On Selberg's trace formula.

Dedicated to Professor Z. Suetuna on his 60th birthday.

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This note is an introduction to A. Selberg's theory on harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces (cf. A. Selberg [3]). We develop the theory in rather abstract form, after R. Godement (cf. R. Godement [2]), and restrict ourselves to the elementary part, so we have to pass by several important items such as unitary representations of class one, Plancherel's formula, Laplacians etc. It must be interesting to extend the theory to the case where discrete groups we consider have non compact homogeneous spaces with finite volume, but we do not discuss the problem either. We treat here only a frame of Selberg's theory, however we can apply it to several problems.

The author wishes to apologize for presenting this rather expository paper. In recent years, at least he believes, the theory of topological groups is playing an important role in the study of the algebraic number theory and non commutative number theory, e.g. the arithmetic theory of classical groups and simple algebras. So we have to prepare as many tools as we can collect. We know that Selberg's theory is one of the biggest guns as it was shown by Selberg himself. In subsequent papers we will apply the theory to several number theoretical problems, e.g. the theory of ζ-functions of a division algebra, Eichler's trace formula of Hecke operators etc. The author wishes to express his thanks to Professor S. Ito who gave him many valuable advices in preparing this paper.

§1. Preliminaries. Throughout this note, a "locally compact group" means a locally compact group with the second axiom of countability satisfying the assumption that a left invariant Haar measure on that group is also right invariant. Let G be a locally compact group and dx the volume element of a Haar measure on G. We will consider the following type of vector spaces of complex valued measurable functions on G.

1. $L(G)$: The space of all continuous functions with compact support.
2. $L^{\infty}(G)$: The completion of $L(G)$ with respect to the norm $\| \|_{\infty}$ defined by
$$\|f\|_{\infty}=\sup_{x\in G}|f|.$$
3. $L_p(G)$ $(1\leq p<\infty)$: The vector space of all f with
$$\|f\|_p=\left\{\int_G |f(x)|^p dx\right\}^{\frac{1}{p}}<\infty.$$

The space $L_p(G)$ is the completion of $L(G)$ with respect to the norm $\| \|_p$.

4. The $C(G)$: The space of all continuous functions on G. $C(G)$ is a Frechet space with the norm

$$\|f\|_C=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|f\|_{\infty,}\ \|f\|_{\infty,n}}{1+\|f\|_{\infty,}\ \|f\|_{\infty,n}}$$

where $K_1 \subset K_2 \subset \cdots$ is a series of compact sets such that every compact set K is contained in some K_i, and $\|f\|_{\infty,\nu}$ is the maximum of $|f(x)|$ on K_ν.

A linear combination of a finite number of locally finite Borel measures on G with complex coefficients will be called a complex valued measure on G. If $d\mu(x)$ is such a measure, then

$$\hat{\mu}(f)=\int_G f(x^{-1})d\mu(x) \tag{1}$$

is a linear functional of $L(G)$ satisfying the following continuity condition C:

(C): If $\{f_n\}$ is a series of functions in $L(G)$ whose supports are contained in a compact set K, then

$$\lim_{n\to\infty}\hat{\mu}(f_n)=0 \quad \text{provided} \quad \lim_{n\to\infty}\|f_n\|_\infty=0.$$

Conversely if $\hat{\mu}$ is a linear functional of $L(G)$ of this property, then there exists a uniquely determined complex valued measure on G such that $\hat{\mu}(f)$ is defined by (1). If $\hat{\mu}$ satisfies stronger condition C_p $(1\leqq p<\infty)$; $\hat{\mu}$ is continuous with respect to $\| \|_p$, then there exists a uniquely determined function $\mu(x)\in L_{p'}(G)$ with $1/p+1/p'=1$ such that

$$\hat{\mu}(f)=\int_G f(x^{-1})\mu(x)dx. \tag{2}$$

In the case $p=1$, $\mu(x)$ is a measurable function such that

$$|\mu(x)|\leqq \sup_{\|f\|_1=1}(\hat{\mu}(f))<\infty$$

for almost all $x\in G$.

For every function f on G, we put

$$\tilde{f}(x)=f(x^{-1}) \qquad \text{and} \qquad f^*(x)=\overline{f(x^{-1})}.$$

If f and g are measurable functions, then the convolution $f*g$ of f and g is defined by

$$\int_G f(xy^{-1})g(y)dy=\int_G f(y^{-1})g(yx)dy=f*g(x)$$

whenever the integral is defined for almost all x. If $f\in L_1(G)$ and $g\in L_p(G)$, then $f*g$ is defined and contained in $L_p(G)$ satisfying the following important inequality:

$$\|f*g\|_p \leqq \|f\|_1 \|g\|_p. \tag{3}$$

If $p=1$ and f and g are $\geqq 0$, then the equality holds in (3). If $f \in L_p(G)$ and $g \in L_{p'}(G)$ with $1/p+1/p'=1$, then $f*g \in L^\infty(G)$ and

$$\|f*g\|_\infty \leqq \|f\|_p \|g\|_{p'}. \tag{4}$$

A function $\varphi \in C(G)$ which is not identically 0 is called positive definite if

$$\int_G \int_G \varphi(xy^{-1}) f(x) \overline{f(x)}\, dx\, dy \geqq 0$$

for all $f \in L(G)$. If φ is positive definite, then we have

$$|\varphi(x)| \leqq \varphi(e) \tag{5}$$

$$\text{and} \quad \varphi(x^{-1}) = \overline{\varphi(x)} \tag{6}$$

for all $x \in G$.

§2. Zonal spherical functions. Let G be a locally compact group and U a compact subgroup of G. The vector space $L(G)$ is an associative algebra over the complex number field C if we define the multiplication of $L(G)$ by the convolution $*$. Let $L(G, U)$ be the subspace of $L(G)$ of all f such that $f(uxu') = f(x)$ for all $x \in G$ and $u, u' \in U$. Then $L(G, U)$ is a subalgebra of $L(G)$. We normalize a Haar measure on U so that

$$\int_G du = 1.$$

If U is an open subgroup of G, we assume that $dx = du$ on U. The algebra $L(G, U)$ does not contain the unit element unless U is open. In this exceptional case, the characteristic function of U, namely the function which takes the value 1 on U and 0 outside U is the unit element of $L(G, U)$. For every $f \in C(G)$ we put

$${}^0f(x) = \int_U f(ux)du, \qquad f^0(x) = \int_U f(xu)du$$

$$\text{and} \quad {}^0f^0(x) = \int_U \int_U f(uxu')du\, du'.$$

Let $C(U\backslash G)$ be the set of all $f \in C(G)$ such that $f(ux) = f(x)$ for all $x \in G$ and $u \in U$, and $C(G/U)$ the set of all $f \in C(G)$ with $f(xu) = f(x)$ for all $x \in G$ and $u \in U$. Then the mappings $f \to {}^0f$ and $f \to f^0$ are projections of $C(G)$ onto $C(U\backslash G)$ and $C(G/U)$ respectively. The mapping $f \to {}^0f^0$ is also a projection of $L(G)$ onto

$L(G, U)$ and it is easy to see that

$$\|f\|_p \geqq \|{}^0f^0\|_p$$

for $1 \leqq p \leqq \infty$. Hence the projection $f \to {}^0f^0$ can be extended to $L_p(G)$ $(1 \leqq p < \infty)$, and the image $L_p(G, U)$ of this projection is a closed subspace of $L_p(G)$ of all f such that $f(uxu') = f(x)$ for all $x \in G$ and u, $u' \in U$.

Now $L_1(G)$ is an associative algebra with the multiplication $*$, and $L_1(G, U)$ is a subalgebra of $L_1(G)$. $L(G, U)$ is everywhere dense in $L_1(G, U)$.

We impose on G and U the following basic assumption (A).

(A). The algebra $L(G, U)$ is commutative, i. e. for every f and $g \in L(G, U)$, we have

$$f * g = g * f.$$

If $L(G, U)$ is commtative, then the following assertions are true.

1. If f and g are in $L_1(G, U)$, then $f * g = g * f \in L_1(G, U)$.

2. If $f \in L(G, U)$ and $g \in C(G, U)$ (the space of all $f \in C(G)$ with $f = {}^0f^0$), then $f * g \in C(G, U)$ and $f * g = g * f$.

A complex valued measure $d\omega(x)$ on G will be called spherical if $d\omega(uxu') = d\omega(x)$ for all $u, u' \in U$ and $x \in G$, and the mapping $\hat{\omega}$ of $L(G, U)$ onto the complex number field C defined by

$$\hat{\omega}(f) = \int_G f(x^{-1}) d\omega(x) \tag{7}$$

is a homomorphism of the algebra $L(G, U)$ onto C, namely

$$\hat{\omega}(f * g) = \int_G \int_G f(x^{-1}y) g(y^{-1}) dy\, d\omega(x) = \hat{\omega}(f)\hat{\omega}(g) \tag{8}$$

for every f, $g \in G$. We exclude the measure identically 0. The measure dx is an example of spherical measure. A spherical measure $d\omega(x)$ is uniquely determined by the homomorphism $\hat{\omega}$ because for every $f \in L(G)$, we have

$$\int_G f(x^{-1}) d\omega(x) = \int_G f(ux^{-1}u') d\omega(x) = \int_G {}^0f^0(x^{-1}) d\omega(x) = \hat{\omega}({}^0f^0).$$

If $\hat{\omega}$ is a homomorphism of $L(G, U)$ onto C such that the continuity assumption (C) holds for the linear functional $f \to \hat{\omega}({}^0f^0)$ of $L(G)$, then there exists a uniquely determined spherical measure $d\omega(x)$ such that $\hat{\omega}(f)$ is given by the integral (7) for $f \in L(G, U)$.

Let $d\omega(x)$ be a spherical measure on G. Then there exists a $f_0 \in L(G, U)$ such that

$$\hat{\omega}(f_0)=\int_G f_0(x^{-1})d\omega(x)\neq 0.$$

For every $f\in L(G, U)$ we have

$$\hat{\omega}(f*f_0)=\int_G\left\{\int_G f(x^{-1}y^{-1})f_0(y)\,dy\right\}d\omega(x)=\int_G f(y^{-1})\left\{\int_G f_0(yx^{-1})\,d\omega(x)\right\}dy$$

Hence we have

$$\hat{\omega}(f)=\int_G f(x^{-1})\left\{\int_G f_0(xy^{-1})\,d\omega(y)/\hat{\omega}(f_0)\right\}dx.$$

Since

$$\int_G f_0(uxu'y^{-1})d\omega(y)=\int_G f_0(xy^{-1})d\omega(y)$$

for all $x\in G$ and $u, u'\in U$, we have

$$d\omega(x)=\omega(x)dx \qquad \text{with} \qquad \omega(x)=\int_G f_0(xy^{-1})\,d\omega(y)/\hat{\omega}(f_0).$$

The function $\omega(x)$ is uniquely determined by $d\omega(x)$ and continuous. A function $\omega(x)\in C(G, U)$ will be called a zonal spherical function on G if $\omega(x)dx$ is a spherical measure. Every spherical measure $d\omega(x)$ is of the form $\omega(x)dx$ with a uniquely determined zonal spherical function $\omega(x)$.

Lemma 1. *If $\varphi(x)$ is a function in $C(G, U)$ such that*

$$\int_G f(x^{-1})\varphi(x)dx=0$$

for all $f\in L(G, U)$. Then we have $\varphi=0$.

Proof. Since $\varphi\in C(G, U)$, we have

$$\int_G f(x)\varphi(x)dx=\int_G {}^0f^0(x)\varphi(x)dx=0$$

for every $f\in L(G)$. Hence we have $\varphi(x)=0$.

Corollary. *Let $\omega(x)$ be a zonal spherical function on G and $\varphi(x)$ a function in $C(G, U)$. If*

$$\int_G f(x^{-1})\varphi(x)dx=c\hat{\omega}(f) \qquad (c\text{ ; a constant})$$

for every $f\in L(G, U)$, we have $\varphi(x)=c\omega(x)$.

Proposition 1. *If $\omega(x)$ is a zonal spherical function, then for every $f\in L(G, U)$ we have*

$$f*\omega=\omega*f=\hat{\omega}(f)\omega. \tag{9}$$

Proof. It suffices to prove $f*\omega=\hat{\omega}(f)\omega$. If $g\in L(G, U)$ we have

$$\int_G g(x^{-1}) f*\omega(x)dx = \int_G g(x^{-1}) \int_G f(xy^{-1})\omega(y)dydx$$
$$= \int_G f*g(y^{-1})\omega(y)dy = \hat{\omega}(g)\hat{\omega}(f).$$

Since $f*\omega \in C(G, U)$, from the Corollary of Lemma 1, we have

$$f*\omega = \hat{\omega}(f)\omega.$$

Proposition 2. *If $\omega(x)$ is a zonal spherical function on G, then we have*

$$\int_U \omega(xuy)\,du = \omega(x)\omega(y) \tag{10}$$

for all $x, y \in G$.

Proof. If $f \in L(G, U)$, we have, from Prop. 1,

$$\hat{\omega}(f)\,\omega(y) = \int_G f(x^{-1})\omega(xy)dx = \int_G f(x^{-1})\omega(xuy)dx = \int_G f(x^{-1})\left\{\int_U \omega(xuy)\,du\right\}dx.$$

Put $\omega_y(x) = \int_U \omega(xuy)du$. Then $\omega_y(x) \in C(G, U)$ and

$$\int_G f(x^{-1})\omega_y(x)dx = \hat{\omega}(f)\omega(y).$$

From the Corollary of Lemma 1, we have $\omega_y(x) = \omega(x)\omega(y)$.

Corollary. *Let e be the unit element of G. If $\omega(x)$ is a zonal spherical function on G, then we have*

$$\omega(e) = 1. \tag{11}$$

Proof. From (10), we have

$$\omega(x)\omega(e) = \int_U \omega(xu)du = \omega(x).$$

Since $\omega(x)$ is not identically 0 on G, we have $\omega(e) = 1$.

Proposition 3. *Let $\varphi(x)$ be a function in $C(G, U)$ being not identically 0, such that*

$$f*\varphi = \lambda_f \varphi$$

for every $f \in L(G, U)$ where λ_f is a constant depending on f. Then we have $\varphi(e) \neq 0$ and $\omega(x) = \varphi(x)/\varphi(e)$ is a zonal spherical function on G.

Proof. From the assumption, we have

$$\lambda_f \varphi(e) = \int_G f(x^{-1})\varphi(x)dx.$$

From Lemma 1, we have $\lambda_f\psi(e)\neq 0$ for some $f\in L(G, U)$, hence $\psi(e)\neq 0$. If f and g are in $L(G, U)$, then we have

$$(f*g)*\psi(x)=\lambda_f\lambda_g\psi(x)=\lambda_{f*g}\psi(x).$$

Hence $(\psi(x)/\psi(e))dx$ is a spherical measure because

$$f*\psi(e)/\varphi(e)=\int_G f(x^{-1})\psi(x)/\psi(e)\,dx=\lambda_f.$$

Proposition 4. *A function $\omega(x)\in C(G, U)$ is a zonal spherical function on G if and only if $\omega(e)=1$ and $f*\omega=\lambda_f\omega$ for all $f\in L(G, U)$.*

Proof. From the Corollary of Prop. 3 and Prop. 4 we can prove our assertion.

It is easy to see that a function $0\neq\omega(x)\in C(G, U)$ is a zonal spherical function if and only if the functional equatiou (10) holds.

Proposition 5. *Let $\chi(x)$ be a function in $C(G, U)$ such that $\chi(xy)=\chi(x)\chi(y)$ for all $x, y\in G$. If $\omega(x)$ is a zonal spherical function on G, then $\chi(x)\omega(x)$ is also a zonal spherical function provided χ is not identically 0.*

Proof. From the assumption we have $\chi(e)=1$. For every $f\in L(G, U)$ we have

$$f*(\chi\omega)=\int_G f(y^{-1})\chi(yx)\omega(yx)dy=\chi(x)\int_G f(y^{-1})\chi(y)\omega(yx)dy=\hat{\omega}(f\tilde{\chi})\chi(x)\omega(x).$$

Since $\chi\omega\in C(G, U)$, $\chi\omega$ is a zonal spherical function.

Remark. $\chi(x)$ itself a zonal spherical function because $\omega(x)=1$ is a zonal spherical function.

A function $\varphi(x)\in C(U\backslash G)$ will be called right spherical if

$$f*\varphi=\lambda_f\varphi$$

for all $f\in L(G, U)$ where λ_f is a constant depending on f. Similarly a function $\psi\in C(G/U)$ will be called left spherical if

$$\psi*f=\lambda_f\psi$$

for all $f\in L(G, U)$. If φ is right spherical, then $\tilde{\varphi}$ is left spherical. If $\varphi\neq 0$ is left and right spherical, then $\varphi/\varphi(e)$ is a zonal spherical function.

Let φ be a right spherical function which is not identically 0. Then there exists a uniquely determined zonal spherical function $\omega(x)$ such that

$$\int_U \varphi(xux_0)\,du=\varphi(x_0)\omega(x) \tag{12}$$

for all $x_0\in G$. For if $f\in L(G, U)$, we have

$$\int_G f(x^{-1})\int_U \varphi(xux_0)du\,dx=\lambda_f\int_U \varphi(ux)du=\lambda_f\varphi(x_0)$$

and $\omega'(x)=\int_U \varphi(xux_0)du\in(G, U)$. Since $\lambda_{f*g}=\lambda_f\lambda_g$, $\omega'(x)/\varphi(x_0)=\omega(x)$ is a zonal spherical function such that $\hat{\omega}(f)=\lambda_f$ provided $\varphi(x_0)\neq 0$.*) Hence $\omega(x)$ does not depend on x_0. If $\varphi(x_0)=0$, then from Lemma 1, we have $\omega'(x)=0$. We will call such φ a right spherical function belonging to ω. Similarly if ψ is a left spherical function, then we have

$$\int_U \psi(x_0ux)du=\psi(x_0)\omega(x) \tag{13}$$

with a zonal spherical function ω. We will call ψ a left spherical function belonging to ω.

Let H be a closed subgroup of G such that $G=UH$ and ψ a continuous homomorphism of H onto the multiplicative group of all complex numbers $\neq 0$ such that $\psi(U\frown H)=\{1\}$. Then for $ut=x$, $u\in U$ and $t\in H$, we put $\varphi(x)=\psi(t)$ and have a function $\varphi\in C(U\backslash G)$. It is easy to see that ψ is a right spherical function, for if $f\in L(G, U)$, then

$$f*\varphi(x)=\int_G f(uy)\varphi(y^{-1}t)dy=\psi(t)\int_G f(y)\varphi(y^{-1})dy=\lambda_f\varphi(x).$$

For every ψ we have a zonal spherical function ω_ψ by the integral

$$\omega_\psi(x)=\int_U \varphi(xu)du.$$

In many cases, we can obtain all zonal spherical functions by this way if we choose a suitable H.

Let s be an element of G. The right translation R_s and the left translation L_s are defined by

$$(R_sf)(x)=f(xs)$$

$$\text{and}\qquad (L_sf)(x)=f(s^{-1})x.$$

If $s_1, s_2\in G$, then we have $R_{s_1}R_{s_2}=R_{s_1s_2}$ and $L_{s_1}L_{s_2}=L_{s_1s_2}$. It is obvious that R_s and L_s are continuous linear operators of $C(G)$.

Lemma 2. *Let f and f' be function in $L(G)$ and g a function in $C(G)$. Then for every compact set K_0 and $\varepsilon>0$, there exist constants $c_1,\cdots, c_N$ and $y_1, \cdots, y_N\in G$ such that*

*) Note that $\omega(e)=1$ if $\varphi(x_0)\neq 0$.

$$|f*g(x)-\sum_{i=1}^{N}c_i f(xy_i^{-1})|<\varepsilon \tag{14}$$

and

$$|f'*g(x)-\sum_{i=1}^{N}c_i f'(xy_i^{-1})|<\varepsilon \tag{15}$$

uniformly on K_0.

Proof. Since f and $f'\in L(G)$, we can find a compact set K such that $f(xy^{-1})=f'(xy^{-1})=0$ for $x\in K$ and $y\notin K$. Then we have

$$f*g(x)=\int_K f(xy^{-1})g(y)dy$$

and

$$f'*g(x)=\int_K f'(xy^{-1})g(y)dy.$$

Since the function $F(x, y)=f(xy^{-1})g(y)$ is uniformly continuous on $K_0\times K$, we can find a decomposition $K=E_1\cup\cdots\cup E_N$ of K into mutually disjoint measurable sets $E_1,\cdots, E_N$ so that

$$|f*g(x)-\sum_{i=1}^{N}f(xy_i^{-1})g(y_i)\int_{E_i}dy|<\varepsilon$$

and $$|f'*g(x)-\sum_{i=1}^{N}f'(xy_i^{-1})g(y_i)\int_{E_i}dy|<\varepsilon$$

for all $x\in K_0$ where $y_1,\cdots, y_N$ are points on $E_1,\cdots, E_N$ respectively. Putting $c_i=g(y_i)\int_{E_i}dy$, we have inequalities (14) and (15).

Let $\mathfrak{c}=(c_1,\cdots, c_N)$ be an ordered set of complex numbers and $\mathfrak{y}=(y_1,\cdots, y_N)$ an ordered set of elements on G. For every $f\in L(G)$, we put

$$f_{\mathfrak{c},\mathfrak{y}}(x)=\sum_{i=1}^{N}c_i f(xy_i^{-1}).$$

Using this notation, we have the following

Lemma 3. *If f, f' are functions in $L(G)$, then for a $g\in C(G)$ there exist $\mathfrak{c}_1, \mathfrak{c}_2,\cdots$ and $\mathfrak{y}_1, \mathfrak{y}_2,\cdots$ such that*

$$\lim_{n\to\infty} f_{\mathfrak{c}_n,\mathfrak{y}_n}(x)=f*g(x)$$

and

$$\lim_{n\to\infty} f'_{\mathfrak{c}_n,\mathfrak{y}_n}(x)=f'*g(x).$$

"$\lim_{n\to\infty}$" *means the limes with respect to the topology of $C(G)$.*

Proof. Let $K_1 \subset K_2 \subset K_3 \subset \cdots$ be a series of compact sets on G such that every compact set K_0 is contained in some K_ν. From Lemma 2, we can find $\mathfrak{c}_n = (c_1^{(n)}, \cdots, c_{N_n}^{(n)})$ and $\mathfrak{y}_n = (y_1^{(n)}, \cdots, y_{N_n}^{(n)})$ such that

$$|f * g(x) - f_{\mathfrak{c}_n, \mathfrak{y}_n}(x)| < 1/n$$

$$\text{and} \qquad |f' * g(x) - f'_{\mathfrak{c}_n, \mathfrak{y}_n}(x)| < 1/n$$

for all $x \in K_n$. Then we have

$$\lim_{n \to \infty} f_{\mathfrak{c}_n, \mathfrak{y}_n} = f * g$$

and

$$\lim_{n \to \infty} f'_{\mathfrak{c}_n, \mathfrak{y}_n} = f * g.$$

Lemma 4. *Let T be a continuous linear operator of $C(G)$ such that*

1) $TR_s = R_s T$ *for all* $s \in G$,

and 2) $T(L(G)) \subset L(G)$.

Then for every $f \in L(G)$ and $g \in C(G)$, we have

$$T(f * g) = (Tf) * g.$$

Proof. Put $Tf = f'$. Then f' is a function in $L(G)$, so we can find, from Lemma 3, $\mathfrak{c}_1, \mathfrak{c}_2, \cdots$ and $\mathfrak{y}_1, \mathfrak{y}_2, \cdots$ such that

$$\lim_{n \to \infty} f_{\mathfrak{c}_n, \mathfrak{y}_n} = f * g$$

and

$$\lim_{n \to \infty} f'_{\mathfrak{c}_n, \mathfrak{y}_n} = f' * g.$$

On the other hand, we have

$$T(f_{\mathfrak{c}_n, \mathfrak{y}_n}) = f'_{\mathfrak{c}_n, \mathfrak{y}_n}.$$

Hence from the continuity of T, we have $T(f * g) = Tf * g$.

Proposition 6. *Let T be a continuous linear operateor of $C(G)$ such that*

1) $TR_s = R_s T$ *for all* $s \in G$,

2) $TL_u = L_u T$ *for all* $u \in U$,

and 3) $T(L(G)) \subset L(G)$.

Then for every right spherical function φ we have

$$T\varphi = \lambda\varphi$$

where λ is a constant depending on T and the zonal spherical function ω to which φ belongs.

Proof. Assume that $\varphi \neq 0$. Then there exists a $f_0 \in L(G, U)$ with $\hat{\omega}(f) \neq 0$. Put $Tf = f_1$. Since $L_u f_0 = f_0 R_u = f_0$, we have, from 1) and 2), $L_u f_1 = f_1 R_u = f_1$, hence

$f_1 \in L(G, U)$. Therefore from Lemma 4 we have

$$T(f_0 * \varphi) = \hat{\omega}(f_0) T\varphi = (Tf_0) * \varphi = \hat{\omega}(f_1)\varphi$$

and

$$T\varphi = \hat{\omega}(Tf_0)\hat{\omega}(f_0)^{-1}\varphi.$$

An operator T satisfying the conditions 1), 2) and 3) maps $C(U\backslash G)$ into itself, so defines a continuous operator T_U of $C(U\backslash G)$. We will call T_U a right invariant operator of $C(U\backslash G)$. We will show that if T_U and S_U are right invariant operators of $C(U\backslash G)$, then $T_U \cdot S_U = S_U \cdot T_U$. To prove our assertion, we have to introduce the notion of approximate identities of $L(G, U)$. Let $\{V_i\}$ $(i=1,2,\cdots)$ be a basis of open neighbourhoods of e such that $V_{i-1} \supset V_i^2$, $V_i^{-1} = V_i$ and $\bar{V}_i$ is compact for all i. Put

$$h_i'(x) = \chi_{V_i} * \chi_{V_i}(x) \Big/ \left\{ \int_{V_i} dx \right\}^2$$

where χ_{V_i} is the characteristic function of V_i. Then $h_i'(x)$ is a continuous positive definite function being equal to 0 for $x \notin V_{i-1}$, and $\|h_i'\|_1 = 1$. It is easy to see that for every $f \in C(G)$ we have

$$\lim_{i\to\infty} h_i' * f = f.$$

Put

$$h_i(x) = \int_U \int_U h_i(uxu')dudu'.$$

Then for $f \in C(U\backslash G)$ we have

$$h_i * f(x) = \int_U \int_U \int_G h_i'(uxu'y^{-1}) f(y) dydudu'$$

$$= \int_U h_i' * f(ux)du = {}^0(h_i' * f)(x).$$

Since ${}^0 f = f$ we have

$$\lim_{i\to\infty} h_i * f = f.$$

We will call $\{h_i\}$ a series of approximate identities of $L(G, U)$. Let T be a continuous operator of $C(G)$ satisfying the conditions 1), 2) and 3). For every $g \in L(G, U)$ we have

$$T(g*f) = T(\lim_{i\to\infty} h_i * g * f) = \lim_{i\to\infty} T(h_i * g * f) = \lim_{i\to\infty} (Th_i) * g * f$$

$$= \lim_{i\to\infty} g * Th_i * f = g * T(\lim_{i\to\infty} h_i * f) = g * Tf.$$

Hence if T_U and S_U are right invariant operators of $C(U/G)$, we have

$$T_U \cdot S_U(g*f)=(T_U g)*(S_U f)=S_U T_U(g*f).$$

Therefore we have

$$\lim_{i\to\infty} T_U S_U(h_i*f)=T_U S_U f=\lim_{i\to\infty} S_U T_U(h_i*f)=S_U T_U f$$

for every $f\in C(U\backslash G)$.

Now we would like to study the behavior of zonal spherical functions on the direct product of two or more groups. Let G_1 and G_2 be a locally compact groups, U_1 and U_2 compact subgroups of G_1 and G_2 respectively. Assume that $L(G_1, U_1)$ and $L(G_2, U_2)$ are commutative. Let $L(G_1, U_1)\otimes L(G_2, U_2)$ be the vector space of all functions $f(x_1, x_2)$ on $G=G_1\times G_2$ of the form

$$f(x_1, x_2)=\sum_\nu f_\nu^{(1)}(x_1)f_\nu^{(2)}(x_2)$$

with $f_\nu^{(1)}\in L(G_1, U_1)$ and $f_\nu^{(2)}\in L(G_2, U_2)$. It is easy to see that $L(G_1, U_1)\otimes L(G_2, U_2)$ is a subset of $L(G, U)$, $U=U_1\times U_2$, and everywhere dense in $L(G, U)$ with respect to the norm $\|\ \|_\infty$. Hence $L(G, U)$ is commutative because $L(G_1, U_1)\otimes L(G_2, U_2)$ is the tensor product of $L(G_1, U_1)$ and $L(G_2, U_2)$.

Proposition 7. *If $\omega^{(1)}(x_1)$ and $\omega^{(2)}(x_2)$ are zonal spherical funtions on G_1 and G_2 respectively, then $\omega(x_1, x_2)=\omega^{(1)}(x_1)\omega^{(2)}(x_2)$ is a zonal spherical function on G. Conversely every zonal spherical function on G can be represented by this form.*

Proof. Let $f(x_1, x_2)$ be a function in $L(G, U)$. Then we have

$$\int_{G_1} f(x_1y_1^{-1}, x_2)\omega^{(1)}(y_1)dy_1=\lambda(x_2)\omega^{(1)}(x_1)$$

with $\lambda(x_2)\in L(G_2, U_2)$. Hence we have

$$\int_{G_1}\int_{G_2} f(x_1y_1^{-1}, x_2y_2^{-1})\omega^{(1)}(y_1)\omega^{(2)}(y_2)dy_1dy_2=\hat{\omega}^{(2)}(\lambda)\omega^{(1)}(x_1)\omega^{(2)}(x_2).$$

Since $\omega^{(1)}(e_1)\omega^{(2)}(e_2)=1$, from Prop. 4, $\omega^{(1)}(x_1)\omega^{(2)}(x_2)=\omega(x_1, x_2)$ is a zonal spherical function on G. Conversely assume that $\omega(x_1, x_2)$ is a zonal spherical function on $G_1\times G_2=G$. For every $f^{(1)}\in L(G_1, U_1)$, we define an operator $T_{f^{(1)}}$ of $C(G)$ by

$$T_{f^{(1)}}g(x_1, x_2)=\int_{G_1} f^{(1)}(x_1y_1^{-1})g(y_1, x_2)dy_1.$$

It is easy to see that $T_{f^{(1)}}$ induces a right invariant operator of $C(U\backslash G)$. Hence from Prop. 6 we have

$$T_{f^{(1)}}\omega(x_1, x_2)=\lambda_{f^{(1)}}\omega(x_1, x_2).$$

Therefore $\omega(x_1, e_2)$ is a zonal spherical function on G_1, and we have $\omega(x_1, x_2)$

$=\omega(x_1, e_2)\omega(e_1, x_2)$. Similarly we see that $\omega(e_1, x_2)$ is a zonal spherical function on G_2. Hence we have proved our assertion.

By using induction, we have the following:

Corollary. *Let $G_1, \cdots, G_r$ be locally compact groups and $U_1, \cdots, U_r$ compact subgroups of $G_1, \cdots, G_r$ respectively such that each algebra $L(G_i, U_i)$ is commutative. Then for $G=G_1\times\cdots\times G_r$ and $U=U_1\times\cdots\times U_r$ the algebra $L(G, U)$ is commutative, and for every zonal spherical function $\omega(x_1, \cdots, x_r)$ on G there exist uniquely determined zonal spherical functions $\omega^{(1)}(x_1), \cdots, \omega^{(r)}(x_r)$ on $G_1, \cdots, G_r$ respectively such that*

$$\omega(x_1, \cdots, x_r)=\omega^{(1)}(x_1)\cdots\omega^{(r)}(x_r).$$

Conversely every function $\omega(x_1, \cdots, x_r)$ of this form is a zonal spherical function on $G=G_1\times\cdots\times G_r$.

Let I' be a countable set of indices $p, q, \cdots$, $G_p, G_q, \cdots$ locally compact groups associated to indices $p, q, \cdots\in I'$ and $U_p, U_q, \cdots$ open compact subgroups of $G_p, G_q, \cdots$ respectively. Assume that $L(G_p, U_p)$ is commutative for every $p\in I'$. Let G' be the subgroup of the direct product $\prod_{p\in I'} G_p$ of all G_p of all elements $x=(\cdots x_p\cdots)$ such that $x_p\in U_p$ for all but a finite number of p. The group G contains the direct product $U'=\prod_p U_p$. Since U_p is compact, U' is a compact group with respect to its natural topology. It is easy to see that we can introduce a uniquely determined topology into G' so that G' is a locally compact group and U' is an open subgroup of G'. Now we see that $L(G', U')$ is the tensor product of all $L(G_p, U_p)$.*) Hence $L(G', U')$ is commutative. If $\hat{\omega}$ is a homomorphism of $L(G', U')$ onto G, then there exists uniquely determined homomorphism $\hat{\omega}_p$ of $L(G_p, U_p)$ onto C such that

$$\hat{\omega}(f_{p_1}\otimes\cdots\otimes f_{p_r})=\hat{\omega}_{p_1}(f_{p_1})\cdots\hat{\omega}_{p_r}(f_{p_r})$$

for every $f_{p_1}\otimes\cdots\otimes f_{p_r}\in L(G_{p_1}, U_{p_1})\otimes\cdots\otimes L(G_{p_r}\otimes U_{p_r})\subset(L(G', U')$. Since U_p and U' are open subgroups of G_p and G' respectively, every homomorphism $\hat{\omega}$ and $\hat{\omega}_p$ define zonal spherical functions ω and ω_p on G and G_p respectively. It is easy to see that

$$\omega(x)=\prod_{p\in I'}\omega_p(x_p) \qquad \text{for } x=(\cdots x_p\cdots). \tag{16}$$

Note that this infinite product is essentially a finite product for every $x\in G'$ because $\omega_p(u_p)=1$ for $u_p\in U_p$. Conversely if we associate a zonal spherical function ω_p to every $p\in I'$, then (16) defines a zonal spherical function $\omega(x)$ on G'.

Now we add a finite number of indices $p_{\infty,1}, \cdots, p_{\infty,r}$ to I', and consider locally

*) Note that $L(G_p, U_p)$ contains the unit element.

compact groups $G_{p\infty,1},\cdots,G_{p\infty,r}$ with compact subgroups $U_{p\infty,1},\cdots,U_{p\infty,r}$ such that each $L(G_{p\infty,i}, U_{p\infty,i})$ is commutative. Put $I=I'\smile\{p_{\infty,1},\cdots,p_{\infty,r}\}$. We will call the group

$$G=G'\times G_{p\infty,1}\times\cdots\times G_{p\infty,r}$$

the restiricted direct product the G_p relative to U_p. Put

$$U=U'\times U_{p\infty,1}\times\cdots\times U_{\infty,r}.$$

Then U is a compact subgroup of G and $L(G, U)$ is commutative. Using Proposition 7 and its Corollary, we have the following

Proposition 8. *Let I be a countable set of indeces, $G_p, G_q,\cdots$ locally compact groups associated to indices $p, q,\cdots\in I$ and $U_p, U_q,\cdots$ compact subgroups of G_p, $G_q,\cdots$ such that U_p is open for all but a finite number of p. Let $G=\Pi' G_p$ the restricted direct product of all G_p relative to U_p and U the compact subgroup $\prod_p U_p$ of G. If $L(G_p, U_p)$ is commutative for all p, then $L(G, U)$ is commutative, and every zonal spherical function ω on G has the form*

$$\omega(x)=\prod_{p\in I}\omega_p(x_p) \qquad \textit{for } x=(\cdots x_p\cdots)$$

where ω_p is a zonal spherical function on G_p determined uniquely by ω. Conversely if we assoctate a zonal spherical function ω_p to each $p\in I$, then the function ω on G defined by this infinite product is a zonal spherical function on G.

This proposition will play an important role in the application of our theory to number theory.

§3. Existence theorem and the space of zonal spherical functions.

We have not yet proved that there exist sufficiently many zonal spherical functions. Using analogous method as in the case of locally compact abelian groups, we can prove the following theorem.

Theorem 1. *For every $0\neq f\in L(G, U)$, there exists a zonal spherical function $\omega(x)$ such that*

$$\hat{\omega}(f)\neq 0$$

$$\textit{and } |\omega(x)|\leqq 1 \textit{ for all } x\in G.$$

Proof. We consider the algebra $L_1(G, U)$. If U is not open, we add an identity 1 to $L_1(G, U)$ and form an algebra $C1\otimes L_1(G, U)=A(G, U)$*) with the norm

$$\|\alpha\mathbf{1}+f\|=|\alpha|+\|f\|_1$$

*) If U is open, we put $L_1(G, U)=A(G, U)$.

Then we get a normed ring*) $A(G, U)$ with this norm $\| \|$. We will prove that if $f\in L_1(G, U)$, $f\neq 0$, then $f*f^*=h$ is not contained in the radical of $A(G, U)$. Let T_h be the operator of the Hilbert space $L_2(G)$ defined by

$$g\to T_h g=h*g.$$

Since

$$\|h*g\|_2\leqq\|h\|_1\ \|g\|_2\,,$$

T_h defines bounded Hermitian positive definite operator of $L_2(G)$ with the norm $\|T_h\|\leqq\|h\|_1$. Put $h^n=\underbrace{h*\cdots*h}_{n}$. Then we have

$$\|T_{h^{2^n}}\|=\|T_h^{2^n}\|=\|T_h\|^{2^n}\leqq\|h^{2^n}\|_1\,.$$

Hence we have

$$\lim_{n\to\infty}\{\|h^{2^n}\|_1\}^{1/2^n}\geqq\|T_h\|>0.$$

Therefore h is not contained in the redical $\mathfrak{R}$ of $A(G, U)$, so f is not in $\mathfrak{R}$ either. Therefore there exists a maximal ideal $\mathfrak{z}$ of $A(G, U)$ such that the value $f(\mathfrak{z})$ of f at $\mathfrak{z}$, namely a complex number such that $f-f(\mathfrak{z})\in\mathfrak{z}$, is not equal to 0. Every maximal ideal $\mathfrak{z}$ of $A(G, U)$ such that $\mathfrak{z}\neq L_1(G, U)$ defines a linear functional

$$f\to f(\mathfrak{z})=\hat{\omega}_{\mathfrak{z}}(f)$$

of $L(G, U)$ which is continuous with respect to the norm $\| \|_1$. Conversely if $\hat{\omega}$ is a linear functional of $L(G, U)$ continuous with respect to $\| \|_1$ such that $\hat{\omega}(f*g)$ $\hat{\omega}(f)\hat{\omega}(g)$, then $\hat{\omega}$ gives a homomorphism of $L(G, U)$ onto C which is extended to $L_1(G, U)$, so there exists a uniquely determined maximal ideal $\mathfrak{z}$ such that $\hat{\omega}=\hat{\omega}_{\mathfrak{z}}$. Since

$$|\hat{\omega}_{\mathfrak{z}}(f)|\leqq\|f\|_1$$

for all $f\in L_1(G, U)$ we have a zonal spherical function $\omega_{\mathfrak{z}}$ uniquely determined by $\mathfrak{z}$ such that $\hat{\omega}_{\mathfrak{z}}(f)=\int_G f(x)^{-1}\omega_{\mathfrak{z}}(x)dx$, and $|\omega_{\mathfrak{z}}(x)|\leq 1$ for all $x\in G$. Thus we have proved our assertion.

From the proof of Theorem 1, we see the following fact:

Proposition 9. *There exists a one-to-one correspondence between the set of all bounded zonal spherical functions and the set of all maximal ideal* $\mathfrak{z}$ *of* $A(G, U)$ *such that* $\mathfrak{z}\neq L_1(G, U)$.

Let $\mathfrak{S}(G)$ be the set of all zonal spherical functions on G, $\mathfrak{S}_1(G)$ the subset of all bounded zonal spherical functions and $\mathfrak{P}(G)$ the subset of all positive definite zonal spherical functions. For every $f\in L(G, U)$ we define a function $\hat{f}$ on $\mathfrak{S}(G)$ by

*) cf. Gelfand [1].

$$\hat{f}(\omega)=\hat{\omega}(f)=\int_G f(x^{-1})\omega(x)dx$$

We will call $\hat{f}$ the Fourier transform of f. We introduce the weakest topology into $\mathfrak{S}(G)$ so that $\hat{f}$ is continuous for all $f\in L(G, U)$. Then $\mathfrak{S}_1(G)$ is a closed subspace of $\mathfrak{S}(G)$, and from Prop. 9, $\mathfrak{S}_1(G)$ is locally compact. If U is open, then $\mathfrak{S}_1(G)$ is compact. It is obvious that $\mathfrak{P}(G)$ is a closed subspace of $\mathfrak{S}_1(G)$, so $\mathfrak{P}(G)$ is locally compact. For every $f\in L_1(G, U)$ we have a function $\hat{f}$ on $\mathfrak{S}_1(G)$ defined by

$$\hat{f}(\omega)=\int_G f(x^{-1})\omega(x)dx.$$

It is easy to see that $\hat{f}$ is continuous on $\mathfrak{S}_1(G)$. It is known that there exists a positive measure $d\mu(\omega)$ on $\mathfrak{P}(\mathfrak{S})$ such that for every $f\in L(G, U)$ we have

$$\int_G |f|^2 d(x)=\int_{\mathfrak{P}(\mathfrak{S})} |\hat{f}|^2 d\mu(x).$$

If $G_1,\cdots, G_r$ are locally compact groups and $U_1,\cdots, U_r$ compact subgroups of $G_1, \cdots, G_r$ respectively satisfying assumptions of the Corollary of Prop. 7, then we have

$$\mathfrak{S}(G)=\mathfrak{S}(G_1)\times\cdots\times\mathfrak{S}(G_r),\ \mathfrak{S}_1(G)=\mathfrak{S}_1(G_1)\times\cdots\times\mathfrak{S}_1(G_r)$$
$$\text{and}\qquad \mathfrak{P}(G)=\mathfrak{P}(G_1)\times\cdots\times\mathfrak{P}(G_r).$$

If $G=\Pi' G_p$ is the restricted direct product of G_p relative to U_p, as we described in Prop. 8, then we have

$$\mathfrak{S}_1(G)=\prod_p \mathfrak{S}_1(G_p)$$
$$\text{and}\qquad \mathfrak{P}(G)=\prod_p \mathfrak{P}(G_p).$$

§4. Automorphic functions.

Let Γ be a discrete subgroup of G suh that homogeneous space G/Γ is compact. Then there exists a uniquely determined G-invariant Borel measure $d\bar{x}$ on G/Γ such that for every $f\in L(G)$ we have

$$\int_G f(x)dx=\int_{G/\Gamma}\Big(\sum_{\Gamma\ni\alpha} f(x\alpha)\Big)d\bar{x}.$$

If $f\in C(G)$ is Γ-right invariant, namely $f(x\alpha)=f(x)$ for all $x\in G$ and $\alpha\in\Gamma$, we can consider f a continuous function on G/Γ. Conversely every continuous functions on G/Γ is considered a Γ right invariant continuous function on G. Let $C(G/\Gamma)$ be the vector space of all continuous function on G/Γ. Since G/Γ is compact, $C(G/\Gamma)$ is a Banach space with respect to the norm $\|\ \|_\infty$. For simplicity we write

$$\int_{G/\Gamma} f(x)d\bar{x}=\int_{G/\Gamma} f(x)dx.$$

Let $L_p(G/\Gamma)$ be the vector space of all measurable Γ-right invariant function such that

$$\left\{\int_{G/\Gamma} |f|^p dx\right\}^{\frac{1}{p}}=\|f\|_{p,\Gamma}<\infty.$$

Then $L_p(G/\Gamma)$ is a Banach space and $C(G/\Gamma)$ is an everywhere dense subspace of $L_p(G/\Gamma)$. Let $C(U\backslash G/\Gamma)$ and $L_p(U\backslash G/\Gamma)$ be subspaces of $C(G/\Gamma)$ and $L_p(G/\Gamma)$ of all f with $f(ux)=f(x)$ for $u\in U$ and $x\in G$ respectively.

Lemma 5. *If $f\in L_1(G)$ and $g\in L_p(G/\Gamma)$, then we have*

$$f*g\in L_p(G/\Gamma)$$
$$\text{and}\qquad \|f*g\|_{p,\Gamma}\leqq\|f\|_1\,\|g\|_{p,\Gamma}$$

Proof. Let $h(x)$ be a function in $L_{p'}(G/\Gamma)$ with $1/p+1/p'=1$. Then we have

$$\left|\int_{G/\Gamma}\left\{\int_G f(xy^{-1})g(y)dy\right\}h(x)dx\right|=\left|\int_G f(y^{-1})\left\{\int_{G/\Gamma} g(yx)h(x)dx\right\}dy\right|$$
$$\leqq\int_G |f(y^{-1})|\left\{\left|\int_{G/\Gamma} g(yx)h(x)dx\right|\right\}dy\leqq\|g\|_{p,\Gamma}\,\|h\|_{p',\Gamma}\,\|f\|_1$$

for all $f\in L_{p'}(G/\Gamma)$. Hence $f*g$ is defined and contained in $L_p(G/\Gamma)$ for all $f\in L_1(G)$ and we have

$$\|f*g\|_{p,\Gamma}\leqq\|f\|_1\,\|g\|_{p,\Gamma}$$

Lemma 6. *Let $\{h_\nu\}$ be a series of approximate identities of $L(G, U)$. Then for $g\in L_p(G/\Gamma)$ we have*

$$\lim_{\nu\to\infty}\|g-h_\nu*g\|_p=0$$

Proof. First we prove that

$$\lim_{\nu\to\infty}\|g-h'_\nu*g\|_p=0$$

for all $g\in L_p(G/\Gamma)$. Let g_0 be a function in $C(G/\Gamma)$ such that $\|g-g_0\|_{p,\Gamma}<\delta$ for any given $\delta>0$. Since G/Γ is compact there exists a neighbourhood V of e such that for every $s\in V$ we have

$$\|g_0(sx)-g_0(x)\|_\infty<\delta.$$

Then we have

$$\|g(sx)-g(x)\|_{p,\Gamma}\leqq\|g(sx)-g_0(sx)\|_{p,\Gamma}+\|g_0(sx)-g_0(x)\|_{p,\Gamma}+\|g_0(x)-g(x)\|_{p,\Gamma}$$

$$<2\delta+\delta\left(\int_{G/\Gamma} dx\right)^{\frac{1}{p}}.$$

Hence for any $\varepsilon>0$ we can choose V such that

$$\|g(sx)-g(x)\|_{p,\Gamma}<\varepsilon$$

for all $s\in V$. Now for sufficiently large i, we have $V_{i-1}\subset V$ and if $k(x)\in L_{p'}(G/\Gamma)$ we have

$$\left|\int_{G/\Gamma} k(x)(h_i'*g(x)-g(x))dx\right|\leqq\int_G h_i'(y^{-1})\left\{\int_{G/\Gamma} k(x)(g(yx)-g(x))dx\leqq\varepsilon\|k\|_{p',\Gamma}.\right.$$

Hence we have

$$\|h_i'*g-g\|_{p,\Gamma}<\varepsilon.$$

If $g\in L_p(U\backslash G/\Gamma)$ then we have

$$h_i*g=\int_U h_i'*g(ux)du$$

and

$$\|h_i*g-g\|_{p,\Gamma}^p=\int_{G/\Gamma} dx\left|\int_U(h_i'*g(ux)-g(ux))du\right|^p dx$$
$$\leqq\int_{G/\Gamma} dx\int_U|h_i'*g(ux)-g(ux)|^p du\leqq\|h_i'*g-g\|_{p,\Gamma}^p<\varepsilon^p.$$

Hence we have proved our assertion.

A function $\varphi\in C(U\backslash G/\Gamma)$ will be called (right) automorphic if φ is a right spherical function. If $\varphi\neq 0$, then φ belongs to a zonal spherical function $\omega(x)$ defined by

$$\omega(x)=\int_U\varphi(xux_0)du/\varphi(x_0)$$

Proposition 10. *If $\varphi(x)$ is an automorphic function which is not identically 0, then the zonal spherical function to which φ belongs is postive definite.*

Proof. Put

$$\omega'(s)=\frac{1}{\|\varphi\|_2^2}\int_{G/\Gamma}\varphi(sx)\overline{\varphi(x)}dx \tag{17}$$

Then it is easy to see that $\omega'(s)\in C(G,U)$ and $\omega'(e)=1$. For every $f\in L(G,U)$ we have

$$f*\omega'(s)=\frac{1}{\|\varphi\|_{2,\Gamma}^2}\int_G f(st^{-1})\left\{\int_{G/\Gamma}\varphi(tx)\overline{\varphi(x)}dx\right\}dt$$
$$=\frac{1}{\|\varphi\|_{2,\Gamma}^2}\int_{G/\Gamma}\left\{\int_G f(st^{-1})\varphi(tx)dt\right\}\overline{\varphi(x)}dx=\hat{\omega}(f)\omega'(s).$$

Hence we have $\omega'(s)=\omega(s)$. The function $\omega'(s)$ defined by the integral (17) is obviously positive definite.

The vector space $L_2(G/T)=H(G/\Gamma)$ is a Hilbert space with the inner product

$$(f, g)_\Gamma=\int_{G/\Gamma} f(x)\bar{g}(x)dx.$$

The space $L_2(U\backslash G/\Gamma)=H(U\backslash G/\Gamma)$ is a closed subspace of $H(G/\Gamma)$. For the sake of simplicity we denote the space by H_0.

Let f be a function in $L(G)$. Then an operator K_f of $H(G/\Gamma)$ defined by

$$K_f g=f*g \qquad g\in H(G/\Gamma).$$

is completely continuous. For we have

$$\int_G f(xy^{-1})g(y)dy=\int_{G/\Gamma}(\sum_{\Gamma\ni\alpha}f(x\alpha y^{-1}))g(y)dy=\int_{G/\Gamma}k_f(x, y)g(y)dy=f*g(x)=K_f g(x).$$

Now the function

$$k_f(x, y)=\sum_{\Gamma\in\alpha}f(x\alpha y^{-1})$$

is continuous because the series converges absolutely and uniformly provided x and y belong to a compact set K. Since G/Γ is conpact, the operator K_f is obviously completely continuous. From Lemma 5, we have $\|K_f\|\leqq\|f\|_1$. If $f\in L_1(G)$ then K_f: $K_f g=f*g$ is a bounded operator of $H(G/\Gamma)$ and we see that K_f is completely continuous. For we can find a series $\{f_\nu\}$, $f_\nu\in L(G)$ such that $\lim\|f-f_\nu\|_1=0$. Then from Lemma 1 we have

$$\lim_{\nu\to\infty}\|K_f-K_{f_\nu}\|=0.$$

Since K_{f_ν} is completely continuous for all ν, K_f is also completely continuous.

Proposition 11. *If $f\in L_1(G)$, the operator K_f of $H(G/L)$ defined by*

$$K_f g=f*g$$

is completely continuous with

$$\|K_f\|=\|f\|_1.$$

Let g and h be functions in $H(G/\Gamma)$. If $f\in L_1(G)$ then we have

$$\begin{aligned}(K_f g, h)&=\int_{G/\Gamma}\left\{\int_G f(xy^{-1})g(y)dy\right\}\overline{h(x)}dx\\&=\int_{G/\Gamma}\left(\int_G f(y^{-1})g(yx)dy\right)\overline{h(x)}dx=\int_G f(y^{-1})\left(\int_{G/\Gamma}g(yx)\overline{h(x)}dx\right)dy\\&=\int_G f(y^{-1})\left(\int_{G/\Gamma}g(x)\overline{h(y^{-1}x)}\right)dy=\int_{G/\Gamma}g(x)\left(\int_G f(y^{-1})\overline{h(y^{-1}x)}\,dy\right)dx.\end{aligned}$$

Hence we have $K_f^* = K_{f^*}$. If $f \in L_1(G, U)$, then we have $K_f K_f^* = K_{f \cdot f^*} = K_f^* K_f$, so K_f is a normal operator. If $f, g \in L_1(G, U)$, then we have

$$K_f K_g = K_{f * g} = K_g K_f.$$

Hence the set

$$\{K_f;\ f \in L_1(G, U)\}$$

is a set of mutually commutative completely continuous normal operators of $H(G/\Gamma)$. It is obvious that

$$K_f[H(G/\Gamma)] \subset H_0$$

for all $f \in L_1(G, U)$.

Let ω be a function in $\mathfrak{P}(G)$. We denote the vector space of all automorphic functions belonging to ω by $\mathfrak{M}(\omega)$. Let $\Lambda(\Gamma)$ be the set of all ω such that $\mathfrak{M}(\omega) \neq \{0\}$. The principal zonal spherical function $\omega_0 = 1$ is contained in $\Lambda(\Gamma)$.

Theorem 2. *$\Lambda(\Gamma)$ is a discrete countable set having no cluster point in $\mathfrak{S}(G)$. For every $\omega \in \Lambda(\Gamma)$, $\mathfrak{M}(\omega)$ is finite dimensional. For the principal zonal spherical function ω_0 we have* $\dim \mathfrak{M}(\omega_0) = 1$, *namely we have $\mathfrak{M}(\omega_0) = C$. If ω, $\omega' \in \Lambda(\Gamma)$ are distinct, then $\mathfrak{M}(\omega)$ and $\mathfrak{M}(\omega')$ are mutually orthogonal. The space H_0 is spanned by all $\mathfrak{M}(\omega)$, $\omega \in \Lambda(\Gamma)$.*

Proof. We prove our assertions in several steps.

1) $\mathfrak{M}(\omega)$ is finite dimensional. Let f be a function in $L(G, U)$ such that $\hat{f}(\omega) \neq 0$. Since K_f is completely continuous, the dimension of the eigen space of K_f belonging to the eigen value $f(\omega)$ is finite. $\mathfrak{M}(\omega)$ is contained in this eigen space, so $\mathfrak{M}(\omega)$ is finite demensional.

2) If $\Lambda(\Gamma) \ni \omega, \omega'$ are distinct, then $\mathfrak{M}(\omega)$ and $\mathfrak{M}(\omega')$ are mutually orthogonal. For if $\hat{f}(\omega) \neq \hat{f}(\omega')$ with $f \in L(G, U)$, then $\mathfrak{M}(\omega)$ and $\mathfrak{M}(\omega')$ are contained in different eigen spaces of K_f. Since K_f is normal, $\mathfrak{M}(\omega)$ and $\mathfrak{W}(\omega')$ are mutually orthogonal.

3) H_0 is spanned by all $\mathfrak{M}(\omega)$, $\omega \in \Lambda(\Gamma)$. Since K_f is normal and completely continuous for $f \in L(G, U)$, the space $K_f H_0$ is spanned by its eigen spaces $\mathfrak{N}_1$, $\mathfrak{N}_2, \cdots$. All $\mathfrak{N}_i$ are finite dimensional, and if $g \in L(G, U)$ we have $K_g \mathfrak{N}_i \subset \mathfrak{N}_i$ because of $K_g K_f = K_f K_g$. Hence we have a completely reducible representation of $L(G, U)$ into the algebra of endomorphisms of $\mathfrak{N}_i$, and since $L(G, U)$ is commutative, every irreducible subspace is 1-dimensional. This means that $\mathfrak{N}_i$ is spanned by automorphic functions. From the proof of 1, we see that $\mathfrak{N}_i$ is the direct sum of a finite number of $\mathfrak{M}(\omega)$'s, $\omega \in \Lambda(\Gamma)$. Hence $K_f H_0$ is spanned by suitable $\mathfrak{M}(\omega)$'s. Let H_0' be a closed subspace of H_0 containing all $\mathfrak{M}(\omega)$. Then we have $H_0' \supset K_f H_0$ for every $f \in L(G, U)$. From Lemma 6, a closed subspace containing

all $K_f H_0$ is equal to H_0. Hence H_0 is spanned by all $\mathfrak{M}(\omega)$'s.

4). Since H_0 is separable, $\Lambda(\Gamma)$ is a countable set. If ω is a cluster point of $\Lambda(\Gamma)$, then there exists a $f \in L(G, U)$ such that $\hat{f}(\omega) \neq 0$, and $\hat{f}(\omega)$ is a cluster point of $\{\hat{f}(\omega_i)\,;\ \omega_i \in \Lambda(\Gamma)\}$. This gives a contradiction because K_f is completely continuous.

5) $\dim \mathfrak{M}(\omega_0)=1$. Let φ be a function in $\mathfrak{M}(\omega_0)$. Since G/Γ is compact, there exists *a* $x_0 \in G$ such that $|\varphi(x)| \leqq \varphi(x_0)$ for all $x \in G$. Then we have

$$\int_U \varphi(xux_0)du = \varphi(x_0)\omega_0(x) = \varphi(x_0).$$

Hence $|\varphi(x)|$ must be constant. Assume that there exists a x_1 such that $\varphi(x_1) \neq \varphi(x_0)$. Put

$$\varphi(x_1 x_0^{-1} u x_0) = \varphi(x_0)\theta(u).$$

Then the function $\theta(u)$ defined on U is continuous and $|\theta(u)|=1$. Furthermore we have

$$\int_U \theta(u)\, du = \omega_0(x_1) = 1.$$

On the other hand, we have $\theta(e) \neq 1$. This gives a contradiction.

Now we have proved all of our assertions.

Let $\omega_0, \omega_1, \omega_2, \cdots$ are elements of $\Lambda(\Gamma)$. Put $\kappa_i = \dim \mathfrak{M}(\omega_i)$. We choose an orthonormal base $\varphi_1^{(\nu)}, \cdots, \varphi_{\kappa_\nu}^{(\nu)}$ of $\mathfrak{M}(\omega_\nu)$ for every ω_ν. For ω_0, we put

$$\varphi_0 = 1\Big/\left\{\int_{G/\Gamma} dx\right\}^{\frac{1}{2}}.$$

The functions $\varphi_0, \varphi_1^{(1)}, \cdots, \varphi_{\kappa_1}^{(1)}, \varphi_1^{(2)}, \cdots, \varphi_{\kappa_2}^{(2)}, \cdots$ form a complete orthonormal basis of H_0. For every $f \in L_1(G, U)$, these functions are eigen functions of K_f with eigen values $\hat{f}(\omega_0)$, $\hat{f}(\omega_1), \cdots$.

§5. The trace formula. Let f be a function in $L_1(G, U) \frown C(G)$. If the series

$$\sum_{\alpha \in \Gamma} f(x\alpha y^{-1}) = k_f(x, y) \tag{18}$$

converges absolutely and uniformly on $K \times K$ where K is any compact set on G, then $k_f(x, y)$ is a continuous function of x and y, and

$$k_f(x\alpha, y\beta) = k_f(x, y)$$

for every $\alpha, \beta \in \Gamma$. For every $g \in H(G/\Gamma)$ we have

$$K_f \cdot g = f * g = \int_{G/\Gamma} k_f(x, y) g(y) dy.$$

From Theorem 2, we have formaly

$$k_f(x, y) \sim \sum_{i=0}^{\infty} \hat{f}(\omega_i) \sum_{j=1}^{\kappa_i} \varphi_j^{(i)}(x)\overline{\varphi_j^{(i)}(y)}. \tag{19}$$

Putting $x=y$ and integrating over G/Γ, we have

$$\int_{G/\Gamma} k_f(x, x)\,dx = \sum_{\Gamma \ni \alpha} \int_F f(x\alpha x^{-1})dx = \sum_{i=0}^{\infty} \kappa_i f(\omega_i) \tag{20}$$

where F is a fundamental domain of Γ.

We will study the condition when the equality holds in exact sense.

A function $f \in L_1(G, U) \frown C(G)$ will be called admissible if the series (18) converges absolutely and uniformly for $(x, y) \in K \times K$ where K is an arbitrary compact set on G. A linear combination of a finite number of admissible functions are also admissible. If $f \in L_1(G, U) \frown C(G)$ and $|f| \leqq |f_1|$ for all $x \in G$ with an admissible f_1, then f is also admissible.

Furthermore if $f \in L_1(G, U)$ and $h \in L(G, U)$ then $h*f$ is admissible. Without loss of generality we prove it in the case $h \geqq 0$ and $f \geqq 0$. Then first we have $h*f \in L_1(G, U) \frown L^\infty(G)$.

If $(x, y) \in K \times K$, then we have

$$\sum_{\alpha \in \Gamma} h*f(xy^{-1}) = \sum_{\alpha \in \Gamma} \int_G h(xz^{-1})\, f(zy)\,dz = \int_G k_h(x, z) f(zy) dz.$$

Since $j_h(x, z)$ is uniformly bounded and continuous on $G \times G$, we can easily check that $h*f$ is admissible. Similarly we can prove if f and g are in $L_1(G, U)$ and f is admissible, then $f*g$ is admissible.

Now if f is admissible and of the form $f = g*g^*$ with $g \in L_1(G, U)$ then $\hat{f}(\omega_i) \geqq 0$ for all $\omega_i \in \Lambda(\Gamma)$, and from Mercer's theorem we have

$$k_f(x, y) = \sum_{i=0}^{\infty} f(\omega_i) \sum_{j=1}^{\kappa_i} \overline{\varphi_j^{(i)}(x)}\varphi_j^{(i)}(y) \tag{21}$$

and

$$\int_{G/\Gamma} k_f(x, x)dx = \sum_{i=0}^{\infty} \kappa_i \hat{f}(\omega_i). \tag{22}$$

From this fact, we see that if f is admissible and of the form $f = h*g$ with admissible h and g, then equations (21) and (22) holds in this case too. For we have

$$h*g = \frac{1}{4}\Big\{(h+g^*)(h^*+g) - (h-g^*)(h^*-g) - \frac{1}{\sqrt{-1}}(h-\sqrt{-1}\,g^*)(h^*+\sqrt{-1}\,g)$$
$$- \frac{1}{\sqrt{-1}}(h+\sqrt{-1}\,g^*)(h^*-\sqrt{-1}\,g)\Big\},$$

so $h*g$ is a linear combination of admissible function of the form $k(x)*k^*(x)$. Hence if $\{h_i\}$ $i=1,2,\cdots$ is a series of approximate identities of $L(G, U)$ and f an admissible function, then for $h_\nu*f=f_\nu$ we have

$$\int_{G/\Gamma} k_{f_\nu}(x, x)dx=\sum_{i=0}^{\infty}\kappa_i\hat{f}_i(\omega_i)=\sum_{i=0}^{\infty}\kappa_i\theta_{\nu,i}\hat{f}(\omega_i)$$

where we put $\theta_{\nu,i}=\hat{h}_\nu(\omega_i)$. From the definitlon of h_ν and since ω_i is positive definite, we have $0\leqq\hat{h}_\nu(\omega_i)\leqq 1$.

Now we have

$$k_{f_\nu}(x, y)=\int_G h_\nu(xz^{-1})k_f(z, y)dz,$$

so $\lim_{\nu\to\infty} k_{f_\nu}(x, y)=k_f(x, y)$ uniformly on $G/\Gamma\times G/\Gamma$. Hence we have

$$\lim_{\nu\to\infty}\{\sum\kappa_i\theta_{i,\nu}\hat{f}(\omega_i)\}=\int_{G/\Gamma} k_f(x, x)dx.$$

If the series $\sum_{i=0}^{\infty}\kappa_i\hat{f}(\omega_i)$ converges absolutely, then we have the equality

$$\sum_{i=0}^{\infty}\kappa_i f(\omega_i)=\int_{G/\Gamma} k_f(x, x)dx$$

because all $\theta_{i\nu}$ are non negative and not greater than 1.

Now we can transform the right side of this equation still further. First we prove the following:

Lemma 7. *Let α be an element of Γ, G_α the closed subgroup of all x with $x\alpha=\alpha x$ and Γ_α the aroup $G_\alpha\frown\Gamma$. Then the homogeneous space G_α/Γ_α is compact.*

Proof. Let C_α be the set of all y such that $y=x^{-1}\alpha x$ with $x\in G$. We consider C_α a subspace of G. Then the mapping ν: $x\to x^{-1}\alpha x$ is a continuous one to one mapping of $G_\alpha\backslash G$ onto C_α, and ν maps $G_\alpha\backslash G_\alpha\Gamma$ into $\Gamma\frown C_\alpha$. Since $\Gamma\frown C_\alpha$ is a discrete set having no cluster point, $G_\alpha\Gamma$ is a closed subset of G. Hence the space $G_\alpha\Gamma/\Gamma=G_\alpha/\Gamma_\alpha$ is a closed subspace of G/Γ, so is compact.

If f is admissible, then we have

$$\int_{G/\Gamma} k_f(x, x)dx=\int_{G/\Gamma}\{\sum_{\Gamma\ni\alpha} f(x\alpha x^{-1})\}dx$$

$$=\int_{G/\Gamma}\{\sum_{\{\alpha\}}\sum_{\{\alpha\}\ni\gamma} f(x\gamma x^{-1})\}dx=\sum_{\{\alpha\}}\int_{G/\Gamma}\{\sum_{\{\alpha\}\ni\gamma} f(x\gamma x^{-1})\}dx.$$

where $\sum_{\{\alpha\}}$ ranges over all conjugate classes of Γ. For every $\{\alpha\}$ we choose a representative α. Then the function $\psi_\alpha(\bar{x})$ defined on G/G_α by $\psi_\alpha(\bar{x})=f(x\alpha x^{-1})$ is continuous, and from Lemma 7 there exists an invariant measure $d_\alpha\bar{x}$ on G/G_α such that

$$dx=d_\alpha\bar{x}dx_\alpha$$

where dx_α is a Haar measure on G_α. From A. Weil's theorem*), we have, from Lemma 7,

$$\int_{G/\Gamma}\{\sum_{\{\alpha\}\ni\gamma} f(x\gamma x^{-1})\}dx=\int_{G/G_\alpha}\psi_\alpha(\bar{x})\,d_\alpha\bar{x}\cdot\int_{G_\alpha/\Gamma_\alpha}dx_\alpha .$$

Hence we have the following:

Theorem 3 (A. Selberg). *If f is admissible function and the series*

$$\sum_{i=0}^{\infty}\kappa_i\hat{f}(\omega_i)$$

is absolutely convergent, then we have

$$\sum_{i=0}^{\infty}\kappa_i\hat{f}(\omega_i)=\sum_{\{\alpha\}}\quad volume(G_\alpha/\Gamma_\alpha)\int_{G/G_\alpha}\psi_\alpha(\bar{x})d_\alpha\bar{x}.$$

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*) cf. A. Weil [4].