On the existence and regularity of the steady-state solutions of the Navier-Stokes equation

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§ 1. Introduction.[1]

The present paper is connected with the boundary value problem posed by the 3-dimensional[2] steady flow problem of a viscous incompressible fluid. It is well known [13] that in the domain R of the 3-dimensional Euclidean space E_3 occupied by the fluid with the density $\rho=1$, the flow velocity $u=\{u_1, u_2, u_3\}$ and the pressure p satisfy the *Navier-Stokes* system

$$-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f\,, \tag{1.1}$$

$$\operatorname{div} u=0\,, \tag{1.2}$$

where the prescribed vector function f represents the external force acting on the fluid and the positive constant ν means the viscosity. If R is a bounded domain we impose on u the boundary condition that

$$u=\beta \qquad \text{on } \partial R \tag{1.3}$$

with β prescribed on the boundary ∂R of R. Then we call the boundary value problem with the unknown functions u and p composed of (1.1), (1.2) and (1.3) the *interior Navier-Stokes boundary value problem* or simply the *interior problem*. On the other hand, suppose that R is an unbounded domain with bounded R^c, the complementary set of R. Then in addition to (1.3) we impose on u the "*boundary condition at the infinity*"

$$u(x)\to u_\infty\,, \qquad (|x|\to\infty), \tag{1.4}$$

where u_∞ is a given constant vector. In such a case we call the boundary value problem with the unknown functions u and p composed of (1.1), (1.2), (1.3) and (1.4) the *exterior Navier-Stokes boundary value problem* or simply the *exterior problem*.

The object of the present paper is to make a functional analysis approach to these boundary value problems, introducing the notion of the generalized solution defined in an appropriate manner. Namely we study the existence, the interior regularity, the regularity at the boundary and, for the exterior problem, the regularity at the infinity of the generalized solutions. As a result

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1) In this section we use some notions and notations whose precise definitions will be found later.

2) 2-dimensional case will be treated elesewhere in consideration of its peculiar difficulty in the exterior problem.

we obtain an existence proof of the strict solutions, i.e., the solutions in the ordinary sense, under reasonable assumptions on the given data.

Here we must refer to J. Leray's famous work [10] published in 1933 as well as O. A. Ladyzhenskaia's interesting work[3] published recently. Under some physically acceptable assumptions Leray established the existence of the strict solutions of the interior problem and of the exterior problem with $u_\infty=0$. For the exterior problem with $u_\infty\neq 0$ his solution satisfied (1.1), (1.2), (1.3) but in place of (1.4) was subjected to a weaker condition

$$(1.4)' \qquad \int_R |\nabla u|^2 dx < \infty \,, \quad \int_R \frac{|u(y)-u_\infty|^2}{|x-y|^2}\, dy \leq K < \infty$$

with a constant K independent of x. Recently R. Finn [3] succeeded in proving that Leray's solution actually satisfies (1.4) also in the case $u_\infty\neq 0$. On the other hand, in [9] Ladyzhenskaia made an approach of a character similar to that of ours. In fact, our study possesses some features common to her study in methods as well as in results. In the course of the present paper the resemblance and the difference between our treatments and those of others will be observed. Here we note only that our approach of seemingly elementary character is hoped to be applicable to problem of a more general type.

The contents of the present paper are as follows. In §2 we introduce the notations used frequently throughout the present paper and then state some basic integro-differential inequalities with some remarks. At the end of §2 we give the definition of the generalized solution, the associated pressure and the strict solution. §3 is devoted to establishing the existence of the generalized solutions of the interior and the exterior problems. There we construct the solution as the limit of approximating solutions obtained by *Galerkin's method*, which was employed with success in the recent investigations of the time-dependent solutions of the Navier-Stokes system by E. Hopf [5] and others [8]. Also we resort to *Brouwer's fixed-point theorem*, whereas Leray's proof as well as Ladyzhenskaia's was carried out essentially with the aid of Leray-Schauder's fixed-point theorem. In our existence proof the results of Odqvist [12] concerning the *Green tensor* of *Stokes' problem* are not needed and no increased difficulties arise even in dealing with the exterior problem. The object of the first half of §4 is to describe some known formulae in connection with Stokes' problem in a form convenient for the second half, where we study the interior regularity of the generalized solution under various regularity conditions on f. It should be remarked that the Navier-Stokes system is *elliptic* according to the definition of Douglis-Nirenberg [2] and hence the regularity properties of u and p are obtained at once by means of the known theorems if u and p are known to be smooth to a certain extent. Therefore the main task of §4 is to improve the regularity properties of u and p to this extent. In §5 we study the regularity at the boundary of the generalized solution making use of the Green tensor of Stokes' problem. §6 and §7 are devoted to the investigations of the

3) It is the author's regret that he did not know the existence of [9] in advance of completing the present work.

regularity at the infinity. We give a simple proof that u really satisfies (1.4) under a weak condition on f. Then assuming more of f we derive more detailed results with the aid of the integral representations with the kernels constructed from the fundamental solutions of Stokes' and Oseen's system. The use of these kernels are common to Leray's or Finn's treatment and ours but otherwise we follow a different way intending to illustrate various methods applicable to problems of this sort.

§ 2. Preliminaries and Definition of Generalized Solutions.

I) *Notations.*

Here we collect some notations used frequently in the sequel.

i) For two points $x=\{x_1, x_2, x_3\}$ and $y=\{y_1, y_2, y_3\}$ in E_3, $|x-y|$ means the distance between x and y.

ii) For any point set A in E_3, $\bar{A}$ and A^c denote respectively the closure and the complementary set of A. We write $A \dashv B$, if $\bar{A} \subset \overset{\circ}{B}$, *the interior of* B. ∂A is the boundary of A. γ being a positive constant, the point set $\omega(\gamma, A)=\omega(\gamma)=\{x;\ x\in A \text{ and dist.}(x, \partial A)<\gamma\}$ is the *boundary strip* of A with width γ. We use the notation $A(\gamma)=A-\bar{\omega}(\gamma, A)$.

iii) $B(x, r)$ means an open sphere with center at x and radius r.

iv) Any function, possibly scalar, vector or tensor, considered in this paper is real-valued. V being a measurable set, the function space $L_p(V)$ is composed of functions u with finite

$$\|u\|_{p,V}=\|u\|_p=\left(\int_V |u|^p dx\right)^{1/p}.$$

Here $|u|$ is interpreted as

$$|u|=\left(\sum_{i=1}^3 |u_i|^2\right)^{1/2} \quad \text{or} \quad |u|=\left(\sum_{i\ j=1}^3 |u_{ij}|^2\right)^{1/2},$$

according as u is a vector $\{u_i\}$ or a tensor $\{u_{ij}\}$. We write simply $\|u\|_V$, $\|u\|$ in place of $\|u\|_{2,V}$, $\|u\|_2$ and introduce the inner product (u, v) by

$$(u, v)=(u, v)_V=\int_V uvdx\ , \qquad (u, v\in L_2(V)),$$

where

$$uv=\sum_{i=1}^3 u_iv_i=u_iv_i \quad \text{or} \quad uv=\sum_{i,j=1}^3 u_{ij}v_{ij}=u_{ij}v_{ij}\ .$$

Here and hereafter we follow the *summation convention*. $L_p^{loc}(V)$ is composed of functions u belonging to $L_p(K)$ for any compact subset $K\dashv V$.

v) V being an open set, the meaning of the function classes $C^n(V)$, $C^{n+h}(V)$ $(n=0, 1, \cdots, \infty;\ 0<h<1)$, is as usual. Then the notations $C^n(\bar{V})$, $C^{n+h}(\bar{V})$ are familiar. $C_0(V)$ is the class of functions u whose *carrier* is compact and $\dashv V$. We put $C_0^{n+h}(V)=C^{n+h}(V)\cap C_0(V)$, $(n=0, 1, \cdots, \infty;\ 0\leq h<1)$.

vi) By $\partial_i u$ or $\partial u/\partial x_i$ we mean the generalized derivative of $u(x)$ in the distribution sense. The differentiation involved in ∇, div and other formal differential operators is interpreted in the generalized sense, if necessary.
vii) A vector function u is said to be solenoidal if $\operatorname{div} u=0$. Let $C_\sigma^1(V)$ be the class of solenoidal vector functions $u\in C^1(V)$. Then we put $C_\sigma^{n+h}(V)=C^{n+h}(V)\cap C_\sigma^1(V)$, $C_{0,\sigma}^{n+h}(V)=C_0^{n+h}(V)\cap C_\sigma^1(V)$, $(n=1, 2, \cdots, \infty;\ 0\leq h<1)$.
viii) ∇u means the vector function $\{\partial_i u\}$ or the tensor function $\{\partial_i u_j\}$ according as u is a scalar or a vector function. $\hat{H}_0^1(V)$ means the Hilbert space obtained by completion of $C_0^1(V)$ with the norm $|||\varphi|||=\|\nabla\varphi\|_{2,V}$. As a set it is identical with the completion of $C_0^1(V)$ with the norm $|||\varphi|||=\|\nabla\varphi\|_{2,V}+\|\varphi\|_{2,V}$ if V is bounded. $\hat{H}_{0,\sigma}^1(V)$ means the Hilbert space obtained by completion of $C_{0,\sigma}^1(V)$ with the norm $|||\varphi|||=\|\nabla\varphi\|_{2,V}$. Any $u\in\hat{H}_{0,\sigma}^1(V)$ possesses generalized derivatives in $L_2(V)$ and satisfies $\operatorname{div} u=0$.
ix) Throughout the present paper R means the domain where the flow problem, i.e., the Navier-Stokes boundary value problem is considered. Concerning the type of R we assume the following. Let $\mathcal{S}$ be a class of bounded domains V whose boundary ∂V is a closed surface of class C^1. In the interior problem $R\in\mathcal{S}$ or $R=V-\bar{V}_1-\bar{V}_2-\cdots-\bar{V}_s$, where $V\in\mathcal{S}$, $V_i\in\mathcal{S}$, $V_i\Subset V$, $(i=1, 2, \cdots, s)$ and $\bar{V}_i$-s are mutually disjoint. In the exterior problem $R=E_3$ or $R=E_3-\bar{V}_1-\cdots-\bar{V}_s$, where $V_i\in\mathcal{S}$ and $\bar{V}_i$-s are mutually disjoint.

II) *Lemmas.*

For the convenience of later reference we here state several lemmas, most of which appear well or essentially well known. Brief proofs are given to those whose available proofs are scattered in various literatures.

Lemma 2.1. *Let V be a bounded domain. Then there exists a positive constant c_1 depending on V such that*

$$\|u\|_{2,V}\leq c_1\|\nabla u\|_{2,V} \tag{2.1}$$

for any $u\in\hat{H}_0^1(V)$.

We omit the proof since it is well known[4].

Lemma 2.2. *Let V be a domain bounded or unbounded. Let u be any function in $\hat{H}_0^1(V)$ and set $u^*(y)=u(y)/|x-y|$ for an arbitrary but fixed x. Then we have*

$$\|u^*\|_{2,V}\leq 2\|\nabla u\|_{2,V}\ . \tag{2.2}$$

Remark. Hereafter we may write $\|u(y)/|x-y|\|$ in place of $\|u^*\|$ in (2.2). From (2.2) it follows immediately that

$$\|u\|_{2,K}\leq 2d\|\nabla u\|_{2,V}\ , \qquad (u\in\hat{H}_0^1(V)), \tag{2.3}$$

for any bounded subdomain $K\subset V$ with diameter less than d. (2.3) implies that the strong convergence in $\hat{H}_0^1(V)$ ensures the strong convergence in $L_2(K)$.

4) e.g., see [1]. Otherwise (2.1) follows from (2.5) in the same way as (2.6).

Proof of Lemma 2.2. It suffices to deal with the scalar case assuming $u\in C_0^1(V)$. We extend u over the whole space E_3 setting $u\equiv 0$ outside V. With any positive constants r_1, r_2, $(r_1<r_2)$ let $A=A(x, r_1, r_2)$ be the annular domain $B(x, r_2)-\overline{B}(x, r_1)$. Substituting $f(y)=\log|y-x|$, $g(y)=u^2(y)$ into the familiar integral identity

$$-\int_A \Delta f g\, dy+\int_{\partial A}\frac{\partial f}{\partial n}g dS=\int_A \nabla f \nabla g\, dy\ ,$$

we have

$$\int_A |u^*|^2 dy=2\int_A u^*(y)\frac{x-y}{|x-y|}\nabla u(y)dy+r_2\mu(r_2)-r_1\mu(r_1)\ , \tag{2.4}$$

where

$$\mu(r)\equiv\mu(u, x, r)\equiv\frac{1}{r^2}\int_{\partial B(x,r)}|u^2|dS\ .$$

Making $r_2\to\infty$, $r_1\to+0$ in (2.4) we arrive at (2.2) on account of Schwarz' inequality.

Lemma 2.3. *Let V be a domain bounded or unbounded. Then there exists an absolute constant c such that*

$$\|u\|_{6,V}\le c\|\nabla u\|_{2,V} \tag{2.5}$$

for any $u\in \hat{H}_0^1(V)$. Moreover, if K is a bounded subdomain of V, then we have for any $u\in\hat{H}_0^1(V)$

$$\|u\|_{4,K}\le c_K\|\nabla u\|_{2,V} \tag{2.6}$$

with a constant c_K dependent on K.

Remark. If V is bounded, we have

$$\|u\|_{4,V}\le c_2\|\nabla u\|_{2,V}\ , \qquad (u\in\hat{H}_0^1(V)), \tag{2.7}$$

with some constant c_2 by putting $K=V$ in (2.6). Hence strong convergence in $\hat{H}_0^1(V)$ implies strong convergence in $L_4(V)$ when V is bounded. (2.6) shows that strong convergence in $\hat{H}_0^1(V)$ ensures locally strong convergence in $L_4(V)$ even if V is unbounded. Also we note that if $u\in\hat{H}_0^1(V)$ and K is a bounded subdomain of V, then we have $u\in L_4(K)$, $\nabla u\in L_2(K)$ and hence $(u\cdot\nabla)u\in L_{4/3}(K)$ by means of Hölder's inequality.

Proof of Lemma 2.3. Take an arbitrary scalar function $u\in C_0^1(V)$ and extend it over the whole space by setting $u\equiv 0$ outside V. Applying the well-known formula

$$v(x)=\frac{1}{4\pi}\int_{E_3}\frac{\partial}{\partial y_i}\frac{1}{|x-y|}\cdot\frac{\partial}{\partial y_i}v(y)dy\ , \qquad (v\in C_0^1(E_3)),$$

to $v=u^4$, we have

$$u^4(x)=\frac{1}{\pi}\int_{E_3}\frac{x_i-y_i}{|x-y|^3}u^3(y)\frac{\partial}{\partial y_i}u(y)dy\ . \tag{2.8}$$

We multiply both the sides of (2.8) by $u^2(x)$ and integrate over E_3 with respect to x. After the interchange of the integration order and the application of (2.2) we obtain

$$\|u\|_6^6=\int_{E_3}u^6(x)dx\leq\frac{4}{\pi}\|\nabla u\|^2\int_{E_3}|u^3(y)|\cdot|\nabla u(y)|dy$$
$$\leq(4/\pi)\|\nabla u\|_2^2\cdot\|u\|_6^3\cdot\|\nabla u\|_2\ .$$

Hence we have (2.5) with $c=(4\pi)^{1/3}$. (2.6) follows from (2.5) by means of Hölder's inequality as

$$\|u\|_{4,K}\leq\|u\|_{6,K}\cdot\|1\|_{12,K}=(\text{mes}\cdot(K))^{1/12}\|u\|_{6,V}\ .$$

We introduce the following notation used also in the later sections:

$$\|u\|_{\Omega(x,r)}=\left(\frac{1}{r^2}\int_{\partial B(x,r)}|u|^2dS\right)^{1/2}.$$

$\|u\|_{\Omega(x,r)}/2\sqrt{\pi}$ is seen to be the spherical square mean of u on the spherical surface $\partial B(x, r)$.

Lemma 2.4.[5] *Let V be an unbounded domain such that V^c is bounded. If a function $u\in C^1(V)$ satisfies*

$$\|\nabla u\|_V\leq K \quad \textit{and} \quad \|u(y)/|x-y|\|_V\leq K \tag{2.9}$$

with a constant K independent of x, then there exists a constant K_1 depending only on K such that

$$\|u\|_{\Omega(x,r)}\leq K_1r^{-1/2}\ , \qquad (r\to\infty). \tag{2.10}$$

Remark. Any $u\in\hat{H}_0^1(V)$ satisfies (2.9) for some K in virture of Lemma 2.2. Conversely, if $u\in C^1(V)$ satisfies (2.9) and vanishes identically near ∂V, then we can show $u\in\hat{H}_0^1(V)$ by a standard procedure. In this sense the condition (2.9) characterizes the behavior at the infinity of a function in $\hat{H}_0^1(V)$.

Proof of Lemma 2.4. We return to (2.4). It is easily seen from (2.4) that $r\mu(r)$ tends to a certain constant α as $r\to\infty$ and then that the value of α must be 0. Therefore we obtain

$$r_1\mu(r_1)\leq2\|u(y)/|x-y|\|_{B'}\|\nabla u\|_{B'}\leq2K^2, \qquad (B'=B(x, r_1)^c),$$

be means of Schwarz' inequality. Thus (2.10) holds with $K_1=\sqrt{2}\ K$.

The following lemmas are concerned with the behavior of a function $u\in\hat{H}_0^1(R)$ near the boundary. We recall the assumptions imposed on R. Take and fix a component ∂R^* of ∂R. γ being a sufficiently small positive number,

5) In connection with this lemma we refer to [14].

$\omega^*(\gamma)$ means the component of $\omega(\gamma)$ adjacent to ∂R^*. Further $\rho^*=\rho^*(x)$ denotes the distance between ∂R^* and the point $x\in R$. Using these notations we have

Lemma 2.5. *There exist positive constants γ_0 and $c_3=c_3(\gamma_0)$ such that*

(2.11) $$\left\|\frac{u}{\rho^*}\right\|_{2,\omega^*(\gamma)}\leq c_3\|\nabla u\|_{2,\omega^*(\gamma)}$$

holds for any $u\in\hat{H}_0{}^1(R)$ and any γ in $0<\gamma\leq\gamma_0$.

Lemma 2.6. *Suppose that $u\in C^1(\omega^*(\delta))$ for some $\delta>0$. If*

(2.12) $$\|\nabla u\|_{\omega^*(\delta)}\leq K \quad \text{and} \quad \left\|\frac{u}{\rho^*}\right\|_{\omega^*(\delta)}\leq K\,,$$

then there exists a constant K_1 depending only on K and ∂R^* such that

$$\mu^*(\gamma)=\int_{\rho^*=\gamma}|u|^2dS\leq K_1\gamma \qquad \text{as } \gamma\to 0.$$

The proofs of these lemmas are similar respectively to those of Lemma 2.3 and Lemma 2.4 and are omitted here.

Finally we state the well-known [7, 15]

Lemma 2.7. *Let*

$$v(x)=\int_{V'}k(x,y)u(y)dy\,, \qquad (x\in V)$$

where the kernel $k(x,y)$ defined in $V\times V'$ is assumed to satisfy

$$\left(\int_V|k(x,y)|^r dy\right)^{1/r}\leq M \quad \text{and} \quad \left(\int_{V'}|k(x,y)|^r dx\right)^{1/r}\leq M$$

for positive constants M and r $(1<r)$. If $1<p<\infty$ and $1/q=1/p+1/r-1>0$, then we have $v\in L_q(V)$ and

$$\|v\|_{q,V}\leq M\|u\|_{p,V'}\,,$$

for any $u\in L_p(V')$.

III) *Definition of Generalized Solutions.*

Suppose that a vector function u and a scalar function p are sufficiently smooth and obey the Navier-Stokes equation

(2.13) $$-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f$$

in a domain V. Multiplying (2.13) by a vector function φ in $C_0{}^1(V)$ and then integrating over V, we obtain

(2.14) $$\nu(\nabla\varphi,\nabla u)+(\varphi,(u\cdot\nabla)u)-(\operatorname{div}\varphi,p)=(\varphi,f)$$

after obvious partial integrations. If $\varphi\in C^1_{0,\sigma}(V)$, i.e., div $\varphi=0$, then (2.14) takes a form not involving the pressure p: namely,

$$W(\varphi, u)=(\varphi, f)\ , \qquad (\varphi\in C^1_{0,\sigma}(V)), \tag{2.15}$$

with

$$W(\varphi, u)=\nu(\nabla\varphi, \nabla u)+(\varphi, (u\cdot\nabla)u)\ .$$

Conversely, it is known that under the assumption of sufficient smoothness of u, (2.15) is equivalent to (2.13) in the following sense [5]. If (2.15) is valid for any $\varphi\in C^1_{0,\sigma}(V)$, then there exists a scalar function (single-valued!) p which satisfies (2.13) in V together with u. Hence we say that a vector function u satisfies (2.13) *weakly* in V or satisfies the *weak equation* (2.15) of (2.13) in V, if ∇u and $(u\cdot\nabla)u$ are locally integrable in V and (2.15) holds for any $\varphi\in C^1_{0,\sigma}(V)$. Here the differentiation involved in ∇u and $(u\cdot\nabla)u$ is interpreted in the generalized sense. φ in (2.15) will be called a *test function* of the weak equation.

Next, we consider the tri-linear integral form

$$\mathcal{S}(u, v, w)=(u, (v\cdot\nabla)w)=\int_V u_i v_k \partial_k w_i dx\ , \tag{2.16}$$

where the arguments u, v, w are required to possess the following properties:

1) each of them is expressible as the sum of a function in $\hat{H}^1_{0,\sigma}(V)$ and a function in $C_\sigma^1(V)$. Hence any one of them, in particular, v is subjected to the condition $\operatorname{div} v=0$.

2) Either u or w belongs to $\hat{H}^1_{0,\sigma}(K)$, K being a bounded subdomain of V.

For these u, v, w the definition of $\mathcal{S}(u, v, w)$ in (2.16) is significant. Indeed, we have $\|u\|_{4,K}<\infty$, $\|v\|_{4,K}<\infty$ by means of Lemma 2.3, whereas noting $|u_i v_j \partial_j w_i|\leq|u|\cdot|v|\cdot|\nabla w|$ we obtain

$$|\mathcal{S}(u, v, w)|\leq\|u\|_{4,K}\|v\|_{4,K}\|\nabla w\|_{2,K}$$

in virtue of Hölder's inequality. Moreover, if $u\in\hat{H}^1_{0,\sigma}(K)$ we have[6]

$$|\mathcal{S}(u, v, w)|\leq C\|\nabla u\|_{2,K}\|v\|_{4,K}\|\nabla w\|_{2,K} \tag{2.17}$$

by virtue of (2.6). In consideration of (2.17) we notice that

$$\lim_{n\to\infty}\mathcal{S}(\varphi^n, v, w)=\mathcal{S}(\varphi, v, w)\ , \tag{2.18}$$

if $\varphi^n\to\varphi$ strongly in $\hat{H}^1_{0,\sigma}(K)$. For any u, v, w admissible as the arguments we have

$$\mathcal{S}(u, v, w)=-\mathcal{S}(w, v, u) \tag{2.19}$$

and, in particular,

$$\mathcal{S}(u, v, u)=0\ , \tag{2.19$'$}$$

which are immediately verified by partial integration taking account of $\operatorname{div} v=0$.

6) Throughout the present paper we may use the symbol C in order to represent positive constants, the value of which may change even in the same context.

We have reached a stage to state the definition of the generalized solution of the Navier-Stokes boundary value problem. Let R, f, β and u_∞ be as in §1. We recall that the boundary value problem is named interior or exterior, according as R is bounded or unbounded.

Definition 2.1. Let R be bounded. Then a vector function u is called a *generalized solution of the interior problem*, if following conditions i) and ii) are both satisfied:

i) $u-b$ belongs to $\hat{H}^1_{0,\sigma}(R)$ for some b such that

$$b\in C_\sigma^1(\bar{R}), \qquad b=\beta \text{ on } \partial R. \tag{2.20}$$

ii) u satisfies (2.13) weakly in R.

Remark. With the aid of (2.18) we can show that the generalized solution u of the interior problem satisfies the weak equation not only for every φ in $C^1_{0,\sigma}(R)$ but also for every φ in $\hat{H}^1_{0,\sigma}(R)$ if $f\in L_2(R)$.

Definition 2.2. Let R be unbounded. Then a vector function u is called a *generalized solution of the exterior problem*, if the following conditions i) and ii) are both satisfied:

i) $u-b$ belongs to $\hat{H}^1_{0,\sigma}(R)$ for some b such that

$$\begin{aligned} &b\in C_\sigma^1(\bar{R}), \qquad b=\beta \text{ on } \partial R, \\ &b(x)-u_\infty=O(|x|^{-1}), \quad \nabla b(x)=O(|x|^{-2}), \qquad (|x|\to\infty). \end{aligned} \tag{2.21}$$

ii) u satisfies (2.13) weakly in R.

Remark. This time u satisfies the weak equation for any $\varphi\in\hat{H}^1_{0,\sigma}(K)$, K being an arbitrary bounded subdomain of R. From (2.21) it follows that $\|\nabla b\|<\infty$, $\|(b(y)-u_\infty)/|x-y|\|=O(|x|^{-1/2})$ in virtue of the obvious inequality

$$\int_{R_3}\frac{1}{|x-y|^2|y|^2}dy=O(|x|^{-1}), \qquad (|x|\to\infty). \tag{2.22}$$

Thus according to Lemma 2.2, the generalized solution u satisfies

$$\|\nabla u\|_R\le K \quad\text{and}\quad \left\|\frac{u(y)-u_\infty}{|x-y|}\right\|\le K, \tag{2.23}$$

where K is a constant independent of x.

Definition 2.3. Let u be a generalized solution of the Navier-Stokes boundary value problem, interior or exterior. Then a scalar function $p\in L_2^{loc}(R)$ is called the *pressure associated with* u, if u and p satisfy (2.14) for any $\varphi\in C_0^1(R)$. (2.14) is called the *defining equation* of p.

Remark. When a generalized solution u is given, the associated pressure is unique except an additive constant. In fact, suppose that p_1 and p_2 are the associated pressure. Then we find $(\operatorname{div}\varphi, p_1-p_2)=0$ for any $\varphi\in C_0^1(R)$. Substituting $\varphi=\nabla h$, $h\in C_0^\infty(R)$, we obtain $(\Delta h, p_1-p_2)=0$ and note that p_1-p_2 satisfies the Laplace equation weakly. According to a theorem of H. Weyl, this implies that p_1-p_2 is harmonic in V. In particular, from $(\operatorname{div}\varphi, p_1-p_2)=0$ follows $(\varphi, \nabla(p_1-p_2))=0$. Thus we have $\nabla(p_1-p_2)=0$ in R and hence $p_1-p_2=$ const. in R.

Definition 2.4. Concerning the interior (exterior) problem a pair of a vector function u and a scalar function p is called the *strict solution* if $u\in C^2(R)\cap C^0(\bar{R})$, $p\in C^1(R)$ and (1.1), (1.2), (1.3) (and (1.4)) are satisfied. However, the vector function u alone is also sometimes called the strict solution.

§ 3. Existence of Generalized Solutions.

We establish in this section some theorems concerning the existence of generalized solutions of the interior as well as the exterior problem. Our existence proof will be accomplished along the following scheme: (1) existence proof of the approximating solutions u_N in Galerkin's method, (2) derivation of a bound of $\|\nabla u_N\|$, (3) construction of a function u^* from u_N by a limiting procedure, (4) verification that u^* is the desired generalized solution. At the stages (1) and (3) we resort to Brouwer's fixed-point theorem and Rellich's choice theorem[7] respectively.

Our main result concerning the interior problem is the following

Theorem 3.1. *Assume that R is bounded and $f\in L_2(R)$. Then there exists a generalized solution of the interior problem, if one of the following conditions* 1) *and* 2) *is fulfilled:*

1) *β is the boundary value of a function $b^*\in C_\sigma^1(\bar{R})$ with small $|b^*|$ or $|\nabla b^*|$ in the sense of* (3.15) *to be given below.*

2) *β is the boundary value of a function $b^*\in C_\sigma^1(\bar{R})$ expressible in the form $b^*=\operatorname{rot} a$, $a\in C^1(\bar{R})$ and ∂R is of class C^2.*

Remark 1. In the theorem we may replace the assumption $f\in L_2(R)$ by a weaker one that $f-\nabla h\in L_2(R)$ for some $h\in C^1$. Furthermore, under the condition 1) R may be an arbitrary bounded open set.

Remark 2. Let m_i be the "out-flow" from the i-th component ∂R_i of ∂R, i.e.,

$$m_i=\int_{\partial R_i}\beta_n dS\,, \qquad (i=1, 2, \cdots, s), \tag{3.1}$$

where $\beta_n=\beta\cdot n$ and $n=\{n_1, n_2, n_3\}$ is the unit outer normal[8] to ∂R. If β is the boundary value of a vector function in $C_\sigma^1(\bar{R})$, then the "total out-flow" $m=m_1+m_2+\cdots+m_s$ must be 0. On the other hand, if β is the boundary value as in 2) of the theorem, then

$$m_i=0\,, \qquad (i=1, 2, \cdots, s). \tag{3.2}$$

Conversely, if β is of class C^2 and ∂R of class C^3 and if (3.2) holds, then we can construct a b^* as stated in 2).

7) (Rellichscher Auswahlsatz). e.g. see [1], II, p. 489 and p. 513.

8) A *normal vector* to ∂V, the boundary of V under consideration, is called *inner* or *outer*, according as it is directed toward V or V^c. For instance, if V is the exterior of a sphere the outer normal to ∂V is directed toward the center.

Proof of Theorem 3.1. Firstly we introduce the notion of Condition (B): namely, a vector function b is said to satisfy Condition (B) if

$$b \in C_\sigma^1(\bar{R}), \qquad b=\beta \text{ on } \partial R \tag{3.3}$$

and if

$$|\mathscr{S}(b, w, w)| \leq \alpha \|\nabla w\|^2 \tag{3.4}$$

is valid for any $w \in C_{0,\sigma}^1(R)$ and with some constant α in $0 \leq \alpha < \nu$.

Now, suppose that we are given a b satisfying Condition (B) and then we seek a generalized solution of the form

$$u=v+b, \qquad (v \in \hat{H}_{0,\sigma}^1(R)).$$

We introduce and fix a sequence $\{\psi^n\}_{n=1}^\infty$ of functions in $C_{0,\sigma}^1(R)$ such that its linear hull $\mathscr{M}$ is dense in $\hat{H}_{0,\sigma}^1(R)$ but for convenience we normalize it as

$$(\psi^i, \psi^j)=\delta_{ij}=\begin{cases} 1, & (i=j), \\ 0, & (i \neq j). \end{cases} \tag{3.5}$$

The linear hull of the first N functions of $\{\psi^n\}$ is denoted by $\mathscr{M}_N$. Referred to this base a function $u_N=v_N+b$ is called an N-th order approximating solution if the following two conditions are fulfilled:

$$v_N=u_N-b \in \mathscr{M}_N. \tag{3.6}$$

$$W(\varphi, u_N) \equiv \nu(\nabla\varphi, \nabla u_N)+\mathscr{S}(\varphi, u_N, u_N)=(\varphi, f) \tag{3.7}$$

holds for any $\varphi \in \mathscr{M}_N$, or what comes to the same thing,

$$W(\psi^i, u_N)=(\psi^i, f), \qquad (i=1, 2, \cdots, N), \tag{3.7$'$}$$

Concerning the existence of the approximating solutions we state

Lemma 3.1. *If b satisfies Condition (B) and $f \in L_2(R)$, then the approximating solution u_N exists for each N.*

Proof. During the proof of this lemma, we fix N and write simply u, v in place of u_N and v_N. We put

$$u=b+v=b+\xi_1\psi^1+\xi_2\psi^2+\cdots+\xi_N\psi^N \tag{3.8}$$

with a numerical coefficient vector $\xi=\{\xi_1, \xi_2, \cdots, \xi_N\}$. Obviously $W(\varphi, b+v) =\nu(\nabla\varphi, \nabla v)+\nu(\nabla\varphi, \nabla b)+\mathscr{S}(\varphi, b, v)+\mathscr{S}(\varphi, v, b)+\mathscr{S}(\varphi, v, v)+\mathscr{S}(\varphi, b, b)$. Hence the substitution of (3.8) into (3.7$'$) gives

$$\sum_{j=1}^N \{\nu(\nabla\psi^i, \nabla\psi^j)+\mathscr{S}(\psi^i, b, \psi^j)+\mathscr{S}(\psi^i, \psi^j, b)+\mathscr{S}(\psi^i, v, \psi^j)\}\xi_j$$
$$=(\psi^i, f)-\nu(\nabla\psi^i, \nabla b)-\mathscr{S}(\psi^i, b, b). \tag{3.9}$$

In order to rewrite (3.9) as an equation with the unknown N-vector ξ, we introduce an $N\times N$-matrix $T(\xi)=\{T_{ij}(\xi)\}$, depending on ξ, and an N-vector $\eta=\{\eta_j\}$ by

$$T_{ij}(\xi)=\nu d_{ij}+a_{ij}^{(1)}+a_{ij}^{(2)}+a_{ij}(\xi)\ ,\quad \eta_i=\eta_i^{(1)}-\eta_i^{(2)}\ , \tag{3.10}$$

where

$$d_{ij}=(\nabla\psi^i, \nabla\psi^j)\ ,\quad a_{ij}^{(1)}=\mathscr{A}(\psi^i, b, \psi^j)\ ,\quad a_{ij}^{(2)}=\mathscr{A}(\psi^i, \psi^j, b)\ ,$$

$$a_{ij}(\xi)=\mathscr{A}\left(\psi^i,\ \sum_{k=1}^{N}\xi_k\psi^k,\ \psi^j\right)\ ,$$

$$\eta_i^{(1)}=(\psi^i, f)\ ,\quad \eta_i^{(2)}=\nu(\nabla\psi^i, \nabla b)+\mathscr{A}(\psi^i, b, b)\ .$$

Then it is easy to verify that (3.4) is reduced to the equation

$$T(\xi)\xi=\eta \tag{3.11}$$

for the unknown ξ. From now on we regard ξ as an element of N-dimensional Euclidean space $E=E_N$ and use the notation $|\xi|$ and $\xi\cdot\eta$ to denote the norm of $\xi\in E$ and the scalar product of $\xi, \eta\in E$. We shall show that $T(\xi)^{-1}$ exists so that (3.11) is reduced to

$$\xi=F(\xi)\ ,$$

where

$$F(\xi)=T(\xi)^{-1}\eta\ .$$

To this end we estimate $\zeta\cdot T(\xi)\zeta$ for arbitrary ξ and ζ in E. Associating the vector functions $v,\ w\in\mathscr{M}_N\subset C_{0,\sigma}^1(R)$ with $\xi, \zeta\in E$ by $v=\xi_1\psi^1+\xi_2\psi^2+\cdots+\xi_N\psi^N$ and $w=\zeta_1\psi^1+\zeta_2\psi^2+\cdots+\zeta_N\psi^N$, we observe that

$$\begin{aligned}\zeta\cdot T(\xi)\zeta&=\nu\|\nabla w\|^2+\mathscr{A}(w, b, w)+\mathscr{A}(w, w, b)+\mathscr{A}(w, v, w)\\&=\nu\|\nabla w\|^2-\mathscr{A}(b, w, w)\end{aligned}$$

by virtue of (2.19) and (2.19)′. Hence we have by (3.4)

$$(\nu-\alpha)\|\nabla w\|^2\le|\zeta\cdot T(\xi)\zeta|\le|\zeta|\cdot|T(\xi)\zeta|\ . \tag{3.12}$$

On the other hand, by means of (3.5) and Lemma 2.1, we obtain

$$|\zeta|=\|w\|\le c_1\|\nabla w\|\ .$$

Combining this with (3.12) we are led to $(\nu-\alpha)|\zeta|^2\le c_1^2|\zeta|\cdot|T(\xi)\zeta|$ and hence to $|\zeta|\le\kappa|T(\xi)\zeta|$, where $\kappa=c_1^2/(\nu-\alpha)$. This implies that $T(\xi)^{-1}$ exists and

$$|T(\xi)^{-1}|\le\kappa \tag{3.13}$$

holds, where the left hand side means the norm of the linear transformation $T(\xi)^{-1}$ in E. Since η is a constant vector independent of ξ, we thus see that the inequalities

$$|F(\xi)|\le|T(\xi)^{-1}|\cdot|\eta|\le\kappa|\eta|=d$$

hold for any $\xi \in E$. In particular, the closed sphere $S(d)$ of E with center at the origin and radius d is mapped by F into itself. Since the continuity of $F(\xi)$ in ξ is obvious, we are able to apply Brouwer's theorem and conclude the existence of a solution ξ of the equation $\xi = F(\xi)$. This proves the lemma.

Concerning the approximating solutions obtained above we have

Lemma 3.2. *There exists a constant K such that $\|\nabla v_N\| \le K$.*

Proof. Making use of (2.19) and (2.19)′ we have easily

$$W(v, v+b) = \nu\|\nabla v\|^2 + \nu(\nabla v, \nabla b) - \mathscr{A}(b, v, v) - \mathscr{A}(b, b, v)$$

for any $v \in C^1_{0,\sigma}(R)$. Hence, setting $\varphi = v_N$ and $u_N = b + v_N$ in (3.7) we immediately obtain

$$\nu\|\nabla v_N\|^2 - \mathscr{A}(b, v_N, v_N) = (v_N, f) - \nu(\nabla v_N, \nabla b) + \mathscr{A}(b, b, v_N)\ ,$$

whence follows by virtue of (3.4) and Schwarz' inequality that

$$(\nu - \alpha)\|\nabla v_N\|^2 \le C'\|v_N\| + C''\|\nabla v_N\| \le C\|\nabla v_N\|$$

with appropriate constants C', C'' and C. Consequently we have $\|\nabla v_N\| \le C/(\nu - \alpha) = K$.

Since $\|\nabla v_N\|$ and, therefore, $\|v_N\|$ are bounded, we can apply Rellich's theorem and are able to choose a subsequence $\{v_{N'}\}$ of $\{v_N\}$ tending to a $v^* \in \hat{H}^1_{0,\sigma}(R)$ in the sense that $v_{N'} \to v^*$ strongly in $L_2(R)$ and $\nabla v_{N'} \to \nabla v^*$ weakly in $L_2(R)$. We shall show that $u^* = b + v^*$ is the desired generalized solution.

If φ is a fixed function in $C^1_{0,\sigma}(R)$, we have

$$(3.14)\qquad (\nabla\varphi, \nabla v_{N'}) \to (\nabla\varphi, \nabla v^*)\ ,\quad \mathscr{A}(\varphi, v_{N'}, v_{N'}) \to \mathscr{A}(\varphi, v^*, v^*)$$

as $N \to \infty$. Fix an arbitrary positive integer n. Then for any $N \ge n$ we have $W(\psi^n, b + v_{N'}) = (\psi^n, f)$, whence follows by making $N \to \infty$ $W(\psi^n, b + v^*) = (\psi^n, f)$. Further, we notice that $W(\varphi, u^*) = (\varphi, f)$ is valid for any φ in $\mathscr{M}$. Then take an arbitrary φ in $C^1_{0,\sigma}(R)$. Since $\mathscr{M}$ is dense in $\hat{H}^1_{0,\sigma}(R)$, we can find a sequence $\{\varphi^n\}$ such that $\varphi^n \in \mathscr{M}$ and φ^n converges to φ strongly in $\hat{H}^1_{0,\sigma}(R)$. Taking the limit of $W(\varphi^n, u^*) = (\varphi^n, f)$ we arrive at $W(\varphi, u^*) = (\varphi, f)$ by virtue of (2.18).

At this stage it has been proved that a generalized solution exists if there exists a function b satisfying Condition (B). We shall show that the condition 1) or 2) in the theorem is sufficient for the existence of such a b. Firstly we deal with 1). Let M_0 and M_1 be constants such that $|b^*(x)| \le M_0$ and $|\nabla b^*(x)| \le M_1$ hold for any $x \in R$. Then by means of Lemma 2.1 and (2.19) we have

$$|\mathscr{A}(b^*, w, w)| \le M_0\|w\| \cdot \|\nabla w\| \le M_0 c_1 \|\nabla w\|^2$$

and

$$|\mathscr{A}(b^*, w, w)| \le M_1\|w\|^2 \le M_1 c_1^2 \|\nabla w\|^2$$

for any $w \in C^1_{0,\sigma}(R)$, where c_1 is the domain constant in (2.1). Therefore b^* itself satisfies Condition (B), if either

(3.15) $$\nu > M_0 c_1 \quad \text{or} \quad \nu > M_1 c_1^2 .$$

We turn to 2). We note that Lemma 2.5 is applicable to each component of ∂R and (2.11) remains valid with an appropriate constant c_3 if we replace $\omega^*(\gamma)$ by $\omega(\gamma)=\omega(\gamma, R)$ and $\rho^*(x)$ by $\rho(x)=\text{dist.}(x, \partial R)$. We construct the desired b in the form of $b^{**}=\text{rot}(h(\rho)a)$ with a scalar function h of a single variable [6] defined by

$$h(t)=1-\left(\int_0^t j(s)ds \Big/ \int_0^\infty j(s)ds\right), \qquad (t>0),$$

where $j(s)$ is a function with the following properties: $j(s)$ involves two parameters γ and κ whose values are contained in $0<\gamma<\gamma_0$, $0<\kappa<1/4$ and i) $j(s)\in C^\infty[0, \infty)$, ii) $0\leq j(s)\leq 1/s$, $(s>0)$, iii) $j(s)\equiv 0$, $(0\leq s\leq \kappa\gamma$, $(1-\kappa)\gamma\leq s)$, iv) $j(s)=1/s$, $(2\kappa\gamma\leq s\leq(1-2\kappa)\gamma)$. Obviously, $h\in C^\infty[0, \infty)$, and

$$h(t)=\begin{cases} 1, & (0\leq t\leq \kappa\gamma) \\ 0, & ((1-\kappa)\gamma<t). \end{cases}$$

Moreover, as $\kappa\to 0$, $t\,h'(t)$ tends to 0 uniformly with respect to γ and t. Therefore, by means of the well known formula

(3.16) $$\text{rot}(ha)=h\,\text{rot}\,a-a\times\nabla h=h\,\text{rot}\,a-a\times h'(\rho)\nabla\rho ,$$

we have $b^{**}=b^*=\beta$ on ∂R, $b^{**}\equiv 0$ outside $\bar{\omega}(\gamma)$ and $b^{**}\in C_\sigma^1(\bar{R})$, because ρ^* is now of class C^2. Furthermore, for any $\varepsilon>0$, we can make $|\rho b^{**}(x)|<\varepsilon$ valid everwhere in R by a suitable choice of κ. Then we have in view of Lemma 2.5

$$|\mathscr{T}(b^{**}, w, w)|\leq|\mathscr{T}(\rho b^{**}, w/\rho, w)|\leq\varepsilon\|w/\rho\|_\omega\cdot\|\nabla w\|_\omega\leq\varepsilon c_3\|\nabla w\|^2$$

for any $w\in C_{0,\sigma}^1(R)$. Thus b^{**} satisfies Condition (B) by a suitable choice of κ. This completes the proof of Theorem 3.1.

Remark on the Uniqueness. The generalized solution of the interior problem is unique, if ∂R is sufficiently smooth and the "Reynolds number" is sufficiently small. For instance, assume that $\beta\equiv 0$, $f\in L_2(R)$ and ∂R is of class C^2. Then we can show the uniqueness of the generalized solution, provided that

$$\nu^2>c_1c_2^2\|f\| ,$$

where the constants c_1 and c_2 are those in Lemma 2.1 and (2.7) respectively. By the way, in such a situation the equation (3.11) can be solved by the iteration $T(\xi^{(n+1)})\xi^{(n)}=\eta$.

Concerning the exterior problem we state

Theorem 3.2. *Let R be unbounded. Assume that $f\in L_2(R)$ and $f'\in L_2(R)$, where $f'(x)=|x|f(x)$. Then there exists a generalized solution of the exterior problem if one of the following conditions* 1) *and* 2) *is fulfilled:*

1) *$\beta-u_\infty$ is the boundary value of a vector function $b^*\in L_2(R)\cap C_\sigma^1(\bar{R})$ with sufficiently small $|b^*|$ or $|\nabla b^*|$ in the sense of (3.22) to be given below.*

2) *β is the boundary value of a vector function $b^*\in C_\sigma^1(R)$ expressible as $b^*=\mathrm{rot}\, a$, $(a\in C^1(\bar{R}))$, and ∂R is of class C^2.*

Proof. Again we introduce the notion of Condition (B′): namely, a vector function b' is said to satisfy Condition (B′) if

$$(3.17)\qquad b'\in C_\sigma^1(\bar{R})\cap L_2(R)\,,\quad b'=\beta-u_\infty \text{ on } \partial R\,,$$
$$b'(x)=O(|x|^{-1})\,,\quad \nabla b'(x)=O(|x|^{-2})\,,\qquad (|x|\to\infty),$$

and if

$$(3.18)\qquad |\mathscr{S}(b',w,w)|\le\alpha\|\nabla w\|^2$$

holds for any $w\in C^1_{0,\sigma}(R)$ and some α in $0\le\alpha<\gamma$. Supposing that we are given a b' satisfying Condition (B′), we seek the generalized solution of the form

$$u=b'+u_\infty+v\,,\qquad (v\in\hat{H}^1_{0,\sigma}(R))$$

in the same way as before. The notations $\{\psi^n\}$, $\mathscr{M}_N$, $\mathscr{M}$ etc. keep their meaning unchanged. But this time the base $\{\psi^n\}$ is chosen so that the following conditions are both satisfied:

i) $\psi^n\in C^1_{0,\sigma}(R)$ for every n and (3.5) holds.

ii) When an arbitrary φ in $C^1_{0,\sigma}(R)$ is given, we can find a sequence of $\varphi^n\in\mathscr{M}$, such that $\varphi^n\to\varphi$ strongly in $\hat{H}^1_{0,\sigma}(K)$ and the carrier of φ^n is included in a fixed bounded subset K.

The existence of such a base is easily shown. A vector function $u_N=b'+u_\infty+v_N$ is called an N-th order approximating solution if the following two conditions are satisfied:

$$(3.19)\qquad v_N\in\mathscr{M}_N\,.$$

$$(3.20)\qquad W(\varphi,u_N)=(\varphi,f)\qquad\text{for every } \varphi\in\mathscr{M}_N.$$

Similarly to Lemma 3.1, we have

Lemma 3.3. *If b' satisfies Condition (B′) and f the assumptions in the theorem, then there exists an approximating solution u_N for each N.*

N being fixed, the carriers of functions in $\mathscr{M}_N$ are contained in a fixed compact set. Hence the proof of Lemma 3.3 is quite parallel to that of Lemma 3.1 and is omitted here. Correspondingly to Lemma 3.2, we have

Lemma 3.4. *There exists a constant K such that $\|\nabla v_N\|\le K$.*

Proof of Lemma 3.4. The substitution of $\varphi=v_N$ into (3.20) gives

$$\nu\|\nabla v_N\|^2-\mathscr{S}(b',v_N,v_N)=(v_N,f)+\mathscr{S}(b',u_\infty,v_N)+\mathscr{S}(b',b',v_N)\,,$$

whereas we can estimate as

$$|(v_N, f)| = \left|\left(\frac{v_N}{|x|},\ |x| f\right)\right| \leq \left\|\frac{v_N}{|x|}\right\| \cdot \|f'\| \leq 2\|f'\| \cdot \|\nabla v_N\| \leq C\|\nabla v_N\|,$$
$$|\mathscr{V}(b', v_N, v_N)| \leq \alpha\|\nabla v_N\|^2,$$
$$|\mathscr{V}(b', b', v_N)| \leq C\|\{b_i' b_j'\}\| \cdot \|\nabla v_N\| \leq C\|\nabla v_N\|,$$
$$|\mathscr{V}(b', u_\infty, v_N)| \leq C\|b'\| \cdot |u_\infty| \cdot \|\nabla v_N\| \leq C\|\nabla v_N\|$$

with resort to Lemma 2.2 and Condition (B′). From these relations follows the inequality $(\nu-\alpha)\|\nabla v_N\|^2 \leq C\|\nabla v_N\|$ and consequently $\|\nabla v_N\| \leq C/(\nu-\alpha) = K$.

We have proved that $\|\nabla v_N\|$ is bounded. However, it should be noted that Rellich's theorem is not applicable directly to R, since R is unbounded. We put $K_i = R \cap B(0, i)$ for every sufficiently large integer i. Then according to (2.3), there exists a constant c_i for each K_i such that $\|v_N\|_{K_i} \leq c_i$. Hence, Rellich's theorem is applicable to each K_i. Furthermore, by means of a standard diagonal procedure we can choose from $\{v_N\}$ a subsequence $\{v_{N'}\}$ tending to $v^* \in \hat{H}^1_{0,\sigma}(R)$ in the sense that $\nabla v_{N'} \to \nabla v^*$ weakly in $L_2(R)$ and $v_{N'} \to v^*$ locally strongly in $L_2(R)$.

Then we turn to verify that $u^* = b' + u_\infty + v^*$ is a generalized solution. The proof that u^* satisfies the weak equation for every φ in $\mathscr{M}$ is entirely the same as before. Furthermore, in view of the condition ii) imposed on the base $\{\psi^n\}$, we can show by an argument similar to the previous one that u^* satisfies the weak equation for every φ in $C^1_{0,\sigma}(R)$ and, hence, is a generalized solution.

It still remains to show the existence of b' satisfying Condition (B′). Firstly we deal with the case 1). Let M_0' and M_1' be constants such that

$$(3.21) \qquad |x-x_0| \cdot |b^*(x)| \leq M_0' \quad \text{and} \quad |x-x_1|^2 \cdot |\nabla b^*(x)| \leq M_1'$$

hold for any $x \in R$, where x_0 and x_1 are some fixed points. Then we have for any $w \in C^1_{0,\sigma}(R)$,

$$|\mathscr{V}(b^*, w, w)| = |\mathscr{V}(r_0 b^*, w/r_0, w)| \leq M_0' \|w/r_0\| \cdot \|\nabla w\| \leq 2M_0' \|\nabla w\|^2,$$
$$|\mathscr{V}(b^*, w, w)| = |\mathscr{V}(w, w, b^*)| \leq M_1' \|w/r_1\|^2 \leq 4M_1' \|\nabla w\|^2,$$

where $r_0 = |x-x_0|$ and $r_1 = |x-x_0|$. Thus b^* itself satisfies Condition (B′), if either

$$(3.22) \qquad \nu > 2M_0' \quad \text{or} \quad \nu > 4M_1'.$$

We turn to the case 2). We note that $\beta - u_\infty$ is the boundary value of $b^{*\prime} \in C_\sigma^1(\bar{R})$ expressible in the form $b^{*\prime} = \mathrm{rot}\, a'$, $a' \in C^1(\bar{R})$, since the constant vector u_∞ can be written as $u_\infty = \mathrm{rot}\,(1/2 \cdot u_\infty \times x)$. Making use of the same device as before, we can construct $b^{**\prime} = \mathrm{rot}\,(h(\rho)a')$ satisfying Condition (B′) for an arbitrarily small $\alpha > 0$. Thus the theorem has been established.

Remark 1. In Theorem 3.2 we may replace the assumptions on f by weaker ones that $f(x) - \nabla h \in L_2(R)$ and $|x|(f(x) - \nabla h) \in L_2(R)$ for some $h \in C^1$. Furthermore, under the condition 1) R may be the exterior domain of an arbitrary bounded set.

Remark 2. If β is of class C^2 and ∂R of class C^3, the condition 2) is equivalent to (3.2).

Remark 3. Concerning the exterior problem we can show the existence of the generalized solution even if the total out-flow m from the boundary ∂R is not 0, although m is assumed to be sufficiently small. As an example, we consider the simplest case. Suppose that R is the exterior domain of a compact closed surface of class C^3 enclosing the sphere $B(0, A)$ and β is of class C^2 and $f\equiv 0$. Introducing

$$\sigma(x)=\frac{m}{4\pi}\nabla\frac{1}{|x|}\ , \qquad \left(m=\int_{\partial R}\beta_n dS\right),$$

we put $\hat{\beta}=\beta-u_\infty-$(*the boundary value of* σ). Then we notice that

$$\int_{\partial R}\hat{\beta}_n dS=0$$

and hence $\hat{\beta}$ is expressible as the boundary value of rot $\hat{a}$, $\hat{a}\in C^2(\overline{R})$. Therefore we can construct a vector function $\hat{b}$ with compact carrier such that $\hat{b}\in C_\sigma{}^1(\overline{R})$, $\hat{b}=\hat{\beta}$ on ∂R and (3.18) is valid for $b'=\hat{b}$ and any fixed $\alpha>0$. We seek the generalized solution of the form $u=\hat{b}+\sigma+u_\infty+v$, $(v\in \widehat{H}^1_{0,\sigma}(R))$, along the way in the proof of Theorem 3.2. Then it is made clear that a generalized solution exists if $\nu>|m|/2\pi A$.

§4. Regularity in the Interior.

I) *Preliminary Considerations on Stokes' System.*

In the first part of this section we describe some formal relations concerning Stokes' system [12]

$$\Delta u-\nabla p=-f\ ,\quad \operatorname{div} u=0\ .$$

For the moment, V means a bounded domain with the smooth boundary ∂V and the functions u, v and p, q are assumed to be sufficiently smooth in $\overline{V}$, unless otherwise stated.

With a pair of a vector function u and a scalar function p we associate a 3×3-matrix function $T[u, p]=\{T_{ij}[u, p]\}$ by

$$T_{ij}[u, p]=-(p+\operatorname{div} u)\delta_{ij}+\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \tag{4.1}$$

Then by means of Gauss' divergence theorem we obtain

$$\int_V\{v(\Delta u-\nabla p)-u(\Delta v-\nabla q)+q\operatorname{div} u-p\operatorname{div} v\}\,dV = \int_{\partial V}\{vT[u, p]n-uT[v, q]n\}dS\ , \tag{4.2}$$

where n represents the outer normal vector with unit length. If $\operatorname{div} u=0$ and $\operatorname{div} v=0$, (4.2) is reduced to

$$(4.2)' \qquad \int_V \{v(\Delta u-\nabla p)-u(\Delta v-\nabla q)\}dV = \int_{\partial V} \{vT[u,\, p]n-uT[v,\, q]n\}dS\,.$$

We introduce the *fundamental solutions* of Stokes' system [12, 13], namely, the following 3×3-matrix function $E=\{E_{ij}\}$ and 3-vector function $e=\{e_j\}$;

$$(4.3) \qquad E_{ij}=\left(\delta_{ij}\Delta-\frac{\partial^2}{\partial x_i \partial x_j}\right)\Phi\,, \quad e_j=-\frac{\partial}{\partial x_j}\Delta\Phi\,, \qquad (i,\, j=1,\, 2,\, 3),$$

where $\Phi=|x-y|/8\pi$. More explicitly,

$$E_{ij}=\frac{1}{8\pi}\left\{\frac{1}{|x-y|}\delta_{ij}+\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}\right\}\,, \quad e_j=\frac{1}{4\pi}\frac{x_j-y_j}{|x-y|^3}\,.$$

We notice immediately $E_{ij}(x-y)=E_{ji}(x-y)=E_{ij}(y-x)=E_{ji}(y-x)$ and $e_j(x-y)=-e_j(y-x)$. Also the following inequalities are obvious:

$$|E_{ij}|\leq C|x-y|^{-1}\,, \quad |e_j|\leq C|x-y|^{-2}\,, \quad \left|\frac{\partial}{\partial x_m}E_{ij}\right|\leq C|x-y|^{-2}\,.$$

We can easily verify that

$$(4.4) \qquad \frac{\partial}{\partial x_i}E_{ij}=0\,, \quad \frac{\partial}{\partial y_j}E_{ij}=0\,, \qquad (x\neq y)$$

and that

$$(4.5) \qquad \Delta_x E_{ij}-\frac{\partial}{\partial x_i}e_j=-\delta_{ij}\delta(x-y)\,, \quad \Delta_y E_{ij}+\frac{\partial}{\partial y_j}e_i=-\delta_{ij}\delta(x-y)\,,$$

where $\delta(x-y)$ means the *dirac delta function*, because

$$\Delta_x E_{ij}-\frac{\partial}{\partial x_i}e_j=\delta_{ij}\Delta^2\Phi$$

and since $-\Phi$ is the fundamental solution of the biharmonic differential operator.

Hereafter we may use the following notations in connection with formal integral transforms, if no fear of confusion arises; let $T=\{T_{ij}\}$ and $t=\{t_j\}$ be a 3×3-matrix function and a 3-vector function defined in $V\times V$ respectively and let ψ be a 3-vector function defined in V. Then vector functions $T\psi$, $T^*\psi$, and scalar functions $t\psi$ and $\hat{t}\psi$ are defined by

$$T\psi(x)=\int_V T(x,\, y)\psi(y)dy\,, \quad T^*\psi(y)=\int_V \psi(x)T(x,\, y)dx\,,$$

$$t\psi(x)=\int_V t(x, y)\psi(y)dy\ , \qquad \hat{t}\psi(y)=\int_V \psi(x)t(x, y)dx\ .$$

Moreover, h being a scalar function a vector function t^*h is given by

$$t^*h(y)=\int_V h(x)t(x, y)dx\ .$$

With these notations we can write

$$\begin{aligned} &\Delta E\psi-\nabla e\psi=-\psi\ , \qquad \operatorname{div} E\psi=0, \\ &\Delta E^*\psi+\nabla\hat{e}\psi=-\psi\ , \qquad \operatorname{div} E^*\psi=0, \end{aligned} \tag{4.6}$$

for any smooth ψ. Denoting the i-th row vector of $\{E_{ij}\}$ by E_{i*}, we can verify that

$$\lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)} u(y)T[E_{i*}, -e_i]ndS_y=u_i(x) \tag{4.7}$$

for any vector function $u=u(y)$ continuous near $y=x$ [12, 13].

We define the *modified* or the *truncated fundamental solutions* of Stokes' system. Fix a scalar function $\eta^{(1)}(t)\in C^\infty(E_3)$ such that $\eta^{(1)}(t)$ depends on t as a function of $|t|$ and is subjected to the conditions

$$\eta^{(1)}(t)=\begin{cases}1\ , & (|t|\le 1), \\ 0\ , & (|t|\ge 2).\end{cases}$$

γ being a positive parameter, a family of functions $\eta^{(\gamma)}(t)$ is given by

$$\eta^{(\gamma)}(t)=\eta^{(1)}(t/\gamma)\ . \tag{4.8}$$

Evidently, $\eta^{(\gamma)}(t)=1$ if $|t|\le\gamma$ and $\eta^{(\gamma)}(t)=0$ if $|t|\ge 2\gamma$. We define the modified fundamental solutions $E^{(\gamma)}=\{E_{ij}^{(\gamma)}\}$ and $e^{(\gamma)}=\{e_j^{(\gamma)}\}$ by

$$E_{ij}^{(\gamma)}=\left(\delta_{ij}\Delta-\frac{\partial^2}{\partial x_i\partial x_j}\right)\Phi^{(\gamma)}\ ,\quad e_j^{(\gamma)}=-\frac{\partial}{\partial x_j}\Delta\Phi^{(\gamma)}\ ,$$

where $\Phi^{(\gamma)}=\eta^{(\gamma)}(x-y)\cdot\Phi=\eta^{(\gamma)}(x-y)\cdot|x-y|/8\pi$. Obviously $E^{(\gamma)}$ and $e^{(\gamma)}$ vanish identically for $|x-y|>2\gamma$ and coincide respectively with E and e for $|x-y|\le\gamma$. In particular, we have for any x, y

$$|E_{ij}^{(\gamma)}|\le C_\gamma|x-y|^{-1}\ ,\quad |e_j^{(\gamma)}|\le C_\gamma|x-y|^{-2}\ ,\quad \left|\frac{\partial}{\partial x_m}E_{ij}^{(\gamma)}\right|\le C_\gamma|x-y|^{-2}\ , \tag{4.9}$$

$(i, j, m=1, 2, 3)$, where C_γ is an constant depending on γ.

In addition, a 3×3-matrix function $H^{(\gamma)}=\{H_{ij}^{(\gamma)}\}$ is defined by $H^{(\gamma)}(0)=0$ and

$$H_{ij}^{(\gamma)}(x-y)=\delta_{ij}\Delta^2\Phi^{(\gamma)}\ , \qquad (x\neq y). \tag{4.10}$$

Since $\Delta^2\Phi=0$ for $x\neq y$ and all derivatives of $\eta^{(\gamma)}(x-y)$ vanish unless $\gamma<|x-y|<2\gamma$, $H^{(\gamma)}(x-y)$ is a function of class C^∞ and vanishes unless $\gamma<|x-y|<2\gamma$. In particular, the function $\boldsymbol{H}^{(\gamma)}u$ is of class C^∞ in the whole space E_3 provided that $u\in L_1$. The formulae obtained for E and e are modified and take the following forms, when E and e are replaced by $E^{(\gamma)}$ and $e^{(\gamma)}$. Similarly to (4.4) and (4.5) we have

$$\frac{\partial}{\partial x_i}E_{ij}^{(\gamma)}=0\,,\qquad \frac{\partial}{\partial y_j}E_{ij}^{(\gamma)}=0\,,\qquad (x\neq y), \tag{4.11}$$

and

$$\begin{aligned} &\Delta_x E_{ij}^{(\gamma)}-\frac{\partial}{\partial x_i}e_j^{(\gamma)}=-\delta_{ij}\delta(x-y)+H_{ij}^{(\gamma)}\,,\\ &\Delta_y E_{ij}^{(\gamma)}+\frac{\partial}{\partial y_j}e_i^{(\gamma)}=-\delta_{ij}\delta(x-y)+H_{ij}^{(\gamma)}\,. \end{aligned} \tag{4.12}$$

Similarly to (4.6) we have

$$\begin{aligned} &\Delta\boldsymbol{E}^{(\gamma)}\psi-\nabla e^{(\gamma)}\psi=-\psi+\boldsymbol{H}^{(\gamma)}\psi\,, &&\operatorname{div}\boldsymbol{E}^{(\gamma)}\psi=0\,,\\ &\Delta\boldsymbol{E}^{(\gamma)*}\psi+\nabla\hat{e}^{(\gamma)}\psi=-\psi+\boldsymbol{H}^{(\gamma)*}\psi\,, &&\operatorname{div}\boldsymbol{E}^{(\gamma)*}\psi=0\,, \end{aligned} \tag{4.13}$$

for any smooth ψ in V, on which the integration is extended. Moreover, as inferred from (4.12) we can derive the integral representation in $V(2\gamma)$

$$\begin{aligned} u&=-\boldsymbol{E}^{(\gamma)}(\Delta u-\nabla p)-e^{(\gamma)*}\operatorname{div}u+\boldsymbol{H}^{(\gamma)}u\\ &=-\boldsymbol{E}^{(\gamma)*}(\Delta u-\nabla p)-e^{(\gamma)*}\operatorname{div}u+\boldsymbol{H}^{(\gamma)*}u\,. \end{aligned} \tag{4.14}$$

In (4.14) we may put $p=0$ because of the relations

$$\int_V E^{(\gamma)}\cdot\nabla p\,dy=-\int_V \operatorname{div}E^{(\gamma)}\cdot p=0\,.$$

Thus (4.14) is reduced to

$$u=-\boldsymbol{E}^{(\gamma)}\Delta u+\boldsymbol{H}^{(\gamma)}u=-\boldsymbol{E}^{(\gamma)*}\Delta u+\boldsymbol{H}^{(\gamma)*}u \tag{4.15}$$

provided that $\operatorname{div}u=0$.

Finally we state the following lemma [7], which seems to be essentially well known.

Lemma 4.1. *Let $g(t)$ stand for any one of $\partial_m E_{ij}^{(\gamma)}(t)$ and $e_j^{(\gamma)}(t)$. Then for any θ contained in $0<\theta<1$, we can find a continuous function $\zeta(t)$ with the following properties; for any $t, t'\in E_3$ the inequality*

$$|g(t)-g(t')|\le|t-t'|^\theta\left|\frac{\zeta(t)}{|t|^{2+\theta}}+\frac{\zeta(t')}{|t'|^{2+\theta}}\right| \tag{4.16}$$

holds and $\zeta(t)\equiv0$ for $|t|>2\gamma$.

II) *Theorems concerning Interior Regularity.*

The remainder of this section is devoted to the following theorems.

Theorem 4.1. *Let u be a generalized solution of the interior or the exterior problem, in which $f\in L_2^{loc}(R)$. Then there exists the pressure p associated with u.*

Theorem 4.2. *Let u be a generalized solution of the interior or the exterior problem with $f\in L_2^{loc}(R)$. Let p be the associated pressure and V an arbitrary open subset of R. Then the following statements are true;*

1) *If f is locally bounded in V, then $u\in C^{1+\theta}(V)$ and $p\in C^{\theta}(V)$ for any θ in $0<\theta<1$.*

2) *If $f\in C^{n+h}(V)$, $(n=0, 1, \cdots, \infty;\ 0<h<1)$, then $u\in C^{n+2}(V)$ and $p\in C^{n+1}(V)$ and the Navier-Stokes system is strictly satisfied in V.*

3) *If f is analytic in V, so are u and p.*

In proving these theorems we assume $\nu=1$ without loss of generality. The first step of deriving the theorems is to prove the following "local" integral representation.

Lemma 4.2. *Let u be a generalized solution of the interior or the exterior problem with $f\in L_2^{loc}(R)$. Let K be an arbitrary bounded subdomain $\ni R$ and γ any positive constant. Then for almost every $x\in K(2\gamma)$, we have*

$$u(x)=-\int_K E^{(\gamma)}(x-y)\tilde{f}(y)dy+\int_K H^{(\gamma)}(x-y)u(y)dy \tag{4.17}$$

and

$$\partial_m u(x)=-\int_K \partial_m E^{(\gamma)}(x-y)\tilde{f}(y)\,dy+\int_K \partial_m H^{(\gamma)}(x-y)u(y)dy, \tag{4.18}$$

$(m=1, 2, 3)$, *where $\tilde{f}=(u\cdot\nabla)u-f$ and ∂_m means $\partial/\partial x_m$.*

Remark. Since the generalized solution u as well as its generalized derivatives is determined up to a null set by its nature, we may and hereafter shall consider that (4.17) and (4.18) are valid for every x in $K(2\gamma)$. Also in what follows we shall agree on the conventions of this sort.

Proof of Lemma 4.2. Take a general element ψ of $C_0^\infty(K(2\gamma))$ and put $\varphi=\boldsymbol{E}^{(\gamma)}*\psi$, $\pi=\hat{e}^{(\gamma)}\psi$. Then according to (4.13) we have $\varphi\in C_{0,\sigma}^\infty(K)\subset C_{0,\sigma}^1(R)$ and $\Delta\varphi+\nabla\pi=-\psi+\boldsymbol{H}^{(\gamma)}*\psi$. The result of substitution of this φ into the weak equation

$$-(\nabla\varphi,\ \nabla u)=(\varphi,\ \tilde{f})$$

is reduced to

$$(\psi,\ u)=(\psi,\ -\boldsymbol{E}^{(\gamma)}\tilde{f}+\boldsymbol{H}^{(\gamma)}u), \qquad (\psi\in C_0^\infty(K(2\gamma))), \tag{4.19}$$

by means of the following relations;

$$(\varphi,\ \tilde{f})=(\boldsymbol{E}^{(\gamma)}*\psi,\ \tilde{f})=(\psi,\ \boldsymbol{E}^{(\gamma)}\tilde{f})$$

$$\begin{aligned}(\nabla\varphi,\ \nabla u)&=-(\Delta\varphi,\ u)=(\psi+\nabla\pi-\boldsymbol{H}^{(\gamma)}*\psi,\ u)=(\psi-\boldsymbol{H}^{(\gamma)}*\psi,\ u)\\&=(\psi,\ u)-(\psi,\ \boldsymbol{H}^{(\gamma)}u).\end{aligned}$$

The fact that (4.19) is valid for every $\psi \in C_0^\infty(K(2\gamma))$ implies (4.17).

In deriving (4.18), differentiation of the integrals of (4.17) directly under the integral sign seems difficult to justify and we prefer an alternative way. Take a general element ψ in $C_0^\infty(K(2\gamma))$ and put

$$\varphi^m = -\boldsymbol{E}^{(\gamma)*}(\partial_m\psi)\ , \qquad \pi^m = -\hat{e}^{(\gamma)}(\partial_m\psi)\ .$$

Again (4.13) gives $\varphi^m \in C_{0,\sigma}^\infty(K)$, $\pi^m \in C_0^\infty(K)$ and

$$\Delta\varphi^m + \nabla\pi^m = \partial_m\psi - \boldsymbol{H}^{(\gamma)*}(\partial_m\psi)\ .$$

Since $H^{(\gamma)}$ is a smooth function and the singularity of $\partial_m E^{(\gamma)}$ is $O(|x-y|^{-2})$, we have by partial integration

$$\boldsymbol{E}^{(\gamma)*}(\partial_m\psi) = -(\partial_m\boldsymbol{E}^{(\gamma)})^*\psi \quad \text{and} \quad \boldsymbol{H}^{(\gamma)*}(\partial_m\psi) = -(\partial_m\boldsymbol{H}^{(\gamma)})^*\psi\ .$$

In view of these relations we can calculate as $-(\nabla\varphi^m, \nabla u) = (\Delta\varphi^m, u) = (\partial_m\psi - \nabla\pi + (\partial_m\boldsymbol{H}^{(\gamma)})^*\psi, u) = (\partial_m\psi, u) + ((\partial_m\boldsymbol{H}^{(\gamma)})^*\psi, u) = -(\psi, \partial_m u) + (\psi, \partial_m\boldsymbol{H}^{(\gamma)}u)$. Evidently $(\varphi^m, \tilde{f}) = (\psi, (\partial_m\boldsymbol{E}^{(\gamma)})\tilde{f})$. Thus substituting $\varphi = \varphi^m$ into the weak equation we obtain

$$(\psi, \partial_m u) = (\psi, -(\partial_m\boldsymbol{E}^{(\gamma)})\tilde{f} + (\partial_m\boldsymbol{H}^{(\gamma)})u)$$

for any $\psi \in C_0^\infty(K(2\gamma))$ and obtain (4.18).

Proof of Theorem 4.1. Let K' be an arbitrary bounded subdomain $\dashv R$. Evidently there exists a bounded subdomain K such that $K' \dashv K(2\gamma) \dashv K \dashv R$ holds for any sufficiently small $\gamma > 0$. By means of (4.17) we have

$$(\nabla\varphi, \nabla u) = (\Delta\varphi, -u) = (\Delta\varphi, \boldsymbol{E}^{(\gamma)}\tilde{f} - \boldsymbol{H}^{(\gamma)}u) = (\boldsymbol{E}^{(\gamma)*}\Delta\varphi, \tilde{f}) - (\varphi, \Delta\boldsymbol{H}^{(\gamma)}u)$$

for any $\varphi \in C_{0,\sigma}^2(K(2\gamma))$. On the other hand, $\boldsymbol{E}^{(\gamma)*}\Delta\varphi = -\varphi + \boldsymbol{H}^{(\gamma)*}\varphi$ by virtue of (4.15). Therefore, $(\nabla\varphi, \nabla u) = -(\varphi, \tilde{f}) + (\boldsymbol{H}^{(\gamma)*}\varphi, \tilde{f}) - (\varphi, \Delta\boldsymbol{H}^{(\gamma)}u)$ is obtained. From this equality and the weak equation it follows that

$$(\varphi, \boldsymbol{H}^{(\gamma)}\tilde{f} - \Delta\boldsymbol{H}^{(\gamma)}u) = 0$$

is valid for any $\varphi \in C_{0,\sigma}^2(K(2\gamma))$, while $\boldsymbol{H}^{(\gamma)}\tilde{f} - \Delta\boldsymbol{H}^{(\gamma)}u$ is of class C^∞ in the whole space. Hence by virtue of a familiar theorem of vector analysis, there exists a scalar function $\pi' \in C^\infty(K(2\gamma))$ determined uniquely modulo an additive constant such that

$$\boldsymbol{H}^{(\gamma)}\tilde{f} - \Delta\boldsymbol{H}^{(\gamma)}u = -\nabla\pi'$$

everywhere in $K(2\gamma)$. Then we introduce

$$p'(x) = \pi'(x) - \int_K e^{(\gamma)}(x-y)\tilde{f}(y)dy = \pi' - e^{(\gamma)}\tilde{f}$$

and note that $e^{(\gamma)}\tilde{f} \in L_2(K(2\gamma))$ in consideration of (4.9), Lemma 2.7 and $\tilde{f} \in L_{4/3}(K)$. We proceed to the verification that $p = p'$ satisfies the defining equation of the pressure

$$(4.20)\qquad (\nabla\varphi,\ \nabla u)-(\operatorname{div}\varphi,\ p)=-(\varphi,\ \tilde{f})$$

for every $\varphi\in C_0^2(K(2\gamma))$ and hence, for any $\varphi\in C_0^2(K')$. According to (4.14), $\boldsymbol{E}^{(\gamma)*}\Delta\varphi=-\varphi-\boldsymbol{e}^{(\gamma)*}\operatorname{div}\varphi+\boldsymbol{H}^{(\gamma)}\varphi$. In view of this relation and (4.17), we can calculate as

$$\begin{aligned}(\nabla\varphi,\ \nabla u)&=-(\Delta\varphi,\ u)=(\Delta\varphi,\ \boldsymbol{E}^{(\gamma)}\tilde{f}-\boldsymbol{H}^{(\gamma)}u)=(\boldsymbol{E}^{(\gamma)*}\Delta\varphi,\ \tilde{f})-(\varphi,\ \Delta\boldsymbol{H}^{(\gamma)}u)\\&=-(\varphi,\ \tilde{f})-(\boldsymbol{e}^{(\gamma)*}\operatorname{div}\varphi,\ \tilde{f})+(\boldsymbol{H}^{(\gamma)*}\varphi,\ \tilde{f})-(\varphi,\ \Delta\boldsymbol{H}^{(\gamma)}u)\\&=-(\varphi,\ \tilde{f})-(\operatorname{div}\varphi,\ \boldsymbol{e}^{(\gamma)}\tilde{f})+(\varphi,\ \boldsymbol{H}^{(\gamma)}\tilde{f}-\Delta\boldsymbol{H}^{(\gamma)}u)\\&=-(\varphi,\ \tilde{f})-(\operatorname{div}\varphi,\ \boldsymbol{e}^{(\gamma)}\tilde{f})-(\varphi,\ \nabla\pi')\\&=-(\varphi,\ f)+(\operatorname{div}\varphi,\ \pi'-\boldsymbol{e}^{(\gamma)}\tilde{f})=-(\varphi,\ \tilde{f})+(\operatorname{div}\varphi,\ p')\ .\end{aligned}$$

Now let δ be an arbitrary positive constant and set $K_\delta=R(\delta)\cap B(0,\ 1/\delta)$. In every K_δ we construct a scalar function p_δ satisfying (4.20) for any $\varphi\in C_0^2(K_\delta)$ just in the same way as we have constructed p' in K'. Evidently $0<\delta''<\delta'$ implies $K_{\delta'}\subset K_{\delta''}\subset R$. Then by the reasoning in the remark below Definition 2.3 it is made clear that $p_{\delta'}-p_{\delta''}\equiv$const. in $K_{\delta'}$. On the other hand, p_δ involves an arbitrary additive constant. Therefore we can choose this additive constant so that $p_{\delta'}=p_{\delta''}$ in $K_{\delta'}$ for any δ', δ'' $(0<\delta''<\delta')$. Then we define $p\in L_2^{loc}(R)$ as the inductive limit of p_δ as $\delta\to 0$. This p satisfies (4.20) not only for any $\varphi\in C_0^2(R)$ but also for any $\varphi\in C_0^1(R)$, because any function in $C_0^1(R)$ can be uniformly approximated together with the first derivatives by functions in $C_0^2(R)$. Thus Theorem 4.1 has been proved.

The following proposition has been established in the course of the proof of Theorem 4.1.

Lemma 4.3. *Under the same assumptions as in Theorem 4.1, the associated pressure p is expressible in the form*

$$(4.21)\qquad p(x)=\pi(x)+\int_K e^{(\gamma)}(x-y)\tilde{f}(y)\,dy$$

for $x\in K(2\gamma)$, where K is an arbitrary bounded subdomain-$\ni R$ and π is a certain function of class C^∞.

Proof of Theorem 4.2. Let K be an arbitrary bounded subdomain-$\ni R$. Then on account of (4.17) and (4.21) we notice that the regularity of u and p in $K(2\gamma)$ is implied respectively by that of

$$(4.22)\qquad v(x)=\int_K E^{(\gamma)}(x-y)\tilde{f}(y)dy\ ,\quad q(x)=\int_K e^{(\gamma)}(x-y)\tilde{f}(y)dy\ .$$

On the other hand, the types of the sigularities of $E^{(\gamma)}$ and $e^{(\gamma)}$, which are identical with those of E and e, are familar in potential theory and the following proposition is essentially well known: if $\tilde{f}\in C^{n+h}(K)$, $(n=0,\ 1,\cdots,\ \infty;\ 0<h<1)$, then $v\in C^{n+2}(K(2\gamma))$ and $q\in C^{n+1}(K(2\gamma))$. From this proposition we immediately arrive at the following

Lemma 4.4. *In addition to the assumptions of Theorem 4.2, assume that* $f\in C^{n+h}(V)$ $(n=0, 1, \cdots, \infty;\ 0<h<1)$ *and* $u\in C^{1+h}(V)$. *Then* $u\in C^{n+2}(V)$ *and* $p\in C^{n+1}(V)$.

We recall (2.23) and note that

$$\|\nabla u\|_R\leq C,\quad \|u(y)/|x-y|\,\|_K\leq C\,, \tag{4.23}$$

K being an arbitrary fixed bounded subdomain of R. Our task is to improve the regularity of u starting from (4.23) so that Lemma 4.4 is applicable. For this purpose, we establish the following series of propositions:

i) $u\in L_\infty(K(2\gamma))$ if $f\in L_2(K)$.

ii) $\nabla u\in L_5(K(2\gamma))$ if $f\in L_2(K)$ and $u\in L_\infty(K)$.

iii) $\nabla u\in L_\infty(K(2\gamma))$, if $f\in L_5(K)$ and $u\in L_\infty(K)$ and $\nabla u\in L_5(K)$.

iv) $\nabla u\in C^\theta(K(2\gamma))$ for any θ in $0<\theta<1$, if u, ∇u and $f\in L_\infty(K)$.

Once these propositions have been established, the wanted improvement of the regularity of u can be achieved by successive application of them in consideration that γ is an arbitrary positive constant.

Proof of i). We put

$$v'(x)=\boldsymbol{E}^{(\gamma)}(u\cdot\nabla)u=\int_K E^{(\gamma)}(x-y)(u\cdot\nabla)u(y)dy$$

and similarly

$$v''=\boldsymbol{E}^{(\gamma)}f\,,\quad h=\boldsymbol{H}^{(\gamma)}u\,.$$

According to (4.17) $u=-v+h=-v'+v''+h$ in $K(2\gamma)$. We note that h is of class C^∞ in the whole space. By virtue of (4.19) and (4.23) we have

$$|v'(x)|\leq C\int_K\frac{|u|\cdot|\nabla u|}{|x-y|}dy\leq C\|u(y)/|x-y|\,\|_K\cdot\|\nabla u\|_K\leq C$$

and

$$|v''(x)|\leq C\|\,|x-y|^{-1}\|_K\cdot\|f\|_K\leq C\|f\|_K\leq C\,.$$

Consequently, $u\in L_\infty(K(2\gamma))$.

Proof of ii). Putting

$$v^m(x)=\int_K\partial_m E^{(\gamma)}(x-y)\tilde{f}(y)dy\,,\qquad (m=1, 2, 3),$$

we have $\partial_m u(x)=-v^m(x)+\partial_m h(x)$ by means of (4.18). $\partial_m h$ is of class C^∞ in the whole space. Under the assumptions of ii) $\tilde{f}=(u\cdot\nabla)u-f\in L_2(K)$. By means of (4.9) we obtain

$$\int_{K(2\gamma)}|\partial_m E^{(\gamma)}|^{10/7}dy\leq C\int_{B(x,2\gamma)}|x-y|^{-20/7}dy\leq C\gamma^{1/7}\,,$$

which enables us to apply Lemma 2.7 with $p=2$, $q=5$, $r=10/7$. As the result we have $\|v^m\|_{5,K(2\gamma)} \leq C\|\tilde{f}\|_{2,K}$, which implies obviously $\nabla u \in L_5(K(2\gamma))$.

Proof of iii). By virtue of (4.9) and Hölder's inequality we have

$$|v^m(x)| \leq C\int_B \frac{|\tilde{f}(y)|}{|x-y|^2}dy \leq C\||x-y|^{-2}\|_{5/4,B} \cdot \|\tilde{f}\|_{5,K} \leq C\|\tilde{f}\|_{5,K}$$

where $B = B(x, 2\gamma)$. Hence $v^m \in L_\infty(K(2\gamma))$ and thus $\nabla u \in L_\infty(K(2\gamma))$.

Proof of iv). It suffices to show that $v^m \in C^\theta(K(2\gamma))$. This follows from $\tilde{f} \in L_\infty(K)$ with the aid of Lemma 4.1. Namely,

$$|v^m(x) - v^m(x')| \leq C|x-x'|^\theta \int_K \left\{ \frac{\zeta(x-y)}{|x-y|^{2+\theta}} + \frac{\zeta(x'-y)}{|x'-y|^{2+\theta}} \right\} dy$$
$$\leq C|x-x'|^\theta \int_{B(x,2\gamma)} |x-y|^{-2-\theta} dy \leq C|x-x'|^\theta .$$

Just in the same way as in the proof of iv) we can show that $q(x)$ in (4.22) as well as $p(x)$ is of class C^θ in $K(2\gamma)$ if $\tilde{f} \in L_\infty(K)$.

From the results so far obtained, 1) and 2) of Theorem 4.2 have been established. Finally we deal with 3). If f is analytic, u, p are of class C^∞ in V according to 2). They satisfy the Navier-Stokes system

$$-\Delta u + (u \cdot \nabla)u + \nabla p = f , \qquad \operatorname{div} u = 0,$$

which forms an elliptic (but not strongly elliptic) system with four unknown functions u_1, u_2, u_3, p. Examining the structure of the Navier-Stokes system we notice that a theorem due to Morrey [11] and Friedman [4] concerning the analyticity of solutions of such a system is applicable and we obtain the desired analyticity of u and p. Thus Theorem 4.2 has been established.

§ 5. Regularity at the Boundary.

This section is concerned with the regularity of a generalized solution u at the boundary of R. We shall prove

Theorem 5.1. *Let u be a generalized solution and p the associated pressure. Let ∂R^* be one of the components of ∂R such that ∂R^* is of class C^{2+h} and β is of class C^{1+h} on ∂R^* for some h in $0<h<1$. If f is bounded near ∂R^*, then ∇u and p are Hölder continuous near and on ∂R^*. Moreover, u assumes the boundary value β on ∂R^*.*

Before entering into the derivation of this theorem we state some propositions concerning the interior problem obtainable by combining this theorem with the preceding theorems. The corresponding propositions concerning the exterior problem are found in the next section.

Theorem 5.2. *In the interior problem we can identify the generalized solution with the strict solution, if f is bounded and Hölder continuous in R and if ∂R and β are of class C^{2+h} and of class C^{1+h} for some h in $0<h<1$, respectively.*

Theorem 5.3. *In addition to the assumptions in Theorem 5.2, assume that* *one of the conditions* 1) *and* 2) *in Theorem 4.1 is satisfied. Then there exists a strict solution of this interior problem.*

As the preliminaries for the proof of Therem 5.1, we describe some results of Odqvist [12] concerning *Stokes' problem.* Let V be a bounded domain with the boundary ∂V of class $C^{2+h}(0<h<1)$. In this domain we consider Stokes' problem composed of *Stokes' system*

$$\Delta v-\nabla q=-f' , \qquad \mathrm{div}\, v=0,$$

and the boundary condition that

$$v=\beta' \qquad \text{on } \partial V$$

for an unknown vector function v and an unknown scalar function q with given f' and β'. Odqvist developed an analogue of potential theory with respect to this problem and deduced the following results:

i) If $f'=0$ and $\beta'\in C^{1+h}$ is subjected to the subsidiary condition

$$\int_{\partial V} \beta_n' dS=0 ,$$

then the solution $\{v, q\}$ exists and is analytic in V. Then it is known also that $v\in C^{1+h}(\bar{V})$ and $q\in C^{h}(\bar{V})$.

ii) There exist a 3×3-matrix function $G(x, y)=\{G_{ij}(x, y)\}$ and a 3-vector function $g(x, y)=\{g_j(x, y)\}$ such that

$$v(x)=\boldsymbol{G}f'(x)=\int_V G(x, y)f'(y)dy , \tag{5.1}$$

$$q(x)=\boldsymbol{g}f'(x)=\int_V g(x, y)f'(y)dy$$

furnish the solution of Stokes' problem with the homogeneous boundary condition $\beta'=0$ for any smooth f'. v and q in (5.1) are known to satisfy the conditions

$$v\in C^2(V)\cap C^{1+h}(\bar{V}) , \quad q\in C^1(V)\cap C^h(\bar{V}) ,$$

provided that f' is Hölder continuous in V and bounded in $\bar{V}$. The kernel $G(x, y)$ is formally symmetric. If $k(x, y)$ stands for any one of $G_{ij}(x, y)$ and $K(x, y)$ stands for any one of $g_j(x, y)$, $\partial G_{ij}(x, y)/\partial x_m$, $(i, j, m=1, 2, 3)$, then

$$|k(x, y)|\leq C|x-y|^{-1} \quad \text{and} \quad |K(x, y)|\leq C|x-y|^{-2} \tag{5.2}$$

for any $x, y\in \bar{V}$ with an appropriate constant C. Moreover,

$$|K(x, y)-K(x', y)|\leq C\left|\frac{d|\log d|}{r^3}+\frac{d|\log d|^2}{r^2}+\frac{d^h}{r}\right| \tag{5.3}$$

for any $x, x', y\in \bar{V}$, where $r=\min\{|x-y|, |x'-y|\}$, $d=|x-x'|$.

Proof of Theorem 5.1. We may assume $\nu=1$ without loss of generality. According to Theorem 4.2 u is of class $C^{1+\theta}$ $(0<\theta<1)$ in an suitable open set surrounding ∂R^*. Draw a smooth closed surface S through this open set so that S encloses and bounds away ∂R^* from the other components of ∂R. Hereafter V means the annular domain between ∂R^* and S. Therefore $\partial V=\partial R^*+S$.

Let u' and p' be the solution of Stokes' problem in V with $f'=0$ and β' such that $\beta'=\beta$ on ∂R^* and $\beta'=u$ on S. Then taking a general element ψ in $C_0^\infty(V)$, we put

$$\varphi=\boldsymbol{G}\psi=\boldsymbol{G}^*\psi\ , \quad \pi=\boldsymbol{g}\psi\ .$$

As stated above, $\varphi\in C_\sigma^{1+h}(\bar{V})$ and $\varphi=0$ on ∂V and we can easily show[9] $\varphi\in \hat{H}^1_{0,\sigma}(V)$. Hence we can substitute $\varphi=\boldsymbol{G}\psi$ into the weak equation

$$-(\nabla\varphi,\ \nabla u)=(\varphi,\ \tilde{f})\ , \qquad (\tilde{f}=(u\cdot\Delta)u-f),$$

after extending φ outside V in the natural way. The results of the substitution are reduced to

$$(5.4)\qquad (\psi,\ u-u'+\boldsymbol{G}\tilde{f})=0$$

in consideration that $\Delta\varphi-\nabla\pi=-\psi$ and by means of the following relations to be justified later[10]:

$$(5.5)\qquad (\varphi,\ \tilde{f})_R=(\boldsymbol{G}^*\psi,\ \tilde{f})_V=(\psi,\ \boldsymbol{G}\tilde{f})_V\ ,$$

$$\begin{aligned}(5.6)\qquad (\nabla\varphi,\ \nabla u)_R&=(\nabla\varphi,\ \nabla u)_V=(\nabla\varphi,\ \nabla(u-u'))+(\nabla\varphi,\ \nabla u')\\ &=-(\Delta\varphi,\ u-u')-(\varphi,\ \Delta u')=(\psi-\nabla\pi,\ u-u')-(\varphi,\ \nabla p')\\ &=(\psi,\ u-u')+(\operatorname{div}\varphi,\ p')=(\psi,\ u-u')\ .\end{aligned}$$

The fact that (5.4) holds for every $\psi\in C_0^\infty(V)$ implies that

$$(5.7)\qquad u(x)=u'(x)-\boldsymbol{G}\tilde{f}=u'(x)-\int_V G(x,\ y)\tilde{f}(y)dy$$

for almost every $x\in V$. We can differentiate (5.7) under the integral sign, for the resulting integral converges absolutely and locally uniformly. Hence we have

$$\partial_m u(x)=\partial_m u'(x)-\int_V \frac{\partial}{\partial x_m}G(x,\ y)\tilde{f}(y)dy\ .$$

By means of these integral representations and with the aid of lemma 2.7, the

9) The proof of this fact is not so trivial because of the subsidiary condition $\operatorname{div}\varphi=0$. It will be treated subsequently with some lemmas of the same sort.

10) In general, $\Delta\boldsymbol{G}\psi$ and $\nabla\boldsymbol{g}\psi$ are not smooth on the closed set $\bar{V}$ even if ψ is smooth there.

following propositions are established in the same way as for the corresponding ones in § 4:

i) $u \in L_\infty(V)$ if $f \in L_2(V)$.

ii) $\nabla u \in L_5(V)$ if $f \in L_2(V)$ and $u \in L_\infty(V)$.

iii) $\nabla u \in L_\infty(V)$ if $f \in L_5(V)$, $u \in L_\infty(V)$ and $\nabla u \in L_5(V)$.

Since we assumed $f \in L_\infty(V)$, we see by successive application of these propositions that $\tilde{f}=(u\cdot\nabla)u-f$ as well as ∇u is bounded in V. In order to show that ∇u is Hölder continuous in the closed set $\bar{V}$, it suffices to show that a function $w(x)$ defined by

$$w(x)=\int_V K(x,\ y)h(y)dy\ , \qquad (h \in L_\infty(V)), \tag{5.8}$$

is Hölder continuous under the assumptions of (5.2) and (5.3). Let x and x' be two arbitrary points in V and set $d=|x-x'|$, $B'=B((x+x')/2,\ d)$, $V'=V-B'$. Evidently we have

$$\begin{aligned}|w(x)-w(x')| &\leq \int_{B'}|K(x,\ y)h(y)|dy+\int_{B'}|K(x',\ y)h(y)|dy\\ &\quad+\int_{V'}|K(x,\ y)-K(x',\ y)|\cdot|h(y)|dy\\ &\equiv I_1(x)+I_2(x')+I_3(x,\ x')\ ,\end{aligned}$$

whereas we easily obtain $I_1(x)\leq Cd$ and $I_2(x')\leq Cd$ and obtain $I(x,\ x') \leq C\{d|\log d|^2+d^h\}\leq Cd^h$. In this way we have

$$|w(x)-w(x')|\leq C|x-x'|^h \tag{5.9}$$

for any $x,\ x' \in V$ with constants $C,\ h$ independent of x and x'.

Concerning p we can give the integral representation

$$p(x)=\text{const.}+p'(x)-\int_V g(x,\ y)\tilde{f}(y)dy \tag{5.10}$$

by verifying that the right member satisfies the defining equation of the pressure

$$(\nabla\varphi,\ \nabla u)+(\varphi,\ \tilde{f})-(\text{div}\ \varphi,\ p)=0$$

for any $\varphi \in C_0^2(V)$. In fact, we have by means of (5.7)

$$\begin{aligned}&(\nabla\varphi,\ \nabla u)+(\varphi,\ \tilde{f})-(\text{div}\ \varphi,\ p'-\boldsymbol{g}\tilde{f})\\ &\quad=-(\Delta\varphi,\ u)+(\varphi,\ \tilde{f})+(\varphi,\ \nabla p')+(\boldsymbol{g}^*\,\text{div}\ \varphi,\ \tilde{f})\\ &\quad=-(\Delta\varphi,\ u'-\boldsymbol{G}\tilde{f})+(\varphi,\ \tilde{f})+(\varphi,\ \nabla p')+(\boldsymbol{g}^*\,\text{div}\ \varphi,\ \tilde{f})\\ &\quad=-(\varphi,\ \Delta u'-\nabla p')+(\boldsymbol{G}^*\Delta\varphi+\boldsymbol{g}^*\,\text{div}\ \varphi+\varphi,\ \tilde{f})\\ &\quad=(\boldsymbol{G}^*\Delta\varphi+\boldsymbol{g}^*\,\text{div}\ \varphi+\varphi,\ \tilde{f})\ .\end{aligned}$$

On the other hand, $G^*\Delta\varphi+g^*\operatorname{div}\varphi+\varphi=0$ because we have $(G^*\Delta\varphi+g^*\operatorname{div}\varphi, h)$ $=(\Delta\varphi, Gh)+(\operatorname{div}\varphi, gh)=(\varphi, \Delta Gh-\nabla gh)=-(\varphi, h)$ for any smooth h. By the way we note that if f is Hölder continuous, then $u\in C^2$, $p\in C^1$ and the verification of the defining equation of pressure is more simply carried out by showing that $-\Delta u+\nabla(p'-g\tilde{f})=-\tilde{f}$.

We see from (5.10) that p is of class C^h if $g\tilde{f}$ is, whereas (5.9) yields the Hölder continuity of $g\tilde{f}$ by virtue of (5.3) and (5.8).

Now it remains to justify the formal calculations in (5.5) and (5.6). (5.5) is legitimate according to Fubini's theorem. Putting $\tilde{u}=u-u'$ we have

$$(5.11)\qquad (\nabla\varphi, \nabla\tilde{u})=(\nabla\varphi, \nabla\tilde{u})_{\omega(\gamma)}+(\nabla\varphi, \nabla\tilde{u})_{V(\gamma)}$$
$$=(\nabla\varphi, \nabla\tilde{u})_{\omega(\gamma)}+(\psi, \tilde{u})_{V(\gamma)}+\int_{\partial V(\gamma)}\left(\frac{\partial\varphi}{\partial n}\tilde{u}-\pi\tilde{u}_n\right)dS$$

with an arbitrary small positive constant γ by means of the necessary partial integrations. Evidently $(\nabla\varphi, \nabla\tilde{u})_{\omega(\gamma)}\to 0$ and $(\psi, \tilde{u})_{V(\gamma)}\to(\psi, \tilde{u})_V$ as $\gamma\to 0$. We recall that S is contained in R and u, φ, π are smooth near S. Noting that $\tilde{u}=0$ on S, we have

$$\int_{S(\gamma)}\left(\frac{\partial\varphi}{\partial n}\tilde{u}-\pi\tilde{u}_n\right)dS\to 0\,, \qquad (\gamma\to 0),$$

where $S(\gamma)$ is the component of $\partial V(\gamma)$ adjacent to S. Next, we note that $\tilde{u}$ is expressible as $\tilde{u}=w+b$ with $w\in\hat{H}^1_{0,\sigma}(V)$ and smooth b vanishing on ∂R^*. Hence using the notations in Lemma 2.5, we have by means of Lemma 2.5

$$\int_{\omega^*(\gamma)}|\tilde{u}/\rho^*|^2dy\leq K \quad\text{and}\quad \int_{\omega^*(\gamma)}|\nabla\tilde{u}|^2dy\leq K$$

$(0<\gamma\leq\gamma_0)$, whence it follows that

$$\tilde{\mu}(\gamma)=\int_{\rho^*=\gamma}|u|^2dS\to 0\,, \qquad (\gamma\to 0)$$

according to Lemma 2.6. Consequently we have

$$\left|\int_{\rho^*=\gamma}\left(\frac{\partial\varphi}{\partial n}\tilde{u}-\pi\tilde{u}_n\right)dS\right|\leq C\sqrt{\tilde{\mu}(\gamma)}\to 0\,, \qquad (\gamma\to 0),$$

since $\partial\varphi/\partial n$ and π are Holder continuous near ∂R^*. In this way we ascertain that the surface integral in (5.11) tends to 0 as $\gamma\to 0$. Thus $(\nabla\varphi, \nabla\tilde{u})=(\psi, \tilde{u})$ is shown by making $\gamma\to 0$ in (5.11). On the other hand, the verification of $(\nabla\varphi, \nabla u')$ $=0$ is carried out by use of an appropriate sequence φ^n such that $\varphi^n\in C^1_{0,\sigma}(V)$ and $\varphi^n\to\varphi$ $(n\to\infty)$ strongly in $\hat{H}^1_{0,\sigma}(V)$.

Combining these results the calculations in (5.6) have been justified and hence the theorem has been established.

§ 6. Regularity at the Infinity.

This and the succeeding section are devoted to the study of the behavior at the infinity of generalized solutions of the exterior problem. The main question is whether the generalized solution u really satisfies the boundary condition at the infinity $u(x)\to u_\infty$, $(|x|\to\infty)$. This question was essentially resolved with the affirmative results under the physically acceptable assumption $f\equiv 0$ for the case $u_\infty=0$ by J. Leray [10] and for the case $u_\infty\neq 0$ recently by R. Finn [3]. We shall give another proof of this result, which has some features distinct from those of the proofs by Leray or Finn and is possibly applicable to problems of more general types. Besides, some results a little more detailed will be obtained. With a view to presenting various devices we adopt different methods according as $u_\infty=0$ or $u_\infty\neq 0$, though the method applied to the latter case is conveniently applicable to the other case. Our main results are the following two theorems, which are essentially independent of each other.

Theorem 6.1. *Let u be a generalized solution of the exterior problem and assume $f\in L_\infty^{loc}(R)\cap L_2(R)$. Then $u(x)\to u_\infty$ as $|x|\to\infty$.*

Theorem 6.2. *Let u be a generalized solution of the exterior problem and p the associated pressure. Let V be an unbounded subdomain of R with smooth compact boundary ∂V. f is assumed to be Hölder continuous in V and to possess the following properties;*

$$(6.1)\qquad f\in L_2(V)\cap L_\infty(V)\,,\quad f'\in L_2(V),\ (f'(x)=|x|f(x))\,,\qquad f(x)\to 0\qquad as\ |x|\to\infty.$$

Then we have

$$(6.2)\qquad u(x)\to u_\infty\,,\quad \nabla u(x)\to 0\,,\qquad (|x|\to\infty),$$

and $p(x)$ tends to a constant p_∞ as $|x|\to\infty$. Furthermore, we have the following integral representations[11]*:*

i) If $\nu=1$ and $u_\infty=0$, we have for $x\in V$

$$(6.3)\qquad u(x)=-\int_V E(x-y)\tilde{f}(y)dy+s(x)\,,$$

$$(6.3)'\qquad p(x)=p_\infty-\int_V e(x-y)\tilde{f}(y)dy+\sigma(x)$$

where $\tilde{f}=(u\cdot\nabla)u-f$,

$$s_i(x)=\int_{\partial V}\{E_{i*}T[u,\ p]n-uT[E_{i*},\ -e_i]n\}dS\,,\qquad (i=1,\ 2,\ 3),$$

$$\sigma(x)=\int_{\partial V}\{eT[u,\ p]n-uT[e,\ 0]n\}dS$$

and where E and e are those given in (4.3).

11) For any $\nu>0$ and u_∞, similar integral representations are possible [13].

ii) If $\nu=1$ *and* u_∞ *is normalized as* $u_\infty=\{2, 0, 0\}$, *then we have for* $x\in V$

$$u(x)=u_\infty-\int_V \mathscr{E}(x-y)\tilde{f}(y)dy+s(x), \tag{6.4}$$

$$p(x)=p_\infty-\int_V e(x-y)\tilde{f}(y)dy+\sigma(x), \tag{6.4'}$$

where $\tilde{f}=(v\cdot\nabla)v-f$, $v=u-u_\infty$,

$$s_i(x)=\int_{\partial V}\{\mathscr{E}_{i*}T[v, p]n-vT[\mathscr{E}_{i*}, -e_i]n-2(\mathscr{E}_{i*}v)n_1\}dS, \quad (i=1, 2, 3),$$

$$\sigma(x)=\int_{\partial V}\{eT[v, p]n-vT[e, 2e_1]-(ev)n_1\}dS$$

and where $\mathscr{E}$ *and* e *are those given later in* (7.3).

Remark 1. The conclusions concerning u and ∇u of the theorems remain valid if the assumptions on f are satisfied by $f-\nabla\Phi$ with some $\Phi\in C^1$.

Combining the theorems so far obtained in connection with the exterior problem we have

Theorem 6.3. *As to the exterior problem we can identify the generalized solution with the strict solution, if* f *is Hölder continuous in* R, *bounded near* ∂R, $f\in L_2(R)$ *and if* ∂R *and* β *are of class* C^{2+h} *and of class* C^{1+h} *for some* h *in* $0<h<1$, *respectively.*

Theorem 6.4. *In addition to the assumptions of the preceding theorem, assume that one of the conditions* 1) *and* 2) *in Theorem 3.2 is satisfied and that* $|x|f(x)\in L_2(R)$. *Then there exists a strict solution of the exterior problem.*

In the remaining part of this section we shall deal exclusively with the case $u_\infty=0$. The other case will be treated in the next section.

Proof of Theorem 6.1 for the case $u_\infty=0$. We assume $\nu=1$ without loss of generality. We know that u satisfies the inequalities

$$\|\nabla u\|_R\leq K \quad\text{and}\quad \|u(y)/|x-y|\|_R\leq K \tag{6.5}$$

for some constant K. On the other hand $u\in C^{1+\theta}$ $(0<\theta<1)$ according to Theorem 4.2. Hence we have by means of Lemma 2.3 and Remark to Lemma 2.4

$$\|u\|_{6,R}\leq K'$$

with another constant K'. Now, by means of Lemma 4.2 we have for $x\in R(2)$

$$u(x)=-\int E^{(1)}(x-y)(u\cdot\nabla)udy+\int E^{(1)}(x-y)f(y)dy+\int H^{(1)}(x-y)u(y)dy$$
$$\equiv -v'(x)+v''(x)+h(x).$$

In view of (4.9) and (6.5) we can estimate as

$$|v'(x)|\leq C\int_B \frac{|u|}{|x-y|}|\nabla u|dy\leq C\|u(y)/|x-y|\|_B\cdot\|\nabla u\|_B\leq C\|\nabla u\|_B ,$$

$$|v''(x)|\leq C\int_B \frac{|f(y)|}{|x-y|}dy\leq C\||x-y|^{-1}\|_B\cdot\|f\|_B\leq C\|f\|_B ,$$

where $B=B(x, 2)$. Hence we have $v'(x)\to 0$, $v''(x)\to 0$ as $|x|\to\infty$. Also in view of the properties of $H^{(\gamma)}$ stated in § 4 we have

$$|h(x)|\leq C\|H^{(1)}(x-y)\|_{6/5,A}\cdot\|u\|_{6,A}\leq C\|u\|_{6,A}$$

where $A=B(x, 2)-B(x, 1)$. Hence in virtue of $u\in L_6(R)$, we have $h(x)\to 0$. Thus $u(x)\to 0$, $(|x|\to\infty)$.

Proof of Theorem 6.2 for the case $u_\infty=0$. Again we assume $\nu=1$ without loss of generality. Under the assumptions of Theorem 6.2, u and p are smooth and satisfy the Navier-Stokes system

$$(6.6)\qquad \Delta u-\nabla p=\tilde{f}=(u\cdot\nabla)u-f , \qquad \operatorname{div} u=0$$

in V as well as the inequalities (6.5). As seen from (4.2)′ and (4.12) the integral representation

$$(6.7)\qquad u(x)=-\int_V E^{(\gamma)}(x-y)\tilde{f}(y)dy+\int_V H^{(\gamma)}(x-y)u(y)dy+s(x)$$

is valid for $x\in V$ and sufficiently large γ.

For the moment we assume $f\equiv 0$ so that $\tilde{f}=(u\cdot\nabla)u$. Then we put

$$v(x)=\int_V E(x-y)\tilde{f}(y)dy=\int_V E(x-y)(u\cdot\nabla)u(y)dy$$

and note that the integral on the right side is absolutely convergent and diminishes when $|x|\to\infty$. In fact, with sufficiently large N we set $V_N=V\cap B(0, N)$, $V_N'=V-B(0, N)=B(0, N)^c$ and split $v(x)$ as $v(x)=v_N(x)+v_N'(x)$, where

$$v_N(x)=\int_{V_N} E(x-y)(u\cdot\nabla)udy , \quad v'(x)=\int_{V_N'} E(x-y)(u\cdot\nabla)udy .$$

Since V_N is a bounded set, the inequalities

$$|v_N(x)|\leq C\int_{V_N}\frac{|(u\cdot\nabla)u|}{|x-y|}dy\leq\frac{C}{|x|-N}\int_{V_N}|(u\cdot\nabla)u|dy\leq\frac{C_N}{|x|-N}$$

hold for some constant C_N and any $x\in V_N'$. On the other hand we have by means of (6.5)

$$|v_N'(x)|\leq C\int_{V_N'}\frac{|u|\cdot|\nabla u|}{|x-y|}dy\leq C\|u/|x-y|\|_{V_N'}\cdot\|\nabla u\|_{V_N'}\leq C\|\nabla u\|_{V_N'} ,$$

whence follows that $v_N'(x)\to 0$ $(N\to\infty)$ uniformly with respect to x. Thus for any positive constant ε we have $|v(x)|\leq\varepsilon$ when we make N large and then make $|x|$ large enough. Namely we have

$$(6.8)\qquad v(x)\to 0\,, \qquad (|x|\to\infty).$$

(6.3) will follow from (6.7) if we can show that

$$(6.9)\qquad \int_V E^{(\gamma)}(x-y)\tilde{f}(y)\,dy \to \int_V E(x-y)\tilde{f}(y)dy = v(x)\,,$$

$$(6.10)\qquad \int_V H^{(\gamma)}(x-y)u(y)dy\to 0 \qquad \text{as } \gamma\to\infty.$$

Putting $K_{ij}^{(\gamma)}=E_{ij}^{(\gamma)}-\eta^{(\gamma)}(x-y)E_{ij}$, we observe that the matrix function $K^{(\gamma)}=\{K_{ij}^{(\gamma)}(x-y)\}$ vanishes identically unless $\gamma<|x-y|<2\gamma$. Concerning $H^{(\gamma)}$ and $K^{(\gamma)}$ we need

Lemma 6.1. *With an appropriate constant C independent of x and y we have*

$$(6.11)\qquad |H_{ij}^{(\gamma)}(x-y)|\leq C\gamma^{-3}\,, \qquad (\gamma\to\infty),$$

$$(6.12)\qquad |K_{ij}^{(\gamma)}(x-y)|\leq C\gamma^{-1}\,, \qquad (\gamma\to\infty).$$

The proof of this lemma is quite easy if we recall (4.9) and (4.10). We need also the relations

$$(6.13)\qquad \|\nabla u\|_{A(x,\gamma)}\to 0\,,\quad \|u\|_{A(x,\gamma)}\leq C\gamma\,, \qquad (\gamma\to\infty)$$

where $A(x,\gamma)=B(x,2\gamma)-B(x,\gamma)$. The first of (6.13) is obvious by means of (6.5). The second follows from (6.5) by virtue of Lemma 2.4. Now, by means of (6.12) and (6.13) we have

$$\left|\int_V K^{(\gamma)}(x-y)\tilde{f}(y)\,dy\right|\leq\frac{C}{\gamma}\int_A |u|\cdot|\nabla u|dy\leq\frac{C}{\gamma}\|u\|_A\cdot\|\nabla u\|_A$$
$$\leq C\|\nabla u\|_A\to 0\,,\qquad (\gamma\to\infty\,;\ A=A(x,\gamma)).$$

Also writing B' in place of $B(x,\gamma)^c$, we have

$$\left|\int_V (E-\eta^{(\gamma)}E)\tilde{f}dy\right|\leq C\int_{B'}|E(x-y)|\cdot|\tilde{f}(y)|dy\leq C\int_{B'}\frac{|u|\cdot|\nabla u|}{|x-y|}dy$$
$$\leq C\|u(y)/|x-y|\|_{B'}\cdot\|\nabla u\|_{B'}\leq C\|\nabla u\|\to 0\,,\qquad (\gamma\to\infty),$$

on account of (6.5). Thus we obtain (6.9) because $E^{(\gamma)}-E=(E^{(\gamma)}-\eta^{(\gamma)}E)-K^{(\gamma)}$. In virtue of (6.11) and (6.13) we have

$$\left|\int_V H^{(\gamma)}(x-y)u(y)\,dy\right|\leq\frac{C}{\gamma^3}\int_A |u|\,dy\leq\frac{C}{\gamma^3}\|1\|_A\cdot\|u\|_A$$
$$\leq C\gamma^{-3}\cdot\gamma^{3/2}\cdot\gamma=C\gamma^{-1/2}\to 0\,,\qquad (\gamma\to\infty),$$

which proves (6.10). In this way we have established the integral representation (6.3) for the case $f=0$. We have also remarked that $v(x)\to 0$. On the other hand, $s(x)=O(|x|^{-1})$ is clear because of $E=O(|x-y|^{-1})$, $T[E, e]=O(|x-y|^{-2})$. Consequently we have proved $u(x)\to 0$ for the case $f=0$.

We turn to ∇u. By differentiation it follows from (6.3) that $\partial_m u(x)=-\partial_m v(x)+\partial_m s(x)$, $(x\in V,\ m=1, 2, 3)$, where $\partial_m v(x)$ and $\partial_m s(x)$ are obtained respectively from $v(x)$ and $s(x)$ by differentiating the integrands under the integral signs. Then $\partial_m s(x)=O(|x|^{-2})$ is apparent in view of

$$(6.14) \qquad \partial_m E=\frac{\partial}{\partial x_m}E(x-y)=O(|x-y|^{-2})$$

and

$$\frac{\partial}{\partial x_m}T[E_{l*},\ -e_l]=O(|x-y|^{-3})\ , \qquad (|x-y|\to\infty).$$

Putting

$$(6.15) \qquad v^m(x)=\int_{B(x,1)}\partial_m E(x-y)\tilde{f}(y)dy\ ,$$

$$v^{*m}(x)=\int_{V'}\partial_m E(x-y)\tilde{f}(y)dy\ , \qquad (V'(x)=V-B(x, 1)),$$

we note that $\partial_m u(x)=-v^m(x)-v^{*m}(x)+\partial_m s(x)$. Since $y\in V'(x)$ implies $|x-y|\geq 1$, (6.14) gives

$$|v^{*m}(x)|\leq C\int_{V'}\frac{|\tilde{f}(y)|}{|x-y|^2}dy\leq C\int_{V'}\frac{|\tilde{f}(y)|}{|x-y|}dy\ .$$

Hence we obtain $v^{*m}(x)\to 0$ similarly to (6.8). Therefore $-v^{*m}+\partial_m s$ is a bounded function which vanishes at the infinity. For convenience we denote such a function by one and the same symbol $d(x)$. Setting

$$k(x-y)=\begin{cases}\partial_m E(x-y)\ , & (|x-y|<1),\\ 0\ , & (|x-y|>1),\end{cases}$$

we consider the integral transformation

$$w'(x)=\int_{B_1}k(x-y)w(y)dy\ , \qquad (x\in B_2).$$

Here B_1 is a sphere and B_2 is another sphere concentric with B_1 such that the radius of B_2 is less by 1 than that of B_1. Then just in the same way as we deduced i), ii) and iii) in § 4, we obtain

$$(6.16) \qquad \|w'\|_{5,B_2}\leq C\|w\|_{2,B_1}\ , \quad \|w'\|_{\infty,B_2}\leq C\|w\|_{5,B_1}\ .$$

Furthermore, a being an arbitrary point sufficiently far from the origin, we put $M(a)=\|u\|_{\infty,B(a,3)}$. Then by virtue of $u(x)\to 0$, we see that $M(a)\to 0$ as $|a|\to\infty$. Noting

$$\|\tilde{f}\|_{B(a,3)} = \|(u\cdot\nabla)u\|_{B(a,3)} \leq M(a)\|\nabla u\|_{B(a,3)}$$

and applying (6.16), we are led to

$$\|v^m(x)\|_{5,B(a,2)} \leq CM(a)\|\nabla u\|_{B(a,3)} \leq CM(a)\ .$$

Taking account of $\partial_m u = -v^m + d$ we thus have

$$\|\nabla u\|_{5,B(a,2)} \leq CM'(a) \quad \text{and} \quad \|\tilde{f}\|_{5,B(a,2)} \leq CM'(a)\ ,$$

where $M'(a)$ is another function of a tending to 0 as $|a|\to\infty$. Again we resort to (6.16) and see that

$$\|v^m\|_{\infty,B(a,1)} \leq C\|\tilde{f}\|_{5,B(a,2)} \leq CM'(a)\ ,$$

whence follows $\|\partial_m u\|_{\infty,B(a,1)} \leq CM'(a)$ and, namely, $\nabla u\to 0$.

We turn to p. Putting

$$\pi(x) = \int_V e(x-y)\tilde{f}(y)dy\ ,$$

we notice in view of (4.5) that $\Delta v - \nabla\pi = -\tilde{f}$. On the other hand we can easily verify $\Delta s - \nabla\sigma = 0$ with the aid of the relations

$$\Delta e_i = 0\ , \quad \Delta_x E_{ij} = \frac{\partial}{\partial x_i}e_j\ , \quad \Delta_x T_y[E_{i*}, -e_i] = \frac{\partial}{\partial x_i}T_y[e, 0]\ .$$

Consequently, $\Delta u - \nabla(-\pi + \sigma) = \tilde{f} \equiv (u\cdot\nabla)u$. Comparing this with (6.6) we have

(6.17) $$p(x) = \text{const.} - \pi(x) + \sigma(x)\ .$$

This implies (6.3)′, provided that

(6.18) $$-\pi(x) + \sigma(x) \to 0\ , \qquad (|x|\to\infty).$$

In fact, $\sigma(x) = O(|x|^{-2})$ is obvious. We ascertain

(6.19) $$\pi(x) \to 0\ , \qquad (|x|\to\infty)$$

as follows. Write $\pi(x) = \pi'(x) + \pi''(x)$ with

$$\pi'(x) = \int_{B(x,1)} e(x-y)\tilde{f}(y)dy\ , \qquad \pi''(x) = \int_{V'(x)} e(x-y)\tilde{f}(y)dy\ .$$

It is easily seen that $\pi''(x)\to 0$ similarly to v^{*m} in (6.15). $\pi'(x)\to 0$ follows from $\tilde{f} = (u\cdot\nabla)u\to 0$ and $|\pi'(x)| \leq C\|\tilde{f}\|_{\infty,B(x,1)}$. In this way we obtain (6.3)′ for the case $f\equiv 0$.

Reexamining the above arguments we notice that the validity of the theorem will be established for the general f subjected to the assumptions in the theorem, once the following relations are proved.

$$\int_V E^{(\gamma)}(x-y)f(y)dy \to \int_V E(x-y)f(y)dy\ , \qquad (\gamma\to\infty), \tag{6.20}$$

$$\int_V E(x-y)f(y)dy \to 0\ , \qquad (|x|\to\infty), \tag{6.21}$$

$$\int_V \partial_m E(x-y)f(y)dy \to 0\ , \qquad (|x|\to\infty), \tag{6.22}$$

$$\int_V e(x-y)f(y)dy \to 0\ , \qquad (|x|\to\infty), \tag{6.23}$$

The proofs of these are as follows. Noting (6.12) and $\||x-y|\cdot|f(y)|\|\leq\|f'\|+|x|\cdot\|f\|=M(x)$, we have with $A=A(x,\gamma)=B(x,2\gamma)-B(x,\gamma)$

$$\left|\int_V K^{(\gamma)}(x-y)f(y)dy\right|\leq C\gamma^{-1}\||x-y|^{-1}\|_A\cdot\||x-y|f(y)\|_A\leq C\gamma^{-1/2}M(x)\ ,$$

$$\left|\int_V (E-\eta^{(\gamma)}E)f(y)dy\right|\leq C\int_{V-B(x,\gamma)}\frac{|f|}{|x-y|}dy$$
$$\leq C\||x-y|^{-2}\|_{V-B}\cdot\||x-y|f(y)\|_{V-B}\leq C\gamma^{-1/2}M(x)\ .$$

These estimates give (6.20). (6.21) is obtained by means of (2.22). Namely,

$$\left|\int_V E(x-y)f(y)dy\right|\leq C\left\|\frac{1}{|x-y|\cdot|y|}\right\|\cdot\|f'\|=O(|x|^{-1/2})\ . \tag{6.24}$$

If we put

$$I(x)=\int_V\frac{|f(y)|}{|x-y|^2}dy=\int_{B(x,1)}\frac{|f|}{|x-y|^2}dy+\int_{V'(x)}\frac{|f|}{|x-y|^2}dy\equiv I'(x)+I''(x)\ ,$$

then the proof of (6.22) and (6.23) is reduced to the proof of $I(x)\to 0$, $(|x|\to\infty)$. On the other hand,

$$|I'(x)|\leq C\|f\|_{\infty,B(x,1)}\to 0\ , \qquad (|x|\to\infty)$$

in virtue of the assumption $f(x)\to 0$. And $|I''(x)|=O(|x|^{-1/2})$ is shown in the same way as (6.24), because $y\in V'(x)$ implies $|x-y|\leq|x-y|^2$.

Thus we have established Theorem 6.2 for the case $u_\infty=0$.

§ 7. Regularity at the Infinity (Continued).

The object of this section is to prove Theorem 6.1 and Theorem 6.2 for the case $u_\infty\neq 0$. We divide this section into two parts. The first part contains some preliminary considerations concerning Oseen's system and its fundamental solution, which are employed in the second part.

I) *Preliminary considerations.*

In proving the theorems in question we may assume $\nu=1$ and $u_\infty=\{2,0,0\}$ without affecting the generality. Indeed, a suitable choice of the coordinate system permits us to take $u_\infty=\{|u_\infty|,0,0\}$ and then the scale transformations

$u'=u/k$, $x'=kx$, $p'=k^2p$, $f'=k^3f$ with $k=2/|u_\infty|$ changes the boundary value at the infinity into the one mentioned above.

For a while, suppose that u and the associated pressure p are smooth and $\nu=1$, $u_\infty=\{2, 0, 0\}$. Then $v=u-u_\infty$ satisfies

$$(7.1) \qquad (\Delta-2\partial_1)v-\nabla p=(v\cdot\nabla)v-f\,, \qquad \operatorname{div} v=0.$$

When the non-linear term in the first equation of (7.1) is dropped we receive

$$(7.2) \qquad (\Delta-2\partial_1)v-\nabla p=-f\,, \qquad \operatorname{div} v=0,$$

which is called *Oseen's system*. The fundamental solution of Oseen's system is explicitly known owing to C.W. Oseen [13]. It consists of a 3×3-matrix function $\mathscr{E}=\{\mathscr{E}_{ij}\}$ and a 3-vector function $e=\{e_j\}$ given by

$$(7.3) \qquad \mathscr{E}_{ij}=\mathscr{E}_{ij}(x-y)=\left(\delta_{ij}\Delta-\frac{\partial^2}{\partial x_i\partial x_j}\right)\Psi\,,$$

$$e_j=e_j(x-y)=-\frac{\partial}{\partial x_j}\left(\Delta-2\frac{\partial}{\partial x_1}\right)\Psi\,,$$

where

$$\Psi=\Psi(x-y)=\frac{1}{8\pi}\int_0^\tau\frac{1-e^{-t}}{t}\,dt \quad \text{and} \quad \tau=|x-y|-(x_1-y_1)\,.$$

Similarly to §4 we sketch the properties of this fundamental solution. The following relations are verified by straight-forward calculations:

$$(7.4) \qquad (\Delta_x-2\partial/\partial x_1)\Psi=1/4\pi|x-y|\,,$$

$$\left(\Delta_x-2\frac{\partial}{\partial x_1}\right)\mathscr{E}_{ij}-\frac{\partial}{\partial x_i}e_j=-\delta_{ij}\delta(x-y)\,,$$

$$\left(\Delta_y+2\frac{\partial}{\partial y_1}\right)\mathscr{E}_{ij}+\frac{\partial}{\partial y_j}e_i=-\delta_{ij}\delta(x-y)\,,$$

$$\frac{\partial}{\partial x_i}\mathscr{E}_{ij}=0 \quad \text{and} \quad \frac{\partial}{\partial y_j}\mathscr{E}_{ij}=0\,,$$

where δ is the dirac delta function. In view of (7.3) and (7.4) we notice

$$(7.5) \qquad e_j=-\frac{\partial}{\partial x_j}\frac{1}{4\pi|x-y|}=\frac{1}{4\pi}\frac{x_j-y_j}{|x-y|^3}$$

which shows that the e under consideration is identical with e introduced in §4. The singularity of $\mathscr{E}$ is similar to that of E in §4. In particular, (4.7) maintains its validity if E_{i*} is replaced by $\mathscr{E}_{i*}$, the i-th row vector of $\mathscr{E}$. Hence we can derive integral representations with the kernels constructed from $\mathscr{E}$ and e by means of a certain integral identity. This integral identity is analogous to and obtained from (4.2) and takes the following form:

(7.6) $$\int_V \{v(\Delta u-2\partial_1 u-\nabla p)-u(\Delta v+2\partial_1 v-\nabla q)+q\,\mathrm{div}\,u-p\,\mathrm{div}\,v\}\,dV$$
$$=\int_{\partial V}\{vT[u,\,p]n-uT[v,\,q]n-2(uv)n_1\}dS\,.$$

Here and for the moment u, v and p, q stand for any smooth vector and scalar functions. We have also the integral representation for $x\in V$

(7.7) $$v(x)=-\int_V\{\mathscr{E}(x-y)(\Delta v-2\partial_1 v-\nabla p)-e(x-y)\,\mathrm{div}\,v\}dy+s(x)\,,$$

where $s(x)$ is formally identical with that in (6.4). The corresponding integral representation for $p(x)$ in the case $\mathrm{div}\,v=0$ is

(7.8) $$p(x)=-\int_V e(x-y)(\Delta v-2\partial_1 v-\nabla p)dy+\sigma(x)\,, \qquad (x\in V),$$

where $\sigma(x)$ is formally identical with that in (6.4)′. As noted above, $\mathscr{E}$ behaves like E when $|x-y|\to 0$ but behaves somewhat differently when $|x-y|\to\infty$. The following inequalities are known:

(7.9) $$|\mathscr{E}_{ij}(x-y)|\le C|x-y|^{-1}\,,\quad |e_j(x-y)|\le C|x-y|^{-2}$$

both for $|x-y|\to 0$ and $|x-y|\to\infty$.

(7.10) $$|\partial_m\mathscr{E}_{ij}(x-y)|\le C|x-y|^{-2}\,, \qquad (|x-y|\to 0),$$
$$|\partial_m\mathscr{E}_{ij}(x-y)|\le C|x-y|^{-3/2}\,, \qquad (|x-y|\to\infty).$$

We introduce the modified or truncated fundamental solutions of Oseen's system similar to those of Stokes' system introduced in §4. Namely, γ being a positive parameter we put

$$\mathscr{E}_{ij}^{(\gamma)}=\Big(\delta_{ij}\Delta-\frac{\partial^2}{\partial x_i\partial x_j}\Big)\Psi^{(\gamma)}\,,\quad e_j^{(\gamma)}=-\frac{\partial}{\partial x_j}\Big(\Delta-2\frac{\partial}{\partial x_1}\Big)\Psi^{(\gamma)}\,,$$

where $\Psi^{(\gamma)}=\eta^{(\gamma)}(x-y)\Psi$ and $\eta^{(\gamma)}$ means the same as in §4. $\mathscr{E}^{(\gamma)}=\{\mathscr{E}_{ij}^{(\gamma)}\}$ and $e^{(\gamma)}=\{e_j^{(\gamma)}\}$ coincide with $\mathscr{E}$ and e for $|x-y|\le\gamma$ and they vanish identically for $|x-y|\ge 2\gamma$. Another matrix function $\mathscr{H}^{(\gamma)}=\{\mathscr{H}_{ij}^{(\gamma)}\}$ is defined by $\mathscr{H}^{(\gamma)}(0)=0$ and

(7.11) $$\mathscr{H}_{ij}^{(\gamma)}=\mathscr{H}_{ij}^{(\gamma)}(x-y)=\delta_{ij}\Delta\Big(\Delta-2\frac{\partial}{\partial x_1}\Big)\Psi^{(\gamma)}\,, \qquad (|x-y|\neq 0).$$

Then we can derive the integral representation

(7.12) $$v(x)=-\int_V\mathscr{E}^{(\gamma)}(x-y)(\Delta v-2\partial_1 v-\nabla p)dy+\int_V\mathscr{H}^{(\gamma)}(x-y)v\,dy\,,$$

$(x\in V(2\gamma))$, provided that $\mathrm{div}\,v=0$.

We need some knowledge about the asymptotic behaviors of $\mathscr{H}^{(\gamma)}$ as $\gamma\to\infty$. This estimation is a little more difficult than the previous one concerning $H^{(\gamma)}$, inasmuch as the formal differential operator appearing in (7.11) is not of homogeneous order and the asymptotic behavior of derivatives of $\tau=|x-y|-(x_1-y_1)$, $|x-y|$ tending to ∞, is a little complicated. We state only the result in the following Lemma 7.1 without proof.

Lemma 7.1. *For any differentiation D^α with order $|\alpha|$, we have*

$$|D^\alpha \mathscr{H}^{(\gamma)}|\leq C_\alpha \gamma^{-(4+|\alpha|)/2}\ , \qquad (\gamma\to\infty), \tag{7.13}$$

uniformly with respect to x and y.

As an application of the preceding lemma we deduce

Lemma 7.2. *Let v and p be of class C^2 and of class C^1 in the whole space and assume that the homogeneous Oseen's system*

$$(\Delta-2\partial_1)v-\nabla p=0\ , \qquad \operatorname{div} v=0$$

is satisfied there. If

$$\|v\|_{\Omega(x,r)}\to 0\ , \qquad (r\to\infty) \tag{7.14}$$

for every x, then $v\equiv 0$ and $p\equiv$ const..

Proof of Lemma 7.2. By means of (7.12) we have

$$v(x)=\int_{E_3}\mathscr{H}^{(\gamma)}(x-y)v(y)dy\ , \qquad (x\in E_3),$$

and then

$$D^\alpha v(x)=\int_{E_3}D^\alpha\mathscr{H}^{(\gamma)}(x-y)v(y)dy\ ,$$

D^α being an arbitrary differentiation with order $|\alpha|=2$. With resort to (7.13) we have

$$\begin{aligned}|D^\alpha v(x)|&\leq\frac{C}{\gamma^3}\int_{A(x,\gamma)}|v(y)|dy\leq\frac{C}{\gamma^3}\int_\gamma^{2\gamma}\|v\|_{\Omega(x,r)}r^2dr\\ &\leq C\max_{\gamma\leq r\leq 2\gamma}\|v\|_{\Omega(x,\gamma)}\ , \qquad (A(x,\gamma)=B(x,2\gamma)-B(x,\gamma))\end{aligned}$$

The last term tends to 0 as $\gamma\to\infty$ by hypothesis. Hence we have $D^\alpha v=0$ for any x and for any D^α with $|\alpha|=2$, which implies that v_i is a polynomial with degree 0 or 1 ($i=1, 2, 3$). Taking account of (7.4), we conclude $v\equiv 0$, whence follows moreover that $\nabla p=0$ and thus $p\equiv$const.. This proves the lemma.

II) *Proof of Theorem 6.1 and Theorem 6.2 for the case $u_\infty\neq 0$.*

Proof of Theorem 6.1 for the case $u_\infty\neq 0$.

We assume without loss of generality $\nu=1$ and $u_\infty=\{2, 0, 0\}$. We put $v=u-u_\infty$ and recall the inequalities

$$\|\nabla v\|_R\leq K \quad\text{and}\quad \|v(y)/|x-y|\|_R\leq K\ ,$$

where K is a constant independent of x. According to Theorem 4.2 $v \in C^{1+\theta}$ $(0<\theta<1)$ and according to Lemma 2.3 and Remark to Lemma 2.4 we have $v \in L_6(R)$. On the other hand, we can derive the integral representation

$$v(x)=-\int_R \mathscr{E}^{(1)}(x-y)(v\cdot\nabla)v\,dy+\int_R \mathscr{E}^{(1)}(x-y)f\,dy+\int_R \mathscr{H}^{(1)}(x-y)v\,dy$$

for $x \in R(2)$ just in the same way as we derived (4.17). Since the sigularity and the regularity of $\mathscr{E}^{(1)}$ and $\mathscr{H}^{(1)}$ are the same as those of $E^{(1)}$ and $H^{(1)}$, we can show $v(x)\to 0$ by the same arguments as in the proof for the case $u_\infty=0$.

Proof of Theorem 6.2 for the case $u_\infty \neq 0$. We assume $\nu=1$ and $u_\infty=\{2, 0, 0\}$ again. We fix a scalar function $h(x)$ of class C^∞ in the whole space such that

$$h(x)=\begin{cases}1\,, & ((x\in V(2\delta)),\\ 0\,, & ((x\in V(\delta)^c=E_3-V(\delta)),\end{cases}$$

where δ is a sufficiently small positive constant. Using $h(x)$ we extend v and p over the whole space by setting

$$\begin{aligned} v'(x)&=h(x)\cdot v(x) \quad\text{and}\quad p'(x)=h(x)\cdot p(x) && \text{in } V,\\ v'(x)&=0 \quad\text{and}\quad p'(x)=0 && \text{outside } V.\end{aligned}$$

Evidently v' and p' are identical in $V(2\delta)$ with v and p respectively. Also we have $\|\nabla v'\|_{E_3}\leq K_1$ and $\|v'(y)/|x-y|\|_{E_3}\leq K_1$ with a constant K_1 independent of x. This implies, according to Lemma 2.4, that

$$\|v'\|_{\Omega(x,r)}\to 0\,, \qquad (r\to\infty). \tag{7.15}$$

In general, v' is no longer solenoidal, though $\operatorname{div} v'$ is a function with compact carrier included between $\partial V(2\delta)$ and $\partial V(\delta)$. Because this situation is inconvenient we introduce

$$v''=v'-\nabla\varphi\,,\quad p''=p'+2\partial_1\varphi-\operatorname{div} v'$$

where

$$\varphi(x)=-\frac{1}{4\pi}\int_{E_3}\frac{\operatorname{div} v'}{|x-y|}dy\,. \tag{7.16}$$

By virtue of $\operatorname{div}\nabla\varphi=\Delta\varphi=\operatorname{div} v'$, v'' is solenoidal everywhere. Hereafter we write G in place of $V(2\delta)$. Now we put

$$\tilde{f}'=(\Delta-2\partial_1)v''-\nabla p''\,. \tag{7.17}$$

Then we notice $\tilde{f}'=\tilde{f}$ in G, since

$$\begin{aligned}\tilde{f}'&=(\Delta-2\partial_1)(v'-\nabla\varphi)-\nabla(p'+2\partial_1\varphi-\operatorname{div} v')\\ &=\Delta v'-2\partial_1 v'-\nabla p'=\Delta v-2\partial_1 v-\nabla p=\tilde{f}=(v\cdot\nabla)v-f\,.\end{aligned}$$

Here use has been made of the fact that $v'=v$, $p'=p$, $\Delta\varphi=\operatorname{div} v'=\operatorname{div} v=0$ in G. We now put

$$w'(x)=-\int_{E_3}\mathscr{E}(x-y)\tilde{f}'(y)dy$$

and observe that

$$w'(x)\to 0\,, \qquad (|x|\to\infty). \tag{7.18}$$

Indeed, putting

$$w_1'(x)=-\int_{G^c}\mathscr{E}(x-y)\tilde{f}'(y)dy\,,\quad w_2'(x)=-\int_G\mathscr{E}(x-y)(v\cdot\nabla)v(y)dy\,,$$
$$w_3'(x)=\int_G\mathscr{E}(x-y)f(y)dy\,,$$

we notice $w'=w_1'+w_2'+w_3'$ in consideration that $\tilde{f}'=f$ in G. $w_1'(x)=O(|x|^{-1})$, since G^c is a bounded set and (7.9) is known. By means of (7.9) and (7.15). $w_2'(x)\to 0$ can be shown similarly to (6.8), and $w_3'(x)\to 0$ can be shown similarly to (6.12). We then prove the identity $w'=v''$. Introducing

$$\pi'(x)=-\int_{E_3}e(x-y)\tilde{f}'(x)dy$$

we note that the equation

$$(\Delta-2\partial_1)w'-\nabla\pi'=\tilde{f}'\,,\qquad \operatorname{div}w'=0$$

are satisfied. Recalling (7.17), the pair $\{w'-v'',\ \pi'-p''\}$ is seen to obey the homogeneous Oseen's system. On the other hand, (7.15) and (7.18) give

$$\|w'-v''\|_{\Omega(x,r)}=\|w'-v'+\nabla\varphi\|_{\Omega(x,r)}\to 0\,, \qquad (r\to\infty),$$

because of $\nabla\varphi=O(|x|^{-2})$. Here Lemma 7.2 is applicable and we obtain $w'-v''\equiv 0$ as well as $\pi'-p''\equiv$const.. In G the equality $w'=v''=v'-\nabla\varphi$ is reduced to $v=w'+\nabla\varphi$. We already remarked that $w'\to 0$ and $\nabla\varphi\to 0$. Consequently we arrive at

$$v(x)=u(x)-u_\infty\to 0\,, \qquad (|x|\to\infty). \tag{7.19}$$

Differentiating $v'=w'+\nabla\varphi$ by x_m we obtain

$$\partial_m v(x)=-w^m(x)+d^m(x)\,, \qquad (x\in G),$$

where

$$w^m(x)=\int_B\partial_m\mathscr{E}(x-y)(v\cdot\nabla)vdy\,, \qquad (B=B(x,\ 1)),$$

$$d^m(x)=\nabla\partial_m\varphi(x)-\int_{G^c}\partial_m\mathscr{E}\tilde{f}'dy+\int_G\partial_m\mathscr{E}f\,dy \tag{7.20}$$
$$-\int_{G-B}\partial_m\mathscr{E}\cdot(v\cdot\nabla)vdy$$

with $\partial_m \mathscr{E}=\partial_m \mathscr{E}(x-y)/\partial x_m$. By means of (7.10) and (7.15) the last integral in (7.20) diminishes at the infinity similarly to v^{*m} in (6.15). $\partial_m \nabla\varphi=O(|x|^{-3})$ is obvious. The second term is $O(|x|^{-3/2})$ in virtue of (7.10). The third term can be dealt with similarly to (6.22). Hence we have $d^m(x)\to 0$. Then we resort to the iterative consideration employed in deriving $\nabla u\to 0$ for the case $u_\infty=0$ and obtain $\partial_m v(x)\to 0$. Thus we have $\nabla u=\nabla v\to 0$, $(|x|\to\infty)$.

We turn to the pressure. Rewriting $\pi'-p''=\text{const.}$, we obtain

$$p'=\operatorname{div} v'-2\partial_1\varphi+\pi'+\text{const.}\,,$$

and hence for $x\in G$ we obtain

$$p(x)-\text{const.}=-2\partial_1\varphi(x)-\int_{E_3} e(x-y)\tilde{f}'(y)dy\,. \tag{7.21}$$

Obviously $\partial_1\varphi=O(|x|^{-2})$. In consideration that $\tilde{f}'=\tilde{f}$ in G the second term of the right hand side of (7.21) is written as

$$\int_{G^c} e(x-y)\tilde{f}'dy+\int_G e(x-y)(v\cdot\nabla)vdy-\int_G e(x-y)fdy$$
$$\equiv q_1+q_2-q_3\,,$$

while it is clear from (7.9) that $q_1=O(|x|^{-2})$. And we can prove $q_2\to 0$, $q_3\to 0$, $(|x|\to\infty)$ by arguments similar to those employed for (6.19) and (6.23). In this way we see that the right hand side of (7.21) tends to 0 as $|x|\to\infty$ and conclude that $p(x)$ tends to a certain constant p_∞ as $|x|\to\infty$.

We turn to the integral representation (6.4) and (6.4)'. We note in view of (7.16) and (7.5)

$$\nabla\varphi=\int_{E_3} e(x-y)\operatorname{div} v'dy\,.$$

Hence the equality $v'=\nabla\varphi+w'$ can be written as

$$v(x)=v'(x)=\int_{E_3}\{e(x-y)\operatorname{div} v'-\mathscr{E}(x-y)\tilde{f}'\}dy$$
$$=\int_{G^c}\{e(x-y)\operatorname{div} v'-\mathscr{E}(x-y)\tilde{f}'\}dy-\int_G \mathscr{E}(x-y)\tilde{f}\,dy$$

for any x in G, because $\tilde{f}'=\tilde{f}$ and $\operatorname{div} v'=\operatorname{div} v=0$ in G. On the other hand, let $s'(x)$ be a function defined by the same expression as $s(x)$ using ∂G in place of ∂V. Then we have

$$\int_{G^c}\{e(x-y)\operatorname{div} v'-\mathscr{E}(x-y)\tilde{f}'\}dy=s'(x)\,, \tag{7.22}$$

whence follows in view of the above obtained equality

$$v(x)=s'(x)-\int_G \mathscr{E}(x-y)\tilde{f}\,dy\ , \qquad (x\in G). \tag{7.23}$$

The derivation of (7.22) is easy if we use two equalities obtained in the following way. One is obtained by applying the integral identity (7.6) to the domain G^c and to the pairs $\{v'', p''\}$, $\{\mathscr{E}_{i*}, -e_i\}$ and the other is obtained by applying (7.6) to the same domain and to the pairs $\{-\nabla\varphi, 2\partial_1\varphi-\operatorname{div} v'\}$ and $\{\mathscr{E}_{i*}, -e_i\}$.

Moreover, G and ∂G in (7.23) may be replaced respectively by V and ∂V, since we can show

$$\int_{V-G} \mathscr{E}(x-y)\tilde{f}(y)dy=s(x)-s'(x)$$

by applying (7.6) to the annular domain $V-G$ and to the pairs $\{v, p\}$ and $\{\mathscr{E}_{i*}, -e_i\}$. Furthermore, the result is true for any x in V and implies (6.4), for δ is arbitrary. In deriving (6.4)′ we put

$$\pi(x)=\int_V e(x-y)\tilde{f}\,dy\ .$$

In order to establish (6.4)′ it suffices to verify that $-\pi+\sigma$ satisfies

$$(\Delta-2\partial_1)v-\nabla(-\pi+\sigma)=\tilde{f} \tag{7.24}$$

and that

$$-\pi(x)+\sigma(x)\to 0\ , \qquad (|x|\to\infty). \tag{7.25}$$

(7.24) is immediately ascertained by means of (6.4), (7.4). (7.25) is shown in the same way as (6.18).

Thus we have established Theorem 6.2 for the case $u_\infty\neq 0$.

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