Spectral sequence in the de Rham cohomology of fibre bundles.

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Introduction.

Leray [6] initiated the theory of spectral sequences in the Čech-Alexander cohomology of fibre bundles. After Leray, Serre [13] developed the theory of singular homology and cohomology of fibre spaces. In both theories, spectral sequences connect the cohomologies of fibre and of base to that of total space. On the other hand, Hochschild-Serre [4] gave the spectral sequence theory in the cohomology of Lie algebras. In this case, the spectral sequence relates the cohomologies of an ideal and of its factor algebra to that of the algebra.

In differentiable fibre bundles we have the de Rham cohomology theory. Results of Leray, Serre and Hochschild-Serre suggest the possibility of getting a spectral sequence theory of the de Rham cohomology of differentiable fibre bundles.

Our purpose in Chapter I is to give a natural filtration in the complex of differential forms of total space to the effect that the resulting spectral sequence relates the cohomologies of base and of fibre to that of total space.

Whereas our results resemble in their final form to those of Leray and Serre, our method is quite similar to that of Hochschild-Serre. This analogy comes from the fact that the complex of forms can be considered as complex of the Lie algebra of vector fields.

Then, in Chapter II, we discuss the case where the spaces involved are all homogeneous spaces of type G/D, where D is a discrete subgroup of a Lie group G. In particular we get a homomorphism of the Hochschild-Serre spectral sequences into ours, which leads to a generalization of a theorem of Nomizu [10]. Nomizu proved that if G/D is a compact homogeneous space of a nilpotent Lie group G by a discrete subgroup D, then the cohomology of G/D is isomorphic to that of the Lie algebra of G. Saito [12] treated a class of solvable groups ("groupes à racines réelles") which have properties very near to nilpotent groups. We shall generalize Nomizu's theorem to the case where G is "à racines réelles" and the cohomology is with local coefficient systems of a special type.

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Contents.

Chapter I.

Terminologies and notations.

In the sequel, we shall denote by R the field of real numbers, and by R^n the vector space of n-tuples $(x_1, \ldots, x_n)$, $x_i \in R$.

Let $(\mathcal{S}, \pi, M)$ be a sheaf on a space M [1]. Then, we put, for a subspace $U \subset M$,

$\mathcal{S}(U)$=the restricted sheaf of $\mathcal{S}$ on U,

$\Gamma(\mathcal{S}, U)=\mathcal{S}_U$=the module of (continuous) cross-sections of $\mathcal{S}$ on U. If $U=M$, we shall write $\Gamma(\mathcal{S})=\Gamma(\mathcal{S}, M)$.

Let M be a C^∞-manifold. By a C^∞-fibre bundle (B, π, M, F, G) on M, we mean a fibre bundle in the sense of Steenrod[1)] [14, §2.4] such that its fibre F is a C^∞-manifold and its structural group G is a Lie group of C^∞-transformations of F and its coordinate transformations are C^∞-maps. The bundle space B has a unique structure of C^∞-manifold such that its coordinate functions are diffeomorphisms. Then the projection $\pi: B \to M$ is a C^∞-map.

For an open submanifold $U \subset M$, set

$\Gamma(B, U)=C^\infty$-cross-sections of B on U.

For $U \subset V$, let $\pi^V_U: \Gamma(B, V) \to \Gamma(B, U)$ be defined by restriction.

Similarly, for $x \in U$, $\pi_x^U : \Gamma(B, U) \to F_x = \pi^{-1}(x)$ will be defined by restriction. The same notation is used in case of sheaf.

When $F = R^n$ for some n, and $G = GL(n, R)$, the C^∞-fibre bundle $(B, \pi, M, R^n, GK(n, R))$ is called C^∞-vector bundle. Note that, in this case, each fibre F_x, $x \in M$, has the structure of vector space over R, and $\Gamma(B, U)$ is a (left) module over the algebra of C^∞-real valued functions on U.

Let $\boldsymbol{G}_\infty$ be the sheaf of germs of C^∞-functions from M to G. Any covering[2] $\{U_\alpha\}$ of M by coordinate neighborhoods of the bundle (B, π, M, F, G) and its coordinate transformations $\{g_{\alpha\beta} : U_\alpha \frown U_\beta \to G\}$ represents a well-defined element $c(B)$ of $H^1(M, \boldsymbol{G}_\infty)$ [3. §3-1].

Let $\boldsymbol{G}_{lc}$ be the sheaf of germs of locally constant function from M to G, and let $h : \boldsymbol{G}_{lc} \to \boldsymbol{G}_\infty$ be the natural injection. h induces a function $h_* : H^1(M, \boldsymbol{G}_{lc}) \to H^1(M, \boldsymbol{G}_\infty)$.

A C^∞-vector bundle $(B, \pi, M, R^n, GL(n, R))$ is called locally constant if there exists a $c \in H^1(M, G_{lc})$, $G = GL(n, R)$, such that $h_*(c) = c(B)$, or equivalently if there exists a covering $\{U_\alpha\}$ of M by coordinate neighborhoods of the bundle such that its coordinate transformations $\{g_{\alpha\beta}\}$ are constant on their domain of definition.

§1. De Rham cohomology with coefficients in a locally constant sheaf.

Now let M be a C^∞, paracompact, connected manifold[3] and $(\mathcal{S}, \pi, M)$ be a locally constant sheaf of finite dimensional vector space over reals on M. We call such a sheaf an *admissible* sheaf. The cohomology groups $H^*(M, \mathcal{S})$ of M with coefficients in $\mathcal{S}$ is obtained as follows [1]. Denoting by $\Omega(M) = \Omega$ the sheaf of germs of differential forms on M, the tensor product sheaf $\Omega \circ \mathcal{S}$ (tensor product is always taken over reals) has a coboundary which is induced by that of Ω. Therefore the vector space $\Gamma(\Omega \circ \mathcal{S})$ can be considered as a (cochain) complex. The derived groups of this complex are exactly the cohomology groups $H^*(M, \mathcal{S})$.

We shall now give an interpretation of this complex which is suited for our purpose.

Let $\mathcal{S}$ be an admissible sheaf on M. By the hypothesis, we see that there exists a covering $\{U_\alpha\}$ of M such that, for each α, there is an isomorphism of sheaf on U_α

$$\varphi_\alpha : \quad U_\alpha \times R^n \to \pi^{-1}(U_\alpha),$$

where $U_\alpha \times R^n$ means the constant sheaf on U_α defined by the vector space R^n. The fact that φ_α are isomorphisms of sheaf of vector space implies that they satisfy the following conditions:

(1.1) *If* $U_\alpha \frown U_\beta \neq \phi$, *and* $x \in U_\alpha \frown U_\beta$, *then*
$$\varphi_\alpha^{-1}\varphi_\beta(x, y) = (x, g_{\alpha\beta}(x)\cdot y),$$
where $$g_{\alpha\beta}(x) \in GL(n, R).$$

(1.2) *The function* $g_{\alpha\beta}: U_\alpha \frown U_\beta \to GL(n, R)$ *is constant on each connected component of* $U_\alpha \frown U_\beta$.

(1.3) *If* $x \in U_\alpha \frown U_\beta \frown U_\gamma$, *then,*
$$g_{\alpha\gamma}(x) = g_{\alpha\beta}(x)\cdot g_{\beta\gamma}(x).$$

In virtue of (1.2), the function $g_{\alpha\beta}$ is a C^∞-map if we endow $GL(n, R)$ with the usual structure of Lie group. Therefore the set of functions $\{g_{\alpha\beta}\}$ gives a C^∞-vector bundle whose bundle space may and will be identified with $\mathcal{S}$ as a set. This is done by giving R^n the usual manifold structure and requiring φ_α to be diffeomorphisms.

We shall denote this bundle by $\boldsymbol{B}(\mathcal{S}, \{\varphi_\alpha\})$. If we take another covering $\{V_\alpha\}$ of M and isomorphisms of sheaf
$$\phi_\lambda: V_\lambda \times R \to \pi^{-1}(V_\lambda),$$
then $g_{\alpha\lambda}: U_\alpha \frown V \to GL(n, R)$ defined by
$$\varphi_\alpha^{-1}\varphi_\gamma(x, y) = (x, g_{\alpha\lambda}(x)\cdot y)$$
is also constant on each connected component of $U_\alpha \frown V_\lambda$. Hence $\{g_{\alpha\beta}\}$ and $\{g_{\lambda\mu}\}$ define the same element $c(\mathcal{S}) \in H^1(M, GL(n, R)_{lc})$. In particular $\boldsymbol{B}(\mathcal{S}, \{\varphi_\alpha\}) = \boldsymbol{B}(\mathcal{S}, \{\phi_\lambda\})$; hence we may write $\boldsymbol{B}(\mathcal{S}) = \boldsymbol{B}(\mathcal{S}, \{\varphi_\alpha\})$. Thus

(1.4) *Every admissible sheaf* $(\mathcal{S}, \pi, M)$ *defines a locally constant* C^∞-*vector bundle* $(\mathbf{B}(\mathcal{S}), \pi, M, R^n, GL(n, R))$ *and an element* $c(\mathcal{S})$ *of* $H^1(M, GL(n, R)_{lc})$ *such that* $h_* c(\mathcal{S}) = c(\boldsymbol{B}(\mathcal{S}))$.

The following two propositions are immediate.

(1.5) *Given a locally constant* C^∞-*vector bundle* $(B, \pi, M, R^n, GL(n, R))$ *and an element* $c \in H^1(M, GL(n, R)_{lc})$ *such that* $h_*(c) = c(B)$, *there exists a unique admisssible sheaf* $(\mathcal{S}, \pi, M)$ such that $\boldsymbol{B}(\mathcal{S}) = B$ and $c(\mathcal{S}) = c$.

(1.6) *Let* $\mathcal{S}$ *and* $\mathcal{S}'$ *be admissible sheaves on* M. $\mathcal{S}$ *and* $\mathcal{S}'$ *are isomorphic, if and only if* $c(\mathcal{S}) = c(\mathcal{S}')$. *In particular* $\boldsymbol{B}(\mathcal{S})$ *and* $\boldsymbol{B}(\mathcal{S}')$ *are isomorphic, i.e., equivalent in* $GL(n, R)$, *if* $c(\mathcal{S}) = c(\mathcal{S}')$.

We use the following notations.

$\mathcal{F}(M) = \mathcal{F} =$ *the algebra of* C^∞-*real valued functions on* M.

$\mathfrak{X}(M) = \mathfrak{X} =$ *the Lie algebra over* R *of* C^∞-*vector fields on* M.

Elements of $\mathfrak{X}(M)$ are derivations of $\mathfrak{F}(M)$ and $\mathfrak{X}(M)$ is a (left) $\mathfrak{F}(M)$-module. For $f\in\mathfrak{F}$, $X\in\mathfrak{X}$, we denote the operation of X on f by $X\cdot f$. By definition,

$$[X, Y]\cdot f=X\cdot(Y\cdot f)-Y\cdot(X\cdot f),\ X,\ Y\in\mathfrak{X},\ f\in\mathfrak{F}.$$

Suppose always that a fixed admissible sheaf $(\mathcal{S}, \pi, M)$ is given. Then $\Gamma=\Gamma(\boldsymbol{B}(\mathcal{S}))$ is a (left) $\mathfrak{F}(M)$-module. Let U be an open set of M such that $\mathcal{S}$ is constant on U. Then $\pi_x^U : \mathcal{S}_U\to\mathcal{S}_x$ is bijective, and $\mathcal{S}_U$ is a finite dimensional vector space over R. Since we have identified $\boldsymbol{B}(\mathcal{S})$ with $\mathcal{S}$ as a set, we have $\mathcal{S}_U\subset\Gamma(\boldsymbol{B}(\mathcal{S}), U)$. Moreover the $\mathfrak{F}(U)$-linear map

$$\sigma_U : \mathfrak{F}(U)\otimes\mathcal{S}_U\to\Gamma(\mathrm{B}(\mathcal{S}), U) \tag{1.7}$$

defined by $\sigma_U(f\otimes s)=f\cdot s$ is bijective. Here the $\mathfrak{F}(U)$-module structure on $\mathfrak{F}(U)\otimes\mathcal{S}_U$ is defined by

$$f'(f\otimes s)=f'f\otimes s,\quad f,\ f'\in\mathfrak{F}(U),\ s\in\mathcal{S}_U.$$

We define the $\mathfrak{X}(U)$-module structure on $\mathfrak{F}(U)\otimes\mathcal{S}_U$ by

$$X\cdot(f\otimes s)=X\cdot f\otimes s,\quad f\in\mathfrak{F}(U),\quad s\in\mathcal{S}_U. \tag{1.8}$$

Then the $\mathfrak{X}(U)$-module structure on $\Gamma(\boldsymbol{B}(\mathcal{S}), U)$ is defined by

$$X\cdot s=\sigma_U(X\cdot\sigma_U^{-1}(s)),\quad s\in\Gamma(\mathrm{B}(\mathcal{S}), U).$$

The following identities are immediate consequences of the definitions:

$$f(X\cdot s)=(fX)\cdot s, \tag{1.9}$$

$$X\cdot(fs)=(X\cdot f)s+f(X\cdot s), \tag{1.10}$$

$$[X, Y]s=X\cdot(Y\cdot s)-Y\cdot(X\cdot s),\ \text{for}\ X,\ Y\in\mathfrak{X}(U),\ f\in\Gamma(\boldsymbol{B}(\mathcal{S}), U). \tag{1.11}$$

If $U\supset V$, then,

$$\pi_V^U(X\cdot s)=\pi_V^U(X)\cdot\pi_V^U(s). \tag{1.12}$$

Take a covering $\{U_\alpha\}$ of M such that $\mathcal{S}$ is constant on each U_α. Let $X\in\mathfrak{X}(M)$ and $s\in\Gamma(\boldsymbol{B}(\mathcal{S}))$. We shall define an element $X\cdot s$ of $\Gamma(\boldsymbol{B}(\mathcal{S}))$ by

$$\pi_{U_\alpha}^M(X\cdot s)=\pi_{U_\alpha}^M(X)\cdot\pi_{U_\alpha}^M(s)$$

for all U_α. This definitions is licit, because we have, by (1.12),

$$\begin{aligned}\pi_{U_\alpha\cap U_\beta}^{U_\alpha}(\pi_{U_\alpha}^M(X)\cdot\pi_{U_\alpha}^M(s))&=\pi_{U_\alpha\cap U_\beta}^M(X)\cdot\pi_{U_\alpha\cap U_\beta}^M(s)\\&=\pi_{U_\alpha\cap U_\beta}^{U_\beta}(\pi_{U_\beta}^M(X)\cdot\pi_{U_\beta}^M(s)),\end{aligned}$$

for any U_α and U_β such that $U_\alpha\cap U_\beta\neq\phi$. It is easily verified that the definition of $X\cdot s$ does not depend on the choice of covering $\{U_\alpha\}$ and that the identities (1.9)—(1.11) hold foy X, $Y\in\mathfrak{X}(M)$, $f\in\mathfrak{F}(M)$, $s\in\Gamma(\mathrm{B}(\mathcal{S}))$.

Note that, for a locally constast C^∞-vector bundle B on M, we may define an operation of $\mathfrak{X}(M)$ on $\Gamma(B, M)$ in virtue of (1.5), but it depends on the choice of $c \in H^1(M, G_{lc})$ such that $h_*(c) = c(B)$.

Now we define a (cochain) complex which will be canonically isomorphic to $\Gamma(\Omega \circ \mathcal{S})$. Set

$$C^0 = C^0(\mathfrak{X}(M), \Gamma(\boldsymbol{B}(\mathcal{S}))) = \Gamma(\boldsymbol{B}(\mathcal{S})) = \Gamma,$$
$$C^q = C^q(\mathfrak{X}(M), \Gamma(\boldsymbol{B}(\mathcal{S}))) = \text{the } \mathfrak{F}(M)\text{-module of}$$
$$\mathfrak{F}(M)\text{-linear } q\text{-alternating functions from } \mathfrak{X}(M) \text{ to } \Gamma,\ q > 0.$$

In the graded module $C = \sum_{q \geq 0} C^q$ we define an R-linear endomorphism d which augments degrees by one. For $w \in C^q$, define namely

$$(1.13) \qquad dw(x_1, \cdots, x_{q+1})$$
$$= \frac{1}{q+1} \{\textstyle\sum_i (-1)^{i+1} X_i(w(X_1, \cdots, \hat{X}_i, \cdots, X_{q+1}))$$
$$+ \textstyle\sum_{i<j} (-1)^{i+j} w([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{q+1})\}.$$

It is easily seen that dw is in C^{q+1} and that $d^2 = 0$. We shall denote by $H^*(\mathfrak{X}(M), \Gamma(\boldsymbol{B}(\mathcal{S})))$ the derived group of C by d.

If U is an open submanifold, then the complex $C(\mathfrak{X}(U), \Gamma(\boldsymbol{B}(\mathcal{S}), U))$ is similarly defined. An element $w \in C^q(\mathfrak{X}(U), \Gamma(\mathbf{B}(\mathcal{S}), U))$ is an $\mathfrak{F}(U)$ linear q-alternating function from $\mathfrak{X}(U)$ to $\Gamma(\boldsymbol{B}(\mathcal{S}), U)$. Hence it follows easily that the element $w(X_1, \cdots, X_q)(x)$ of $\boldsymbol{B}(\mathcal{S})$ for $x \in U$ depends only on the tengent vectors $X_{1x}, \cdots, X_{qx}$ determined by the vector fields $X_1, \cdots, X_q$ at x. Thus w may be identified with a collection $\{w_x\}_{x \in U}$ of R-linear q-alternating functions w_x from the tangent space T_x of M at x to $\boldsymbol{B}(\mathcal{S})_x = \mathcal{S}_x$ such that, if $X_1, \cdots, X_q$ are vector fields on an open set U' of U, then the map $x \to w_x(X_{1q}, \cdots, X_{qx})$, $x \in U'$, is a C^∞-cross section of $\boldsymbol{B}(\mathcal{S})$ on U'. For $U \supset V$ define a map $\pi_V^U : C^q(\mathfrak{X}(U), \Gamma(\boldsymbol{B}(\mathcal{S}), U)) \to C^q(\mathfrak{X}(V), \Gamma(\boldsymbol{B}(\mathcal{S}), V))$ by

$$\pi_V^U\{w_x\}_{x \in U} = \{w_x\}_{x \in V}.$$

It is easily checked that $\pi_V^U w$ is an element of $C^q(\mathfrak{X}(V), \Gamma(\boldsymbol{B}(\mathcal{S}), V))$ and that π_V^U commutes with the coboundary; $d\pi_V^U = \pi_V^U d$.

Proposition 1.1. *The complex* $C = C(\mathfrak{X}(M), \Gamma(\boldsymbol{B}(\mathcal{S}))$ *is canonically isomorphic to* $\Gamma(\Omega \circ \mathcal{S})$.

Proof. Let U be an open submanifold of M. Then we know that Ω_U^q is equal to $C^q(\mathfrak{X}(U), \mathfrak{F}(U))$, i.e., to the $\mathfrak{F}(U)$-module of $\mathfrak{F}(U)$-linear q-alternating functions from $\mathfrak{X}(U)$ to $\mathfrak{F}(U)$.

If $\mathcal{S}$ is constant on U, then it is evident that we have $\Gamma(\Omega^q \circ \mathcal{S}, U) = \Omega_U^q \otimes \mathcal{S}_U$

$=C^q(\mathfrak{X}(U), \mathfrak{F}(U))\otimes S_U = C^q(\mathfrak{X}(U), \mathfrak{F}(U)\otimes S_U)$.

Then the isomorphism $\sigma_U : \mathfrak{F}(U)\otimes S_U \to \Gamma(\boldsymbol{B}(S), U)$ induces an isomorphism

(1.14) $\quad \sigma_U : \Gamma(\Omega^q \circ S, U) = C^q(\mathfrak{X}(U), \mathfrak{F}(U)\otimes S_U) \to C^q(\mathfrak{X}(U), \Gamma(\boldsymbol{B}(S), U))$.

Take an open covering $\{U_\alpha\}$ of M such that S is constant on each U_α. For $w \in C^q(\mathfrak{X}(M), \Gamma(\boldsymbol{B}(S)))$, set

(1.15) $$w_\alpha = \pi^M_{U_\alpha} w \in C^q(\mathfrak{X}(U_\alpha), \Gamma(\boldsymbol{B}(S), U_\alpha)).$$

Then w_α's satisfy the following condition:

(1.16) $$\pi^{U_\alpha}_{U_\alpha \frown U_\beta}(w_\alpha) = \pi^{U_\beta}_{U_\alpha \frown U_\beta}(w_\beta)$$

for any U_α and U_β such that $U_\alpha \frown U_\beta \neq \phi$. Conversely the collection $\{w_\alpha\}$ of elements $w_\alpha \in C^q(\mathfrak{X}(U_\alpha), \Gamma(\boldsymbol{B}(S), U_\alpha))$ which satisfy (1.16) determines a unique element $w \in C^q$ satisfying (1.15).

For $s \in \Gamma(\Omega^q \circ S)$, consider the elements

$$w_\alpha = \sigma_{U_\alpha} \pi^M_{U_\alpha}(s).$$

Since the relation $\pi^{U_\alpha}_{U_\alpha \frown U_\beta} \sigma_{U_\alpha} = \sigma_{U_\alpha \frown U_\beta} \pi^{U_\alpha}_{U_\alpha \frown U_\beta}$ holds, these π_α's satisfy (1.16). Define $\sigma(s) \in C^q$ by $\pi^M_{U_\alpha}(\sigma(s)) = w_\alpha = \sigma_{U_\alpha} \pi^M_{U_\alpha}(s)$. Since each σ_{U_α} is an isomorphism, σ is an isomorphism (of module).

To show that $d\sigma = \sigma d$, it suffices to prove $d\sigma_{U_\alpha} = \sigma_{U_\alpha} d$, because $\pi^M_{U_\alpha} d = d \pi^M_{U_\alpha}$ for both sides of (1.14). On $\Gamma(\Omega^q \circ S, U_\alpha) = \Omega^q_{U_\alpha} \otimes S_{U_\alpha} = C^q(\mathfrak{X}(U_\alpha), \mathfrak{F}(U_\alpha)) \otimes S_{U_\alpha}$ the differential d is defined by

$$d(w\otimes s) = dw \otimes s, \quad w \in C^q(\mathfrak{X}(U_\alpha), \mathfrak{F}(U_\alpha)), \quad s \in S_{U_\alpha}.$$

It is well known that the formula which defines dw is given by (1.11). Thus, by the definition of σ_{U_α} (cf. (1.7), (1.8) and (1.14)), we have

$$\begin{aligned}
\sigma_{U_\alpha} d(w\otimes s)(X_1, \cdots, X_{q+1}) &= \sigma_{U_\alpha}(dw \otimes s)(X_1, \cdots, X_{q+1}) \\
&= dw(X_1, \cdots, X_{q+1}) s \\
&= \frac{1}{q+1} \{ \textstyle\sum_i (-1)^{i+1} X_i(w(X_1, \cdots, \hat{X}_i, \cdots, X_{q+1})) s \\
&\quad + \textstyle\sum_{i<j} (-1)^{i+j} w([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots \hat{X}_j, \cdots, X_{q+1})) s \} \\
&= \frac{1}{q+1} \{ \textstyle\sum_i (-1)^{i+1} X_i \cdot \sigma_{U_\alpha}(w(X_1, \cdots, \hat{X}_i, \cdots, X_{q+1}) \otimes s) \\
&\quad + \textstyle\sum_{i<j} (-1)^{i+j} \sigma_{U_\alpha}(w([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{q+1}) \otimes s) \} \\
&= d\sigma_{U_\alpha}(w\otimes s)(X_1, \cdots, X_{q+1}).
\end{aligned}$$

Hence $\sigma_{U_\alpha} d = d\sigma_{U_\alpha}$; this proves the proposition.

Corollary 1.2. *$H^*(\mathfrak{X}(M), \Gamma(\boldsymbol{B}(\mathcal{S})))$ is canonically isomorphic to $H^*(M, \mathcal{S})$.*

Remark. (cf. §2.6). If $\mathcal{S}$ is a sheaf of algebra over R, then $\Gamma(\boldsymbol{B}(\mathcal{S}))$ is an algebra over $\mathfrak{F}(M)$. The usual Grassman multiplication makes $C(\mathfrak{X}(M), \Gamma(\boldsymbol{B}(\mathcal{S})))$ an algebra over $\mathfrak{F}(M)$. $\Gamma(\Omega\circ\mathcal{S})$ is also an algebra over $\mathfrak{F}(M)$. The canonical isomorphism σ of $\Gamma(\Omega\circ\mathcal{S})$ onto $C(\mathfrak{X}(M), \Gamma(\boldsymbol{B}(\mathcal{S})))=C$ is an isomorphism of algebra, as is easily seen. Note that, in this case, the following formulae hold.

$$X\cdot(w\wedge w')=(X\cdot w)\wedge w'+w\wedge(X\cdot w')$$
$$d(w\wedge w')=(dw)\wedge w'+(-1)^p w\wedge(dw'),$$

where $w\in C^p$, $w'\in C^q$ and $X\in\mathfrak{X}(M)$.
Thus

Corollary 1.3. *If $\mathcal{S}$ is an admissible sheaf of algebra over R, then the algebra (over R) $H^*(\mathfrak{X}(M), \Gamma(\boldsymbol{B}(\mathcal{S})))$ is canonically isomorphic to the cohomology algbra $H^*(M, \mathcal{S})$ of M with coefficients in $\mathcal{S}$.*

Hereafter we shall identify $H^*(\mathfrak{X}(M), \Gamma(\boldsymbol{B}(\mathcal{S})))$ with $H^*(M, \mathcal{S})$.

§2. De Rham cohomology in fibre bundles.

1. *Preliminaries.*

First we recall elementary parts of theory of connections in fibre bundles [9].

Let (B, π, M, F, G) be a C^∞-fibre bundle. A connection in this bundle is by definition a C^∞-distribution $Q: x\to Q_x$ of dimension $n(=\dim M)$ on B which satisfies the following conditions:

(2.1) $B_x=F_x+Q_x$ at each $x\in B$, where B_x is the tangent space at x to B and F_x is the subspace tangent to the fibre through x;

(2.2) For any C^∞-curve $u(t)$ in M, there is an integral curve $\tilde{u}(t; x)$ (called a lift of $u(t)$) of the distribution Q which starts at any given point x of the fibre $\pi^{-1}(u(o))$ and which covers the curve $u(t)$. Moreover, for fixed t, the correspondence $u_t: x\to u(t; x)$ defines an isomorphism of $\pi^{-1}(u(o))$ onto $\pi^{-1}(u(t))$, i.e., if $\xi: F\to\pi^{-1}(u(o))$ and $\xi': F\to\pi^{-1}(u(t))$ are any admissible maps [13] then $\xi'^{-1}u_t\xi$: $F\to F$ is an element of G; and the function $(t, x)\to\tilde{u}(t; x)\in B$ is C^∞.

In particular, for a vector bundle, u_t is a linear map of the vector space $\pi^{-1}(u(o))$ onto $\pi^{-1}(u(t))$.

If $u(t)$ is a closed curve, it defines an automorphism of $\pi^{-1}(u(o))$. The totality of such automorphisms forms the holonomy group $\Phi_{u(o)}$ at $u(o)$ of this connection which can be considered as a subgroup of G.

Let $(\mathcal{S}, \pi, M)$ be an admissible sheaf. We shall define a special connection in

$\mathbf{B}(\mathcal{S})$ which will be called the *connection defined by* $\mathcal{S}$. Let x be a point of $\mathbf{B}(\mathcal{S})$. There exists a neighborhood U of $\pi(x)\in M$ on which the unique cross-section s of $\mathcal{S}$ through x is defined. s can be considered as a C^{∞}-cross-section of U into $\mathbf{B}(\mathcal{S})$. Define $Q_x=s_*(M_{\pi(x)})$, where s_* denotes the differential of s and $M_{\pi(x)}$ denotes the tangent space at $\pi(x)$ to M. The distribution $x\to Q_x$ defines the desired connection. In this case, if $u(t)$ is a curve in M and $\pi(x)=u(o)$, then its lift $\tilde{u}(t\,;x)$ is yet a continuous curve in $\mathcal{S}$, and is the unique cross-section image of $u(t)$ through x. $\tilde{u}(\mathrm{t}\,;x)$ considered as a curve in $\mathcal{S}$ will also be called the lift of $u(t)$ through x.

Consider again the bundle (B,π,M,F,G). The vectors of F_x are called *vertical*. A C^{∞}-vector field X on B is called vertical if X_x is vertical for every $x\in B$. Let $\mathfrak{X}_v(B)=$the totality of vertical vector fields on B. $\mathfrak{X}_v(B)$ is an $\mathfrak{F}(B)$-submodule and R-Lie subalgebra of $\mathfrak{X}(B)$. If we imbed $\mathfrak{F}(M)$ in $\mathfrak{F}(B)$ through the homomorphism π^*, then $\mathfrak{X}_v(B)$ is characterized as the annihilator of $\mathfrak{F}(M)$ in $\mathfrak{X}(B)$.

Let $Q:x\to Q_x$ be a connection of the bundle. Vectors of Q_x are called *horizontal* (with respect to this connection). A C^{∞}-vector field X on B is called horizontal if X_x is horizontal for every $x\in B$. The totality $\mathfrak{X}_h(B)$ of horizontal vector fields form an $\mathfrak{F}(B)$-submodule of $\mathfrak{X}(B)$.

$\mathfrak{X}(B)$ is a direct sum of $\mathfrak{X}_v(B)$ and $\mathfrak{X}_h(B)$. We shall denote by $v:\mathfrak{X}(B)\to x_v(B)$ and $h:\mathfrak{X}(B)\to\mathfrak{X}_h(B)$ respective projections.

Let X be a C^{∞}-vector field on M. There exists a unique horizontal vector field $\tilde{X}$ on B, called the life of X, such that $\pi_x\tilde{X}=X$ ($\tilde{X}$ and X are π-related). The correspondence $l:X\to l(X)=\tilde{X}$ gives an $\mathfrak{F}(M)$-linear map $l:\mathfrak{X}(M)\to\mathfrak{X}(B)$ which we shall call the lift map. Moreover we have $h([\tilde{X},\tilde{Y}])=[X,Y]$.

In the case of the connection defined by $\mathcal{S}$ in $\mathbf{B}(\mathcal{S})$, $[\tilde{X},\tilde{Y}]=[X,Y]$ i.e., l is a homomorphism of Lie algebra, as we see easily.

Proposition 2.1. *Suppose that a connection is given in a C^{∞}-fibre bundle (B,π,M,F,G). Let $X\in\mathfrak{X}_v(B)$ and $Y\in\mathfrak{X}(M)$. Then, we have $[X,\tilde{Y}]\in\mathfrak{X}_v(B)$, where $\tilde{Y}$ is the lift of Y with respect to the given connection.*

Proof. Let $f\in\mathfrak{F}(M)$. It suffices to show that

$$[X,\tilde{Y}]\pi^*f=0.$$

Now
$$[X,\tilde{Y}]\pi^*f=X(\tilde{Y}(\pi^*f))-\tilde{Y}(X(\pi^*f))=X(\tilde{Y}(\pi^*f)).$$

Since
$$Y_{\pi(x)}f=(\pi_*\tilde{Y}_x)f=\tilde{Y}_x(\pi^*f),\ \text{i.e.,}$$
$$\tilde{Y}(\pi^*f)=\pi^*(Yf),$$

we have $[X,\tilde{Y}]\pi^*f=X(\pi^*(Yf))=0$.

Let M be a manifold. A collection $\{X_i\}$ of elements X_i of $\mathfrak{X}(M)$ is said locally finite if the carriers of X_i's form a locally finite family, i.e., if for each $x \in M$, there exists a neighborhood U of x such that only a finite number of X_i's exists whose carriers meet U. (The carrier of $X = \{x; X_x \neq 0, x \in M\}$). If $\{X_i\}$ is locally finite then $\sum_i X_i$ is a well defined element of $\mathfrak{X}(M)$.

Proposition 2.2. *Suppose that a connection is given in the fibre bundle* (B, π, M, F, G). *Let* $l: \mathfrak{X}(M) \to \mathfrak{X}(B)$ *be the corresponding lift map. Then we have*

$$\mathfrak{X}_h(B) = \mathfrak{F}(B) \cdot l(\mathfrak{X}(M)),$$
$$= \{\textstyle\sum_i f_i l(X_i);\ f_i \in \mathfrak{F}(B),\ X_i \in \mathfrak{X}(M),\ \{X_i\} \text{ locally finite}\}.$$

Proof. Note that $X \in \mathfrak{X}_h(B)$ is characterized by $X_x \in Q_x$ for any $x \in B$. Since $l(X) \in \mathfrak{X}_h(B)$, $\mathfrak{X}_h(B) \supset \mathfrak{F}(B) \cdot l(\mathfrak{X}(M))$ is obvious. Suppose that U is a coordinate neighborhood of some point of the manifold M. Then there exist $X_1, \cdots, X_n \in \mathfrak{X}(U)$, $n = \dim M$, such that $\{X_{i,x}\}$ span T_x for any $x \in U$. Then $l(X_1)_{x^*}, \cdots, l(X_n)_{x^*}$ span Q_{x^*} for any $x^* \in \pi^{-1}(U)$. Take a locally finite covering $\{U_i\}$ of M by local coordinate neighborhoods. Let $\{V_i\}$ be a covering of M such that $\bar{V}_i \subset U_i$. Let $\{\lambda_i\}$ be a partition of unity attached to $\{V_i\}$. Set $\lambda_i^* = \pi^* \lambda_i$. If $X \in \mathfrak{X}_h(B)$, then the carrier of $\lambda_i X$ is contained in $\pi^{-1}(V_i)$. Take $X_{i,1}, \cdots, X_{i,n} \in \mathfrak{X}(M)$ such that $\{X_{i,j,x}\}$ span T_x for any $x \in V_i$ and $X_{i,j,x} = 0$ for $x \notin U_i$. Then, we have $\lambda_i^* X = \sum_{j=1}^{n} f_{ij} l(X_{i,j})$, $f_{ij} \in \mathfrak{F}(B)$; $\{X_{ij}\}$ is obviously locally finite, therefore

$$X = \sum \lambda_i^* X = \sum_{i,j} f_{i,j} l(X_{i,j}) \in \mathfrak{F}(B) \cdot l(\mathfrak{X}(M)).$$

Definition 2.3. *Let M be a connected manifold and let Γ be a (left) $\mathfrak{F}(M)$-module. Further, let $\mathfrak{X}'$ be an R-Lie subalgebra of $\mathfrak{X}(M)$ which is, at the same time, an $\mathfrak{F}(M)$-submodule of $\mathfrak{X}(M)$. We shall say that $\mathfrak{X}'$ operates on Γ, if the following conditions are satified:*

(a) *Γ is a (left) $\mathfrak{X}'$-module. We shall denote the operation of $X \in \mathfrak{X}'$ on* $c \in \Gamma$ by $X \cdot c$;

(b) $(fX) \cdot c = f(X \cdot c)$, $f \in \mathfrak{F}(M)$, $X \in \mathfrak{X}'$, $c \in \Gamma$;

(c) $X \cdot (fc) = (X \cdot f)c + f(X \cdot c)$, $f \in \mathfrak{F}(M)$, $X \in \mathfrak{X}'$, $c \in \Gamma$;

(d) $[X, Y] \cdot c = X \cdot (Y \cdot c) - Y \cdot (X \cdot c)$, $X, Y \in \mathfrak{X}'$ $c \in \Gamma$.

In particular $\mathfrak{X}(M)$ operates on $\mathfrak{F}(M)$ in this sense (cf. (1.9)-(1.11)), and $\mathfrak{X}'$ operates on itself by the operation $X \cdot Y = [X, Y]$.

Definition 2.4. *Let Γ and Γ' be $\mathfrak{F}(M)$-modules. Define a graded $\mathfrak{F}(M)$-module* $C(\Gamma, \Gamma') = \sum_{q \geq 0} C(\Gamma, \Gamma')$ by

$C^0(\Gamma, \Gamma')=\Gamma'$,

$C^q(\Gamma, \Gamma')=\mathcal{F}(M)$-*module of* $\mathcal{F}(M)$ *linear q-alternating functions from* Γ *to* Γ'.

For $f\in\mathcal{F}(M)$ *and* $w\in C^q(\Gamma, \Gamma')$, fw *is defined by*

$$(fw)(C_1,\cdots,C_q)=f(w(c_1,\cdots,c_q)).$$

Definition 2.5. *Let* $\mathcal{X}'$ *and* Γ *be as in Definition* 2.3. *Let* $X\in\mathcal{X}'$. *We define three R-linear endomorphisms* $\theta(X)$, $i(X)$ *and* d *in* $C(\mathcal{X}', \Gamma)$ by[4)]

(a) $\theta(X)\cdot c=X\cdot c,\ c\in C^0(\mathcal{X}', \Gamma)=\Gamma$,

$$(\theta(X)w)(X_1,\cdots,X_q)=X\cdot(w(P_1,\cdots,X_q))-\sum_i w(X_1,\cdots,[X, X_i],\cdots,X_q),$$
$$w\in C^q(\mathcal{X}', \Gamma).$$

(b) $i(X)c=0,\quad c\in C^0(\mathcal{X}', \Gamma)$,

$$i(X)w(X_1,\cdots,X_{q-1})=w(X, X_1,\cdots,X_{q-1}),\ w\in C^q(\mathcal{X}', \Gamma).$$

(c) $$dw(X_1,\cdots,X_{q+1})=\sum_i(-1)^{i+1}X_i\cdot(w(X_1,\cdots,\hat{X}_i,\cdots,X_{q+1}))$$
$$+\sum_{i<j}(-1)^{i+j}w([X_i, X_j], X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{q+1}).$$

$\theta(X)$ preserves the degree, and by the action $w\to\theta(X)\cdot w$ $\mathcal{X}'$ operates on $C(\mathcal{X}', \Gamma)$ (Definition 2.3). $i(X)$ diminuates the degree by one and is $\mathcal{F}(M)$-linear. d augments the degree by one. The following identities are easy to verify (see [4]):

(2.1.1) $\quad i(X)\theta(Y)-\theta(Y)i(X)=i([X, Y]),\ X, Y\in\mathcal{X}'$,

(2.1.2) $\quad i(X)d+di(X)=\theta(X),\qquad X\in\mathcal{X}'$,

(2.1.3) $\quad \theta(X)d=d\theta(X),\qquad X\in\mathcal{X}'$,

(2.1.4) $\quad d^2=d\circ d=0.$

The derived R-module of $C(\mathcal{X}', \Gamma)=\sum_q C^q(\mathcal{X}', \Gamma)$ with respect to d is denoted by $H^*(\mathcal{X}', \Gamma)=\sum_{q\geq 0}H^q(\mathcal{X}', \Gamma)$[5)].

2. $H^*(\mathcal{X}_v(B), \Gamma)$.

Now let (B, p, M, F, G) be a C^∞-bundle, and let Γ be an $\mathcal{F}(B)$-module on which $\mathcal{X}(B)$ operates. As remarked in §1, the $\mathcal{F}(B)$-submodule $\mathcal{X}_v(B)$ of vertical vector fields ia also an R-Lie subalgebra of $\mathcal{X}(B)$. We imbed $\mathcal{F}(M)$ in $\mathcal{F}(B)$ through π^*. Since $\mathcal{X}_v(B)$ is the annihilator of $\mathcal{F}(M)$ in $\mathcal{F}(B)$, the following proposition follows directly from the definition of d in $C(\mathcal{X}_v(B), \Gamma)$.

(2.2.1) *In* $C(\mathcal{X}_v(B), \Gamma)$, d *commutes with the operations of* $\mathcal{F}(M)$, i.e., *for* $w\in C^q$ *and* $f\in\mathcal{F}(M)$, *we have* $dfw=dfw$. *Thus* $\mathcal{F}(M)$-*module strucure of* $C(\mathcal{X}_v(B), \Gamma)$ *is inherited by* $H^*(\mathcal{X}_v(B), \Gamma)$.

Let $l:\mathcal{X}(M)\to\mathcal{X}(B)$ be the lift may with respect to a connection of the bundle.

Let $X\in\mathfrak{X}(M)$ and $c\in C^q(\mathfrak{X}_v(B),\Gamma)$. Define $X\cdot c\in C^q(\mathfrak{X}_v(B),\Gamma)$ by

$$(X\cdot c)(X_1,\cdots,X_q)=l(X)\cdot(c(X_1,\cdots,X_q)-\sum_i c(X_1,\cdots,[l(X),X_i],\cdots,X_q).$$

Since $[l(X),X_i]\in\mathfrak{X}_v(B)$ by Proposition 2.1, the above definition has a sense.

We shall prove the following identities: for $c\in C^q(\mathfrak{X}_v(B),\Gamma)$, $X\in\mathfrak{X}(M)$ and $f\in\mathfrak{F}(M)$,

$$d(X\cdot c)=X\cdot(dc) \tag{2.2.2}$$

$$(fX)\cdot c=f(X\cdot c) \tag{2.2.3}$$

$$X\cdot(fc)=(X\cdot f)c+f(X\cdot c) \tag{2.2.4}$$

First we note the following formulas analogous to (2.1.1) and to (d) in Definition 2.3, whose proof is immediate.

(2.2.5) $i(Y)(X\cdot c)-X\cdot(i(Y)c)=i([Y,l(X)])c$
for $Y\in\mathfrak{X}_v(B)$, $X\in\mathfrak{X}(M)$ and $c\in C^q(\mathfrak{X}_v(B),\Gamma)$.

(2.2.6) $\theta(Y)(X\cdot c)-X\cdot(\theta(Y)c)=\theta([Y,l(X)])c$
for $Y\in\mathfrak{X}_v(B)$, $X\in\mathfrak{X}(M)$ and $c\in C^q(\mathfrak{X}_v(B),\Gamma)$.

We suppose that the formula (2.2.2) is proved for c of dimension less than q, and we shall prove it for $c\in C^q(\mathfrak{X}_v(B),\Gamma)$. Calculate $i(Y)d(Xc)$ for $Y\in\mathfrak{X}_v(B)$;

$$\begin{aligned}
i(Y)d(X\cdot c)&=\theta(Y)(X\cdot c)-d(i(Y)(X\cdot c)) && \text{by (2.1.2),}\\
&=\theta(Y)(X\cdot c)-d(X\cdot(i(Y)c))-d(i([Y,l(X)])c) && \text{by (2.2.5),}\\
&=\theta(Y)(X\cdot c)-X\cdot(di(Y)c)-d(i([Y,l(X)])c)\\
&\qquad\qquad\text{by the inductive assumption,}\\
&=\theta(Y)(X\cdot c)-X\cdot(\theta(Y)c)+X\cdot(i(Y)dc)\\
&\qquad-([Y,l(X)])c+i([Y,l(X)])dc && \text{by (2.1.2)}\\
&=X\cdot(i(Y)dc)+i([Y,l(X)])dc && \text{by (2.2.6)}\\
&=i(Y)(X\cdot(dc)) && \text{by (2.2.5).}
\end{aligned}$$

Thus $i(Y)d(X\cdot c)=i(Y)(X\cdot dc)$ for any $Y\in\mathfrak{X}_v(B)$; this implies that $d(X\cdot c)=X\cdot(dc)$.

Let $f\in\mathfrak{F}(M)$. Then we have $l(fX)=fl(X)$, and therefore

$$fX\cdot c(X_1,\cdots,X_q)=fl(X)\cdot(c(X_1,\cdots,X_q))-\sum_i c(X_1,\cdots[fl(X),X_i],\cdots,X_q).$$

Since $X_if=0$, we have $[fl(X),X_i]=f[l(X),X_i]$. Hence

$$(fX\cdot c)(X_1,\cdots,X_q)=f(X\cdot c)(X_1,\cdots,X_q).$$

Thus, (2.2.3) is proved.

Noting that $l(X)f=Xf$, we can prove (2.2.4) in a similar manner.

In virtue of (2.2.2), the action of $\mathfrak{X}(M)$ on $C^q(\mathfrak{X}_v(B),\Gamma)$ transfers on $H^q(\mathfrak{X}_v(B),\Gamma)$. If $\bar{c}\in H^q(\mathfrak{X}_v,\Gamma)$ and if c is a representative cocycle of $\bar{c}$, then $X\cdot\bar{c}$ is defined as the class of $X\cdot c$.

Proposition 2.6. *Let $X, Y \in \mathfrak{X}(M)$, and $c \in H^q(\mathfrak{X}_v(B), \Gamma)$. Then*

$$[X, Y]\cdot c = X\cdot(Y\cdot c) - Y\cdot(X\cdot c).$$

Proof. Let $w \in C^q(\mathfrak{X}_v(B), \Gamma)$ be a cocycle representing c. Then direct calculation shows that

$$(X\cdot(Y\cdot w))(X_1, \cdots, X_q) - (Y\cdot(X\cdot w))(X_1, \cdots, X_q)$$
$$= [l(X), l(Y)](w(X_1, \cdots, X_q)) - \sum_{i=1}^q w(X_1, \cdots, [[l(X), l(Y)], X_i], \cdots, X_q), \ X_i \in \mathfrak{X}_q(B).$$

Since
$$[l(X), l(Y)] = h([l(X), l(Y)]) + v([l(X), l(Y)])$$
$$= l([X, Y]) + v([l(X), l(Y)]),$$

it follows that

$$X\cdot(Y\cdot w) - Y\cdot(X\cdot w) = [X, Y]\cdot w + \theta(v([l(X), l(Y)]))w.$$

But $dx = 0$; therefore $\theta(v([l(X), l(Y)]))w = di(v([l(X), l(Y)]))w$ by (2.1.2). It follows that $X\cdot(Y\cdot w) - Y\cdot(X\cdot w)$ is cohomologous to $[X, Y]\cdot w$. Hence

$$X\cdot(Y\cdot c) - Y\cdot(X\cdot c) = [X, Y]\cdot c$$

Proposition 2.6 together with (2.2.3) and (2.2.4) shows that $\mathfrak{X}(M)$ *operates on the $\mathfrak{F}(M)$-module $H^*(\mathfrak{X}_v(B), \Gamma)$*

Proposition 2.7. *The operation of $\mathfrak{X}(M)$ on $H^*(\mathfrak{X}_v(B), \Gamma)$ defined above is independent of the choice of connections used in the definition.*

Proof. Let l_1 and l_2 be the lift maps corresponding to any two connections given. Then $l_1(X) - l_2(X)$ is a vertical vector field for $X \in \mathfrak{X}(M)$. Therefore

$$l_1(X)(w(X_1, \cdots, X_q)) - \sum_i w(X_1, \cdots, [l_1(X), X_i], \cdots, X_q)$$
$$- \{l_2(X)(w(X_1, \cdots, X_q)) - \sum_i w(X_1, \cdots, [l_2(X), X_i], \cdots, X_q)\}$$
$$= \theta(l_1(X) - l_2(X))w(X_1, \cdots, X_q), \quad X_i \in \mathfrak{X}_v(B), \quad w \in C^q(\mathfrak{X}_v(B(, \Gamma)).$$

If $dw = 0$, then $\theta(l_1(X) - l_2(X))w = di(l_1(X) - l_2(X))w$ is a coboundary, so that our assertion is proved.

3. *Sheaf $\mathcal{H}(\mathcal{S})$.*

Now let (B, p, M, F, G) be a C^∞-fibre bundle in which B, M and F are connected. Let $(\mathcal{S}, \pi, B)$ be an admissible sheaf on B. Let x be a point of M and let U be a neighborhood of x in M which is mapped diffeomorphically onto the unit spherical domain of a Euclidean space. Take a point y in U and let $u(t)$, $0 \leqq t \leqq 1$, be a C^∞-simple curve in U with $u(0) = x$, $u(1) = y$. Using a fixed connection of the bundle, lifts of $u(t)$ define a diffeomorphism u of $p^{-1}(x)$ onto $p^{-1}(y)$:

$$u(x^*) = u_1(x^*) = \tilde{u}(1\,; x^*), \ x^* \in p^{-1}(x).$$

Let $x^{**} \in \pi^{-1}p^{-1}(x)$ and $\pi(x^{**}) = x^*$. Denote by $\tilde{\tilde{u}}(t\,; x^{**})$ the lift of $\tilde{u}(t\,; x^*)$

through x^{**}. Then the correspondence $\tilde{u}: x^{**} \rightarrow \tilde{\tilde{u}}(t; x^{**})$ gives a 1-1 map of $\pi^{-1}p^{-1}(x)$ onto $\pi^{-1}p^{-1}(y)$. Both $(\pi^{-1}p^{-1}(x), \pi, p^{-1}(x))$ and $(\pi^{-1}p^{-1}(y), \pi, p^{-1}(y))$ are admissible sheaves induced by $(\mathcal{S}, \pi, M)$.

Proposition 2.8. *The pair of maps $(\tilde{u}, u)$ is compatible with sheaf structure, that is, it satisfies the following conditions.*

(2.3.1) $$\pi\tilde{u} = u\pi.$$

(2.3.2) $$\tilde{u}|\pi^{-1}(x^*) \text{ is a linear map for any } x^* \in p^{-1}(x).$$

(2.3.3) $$\tilde{u} \text{ is a homeomorphism.}$$

Proof. (2.3.1) is immediate from the definition.

Since $\tilde{u}(x^{**}) = \tilde{\tilde{u}}(1, x^{**})$, and since $\tilde{\tilde{u}}(t, x^{**})$, $0 \leqq t \leqq 1$, is the cross-section image of $\tilde{u}(t; x^*)$, $x^* = \pi(x^{**})$, $0 \leqq t \leqq 1$, in a locally constant sheaf, (2.3.2) holds.

To prove (2.3.3) it is sufficient to show that any cross-section image is mapped by $\tilde{u}$ on a cross-section image. But this follows also from the fact that $\tilde{\tilde{u}}(t; x^{**})$, $0 \leqq t \leqq 1$, is a cross-section image for every $x^{**} \in \pi^{-1}p^{-1}(x)$.

Let $(\mathcal{S}, \pi, V)$ and $(\mathcal{S}', \pi', V')$ be admissible sheaves on connected, paracompact C^∞-manifolds V and V' respectively. Let $\tilde{u}: \mathcal{S} \rightarrow \mathcal{S}'$ and $u: V \rightarrow V'$ be maps compatible with sheaf structure, i.e. satisfying (2.3.1)—(2.3.3). (In (2.3.2), $p^{-1}(x)$ will be replaced by V). Then the map $\tilde{u}: \boldsymbol{B}(\mathcal{S}) \rightarrow \boldsymbol{B}(\mathcal{S}')$ is a C^∞ bundle map; hence it induces $\tilde{u}: \Gamma(\boldsymbol{B}(\mathcal{S})) \rightarrow \Gamma(\boldsymbol{B}(\mathcal{S}'))$. If $X \in \mathfrak{X}(V)$, and $s \in \Gamma(\boldsymbol{B}(\mathcal{S}))$, then, as is easily verified, we have

(2.3.4) $$\tilde{u}(X \cdot s) = u_* X \cdot \tilde{u}(s).$$

Define

$$u^\sharp: C^q(\mathfrak{X}(V'), \Gamma(\boldsymbol{B}(\mathcal{S}'))) \rightarrow C^q(\mathfrak{X}(V), \Gamma(\boldsymbol{B}(\mathcal{S})))$$

by

$$u^\sharp w(X_1, \cdots, X_q) = \tilde{u}^{-1}(w(u_* X_1, \cdots, u_* X_q)),$$
$$X_i \in \mathfrak{X}(V), \ x \in V.$$

$u^\sharp$ is R-linear. From (2.3.4) it follows that $du^\sharp = u^\sharp d$. Thus, $u^\sharp$ induces an isomorphism

$$u^*: H^*(\mathfrak{X}(V'), \Gamma(\boldsymbol{B}(\mathcal{S}'))) \rightarrow H^*(\mathfrak{X}(V), \Gamma(\boldsymbol{B}(\mathcal{S}))).$$

Applying this to the Proposition 2.8, we have

Proposition 2.9. *The pair of maps $(\tilde{u}, u)$ induces an isomorphism*

$$u^*: H^*(\mathfrak{X}(p^{-1}(y)), \Gamma(\boldsymbol{B}(\mathcal{S}), p^{-1}(y))) \rightarrow H^*(\mathfrak{X}(p^{-1}(x)), \Gamma(\boldsymbol{B}(\mathcal{S}), p^{-1}(x))),$$

or equivalently, $u^*: H^*(p^{-1}(y), \mathcal{S}(p^{-1}(y))) \rightarrow H^*(p^{-1}(x), \mathcal{S}(p^{-1}(x)))$.

Proposition 2.10. *The isomorphism of Proposition 2.9 does not depend on the choice of curves in U joining x to y.*

Proof. Let $u^{(0)}$ and $u^{(1)}$ be two curves in U joining x to y. Then there exists a C^∞-homotopy $u^{(\tau)}$, $0\leqq\tau\leqq1$, between $u^{(0)}$ and $u^{(1)}$. Precisely, $u^{(\tau)}(t)$, $0\leqq t\leqq1$, is a C^∞-curve in U joining x to y and the function $I\times I\to U$ defined by $(\tau, t)\to u^{(\tau)}(t)$ is C^∞. Then the family of curves $\tilde{u}^{(\tau)}(t\,;x^*)$, $x^*\in p^{-1}(x)$, is a C^∞-homotopy between $\tilde{u}^{(0)}(t\,;x^*)$ and $\tilde{u}^{(1)}(t\,;x^*)$ (cf. [9]). It follows that $u^{(\tau)}$: $p^{-1}(x)\to p^{-1}(y)$, $0\leqq\tau\leqq1$, is a C^∞-isotopy between $u^{(0)}$ and $u^{(1)}$. Similarly $\tilde{u}^{(\tau)}$: $\pi^{-1}p^{-1}(x)\to\pi^{-1}p^{-1}(y)$ is a C^∞-isotopy between $\tilde{u}^{(0)}$ and $\tilde{u}^{(1)}$, where we consider $\pi^{-1}p^{-1}(x)$ and $\pi^{-1}p^{-1}(y)$ as C^∞-manifolds. Our proposition follows, therefore, from the following lemma.

Lemma 2.11. *Let M and M' be manifolds. Let $(\mathcal{S}, \pi, M)$ and $(\mathcal{S}', \pi', M')$ be admissible sheaves on M and M' respectively. Suppose that there are maps*

$$\tilde{\phi}:\mathcal{S}\times I\to\mathcal{S}'$$

and

$$\phi: M\times I\to M'.$$

Define maps $\tilde{\phi}_\tau:\mathcal{S}\to\mathcal{S}'$ *and* $\phi_\tau: M\to M'$ *by*

$$\tilde{\phi}_\tau(x)=\tilde{\phi}(x, \tau), \quad x\in\mathcal{S}$$

and

$$\phi_\tau(x)=\phi(x, \tau) \quad x\in M.$$

Suppose moreover that the following condition are satisfied:

(a) $\pi'\tilde{\phi}(x, \tau)=\phi(\pi(x), \tau)$,

(b) *ϕ_τ is a diffemorphism for each $\tau\in I$.*

(c) *$\tilde{\phi}_\tau$ is a homeomorphism compatible with sheaf structures.*

(d) *If we consider $\tilde{\phi}$ as a map of $\boldsymbol{B}(\mathcal{S})\times I$ into $\boldsymbol{B}(\mathcal{S})$, then $\tilde{\phi}$ is C^∞.*

Under the above assumptions, $\tilde{\phi}_0$ and $\tilde{\phi}_1$ induce the same isomorphism of $H^*(M', \mathcal{S}')$ onto $H^*(M, \mathcal{S})$.

Proof. It is sufficient to show the existence of a homotopy operator

$$D_q: C^q(\mathfrak{X}(M'), \Gamma(\boldsymbol{B}(\mathcal{S}')))\to C^{q-1}(\mathfrak{X}(M), \Gamma(\boldsymbol{B}(\mathcal{S})))$$

such that

$$dD_q+D_{q+1}d=\phi_1^{\#}-\phi_0^{\#}$$

where $\phi_\tau^{\#}: C^q(\mathfrak{X}(M'), \Gamma(\boldsymbol{B}(\mathcal{S})))\to C^q(\mathfrak{X}(M), \Gamma(\boldsymbol{B}(\mathcal{S})))$

is defined by

$$(\phi_\tau^{\#}w)(X_1, \cdots, X_q)(x)=\tilde{\phi}_\tau^{-1}(w(\phi_\tau X_1, \cdots, \phi_\tau X_q)(\phi_\tau(x))).$$

The tangent space $T_{(x,\tau)}$ at (x,τ) of $M\times I$ is cannonically identified with $T_x\oplus R$, where T_x is the tangent space at x of M. (If $\tau=0$ (or 1) it is identified with $T_x\oplus[0,\infty)$ (or $T_x\oplus(-\infty,0]$).)

For $X\in\mathfrak{X}(M)$, we shall define $\tilde{X}\in\mathfrak{X}(M\times I)$ by $\tilde{X}_{(x,\tau)}=(X,0)$ in $T_{(x,\tau)}=T_x\oplus R$. Let $Y_0\in\mathfrak{X}(M\times I)$ be defined by

$$Y_0 f(x,t)=\frac{\partial}{\partial t}f(x,t).$$

Let $(\tilde{\mathcal{S}},\tilde{\pi},M\times I)$ be the tensor product sheaf of $(\mathcal{S},\pi,M)$ and the constant sheaf $(I\times R,p,I)$ (cf. [1]). Since the fibre of $\boldsymbol{B}(\tilde{\mathcal{S}})$ over (x,t) is canonically identified with $\mathcal{S}_x$, an element $s\in\Gamma(\boldsymbol{B}(\tilde{\mathcal{S}}))$ can be considered as a C^∞-function $s:M\times I\to\boldsymbol{B}(\mathcal{S})$ such that $s(x,t)\in\boldsymbol{B}(\mathcal{S})_x$.

Now define

$$h_q: C^q(\mathfrak{X}(M\times I),\ \Gamma(\boldsymbol{B}(\tilde{\mathcal{S}})))\to C^{q-1}(\mathfrak{X}(M),\ \Gamma(\boldsymbol{B}(\mathcal{S})))$$

by

$$h_q w=0 \qquad\qquad \text{if } q=0,$$
$$h_q w(X_1,\cdots,X_{q-1})(x)=\int_0^1 w(Y_0,\tilde{X}_1,\cdots,\tilde{X}_{q-1})(x,t)dt, \text{ if } q>0.$$

Since $w(Y_0,\tilde{X}_1,\cdots,\tilde{X}_{q-1})(x,t)\in\mathrm{B}(\mathcal{S})_x$ for all $t\in I$, the integral of the right hand side is well defined.

Using the fact that

$$[Y_0,\tilde{X}]=0 \text{ and } X(h_q w(X_1,\cdots,X_{q-1}))(x)=\int_0^1\tilde{X}(w(Y_0,\tilde{X}_1,\cdots,\tilde{X}_{q-1}))(x,t)dt,$$

the following identity is easily verified:

$$\begin{aligned}(dh_q+h_{q+1}d)&w(X_1,\cdots,X_q)(x)\\ &=\int_0^1 Y_0(w(\tilde{X}_1,\cdots,\tilde{X}_q))(x,t)dt\\ &=w(\tilde{X}_1,\cdots,\tilde{X}_q)(x,1)-w(\tilde{X}_1,\cdots,\tilde{X}_q)(x,0).\end{aligned}$$

Define $\phi^\#: C^q(\mathfrak{X}(M'),\Gamma(\boldsymbol{B}(\mathcal{S}')))\to C^q(\mathfrak{X}(M\times I),\ \Gamma(\boldsymbol{B}(\mathcal{S})))$ by

$$\phi^\# w(Y_1,\cdots,Y_q)(x,t)=\tilde{\phi}_t^{-1}(w(\phi(Y_1),\cdots,\phi(Y_q))(\phi(x,t)).$$

Then we have

$$\begin{aligned}\phi^\# w(\tilde{X}_1,\cdots,\tilde{X}_q)(x,1)&-\phi^\# w(\tilde{X}_1,\cdots,\tilde{X}_q)(x,0)\\ &=\phi_1^\# w(X_1,\cdots,X_q)(x)-\phi_1^\# w(X_1,\cdots,X_q)(x).\end{aligned}$$

Set $D_q=h_q\circ\phi^\#$. Then, for $w\in C^q(\mathfrak{X}(M'),\Gamma(\mathrm{B}(\mathcal{S}')))$, we have

$$\begin{aligned}(dD_q+D_{q+1}d)&w(X_1,\cdots,X_q)(x)\\ &=\phi^\# w(\tilde{X}_1,\cdots,\tilde{X}_q)(x,1)-\phi^\# w(\tilde{X}_1,\cdots,\tilde{X}_q)(x,0)\\ &=\phi_1^\# w(X_1,\cdots,X_q)(x)-\phi_0^\# w(X_1,\cdots,X_q)(x);\end{aligned}$$

thus,

$$dD_q + D_{q+1}d = \phi_1^{\#} - \phi_0^{\#}.$$

The isomorphism u^* is defined by means of a fixed connection. However, we have

Proposition 2.12. *The isomorphism u^* of Proposition 2.9 does not depend on the choice of connections.*

Proof. Let Q_0 and Q_1 be two connections of (B, p, M). We know by [9] that there exists a family Q_τ of connections $(0 \leqq \tau \leqq 1)$ such that the lift of an $X \in \mathfrak{X}(M)$ with respect to Q_τ is given by $(1-\tau)l_0(X) + \tau l_1(X)$, where l_0 and l_1 are lifts operator with respect Q_0 and Q_1 respectively. Let $\tilde{u}_\tau(t\,;x^*)$, $x^* \in p^{-1}(x)$, be the lift of u with respect to Q_τ starting from x. Then it is easily checked that the curves $\tilde{u}_\tau(t:x^*)$ give a homotopy between $\tilde{u}_0(t\,;x^*)$ and $\tilde{u}_1(t\,;x^*)$. It follows that $u_\tau: p^{-1}(x) \to p^{-1}(y)$ is a C^∞-isotopy between u_0 and u_1; $(u_\tau(x^*) = \tilde{u}_\tau(1, x^*))$. Similarly $\tilde{u}_\tau: \pi^{-1}p^{-1}(x) \to \pi^{-1}p^{-1}(y)$ is an isotopy between $\tilde{u}_0$ and $\tilde{u}_1$. The situation is the same as in the proof of Proposition 2.10, and the Proposition 2.12 follows from Lemma 2.11.

Set $\mathcal{H}(\mathcal{S})_x = H^*(p(x), \mathcal{S}(p^{-1}(x)))$ and $\mathcal{H}(\mathcal{S}) = \bigcup_{x \in M} \mathcal{H}(\mathcal{S})_x$. Define a function $\pi: \mathcal{H}(\mathcal{S}) \to M$ by $\pi(x^*) = x$ for $x^* \in \mathcal{H}(\mathcal{S})_x$. Each $\mathcal{H}(\mathcal{S})_x$ is a vector space over R. Let U be an open set of M diffeomorphic with a Euclidean space. In virtue of Propositions 2.11 and 2.12, there exists a well defined isomorphism $\sigma(x, y): \mathcal{H}(\mathcal{S})_y \to \mathcal{H}(\mathcal{S})_x$ for any $x, y \in U$, such that $\sigma(x, y) \circ \sigma(y, z) = \sigma(x, z)$ for $x, y, z \in U$. Let $U \subset M$ be as above, and let $x^* \in \mathcal{H}(\mathcal{S})_x$, $x \in U$. Define $\tilde{U}(x^*)$ to be the set of all $\sigma(x, y)^{-1}x^*$, $y \in U$. Since $y^* \in \tilde{U}(x^*)$ implies that $\tilde{U}(y^*) = \tilde{U}(x^*)$, $\{\tilde{U}(x^*)\}$ is a neighborhood system of a unique topology in $\mathcal{H}(\mathcal{S})$, where $x^* \in \mathcal{H}(\mathcal{S})$ and U runs over Euclidean open sets of M. It is easily checked that

(2.3.5) *$(\mathcal{H}(\mathcal{S}), \pi, M)$ with the topology of $\mathcal{H}(\mathcal{S})$ defined above is a locally constant sheaf of vector space.*

Generally, $\mathcal{H}(\mathcal{S})$ is not admissible, i.e., is not finite dimensional. However

(2.3.6) *If the fibre F of the bundle (B, p, M) is compact, then the sheaf $\mathcal{H}(\mathcal{S})$ is admissible.*

This is an immediate consequence of the known fact that over compact X, cohomology groups $H^*(x, \mathcal{S})$ with coefficients in an admissible sheaf are finite dimensional. We have only to apply it on $p^{-1}(x)$ and $\mathcal{S}(p^{-1}(x))$.

4. $\Gamma(\boldsymbol{B}(\mathscr{H}(\mathcal{S})))$.

We continue with the situation of §2.3. Let now $\Gamma=\Gamma(\boldsymbol{B}(\mathcal{S}))$, where $\boldsymbol{B}(\mathcal{S})$ is the vector bundle defined in §1. *Assuming that* $\mathscr{H}(\mathcal{S})$ *is admissible*, we shall prove the existence of a canonical isomorphism of $H^*(\mathfrak{X}_v(\boldsymbol{B}), \Gamma)$ onto $\Gamma(\boldsymbol{B}(\mathscr{H}(\mathcal{S})))$.

First we note the following fact. Let T_v denote the bundle of tangent vectors along the fibres of B, and let $\wedge^{*q}(T_v)$ be the dual of q-th exterior product bundle of T_v. Then $C^q(\mathfrak{X}_v(\boldsymbol{B}), \Gamma)=\Gamma(\wedge^{*q}(T_v)\otimes\boldsymbol{B}(\mathcal{S}))$, so that, for $c\in C^q(\mathfrak{X}_v(\boldsymbol{B}), \Gamma)$ and $x\in B$, $c(x)$ is an element of the fibre over x of $\wedge^{*q}(T_v)\otimes\boldsymbol{B}(\mathcal{S})$; $c(x)$ may also be considered as a q-linear alternating function from the vector space of tangent vectors at x along the fibre through x to the vector space $\mathcal{S}_x$ (cf. the analoguous statement in §1 preceding Proposition 1.1).

Let N be a submanifold (regularly imbedded) of M. If we consider $c(x)$ only for such x that $p(x)\in N$, we get an element $\pi_N c$ of $C^q(\mathfrak{X}_v(p^{-1}(N)), \Gamma(\boldsymbol{B}(\mathcal{S}), p^{-1}(N)))$. π_N is a function $C^q(\mathfrak{X}_v(\boldsymbol{B}), \Gamma)\to C^q(\mathfrak{X}_v(p^{-1}(N)), \Gamma(\boldsymbol{B}(\mathcal{S}), p^{-1}(N)))$. π_N sends cocycles into cocycles and coboundaries into coboundaries. We shall write

$$C_N^q=C^q(\mathfrak{X}_v(p^{-1}(N)), \Gamma(\boldsymbol{B}(\mathcal{S}), p^{-1}(N))).$$

In particular $C_M^q=C^q(\mathfrak{X}_v(\boldsymbol{B}), \Gamma)$ and $C_x^q=C_{\{x\}}^q=C^q(\mathfrak{X}(p^{-1}(x)), \Gamma(\boldsymbol{B}(\mathcal{S}), p^{-1}(x)))$. Let Z_N^q and B_N^q be cocycles and coboundaries of C_N^q. Set $H_N^q=Z_N^q/B_N^q$. Then π_N induces a map $H_M^q\to H_N^q$ which we denote also by π_N.

For $c\in H_M^q$, let the cross-section $\rho(c)$ of $\boldsymbol{B}(\mathscr{H}(\mathcal{S}))$ be defined by

$$\rho(c)(x)=\pi_x c\in H_x^q=\mathscr{H}(\mathcal{S})_x.$$

We shall show that $\rho(c)\in\Gamma(\boldsymbol{B}(\mathscr{H}(\mathcal{S})))$, i.e., $\rho(c)$ is a C^∞-cross section of $\boldsymbol{B}(\mathscr{H}(\mathcal{S}))$, and that $\rho: H^q(\mathscr{H}_v(\boldsymbol{B}), \Gamma)\to\Gamma(\boldsymbol{B}(\mathscr{H}^q(\mathcal{S})))$ is bijective.

Let x_0 be a point of M, and let U be a neighborhood of x_0 such that there exists a diffeomorphism $\varphi: U\to\{(t_1,\cdots,t_n);\ t_1^2+\ \ +t_n^2<1, t_i\in R\}$; $\varphi(x_0)=(0,\cdots,0)$. A C^∞-curve $u(\tau)$, $0\leqq\tau$, $u(0)=x_0$, is called straight line if $\varphi u(\tau)$ is a straight line in the Euclidean space R^n. If $x\in U$ then there exists a unique straight line u joining x_0 to x. As explained earlier, u induces, by means of a connection, an isomorphism $\alpha_x: C_x^q\to C_{x_0}^q$ which commutes with d. If $V\subset p^{-1}(x_0)$ is a sufficiently small neighborhood, then we may identify canonically $B(\mathcal{S})_{x^*}$ with R^k for $x^*\in V$. If $c\in C_U^q$, then we see easily that the function $(x, x^*)\to\alpha_x\pi_x c(X_1,\cdots,X_q)(x^*)\in R^k$ is a C^∞ function $U\times V\to R^k$ for any $X_i\in\mathfrak{X}(p^{-1}(x_0))$ and for any sufficiently small $V\subset p^{-1}(x_0)$. We shall express this fact by saying that the function $U\to C_{x_0}^q$ defined by $x\to\alpha_x\pi_x c$ is C^∞. Conversely a C^∞-function $c': U\to C_{x_0}^q$ defines a unique element $c\in C_U^q$ such that $c'(x)=\alpha_x\pi_x c$.

Thus we established a 1-1 correspondence between C_U^q and the set of all C^∞-functions $U\to C_{x_0}^q=C^q(\mathfrak{X}(p^{-1}(x_0)), \Gamma(\boldsymbol{B}(\mathcal{S}), p^{-1}(x_0)))$. The latter is an $\mathcal{F}(U)$-module in an ovbious way, and may be identified with $\mathcal{F}(U)\hat{\otimes}C_{x_0}^q$, the topological tensor product of Grothendieck of nuclear spaces $\mathcal{F}(U)$ and $C_{x_0}^q$ [2, Ch. II, §3.1, Example 1]. It is clear that, by the isomorphism $C_U^q\to\mathcal{F}(U)\hat{\otimes}C_{x_0}^q$, the differential d in C_U^q is translated into $1\hat{\otimes}d$ in $\mathcal{F}(U)\hat{\otimes}C_{x_0}^q$. Since d is a topological homomorphism of the topological vector space $C_{x_0}^q$, the derived module of $\mathcal{F}(U)\hat{\otimes}C_{x_0}^q$ with respect to $1\hat{\otimes}d$ is $\mathcal{F}(U)\hat{\otimes}H_{x_0}^q$ (cf. [12, Exp. 24, Lemma]). Since $H_{x_0}^q=\mathcal{H}^q(\mathcal{S})_{x_0}$, it is finite dimensional by the assumption, so that $\mathcal{F}(U)\hat{\otimes}H_{x_0}^q=\mathcal{F}(U)\otimes H_{x_0}^q$.

Thus we have an isomorphism

$$\alpha : H_U^q\to\mathcal{F}(U)\otimes H_{x_0}^q.$$

On the other hand, from the difinition of $\mathcal{H}(\mathcal{S})$ it follows that there exists an isomorphism

$$\beta : \Gamma(\boldsymbol{B}(\mathcal{H}(\mathcal{S})), U))\to\mathcal{F}(U)\otimes H_{x_0}^q$$

such that the map $\beta^{-1}\circ\alpha : H_U^q\to\Gamma(\boldsymbol{B}(\mathcal{H}(\mathcal{S})), U)$ is just the map ρ defined on H_U^q.

Consider the following commutative diagram

$$\begin{array}{ccc} H_M^q & \xrightarrow{\rho} & \Gamma'(\boldsymbol{B}(\mathcal{H}(\mathcal{S})), M) \\ \pi_U\Big\downarrow & & \Big\downarrow\pi_U \\ H_U^q & \xrightarrow{\rho} & \Gamma'(\boldsymbol{B}(\mathcal{H}(\mathcal{S})), U) \end{array}$$

where Γ' means the set of all (not necessarily continuous) cross-sections of bundle. Then, $\rho(H_M^q)\subset\Gamma(\boldsymbol{B}(\mathcal{H}(\mathcal{S})), M)$ if and only if $\pi_U\rho(H_M^q)\subset\Gamma(\boldsymbol{B}(\mathcal{H}(\mathcal{S})), U)$ for sufficiently small neighborhoods U of any points of M. Since $\rho(H_U^q)\subset\Gamma(\boldsymbol{B}(\mathcal{S})), U)$, we have proved that $\rho(H_M^q)\subset\Gamma(\boldsymbol{B}(\mathcal{H}(\mathcal{S})))$.

Let V be an open set such that $\overline{V}\subset U$. Let $c\in\Gamma(\boldsymbol{B}(\mathcal{H}(\mathcal{S})), U)$ be such that $c(x)=0$ for $x\notin V$. If we write $\beta(c)=\sum f_i\otimes c_i$, where $f_i\in\mathcal{F}(U)$ and $\{c_i\}$ is a base of $H_{x_0}^q$, then, $f_i(x)=0$ for $x\notin V$. If $z_i\in Z_{x_0}^q$ is a representative of c_i, then $\sum f_i\otimes z_i=z\in\mathcal{F}(U)\otimes C_{x_0}^q\subset\mathcal{F}(U)\hat{\otimes}C_{x_0}^q$ is a representative of $\beta(c)$. Since $(\alpha^{-1}z)(x)=0$ for $x\notin V, (x\in U)$, $\alpha^{-1}(z)$ extends uniquely to an element $z'\in Z_M^q$ such that $z'(x)=0$ for $x\notin V$, $x\in M$.

Let $\{V_i\}$ and $\{U_i\}$ be locally finite coverings of M, such that $\overline{V}_i\subset U_i$. Let $\{\lambda_i\}$ be a partition of unity belonging to $\{V_i\}$.

Now if $c\in\Gamma(\boldsymbol{B}(\mathcal{H}(\mathcal{S})))$, then $\lambda_i c\in\Gamma(\boldsymbol{B}(\mathcal{H}(\mathcal{S})))$ is zero outside of V_i. Therefore, by the above remark, there exists a $z_i\in Z_M^q$ such that $z_i(x)=0$ for $x\notin V_i$ and that ρ(class of z_i)$=\lambda_i c$. Set $z=\sum' z_i$. Then ρ(class of z)$=\sum_i \rho$(class of z_i)$=\sum_i \lambda_i c=c$. This proves that ρ is surjective.

Let $z\in Z_M^q$ be such that ρ(class of z)$=0$. Then, ρ(class of $\lambda_i z$)$=\lambda_i \rho$(class of z)$=0$. Since ρ is injective on H_U^q, it follows that π_U(class of $\lambda_i z$)$=0$. Therefore there exists a $z_i'\in C_U^{q-1}$ such that $dz_i'=\pi_U(\lambda_i z)$. Take an open set W_i such that $\bar{V}_i\subset W_i$, $\bar{W}_i\subset U_i$, and let μ_i be a C^∞-function on U_i such that $\mu_i(x)=1$ for $x\in V$ and $\mu_i(x)=0$ for $x\notin W_i$. Then, $d(\mu_i z_i')=\pi_U(\lambda_i z)$ and $\mu_i z_i'=0$ outside of W_i. $\mu_i z_i'$ extends to z_i'' over whole C_M^{q-1} such that $dz_i''=\lambda_i z$ and $z_i''=0$ outside of W_i. Then we have $d(\sum' z_i'')=\sum' \lambda_i z=z$. Thus (class of z)$=0$; this proves that ρ is injective.

We defined an operation of $\mathfrak{X}(M)$ on H_M^q in §2.2. $\mathfrak{X}(M)$ operates also on $\Gamma(\boldsymbol{B}(\mathcal{H}(S)))$. It is easily checked that ρ commutes with the operations of $\mathfrak{X}(M)$.

We summarize these in

Proposition 2.13 *Assume that $\mathcal{H}(S)$ is admissible. Then there exists a canonical $\mathcal{F}(M)$-isomorphism*

$$\rho: H^*(\mathfrak{X}_v(B), \Gamma(\boldsymbol{B}(S)))\to \Gamma(\boldsymbol{B}(\mathcal{H}(S))),$$

which commutes with the operations of $\mathfrak{X}(M)$.

5. *A filtration of $C(\mathfrak{X}(B), \Gamma)$ and its spectral sequence.*

Let (B, p, M, F, G) be a C^∞-fibre bundle with B and F connected, and let Γ be an $\mathcal{F}(B)$-module on which $\mathfrak{X}(B)$ operates. Let $C=\sum_{q\geq 0} C^q$, $C^q=C^q(\mathfrak{X}(B),\Gamma)$. We define a filtration (cf. [6], [13]) $\{A^p\}$ of C as follows: set

$$A^{p,q}=\{w\,;\, w\in C^{p+q}, i(X_1)\cdots, i(X_{q+1})w=0, \text{ for any } X_1,\cdots, X_{q+1}\in \mathfrak{X}_v(B)\}, \text{ for } p\geqq 0,$$
$$A^{p,q}=C^q, \text{ for } p<0,$$

and

$$A^p=\sum_{q\geq 0} A^{p,q}.$$

$A^{p,q}$ is an $\mathcal{F}(B)$-submodule of C^{p+q}, so that A^p is an $\mathcal{F}(B)$-submodule of C. Following properties are easily checked.

$$C=A^0\supset A^1\supset\cdots\supset A^p\supset A^{p+1}\cdots.$$

and $\bigcap_p A^p=0$.

$$i(X)(A^{p,q})\subset A^{p,q-1} \qquad \text{for } X\in \mathfrak{X}_v(B).$$
$$\theta(X)(A^{p,q})\subset A^{p,q} \qquad \text{for } X\in \mathfrak{X}_v(B).$$
$$d(A^{p,q})\subset A^{p,q+1}.$$

In particular $\{A^p\}$ defines a filtration of C.

We shall examine the spectral sequence induced by this filtration.

First define an $\mathcal{F}(B)$-linear map

$$\varphi^{(p,q)}: C^{p+q}(\mathcal{X}(B), \Gamma) \to C^{p}(\mathcal{X}(B),\ C^{q}(\mathcal{X}(B), \Gamma)),\ p \geqq 0,\ q \geqq 0,$$

by

$$((\varphi^{(p,q)}w)(X_1, \cdots, X_p))(X_{p+1}, \cdots, X_{p+q}) = w(X_1, \cdots, X_p, X_{p+1}, \cdots, X_{p+q})$$

for $w \in C^{p+q}$, $X_i \in \mathcal{X}(B)$, $p > 0$,

and

$$\varphi^{(p,q)}w = w, \quad \text{for } p = 0.$$

Then we have (cf. [4])

$$(2.5.1) \qquad (\varphi^{(p,q)}(dw))(X_1, \cdots, X_p)$$
$$= (d(\varphi^{(p-1,q)}w))(X_1, \cdots, X_p) + (-1)^p d(\varphi^{(p,q-1)}w(X_1, \cdots, X_p)),$$

for $w \in C^{p+q-1}$, $p \geqq 0$, $q \geqq 0$. *(We set* $\varphi^{(m,n)} = 0$ *if* m *or* n *is negative.)*

Let Rest : $C^q(\mathcal{X}(B), \Gamma) \to C^q(\mathcal{X}_v(B), \Gamma)$ be defined by restricting arguments to $\mathcal{X}_v(B)$, and we shall denote by $\mathrm{Rest}_{\#} : C^p(\mathcal{X}(B), C^p(\mathcal{X}(B), \Gamma)) \to C^p(\mathcal{X}(B), C^q(\mathcal{X}_v(B), \Gamma))$ the map Rest induces.

Define

$$\bar{\varphi}^{(p,q)}: A^{p,q} \to C^p(\mathcal{X}(B), C^q(\mathcal{X}_v(B), \Gamma))$$

as the composition of $\varphi^{(p,q)}|A^{p,q}$ and $\mathrm{Rest}_{\#}$.

Proposition. 2.14. *The image of* $\bar{\varphi}^{(p,q)}$ *is* $C^p(\mathcal{X}(B)/\mathcal{X}_v(B), C^q(\mathcal{X}_v(B), \Gamma))$ *which is imbedded in* $C^p(\mathcal{X}(B), C^q(\mathcal{X}_v(B), \Gamma))$ *through the dual map of the canonical projection* $\mathcal{X}(B) \to \mathcal{X}(B)/\mathcal{X}_v(B)$. *(An elemnt* $w \in C^p(\mathcal{X}(B), C^q(\mathcal{X}_v(B), \Gamma))$ *belongs to* $C^p(\mathcal{X}(B)/\mathcal{X}_v(B), C^q(\mathcal{X}_v(B), \Gamma))$ *if and only of* $w(X_1, \cdots, X_p) = 0$ *whenever one of* $X_i \in \mathcal{X}(B)$ *is contained* in $\mathcal{X}_v(B)$.)

The kernel of $\bar{\varphi}^{(p,q)}$ *is* $A^{p+1,q-1}$.

Proof.

If $w \in A^{p,q}$, then $(\bar{\varphi}^{(p,q)}w)(X_1, \cdots, X_p) = 0$ whenever one of arguments X_i is in $\mathcal{X}_v(B)$, that is, $\bar{\varphi}^{(p,q)}w$ is contained in $C^p(\mathcal{X}(B)/\mathcal{X}_v(B), C^q(\mathcal{X}_v(B), \Gamma))$.

Conversely, suppose that w' is contained in $C^p(\mathcal{X}(B)/\mathcal{X}_v(B), C^q(\mathcal{X}_v(B), \Gamma))$. Take a connection in the bundle, and let $v : \mathcal{X}(B) \to \mathcal{X}_v(B)$ be the corresponding projection. Define $w \in C^{p+q}$ by

$$w(X_1, \cdots, X_p, X_{p+1}, \cdots, X_{p+q})$$
$$= \sum (-1)^{sign(S)} (w'(\bar{X}_{s_1}, \cdots, \bar{X}_{s_p}))(v(X_{t_{p+1}}), \cdots, v(X_{t_{p+q}})),\ X_i \in \mathcal{X}(B),$$

where $\bar{X}_i$ is the projection of X_i in $\mathcal{X}(B)/\mathcal{X}_v(B)$, $S = (s_1, \cdots, s_p, t_{p+1}, \cdots, t_{p+q})$ is a permutation of $(1, \cdots, p, p+1, \cdots, p+q)$ such that $s_i < s_{i+1}$, $t_j < t_{j+1}$. The summation is taken over all such permutations. Obviously, $w \in A^{p,q}$. If $X_i \in \mathcal{X}(B)$, $1 \leqq i \leqq p$, and $X_j \in \mathcal{X}_v(B)$, $p+1 \leqq j \leqq p+q$, then we have

$$w(X_1,\cdots,X_p,X_{p+1},\cdots,X_{p+q})=(w'(\overline{X}_1,\cdots,\overline{X}_p))(X_{p+1},\cdots,X_{p+q}).$$

This show that $\overline{\varphi}^{(p,q)}w=w'$. Hence $C^p(\mathfrak{X}(B)/\mathfrak{X}_v(B), C^q(\mathfrak{X}_v(B),\Gamma))$ is just the image of $\overline{\varphi}^{(p,q)}$.

It is evident that the kernel of $\overline{\varphi}^{(p,q)}$ is $A^{p+1,q-1}$.

Since the term $E_0^{p,q}$ of the spectral sequence is equal to $A^{p,q}/A^{p+1,q-1}$ by definition, we have

Corollary 2.15. *$\overline{\varphi}^{(p,q)}$ induces an $\mathfrak{F}(B)$-isomorphism*

$$\varphi_0^{(p,q)}: E_0^{p,q}\to C^p(\mathfrak{X}(B)/\mathfrak{X}_v(B), C^q(\mathfrak{X}_v(B),\Gamma)).$$

Define

$$d': C^p(\mathfrak{X}(B)/\mathfrak{X}_v(B), C^q(\mathfrak{X}_v(B),\Gamma))\to C^p(\mathfrak{X}(B)/\mathfrak{X}_v(B), C^{q+1}(\mathfrak{X}_v(B),\Gamma))$$

by

$$(d'w)(X_1,\cdots,X_p)=(-1)^p d(w(X_1,\cdots,X_p)).$$

d' is a differential operator in $C(\mathfrak{X}(B)/\mathfrak{X}_v(B), C(\mathfrak{X}_v(B),\Gamma))=\sum_{p,q}C^p(\mathfrak{X}(B)/\mathfrak{X}_v(B), C^q(\mathfrak{X}_v(B),\Gamma))$.

Proposition 2.16. *Let $d_0: E_0^{p,q}\to E_0^{p,q+1}$ be the differential of E_0 in the spectral sequence. Then*

$$\varphi_0^{(p,q+1)}d_0=d'\varphi_0^{(p,q)}.$$

Proof. d_0 is induced by $d: A^{p,q}\to A^{p,q+1}$. Let $w\in A^{p,q}$. Then (2.5.1) gives

(2.5.1)′
$$(\varphi^{(p,q+1)}(dw))(X_1,\cdots,X_p)$$
$$=(d(\varphi^{(p-1,q+1)}w))(X_1,\cdots,X_p)+(-1)^p\, d(\varphi^{(p,q)}w(X_1,\cdots,X_p)).$$

But since $w\in A^{p,q}$, we have $\mathrm{Rest}_\sharp\,(\varphi^{(p-1,q+1)}w)=0$; therefore $\mathrm{Rest}_\sharp\,(d\varphi^{(p-1,q+1)}w)=0$. Applying $\mathrm{Rest}_\sharp$ on both sides of (2.5.1)′ we obtain

$$(\overline{\varphi}^{(p,q+1)}(dw))(X_1,\cdots,X_p)=d'\overline{\varphi}^{(p,q)}w'(X_1,\cdots,X_p).$$

Hence
$$\varphi_0^{(p,q+1)}d=d'\varphi_0^{(p,q)}.$$

Let $l:\mathfrak{X}(M)\to\mathfrak{X}(B)$ the lift map of a connection. Let $\overline{l}:\mathfrak{X}(M)\to\mathfrak{X}(B)/\mathfrak{X}_v(B)$ be defined as the composition of l and the canonical projection. $\overline{l}$ is $\mathfrak{F}(M)$-linear. $\overline{l}$ induces a dual map

$$l^\sharp: C^p(\mathfrak{X}(B)/\mathfrak{X}_v(B), C^q(\mathfrak{X}_v(B),\Gamma))\to C^p(\mathfrak{X}(M), C^q(\mathfrak{X}_v(B),\Gamma)).$$

Here $C^p(\mathfrak{X}(M), C^q(\mathfrak{X}_v(B),\Gamma))$ consists of $\mathfrak{F}(M)$-linear q-alternating functions from $\mathfrak{X}(M)$ to $C^q(\mathfrak{X}_v(B),\Gamma)$, and an $\mathfrak{F}(B)$-module structure is induced in $C^p(\mathfrak{X}(M), C^q(\mathfrak{X}_v(B),\Gamma))$ by that of $C^q(\mathfrak{X}_v(B),\Gamma)$. Then $l^\sharp$ is $\mathfrak{F}(B)$-linear.
We have

(2.5.2) *$l^\sharp$ is an isomorphism.*

In fact, $\mathfrak{X}(B)$ is the direct sum of $\mathfrak{X}_v(B)$ and $\mathfrak{X}_h(B)$, and the canonical projection restricted on $\mathfrak{X}_h(B)$ is an isomorshism onto $\mathfrak{X}(B)/\mathfrak{X}_v(B)$. Therefore we have

$$(2.5.3) \qquad w(X_1, \cdots, X_p)=w(h(X_1), \cdots, h(X_p)),\ X_i \in \mathfrak{X}(B), \quad \text{for } w\in C^p(\mathfrak{X}_v(B), C^q(\mathfrak{X}_v(B), \Gamma)).$$

Suppose that $l^{\#}w=0$, that is, $w(l(X_1),\cdots, l(X_p))=0$, for any $X_i \in \mathfrak{X}(M)$. Since $\mathfrak{X}_h(B)=\mathfrak{F}(B)\cdot l(\mathfrak{X}(M))$ by Proposition 2.2, it follows that $w(X_1,\cdots, X_p)=0$, if X_i are all in $\mathfrak{X}_h(B)$. Hence $w=0$, by (2.5.3) ; $l^{\#}$ is therefore injective.

If $w'\in C^p(\mathfrak{X}(M), C^p(\mathfrak{X}_v(B),\Gamma))$, then an element $w\in C^p(\mathfrak{X}(B)/\mathfrak{X}_v(B), C^q(\mathfrak{X}_v(B),\Gamma))$ such that $l^{\#}w=w'$, if exist, must satisfy

$$w(l(X_1),\cdots, l(X_p))=w'(X_1,\cdots, X_p),\ X_i \in \mathfrak{X}(M).$$

Since $\mathfrak{X}_h(B)=\mathfrak{F}(B)\cdot l(\mathfrak{X}(M))$, w defined on $l(\mathfrak{X}(M))\times\cdots\times l(\mathfrak{X}(M))$ (p-times product) by the above expression extends uniquely on the p-times product of $\mathfrak{F}(B)\cdot l(\mathfrak{X}(M))$ as is easily checked. This proves that $l^{\#}$ is surjective.

If we choose another connection, and if l' is its lift map, then $l(X)$-$l'(X)$ is vertical for $X\in\mathfrak{X}(M)$. It follows that

$$(2.5.4) \qquad l^{\#}=l'^{\#}$$

By (2.5.2) and (2.5.4) *there exists a canonical isomorphism* $C^p(\mathfrak{X}(B)/\mathfrak{X}_v(B), C^q(\mathfrak{X}_v(B), \Gamma))\to C(\mathfrak{X}(M), C^q(\mathfrak{X}_v(B), \Gamma))$ through which *we shall identify boths.*

Then Corollary 2.15 and Proposition 2.16 yield

Proposition 2.17. $\bar{\varphi}^{(p,q)}$ *induces an isomorphism*

$$\varphi_0^{(p,q)}: E_0^{p,q}\to C^p(\mathfrak{X}(M), C^p(\mathfrak{X}_v(B), \Gamma)).$$

Explicitly it is given by

$$(\varphi_0^{(p,q)}c)(X_1,\cdots, X_p)=(\bar{\varphi}^{(p,q)}w)(l(X_1),\cdots, l(X_p)),\ X_i\in\mathfrak{X}(M),$$

where $w\in A^{p,q}$ *is a representative of* c, *and* l *is the lift map of a connection.*

Let d' *be the differential of* $\sum_{p,q}C^p(\mathfrak{X}(M), C^q(\mathfrak{X}_v(B), \Gamma))$ *defined by*

$$(d'w)(X_1,\cdots, X_p)=(-1)^p\ d(w(X_1,\cdots, X_p)),\ X_i\in\mathfrak{X}(M),$$

then

$$\varphi_0^{(p,q+1)}d_0=d'\varphi_0^{(p,q)}.$$

Set

$Z^q=Z^q(\mathfrak{X}_v(B), \Gamma)$=the module of cocycles in $C^q(\mathfrak{X}_v(B), \Gamma)$,

$B^q=B^q(\mathfrak{X}_v(B), \Gamma)$=the module of coboundaries in $C^q(\mathfrak{X}_v(B), \Gamma)$,

so that we have $Z^q/B^q=H^q(\mathfrak{X}_v(B), \Gamma)$. Set also

$Z'^{p,q}=Z'^{p}(\mathfrak{X}(M), C^{q}(\mathfrak{X}'_{v}(B), \Gamma))$=the module of cocycles (with respect to d') in $C^{p}(\mathfrak{X}(M), C^{q}(\mathfrak{X}'_{v}(B), \Gamma))$,

$B'^{p,q}=B'(\mathfrak{X}(M), C^{q}(\mathfrak{X}'_{v}(B), \Gamma))$=the module of coboundaries (with respect to d') in $C^{p}(\mathfrak{X}(M), C^{q}(\mathfrak{X}'_{v}(B)), \Gamma))$,

and set

$H'^{p,q}=Z'^{p,q}/B'^{p,q}$.

Then, by Proposition 2.17, $\bar{\varphi}^{(p,q)}$ *induces an* $\mathfrak{F}(M)$-*isomorphism*

$$\bar{\varphi}_1^{(p,q)}: E_1^{p,q}\rightarrow H'^{p,q}$$

Note that, since d' commutes with the operation of $\mathfrak{F}(M)$, d_0 does so, and $E_1^{p,q}$ and $H'^{p,q}$ are $\mathfrak{F}(M)$-modules.

Since B^{q} and Z^{q} are both $\mathfrak{F}(M)$-submodules of $C^{q}(\mathfrak{X}'_{v}(B), \Gamma)$, we may imbed $C^{p}(\mathfrak{X}(M), B^{q})$ and $C^{p}(\mathfrak{X}(M), Z^{q})$ as submodules of $C^{p}(\mathfrak{X}(M), C^{q}(\mathfrak{X}'_{v}(B), \Gamma))$. Then, it follows from the definition of d' that

$$Z'^{p,q}=C^{p}(\mathfrak{X}(M), Z^{q}),$$
$$B'^{p,q}\subset C^{p}(\mathfrak{X}(M), B^{q}).$$

Since the kernel of the map $\lambda': C^{p}(\mathfrak{X}(M), Z^{q})\rightarrow C^{p}(\mathfrak{X}(M), H^{q}(\mathfrak{X}'_{v}(B), \Gamma))$ induced by the natural projection $Z^{q}\rightarrow H^{q}(\mathfrak{X}'_{v}(B), \Gamma)$ is exactly $C^{p}(\mathfrak{X}(M), B^{q})$, we get a natural $\mathfrak{F}(M)$-linear map

$$\lambda: H'^{p,q}\rightarrow C^{p}(\mathfrak{X}(M), H^{q}(\mathfrak{X}'_{v}(B), \Gamma)).$$

Proposition. 2.18. *Let* $(\mathcal{S}, \pi, B)$ *be an admissible sheaf such that* $\mathfrak{H}(\mathcal{S})$ *is also admissible and set* $\Gamma=\Gamma(\boldsymbol{B}(\mathcal{S}))$ *where* $\boldsymbol{B}(\mathcal{S})$ *is a locally constant vector bundle constructed from* $\mathcal{S}$ *in* §1.
Then λ *is bijective.*

Proof. λ is injective if and only if $B'^{p,q}=C^{p}(\mathfrak{X}(M), B^{q})$, that is, if and only if $d': C^{p}(\mathfrak{X}(M), C^{q-1}(\mathfrak{X}'_{v}(B), \Gamma))\rightarrow C^{p}(\mathfrak{X}(M), B^{q}(\mathfrak{X}'_{v}(B), \Gamma))$ is onto.

λ is onto if and only if $\lambda': C^{p}(\mathfrak{X}(M), Z^{q})\rightarrow C^{p}(\mathfrak{X}(M), H^{q}(\mathfrak{X}'_{v}(B), \Gamma))$ is onto.

We defined a function $\pi_x: C^{q}(\mathfrak{X}'_{v}(B), \Gamma(\boldsymbol{B}(\mathcal{S})))\rightarrow C^{q}(\mathfrak{X}(p^{-1}(x)), \Gamma(\boldsymbol{B}(\mathcal{S}), p^{-1}(x)))$ in §2.4. Then the carrier of $w\in C^{p}(\mathfrak{X}(M), C^{q}(\mathfrak{X}'_{v}(B), \Gamma))$ is by definition the set $\{x\in M;\ \pi_x w\neq 0\}$. If $\{w_i\}$,, $w_i\in C^{p}(\mathfrak{X}(M), C^{q}(\mathfrak{X}'_{v}(B), \Gamma))$, are such that their carriers form a locally finite family, then their sum $\sum w_i$ is a well defined element of $C^{p}(\mathfrak{X}(M), C^{q}(\mathfrak{X}'_{v}(B), \Gamma))$. Moreover if all w_is belong to $C^{p}(\mathfrak{X}(M), Z^{q})$, then $\sum w_i$ belongs to $C^{p}(\mathfrak{X}(M), Z^{q})$, since d (in $C(\mathfrak{X}'_{v}(B), \Gamma)$) is $\mathfrak{F}(M)$-linear.

Let $\{\lambda_i\}$ be a partition of unity belonging to a locally finite covering of M. Then, for $w'\in C^{p}(\mathfrak{X}(M), B^{q})$, the sum $\sum\lambda_i w'$ is well defined and is equal to w', since the values of $\lambda_i w'$ are in B^{q} and summable.

Lel $\{U_i\}$ and $\{V_i\}$ be locally finite coverings of M consisting of local coordinate neighborhoods such that $\bar{V}_i \subset U_i$ and let $\{\lambda_i\}$ belong to $\{V_i\}$.

Let $X_{i,1}, \cdots, X_{i,n}$, $n = \dim M$, be elements of $\mathfrak{X}(M)$ such that $X_{i,1\,x}, \cdots, X_{i,n\,x}$ span T_x (the tangent space at x of M) for $x \in V_i$, and $X_{i,j\,x} = 0$ for $x \notin U_i$.

If $w'(X_{i,j_1}, \cdots, X_{i,j_p}) = (-1)^p\, dc_{i,j_1,\cdots,j_p}$, $c_{i,j_1,\cdots,j_p} \in C^{q-1}(\mathfrak{X}_v(B), \Gamma)$,

set

$$w_i(X_{i,j_1}, \cdots, X_{i,j_p}) = \lambda_i c_{i,j_1,\cdots,j_p}$$

for any subsets $\{j_1, \cdots, j_p\}$ of $\{1, \cdots, n\}$.

Then w_i determines uniquely an element of $C^p(\mathfrak{X}(V_i), C^{q-1}(\mathfrak{X}_v(B), \Gamma))$ which we shall denote also by w_i. Since w_i is zero on the boundary of V_i, it extends to an element w_i' of $C^p(\mathfrak{X}(M), C^{q-1}(\mathfrak{X}_v(B), \Gamma))$ if we define it to be zero outside of V_i, *i.e.*, $\pi_x w_i = 0$ for $x \notin V_i$. It is clear that

$$d'w_i' = \lambda_i w'$$

Then, by the earlier remark, we have

$$d'(\textstyle\sum_i w_i') = \sum d'w_i' = \sum \lambda_i w' = w'.$$

Theaefore $d' : C^p(\mathfrak{X}(M), C^{q-1}(\mathfrak{X}_v(B), \Gamma)) \to C^p(\mathfrak{X}(M), B^q)$ is onto, and λ is injective.

A similar argument proves the surjectivity of λ, using the fact (Prop. 2. 13) that $H^q(\mathfrak{X}_v(B), \Gamma)$ is canonically isomorphic to $\Gamma(\boldsymbol{B}(\mathscr{H}(\mathcal{S})))$.

The composition of $\bar{\varphi}^{(p,q)}$ and λ defines an $\mathcal{F}(M)$-linear map

$$\varphi_1^{(p,q)} : E_1^{p,q} \to C^p(\mathfrak{X}(M), H^q(\mathfrak{X}_v(B), \Gamma)).$$

Let $w \in A^{p,q}$ be such that $dw \in A^{p+1,q}$. Then,

$$\bar{\varphi}^{(p,q)} w(l(X_1), \cdots, l(X_p)) \in Z^q, \text{ for any } X_1, \cdots, X_p \in \mathfrak{X}(M),$$

as can be seen from (2.5.1), where l is a lift map.

Thus, we have

Proposition 2.19. *$\bar{\varphi}^{(p,q)}$ induces an $\mathcal{F}(M)$-homomorphism*

$$\varphi_1^{(p,q)} : E_1^{p,q} \to C^p(\mathfrak{X}(M), H^q(\mathfrak{X}_v(B), \Gamma))$$

which is bijective if $\Gamma = \Gamma(\boldsymbol{B}(\mathcal{S}))$ for some admissible sheaf $\mathcal{S}$ such that $\mathscr{H}(\mathcal{S})$ is also admissible.

Let $c \in E_1^{p,q}$ and let $w \in A^{p,q}$ respresent c. Then,

$(\varphi_1^{(p,q)} c)(X_1, \cdots, X_p)$ = cohomology class (relative to d) of $\bar{\varphi}^{(p,q)} w(l(X_1), \cdots, l(X_p))$, where l is the lift map of a connection.

(dw must belong to $A^{p+1,q}$ by the very definition of $E_1^{p,q}$)

$\mathcal{X}(M)$ operates on $H^q(\mathcal{X}_v(B), \Gamma)$ by Proposition 2.7, so that the differential d is defined in $\sum_p' C^p(\mathcal{X}(M), H^q(\mathcal{X}_v(B), \Gamma))$.

Proposition 2.20. *Let d_1 be the differential in the term E_1 of the specrtal sequence. Then, we have*

$$\varphi_1^{(p+1,q)} d_1 = d\varphi_1^{(p,q)}$$

Proof. Let $c \in E_1^{p,q}$ and let $w \in A^{p,q}$ represent c, $dw \in A^{p+1,q}$. Then $d_1 c$ is represented by dw. By Proposition 2.19,

$$(2.5.5) \quad \begin{aligned} &\varphi_1^{(p,q)} c(X_1, \cdots, X_p) = \text{class of } \bar{\varphi}^{(p,q)} w(l(X_1), \cdots, l(X_p)), \\ &\varphi_1^{(p+1,q)} d_1 c(X_1, \cdots, X_{p+1}) = \text{class of } \bar{\varphi}^{(p+1,q)} dw(l(X_1), \cdots, l(X_{p+1})). \end{aligned}$$

But by (2.5.1),

$$\begin{aligned} &\varphi^{(p+1,q)} dw(l(X_1), \cdots, l(X_{p+1})) \\ &= (d\varphi^{(p,q)} w)(l(X_1), \cdots, l(X_{p+1})) + (-1)^{p+1}\ d(\varphi^{(p+1,q-1)} w(l(X_1), \cdots, l(X_{p+1})). \end{aligned}$$

Therefore

$$(2.5.6) \quad \begin{aligned} &\text{class of } (\bar{\varphi}^{(p+1,q} dw(l(X_1), \cdots, l(X_{p+1})) \\ &= \text{class of } (\mathrm{Rest}_\sharp (d\varphi^{(p,q)} w(l(X_1), \cdots, l(X_{p+1}))). \end{aligned}$$

Since $w \in A^{p,q}$ and $l([X_i, X_j]) \equiv [l(X_i), l(X_j)] \bmod \mathcal{X}_v(B)$, we have

$$\begin{aligned} d\varphi^{(p,q)} w(l(X_1), \cdots, l(X_{p+1})) &= \textstyle\sum_i (-1)^{i+1} l(X_i)(\varphi^{(p,q)} w(l(X_1), \cdots, l(\hat{X}_i), \cdots, l(X_{p+1})) \\ &+ \textstyle\sum_{i<j} (-1)^{i+j} \varphi^{(p,q)} w(l[(X_i, X_j]), l(X_1), \cdots, l(\hat{X}_i), \cdots, l(\hat{X}_j), \cdots, l(X_{p+1})). \end{aligned}$$

Hence

$$\begin{aligned} &\text{class of } (\mathrm{Rest}_\sharp (d\varphi^{(p,q)} w)(l(X_1), \cdots, l(X_{p+1}))) \\ &= \textstyle\sum_i (-1)^{i+1} X_i (\text{class of } \bar{\varphi}^{(p,q)} w(l(X_1), \cdots, l(\hat{X}_i), \cdots, l(X_{p+1}))) \\ &\quad + \textstyle\sum_{i<j} (-1)^{i+j} (\text{class of } \bar{\varphi}^{(p,q)} w(l([X_i, X_j]), l(X_1), \cdots, l(\hat{X}_i), \cdots, l(\hat{X}_j), \cdots\ l(X_{p+1}))). \end{aligned}$$

Comparing this with (2.5.5) and (2.5.6), we get

$$(d\varphi_1^{(p,q)} c)(X_1, \cdots, X_{p+1}) = (\varphi_1^{(p+1,q)} d_1 c)(X_1, \cdots, X_{p+1}).$$

Corollary 2.21. *$\bar{\varphi}^{(p,q)}$ induces a homomorphism*

$$\varphi_2^{(p,q)} : E_2^{p,q} \to H^p(\mathcal{X}(M), H^q(\mathcal{X}_v(B), \Gamma)).$$

Moreover $\varphi_2^{(p,q)}$ is bijective if $\Gamma = \Gamma(\boldsymbol{B}(\mathcal{S}))$ for some admissible sheaf such that $\mathcal{H}(\mathcal{S})$ is also admissible. In this case, we have

$$H^p(\mathcal{X}(M), H^q(\mathcal{X}_v(B), \Gamma)) \cong H^p(\mathcal{X}(M), \Gamma(\mathcal{H}^q(\mathcal{S}))).$$

Remark. Since $d(A^p)\subset A^p$ the cohomology group $H(A^p)=\sum_q H(A^{p,q})$ is defined. Let $H^{p,q}\subset H^{p+q}(\mathfrak{X}(B),\Gamma)$ be the image of $H(A^{p,q})$ by the natural homomorphism induced by the injection $A^{p,q}\to C^{p+q}(\mathfrak{X}(B),\Gamma)$, and let $H^p=\sum_q H^{p,q}$. Then H^p defines a filtration of $H^*(\mathfrak{X}(B),\Gamma)$. By the standard theory of spectral sequences, E^∞ is the graded module associated to this filtration:

$$E_\infty^{p,q}=H^{p,q}/H^{p+1,q-1}.$$

6. *Ring structure.*

Let M be a C^∞-manifold.

Let Γ be an $\mathfrak{F}(M)$-module satisfying the following conditions:

(a) $\Gamma=\sum_{q\geq 0}\Gamma^q$ (direct sum of $\mathfrak{F}(M)$-modules Γ^q).

(b) Γ is an $\mathfrak{F}(M)$-algebra.

(c) $\Gamma^p\cdot\Gamma^q\subset\Gamma^{p+q}$.

We call such an $\mathfrak{F}(M)$-module a graded $\mathfrak{F}(M)$-*algebra*.

Let $\mathfrak{X}'$ be a Lie subalgebra of $\mathfrak{X}(M)$ which is an $\mathfrak{F}(M)$-submodule of $\mathfrak{X}(M)$. We shall say that $\mathfrak{X}'$ *operates on* Γ *if* $\mathfrak{X}'$ *operates on each* Γ^q *and if it satisfies the following condition*:

$$X\cdot(c\cdot c')=(X\cdot c)\cdot c'+c\cdot(X\cdot c'),\quad X\in\mathfrak{X}',\ c,\ c'\in\Gamma.$$

Let Γ and $\mathfrak{X}'$ be as above. We define the structure of (*bi-*) graded $\mathfrak{F}(M)$-algebra on $C(\mathfrak{X}',\Gamma)$ by

$$\begin{aligned} &w\cdot w'(X_1,\cdots,X_{p+q})\\ &=\sum(-1)^{\mathrm{sign}(S)}w(X_{s_1},\cdots,X_{s_p})\cdot w'(X_{t_{p+1}},\cdots,X_{t_{p+q}}),\end{aligned}$$

for $w\in C^p(\mathfrak{X}',\Gamma)$ and $w'\in C^q(\mathfrak{X}',\Gamma)$, where $S=(s_1,\cdots,s_p,t_{p+1},\cdots,t_{p+q})$ is a permutation of $(1,\cdots,p+q)$ with $s_1<\cdots<s_p$, $t_{p+1}<\cdots<t_{p+q}$, and the summation is extended over all such permutations.

Following series of statements can be easily verified.

(2.6.1) $\quad i(X)(w\cdot w')=(i(X)w)\cdot w'+(-1)^p w\cdot i(X)w',$
for $w\in C^p(\mathfrak{X}',\Gamma)$ and $w'\in C(\mathfrak{X}',\Gamma)$, $X\in\mathfrak{X}'$.

(2.6.2) $\quad \theta(X)(w\cdot w')=(\theta(X)w)\cdot w'+w\cdot(\theta(X)w'),$
for $w,\ w'\in C(\mathfrak{X}')$, $X\in\mathfrak{X}'$.

(2.6.3) $\quad d(w\cdot w')=dw\cdot w'+(-1)^p w\cdot dw',$
for $w\in C^p(\mathfrak{X}',\Gamma)$ and $w\in C(\mathfrak{X}',\Gamma)$.

In particular $\mathfrak{X}'$ *operates on* $C(\mathfrak{X}',\Gamma)$ (*with respect to each of two graded structures*: $C(\mathfrak{X}',\Gamma)=\sum_{p\geq 0}C_p(\mathfrak{X}',\Gamma)$ and $C(\mathfrak{X}',\Gamma)=\sum_{q\geq 0}C(\mathfrak{X}',\Gamma^q)$) by $(X,w)\to\theta(X)\cdot w$.

$H^*(\mathfrak{X}', \Gamma)$ *is a (bi-) graded algebra* over R by the multiplication:
(class of w)$\cdot$(class of w')$=$class of $w\cdot w'$

Let (B, p, M, F, G) be a C^∞-fibre bundle, and Γ a graded $\mathfrak{F}(B)$-algebra on which $\mathfrak{X}(B)$ operates. Let $A^{p,q}$ be defined as in §2.5. Then, from (2.6.1), it follows that

$$A^{p,q}\cdot A^{p',q'}\subset A^{p+p',q+q'}. \tag{2.6.4}$$

From this and from (2.6.3), it folows that E_r *is an algebra which has natural three structures of graduation* (the first two are that of the spectral sequence and the third one is induced by that of Γ); E_0 is an $\mathfrak{F}(B)$-algebra and E_1 is an $\mathfrak{F}(B)$-algebra and E_1 is an $\mathfrak{F}(M)$-algebra;

$$E_r^{p,q}\cdot E_r^{p',q'}\subset E_r^{p+q',q+q'} \tag{2.6.5}$$

Moreover, we have

$$d_r(a\cdot b)=d_r a\cdot b+(-1)^{p+q}a\cdot d_r b \tag{2.6.6}$$
$$\text{for } a\in E_r^{p,q} \text{ and } b\in E_r.$$

On the other hand, $C(\mathfrak{X}_v(B), \Gamma)$ is a (bi)-graded $\mathfrak{F}(B)$-algebra. $H^*(\mathfrak{X}_v(B), \Gamma)$ is a (bi)-graded $\mathfrak{F}(M)$-algebra and $\mathfrak{X}(M)$ *operates* on $H^*(\mathfrak{X}_v(B), \Gamma)$. Then, $\sum_{p,q}'E_0^{p,q}=\sum_{p,q}C^p(\mathfrak{X}(M), C^q(\mathfrak{X}_v(B), \Gamma))$, $\sum_{p,q}'E_1^{p,q}=\sum_{p,q}C^p(\mathfrak{X}(M), H^q(\mathfrak{X}_v(B), \Gamma))$ and $\sum_{p,q}'E_2^{p,q}=\sum_{p,q}H^p(\mathfrak{X}(M), H^q(\mathfrak{X}_v, \Gamma))$ are $(bi$-$)$ graded algebras (they have one more structure of graduation induced by that of Γ).

The maps $\varphi_\nu^{(p,q)}: E_\nu^{p,q}\to'E_\nu^{p,q}$ $(\nu=0, 1, 2)$ defined in §2.5 preserve the last graduation and satisfy the following[6]

$$\varphi_\nu^{(p+p',q+q')}(a\cdot b)=(-1)^{p'q}(\varphi_\nu^{(p,q)}a)\cdot(\varphi_\nu^{(p',q')}b) \tag{2.6.7}$$
$$\text{for } a\in E_\nu^{p,q} \text{ and } b\in E_\nu^{p',q'}.$$

Let $(\mathcal{S}, \pi, B)$ be an admissible sheaf which is, at the same time, a sheaf of graded algebra over R, *Then* $\Gamma(\boldsymbol{B}(\mathcal{S}))$ *is a graded* $\mathfrak{F}(B)$- *algebra on which* $\mathfrak{X}(B)$ *operates*. Therefore $H^*(\mathfrak{X}(p^{-1}(x)), \Gamma(\boldsymbol{B}(\mathcal{S}), p^{-1}(x)))$ *is a (bi)-graded R-algebra*. If $\mathcal{H}(\mathcal{S})$ is admissible, then $\Gamma(\boldsymbol{B}(\mathcal{H}(\mathcal{S})))$ is a (bi)-graded $\mathfrak{F}(M)$-algebra. In this case the map $\rho: H^*(\mathfrak{X}_v(B), \Gamma(\boldsymbol{B}(\mathcal{S})))\to\Gamma(\boldsymbol{B}(\mathcal{H}(\mathcal{S})))$ is an isomorphism of graded $\mathfrak{F}(M)$-algebra.

Chapter II.

§3. De Rham cohomology of homogeneous spaces.

Let G be a connected, simply connected Lie group. Let D be a discrete subgroup of G. The right coset space $B=G/D$ of G by D is a C^∞-manifold.

Define a map

$$h: G\times B\to B$$

by

$$h(g,g'D)=gg'D, \text{ for } g\in G,\ g'D\in B.$$

h is a C^∞-map satisfying the following conditions:

(3.1) $\qquad h(1,x)=x$, for all $x\in B$.

(3.2) $\qquad h(g_1, h(g_2, x))=h(g_1g_2, x)$, for all $g_1, g_2\in G$ and for all $x\in B$.

In the usual terminology, h defines a C^∞-operation of G on B. We shall write $g\cdot x$ for $h(g\cdot x)$.

Proposition 3.1. *Let $(\mathcal{S}, \pi, B)$ be an admissible sheaf on $B=G/D$. Then there exists a unique continuous map*

$$\tilde{h}: G\times \mathcal{S}\to\mathcal{S}$$

satisfying the following conditions:

a) $\pi\tilde{h}(g, x)=h(g, \pi(x))$, *for* $g\in G$, $x\in\mathcal{S}$.

b) *For fixed* $g\in G$, *the map* $x\to h(g, x)$, $x\in\mathcal{S}$, is an isomorphism of sheaf.

c) $\tilde{h}(1, x)=x$, *for all* $x\in\mathcal{S}$,

and

$$\tilde{h}(g_1, \tilde{h}(g_2, x))=\tilde{h}(g_2g_2, x), \textit{ for all } g_1, g_2\in G \text{ and } x\in\mathcal{S}.$$

Proof. Let U be an open neighborhood of 1 in G which is diffeomorphic to a Euclidean space and which meets D only at 1. Let V be a neighborhood of 1 such that $V^2\subset U$. For any $x\in B$, $U\cdot x=\{g\cdot x\,;\, g\in U\}$ is a neighborhood of x diffeomorphic to a Euclidean space. Therefore $\mathcal{S}$ is constant on $U\cdot x$; we have an isomorphism of sheaf.

$$\varphi_x: U\cdot x\times R^n\to\pi^{-1}(U\cdot x).$$

Suppose that a continuous map $\tilde{h}: G\times\mathcal{S}\to\mathcal{S}$ exists which satisfies the conditions *a*), *b*) and *c*). Consider the following diagram:

$$\begin{array}{ccc} V\times V\cdot x\times R^n & \xrightarrow{id.\times\varphi_x} & V\times\pi^{-1}(V\cdot x) \\ \bar{h}\downarrow & & \downarrow\tilde{h}_x \\ U\cdot x\times R^n & \xrightarrow{\varphi_x} & \pi^{-1}(U\cdot x), \end{array}$$

where $\tilde{h}_x$ is the restriction of $\tilde{h}$, and $\bar{h}$ is defined by

$$\bar{h}(g, g'\cdot x, c)=(g\,g'\cdot x, c),\ \ g, g'\in V,\ c\in R^n.$$

In virtue of conditions a), b) and c), we must have

$$\varphi_x^{-1}\circ\tilde{h}_x\circ(id.\times\varphi_x)(g, g'\cdot x, c)=(g\,g'\cdot x, k(g, g')c),$$

where $k(g, g')$ is an element of $GL(n, R)$. The continuity of $\tilde{h}$ implies that the function $V\times V\to GL(n, R)$ defined by

$$(g, g')\to k(g, g')$$

is continuous with respect to the discrete topology of $GL(n, R)$. Therefore $k(g, g')$ is a constant. Since $k(1, g')=1$, we must have $k(g, g')=1$; it follows that

$$\tilde{h}_x=\varphi_x\circ\bar{h}\circ(id\times\varphi_x)^{-1}, \tag{3.3}$$

i.e. that the above diagram commutes.

Conversely the map $\tilde{h}_x$: $V\times\pi^{-1}(V\cdot x)\to\pi^{-1}(U\cdot x)$ defined by (3.3) is independent of the choice of local product expression φ_x. Moreover if $V\cdot x\frown V\cdot y\neq\phi$, then $\tilde{h}_x$ and $\tilde{h}_y$ are identical on $V\times\pi^{-1}(V\cdot x\frown V\cdot y)$. Therefore we may define a continuous map

$$\tilde{h}\ :\ V\times\mathcal{S}\to\mathcal{S}$$

by

$$\tilde{h}(g, x^*)=\tilde{h}_x(g, x^*),\ \text{foy}\ x^*\in\pi^{-1}(V\cdot x).$$

Thus we have shown the uniqueness and the existence of continuous map

$$\tilde{h}\ :\ V\times\mathcal{S}\to\mathcal{S}$$

which satisfy the conditions a), b) together with:

c)′ $\tilde{h}(g_1, \tilde{h}(g_2, x))=\tilde{h}(g_1g_2, x)$, whenever $g_1, g_2\in V$ and $g_1g_2\in V$.

Since the group G is simply connected, the existence and the uniqueness of continuous map $\tilde{h}: G\times\mathcal{S}\to\mathcal{S}$ follow from the monodromy theorem.

Let $\bar{1}\in G/D$ denote the coset of D and set $\gamma=\pi^{-1}(\bar{1})$. Since $\tilde{h}$ maps $D\times\gamma$ on γ, we have a linear representation $\Lambda_{\mathcal{S}}: D\to GL(\gamma)$; $\Lambda_{\mathcal{S}}(d)\cdot x=\tilde{h}(d, x)$, $d\in D, x\in\gamma$. The natural projection p: $G\to G/D$ defines a principal bundle over G/D with group D which we denote by $c\in H^1(G/D, \boldsymbol{D}_{lc})$. Let $\Lambda_{\mathcal{S}*}$: $H^1(G/D, \boldsymbol{D}_{lc})\to H^1(G/D, GL(n, R)_{lc})$ be the map induced by $\Lambda_{\mathcal{S}}$ (we have identified γ with R^n). Proposition 3.1 implies that $\Lambda_{\mathcal{S}*}(c)=c(\mathcal{S})$. Moreover if $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic then $\Lambda_{\mathcal{S}}$ and $\Lambda_{\mathcal{S}'}$ are equivalent, *i.e.*, there exists an element $g\in GL(n, R)$ such that $\Lambda_{\mathcal{S}'}=g\Lambda_{\mathcal{S}}g^{-1}$. Conversely a linear representation $\Lambda: D\to GL(n, R)$ determines a unique admissible sheaf $\mathcal{S}$ such that $\Lambda_*(c)=c(\mathcal{S})$ and $\Lambda_{\mathcal{S}}=\Lambda$. To equivalent representations correspond isomorphic sheaves. Thus

Proposition 3.2. *The map $A \to A_*(c)$ establishes a 1-1 correspondence between the classes of linear representations of D and the isomorphism classes of admissible sheaves over G/D.*

If we consider the map h as a function from $G \times \boldsymbol{B}(\mathcal{S})$ into $\boldsymbol{B}(\mathcal{S})$, then h is differentiable. We shall write $g \cdot x^*$ for $h(g, x^*)$. Let $c \in \Gamma(\boldsymbol{B}(\mathcal{S}))$ and $g \in G$. Define an element $g \cdot c$ of $\Gamma(\boldsymbol{B}(\mathcal{S}))$ by

$$(g \cdot c)(x) = g \cdot (c(g^{-1} \cdot x)), \quad x \in B. \tag{3.4}$$

By the map $G \times \Gamma(\boldsymbol{B}(\mathcal{S})) \to \Gamma(\boldsymbol{B}(\mathcal{S}))$ defined by $(g, c) \to g \cdot c$, G operates on $\Gamma(\boldsymbol{B}(\mathcal{S}))$.

Since D is a discrete subgroup of G, $\mathfrak{X}(B)$ is imbedded in $\mathfrak{X}(G)$ as the set of vector fields on G invariant by the right translations of elements of D. Then, the Lie algebra $\mathfrak{g}$ of G is imbedded in $\mathfrak{X}(B)$ as the set of right invariant vector fields on G. Moreover the $\mathfrak{F}(B)$-linear map

$$\rho : \mathfrak{F}(B) \otimes \mathfrak{g} \to \mathfrak{X}(B)$$

defined by

$$\rho(f \otimes X) = fX, \quad f \in \mathfrak{F}(B), \quad g \in \mathfrak{g} \subset \mathfrak{X}(B),$$

is bijective.

Note that, if $f \in \mathfrak{F}(B)$ and $X \in \mathfrak{g}$, then we have

$$(X \cdot f)(x) = \lim_{t \to 0} \frac{1}{t} \{ f((\exp tX) \cdot x) - f(x) \}.$$

for all $x \in B$. It follows easily that, for $c \in \Gamma(\boldsymbol{B}(\mathcal{S}))$,

$$(X \cdot c)(x) = \lim_{t \to 0} \frac{1}{t} \{ ((\exp tX)^{-1} \cdot c)(x) - c(x) \}. \tag{3.4$'$}$$

Now let $\mathfrak{g}$ be a finite dimensional Lie algebra over R, and let Γ be a $\mathfrak{g}$-module [4]. We shall denote the cochain complex of $\mathfrak{g}$ with coefficients in Γ by $C(\mathfrak{g}, \Gamma) = \sum_{q \geq 0} C^q(\mathfrak{g}, \Gamma)$, where $C^q(\mathfrak{g}, \Gamma)$ is the module of q-linear alternating functions from $\mathfrak{g}$ to Γ. In $C(\mathfrak{g}, \Gamma)$ the operators $\theta(X)$, $i(X)$ and d are defined ($X \in \mathfrak{g}$). Their defining formulae are the same as those in Definition 2.5. The cohomology module of $C(\mathfrak{g}, \Gamma)$ is denoted by $H^*(\mathfrak{g}, \Gamma) = \sum_{q \geq 0} H^q(\mathfrak{g}, \Gamma)$.

Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$, and let B be the coset space of G by a discrete subgroup of G. If Γ is an $\mathfrak{F}(B)$-module on which $\mathfrak{X}(B)$ operates, then Γ is a $\mathfrak{g}$-module since $\mathfrak{g} \subset \mathfrak{X}(B)$, and we have a homomorphism (by restricting arguments to $\mathfrak{g}$)

$$C(\mathfrak{X}(B), \Gamma) \to C(\mathfrak{g}, \Gamma),$$

which is bijective because $\rho : \mathfrak{F}(B) \otimes \mathfrak{g} \to \mathfrak{X}(B)$ is bijective[7)]. The inverse homomorphism

$$\alpha_{\#} : C(\mathfrak{g}, \Gamma) \rightarrow C(\mathfrak{X}(B), \Gamma)$$

is given by

$$(\alpha_{\#} w)(f_1 X_1, \cdots, f_q X_q) = f_1 \cdots f_q w(X_1, \cdots X_q), f_i \in \mathfrak{F}(B), X_i \in \mathfrak{g}, \text{ for } w \in C^q(\mathfrak{g}, \Gamma).$$

More generally, if γ is a $\mathfrak{g}$-module and if $\alpha : \gamma \rightarrow \Gamma$ is a $\mathfrak{g}$-homomorphism, then α induces a homomorphism $\alpha_{\#}$: $C(\mathfrak{g}, \gamma) \rightarrow C(\mathfrak{X}(B), \Gamma)$ which is defined by

$$(\alpha_{\#} w)(f_1 X_1, \cdots, f_q X_q) = f_1 \cdots f_q (\alpha(X_1, \cdots, X_q)), (f_i \in \mathfrak{F}(B), X_j \in \mathfrak{X}(B)),$$

for $w \in C^q(\mathfrak{g}, \gamma)$. $\alpha_{\#}$ commutes with d; it induces a homomorphism

$$\alpha_{*}: H^*(\mathfrak{g}, \gamma) \rightarrow H^*(\mathfrak{X}(B), \Gamma).$$

Now let G and D be as above, and let K be a connected, simply connected, closed normal Lie subgroup of G such that KD is *closed* in G. The projection $G/D \rightarrow G/K \cdot KD/K$ defines a C^∞-fibre bundle whose fibre is $KD/D = K/K \frown D$. Set $B = G/D$, $M = G/K \cdot KD/K$ and $F = KD/D$. The imbedding $\mathfrak{g} \subset \mathfrak{X}(B)$ carries the Lie algebra $\mathfrak{k}$ of K (which is an ideal of $\mathfrak{g}$) into $\mathfrak{X}_v(B)$, and the map $\rho : \mathfrak{F}(B) \otimes \mathfrak{k} \rightarrow \mathfrak{X}_v(B)$ is bijective. Similarly we have an imbedding $\mathfrak{g}/\mathfrak{k} \subset \mathfrak{X}(M)$ such that $\rho : \mathfrak{F}(M) \otimes \mathfrak{g}/\mathfrak{k} \rightarrow \mathfrak{X}(M)$ is bijective.

In this situation, $\alpha : \gamma \rightarrow \Gamma$ induces further

$$\alpha'_{\#} : C(\mathfrak{k}, \gamma) \rightarrow C(\mathfrak{X}_v(B), \Gamma),$$

and hence

$$\alpha'_{*}: H^*(\mathfrak{k}, \gamma) \rightarrow H^*(\mathfrak{X}_v(B), \Gamma).$$

Let $\bar{X} \in \mathfrak{g}/\mathfrak{k}$ and $\bar{c} \in H^q(\mathfrak{k}, \gamma)$ be reprsented by $X \in \mathfrak{g}$ and $c \in Z^q(\mathfrak{k}, \gamma)$ respectively. Then the element $X \cdot c \in Z^q(\mathfrak{k}, \gamma)$ will be defined by

$$(3.5) \qquad X \cdot c(X_1, \cdots, X_q) = X(c(X_1, \cdots, X_q)) - \sum_i c(X_1, \cdots, [X, X_i], \cdots, X_q), X_i \in \mathfrak{k}.$$

The cohomology class of $X \cdot c$ depends only on $\bar{X}$ and $\bar{c}$ and is denoted by $\bar{X} \cdot \bar{c}$. $H^*(\mathfrak{k}, \gamma)$ is a $\mathfrak{g}/\mathfrak{k}$-module by the operation $(X, c) \rightarrow X \cdot c$ (cf. [4]).

On the other hand, $\mathfrak{X}(M)$ operates on $H^*(\mathfrak{X}_v(B), \Gamma)$. If $\bar{c} \in H^q(\mathfrak{X}_v(B), \Gamma)$ is represented by $c \in Z^q(\mathfrak{X}_v(B), \Gamma)$ and if $\bar{X} \in \mathfrak{X}(M)$, then $\bar{X} \cdot \bar{c}$ is represented by $\bar{X} \cdot c$ which is defined by

$$(3.6) \quad \bar{X} \cdot c(X_1, \cdots, X_q) = l(\bar{X})(c(X_1, \cdots, X_q) - \sum_i c(X_1, \cdots, [l(\bar{X}), X_i], \cdots, X_q), \quad X_i \in \mathfrak{X}_v(B),$$

where $l : \mathfrak{X}(M) \rightarrow \mathfrak{X}(B)$ is a lift map (cf. §2.2).

If $\bar{X} \in \mathfrak{g}/\mathfrak{k}$, then it is easily seen that $l(\bar{X})$ belongs to $\mathfrak{g}$ and the projection of $l(\bar{X})$ is $\bar{X}$. Hence, comparing (3.5) and (3.6), we have

(3.7) $\qquad \alpha'_*: H^*(\mathfrak{k}, \gamma) \to H_v(B), \Gamma)$ *is a* $\mathfrak{g}/\mathfrak{k}$*-homomorphism.*

Finally, α'_* induces

$$\alpha_{*1}: C(\mathfrak{g}/\mathfrak{k}, H^*(\mathfrak{k}, \gamma)) \to C(\mathcal{X}(M), H^*(\mathcal{X}_v(B), \Gamma))$$

and

$$\alpha_{*2}: H^*(\mathfrak{g}/\mathfrak{k}, H^*(\mathfrak{k}, \gamma)) \to H^*(\mathcal{X}(M), H^*(\mathcal{X}_v(B), \Gamma)).$$

In $C(\mathfrak{g}, \gamma)$ we shall define a filtration $\{A^p(\mathfrak{g})\}$ as follows:

$$A^{p,q}(\mathfrak{g}) = \{w\,;\, w \in C^{p+q}(\mathfrak{g}, \gamma),\ i(K_1)\cdots i(X_{q+1})w = 0,\ \text{for any } X_1, \cdots, X_{q+1} \in \mathfrak{k}\}$$
$$A^p(\mathfrak{g}) = \textstyle\sum_q A^{p,q}(\mathfrak{g}).$$

We shall denote the corresponding spectral sequence by $E_r(\mathfrak{g})$.

Let $C^p(\mathfrak{g}/\mathfrak{k}, C^p(\mathfrak{k}, \gamma))$ be the module of p-linear alternating functions from $\mathfrak{g}/\mathfrak{k}$ to $C^q(\mathfrak{k}, \gamma)$. Define a linear map

$$\varphi^{(p,q)}: A^{p,q}(\mathfrak{g}) \to C^p(\mathfrak{g}/\mathfrak{k}, C^q(\mathfrak{k}, \gamma))$$

by

$$\varphi^{(p,q)} w(\overline{X}, \cdots, \overline{X}_p))(Y_1, \cdots Y_q) = w(X_1, \cdots, X_p, Y_1, \cdots, Y_q),$$

for $w \in A^{p,q}(\mathfrak{g})$, $\overline{X}_i \in \mathfrak{g}/\mathfrak{k}$, $Y_i \in \mathfrak{k}$, where X_i is a representative of $\overline{X}_i$ in $\mathfrak{g}$. Then we know [4] that

(3.8) $\qquad \varphi^{(p,q)}$ *induces isomorphisms*

$$\varphi_0^{(p,q)}: E_0^{p,q}(\mathfrak{g}) \to C^p(\mathfrak{g}/\mathfrak{k}, C^q(\mathfrak{k}, \gamma)),$$
$$\varphi_1^{(p,q)}: E_1^{p,q}(\mathfrak{g}) \to C^p(\mathfrak{g}/\mathfrak{k}, H^q(\mathfrak{k}, \gamma)),$$
$$\varphi_2^{(p,q)}: E_2^{p,q}(\mathfrak{g}) \to H^p(\mathfrak{g}/\mathfrak{k}, H^q(\mathfrak{k}, \gamma)).$$

Let $\{A^{p,q}\}$ be the filtration of $C(\mathcal{X}(B), \Gamma)$ and $\{E_r\}$ its spectral sequence introduced in §2.5. Γ and γ being as above, $\alpha_\#: C(\mathfrak{g}, \gamma) \to C(\mathcal{X}(B), \Gamma)$ preserves the filtration:

$$\alpha_\#(A^{p,q}(\mathfrak{g})) \subset A^{p,q}.$$

Hence it induces homomorphisms

$$\alpha_r: E_r(\mathfrak{g}) \to E_r.$$

Proposition 3.3. *The following diagrams are commutative*:

$$\begin{array}{ccc}
E_0^{p,q}(\mathfrak{g}) & \xrightarrow{\alpha_0} & E_0^{p,q} \\
\varphi_0^{(p,q)}\Big\downarrow & & \Big\downarrow\varphi_0^{(p,q)} \\
C^p(\mathfrak{g}/\mathfrak{k}, C^q(\mathfrak{k}, \gamma)) & \xrightarrow{\alpha_{\#0}} & C^p(\mathcal{X}(M), C^q(\mathcal{X}_v(B), \Gamma))
\end{array}$$

where the second vertical map $\varphi_0^{(p,q)}$ *is that of* §2.5 *and* $\alpha_{\sharp 0}$ *is* induced by $\alpha'_{\sharp}: C^q(\mathfrak{k}, \gamma) \to C^q(\mathfrak{X}_v(B), \Gamma)$;

$$\begin{array}{ccc} E_1^{p,q}(\mathfrak{g}) & \xrightarrow{\alpha_1} & E_1^{p,q} \\ \varphi_1^{(p,q)}\Big\downarrow & & \Big\downarrow\varphi_1^{(p,q)} \\ C^p(\mathfrak{g}/\mathfrak{k}, H^q(\mathfrak{k}, \gamma)) & \xrightarrow{\alpha_{*1}} & C^q(\mathfrak{X}(M), H^q(\mathfrak{X}_v(B), \Gamma)); \end{array}$$

$$\begin{array}{ccc} E_2^{p,q}(\mathfrak{g}) & \xrightarrow{\alpha_2} & E_2^{p,q} \\ \varphi_2^{(p,q)}\Big\downarrow & & \Big\downarrow\varphi_2^{(p,q)} \\ H^p(\mathfrak{g}/\mathfrak{k}, H^q(\mathfrak{k}, \gamma)) & \xrightarrow{\alpha_{*2}} & H^p(\mathfrak{X}(M), H^q(\mathfrak{X}_v(B), \Gamma)). \end{array}$$

(The proof is imediate and is omitted).

Remark. If γ is a graded algebra, $\gamma = \sum_{q \geq 0} \gamma^q$ such that, for $X \in \mathfrak{g}$ and $c, c' \in \gamma$ we have

$$X(c \cdot c') = (X \cdot c) \cdot c' + c \cdot (X \cdot c'),$$

then we shall call γ a *graded* $\mathfrak{g}$*-algebra*. If γ is a graded $\mathfrak{g}$-algebra, then E_r, $\sum_{p,q} C^p(\mathfrak{g}/\mathfrak{k}, C^q(\mathfrak{k}, \gamma))$, $\sum_{p,q} C^p(\mathfrak{g}/\mathfrak{k}, H^q(\mathfrak{k}, \gamma))$ and $\sum_{p,q} H^p(\mathfrak{g}/\mathfrak{k}, H^q(\mathfrak{k}, \gamma))$ are algebras with three structures of graduation. If Γ is a graded $\mathfrak{F}(B)$-algebra on which $\mathfrak{X}(B)$ operates, and if $\alpha: \gamma \to \Gamma$ is a $\mathfrak{g}$-homomorphism preserving the structure of graduation, then it follows easily that $\alpha_0, \alpha_1, \alpha_2$ and $\alpha_{\sharp 0}, \alpha_{*1}, \alpha_{*2}$ are homomorphisms of graded algebras.

Let again $B = G/D$ where G is a simply connected Lie group with Lie algebra $\mathfrak{g}$ and D is a discrete subgroup of G. Let Γ be an $\mathfrak{F}(B)$-module on which $\mathfrak{X}(B)$ operates and let γ be a $\mathfrak{g}$-module. We are particularly interested in the case where

(3.9) γ *is finite dimensional*, and

(3.10) *the map* $\bar{\alpha}: \mathfrak{F}(B) \otimes \gamma \to \Gamma$ *defined by* $\bar{\alpha}(f \otimes c) = f\alpha(c)$,

is bijective.

If we define the operation of $\mathfrak{X}(B)$ on $\mathfrak{F}(B) \otimes \gamma$ by

(3.11) $(fX)(f' \otimes c) = f(Xf' \otimes c + f' \otimes Xc), \ f, f' \in \mathfrak{F}(B), \ X \in \mathfrak{g}, \ c \in \gamma,$

then $\bar{\alpha}$ is an $\mathfrak{F}(B)$-homomorphism commuting with the operation of $\mathfrak{X}(B)$. Obviously, when γ is given, such a Γ is unique up to isomorphism. Note however that, given a Γ, such a γ does not always exist even if $\Gamma = \Gamma(\boldsymbol{B}(\mathcal{S}))$ for some admissible sheaf $\mathcal{S}$.

Proposition 3.4. *Let $\mathcal{S}$ be an admissible sheaf over $B=G/D$ and set $\Gamma=\Gamma(\boldsymbol{B}(\mathcal{S}))$. Then there exist a finite dimensional $\mathfrak{g}$-module γ and a $\mathfrak{g}$-homomophism $\alpha:\gamma\to\Gamma$ such that $\bar{\alpha}:\mathfrak{F}(B)\otimes\gamma\to\Gamma$ is bijective, if and only if the linear representation $\Lambda_{\mathcal{S}}: D\to GL(n, R)$ extends to a linear representation $\Lambda: G\to GL(n, R)$; $\Lambda|D=\Lambda_{\mathcal{S}}$.*

Proof. Let $\pi_x:\Gamma(\boldsymbol{B}(\mathcal{S}))\to\boldsymbol{B}(\mathcal{S})_x$, $x\in B$, be the restriction. Let γ be a $\mathfrak{g}$-module and $\alpha:\gamma\to\Gamma$ be a $\mathfrak{g}$-homomorphism. Note that

(3.12) $\bar{\alpha}:\mathfrak{F}(B)\otimes\gamma\to\Gamma$ *is bijective if and only if* $\pi_x\alpha:\gamma\to\boldsymbol{B}(\mathcal{S})_x$ *is bijective for any* $x\in B$.

Suppose that there are γ and $\alpha:\gamma\to\Gamma$ with $\bar{\alpha}:\mathfrak{F}(B)\otimes\gamma\to\Gamma$ bijective. Since γ is a $\mathfrak{g}$-module, that is, there is given a homomorphism $\lambda:\mathfrak{g}\to\mathfrak{gl}(n, R)$, and since G is simply connected, there is a unique homomorphism $\Lambda: G\to GL(n, R)$ such that $d\Lambda=\lambda$. Since $\alpha:\gamma\to\Gamma$ is a $\mathfrak{g}$-homomorphism and since the operation of $\mathfrak{g}$ on Γ is given by (3.4)′, it follows that the operations of G on γ and on Γ is equivariant through α; $\alpha(\Lambda(g)\cdot x)=g\cdot\alpha(x)$, $g\in G$, $x\in\gamma$. Therefore if $d\in D$, we have $\pi_{\bar{1}}\alpha(\Lambda(d)x)=\pi_{\bar{1}}(d\cdot\alpha(x))=\Lambda_{\mathcal{S}}(d)\cdot\pi_{\bar{1}}(x)$ (cf. (3.4)). It follows that $\Lambda|D$ and $\Lambda_{\mathcal{S}}$ is equivalent representations and that $\Lambda_{\mathcal{S}}$ is extendable to whole G.

Conversely, suppose that there exists a representation $\Lambda: G\to GL(\gamma)$ extending $\Lambda_{\mathcal{S}}: D\to GL(\gamma)$, $\gamma=\pi^{-1}(\bar{1})$. The differential $d\Lambda:\mathfrak{g}\to\mathfrak{gl}(\gamma)$ defines a $\mathfrak{g}$-module structure on γ.

Let P be the principal bundle belonging to $\Lambda_{\mathcal{S}*}(c)\in H^1(B, GL(n, R)_{lc})$. Elements of P ore admissible maps $\xi:\gamma\to\mathcal{S}$, and every $x\in\mathcal{S}$ can be written as $x=\xi(c)$, $\xi\in P$, $c\in\gamma$. Since coordinate transformations of P are locally constant maps, we may operate[8)] G on P in such a way that, if $g\cdot\xi$ denotes the transform of ξ by g, then we have $(g\cdot\xi)(c)=g\cdot(\xi(c))$.

Since P belongs to $\Lambda_S(c)$, we have a fibre preserving map $\tilde{\Lambda}: G\to P$ such that $\tilde{\Lambda}(g, g')=\tilde{\Lambda}(g)\cdot\Lambda(g')$ for $g\in G$, $g'\in D$. Moreover, we have $\tilde{\Lambda}(g\cdot g')=g\cdot\tilde{\Lambda}(g')$ for $g, g'\in G$.

Let $\xi_0\in P$ be defined by $\xi_0=\tilde{\Lambda}(1)$. ξ_0 is in γ. If $g\in D$, then we have

$$\tilde{\Lambda}(g)=\tilde{\Lambda}(g\cdot 1)=g\cdot\tilde{\Lambda}(1)=g\cdot\xi_0,$$

and

$$\tilde{\Lambda}(g)=\tilde{\Lambda}(1\cdot g)=\tilde{\Lambda}(1)\cdot\Lambda(g)=\xi_0\cdot\Lambda_S(g).$$

It follows that

(3.13) $g\cdot\xi_0=\xi_0\cdot\Lambda_S(g)$ or $g(\xi_0(c))=\xi_0(\Lambda_S(g)\cdot c)$ for any $g\in D$ and $c\in\gamma$.

Now we define a $\mathfrak{g}$-homomorphism $\alpha: \mathcal{V}\to\Gamma(\boldsymbol{B}(\mathcal{S}))$ by $\alpha(c)(gD)=g\cdot(\xi_0(\Lambda(g^{-1})\cdot c))$ for $c\in\mathcal{V}$, $g\in G$. Using (3.13) it is easily shown that the right hand side depends only on the coset gD.

To prove that α commutes with the operation of $\mathfrak{g}$, it is sufficient to show that

$$(3.14) \qquad g\cdot\alpha(c)=\alpha(\Lambda(g)\cdot c), \text{ for } g\in G \text{ and } c\in\mathcal{V},$$

since $\mathfrak{g}$ operates on $\mathcal{V}$ through the differential λ of Λ, and $\mathfrak{g}$ operates on $\Gamma(\boldsymbol{B}(\mathcal{S}))$ by the formula (3.4)$'$.

Now, for $g, g'\in G$, we have

$$\begin{aligned}(g\cdot\alpha(c))(g'\cdot D)&=g\cdot(\alpha(c)(g^{-1}g'\cdot D)),\\ &=g\cdot\{g^{-1}g'(\xi_0(\Lambda(g^{-1}g')\cdot c))\},\\ &=g'\cdot(\xi_0(\Lambda(g'^{-1})\Lambda(g)c)),\\ &=\alpha(\Lambda(g)c)(g'D).\end{aligned}$$

Thus (3.14) holds.

$\pi_x\alpha: \mathcal{V}\to\boldsymbol{B}(\mathcal{S})_x$ is bijective for any $x\in B$, because $\pi_1\alpha=\xi_0$ is bijective and G operates on $B(\mathcal{S})$ as a group of isomorphisms of bundle. It follows from (3.12) that $\bar{\alpha}: \mathcal{F}(B)\otimes\mathcal{V}\to\Gamma(\boldsymbol{B}(\mathcal{S}))$ is bijective.

Remark. The proof given above shows that *the structure of $\mathfrak{g}$-module of $\mathcal{V}$ is determined by the differential of an extension* of $\Lambda_{\mathcal{S}}$.

Proposition 3.5. *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$, and D a discrete subgroup of G. If $\mathcal{V}$ is a finite dimensional $\mathfrak{g}$-module, then there is a unique (up to isomorphisms) admissible sheaf $\mathcal{S}$ over $B=G/D$ and a $\mathfrak{g}$-homomorphism $\alpha: \mathcal{V}\to\Gamma(\boldsymbol{B}(\mathcal{S}))$ such that $\bar{\alpha}: \mathcal{F}(B)\otimes\mathcal{V}\to\Gamma(\boldsymbol{B}(\mathcal{S}))$ is bijective.*

Let $\Lambda: G\to GL(\mathcal{V})$ be the representation whose differential defines the $\mathfrak{g}$-module structure of $\mathcal{V}$. Then a desired sheaf must belong to $(\Lambda|D)_*(c)\in H^1(B, GL(\mathcal{V})_k)$ where $c\in H^1(B, \boldsymbol{D}_{(c)})$ is the bundle $G\to G/D$. Conversely a sheaf belonging to $(\Lambda/D)_*(c)$ satisfies the desired condition (see the Remark above).

Remark. If $\mathcal{V}$ in Proposition 3.3 is a graded $\mathfrak{g}$-algebra, then we may take as $\mathcal{S}$ sheaf of graded algebra; the isomorphism $\bar{\alpha}: \mathcal{F}(B)\otimes\mathcal{V}\to\Gamma(\boldsymbol{B}(\mathcal{S}))$ then becomes an isomorphism of graded algebra.

§4. Cohomology of certain solvmanifolds.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\mathcal{V}$ a finite dimensional vector space over R. A representation $\lambda: \mathfrak{g}\to\mathfrak{gl}(\mathcal{V})$ is called *triangular* if there exists a sequence $\gamma=\gamma_0\supset\gamma_1\supset\cdots\supset\gamma_n=0$ of subspaces of γ such that $\dim \gamma_i = \dim \gamma_{i+1}+1$, $0\leqq i<n$, and that each γ_i is invariant under $\lambda(\mathfrak{g})$. In this case, the $\mathfrak{g}$-module γ is called *triangular*.

A Lie algebra $\mathfrak{g}$ is called *triangular* or "*à racines réelles*" if its adjoint representation is triangular. A triangular Lie algebra is solvable. A Lie group is called triangular if its Lie algebra is triangular.

Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$, and let D be a discrete subgroup of G. Set $B=G/D$. If γ is a finite dimensional $\mathfrak{g}$-module, then $\Gamma=\mathcal{F}(B)\otimes\gamma$ is an $\mathcal{F}(B)$-module on which $\mathfrak{X}(B)$ operates by the formula (3.11). Let $\alpha:\gamma\to\Gamma$ be defined by $\alpha(c)=1\otimes c$. Thus α induces a cochain homomorphism $\alpha_\sharp ; C(\mathfrak{g},\gamma)\to C(\mathfrak{X}(B),\Gamma)$ (cf. §3), which is injective.

Theorem 4.1. *Let G be a simply connected, triangular Lie group with Lie algebra $\mathfrak{g}$ and let D be a discrete subgroup of G such that $B=G/D$ is compact. Let γ be a triangular $\mathfrak{g}$-module. Then the injection $\alpha_\sharp : C(\mathfrak{g},\gamma)\to C(\mathfrak{X}(B),\Gamma)$ induces an isomorphism*

$$\alpha_*: H^*(\mathfrak{g},\gamma)\to H^*(\mathfrak{X}(B),\Gamma).$$

Proof. We proceed by induction.

1) The case: $\dim G=1$. In this case, $\mathfrak{g}$ is generated by an $X\neq 0$. Imbed R in $\mathcal{F}(B)$ as constant functions. We need the following lemma.

Lemma 4.2. **a)** Let $f\in\mathcal{F}(B)$. *Then there exist a unique $h\in\mathcal{F}(B)$ and a unique $b\in R$, such that*

$$X\cdot h+f=b.$$

b) *Let $f\in\mathcal{F}(B)$, and let $a, b\in R$, $a\neq 0$. Then there exists a unique $h\in\mathcal{F}(B)$, such that*

$$X\cdot h+ah+f=b.$$

Proof of Lemma. Since $\dim \mathfrak{g}=1$, we have $G=R$, and we may assume that $D=Z=$the group of integers. (B is the circle). We imbed $\mathcal{F}(B)$ in $\mathcal{F}(R)$ as functions with period 1. $\mathfrak{X}(B)$ is imbedded in $\mathfrak{X}(R)$ and we may assume without loss of generality that $X=\dfrac{d}{dx}$.

Now if $h, h'\in\mathcal{F}(B)$ and $b, b'\in R$ are such that

$$X\cdot h+f=b \quad\text{and}\quad X\cdot h'+f=b',$$

then we have

$$\frac{d}{dx}(h-h')=b-b',$$

or

$$h-h'=(b-b')x+\text{constant}.$$

Since h and h' have period 1, we must have $b=b'$ and $h=h'$. Hence the uniqueness of a) follows.

For $f\in\mathcal{F}(B)$, set $b=\int_0^1 f\,dx$, and define h by

$$h(x)=\int_0^x (b-f)dx.$$

Then

$$\begin{aligned} h(x+1)-h(x) &= \int_x^{x+1}(b-f)dx, \\ &= b-\int_x^{x+1} f\,dx, \\ &= b-\int_0^1 f\,dx, \quad \text{(since } f \text{ has period 1)}, \\ &= 0. \end{aligned}$$

If follows that $h\in\mathcal{F}(B)$ and $X\cdot h+f=b$, proving a).

Let $h, h'\in\mathcal{F}(B)$ be such that

$$X\cdot h+ah+f=b \quad \text{and} \quad X\cdot h'+ah'+f=b.$$

Then we have

$$\frac{d}{dx}(h-h')=-a(h-h'),$$

or

$$h-h'=(\text{constant})\ e^{-ax}.$$

Since h and h' has period 1 and $a\neq 0$, we must have $h=h'$, proving the uniqueness of b).

Let $f\in\mathcal{F}(B)$ and $a, b\in R$, $a\neq 0$. Define h by

$$h(x)=\frac{b}{a}+ce^{-ax}+F(x)e^{-ax},$$

where $$c=\frac{1}{e^{-a}-1}\int_{-1}^0 f(x)e^{ax}dx \quad \text{and} \quad F(x)=-\int_0^x f(x)e^{ax}dx.$$

Since f has period 1, we have

$$\begin{aligned} F(x+1) &= -e^{-a}\int_{-1}^x f(x)e^{ax}dx \\ &= -e^{-a}\int_{-1}^0 f(x)e^{ax}dx+e^{-a}F(x). \end{aligned}$$

It follows that

$$h(x+1)-h(x)=c(e^{-a}-1)e^{-ax}-e^{-ax}\int_{-1}^{0}f(x)e^{ax}dx+e^{-ax}F(x)-e^{-ax}F(x)$$
$$=0.$$

Hence h belongs to $\mathcal{F}(B)$. An easy calculation shows that

$$X\cdot h+ah+f=b,$$

completing the proof of b).

Now it follows easily from the definition that

$$\begin{aligned}
&H^0(\mathfrak{g},\gamma)=\gamma^X=\{c\in\gamma,\ X\cdot c=0\},\\
&H^0(\mathfrak{X}(B),\Gamma)=\Gamma^X=\{c\in\Gamma\,;\ X\cdot c=0\},\\
&H^1(\mathfrak{g},\gamma)=\gamma/X\cdot\gamma,\ \text{where } X\cdot\gamma=\{c\in\gamma\,;\ c=X\cdot c' \text{ for some } c'\in\gamma\},\\
&H^1(\mathfrak{X})B(,\Gamma)=\Gamma/X\cdot\Gamma,\ \text{where } X\cdot\Gamma=\{c\in\Gamma\,;\ c=X\cdot c' \text{ for some } c'\in\Gamma\},\\
&H^q(\mathfrak{g},\gamma)=H^q(\mathfrak{X}(B),\Gamma)=0,\ \text{for } q>1.
\end{aligned}$$

The homomorphisms $\alpha_*\,;\,H^q(\mathfrak{g},\gamma)\to H^q(\mathfrak{X}(B),\Gamma)$ are then the natural maps

$$\alpha_*:\gamma^X\to\Gamma^X \quad\text{and}\quad \alpha_*:\Gamma/X\cdot\gamma\to\Gamma/X\cdot\Gamma,$$

induced by the injection $\alpha:\gamma\to\Gamma$.

Now since γ is a triangular $\mathfrak{g}$-module, there exists a base $\{e_i\}_{i=1}^n$ of γ such that we have

$$\begin{aligned}
&X\cdot e_1=a_{11}e_1,\\
&X\cdot e_2=a_{21}e_1+a_{22}e_2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&X\cdot e_n=a_{n1}e_1+\cdots+a_{nn}e_n,\quad a_{ij}\in R.
\end{aligned}\tag{4.1}$$

Then elements of Γ are written uniquely in the form $\sum_i f_ie_i$ with $f_i\in\mathcal{F}(B)$, and $\sum_i f_ie_i$ belongs to γ if and only if all f_i are in R.

Now $\alpha_*:\gamma^X\to\Gamma^X$ is obviously injective. Suppose that $\sum_i f_ie_i$ is annihilated by X. Since $X\cdot(\sum f_ie_i)=\sum((X\cdot f_i)e_i+f_iX\cdot e_i)$, if follows from (4.1) that $X\cdot f_n+a_{nn}f_n=$ coefficient of e_n of $X\cdot(\sum f_ie_i)=0$. Then in virtue of Lemma 4.2. we hae $f_n\in R$.

Assume that we have $f_i\in R$, for $0\leqq k<i\leqq n$. Since

$$\begin{aligned}
&Xf_k+a_{kk}f_k+a_{k+1,k}f_{k+1}+\cdots+a_{n,k}f_n\\
&=\text{coefficients of } e_k \text{ in } X(\textstyle\sum f_ie_i)\\
&=0,
\end{aligned}$$

and since f_i, $k+1\leqq i\leqq p$, belongs to R, we must have $f_k\in R$ in virtue of Lemma 4.2. We have proved that all f_i are in R and $\sum f_ie_i\in\gamma$. If follows that $\alpha_*:\gamma^X\to\Gamma^X$ is bijective.

To prove the bijectivity of $\alpha_*: \gamma/X\cdot\gamma \to \Gamma/X\cdot\Gamma$, it suffices to prove the following two statements:

(4.2) for any $f\in\mathcal{F}(B)$ and k, $1\leq k\leq p$, there exist $h_i\in\mathcal{F}(B)$ and $c_i\in R$ such that

$$fe_k = X\cdot(\sum_{i=1}^{k} h_i e_i) + \sum_{i=1}^{k} c_i e_i.$$

(4.3) If $X(\sum f_i e_i)=\sum c_i e_i$, with $f_i\in\mathcal{F}(B)$ and $c_i\in R$, then $f_i\in R$.

Now since we have

$$\begin{aligned}X\cdot(\sum_{i=1}^{k} h_i e_i)+\sum_{i=1}^{k} c_i e_i\\ &=(Xh_k+a_{kk}h_k+c_k)e_k\\ &+(Xh_{k-1}+a_{k-1\,k-1}h_{k-1}+a_{k\,k-1}h_k+c_{k-1})\,e_{k-1}\\ &+\cdots\cdots\cdots\cdots\cdots,\end{aligned}$$

we can determine (by Lemma 4.2) $h_k, c_k, h_{k-1}, c_{k-1}\cdots\cdots$ inductively to satisfy

$$\begin{aligned}&X\cdot h_k+a_{kk}h_k+c_k=f,\\ &X\cdot h_{k-1}+a_{k-1\,k-1}h_{k-1}+a_{k\,k-1}h_k+c_{k-1}=0\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

This proves (4.2).

(4.3) is proved by a similar argument using Lemma 4.2.

2) We assume that the theorem is proved for dim $G<k$, and we shall prove it for dim $G=k$. We know that there exists a connected, simply connected, closed normal Lie subgroup $K(0<\dim K<\dim G=k)$ of G such that KD is closed in G. In fact, if G is not nilpotent we can take the Lie subgroup corresponding to maximum nilpotent ideal of $\mathfrak{g}$ [9], and if G is nilpotent we can take such a K in the center of G [10]. Let $\mathfrak{k}$ be the Lie algebra of K.

We shall show that

(4.4) $H^*(\mathfrak{k}, \gamma)$ is a triangular $\mathfrak{g}/\mathfrak{k}$-module.

We have a canonical isomorphism $C^q(\mathfrak{k}, \gamma)\cong \Lambda^q(\mathfrak{k})\otimes\gamma$, where $\Lambda^q(\mathfrak{k})$ is the q-th exterior product of the dual of $\mathfrak{k}$. The operation of $\mathfrak{g}$ on $C^q(\mathfrak{k}, \gamma)$ translates then to the operation on $\Lambda^q(\mathfrak{k})\otimes\gamma$ which is the Kronecker sum of the q-th Kronecker sum of the adjoint represantation of $\mathfrak{g}$ on the dual of the ideal $\mathfrak{k}$ and of the given operation of $\mathfrak{g}$ on γ. Since both operations are triangular by the assumption, their Kronecker sum is triangular; $\mathfrak{g}$ operates triangularly on

$C^q(\mathfrak{k}, \gamma)$. Since $Z^q(\mathfrak{k}, \gamma)$ is an invariant subspace under this operation, $\mathfrak{g}$ operates triangularly on $Z^q(\mathfrak{k}, \gamma)$ and hence on $H^q(\mathfrak{k}, \gamma)$. Since $\mathfrak{k}$ operates trivially on $H^q(\mathfrak{k}, \gamma)$, $\mathfrak{g}/\mathfrak{k}$ operates triangularly on $H^q(\mathfrak{k}, \gamma)$; (4.4) is proved.

Now consider the fibre bundle $p: G/D \to G/K\ KD/D$. Since $\mathfrak{g}$ is triangular, $\mathfrak{k}$ and $\mathfrak{g}/\mathfrak{k}$ are also triangular. The fibre $F = KD/D = K/K \frown D$ and the base $M = G/K\ KD/K$ are compact coset spaces of simply connected triangular groups by discrete subgroups.

In the spectral sequence attached to this bundle, we have a commutative diagram (cf. §3):

$$\begin{array}{ccc} E_2^{p,q}(\mathfrak{g}) & \xrightarrow{\alpha_2} & E_2^{p,q} \\ \downarrow & & \downarrow \\ H^p(\mathfrak{g}/\mathfrak{k}, H^q(\mathfrak{k}, \gamma) & \xrightarrow{\alpha_{*2}} & H^p(\mathfrak{X}(M), H^q(\mathfrak{X}_v(B), \Gamma)). \end{array}$$

The first vertical map is bijective. The second one is olso bijective in virtue of Corollary 2.21, since $\Gamma = \Gamma(\boldsymbol{B}(\mathcal{S}))$ for some admissible sheaf $\mathcal{S}$ by Proposition 3.3, and since $\mathcal{H}(\mathcal{S})$ is admissible because the fibre F is compact (cf. (2.3.6)). Regarding $H^q(\mathfrak{X}_v(B), \Gamma)$ as $\Gamma(\boldsymbol{B}(\mathcal{H}^q(\mathcal{S})))$ (cf. §2.4), we consider the map

$$\pi_x \alpha'_* : H^q(\mathfrak{K}, \gamma) \to \boldsymbol{B}(\mathcal{H}^q(\mathcal{S}))_x, \quad x \in M,$$

which is the composition of $\alpha'_* : H^q(\mathfrak{k}, \gamma) \to \Gamma(\boldsymbol{B}(\mathcal{H}^q(\mathcal{S})))$ and $\pi_x : \Gamma(\boldsymbol{B}\mathcal{H}^q(\mathcal{S}))) \to \boldsymbol{B}(\mathcal{H}^q(\mathcal{S}))_x$. Since $\boldsymbol{B}(\mathcal{H}^q(\mathcal{S}))_x = H^q(\mathfrak{X}(p^{-1}(x)), \Gamma(\boldsymbol{B}(\mathcal{S}), p^{-1}(x)))$ and since $\Gamma(\boldsymbol{B}(\mathcal{S}), p^{-1}(x)) = \mathcal{F}(p^{-1}(x) \otimes \gamma$ as a submodule of $\mathcal{F}(B) \otimes \gamma$, it follows from the inductive assumption that $\pi_x \alpha'_*$ is bijective. (Note that γ is obuiously a triangular $\mathfrak{k}$-module).

Since $\pi_x \alpha'_*$ is bijective for any $x \in M$, it follows from (3.12) that the map $\bar{\alpha}'_*: \mathcal{F}(M) \otimes H^q(\mathfrak{k}, \gamma) \to H^q(\mathfrak{X}_v(B), \Gamma)$ is bijective. Therefore, again from the inductive assumption and from (4.4), we conclude thac α_{*2} is bijective; hence $\alpha_2 : E_2(\mathfrak{g}) \to E_2$ is also bijective. Then by a theorem of Leray [6], $\alpha_* : H^*(\mathfrak{g}, \gamma) \to H^*(\mathfrak{X}(B), \Gamma)$ is bijective. This completes the proof of the theorem.

Remark. If γ is a graded $\mathfrak{g}$-algebra, then α_* is a homomorphism of graded algebra.

Corollary 4.2. *Let $\mathfrak{g}$ be a triangular Lie algebea. Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$, and let D be a discrete subgroup of G such that G/D is compact. Then the injection*

$$\alpha : C(\mathfrak{g}, R) \to C(\mathfrak{X}(G/D), \mathcal{F}(G/D))$$

induced an isomorphism

$$\alpha_* : H^*(\mathfrak{g}, R) \to H^*(\mathfrak{X}(G/D), \mathcal{F}(G/D)).$$

(R is considered as a $\mathfrak{g}$-module in which the operation of $\mathfrak{g}$ is trivial. Note

that $H^*(\mathfrak{A}(G/D), \mathfrak{J}(G/D)) = H^*(G/D, R)$ (cf. §1)).

Remark. Let M be a connected manifold on which a triangular Lie group operates transitively. Then *M is homeomorphic to a product of a Euclidean space and a compact coset space G/D where G is a simply connected triangular group and D is a discrete subgroup of G* (cf. [11]). In particular *M is an orientable manifold*, since a coset space of a Lie group by a *discrete subgroup* is always an orientable manifold.

Corollary 4.3. *Let M be a compact manifold on which a connected triangular group operates transitively. Then the Euler-Poincaré characteristic $\chi(M)$ of M and the index* (see [3]) *$\tau(M)$ of M are equal to zero.*

Proof. We may assume that $M = G/D$, where G is a simply connected triangular Lie group and D a discrete subgroup of G. Then $H^*(G/D, R)$ is isomorphic to $H^*(\mathfrak{g}, R)$ where $\mathfrak{g}$ is the Lie algebra of G. Since G/D is compact orientable manifold $H^*(G/D, R) \cong H^*(\mathfrak{g}, R)$ has the Poincaré duality. Therefore $C(\mathfrak{g}, R)$ is a Poincaré ring with a differential (see *S.S. Chern, F. Hirfiebruch, J.-P. Serre, On the indix of a fibred manifold, Proc. Amer. Math. soc., vol.* 8 (1957), for the definition of a Poincare ring with a differential and for its behavior with respect to the Euler-Poincaré characteristic and the index). Therefore we have $\chi(G/D) = \chi(H^*(\mathfrak{g}, R)) = \chi(C(\mathfrak{g}, R))$ and $\tau(G/D) = \tau(H^*(\mathfrak{g}, R)) = \tau(C(\mathfrak{g}, R))$.

We know that $\chi(C(\mathfrak{g}, R)) = \sum_{i=0}^{n}(-1)^i \binom{n}{i} = 0$. where $n = \dim \mathfrak{g}$.

To compute $\tau(C(\mathfrak{g}, R))$, we may assume that $\dim \mathfrak{g} = 4k$. Let $\{X_1, \cdots, X_{4k}\}$ be a base of $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. Then $\{X_{i_1} \wedge \cdots \wedge X_{i_{2k}}\}_{i_1 < \cdots < i_{2k}}$, is a base of $C^{2k}(\mathfrak{g}, R) = \Lambda^{2k}(\mathfrak{g})$, where $\Lambda^{2k}(\mathfrak{g})$ is the $2k$-th exterior product of $\mathfrak{g}^*$. $C^{4k}(\mathfrak{g}, R) = \Lambda^{4k}(\mathfrak{g})$ is spanned by $\xi = X_1 \wedge \cdots \wedge X_{4k}$. With respect to a suitable ordering of elements of the base of $C^{2k}(\mathfrak{g}, R)$ given above, the symmetric bilinear form on $C^{2k}(\mathfrak{g}, R)$ defined by $(x, y) \to \langle x, y \rangle$, $\langle x, y \rangle \xi = x \wedge y$, may be expressed by a symmetric matrix of the form

$$A = \begin{pmatrix} 0 & & \varepsilon_1 \\ & \iddots & \\ \varepsilon_N & & 0 \end{pmatrix}, \quad N = \binom{4k}{2k}, \varepsilon_i = \pm 1.$$

It follows that $\tau(C(\mathfrak{g}, R)) = \tau(A) = 0$.

Notes.

1) (in p. 290) To define the equivalence in the strict sense of coordinate bundles we require C^∞-differentiability of transition function $\bar{g}_{kl}: V_j \frown V_k' \to G$ instead of continuity in [14, §2.4].

2) (in p. 291) In the sequel, we shall use the word "covering" to mean an open covering.

3) (in p. 291) Hereafter we shall assume the paracompactness of manifolds and of Lie groups.
4) (in p. 299) In [2], the notation $C(\mathfrak{X}, \Gamma)$ is used in the case where $\mathfrak{X}$ is a Lie algebra and Γ is a $\mathfrak{X}$-module to mean the $\mathfrak{X}$-module of R-multilinear alternating functions from $\mathfrak{X}$ to Γ, and operations corresponding to our $\theta(X)$, $i(X)$ are introduced. Our $C(\mathfrak{X}, \Gamma)$ is a submodule of Hochild-Serre's $C(\mathfrak{X}, \Gamma)$ invariant under $\theta(X)$, $i(X)$ and d. (We can check the invariance by calculations.)
5) (in p. 299) The difinition of d in $C(\mathfrak{X}(M), \Gamma)$ in (1.13) differs from the present one only up to a multiplicative constant. Therefore the derived module $H^*(\mathfrak{X}(M), \Gamma)$ is the same in both definitions.
6) (in p. 316) See [4]. Our definition of $\varphi_{\xi}^{(p,q)}$ differs slightly from that of [4]; hence the difference of the multiplicative constant in (2.6.7).
7) (in p. 319) In particular $H^*(\mathfrak{X}(B), \Gamma)$ is isomorphic to $H^*(\mathfrak{g}, \Gamma)$.
8) (in p. 323) The proof of this fact is similar to that of Proposition 3.1.

Bibliography.

[1] H. Cartan, Séminaire E.N.S., 1950-1951, Paris.
[2] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Memoir Amer. Math. Soc., **16**, 1955.
[3] F. Hirzebruch, Neue topologische Methoden in der algebraischen Geometrie, Erg. Math. u. Grenzgebiete (N.F.) **9**, 1956.
[4] G. Hochschild and J.-P. Serre, Cohomology of Lie algebras, Ann. of Math., **57** (1953), 591-603.
[5] J.-L. Koszul, Homologie et cohomologie des algèbres de Lie, Bull. Soc. Math. France, **78** (1950), 65-127.
[6] J. Leray, L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue, J. Math. Pures Appl., **29** (1950), 1-139.
[7] Y. Matsushima, On the discrete subgroups and homogeneous spaces of nilpotent Lie groups, Nagoya Math. J., **2** (1952), 95-110.
[8] G.D. Mostow, Factor spaces of solvable groups, Ann. of Math., **60** (1954), 1-27.
[9] K. Nomizu, Lie groups and differential geometry, Publications Math. Soc. Japan, 1956.
[10] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, Ann. of Math., **59** (1954), 531-538.
[11] M. Saito, Sur certains groupes de Lie résolubles II, Scientific Papers Coll. Gen. Edu., Univ. Tokyo, **7** (1957), 157-168.
[12] L. Schwartz, Séminaire, 1953-1954, Paris.
[13] J.-P. Serre, Homologie singulière des espaces fibrés, Ann. of Math., **54** (1951), 425-505.
[14] N. Steenrod, Topology of fibre bundles, Princeton Math. Series, **14**, 1951.