=\{\xi=(\xi_1, \cdots, \xi_m);\ \xi_i\in[-\pi, 0)\cup(0, \pi) \text{ for every } i\}$. Let $R_n=\{\xi=(\xi_1, \cdots, \xi_m);\ \xi_i\in[-\pi, -1/n)\cup(1/n, \pi) \text{ for every } i\}$. We assume the following three conditions:

(i) $a_{ij}$, $b_i$ and $c$ belong to $C^\infty$, $a_{ij}(x)$ is symmetric and non-negative definite at every point.

(ii) $g(x, \xi)$ is non-negative, belongs to $C^\infty$ as a function of $x$, and is measurable in $\xi$. $G(d\xi)$ is a measure on $R_0$. For $k=0, 1, 2, \cdots$, there exists a function $g_k(\xi)$ majorizing the absolute values of the $k$-th order derivatives of $g(x, \xi)$ with respect to $x$ and satisfying

$$\int_{R_n} g_k(\xi)G(d\xi)<+\infty \qquad (n=1, 2, \cdots,\ k=0, 1, 2, \cdots),$$
$$\int_{R_0}\sum_{i=1}^{m}\xi_i^2 g_k(\xi)G(d\xi)<+\infty \qquad (k=0, 2),$$

and

$$\int_{R_0}\sum_{i=1}^{m}|\xi_i|g_1(\xi)G(d\xi)<+\infty\ .$$[4]

(iii) For $i, j\neq 1$, $a_{ii}$ depends only on $x_i$, $a_{ij}$ only on $x_i$ and $x_j$, $b_i$ only on $x_2, x_3, \cdots, x_m$, and $g$ only on $x_2, x_3, \cdots, x_m$ and $\xi$.

We take $C^2$ as $D(A)$.

THEOREM 3.1. *Under the above assumptions $A$ has the smallest closed extension, which is the infinitesimal generator of a uniquely determined strongly continuous semi-group of bounded linear operators on $C$. This semi-group is*

4) It is easy to weaken $C^\infty$ assumptions in (i) and (ii) by $C^k$ assumptions of some finite $k$.

*positivity-preserving and its norm satisfies* (1.8).

PROOF. We follow the same idea as in the proofs of Theorems 1 and 2. Define $A_n$ similarly to (1.16), that is, $A_n=D_n+J_n$ and

$$D_n u(x)=\sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j}+\frac{1}{n}\sum_{i=1}^{m} a_{ii}(x)\frac{\partial^2 u(x)}{\partial x_i^2}$$
$$+\sum_{i=1}^{m}\left(b_i(x)-\int_{R_n}\xi_i g(x,\xi)G(d\xi)\right)\frac{\partial u(x)}{\partial x_i},$$
$$J_n u(x)=c(x)u(x)+\int_{R_n}[u(x+\xi)-u(x)]g(x,\xi)G(d\xi).$$

Concerning the smallest closed extensions $\bar{A}_n$ and $\bar{D}_n$ of $A_n$ and $D_n$, respectively, we have

$$\bar{A}_n=\bar{D}_n+J_n \tag{3.1}$$

since $J_n$ is a bounded operator. Apply the result of K. Yosida [13, 14], S. Itô [7, 8] or E. Nelson [10] and use the analogue to Lemma 1.3. Then we can find, for any $f\in C$ and $\alpha>\gamma$, the solution $u_n$ of

$$(\alpha-\bar{A}_n)u_n=f. \tag{3.2}$$

Let $H^k$ be the aggregate of all functions in $L_2(R)$ ($R=\{\xi=(\xi_1,\cdots,\xi_m);\ \xi_i\in[-\pi,\pi)$ for every $i\}$) that have strong derivatives of order $\leqq k$.[5] $H^\infty$ denotes $\bigcap_k H^k$. From our assumption (ii), we prove that $u\in H^k$ implies $J_n u\in H^k$. In fact, there exists a sequence $v_p\in C^k$ converging to $u$ in $L_2(R)$ and $Dv_p$ is a Cauchy sequence in $L_2(R)$ for any differentiation $D$ of order $\leqq k$. Since $J_n$ is a bounded operator on $L_2(R)$, $J_n v_p\to J_n u$ ($p\to\infty$) in $L_2(R)$. It is easily seen that $J_n v_p\in C^k$ and that $DJ_n v_p$ is a Cauchy sequence in $L_2(R)$. Hence, we have $J_n u\in H^k$.

Suppose that $f$ is in $C^\infty$. Let us show that the solution $u_n$ of (3.2) also belongs to $C^\infty$. For this purpose, it is sufficient to show that $u_n\in H^\infty$ (cf. Соболев [16]). By (3.1), $u_n$ is the weak solution of the equation

$$(\alpha-D_n)u_n=f+J_n u_n. \tag{3.3}$$

Apply the differentiability theorem in [11] and we obtain $u_n\in H^2$. This implies $J_n u_n\in H^2$ and hence the right side of (3.3) belongs to $H^2$. Using the differentiability theorem again, we see $u_n\in H^4$. Repeating this procedure we finally obtain that $u_n\in H^\infty$.[6]

For brevity we use the notation

5) We say that $u$ has strong derivatives of order $\leqq k$, if and only if there exists such a sequence $v_p\in C^k$ that $v_p$ coverges to $u$ in $L_2(R)$ and $Dv_p$ is a Cauchy sequence in $L_2(R)$ for all differentiations $D$ of order $\leqq k$ (cf. L. Nirenberg [11]).

6) In one dimension we do not need these considerations, since $\bar{D}_n=D_n$ and $\bar{A}_n=A_n$.

$$\|f\|_k = \sum_{|p|\leq k} \left\| \frac{\partial^{|p|} f}{\partial x_1^{p_1} \cdots \partial x_m^{p_m}} \right\|$$

where $p=(p_1, \cdots, p_m)$ and $|p|=p_1+\cdots+p_m$. Take $\alpha$ and a constant $K$ sufficiently large (the largeness needed does not depend on $n$). Then, by the same argument used in the proofs of Theorems 1 and 2, we can prove that

$$\left\| \frac{\partial^{p_1} u_n}{\partial x_1^{p_1}} \right\| \leq \sum_{l=0}^{p_1} \left\| \frac{\partial^l f}{\partial x_1^l} \right\|$$

for $p_1=0, 1, \cdots, 4$. In the proof, however, we must make use of the assumption (iii) or the like, since we are dealing with higher dimensional case. Similarly we get the following estimates for $i, j \neq 1$:

$$\left\| \frac{\partial u_n}{\partial x_i} \right\| \leq \frac{1}{\alpha}\left[ \left\| \frac{\partial f}{\partial x_i} \right\| + K\left( \left\| \frac{\partial^2 u_n}{\partial x_1^2} \right\| + \|u_n\|_1 \right) \right],$$

$$\left\| \frac{\partial^2 u_n}{\partial x_1 \partial x_i} \right\| \leq \frac{1}{\alpha}\left[ \left\| \frac{\partial^2 f}{\partial x_1 \partial x_i} \right\| + K\left( \left\| \frac{\partial^3 u_n}{\partial x_1^3} \right\| + \|u_n\|_2 \right) \right],$$

$$\left\| \frac{\partial^3 u_n}{\partial x_1^2 \partial x_i} \right\| \leq \frac{1}{\alpha}\left[ \left\| \frac{\partial^3 f}{\partial x_1^2 \partial x_i} \right\| + K\left( \left\| \frac{\partial^4 u_n}{\partial x_1^4} \right\| + \sum_{k=1}^{m} \left\| \frac{\partial^3 u_n}{\partial x_1^2 \partial x_k} \right\| + \|u_n\|_2 \right) \right],$$

and

$$\left\| \frac{\partial^2 u_n}{\partial x_i \partial x_j} \right\| \leq \frac{1}{\alpha}\left[ \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\| + K\left( \left\| \frac{\partial^3 u_n}{\partial x_1^2 \partial x_i} \right\| + \left\| \frac{\partial^3 u_n}{\partial x_1^2 \partial x_j} \right\| + \|u_n\|_2 \right) \right].$$

These estimates together yield the estimate

$$\|u_n\|_2 \leq \|f\|_4 ,$$

which is essentially important to our proof. We omit the rest of the proof, for it is quite similar to that of Theorem 1.

If we replace the assumption (iii) by the following (iii′), the theorem remains still true.

(iii′) $a_{ij}$ are constants.

THEOREM 3.2. *Under the assumptions* (i), (ii) *and* (iii′), *the conclusion of* Theorem 3.1 *is true.*

PROOF. It is easier than that of Therem 3.1, for, in this case, we get the estimates

$$\left\| \frac{\partial u_n}{\partial x_i} \right\| \leq \frac{1}{\alpha}\left( \left\| \frac{\partial f}{\partial x_i} \right\| + K\|u_n\|_1 \right)$$

and

$$\left\| \frac{\partial^2 u_n}{\partial x_i \partial x_j} \right\| \leq \frac{1}{\alpha}\left( \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\| + K\|u_n\|_2 \right)$$

for all $i$ and $j$.

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