On homotopy groups of certain complexes.

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Introduction.

Let X and Y be given topological spaces and $f\colon A\rightarrow Y$ a map of a closed subset A of X into Y. X and Y may have common points, but in that case we assume that $X\frown Y\subset A$, $X\frown Y$ is a closed subset both of A and of Y, and that $f|A\frown Y$[1] is the identity map. Then we can construct a new topological space Z as follows: the points of Z consist of the points in $X-A$ and in Y, and its topology is the identification topology determined by the map F of $Y\smile Y$[2] into Z which is defined by:

$$F(x)=x \text{ for } x\in X-A,\ F(y)=y \text{ for } y\in Y \text{ and } F(x)=f(x) \text{ for } x\in A.$$

Notice that Y keeps its own topology and is a closed subset of Z and also $X-A$ keeps its topology. We shall say that Z is formed by attaching X to Y by means of the map f, and we shall call f the attaching map. In this paper, we shall be concerned with the particular case that X is a closed n-cell[3], V^n, and A is its boundary sphere S^{n-1}. In this case we shall denote the above Z with $Y\overset{f}{\smile}e^n$, where e^n is the interior of V^n. Now define the map $\bar{f}$ of V^n into $Y\overset{f}{\smile}e^n$ as follows: $\bar{f}(x)=x$ for $x\in e^n$ and $\bar{f}(x)=f(x)$ for $x\in S^{n-1}$.

Clearly $\bar{f}$ is continuous and $\bar{f}|e^n$ is homeomorphic. Were gard the map $\bar{f}$ as a map of the pair: $(V^n, S^{n-1})\rightarrow(Y\overset{f}{\smile}e^n, Y)$ and call it the characteristic map for e^n. Consider two maps $f, g\colon S^{n-1}\rightarrow Y$ such that f is homotopic to g. Then $Y\overset{f}{\smile}e^n$ has the same homotopy type as $Y\overset{g}{\smile}e^n$. (cf. [23][4], Lemma 5, p. 219). Therefore from the stand-point of homotopy theory we may identify $Y\overset{f}{\smile}e^n$ with $Y\overset{g'}{\smile}e^n$. We shall denote $Y\overset{f}{\smile}e^n$ with $Y\overset{\alpha}{\smile}e^n$, where α is the homotopy class of f.

One of our purpose is to describe the homotopy groups of $S^n\overset{\alpha}{\smile}e^r$ by homotopy groups of spheres and properties of the class α. For example, if $\alpha=0$ we may regard $S^n\overset{\alpha}{\smile}e^r$ as the subset of $S^n\times S^r$[5] consisting of the points in $e^0_n\times S^r\smile S^n\times e^0_r$

(1) $f|A\frown Y$ denote the map obtained from f whose domain is restricted on $A\frown Y$.

(2) $X\smile Y$ denotes the space, which consists of the points of X and Y, and in which a subset K is closed if and only if $K\frown X$ and $K\frown Y$ are closed subsets of X and Y, respectively.

(3) V^n is the subset of n-dimensional euclidean space consisting of the points $(x_1, x_2, \cdots, x_n)$ such that $\sum_{i=1}^{n} x_i^2\leqq 1$.

(4) The number of brackets refer to the bibliography at the end of this paper.

(5) $S^n\times S^r$ denotes the cartesian product of S^n and S^r.

$=S^n\vee S^r$, where e_i^0 denotes the base point of S^i. Then we have $\pi_i(S^n\overset{\alpha}{\cup}e^r)\cong\sum_{n_\alpha}\pi_i(S^{n_\alpha})$, where n_α is the dimension of a basic product of weight n, and the summation runs over all basic products. (cf. [7], Theorem A, p. 155).

Since $\pi_i(S^n)$ is zero-group for $i<n$ the above fact means that it is sufficient for our purpose to consider the case $r>n$.

The case $r=n+1$ presents different features from the other cases. For example,

if $r>n+1$ $H_0(S^n\overset{\alpha}{\cup}e^r, Z)\cong H_n(S^n\overset{\alpha}{\cup}e^r, Z)\cong H_r(S^n\overset{\alpha}{\cup}e^r, Z)\cong Z$

and $H_i(S^n\overset{\alpha}{\cup}e^r, Z)\cong 0$ for $i=0, n, r$.

if $r=n+1$ $H_0(S^n\overset{\alpha}{\cup}e^r, Z)\cong Z$, $H_n(S^n\overset{\alpha}{\cup}e^r, Z)\cong Z_m$, where m

is the degree of $\alpha\in\pi_n(S^n)$, and $H_i(S^n\overset{\alpha}{\cup}e^r, Z)\cong 0$ for $i\neq 0, n$.

So we shall distinguish these two cases in our treatment: Chapter I will be devoted to the case $r>n+1$ and Chapter II to the case $r=n+1$.

Suppose that $r>n+1>3$ and α is an element of a finite order m. Let x_r, y_r be generators of $H_r(S^r, Z)$ and $H_r(S^n\overset{\alpha}{\cup}e^r, Z)$, respectively. Then there exists a map $g: S^r\to S^n\overset{\alpha}{\cup}e^r$ such that $g_*(x_r)=y_r$, where g_* denotes the homomorphism of homology groups induced by g.

Define a map $\bar{g}: S^n\vee S^r\to S^n\overset{\alpha}{\cup}e^r$ such that $\bar{g}(x)=x$ for $x\in S^n$ and $\bar{g}(x)=g(x)$ for $x\in S^r$. Then $\bar{g}_*$ is an C_m[6]-isomorphism and therefore $\pi_i(S^n\overset{\alpha}{\cup}e^r)$ is C_m-isomorphic to $\pi_i(S^n\vee S^r)$ for all i. (cf. [17], Theorem 3, p. 276). This leads to the conclusion that if a primary number p is not a prime factor of m the p-primary component[7] of $\pi_i(S^n\overset{\alpha}{\cup}e^r)$ is isomorphic to the p-primary component of $\pi_i(S^n\vee S^r)$, and also $\pi_i(S^n\overset{\alpha}{\cup}e^r)$ contains an infinite cyclic group as a subgroup for infinite many i.

Now the groups $\pi_i(S^n)$ $(i>n)$ are the finite groups except for $\pi_{2n-1}(S^n)$, n even. (cf. [16], Prop. 5, p. 498). Thus we are interested in the case $S^n\overset{\alpha}{\cup}e^{2n}$, n even. In the case, α is an element of $\pi_{2n-1}(S^n)$ and therefore α has the Hopf invariant, say m. Let K be the order of $[\alpha, \iota_n]$[8] in the quotient group of $\pi_{3n-2}(S^n)$ by the image of the homomorphism $\alpha_*: \pi_{3n-2}(S^{2n-1})\to\pi_{3n-2}(S^n)$. Then as the main Theorem I of Chapter I, we shall show that $\pi_i(S^n\overset{\alpha}{\cup}e^{2n})$ is C_{km}-isomorphic to the direct sum of $\pi_{i-1}(S^{n-1})$ and $\pi_i(S^{3n-1})$ for all i. For example, let σ_n be the Hopf i map for $n=4, 8$. (See [19]). It is known that $S^4\overset{\sigma_4}{\cup}e^8$, $S^8\overset{\sigma_8}{\cup}e^{16}$ are the quotanion projective plane Ω_2, and the Cayley projective plane Π respectively. In the former case, it is known that $m=k=1$. Hence from our theorem we can obtain that $\pi_i(\Omega_2)$ is isomorphic to the direct sum of $\pi_{i-1}(S^3)$ and $\pi_i(S^{11})$ for all i. In the latter case,

(6) C_m denotes the class of finite abelian groups whose order are divisible by only prime factors of m. (See [17])

(7) See page 265 of [17].

(8) $[\alpha, \iota_n]$ denotes Whitehead product of α and a generator of $\pi_n(S^n)$.

m is 1, and we shall prove $k=2^i\cdot 3$ $(1\leqq i\leqq 3)$. Hence we see that if a prime number p is neither 2 nor 3 the p-primary component of $\pi_i(H)$ is isomorphic to the direct sum of the p-primary component of $\pi_{i-1}(S^7)$ and the p-primary component of $\pi_i(S^{23})$.

Next if $r=n+1$ we shall denote $S^n\overset{\alpha}{\smile}e^r$ with X_p^n, where p is the degree of $\alpha\in\pi_n(S^n)$. X_p^n is a kind of Moore space and its homotopy group $\pi_i(X_p^n)$ is a p-group for all i. Since we may regard X_p^{n+1} as the suspended space of X_p^n the suspension homomorphism E is defined so that $E:\pi_i(X_p^n)\to\pi_{i+1}(X_p^{n+1})$. It is known that E is an isomorphism onto for $i\leqq 2n-2$ and a homomorphism onto for $i\leqq 2n-1$. (See [11]). On the other hand Serre has proved that the sequence,

$$0\to\pi_i(S^n)\otimes Z_p\to\pi_i(X_p^n)\to\pi_{i-1}(S^n)*Z_p\to 0^{(9)}$$

is exact for $i\leqq 2n-2$. (Cf. [17], Prop. 9, p. 283). Barratt has then proved that the above sequence is splitting for odd prime number p. Namely if $i\leqq 2n-2$, $\pi_i(X_p^n)$ is isomorphic to the direct sum of ${}^p[\pi_i(S^n)]$ and $[\pi_{i-1}(S^n)]_p$, where ${}^p[G]$, $([G]_p)$ denotes the kernel (cokernel) of the endomorphism $\varphi_p: G\to G$ which $\varphi_p(g)=pg$. (See [1]). In Chapter II, we shall define two homomorphisms H_1, H_2 such that

a) $H_1:\ \pi_{2n+1}(X_p^{n+1})\to Z_p$,
   $H_2:\ \pi_{2n+2}(X_{pu}^{+1})\to Z_p$,

b) the sequence, $0\to\pi_{2n+1}(X_p^n)\xrightarrow{E}\pi_{2n+2}(X_p^{n+1})\xrightarrow{H_2}Z_p\to\pi_{2n}(X_p^n)\xrightarrow{E}\pi_{2n+1}(X_p^{n+1})\xrightarrow{H_1}Z_p\to\pi_{2n-1}(X_p^n)\xrightarrow{E}\pi_{2n}(X_p^{n+1})\to 0$. is exact for odd prime number p.

Then the Theorem II of Chapter II will assert that if n and p are both odd numbers H_1 is onto and H_2 is trivial, if n is even and p is an odd prime number H_1 is trivial and H_2 is onto. From these facts we can deduce that

1) $E:\pi_{2n+1}(X_p^n)\to\pi_{2n+2}(X_p^{n+1})$ is an isomorphism onto if n and p are both odd numbers.

2) the group $\pi_{2n}(X_p^n)$ contains Z_p as a subgroup if n and p are both odd numbers.

3) $E:\pi_{2n}(X_p^n)\to\pi_{2n+1}(X_p^{n+1})$ is an isomorphism into if n is even and p is odd prime.

4) the sequence, $0\to\pi_{2n+1}({}_p{}^n)\to\pi_{2n+2}(X_p^{n+1})\to Z_p\to 0$, is exact if n is even and p is odd prime.

In Chapter III, we shall consider complexes K of the form $S^n\overset{\alpha}{\smile}e^r\overset{\beta}{\smile}e^{n+r(11)}$. For example, the n-sphere bundles over r-spheres W are complexes of this type. (cf. [14]). By the homotopy theory of fibre boundles, $\pi_i(W,S^n)$ is isomorphic to

(9) $\otimes$ denotes the tensor product and $*$ the torsion product.
(10) See page 265 of [17].
(11) We denote $(S^n\overset{\alpha}{\smile}e^r)\overset{\beta}{\smile}e^{n+r}$ with $S^n\overset{\alpha}{\smile}e^r\overset{\beta}{\smile}e^{n+r}$.

$\pi_i(S^r)$ for all i. As a generalization of this fact, Theorem III of Chapter III will assert that if $r<n+1$ there exist two integers m and k such that $\pi_i(K, S^n)$ is C_{km}-isomorphic to $\pi_i(S^r)$ for all i. This implies that if a prime number p is neither a prime factor of m nor k the p-primary component of $\pi_{i-1}(K, S^n)$ is isomorphic to the p-primary component of $\pi_{i-1}(S^r)$. Moreover Theorem III of Chapter III will give a solution to a problem proposed by I.M. James. (cf. [13], p. 376).

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Chapter I. The group $\pi_i(S^n \overset{\alpha}{\smile} e^{2n})$.

In this chapter, $L(\alpha)$ denote a complex $S^n \overset{\alpha}{\smile} e^{2n}$, $(n>2)$ and k be the Hopf invariant of α. It is known that $x_n \smile x_n = kx_{2n}$[12], where x_n and x_{2n} are generators of $H^n(L(\alpha), Z)$ and $H^{2n}(L(\alpha), Z)$ respectively. (See [18]).

§1. Preliminary.

Let $\bar{\alpha}$ be the characteristic map for e^{2n}, i.e. $\bar{\alpha} \in \pi_{2n}(L(\alpha), S^n)$. Since $\pi_{3n-1}(L(\alpha), S^n)$ is the direct sum of $\bar{\alpha}.\ \pi_{3n-1}(V^{2n}, S^{2n-1})$[13] and an infinite cyclic group generated by $[\bar{\alpha}, \iota_n]_r$, (See [8]), every element β of $\pi_{3n-1}(L(\alpha))$ determines an integer m and an element ρ of $\pi_{3n-1}(V^{2n}, S^{2n-1})$ by the formula: $j_*(\beta) = m[\bar{\alpha}, \iota_n]_r + \bar{\alpha}\circ\rho$ where j_* denotes the inclusion homomorphism: $\pi_i(L(\alpha)) \to \pi_i(L(\alpha), S^n)$.

Let a map f: $S^{3n-1} \to L(\alpha)$ be a representative map of the class β. Denote the mapping cylinder of f by B, so that $S^{3n-1} \subset B$.

Let E be the space of maps λ: $I \to B$ such that $\lambda(0) \in S^{3n-1}$, where I is the interval $0 \leqq t \leqq 1$. We embed S^{3n-1} in E so that a point $x \in S^{3n-1}$ corresponds to the path $t \to (x, t)$ under the identification map of the mapping cylinder. Consider the projection P: $E \to B$ which is given by $P(\lambda) = \lambda(1)$. Let $F = P^{-1}(e)$, where e is the base point of B. By a result of Serre (cf. [16], Prop. 4, p. 479) P is a fibre mapping. Since the inclusion map u: $S^{3n-1} \to E$ and v: $L(\alpha) \to B$ are both homotopy equivalences they induce isomorphisms of homotopy and homology groups. Hence it is sufficient to consider groups $\pi_i(E)$ and $\pi_i(B)$ instead of groups $\pi_i(L(\alpha))$ and $\pi_i(S^{3n-1})$. Thus the following lemma (1.1) is the key to the proof of the main Theorem.

LEMMA (1.1) F is $(n-2)$-connected, $H^{n-1}(F, Z) \cong Z$ and $H^i(F, Z) \in C_{km}$ for $i \geqq n$.

(12) $x_n \smile x_n$ denotes the cup product of x_n and x_n.
(13) $\bar{\alpha}\circ\rho$ denotes the composition of $\bar{\alpha}$ and ρ.

This lemma will be proved in the next section. Now we shall suppose that Lemma (1.1) is true, and derive Theorem I. By virtue of Lemma (1.1) $\pi_{n-1}(F)$ is isomorphic to Z by Hurewicz Theorem and the universal coefficient theorem. Now let a map $g: S^{n-1}\to F$ be a representative map of a generator of $\pi_{n-1}(F)$. It follows from $\pi_{n-1}(E)\cong\pi_{n-1}(S^{3n-1})\cong 0$ that $i\circ g$ is contractible to a constant map, where $i: F\to E$ is the inclusion map. Let $\bar{E}$ be a mapping cylinder of $i\circ g$, so that $S^{n-1}\subset\bar{E}$, and let G be a homotopy between $i\circ g$ and the constant map. Define a map $H: \bar{E}\to E$ as follows:

$$H(x)=x \text{ for } x\in E \text{ and } H(x,t)=G(x,t) \text{ for } x\in S^{n-1} \text{ and } t\in I.$$

It is clear that H is well defined and continuous, and moreover the image of S^{n-1} by H is a point. Thus we may regard H as a map of the pair: $(\bar{E}, S^{n-1})\to(E,*)$.

LEMMA (1.2). $\pi_i(\bar{E}, S^{n-1})$ is isomorphic to the direct sum of $\pi_i(\bar{E})$ and $\pi_{i-1}(S^{n-1})$ for all i.

For let w denote the inclusion map: $E\to\bar{E}$, and j the inclusion map of the pair $(\bar{E}, *)\to(\bar{E}, S^{n-1})$. Then the following diagram is clearly commutative.

$$\begin{array}{ccc} \pi_i(E) & \xrightarrow{(H|E)_*} & \pi_i(E) \\ \big\downarrow{\scriptstyle w_*} & & \big\downarrow{\scriptstyle w_*} \\ \pi_i(\bar{E}) & \xrightarrow[H_*j_*]{} & \bar{\alpha}_i(\bar{E}) \end{array}$$

Since $(H|E)_*$ is the identity and w_* is an isomorphism H_*j_* is an automorphism. Hence j_* is an isomorphism into and its image is a direct summand. The Lemma (1.2) follows from the homotopy sequence of the pair $(\bar{E}, S^{n-1})$.

LEMMA (1.3). $\pi_i(\bar{E}, S^{n-1})$ is C_{km}-isomorphic to $\pi_i(E, F)$ for all i.

For, let r be the retraction: $\bar{E}\to E$ such that $r(x)=x$ for $x\in E$ and $r(x,t)=g(x)$ for $x\in S^{n-1}$. Thus r is a map of the pair: $(\bar{E}, S^{n-1})\to(E,F)$ such that $r|S^{n-1}=g$. Since r_* is an isomorphism onto and g_* is an C_{km}-isomorphism by Lemma (1.1) we have Lemma (1.3) from the five lemma of C-theory. (See [17]).

Combining Lemma (1.2) and Lemma (1.3) we have

THEOREM I. If there is a map $f: S^{3n-1}\to L(\alpha)$ such that $j_*(f)=m[\bar{\alpha}, \iota_n]+\bar{\alpha}\circ\rho$, where ρ is some element of $\pi_{3n-1}(V^{2n}, S^{2n-1})\pi_i(L(\alpha))$ is C_{km}-isomorphic to the direct sum of $\pi_i(S^{n-1})$ and $\pi_{i-1}(S^{3n-1})$ for all i.

In the next section we shall prove Lemma (1.1) as a consequence of a series of lemmas.

§2. The cohomology of F.

Consider the cohomology spectral sequence, $(E_r^{p,q}, d_r)$, associated with the fibre space (E, P, B) in §1, as defined in [15]. The cohomology group of B is free, and B is simply connected, since since B has the same homotopy type as $L(\alpha)$.

Hence the first invariant terms of the spectral sequence, $E_2^{p,q}$, can be expressed in the form $H^p(B, Z)\otimes H^q(F, Z)$ in view of Prop. 8 of [16]. (p. 458).

LEMMA (2.1). $H^0(F, Z)\cong H^{n-1}(F, Z)\cong Z$

$H^i(F, Z)\cong 0$ for $i\neq 0,\ n-1$ and $\leqq 2n-3$.

PROOF. Since E has the same homotopy type as S^{3n-1} we have

LEMMA (2.2). $H^0(E, Z)\cong H^{3n-1}(E, Z)\cong Z$

$H^0(E, Z)\cong 0$ for $i\neq 0,\ 3n-1$.

$H^0(B, Z)\cong H^n(B, Z)\cong H^{2n}(B, Z)\cong Z$

$H^i(B, Z)\cong 0$ for $i\neq 0,\ n,\ 2n$.

Since we can easiiy see that $E_r^{0,i}$ has no-coboundary other than zero and $d_r E_r^{0,i}=0$ for $0<i<n-1$, $r\geqq 2$, $E_\infty^{0,i}$ is isomorphic to $E_2^{0,i}$ for $i<n-1$. On the other hand, $E_\infty^{0,i}$ is a quotient group of a subgroup of $H^i(E, Z)$. Hence we have $E_\infty^{0,i}=0$ for $i<n-1$ by (2.2). This means $H^i(F, Z)=0$ for $0<i<n-1$.

Next consider the following sequence,

$$E_{n+1}^{0,n-1} \longrightarrow E_n^{0,n-1} \xrightarrow[d_n]{} E_n^{n,0} \longrightarrow E_{n+1}^{n,0}$$

the homomorphism $E_{n+1}^{0,n-1}\to E_n^{0,n-1}$ is an injection by $d_n E_n^{-n,2n-2}=0$ and the homomorphism $E_n^{n,0}\to E_{n+1}^{n,0}$ is a projection by $d_n E_n^{n,0}=0$. Therefore this sequence is exact. Since $d_r E_r^{-r,n-r}=0$ we have

$$E_{n+1}^{0,n-1}\cong E_\infty^{0,n-1}\cong 0 \quad\text{and}\quad E_{n+1}^{n,0}\cong E_\infty^{n,0}\cong 0.$$

Thus we obtain $H^{n-1}(F, Z)\cong E_2^{0,n-1}\cong E_n^{0,n-1}\cong E_n^{n,0}\cong Z$.

By the same technique we can easily obtain $H^i(F, Z)\cong 0$ for $n\leqq i\leqq 2n-3$. This completes the proof.

Let $\kappa_s^r(p, q)\colon {}^sE_r^{p,q}\to E_s^{p,q}$ be the canonical homomorphism defined on the subset, ${}^sE_r^{p,q}$, of $E_r^{p,q}$ on which $d_r, d_{r+1}, \cdots, d_{s-1}$ are all zero, $(\infty\geqq s>r)$. Then it is easily seen that the following diagram is commutative,

$$\begin{array}{ccc} {}^sE_r^{p,q}\otimes {}^sE_r^{p',q'} & \xrightarrow[\smile]{} & {}^sE_r^{p+p',q+q'} \\ \Big\downarrow \kappa_s^r(p, q)\otimes\kappa_s^r(p', q') & & \Big\downarrow \kappa_s^r(p+p', q+q') \\ E_s^{p,q}\otimes E_s^{p,q} & \xrightarrow[\smile]{} & E_s^{p+p',q+q} \end{array} \tag{2.3}$$

where $\smile$ denotes the cup product of the cohomology spectral sequence. (See [4]).

Since we can easily see that

$$d_r E_r^{n,n-1}=d_r E_r^{n-r,n+r-2}=d_r E_r^{n,0}=d_r E_r^{n+r,r-1}=d_r E_r^{0,r-1}=d_r E_r^{2n-r,r-1}=0,$$

for $2\leqq r<n$, we see that $\kappa_n^2(n, 0)$, $\kappa_n^2(n, n-1)$, $\kappa_n^2(0, n-1)$ are all isomorphisms onto. On the other hand, we can show that the cup product $\smile$: $E_2^{n,0}\otimes E_2^{0,n-1}\to E_2^{n,n-1}$ is an isomorphisms onto. (See [16]). Hence we see that $E_n^{n,n-1}$ is an infinite cyclic group generated by $\kappa_n^2(n, n-1)((x_n\otimes 1)\smile(1\otimes y_{n-1}))$, where y_{n-1} is a generator of

$H^{n-1}(F, Z)$.

LEMMA (2.4). $d_n(\kappa_n^2(n, n-1)((x_n\otimes 1)\smile(1\otimes y_{n-1})))=\pm k\kappa_n^2(2n, 0)(x_{2n}\otimes 1)$.

For, $d_n(\kappa_n^2(n, n-1)((x_n\otimes 1)\smile(1\otimes y_{n-1})))=d_n(\kappa_n^2(n, 0)(x_n\otimes 1)\smile\kappa_n^2(0, n-1)(1\otimes y_{n-1}))$

$$=\pm\kappa_n^2(n, 0)(x_n\otimes 1)\smile\kappa_n^2(n, 0)(x_n\otimes 1)=\pm\kappa_n^2(n, 0)((x_n\smile x_n)\otimes 1)$$
$$=\pm k\kappa_n^2(2n, 0)(x_{2n}\otimes 1).$$

LEMMA (2.5). $H^{2n-2}(F, Z)\cong 0$, $H^{2n-1}(F, Z)\cong Z_k$ and
$H^i(F, Z)\cong 0$ for $2n\leqq i\leqq 3n-3$.

PROOF. Since $d_nE_n^{2n,0}$, $d_nE_n^{-n,3n-3}$ are both zero we see that ${}^{n+1}E_n^{2n,0}=E_n^{2n,0}$ and $\kappa_{n+1}^n(0, 2n-2)$ is an isomorphism onto. And moreover it follows from $d_rE_r^{n,n-1}=d_rE_r^{n-r,n-r-2}=0$, $(r\geqq n+1)$, that $\kappa_\infty^{n+1}(n, n-1)$ is an isomorphism onto. Hence we have $E_{n-1}^{n,n-1}=0$ by $H^{2n-1}(E, Z)\cong 0$. Thus there exists an exact sequence,

$$E_{n+1}^{0,2n-2}\longrightarrow E_n^{0,2n-2}\longrightarrow E_n^{n,n-1}\longrightarrow E_n^{2n,0}\longrightarrow E_{n-1}^{2n,0}\longrightarrow 0. \tag{2.6}$$

On the other hand, we have $E_{n+1}^{0,2n-2}=0$ and $E_n^{0,2n-2}\cong E_2^{0,2n-2}\cong H^{2n-2}(F, Z)$. For, $\kappa_\infty^{n+1}(0, {}^{2n-2})$: $E_{n+1}^{0,2n}\to E_\infty^{0,2n-2}$ is an isomorphism onto by $d_rE_r^{0,2n-2}\cong 0$, $d_rE_r^{-r,2n+r-3}=0$, $(r\geqq n-1)$, and $E_\infty^{0,2n-2}=0$ by $H^{2n-2}(E, Z)=0$. Hence $E_{n+1}^{0,2n-2}=0$. By the same way $\kappa_n^2(0, 2n-2)$ is an isomorphism.

Thus the sequence (2.6) is transformed into the exact sequence

$$0\longrightarrow H^{2n-2}(F, Z)\longrightarrow Z\xrightarrow[d_n]{} Z\longrightarrow E^{2n,0}\longrightarrow 0. \tag{2.7}$$

Combining (2.7) and Lemma (2.4) we see that $H^{2n-2}(F, Z)\cong 0$ and $E_{n+1}^{2n,0}\cong Z_k$. Next it is proved in the same way that there exists the following exact sequence,

$$E_{2n+1}^{0,2n-1}\longrightarrow E_{2n}^{0,2n-1}\longrightarrow E_{2n}^{2n,0}\longrightarrow E_{2n+1}^{2n,0}. \tag{2.8}$$

Here we have

$$E_{2n+1}^{0,2n-1}=0 \text{ from } d_rE_r^{0,2n-1}=0,\ (r\geqq 2n+1) \text{ and } E_\infty^{0,2n-1}=0$$
$$E_{2n+1}^{2n,0}=0 \text{ from } d_rE_r^{2n-r,r-1}=0,\ (r\geqq 2n+1) \text{ and } E_\infty^{2n,0}=0$$
$$E_{2n}^{2n,0}\cong E_{n+1}^{2n,0} \text{ from } d_rE_r^{2n-r,r-1}=0,\ (2n>r\geqq n+1).$$

Hence d_{2n} is an isomorphism onto and we have $E_{2n}^{0,2n-1}\cong Z_k$. Since it is easily shown that $\kappa_{2n}^2(0, 2n-1)$: $E_2^{0,2n-1}\to E_{2n}^{0,2n-1}$ is an isomorphism onto we have $H^{2n-1}(F, Z)\cong Z_k$. Lastly we can easily obtain $H^i(F, Z)\cong 0$ for $2n\leqq i\leqq 3n-3$ by making use of (2.2). This completes the proof.

Let $(''E_r^{p,q}, d_r'')$ be the spectral sequence of the cohomology group associated with the relative fibre space p: $(E, S)\to(B, S^{3n-1})$, $(S=p^{-1}(S^{3n-1}))$, as defined in [17], and let $''\kappa_s^r(p, q)$ be the homomorphism defined in the same way as $\kappa_s^r(p, q)$. Since we can express, $''E_2^{p,q}$, the first invariant terms of the sequence, in the form $H^p(B, S^{3n-1}, Z)\otimes H^q(F, Z)$ (see [17]) we have $''E_2^{p,q}=0$ for $p\neq n$, $2n$, $3n$.

Hence, $''\kappa_n^2(p, q)$ is an isomorphism onto for all p, q, and we see that $''E_n^{2n,n-1}$ and $''E_n^{3n,0}$ are infinite cyclic group generated by $''\kappa_n^2(2n, n-1)(\bar{x}_{2n}\otimes y_{n-1})$[14] and $''\kappa_n^2(3n, 0)(\bar{x}_{3n}\otimes 1)$ respectively. Then we have

LEMMA (2.9). $d_n''\kappa_n^2(2n, n-1)(\bar{x}_{2n}\otimes y_{n-1})=\pm m''\kappa_n^2(3n, 0)(\bar{x}_{3n}\otimes 1)$.

For, $d_n'' \, ''E_n^2(2n, n-1)(\bar{x}_{2n}\otimes y_{n-1})=d_n''\pm(''\kappa_n^2(2n, n-1)((\bar{x}_{2n}\otimes 1)\smile(1\otimes y_{n-1})))$[15]

$$=\pm d_n''(''\kappa_n^2(2n, 0)(\bar{x}_{2n}\otimes 1)\smile''\kappa_n^2(0, n-1)(1\otimes y_{n-1}))$$
$$=\pm''\kappa_n^2(2n, 0)(\bar{x}_{2n}\otimes 1)\smile d_n\kappa_n^2(0, n-1)(1\otimes y_{n-1})$$
$$=\pm''\kappa_n^2(3n, 0)((\bar{x}_{2n}\smile\bar{x}_n)\otimes 1)=\pm m''\kappa_n^2(3n, 0)(\bar{x}_{3n}\otimes 1)$$

where the last equality is the result of Lemma 2 of [15]. From $''\kappa_n^2(''E_2^{n,2n-2})=0$ we obtain an exact sequence,

$$0\longrightarrow ''E_{2n+1}^{2n,n-1}\longrightarrow ''E_n^{2n,n-1}\longrightarrow ''E_n^{3n,0}\longrightarrow ''E_{n+1}^{3n,0}\longrightarrow 0. \tag{2.10}$$

It is clear that $''\kappa_\infty^{n+1}(2n, n-1)$ and $''\kappa_{2n}^{n+1}(3n, 0)$ are isomorphisms onto. Hence, from Lemma (2.9) we have

LEMMA (2.11). $''E_\infty^{2n,n-1}=0$ and $''E_{2n}^{3n,0}\cong Z_m$.

And we have moreover

LEMMA (2.12). $H^{3n-1}(E, S; Z)\in\mathcal{C}_k$ and $H^{3n}(E, S; Z)$ is $\mathcal{C}_k$-isomorphic to Z_m.

For it is easily proved that two sequences,

$$0\rightarrow ''E_\infty^{2n,n-1}\rightarrow H^{3n-1}(E, S; Z)\rightarrow ''E_\infty^{n,2n-1} \text{ and } ''E_{2n}^{n,2n-1}\rightarrow ''E_{2n}^{3n,0}\rightarrow ''E_{2n+1}^{3n,0}\rightarrow 0$$

are exact. On the other hand, $E_2^{n,2n-1}\cong H^n(B, S^{3n-1}; Z)\otimes H^{2n-1}(F, Z)\cong Z\otimes Z_k\cong Z_k$. Therefore $''E_\infty^{n,2n-1}$ and $''E_{2n}^{n,2n-1}$ are belongs to the class $\mathcal{C}_k$ and moreover $''\kappa_\infty^{2n+1}(3n, 0)$ is an isomorphism onto by $d_r''\,''E_r^{3n,0}=d_r''\,''E_r^{3n-r,r-1}=0$, $(r\geqq 2n+1)$, i.e. $''E^{3n,0}\cong ''E_{2n+1}^{3n,0}$. From $''E_\infty^{i,j}=0$ for $i+j=3n$, $i>3n$ we have $''E_\infty^{3n,0}\cong H^{3n}(E, S; Z)$.

Hence the proof is completed by combining these results and Lemma (2.11). For our purpose we need the following algebraic

LEMMA (2.13) In the following diagram we suppose that

1) A, B, C, D are all abelian groups and Z is the group of integers
2) maps are all homomorphisms, and $i=jk$, $\partial_1 p=P\partial_2$, $P=rq$
3) ρk, ρP is identity, and i is onto
4) the sequences: $0\longrightarrow Z\xrightarrow[p]{} B\xrightarrow[j]{} A$ and

(14) $\bar{x}_{in}$ denotes a generator of $H^{in}(B, S^{3n-1}; Z)$ for $i=1, 2, 3$.

(15) $\smile$ means the cup product between the relative cohomology and the absolute cohomology.

$$0 \longrightarrow Z \xrightarrow[k]{} B \xrightarrow[\delta_1]{} C \quad \text{and} \quad D \xrightarrow[t]{} Z_m \xrightarrow[r]{} 0 \qquad \text{are all exact.}$$

Then if $D \in \mathcal{C}$ A is $\mathcal{C}$-isomorphic to Z_m.

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
A & Z \arrow[l, "i"'] \arrow[d, "k"] & & \\
 & B \arrow[ul] \arrow[r, shift left, "p"] \arrow[d, "\delta_1"] & Z \arrow[l, shift left, "\rho"] \arrow[d, "\delta_2"] & 0 \arrow[l] \\
0 & C \arrow[l] & Z \arrow[l, "P"] \arrow[dl, "q=\mathrm{mod}\, m"] & \\
 & Z_m \arrow[u, "r"] & & \\
 & & D \arrow[ul, "t"] &
\end{tikzcd}$$

PROOF. First we have $mZ \subset i^{-1}(0)$. For, since B is isomorphic to the direct sum of $\rho^{-1}(0)$ and $p(Z)$ from 3) we can uniquely express $k(1)$ in the form $x+sp(1)$, where s is an integer and $\rho x=0$. Then $s=1$ and $mx=0$, for $1=\rho k(1)=\rho x+s\rho p(1)$ $=s$ and $\delta_1(mp(1))=p\delta_2(m)=rq\delta_2(m)=mrq\delta_2(1)=0$, i.e. there exists an integer μ such that $mp(1)=\mu k(1)=\mu\{x+p(1)\}$. Thus we obtain $\mu=m$ and $\mu x=0$. Now, $i(mZ)$ $\subset jk(mZ) \subset j(mx+mp(1))=jp(1)=0$, i.e. $mZ \subset i^{-1}(0)$.

From the sequence: $i^{-1}(0) \cong \delta_1^{-1}(0) \frown p(Z) \cong (P\delta_2)^{-1}(0) \cong P^{-1}(0)$ we see that the homomorphisms $\delta_2 \rho k | i^{-1}(0)$ is an isomorphism of the pair $(i^{-1}(0), mZ) \to P^{-1}(0), mZ)$. Consider the following exact sequence, $0 \to i^{-1}(0)/mZ \to Z/mZ \to Z/i^{-1}(0) \to 0$. Then it is easily seen that $i^{-1}(0)/mZ \cong P^{-1}(0)/q^{-1}(0) \cong r^{-1}(0)$. Hence we have $A \cong Z/i^{-1}(0)$ $\underset{\mathcal{C}_h}{\cong} Z_m$.

LEMMA (2.14). $H^{3n-1}(F, Z)$ is $\mathcal{C}_k$-isomorphic to Z_m.

PROOF. Consider the following diagram

$$\begin{tikzcd}
 & H^{3n-1}(E, S; Z) \arrow[d, "j^*"] & \\
H^{3n-1}(F, Z) & H^{3n-1}(E, Z) \arrow[l, "i^*"'] \arrow[d, "i^*"] & \\
 & H^{3n-1}(S, Z) \arrow[ul, "j^*"] \arrow[r, shift left, "\rho^*"] \arrow[d, "\delta"] & H^{3n-1}(S^{3n-1}, Z) \arrow[l, shift left, "(P|S)^*"] \arrow[d] \\
0 & H^{3n}(E, F; Z) \arrow[l] & H^{3n}(B, S^{3n-1}; Z) \arrow[l, "P^*"] \\
 & Z_m \arrow[u] \arrow[ur] & \\
 & & ''E_{2n}^{n, 2n-1} \arrow[ul]
\end{tikzcd}$$

where ρ^* denotes the homomorphim induced by a map $\rho: S^{3n-1}\to S$ defined by the formula: $\rho(x, 0)(t)=(x, 0)$ for $t\in I$, $x\in S^{3n-1}$ and i^*, j^* are appropriate inclusion maps. Then it is easily verified that this diagram satisfies the conditions of Lemma (2.13).

In particular, from $H^{3n-1}(E, Z)\cong Z$ and $H^{3n-1}(E, S: Z)\in C_k$ we see that $j^*: H^{3n-1}(E, S; Z)\to H^{3n-1}(E, Z)$ is a zero-homomorphism. Thus Lemma (2.14) follows from Lemma (2.13).

LEMMA (2.15). $H^{3n-2}(F, Z)\in C_k$.

PROOF. First we have

LEMMA (2.16). $H^{3n-2}(F, Z)\cong E_2{}^{0,3n-2}\cong E_n{}^{0,3n-2}\cong E_{2n}{}^{0,3n-2}$.

For, this isomorphism $\kappa_n^2(0, 3n-2)$ follows from $d_rE_r{}^{0,3n-2}=0$, $(n>r\geqq 2)$. Then consider the following sequence

$$0\longrightarrow E_{n+1}^{0,3n-2}\longrightarrow E_n{}^{0,3n-2}\longrightarrow E_n{}^{n,2n-1}.$$

This sequence is clearly exact and it follows from $E_2{}^{n,2n-1}\cong Z_k$ that $E_n{}^{n,2n-1}$ is belonging to the class C_k. Hence $E_{n+1}^{0,3n-2}$ is C_k-isomorphic to $E_n{}^{0,3n-2}$. On the other hand, we see that $\kappa_{2n}^{n+1}(0, 3n-2)$ is an isomorphism onto by $d_rE_r{}^{0,3n-2}=0$, $(2n>r>n)$. Thus we obtain (2.16).

Secondly we have

LEMMA (2.17). $E_{2n}{}^{2n,n-1}\cong Z\cong E_\infty{}^{2n,n-1}$.

For, the first part $E_{2n}{}^{2n,n-1}\cong Z$ follows from $d_rE_r{}^{2n-r,n-r-2}=0$, $(2n>r\geqq 2)$, and $d_rE_r{}^{2n,n-1}=0$. Let $D^{p,q}$ denote the subgroup of $H^{p+q}(E, Z)$ consisting of elements whose filtration does not exceed p. It can be easily seen that $E_\infty{}^{2n,n-1}$ coincides with $D^{2n,n-1}$ and two sequences;

$$0\longrightarrow E_\infty^{2n,n-1}\longrightarrow D^{n,2n-1}\longrightarrow E_\infty^{n,2n-1}\longrightarrow 0$$

and

$$0\longrightarrow D^{n,2n-1}\longrightarrow H^{3n-1}(E, Z)\longrightarrow H^{3n-1}(F, Z)\longrightarrow 0$$

are exact. Hence if $E_\infty^{2n,n-1}=0$, $D^{n,2n-1}$ is isomorphic to $E_\infty^{n,2n-1}$, but this contradicts to $H^{3n-1}(E, Z)\cong Z$, since $E_\infty^{n,2n-1}$ is a finite group by $E_2{}^{n,2n-1}\cong Z_k$ and also $H^{3n-1}(F, Z)$ is a finite group by Lemma (2.14). Thus we obtain $Z\cong E_\infty^{2n,n-1}$ from $E_\infty^{2n,n-1}\cong D^{2n,n-1}\subset Z$. It is easily shown that the sequence,

$$E_{2n-1}^{0,3n-2}\longrightarrow E_{2n}^{0,3n-2}\longrightarrow E_{2n}^{2n,n-1}\longrightarrow E_{2n+1}^{2n,n-1}\longrightarrow 0 \tag{2.18}$$

is exact.

Hence, since we can obtain $E_{2n+1}^{0,3n-2}=0$ by $H^{3n-2}(E, Z)=0$ and $d_rE_r{}^{0,3n-2}=0$ for $r\geqq 2n+1$ it follows from (2.17) and (2.18) that there exists an exact sequence,

$$0\longrightarrow E_{2n}^{0,3n}\longrightarrow Z\longrightarrow Z\longrightarrow 0.$$

Thus we have $E_{2n}^{0,3n}=0$ and Lemma (2.15) follows from (2.16).

LEMMA (2.19). $H^i(F, Z)\in\mathcal{C}_{km}$ for $i\geqq 3n$.

PROOF. Making use of (2.2) we can inductively construct an exact sequence, $A_i\to H^i(F, Z)\to B_i$, where A_i and B_i are both groups of the class $\mathcal{C}_{km}$ and $i\geqq 3n$. Now Lemma (2.19) follows from the definition of $\mathcal{C}_{km}$.

Thus Lemma (1.1) follows from a series of Lemmas (2.1), (2.5), (2.14), (2.15), (2.19).

In particular we have

Corollary (2.20). $\pi_i(L(\alpha))$ is $\mathcal{C}_k$-isomorphic to $\pi_{i-1}(S^{n-1})$ for $i\leqq 3n-4$, where k is the Hopf invariant of α.

§3. Applications of Theorem I.

We shall consider the Cayley projective plane Π. It is known that Π has the cell decomposition[16] $S^8\overset{\sigma_8}{\cup}e^{16}$, where σ_8 denotes the Hopf map: $S^{15}\to S^8$. Since the Hopf invariant of σ_8 is 1 it is important for us to find the least positive integer m such that $m[\sigma_8, \iota_8]\in\sigma_8\circ\pi_{22}(S^{15})$.

By Theorem (1.8) of [13] m must be divisible by 3. Moreover we can prove

LEMMA (3.1). m must be divisible by 6 and must be a divisor of 24.

PROOF. Let E^k denote the k-fold suspension homomorphism, and let η_i, σ_i denote $E^{i-2}(\eta_2)$[17], $E^{i-8}(\sigma_8)$, respectively. Then we have

$$\begin{aligned} m[\sigma_8, \iota_8] &= m[\iota_8, \iota_8]\circ(-E^7(\sigma_8)\circ E^{14}(\iota_8)) = m[\iota_8, \iota_8]\circ(-\sigma_{15}\circ\iota_{22})^{(18)} \\ &= -m((2\sigma_8-E\tau_7)\circ\sigma_{15})^{(19)} \\ &= -m(2\sigma_8\sigma_{15}-E\tau_7\circ\sigma_{15}) = mE\tau_7\circ\sigma_{15}-2m\sigma_8\circ\sigma_{15} \\ &= E\tau_5\circ m\sigma_{15}-\sigma_8(\circ-2\,m\sigma_{15}). \end{aligned}$$

Since this expression is unique by Prop. 5 of [17] (p. 281), m coincides with the order of $\tau_7\circ\tau_{14}$. Hence it follows from $E\pi_{21}(S^7)$ being isomorphic to Z_2+Z_{24} that m is a divisor of 24. (See [20]).

Now consider the generalized Hopf invariant,

$$H:\ \pi_i(S^n)\longrightarrow\pi_i(S^{2n-1})$$

as defined in [10]. Then we have

LEMMA (3.2). $H(\tau_7)=\eta_{13}$, where H: $\pi_{14}(S^7)\to\pi_{14}(S^{13})$.

(16) See [25].
(17) η_2: $S^3\to S^2$ is the Hopf map.
(18) See [12].
(19) τ_7 denotes a generator of $\pi_{14}(S^7)\cong Z_{120}$. (See [6]).

For, it is known that the sequence,

$$\pi_{13}(S^6) \xrightarrow[E]{} \pi_{14}(S^7) \xrightarrow[H]{} \pi_{14}(S^{13}) \xrightarrow[P]{} \pi_{12}(S^6) \xrightarrow[E]{} \pi_{13}(S^7)$$

is exact. (See [20])

By H. Toda $E: \pi_{12}(S^6) \to \pi_{13}(S^7)$ is an isomorphism onto (see [20]). Hence H is onto. Since $\pi_{14}(S^{13})$ is isomorphic to Z_2 generated by η_{13} we have $H(\tau_7) = \eta_{13}$. Thus we obtain $H(\tau_7 \circ \sigma_{14}) = H(\tau_7 \circ E\sigma_{13}) = H(\tau_7) \circ \sigma_{14} = \eta_{13} \circ \sigma_{14}$ from (3.2) and the formula; $H(x \circ Ey) = H(x) \circ Ey$. (see [11]). And moreover it is known that $\eta_{13} \circ \sigma_{14}$ is not zero. Therefore we obtain $H(\tau_7 \circ \sigma_{14}) \neq 0$. Since $\pi_{21}(S^{13})$ is isomorphic to $Z_2 + Z_2$ m must be even. This completes the proof.

Combining Lemma (3.1) with Theorem I and Corollary (2.20) we have

COROLLARY (3.3). $\pi_i(H)$ is C_6-isomorphic to the direct sum of $\pi_{i-1}(S^7)$ and $\pi_i(S^{23})$ for all i.

COROLLARY (3.4). $\pi_i(H)$ is isomorphic to $\pi_{i-1}(S^7)$ for $i \leqq 20$.

REMARK. If we notice that $\pi_i(H, S^8) = \bar{\sigma}_8 \circ \pi_i(V^{16}; S^{15})$ for $i \leqq 22$ and $\pi_{23}(H, S^8) \cong (Z[\bar{\sigma}, L_8]_r) + \bar{\sigma}_8 \circ \pi_{23}(V^{16}, S^{15})$ it is seen by making use of the homotopy sequence of the pair (H, S^8) that

$$\pi_i(H) \cong \pi_{i-1}(S^{n-1}) \quad \text{for} \quad i \leqq 21,$$

$$\pi_{22}(H) \cong \pi(S)/(\tau_7 \circ \sigma_{14}),$$

and $$\pi_{23}(H) \cong Z\text{—group of order } 480.$$

Next S^n_∞ denote the reduced product complex of S^n, and S^n_2 the $(3n-1)$-skeleton of S^m_∞.

It is easily shown that S^n_2 has the cell decomposition, $S^n \cup_\alpha e^{2n}$ such that α is the Whitehead product $[\iota_n, \iota_n]$. It is known that if n is even $[\iota_n, \iota_n]$ has the Hopf invariant -2. (see [22]). By means of Jacobi identity of Whitehead product, (cf. [21]), we see that $3([[\iota_n, \iota_n], \iota_n]) = 0$ for even n. Consider the part of the homotopy sequence of the pair (S^n_2, S^n),

$$\pi_{3n-1}(S^n_2) \xrightarrow[j_*]{} \pi_{3n-1}(S^n_2, S^n) \xrightarrow[\partial]{} \pi_{3n}(S^n).$$

Since $\partial(3[\bar{\alpha}, \iota_n]_r) = 3\partial[\bar{\alpha}, \iota_n]_r = 3[[\iota_n, \iota_n], \iota_n] = 0$ there exists an element of $\pi_{3n-1}(S^n_2)$, say β, such that $j_*(\beta) = 3[\bar{\alpha}, \iota_n]_r$. Thus we have

COROLLARY (3.5). If n is even $\pi_i(S^n_2)$ is C_6-isomorphic to the direct sum of $\pi_{i-1}(S^{n-1})$ and $\pi_i(S^{3n-1})$ for all i.

COROLLARY (3.6). If n is even $\pi_i(S^n_2)$ is C_2-isomorphic to $\pi_{i-1}(S^{n-1})$ for $i \leqq 3n-4$.

REMARK: Since it is known that $\pi_i(S^n_2)$ is isomorphic to $\pi_{i+1}(S^{n+1})$ for $i \leqq 3n-4$ we see from Corollary (3.6) that $\pi_{i-1}(S^{n-1})$ is C_2-isomorphic to $\pi_{i+1}(S^{n+1})$ for $i \leqq 3n-4$. Of course this is a special case $p=3$ of Prop. 4 of [17]. (p. 280).

Chapter II. The group $\pi_i(X_p^n)$.

§1. Reduced product complexes.

In this section, we shall give a résumé of the theory of James (cf. [10]) for convenience of reference. Let A be a countable CW-complex in which a^0 is the only one vertex. The m-fold topological product A^m, $m=1, 2, 3, \cdots$, is represtnted by infinite sequence in A, $a^m=(a_1, a_2, \cdots, a_r, \cdots)$, such that $a_r=a^0$ if $r>m$. We identify A^0 with a^0 and A^1 with A in the obvious way.

If $a^m \in A^m$ let $p_m(a^m)$ be the (finite) subsequence of a^m consisting of those term which lie in $A-a^0$, with the same relative order as they have in a^m. Let A_m be the set of sequences in $A-a^0$ which have not more than m terms, with the identification topology determined by $p_m: A^m \rightarrow A_m$. It is clear that inclusion function $i_m: A_m \rightarrow A_{m+1}$ is continuous and that $i_m A_m$ is a closed subset in A_{m+1}. Then the reduced product complex A_∞ of A is defined as

$$A_\infty = A_0 \smile A_1 \smile A_2 \smile \cdots \smile A_m \smile \cdots$$

where a set $F \subset A_\infty$ is closed, if and only if $F \frown A_m$ is closed for every $m \geqq 0$. Thus points of A_∞ are the finite sequence of points in $A-a$. Clearly A^m is the union of disjoint product of cells from A, $e_1 \times e_2 \times \cdots \times e_r \times \cdots$, such that e_r is the vertex if $r>m$. Each of such product cells is mapped homomorphically into A_m by p_m, and the image of two of them either coincide or are disjoint. Hence A_∞ is a complex whose cells are the products of cells of $A-a^0$,

$$e_1 \times e_2 \times \cdots \times e_r, \qquad (r=0, 1, 2, \cdots, m).$$

Since A_m is closure finite and p_m is cellular, it follows that the reduced product complex A_∞ of A has this cells structure. (See [24]).

James has proved following two theorems about the homology groups and homotopy groups of A_∞;

(1.1). $\sum_{r=1}^{\infty} H_r(A_\infty, Z) \cong \sum_{m=1}^{\infty} G^m$, where G^m is inductively defined by $G^m = G \otimes G^{m-1} + G * G^{m-1}$, $G^1 = \sum_{r=1}^{\infty} H^r(A, Z)$.

(1.2). there exists the canonical isomorphism, between the homotopy sequence of the pair (A_∞, A) and that of the pair (Ω, A), where Ω denotes the loop space of the suspended space of A.

§2. On the generators of $H_{2n}((X_p^n)_\infty, X_p^n)$ and $\pi_{2n+1}((X_p^n)_\infty, X_p^n)$.

As stated in §1, $(X_p^n)_\infty$ has the cell decompositions;

$$(X_p^n)_\infty = e^0 \smile e^n \smile e^{n+1} \smile e^{2n} \smile e_1^{2n+1} \smile e_2^{2n+1} \smile e^{2n+2} \smile e^{3n} \smile \cdots$$

and the homology groups in lower dimensions are given by (1.1) as follows:

LEMMA (2.1). $H_0((X_p^n)_\infty, Z)\cong Z$
$H_n((X_p^n)_\infty, Z)\cong H_{2n}((X_p^n)_\infty, Z)\cong H_{2n+1}((X_p^n)_\infty, Z)\cong Z_p$
$H_i((X_p^n)_\infty, Z)\cong 0$ for other $i<3n$.

From (2.1) we obtain $H_i((X_p^n)_\infty, X_p^n; Z)=0$ for $i<2n$ and $H_{2n}((X_p^n)_\infty, X_p^n; Z)\cong Z_p$. By Theorem 2 of [17], p. (274). We have

LEMMA $(2.1)_0$ $\pi_{2n}((X_p^n)_\infty, X_p^n)\cong Z_p$.

Now let f_{n+1}, f^1_{2n+1}, f^2_{2n+1} be characteristic maps for e^{n+1}, e_1^{2n+1}, e_2^{2n+1}, respectively, and φ_n: $V^n\to S^n$ a map such that $\varphi_n|e^n$ is a homomorphism and $\varphi_n|S^{n-1}$ is a constant map. Then f^1_{2n+1}, f^2_{2n+1} are represented by tne formula

$$f^1_{2n+1}(t_1, t_2, \cdots, t_{2n+1})=p_2(\varphi_n(t_1, t_2,\cdots, t_n), f_{n+1}(t_{n+1}, \cdots, t_{2n+1}))$$
$$f^2_{2n+1}(t_1, t_2, \cdots, t_{2n+1})=p_2(f_{n+1}(t_1, \cdots, t_{n+1}), \varphi_n(t_{n+2}, t, \cdots, t_{2n+1})).$$

Define two maps v: $V^{2n+1}\to V^{2n+1}$ and τ: $(X_p^n)_\infty\to(X_p^n)$ by

$$v(t_1, \cdots, t_{2n+1})=(t_{n+2}, t_{n+3}, \cdots, t_{2n+1}, t_1, \cdots, t_{n+1})$$
$$\tau(x_1, x_2, \cdots, x_m)=(x_m, x_{m-1}, \cdots, x_2, x_1).$$

It is easily seen that τ is well defined and continuous. From definition we have

LEMMA (2.2). $\tau\tau=$identity and $f^2_{2n+1}=\tau\circ f^1_{2n+1}\circ v$.

Since v is a map of degree $(-1)^{n(n+1)}$ and $\tau|e^{2n}$ is a map of degree $(-1)^{n^2}$ we see from (2.2) that $\partial e_1^{2n+1}=(-1)^n\partial e_2^{2n+1}=\pm pe^{2n}$, where ∂e^i is the homology boundary of the cell e^i. This means that the group of cycles, $Z_{2n+1}((X_p^n)_\infty, Z)$ is an infinite cyclic group generated by $e_1^{2n+1}+(-1)^n e_2^{2n+1}$.

By (2.1) $H_{2n+1}((X_p^n)_\infty, Z)$ is isomorphic to Z_p. Hence we obtain $\partial e^{2n+2}=\pm p(e_1^{2n+1}+(-1)^n e_2^{2n+1})$.

Describing these result in terms of the homotopy theory we have the following

LEMMA (2.3) $\pi_{2n+1}((X_p^n), X_p^n\smile e^{2n})$ is the abelian group generated by $\{f^1_{2n+1}\}$ and $\{f^2_{2n+1}\}$(21) in which only relation $mp(\{f^1_{2n+1}\}+(-1)^n\{f^2_{2n+1}\})$ being zero holds for all integers m.

LEMMA (2.4). $\partial^{-1}(0)$ is isomorphic to the cyclic group of order p generated by $(\{f^1_{2n+1}\}+(-1)^n\{f^2_{2n+1}\})$, where $\partial^{-1}(0)$ denotes the kernel of the boundary homomorphism of the homotopy sequence of the triple $\{(X_p^n)_\infty, X_p^n\smile e^{2n}, X_p^n\}$.

And moreover wehave

LEMMA (2.5). Let i_* be the inclusion homomorphism: $\pi_{2n+1}((X_p^n)_\infty, X_p^n)\to\pi_{2n+1}((X_p^n)_\infty, X_p^n\smile e^{2n})$. Then if p is odd $\pi_{2n+1}((X_p^n)_\infty, X_p^n)$ is isomorphic to the cyclic group of order p generated by η such that $i_*(\eta)=(\{f^1_{2n+1}\}$

(21) $\{f\}$ denotes the homotopy class of f.

$+(-1)^n\{f^2_{2n+1}\})$.

PROOF. Consider the homotopy sequence of the triple as above,

$$\to \pi_{2n+1}(X^n_p \smile e^{2n}, X^n_p) \to \pi_{2n+1}((X^n_p)_\infty, X^n_p) \xrightarrow[i_*]{} \pi_{2n+1}((X^n_p)_\infty, X^n_p \smile e^{2n}) \xrightarrow[\partial]{} \pi_{2n}(X^n_p \smile e^{2n}, X^n_p) \to .$$

Since $\pi_{2n+1}((X^n_p)_\infty, X^n_p)$ is the p-group by Theorem 2 of [17] (p. 274) and $\pi_{2n+1}(X^n_p \smile e^{2n}, X^n_p)$ is isomorphic to $\pi_{2n+1}(S^{2n}) \cong Z_2$ by Theorem II of [2] (p. 198), i_* is an isomorphism into, i.e. $\pi_{2n+1}((X^n_p)_\infty, X^n_p)$ is isomorphic to $\partial^{-1}(0)$. Hence (2.5) is obtained from Lemma (2.4).

§3. Definitions of H_1 and H_2.

In this section, let p be odd. Consider the part of the homotopy sequence of the pair $((X^n_p)_\infty, (X^n_p)$, $\pi_{2n+2}((X^n_p)_\infty, X^n_p) \longrightarrow \pi_{2n+1}(X^n_p) \xrightarrow[i_*]{} \pi_{2n+1}((X^n_p)_\infty) \xrightarrow[j^1_*]{} \pi_{2n+1}((X^n_p)_\infty, X^n_p) \longrightarrow \pi_{2n}(X^n_p) \xrightarrow[i_*]{} \pi_{2n}((X^n_p)_\infty) \xrightarrow[j^2_*]{} \pi_{2n}((X^n_p)_\infty, X^n_p) \longrightarrow \pi_{2n-1}(X^n_p) \xrightarrow[i^*]{} \pi_{2n-1}((X^n_p)_\infty)$. On the other hand,

By (1.2), $\pi_{2n+1}((X^n_p)_\infty) \cong \pi_{2n+2}(X_p^{n+1})$ and $\pi_{2n}((X^n_p)_\infty) \cong \pi_{2n+1}(X_p^{n+1})$. By Lemma (2.5) and (2.1), $\pi_{2n+1}((X^n_p)_\infty, X^n_p) \cong \pi_{2n}((X^n_p)_\infty, X^n_p) \cong Z_p$. And moreover by another method we can prove

$$\pi_{2n+2}((X^n_p)_\infty, X^n_p) \cong \pi_{2n+2}(X_p^{2n+1}) \cong 0.$$

Thus we can define H_1 and H_1 by j^1_* and j^2_* in the obvious fashion in the above exact sequence. Then, since it is known that i_* is equivalent to the suspension homomorphism, we have the exact sequence as stated in the introduction. Now it is sufficient for us to determine whether $\partial\eta = 0$ or not, where η is the generator of $\pi_{2n+1}((X^n_p)_\infty, X^n_p)$ metioned in Lemma (2.5).

Consider the following commutative diagram

$$\begin{tikzcd}
\pi_{2n+1}((X_p{}^n)_\infty, X_p{}^n \smile e^{2n}) \arrow[r, "\partial"'] & \pi_{2n}(X_p{}^n \smile e^{2n}) \\
\pi_{2n+1}((X_p{}^n)_\infty, X_p{}^n) \arrow[u, "i_*"] \arrow[r, "\partial"'] & \pi_{2n}(X_p{}^n) \arrow[u, "i_*"'] \\
& \pi_{2n+1}(X_p{}^n \smile e^{2n}, X_p{}^n) \arrow[u, "\partial"'] \arrow[ul, "j_*"]
\end{tikzcd}$$

The vertical sequence of the right hand side is exact and $\pi_{2n+1}(X^n_p) \smile e^{2n}, X^n_p)$ is isomorphic to Z_2. Therefore i_* is an isomorphism into. On the other hand, $i_*\partial\eta = \partial i_*\eta = \partial\{f^1_{2n+1}\} + \partial(-1)^{n+1}\{f^2_{2n+1}\}$ and $\partial\{f^1_{2n+1}\} = \tau_*\partial\{f^2_{2n+1}\}$,[22] where τ_* is the homomorphism induced by the map $\tau | X^n_p \smile e^{2n}$. Thus we denote $\partial\{f^1_{2n+1}\}$ by α and we have

(3.1) $$i_*\partial\eta = \alpha + (-1)^{n+1}\tau_*\alpha .$$

(22) See (2.2).

Now apply τ_* to (3.1)

$$\tau_*\alpha+(-1)^{n+1}\tau_*\tau_*\alpha=\tau_*i_*\partial\eta=i_*\partial\eta^{(23)}=\alpha+(-1)^{n+1}\tau_*\alpha$$

i. e.
$$(-1)^{n+1}\alpha+\tau_*\alpha=\alpha+(-1)^{n+1}\tau_*\alpha \qquad (\tau_*\tau_*=1)$$
$$\alpha+(-1)^{n+1}\tau_*\alpha=(-1)^{n+1}(\alpha+(-1)^{n+1}\tau_*\alpha).$$

Thus we have

LEMMA (3.2). $$(1-(-1)^{n+1})(\alpha+(-1)^{n+1}\tau_*\alpha)=0\,.$$

Hence if n is even we obtain $2(\alpha-\tau_*\alpha)=0$, i.e. $2i_*\partial\eta=0$. Since i_* is an isomorphism into and $\pi_{2n}(X''_p)$ is the p-group we have $\partial\eta=0$. From these arguments we obtain

LEMMA (3.3). If n is even H_2 is onto.

In section 5 we shall consider the case where n is odd.

§4. Separation elements.

In the following §5, we shall need some results on separation elements as introduced by James [9], of which we shall give a résumé.

Let (K, L) be a pair of CW-complex such that the complementary of L is an open n-cell e^n. That is to say, there is a map of the pair (V^n, S^{n-1}) into the pair (K, L) so that $f|V^n-S^{n-1}$ is a topological map onto e^n. Now let S^{n-1} be the equator of S^n which devide S^n into two hemispheres, E^n_+, E^n_-, i.e. E^n_+, (E^n_-) is the subset of euclidean $(n+1)$-space consisting of the points whose coordinates, $(x_1, x_2, \cdots, x_{n+1})$ satisfy the conditions, $\sum x_i^2\leqq 1$ and $x_{n+1}\geqq 0$ $(x_{n+1}\leqq 0)$. Let p_+, p_- be the orthogonal projection defined by the formula,

$$p_+(x_1, x_2,\cdots, x_{n+1})=(x_1, x_2, \cdots, x_n) \text{ for } x_{n+1}\geqq 0$$
$$p_-(x_1, x_2,\cdots, x_{n+1})=(x_1, x_2, \cdots, x_n) \text{ for } x_{n-1}\leqq 0.$$

Clearly we may consider $p_+(p_-)$ as a map of the pair $(E^n_{+(-)}, S^{n-1})$ into the pair (V^n, S^{n-1}) such that $p_-|S^{n-1}=p_+|S^{n-1}$ is the identity map of S^{n-1}.

Let X be a topological space, and let u, v be two maps: $K\rightarrow X$ which coincide with each other on L.

Define a map κ of S^n into X as follows:

$$k(x_1, x_2,\cdots, x_{n+1})=ufp_+(x_1, x_2,\cdots, x_{n+1}) \text{ for } x_{n+1}\geqq 0$$
$$k(x_1, x_2,\cdots, x_{n+1})=vfp_-(x_1, x_2,\cdots, x_{n+1}) \text{ for } x_{n+1}\leqq 0.$$

Since $p_+|S^{n-1}=p_-|S^{n-1}$ and $u|L=v|L$ k is well defined and continuous.

We shall denote the homotopy class of k by $d(u, v)$ so that $d(u, v)$ is an element of the group $\pi_n(X)$, and we shall call $d(u, v)$ the separation element of

(23) Notice that τ_* is identity on i_*-image from definitions.

u and v. Now we have a series of lemmas.

LEMMA (4.1). Let u, v be maps of K into X such that $u|L=v|L$. Then $d(u, v)=0$, if and only if u is homotopic to v with relative to L.

LEMMA (4.2). Let u, v and w be maps of K into X such that $u|L=v|L=w|L$. Then $d(u, v)=d(v, w)+d(u, w)$.

LEMMA (4.3). Let u be a map of K into X, and let δ be an element of $\pi_n(X)$. Then there exists a map of K into X, say v, such that $u|L=v|L$ and $d(u, v)=\delta$.

LEMMA (4.4). Let u, v be maps of K into X such that $u|L=v|L$, and let h be a map of X into a topological space Y. Then $d(hu, hv)$ is the image of $d(u, v)$ under the homomorphism induced by h.

LEMMA (4.5). Let (K, L) and (K', L') be pairs of CW-complexes such that the complementary of L and L' are both open cells e^n in K and K', and let g be a map of the prirs (K, L) into the pair (K', L') which maps the complementary of L onto that of L' with degree p. Let u, v be maps of K' into X such that $u|L'=v|L'$. Then $d(ug, vg)=pd(u, v)$, i.e. $d(ug, vg)$ is a p-multiple of $d(u, v)$.

Lemma (4.1) and (4.2) are easily obtained from definitions, and the usual construction of homotopy.

New let u be a map of K into X and δ an element of $\pi_n(X)$ and a map $g: S^n \to X$ a representative map of δ. Since E^n_+ is contractible there is a map $G: S^n \to X$ such that G is homotopic to g and $G|E^n_+=ufp_+$. Define a map $v: K \to X$ by the formula,

$$v(x)=u(x) \qquad \text{for } x \in L$$
$$v(y)=Gp_-^{-1}f^{-1}(y) \qquad \text{for } y \in f(V^n).$$

If $f(x')=f(y')=x$ and $x' \neq y'$ $f(x')$ is contained in L and x', y' are points of S^{n-1}. Hence $Gp_-^{-1}(x')=ufp_+(x')=uf(x')=u(x)$ and also $Gp_-^{-1}(y')=ufp_+(y')=uf(y')=u(x)$, i.e. v is well defined and therefore v is continuous. On the other hand, we have by definition

$$G(x)=ufp_+(x) \qquad \text{for } x \in E^n_+$$
$$G(x)=vfp_-(x) \qquad \text{for } x \in E^n_- .$$

Since the homotopy class of G is δ we have $d(u, v)=\delta$. This shows Lemma (4.3).

Next, let K_i, $(i=1, 2)$ be copies of K and $\bar{K}$ the complex obtained by identifying points of $L \subset K_1$ with those of $L \subset K_2$. Since S^n has the cell decomposition, $E^n_+ \cup S^{n-1} \cup E^n_-$ we can define a map $F: S^n \to \bar{K}$ as follows: $F(x)=f_1p_+(x)$ $x \in E^n_+$ and $F(x)=f_2p_-(x)$ $x \in E^n_-$. Let u, v be maps of K into X such that $u|L=v|L$. Define a map $k(u, v)$ of K into X by the formula,

$$k(u, v)(x)=u(x) \qquad \text{for } x \in K_1$$
$$k(u, v)(y)=v(y) \qquad \text{for } y \in K_2.$$

Then it is clear that the homotopy class of $k(u, v)\circ F$ is $d(u, v)$. Namely $d(u, v) = k(u, v)_*\{F\}$, where $\{F\}$ denotes the homotopy class of F. Let h be a map of X into Y. Since we have $k(hu, hv) = h\cdot k(u, v)$ by definitions $d(hu, hv) = K(hu, hv)_*\{F\} = h_* k(u, v)_*\{F\} = h_* d(u, v)$. This shows Lemma (4.4). We shall proceed to Lemma (4.5).

By assumption it is easily shown that there existsa map φ_p of the pair (V^n, S^{n-1}) into the pair (V^n, S^{n-1}) with degree p and the following diagram is homotopy-commutative.

$$\begin{array}{ccc} (K, L) & \xrightarrow[g]{} & (K', L') \\ \downarrow f & & \downarrow f' \\ (V^n, S^{n-1}) & \xrightarrow[\varphi_p]{} & (V^n, S^{n-1}) \end{array}$$

Define a map φ'_p: $S^n \to S^n$ as follows:

$$\varphi'_p(x) = p_+^{-1}\varphi_p p_-(x) \qquad \text{for } x \in E_+^n$$
$$\varphi'_p(x) = p_-^{-1}\varphi_p p_-(x) \qquad \text{for } x \in E_-^n .$$

Clearly, φ'_p is a map of degree p and the following diagram is homotopy-commutative.

$$\begin{array}{ccc} \bar{K} & \xrightarrow[g']{} & \bar{K}' \\ F \uparrow & & \uparrow F' \\ S^n & \xrightarrow[\varphi'_p]{} & S^n \end{array}$$

where g' is defined by $g|K_1 = g = g'|K_2$. Then we have

$$d(ug, vg) = k(ug, vg)_*\{F\} = k(u, v)_* g'_*\{F\}\circ\{\varphi'_p\}$$
$$= k(u, v)_*(p\{F\}) = pk(u, v)_*\{F\} = pd(u, v).$$

This completes the proof of Lemma (4.5).

§5. The proof of Theorem II.

LEMMA (5.1). $[\iota'_n, \iota'_n] = 0$, where ι'_n denotes a generator of $\pi_n(X_p^n)$.

PROOF. Since n is odd we have $2[\iota'_n, \iota'_n] = 0$. However $\pi_{2n-1}(X_p^n)$ has no 2-primary component other than zero. i.e. $[\iota'_n, \iota'_n] = 0$.

This lemma means that X_p^n is a retract of $X_p^n \smile e^{2n}$. Now we have

LEMMA (5.2). There exists a retraction r: $X_p^n \smile e^{2n} \to X_p^n$ such that $r_* = r_* \tau_*$, where r_* denotes the homomorphism of the homotopy groups induced by r.

PROOF. Let ρ, $\bar{\rho}$ be any two retractions, and let $d(\rho, \bar{\rho})$ be the separation element defined in §4. If we notice that ρ, $\bar{\rho}$, $\rho\tau$, and $\bar{\rho}\tau$, agree on X_p^n we have

$$d(\rho, \rho\tau) = d(\rho, \bar{\rho}) + d(\bar{\rho}, \rho\tau) \qquad \text{from (4.2)}$$
$$-d(\bar{\rho}, \bar{\rho}\tau) = d(\bar{\rho}\tau, \rho\tau) + d(\rho\tau, \bar{\rho}) \qquad \text{from (4.2)}.$$

Thus $d(\rho, \rho\tau)-d(\bar\rho, \bar\rho\tau)=d(\rho, \bar\rho)+d(\bar\rho, \rho\tau)+d(\bar\rho\tau, \rho\tau)+d(\rho\tau, \bar\rho)$

$$=d(\bar\rho, \rho)+d(\bar\rho\tau, \rho\tau)+d(\bar\rho, \bar\rho)=d(\rho, \bar\rho)+d(\bar\rho\tau, \rho\tau)$$
$$=d(\rho, \bar\rho)+(-1)^n d(\bar\rho, \rho)=2d(\rho, \bar\rho)$$

i.e.
$$d(\rho, \rho\tau)=d(\bar\rho, \bar\rho\tau)+2d(\rho, \bar\rho).$$

On the other hand, since $\pi_{2n}(X_p^n)$ is the p-group, and element of $\pi_{2n}(X_p^n)$ is divisible by 2. Let x be an element of $\pi_{2n}(X_p^n)$ such that $2x=-d(\bar\rho, \bar\rho\tau)$. Now if $d(\bar\rho, \bar\rho\tau)=0$ $\bar\rho$ is homotopic to $\rho\tau$ by (4.1). i.e. $\bar\rho_*=\bar\rho_*\tau_*$. Hence it is sufficient to put $r=\bar\rho$. If $d(\bar\rho, \bar\rho\tau)\neq 0$ we can choose a retraction ρ such that $d(\rho, \bar\rho)=x$. (By Lemma (4.3)). Therefore we have

$$d(\rho, \rho\tau)=d(\bar\rho, \bar\rho\tau)+2d(\rho, \bar\rho)=-2x+2x=0.$$

Thus in this case it is sufficient to put $r=\rho$. This completes the proof of Lemma (5.2).

Let r be a retraction such that $r_*=r_*\tau_*$. From $i_*\partial\eta=\alpha+\tau_*\alpha$ we obtain $r_*i_*\partial\eta=r_*\alpha+r_*\tau_*\alpha=2r_*\alpha$, i.e. $\partial\eta=2\tau_*\alpha$. Thus $\partial\eta=0$ is equivalent to $r_*\alpha=0$, i.e. X_p^n is a retract of the complex $L=X_p^n\smile e^{2n}\smile e_1^{2n+1}\smile e_2^{2n+1}$. However we hove

LFMMA (5.3). If p is a prime number, there exists no retraction of L onto X_p^n.

PROOF. If such retraction ρ exists, the cup product of x_{n+1} and x_n must be zero[(24)], where x_{n+1}, x_n denote generators of the cohomology groups, $H^{n+1}((X_p^n)_\infty, Z_p)$ and $H^n((X_p^n)_\infty, Z_p)$ respectively.

Let δ_p be the Bockstein operator mod p, (see [5]). It is easily seen that $\delta_p(x_n)=x_{n+1}$ and $\delta_p(x_{n+1})=0$. Hence we have $x_{n+1}\smile x_{n+1}=0$. Now let φ be a map: $X_p^n\to S^{n+1}$ such that $\varphi|S^n$ is the constant map and $\varphi|e^{n+1}$ is a map of degree $+1$. Clearly φ induces a map φ_∞ of $(X_p^n)_\infty$ into S_∞^{n+1} such that $\varphi_\infty(x_1, x_2, \cdots, x_m)=(\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_m))$. Then from cell structure of $(X_p^n)_\infty$ and S_∞^{n+1} we see that the homomorphism φ_∞^*: $H^*(S_\infty^{n+1}, Z_p)\to H^*((X_d^n)_\infty, Z_p)$ transforms generators of the former into generators of the latter at each appropriate dimensions. Of course $\varphi_\infty^*(y_{n+1})=x_{n+1}$, where y_{n+1} denotes a generator of $H^{n+1}(S_\infty^{n+1}, Z_p)$. Hence from naturality of the cup product we have $x_{n+1}\smile x_{n+1}=\varphi_\infty^*(y_{n+1})\smile\varphi_\infty^*(y_{n+1})=\varphi_\infty^*(y_{n+1}\smile y_{n+1})=2\varphi_\infty^*(y_{2n+2})$[(25)] $\neq 0$. Thus $x_{n+1}\smile x_{n+1}=0$ contradics to this fact. This completes the proof. In particular we have

COROLLARY (5.4). If p is an odd prime number X_p^n is not a H-space for all $n>1$.

PROOF. If n is odd this is an immediate consequence of Lemma (5.3) and

(24) Consider the exact sequence, $0\longrightarrow H^{2n+1}((X_p{}^n)_\infty, Z_p)\longrightarrow H^{2n+1}(L, Z_p)$.
(25) See page 488 of [16].

Theorem (1.8) of [10]. If n is even this fact follows from $[\iota'_n, \iota'_n]=0$. This completes the proof.

From Lemma (5.3) we can easily obtain

LEMMA (5.5). If p is an odd prime number H_2 is trivial.

Now combining these lemmas we have

Theorem II. If p is an odd prime number

H_1 is trivial and H_2 is onto for even n,

and H_1 is onto and H_2 is trivial for odd n.

REMARK. The part concerning H_1 is an easy consequence of properties of $[\iota'_n, \iota'_n]$, and H_1 corresponds to the Hopf invariant of the group $\pi_{2n-1}(S^n)$. (See [11]).

Chapter III. The groups $\pi_i(K, S^n)$.

§1. Preliminary.

Let K denote a complex $S^n \overset{\alpha}{\cup} e^r \overset{\beta}{\cup} e^{n+r}$, $(r>n+1)$, and let j_* denote the inclusion homomorphism: $\pi_i(S^n \overset{\alpha}{\cup} e^r) \to \pi_i(S^n \overset{\alpha}{\cup} e^r, S^n)$. As stated in §1 of Chapter I, there is a unique integer m such that $j_*(\beta)=m[\bar{\alpha}, \iota_n]_r+\bar{\alpha}\circ\rho$ $(\rho \in \pi_{n+r-1}(V^r, S^{r-1}))$. Let P be a map of $S^n \overset{\alpha}{\cup} e^r$ into S^r which transforms S^n into e^0_r, and e^r homemorphically onto $S^r - e^0_r$. Then we have

LEMMA (1.1). $P_*\beta = E(\partial\rho)$, where ∂ is the boundary homomorphism and E denotes the suspension homomorphism.

PROOF. Consider the following diagram

(1.2)
$$\begin{tikzcd}
\pi_{n+r-1}(S^n \overset{\alpha}{\cup} e^r) \arrow[rr, "P_*"] \arrow[d, "j_*"] & & \pi_{n+r-1}(S^r) \\
\pi_{n+r-1}(S^n \overset{\alpha}{\cup} e^r, S^n) \arrow[urr, "P'_*"] & & \\
\pi_{n+r-1}(V^r, S^{r-1}) \arrow[u, "\bar{\alpha}_*"] \arrow[uurr, "\chi_*"] \arrow[rr] & & \pi_{n+r-2}(S^{r-1}) \arrow[uu, "E"']
\end{tikzcd}$$

where P'_* is the homomorphism induced by the map determined by P in obvious way, and χ_* is defined by $\chi_* = P'_*\bar{\alpha}$.

In the diagram (1.2) we can easily prove that $P_* = P'_* j_*$ and $\chi_* = E\partial$. Then

$$\begin{aligned} P_*(\beta) = P'_* j_*(\beta) = P'_*(m[\bar{\alpha}, \iota_n]_r + \bar{\alpha}\circ\rho) = P'_*\bar{\alpha}_*(\rho) \qquad \text{from } P'_*[\bar{\alpha}, \iota_n]_r = 0 \\ = \chi_*(\rho) = E(\partial\rho). \end{aligned}$$

This completes the proof.

Since $\pi_{n+r-1}(S^r)$ is a finite group by $r>n+1$ and Prop. 5 of [16], $P_*(\beta)$ has the finite order, say k. Then we have

LEMMA (1.3). There is a map φ_k of K into S^r such that $\varphi_k(S^n)=e^0_r$ and $\varphi_k|e^r$ is degree k.

PROOF. Let ψ be a map: $S^r \to S^r$ of degree k and consider the map $\psi \circ P$: $S^n \overset{\alpha}{\cup} e^r \to S^r$. Then

$$(\psi \circ P)_*(\beta) = \psi_* \circ P_*(\beta) = \psi_*(E(\partial \rho)) = kE(\partial \rho) = kP_*(\beta) = 0.$$

Hence $\psi \circ P$ is extendable over K by Lemma 7 of [23], (p. 225). This completes the proof.

Now let φ_k be a map whose existence has been just proved by Lemma (1.3). Clearly we may regard φ_k as a map of the pair (K, S^n) into the pair (S^r, e_r). Let B be the mapping cylinder of φ_k, and E the space of maps λ: $I \to B$ such that $\lambda(0) \subset K$, where I is the interval $0 \leqq t \leqq 1$. In §1 of Chapter 1, we have defined the projection p: $E \to B$ in case where B is the mapping cylinder of the map f: $S^{3n-1} \to L$. We now consider the corresponding projection p: $E \to B$ for φ_k and the fibre space associated with p: $E \to B$.

Since $p(S^n) = \varphi_k(S^n) = e$, the embedding of K in E carries S^n into F. Thus we have the following commutative diagram (1.4), in which p' is the map determined by p in obvious way, and u, v are inclusion maps.

$$(1.4) \qquad \begin{array}{ccc} (K, S^n) & \xrightarrow{p} & (S^r, e_r^0) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ (E, S^n) & \xrightarrow[p']{} & (B, e) \end{array}$$

The inclusion maps u, v are homotopy equivalences. Hence they induce isomorphisms u_*, v_* which transfer φ_{k*} to p'_*, as shown in the following diagram.

$$(1.5) \qquad \begin{array}{ccc} \pi_i(K, S) & \longrightarrow & \pi_i(S) \\ {\scriptstyle u_*}\downarrow & & \downarrow{\scriptstyle v_*} \\ \pi_i(E, S) & \xrightarrow[p]{} & \pi_i(B) \end{array}$$

Consider the homotopy sequence of the triple (E, F, S^n),

$$(1.6) \qquad \longrightarrow \pi_i(F, S) \xrightarrow[j_*]{} \pi_i(E, S) \xrightarrow[i_*]{} \pi_i(E, F) \xrightarrow[\partial]{} \pi_{i-1}(F, S) \longrightarrow.$$

Since p'_*, in (1.5), can be factared as follows

$$\pi_i(E, S) \xrightarrow[i_*]{} \pi_i(E, F) \xrightarrow[p_*]{} \pi_i(B)$$

where p_* is the homomorphism induced by the fibre map, it follows that the sequence (1.6) can be transformed into an exact sequence,

$$(1.7) \qquad \longrightarrow \pi_i(F, S^n) \longrightarrow \pi_i(K, S^n) \longrightarrow \pi_i(B) \longrightarrow.$$

Thus the study of $\pi_i(F, S^n)$ is important for us, on which we shall prove the following key lemma to our Theorem III.

LEMMA (1.8). $\pi_i(F, S^n)$, $(r > n+1)$, belongs to the class C_{km} for all i.

If we assume that Lemma (1.8) is true we can easily obtain

Theorem III. $\pi_i(K, S^n)$, $(r>n+1)$, is C_{km}-isomorphic to $\pi_i(S^r)$ for all i.

COROLLARY (1.9). If a prime number p is relatively prime to mk, the p-primary component of $\pi_i(K, S^n)$ is isomorphic to the p-primary component of $\pi_i(S^r)$ for all i.

By Theorem 2 of [17] (p. 274) Lemma (1.8) is equivalent to the following lemma.

LEMMA (1.10). $H_i(F, S^n; Z)$ belongs to the class C_{km} for all i. By the universal coefficient theorem this is again equivalent to

LEMMA (1.11). $H^i(F, S; Z)$ belongs to the class C_{km} for all i. We shall this lemma in the last §3 after some preparations in §2.

§2. Spectral sequence.

Let $A=\sum_n {}^nA$ be a filtered graded algebra over integers with a differential operator of degree $+1$, and let $A^{p,q}$ denote the subset of A consisting of elements whose filtration does not exceed p and whose degree is $p+q$. Suppose that $A^{p,q}=0$ if $q<0$ and $A^{0,n}={}^nA$. We use the usual notations (r denotes positive integers):

$C_r^{p,q}$: the subset of elements of $A^{p,q}$ whose coboundaries are included in $A^{p+r,q+r-1}$.
$B_r^{p,q}$: the subset of elements of $A^{p,q}$ which are coboundaries of $A^{p-r,q+r-1}$.
$C_\infty^{p,q}$: the intersection of cocycle of A and $A^{p,q}$.
$B_\infty^{p,q}$: the intersection of coboundaries of A and $A^{p,q}$.

$$E_r^{p,q}=C_r^{p,q}/(C_r^{p+1,q-1}+B_{r-1}^{p,q}) \quad \text{and} \quad E_\infty^{p,q}=C_\infty^{p,q}/(C_\infty^{p+1,q-1}+B_\infty^{p,q}).$$

If we put $D^{p,q}=C_\infty^{p,q}/B_\infty^{p,q}$ then $(D^{p,q})$ defines a filtration of $H^*(A, Z)$, and we have $E_\infty^{p,q}\cong D^{p,q}/D^{p+1,q-1}$. Now we denote by $\mu^{p,q}$: $D^{p,q}\to E_\infty^{p,q}$ and $i^{p,q}: D^{p,q}\to H^{p+q}(A, Z)$ the projection and injection mentioned above, respectively. The cup product, $\smile$, in E_q is defined by the multiplication of representatives in each class so that $E_r^{p,q}\otimes E_r^{p',q'}\to E_q^{p+p',q+q'}$. The cup product, $\smile^A$, in $H^*(A, Z)$ in the same way implies the cup product $\smile^D$: $D^{p,q}\otimes D^{p',q'}\to D^{p+p',q+q'}$. Thus we have the following diagrams of which commutativity is almost evident.

$$(2.1)\qquad \begin{array}{ccc} D^{p,q}\otimes D^{p',q'} & \xrightarrow{\smile D} & D^{p+p',q+q'} \\ \downarrow i^{p,q}\otimes i^{p',q'} & & \downarrow i^{p+p',q+q'} \\ H^{p+q}(A, Z)\otimes H^{p'+q'}(A, Z) & \xrightarrow[\smile A]{} & H^{p+p'+q+q'}(A, Z) \end{array}$$

$$(2.2)\qquad \begin{array}{ccc} D^{p,q}\otimes D^{p',q'} & \xrightarrow{\smile D} & D^{p+p',q+q'} \\ \downarrow \mu^{p,q}\otimes \mu^{p',q'} & & \downarrow \mu^{p+p',q+q'} \\ E_\infty^{p,q}\otimes E_\infty^{p',q'} & \xrightarrow[\smile]{} & E_\infty^{p+p',q+q'} \end{array}$$

Let κ_s^r: ${}^sE_r^{p,q}\to E_s^{p,q}$ be the canonical homomorphism defined on ${}^sE_r^{p,q}$, elements of $E_r^{p,q}$ whose differentials, $d_r, d_{r+1}, \cdots, d_{s-1}$ are all zeros ($\infty\geqq s>r$). Since

$\smile|{}^sE_r^{p,q}\otimes{}^sE_r^{p',q'}\to{}^sE_r^{p+p',q+q'}$ is assured by the formula $d_i(x\smile y)=(d_ix)\smile y+(-1)^{\deg x}x\smile(d_iy)$, $(i=1, 2, \cdots)$. Thus we have the following diagram.

$$
\begin{array}{ccc}
{}^sE^{p,q}\otimes{}^sE^{p',q'} & \xrightarrow{\smile} & {}^sE^{p+p',q+q'} \\
\downarrow{\scriptstyle \kappa_s^r\otimes\kappa_s^r} & & \downarrow{\scriptstyle \kappa_s^r} \\
E_s^{p,q}\otimes E_s^{p',q'} & \xrightarrow{\smile} & E_s^{p+p',q+q'}
\end{array}
\tag{2.3}
$$

Now we suppose that $D^{p+1,q-1}=D^{p'+1,q'-1}=D^{p+p'+1,q+q'-1}=0$. Then it is clear that $\mu^{p,q}$, $\mu^{p',q'}$ and $\mu^{p+p',q+q'}$ are all isomorphisms. Hence we obtain the following homomorphisms,

$$h(p, q)=i^{p,q}(\mu^{p,q})^{-1}:\ E_\infty^{p,q}\to H^{p+q}(A, Z)$$
$$h(p', q')=i^{p',q'}(\mu^{p',q'})^{-1}:\ E_\infty^{p',q'}(A, Z)\to H^{p'+q'}(A, Z)$$
$$h(p+p', q+q')=i^{p+p',q+q'}(\mu^{p+p',q+q'})^{-1}:\ E_\infty^{p+p',q+q'}\to H^{p+p'+q+q'}(A, Z)\,.$$

Combining two diagrams (2.1) and (2.2) we have the commutative diagram.

$$
\begin{array}{ccc}
E_\infty^{p,q}\otimes E_\infty^{p,q} & \xrightarrow{\smile} & E_\infty^{p+p',q+q'} \\
\downarrow{\scriptstyle h(p,\,q)\otimes h(p',\,q')} & & \downarrow{\scriptstyle h(p+p',\,q+q')} \\
H^{p+q}(A, Z)\otimes H^{p+q}(A, Z) & \xrightarrow{\smile_A} & H^{p+p'+q+q'}(A, Z)
\end{array}
\tag{2.4}
$$

Let (E, B, P) be any fibre space, and $(E_r^{p,q})$ the associated spectral seqnence of the cohomology ring over the coefficient group Z, suppose that

a) $H^0(B, Z)\cong H^n(B, Z)\cong Z$ and $H^i(B, Z)\cong 0$ for other i,

b) $n>q_0+1>3$ for some q_0.

It is easily proved that $E_2^{n,0}={}^\infty E_2^{n,0}$, $E_2^{0,q_0}={}^\infty E_2^{0,q_0}$ and $E_2^{n,q_0}={}^\infty E_2^{n,q_0}$ and $D^{n+1,-1}=D^{1,q_0-1}=D^{n+1,q_0-1}=0$. Then from the preceeding argument we obtain the following diagram

$$
\begin{tikzcd}
E_2^{n,0}\otimes E_2^{0,q_0} \arrow[r, "\smile"'] \arrow[d, "\kappa_\infty^2\otimes\kappa_\infty^2"] & E_2^{n,q_0} \arrow[r, "\kappa_\infty^2"] & E_\infty^{n,q_0} \arrow[d, "h(n,\,q_0)"] \\
E_\infty^{n,0}\otimes E_\infty^{0,q_0} \arrow[rru, "\smile"'] \arrow[d, "h(n,\,0)\otimes h(0,\,q_0)"] & & H^{n+q_0}(E, Z) \\
H^n(E, Z)\otimes H^{q_0}(E, Z) \arrow[rru, "\smile A"'] & &
\end{tikzcd}
$$

It follows from diagrams (2.3) and (2.4) that

$$\kappa_\infty^2\circ\smile=\smile\circ(\kappa_\infty^2\otimes\kappa_\infty^2)\ \smile^A\circ(h(n, 0)\otimes h(0, q_0)=h(n, q_0)\circ\smile.$$

On the other hand it is seen that $h(n, 0)\circ\kappa_\infty^2$ is equivalent to[26] $p^*:\ H^n(B, Z)\to H^n(E, Z)$ and $h(0, q_0)\circ\kappa_\infty^2$ is the inverse homomorphism of the inclusion homomorphism: $H^{q_0}(E, Z)\to H^{q_0}(F, Z)$. Since B is a homological n-sphere we have the Wangs sequence,[27]

(26) See [16].

(27) See p. 471 of [16].

$$\longrightarrow H^{i-n}(F, Z) \xrightarrow[h_{i-1}]{} H^{i}(E, Z) \xrightarrow[i^*]{} H^{i}(F, Z) \xrightarrow[\theta_i]{} H^{i-n+1}(F, Z) \longrightarrow .$$

Since we can express elements of E_2^{n, q_0} in the form $1 \otimes x$ for x of $H^{q_0}(F, Z)$, we have

$$h_{n+q_0-1}(x) = h_{n+q_0-1}(1 \otimes x) = h(n, q_0) \circ \kappa_{\sim}^2(1 \otimes x)^{(28)}$$
$$= \smile^{A}(p^*(1) \smile i^{*-1}(x)) = p^*(1) \smile i^{*-1}(x) .$$

Namely we have

LEMMA (2.6). $h_{n+q_0-1}(x) = p^*(1) \smile i^{*-1}(x)$.

§3. The cohomology of F.

In this section we use the same notations as in §1. We shall calculate the cohomology of F and prove Lemma (1.11).

Since E and B are homotopy equivalent to K and S respectively, their cohomology are as follows:

(3.1) $$H^0(E, Z) \cong H^n(E, Z) \cong H^r(E, Z) \cong H^{n+r}(E, Z) \cong Z$$
$$H^i(E, Z) \cong 0 \qquad \text{for other } i.$$

(3.2) $$H^0(B, Z) \cong H^n(B, Z) \cong Z$$
$$H^i(B, Z) \cong 0 \qquad \text{for other } i.$$

Since B is a homological r-sphere we have Wangs sequence associated with the fibre space (E, B, p),

(3.3) $$\longrightarrow H^i(E, Z) \xrightarrow[i^*]{} H^i(F, Z) \xrightarrow[\theta_i]{} H^{i-r+1}(F, Z) \xrightarrow[h_i]{} H^{i+1}(E, Z) \longrightarrow .$$

Then it follows from (3.1) that

LEMMA (3.4). $H^i(F, Z) \cong H^{i-r+1}(F, Z)$ for $i > n+r$.

LEMMA (3.5). i^*: $H^i(E, Z) \to H^i(F, Z)$ is an isomorphism onto for $i < r-1$.

From these we can obtain

LEMMA (3.6). $H^0(F, Z) \cong H^n(F, Z) \cong Z$
$H^i(F, Z) \cong 0 \qquad$ for $r+1 \leqq i \leqq n+r-2$

and $i < r-1$, $i \neq 0$, n.

Thus, by (3.4) it is sufficient for us to calculate $H^{r-1}(F, Z)$, $H^r(F, Z)$, $H^{n+r-1}(F, Z)$ and $H^{n+r}(F, Z)$. For these cases we have two exact sequence which are the part of Wangs sequence,

(3.7) $$0 \to H^{r-1}(F, Z) \to H^0(F, Z) \xrightarrow[h_{r-1}]{} H^r(E, Z) \to H^r(F, Z) \to 0,$$
$$0 \to H^{n+r-1}(F, Z) \to H^n(F, Z) \xrightarrow[h_{r+n-1}]{} H^{r+n}(E, Z) \to H^{r+n}(F, Z) \to 0.$$

It is easily seen that the homomorphism h_{r-1} is equivalent to the homomorphism

(28) 1 denotes a generator of $H(B, Z) \approx Z$.

p^*: $H^r(B, Z) \to H^r(E, Z)$. (See p. 456 of [16]). Since p^* is equivalent to φ_k^*: $H^r(S^r, Z) \to H^r(K, Z)$ by the argument of §1 of this Chapter we see that h_{r-1} transforms a generator of $H^0(F, Z)$ into k-times of a generator of $H^r(E, Z)$. Thus we obtain

LEMMA (3.9). $H^{r-1}(F, Z) \cong 0$ and $H^r(F, Z) \cong Z_k$.

As to h_{n+r-1} we notice that the argument of §2 of this Chapter is applicable to this spectral sequence and we have by Lemma (2.6)

$$h_{r+n-1}(x_n) = p^*(y_r) \smile i^{*-1}(x_n) = k(\omega_r \smile \omega_n) \tag{3.10}$$

where x_n and y_r denotes generators of $H^n(F, Z)$ and $H^r(B, Z)$ respectively, and ω_i denotes a generator of $H^i(E, Z)$ $(i = n, r, n+r)$.

Since by Lemma 2 of [15] we have $\omega_r \smile \omega_n = m\omega_{n+r}$ we obtain

LEMMA (3.11). $H^{r+n-1}(F, Z) \cong 0$ and $H^{n+r}(F, Z) \cong Z_{km}$.

Combining lemmas (3.4), (3.6), (3.9), (3.11) we have

LEMMA (3.12).
$$\begin{aligned} &H^0(F, Z) \cong H^n(F, Z) \cong Z \\ &H^{s(r-1)+r}(F, Z) \cong Z_k \quad \text{for all integers } s \geqq 0 \\ &H^{s(r-1)+n+r}(F, Z) \cong Z_{km} \quad \text{for all integers } s \geqq 0 \\ &H^i(F, Z) \cong 0 \quad \text{for other } i. \end{aligned}$$

Now Lemma (1.11) can be obtained from Lemma (3.12) and the cohomology sequence of the pair (F, S^n).

COROLLARY (3.13). ψ_{k*}: $\pi_i(K, S^n) \longrightarrow \pi_i(S^r)$ is C_k-isomorphism onto for $i < r+n-2$ and is C_k-homomrphism onto for $i \leqq r+n-2$.

PROOF. From Lemma (3.12) and the universal coefficient theorem we have $H_i(F, S^n; Z) \in C_k$ for $i < r-1+n$. Hence $\pi_i(F, S^n) \in C_k$ by Theorem 2 of [17]. (p. 274). Then Corollary (3.13) is a consequence of the sequence (1.7).

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