Theory of Hyperfunctions, I.

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This is the first of the series of papers, which we intend now to publish to give a full account of the theory, which we outlined in our previous papers ([1], [2], [3]). We have shown in these papers how we can introduce a generalized concept of functions—that of "*hyperfunctions*"—on C^ω-manifolds, and how we can deal with them. Hyperfunctions include Schwartz's distributions, they form a module (=additive group), can be multiplied by analytic functions, can be differentiated and integrated. In a certain sense, they can be regarded as "boundary values" of analytic functions. (Analogous ideas have been also explored by G. Köthe [6], [7], H. G. Tillmann [8], A. Grothendieck [9], J. S. Silva [10])*).

We should like to call attention of the reader to a certain class of hyperfunctions, which we shall call *analytic hyperfunctions*. They are those which satisfy analytic linear differential equations in the neighborhood of each point. They may have isolated singularities, which we shall call *thresholds*. Almost all of the functions of frequent use in the applied analysis can be considered as hyperfunctions of this category, and from this point of view, many of the well-known integral formulas can be deduced in a unified manner. Also Hadamard's "finite part" of a divergent integral can be obtained as the value of the integral of an analytic hyperfunction.

On the other hand, we have the well-known "decomposition-of-unity-theorem" on paracompact C^∞-manifolds M; i. e. for any locally finite open covering $\{U_\nu;\ \nu=1, 2, \cdots\}$ of M, there exist C^∞-functions $\eta_\nu(p)$, $\nu=1, 2, \cdots$, on M with $1=\sum_\nu \eta_\nu(p)$, the carrier of $\eta_\nu(p)$ being contained in U_ν. This implies that any Schwartz's distribution T on M can be decomposed in a form: $T=\sum_\nu T_\nu$, the carrier of T_ν being contained in U_ν ("Decomposition theorem" for Schwartz's distributions). This has again the following important consequence: Let $\{U_\alpha;\ \alpha\in N\}$ be any open covering of M, and T_α a Schwartz's distribution on U_α, such that $T_\alpha|U_\alpha\frown U_\beta$ (the restriction of T_α on $U_\alpha\frown U_\beta$) coincides with $T_\beta|U_\alpha\frown U_\beta$ for any $\alpha, \beta\in N$, then there exists a distribution T on M, with $T|U_\alpha=T_\alpha$ for any $\alpha\in N$ ("Localization theorem" for Schwartz's distributions).—Both

*) See also H. G. Tillman: Die Fortsetzung analytischer Funktionale, Abh. Math. Sem. Univ. Hamburg **21** (1957), 139, and literatures quoted there.

these "decomposition theorem" and "localization theorem" are also valid for our hyperfunctions as we shall show in §§20-25 (cf. [1] §3, [3] §§3, 7, 8; for hyperfunctions of several variables, cf. [2], [4]). We shall call this property of hyperfunctions the "*localizability*."

Our hyperfunctions have another remarkable property, which is not shared by Schwartz's distributions. Namely, let M' be any open set of a paracompact C^ω-manifold M. Then for any hyperfunction g' on M', there exists a hyperfunction g on M, such that $g|M'=g'$. We shall call this fact the "completeness theorem" for the hyperfunctions (see §22, cf. also [1] §3, [2] proposition 2, [3] §§3, 7, 8).

Sheaf theory provides us with a neat language to express the above facts. The "germs" of hyperfunctions at each point on M form a sheaf $\mathfrak{B}$ of modules on M. The localization theorem means that each hyperfunction on M can be defined as a section of $\mathfrak{B}$ over M, while the completeness theorem means that $\mathfrak{B}$ is a "complete (or hyperfine) sheaf" as defined in [2]. In case of one variable, these facts can be deduced also from Oka's principle for analytic fibre bundles or from Cartan-Serre's theorem on coherent analytic sheaves*) (see [3] p. 11 and comments (19), (20) on p. 26; cf. also forthcoming paper II).

In this paper I and also in the forthcoming II, we shall be solely concerned with the "hyperfunctions of one variable." To develop fully the theory of hyperfunctions on manifolds of higher dimensions, as we intend to do in our subsequent papers, we shall need the *relative cohomology theory with sheaf-coefficients* (cf. [2], and [3] chapter III). This latter theory will be expounded in another series of papers of ours ([4]).

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Contents of I.

*) H. Cartan et al.: Séminaire sur les fonctions de plusieurs variables, École Normale Supérieure (1951-1952). Exposés XVIII et XIX.

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Note.—The formulas are numbered serially in the same paragraph. We shall quote the formula (n) in §m with (n, m), but simply with (n) when we quote it in the same paragraph.

Basic Concepts.

We shall first give in §§1–8 the definition and simple properties of hyperfunctions of one variable*) (cf. [3], §1).

§1. *Preliminaries.*

We shall use the following notations throughout the paper.

$\boldsymbol{R}$ will mean the real axis $(-\infty, \infty)$ which we consider as lying in the complex plane $\boldsymbol{C}$. $\boldsymbol{C}^+$ and $\boldsymbol{C}^-$ denote the upper and the lower half-plane:

$$(1)\qquad \begin{cases} \boldsymbol{C}^+=\{z\,;\ \Im z>0\}, \\ \boldsymbol{C}^-=\{z\,;\ \Im z<0\}, \\ \boldsymbol{C}=\boldsymbol{C}^+ + \boldsymbol{R} + \boldsymbol{C}^-. \end{cases}$$

For any open subset D of $\boldsymbol{C}$, we shall denote $D^{\pm}=D\frown\boldsymbol{C}^{\pm}$.

We shall denote with $\mathfrak{A}(D)$ the ring of all holomorphic ($=$single-valued regular analytic) functions in D.

S will always denote a locally closed subset**) of $\boldsymbol{R}$. We denote with $\mathfrak{D}(S)$ the family of all the "complex neighborhoods of S," i.e. the open subsets of $\boldsymbol{C}$ which contain S as a closed subset. We shall write

$$\mathfrak{D}^{\pm}(S)=\{D^{\pm};\ D\in\mathfrak{D}(S)\}.$$

$\mathfrak{D}^*(S)$ will mean the family of "symmetric neighborhoods of S", i.e. the subfamily of $\mathfrak{D}(S)$ consisting of elements which are symmetric with respect to $\boldsymbol{R}$.

We now consider the family of all the couples (f, D) with $f\in\mathfrak{A}(D)$, $D\in\mathfrak{D}(S)$, and introduce an equivalence relation in this family as follows: two of such couples (f_j, D_j), $f_j\in\mathfrak{A}(D_j)$, $D_j\in\mathfrak{D}(S)$, $j=1, 2$, are equivalent (in notation: $(f_2, D_2)\sim(f_1, D_1)$) if and only if

$$f_2|D=f_1|D^{***)}$$

i.e.

$$(2)\qquad f_2(z)=f_1(z) \qquad (z\in D)$$

*) In the following, we adopt notations used in [3]. (In [1], we wrote N, M, $\mathcal{A}_N$, $\widetilde{\mathcal{A}}_N$, $\mathcal{B}_N$ in place of S, I, $\mathfrak{A}(S)$, $\widetilde{\mathfrak{A}}(S)$, $\mathfrak{B}(S)$ in the present paper.)

**) A subset Y of a topological space X is called locally closed if we can find for each $p\in Y$ an (open) neighborhood $U_p\ni p$ such that $Y\frown U_p$ is (relatively) closed in U_p, or equivalently, if we can find an open set $U\subset X$ which contain Y as a closed subset. Clearly Y is locally closed, if and only if Y is the intersection of an open set U and $\bar{Y}$ (the closure of Y), i.e. if Y is open in $\bar{Y}$. If X is a locally compact space (which is the case for $X=\boldsymbol{C}$), the condition "Y is locally closed in X" is equivalent with "Y is locally compact."

***) For a function f defined on a space X (or a map f from X into a certain space) and a subset Y of X, $f|Y$ means the restriction of f onto Y. For a function (or a map) f' defined on a *subset* X' of X, $f'|Y$ means the restriction of f' onto $X'\frown Y$; i.e. $f'|Y=f'|(X'\frown Y)$.

with some $D\in\mathfrak{D}(S)$, $D\subset D_1\frown D_2$. The equivalence class determined by the couple (f, D) will be denoted with $[f, D]$ or simply with f, and called an *holomorphic function on* S. The addition and multiplication of holomorphic functions on S are naturally defined, and these functions form a ring, which will be denoted with $\mathfrak{A}(S)$. It is obvious that

$$(3)\quad \begin{cases}\text{From } S=\bigcup_\nu S_\nu,\ S_\nu:\ \text{(relatively) open closed subsets of } S \text{ such that}\\ S_\nu\frown S_\mu=\phi\ (\nu\neq\mu),\ \text{follows } \mathfrak{A}(S)=\prod_\nu \mathfrak{A}(S_\nu)\ \text{(direct product).}\end{cases}$$

§2. *The concept of hyperfunctions.*

In the next place, we consider the family of all of the couples (φ, D) with $\varphi\in\mathfrak{A}(D-S)$, $D\in\mathfrak{D}(S)$, and introduce an equivalence relation in this family as follows: two of such couples (φ_j, D_j), $\varphi_j\in\mathfrak{A}(D_j-S)$, $D_j\in\mathfrak{D}(S)$, $j=1, 2$, are equivalent (in notation: $(\varphi_2, D_2)\sim(\varphi_1, D_1)$) if and only if

$$(1)\qquad \varphi_2(z)=\varphi_1(z)+f(z)\qquad (z\in D-S)$$

with some $D\in\mathfrak{D}(S)$, $D\subset D_1\frown D_2$ and some $f\in\mathfrak{A}(D)$. It is easy to see that this relation $\sim$ is actually an equivalence relation. The equivalence class determined by a couple (φ, D), $\varphi\in\mathfrak{A}(D-S)$, $D\in\mathfrak{D}(S)$, will be denoted with $[\varphi, D]$ or simply with $[\varphi]$, and called *hyperfunction* on S defined by (φ, D) or simply by φ. We shall often denote hyperfunctions by letters like $g, h, \cdots$, and write e. g.

$$(2)\qquad g=[\varphi, D]=[\varphi],$$
$$\text{or: }\ g(x)=[\varphi(z), D]_{z=x}=[\varphi(z)]_{z=x}.$$

We shall call φ (or more precisely (φ, D)) a defining function of the hyperfunction g. The set of all hyperfunctions on S will be denoted by $\mathfrak{B}(S)$. It is easy to verify that $\mathfrak{B}(S)$ forms an $\mathfrak{A}(S)$-module if we define, for any $f_j=[f_j, D_j]$ $\in\mathfrak{A}(S)$ and $g_j=[\varphi_j, D'_j]\in\mathfrak{B}(S)$, $j=1, \cdots, n$,

$$(3)\qquad \sum_{j=1}^n f_j g_j=[\sum_{j=1}^n f_j\varphi_j, D]\ \left(\text{or: } \sum_{j=1}^n f_j(x)g_j(x)=[\sum_{j=1}^n f_j(z)\varphi_j(z)]_{z=x}\right)$$

with $D\in\mathfrak{D}(S)$, $D\subset\bigcap_{j=1}^n(D_j\frown D'_j)$.

It is obvious that

$$(4)\quad \begin{cases}\text{From } S=\bigcup S_\nu,\ S_\nu:\ \text{(relatively) open closed subsets of } S \text{ such that}\\ S_\nu\cap S_\mu=\phi\ (\nu\neq\mu),\ \text{follows } \mathfrak{B}(S)=\prod_\nu \mathfrak{B}(S_\nu)\ \text{(direct product).}\end{cases}$$

To sum up, the above definition of $\mathfrak{A}(S)$ and $\mathfrak{B}(S)$ can be expressed in the following way. For any $D\supset D'$, D, D' being open sets of $\boldsymbol{C}$, we have a natural homomorphism

$$(5)\qquad \rho_{D'D}:\ \mathfrak{A}(D)\longrightarrow\mathfrak{A}(D')$$

by restriction of the domain. The ring $\mathfrak{A}(S)$ defined above is nothing but the inductive limit of $\{\mathfrak{A}(D);\ D\in\mathfrak{D}(S)\}$ by $\rho_{D'D}$.*) On the other hand, the inductive limit of $\{\mathfrak{A}(D-S);\ D\in\mathfrak{D}(S)\}$ by $\rho_{D'-S,D-S}$ likewise constitutes a ring $\tilde{\mathfrak{A}}(S)$. We can consider $\tilde{\mathfrak{A}}(S)$ as an extension ring of $\mathfrak{A}(S)$ (and so as an $\mathfrak{A}(S)$-module, too) in a natural manner, and we can now put

(6) $$\mathfrak{B}(S)=\tilde{\mathfrak{A}}(S) \bmod \mathfrak{A}(S).$$

The operations in $\mathfrak{B}(S)$ in (6) as the $\mathfrak{A}(S)$-module are just given by (3).

§3. *Complex conjugate hyperfunction.*

The *complex conjugate* of $g(x)=[\varphi(z), D]_{z=x}\in\mathfrak{B}(S)$, $D\in\mathfrak{D}(S)$, is defined by

(1) $$\bar{g}(x)=-[\bar{\varphi}(z), D^*]_{z=x}\in\mathfrak{B}(S)$$

with $D^*\in\mathfrak{D}(S)$, the reflection of D with respect to $\boldsymbol{R}$, and $\bar{\varphi}(z)=\overline{\varphi(\bar{z})}\in\mathfrak{A}(D^*-S)$. It is easy to see that this definition of $\bar{g}(x)$ is independent of the choice of the defining function $(\varphi(z), D)$.

We call a hyperfunction $g(x)\in\mathfrak{B}(S)$ *real valued* if $\bar{g}(x)=g(x)$. Thus, each $g\in\mathfrak{B}(S)$ is uniquely described in the form:

(2) $$g=g_1+ig_2 \qquad (g_1, g_2:\ \text{real-valued})$$

where g_1 and g_2 are given by

(3) $$\begin{aligned} g_1(x)&=\Re g(x)=(g(x)+\bar{g}(x))/2,\\ g_2(x)&=\Im g(x)=(g(x)-\bar{g}(x))/2i.\end{aligned}$$

A hyperfunction $g\in\mathfrak{B}(S)$ is real-valued if and only if it is expressible in the form

(4) $$g=[\varphi, D]$$

with some symmetric $D\in\mathfrak{D}^*(S)$ and some $\varphi\in\mathfrak{A}(D-\boldsymbol{R})$ such that

(5) $$\varphi(\bar{z})=-\overline{\varphi(z)} \qquad (z\in D-\boldsymbol{R}).$$

§4. *Representation of a hyperfunction as the "boundary value" of the defining function.*

For any locally closed subset $S\subset\boldsymbol{R}$ and any $D\in\mathfrak{D}(S)$, $I=D\frown\boldsymbol{R}$ is a *real neighborhood* of S, i.e. an open subset (and hence a locally closed subset) of $\boldsymbol{R}$ which contains S as a closed subset. Clearly we have $D\in\mathfrak{D}(I)$. Therefore, for any $g=[\varphi, D]\in\mathfrak{B}(S)$, $\varphi\in\mathfrak{A}(D-S)$, we have $[\varphi|(D-I), D]\in\mathfrak{B}(I)$, where $\varphi|(D-I)=\rho_{D-I,D-S}(\varphi)$ is the restriction of $\varphi\in\mathfrak{A}(D-S)$ onto $D-I$. Thus, identifying $g\in\mathfrak{B}(S)$ with the hyperfunction $\in\mathfrak{B}(I)$ thus obtained, we have

Proposition 4.1. A hyperfunction $g(x)$ on S can always be considered a hyperfunction on some real neighborhood I of S:

*) More generally we define $\mathfrak{A}(E)$ = inductive limit of $\{\mathfrak{A}(D); D=\text{open set}\supset E\}$ by restriction, for *any subset* E of $\boldsymbol{R}$.

$$(1) \qquad g(x)\in\mathfrak{B}(I)^{*)}.$$

Now define $\varphi(x\pm i0)\in\mathfrak{B}(I)$ $(I=D\frown R)$, the "boundary values" of $\varphi(z)\in\mathfrak{A}(D-I)$, by

$$(2) \qquad \begin{aligned}\varphi(x+i0)&=[\varepsilon\varphi, D],\\ \varphi(x-i0)&=-[\bar{\varepsilon}\varphi, D],\end{aligned}$$

where

$$(3) \qquad \varepsilon(z)=\begin{cases}1 & (z\in C^{+})\\ 0 & (z\in C^{-}),\end{cases} \qquad \bar{\varepsilon}(z)=\begin{cases}0 & (z\in C^{+})\\ 1 & (z\in C^{-}),\end{cases}$$
$$\varepsilon(z),\ \bar{\varepsilon}(z)\in\mathfrak{A}(C-I).$$

Then we have immediately the following representation of $g=[\varphi, D]$:

$$(4) \qquad g(x)=\varphi(x+i0)-\varphi(x-i0).$$

§5. *Holomorphic functions as hyperfunctions.*

On the other hand, we shall define a special hyperfunction $1=1_I\in\mathfrak{B}(I)$ by

$$(1) \qquad 1=[\varepsilon, D]=-[\bar{\varepsilon}, D]^{**)}$$

whence we obtain, for any $f\in\mathfrak{A}(I)$,

$$(2) \qquad f(x)\cdot 1=f(x+i0)=f(x-i0)\in\mathfrak{B}(I).$$

The correspondence $f(x)\to f(x)\cdot 1$ induces a natural homomorphism:

$$\mathfrak{A}(I)\longrightarrow\mathfrak{B}(I)$$

which is injective because we have $f\cdot 1=0$ if and only if $f=0$. Therefore, we can hereafter consider $\mathfrak{A}(I)$ as canonically embedded in $\mathfrak{B}(I)$:

$$(3) \qquad \mathfrak{A}(I)\subset\mathfrak{B}(I).$$

On account of (3), a hyperfunction $g\in\mathfrak{B}(I)$ will be called *holomorphic* or *regular* if $g\in\mathfrak{A}(I)$. A hyperfunction $g=[\varphi, D]\in\mathfrak{B}(I)$ is holomorphic if and only if

$$(4) \qquad \varphi(z)=\varepsilon(z)f_1(z)+\varepsilon(z)f_2(z) \qquad (z\in D'-I)$$

with some $D'\in\mathfrak{D}(I)$, $D'\subset D$, and $f_1, f_2\in\mathfrak{A}(D')$.

We shall moreover say that a hyperfunction $g\in\mathfrak{B}(I)$ is *upper* (or *lower*) *semi-holomorphic* if we have

$$(5) \qquad g(x)=\varphi(x+i0) \qquad (\text{or } =\varphi(x-i0))$$

with some $\varphi\in\mathfrak{A}(D^{+})$ (or $\mathfrak{A}(D^{-})$), $D\in\mathfrak{D}(I)$. A hyperfunction $g=[\varphi, D]\in\mathfrak{B}(I)$ is upper (or lower) semi-holomorphic if and only if

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*) Here I depends on each $g\in\mathfrak{B}(S)$. (Cf. §22, proposition 22.1.)

**) $\varepsilon(z)+\bar{\varepsilon}(z)=1\in\mathfrak{A}(D)$, hence $\varepsilon(z)\equiv-\bar{\varepsilon}(z)$ (mod $\mathfrak{A}(D)$).

(6) $\varphi(z)=\varepsilon(z)\varphi_1(z)+\bar{\varepsilon}(z)f_2(z)$ (or : $\varphi(z)=\varepsilon(z)f_1(z)+\bar{\varepsilon}(z)\varphi_2(z)$)

with some $\varphi_1\in\mathfrak{A}(D^+)$ and $f_2\in\mathfrak{A}(D')$ (or : $f_1\in\mathfrak{A}(D')$ and $\varphi_2\in\mathfrak{A}(D^-)$), $D'\in\mathfrak{D}(I)$. A hyperfunction $g\in\mathfrak{B}(I)$ is holomorphic if and only if it is both upper and lower semi-holomorphic.

§6. *Restriction of the domain of hyperfunctions.*

For any locally closed set $S\subset \boldsymbol{R}$ and open (=relatively open) subset S' of S, we shall define the *restriction* $g|S'$ *onto* S' of each $g\in\mathfrak{B}(S)$ as follows. Let $g=[\varphi, D]$, $\varphi\in\mathfrak{A}(D-S)$, $D\in\mathfrak{D}(S)$. There exists a $D'\in\mathfrak{D}(S')$ such that $D'-S'\subset D-S$ (e.g. $D'=(D-S)^{\smile}S'$). The restriction $g|S'$ is defined by

(1) $$g|S'=[\varphi|D', D'](=[\varphi|(D'-S'), D'])\in\mathfrak{B}(S').$$

It is easy to see that the definition of $g|S'$ is independent of the choice of (φ, D) and D'.

Definition. We say $g\in\mathfrak{B}(S)$ *is* $=0$, *is holomorphic, or is real-valued, etc. on* S' *according as* $g|S'$ *is* $=0$, *is holomorphic, or is real-valued etc.*

If $g(x)$ is $=0$ (or holomorphic) in the neighborhood of each point of S, then $g(x)$ itself is $=0$ (or holomorphic). Therefore, there exists a uniquely determined maximal (open) subset$S'\subset S$ such that $g|S'=0$ (or holomorphic). We call the complementary set $S-S'$ the *carrier* (or the *carrier of singularity**)) of $g(x)$, and denote it with car g (or : sing.car g). car g is a closed subset of S, and sing.car g is a closed subset of car g.

§7. *Transformation of variable.*

Let I and I' be open sets of $\boldsymbol{R}$, $\xi(x)$ a real-valued holomorphic function defined on I' and taking the value in I such that

$$\frac{d\xi(x)}{dx}>0 \quad \text{(or } <0\text{) on } I'.$$

Definition. For each $g(x)=[\varphi(z)]_{z=x}\in\mathfrak{B}(I)$, *we define* $g(\xi(x))\in\mathfrak{B}(I')$ *by*

(1) $$g(\xi(x))=[\varphi(\xi(z))]_{z=x} \quad (\textit{or}: =-[\varphi(\xi(z))]_{z=x}).$$

In particular, we have, for any c, $c'\in\boldsymbol{R}$, $c\neq 0$,

(2) $$g(cx+c')=\pm[\varphi(cz+c')]_{z=x} \quad \text{according as} \quad c\gtrless 0.$$

It is easy to verify that the above definition of $g(\xi(x))$ is independent of the choice of $\varphi(z)$, and coincides with the usual concept of the transformation of variable if the hyperfunction $g(x)$ is holomorphic.

A hyperfunction $g\in\mathfrak{B}(I)$, $I=(-a, a)\subset\boldsymbol{R}$, will be called *even* or *odd* according

*) In [1], we named it *carrier of irregularity* of $g(x)$.

as

$$(3) \qquad g(-x)=\pm g(x).$$

Each $g\in\mathfrak{B}(I)$ is uniquely decomposed into the sum

$$(4) \qquad g(x)=g_e(x)+g_0(x), \quad g_e(x): \text{even}, \quad g_0(x): \text{odd},$$

with $g_e(x)=(g(x)+g(-x))/2$, $g_0(x)=(g(x)-g(-x))/2$. A hyperfunction $g\in\mathfrak{B}(I)$ is even (or odd) if and only if it is expressible in the form

$$(5) \qquad g=[\varphi, D], \qquad d\in\mathfrak{D}^*(I)$$

with some odd (or even) holomorphic function $\varphi\in\mathfrak{A}(D-I)$:

$$\varphi(-z)=\mp\varphi(z), \qquad z\in D-I.$$

§8. *Derivatives and indefinite integrals of a hyperfunction.*

Let $g=[\varphi, D]\in\mathfrak{B}(S)$, and let S be a locally closed set of $\boldsymbol{R}$. The *derivatives* $g^{(n)}(x)=\dfrac{d^n g(x)}{dx^n}$ are defined by means of $\varphi^{(n)}(z)=\dfrac{d^n\varphi(z)}{dz^n}\in\mathfrak{A}(D-S)$ as follows:

$$(1) \qquad g^{(n)}(x)=[\varphi^{(n)}(z), D]_{z=x}\in\mathfrak{B}(S)$$

which are clearly independent of the choice of (φ, D).

Proposition 8.1. For a hyperfunction $g\in\mathfrak{B}(I)$ on an open interval $I=(a, b)\subset\boldsymbol{R}$, $-\infty\leqslant a<b\leqslant\infty$, we have

i) $\dfrac{dg(x)}{dx}=0$ *if and only if* $g(x)=$*constant.*

More generally,

ii) $\dfrac{d^n g(x)}{dx^n}=0$ *if and only if* $g(x)=$*polynomial in x of degree $<n$.*

Proof. The condition $\dfrac{d^n g(x)}{dx^n}=0$ yields, by definition,

$$(2) \qquad \varphi^{(n)}(z)\in\mathfrak{A}(D),$$

where we may assume that D is connected and simply connected (by replacing D by a suitable $D'\in\mathfrak{D}(I)$, $D'\subset D$, if necessary). Therefore there exists by the monodromy theorem an holomorphic function $f\in\mathfrak{A}(D)$ for which $f^{(n)}=\varphi^{(n)}$ holds. Then we have

$$(3) \qquad \frac{d^n}{dz^n}(\varphi(z)-f(z))=0 \qquad (z\in D-I)$$

and obtain the expression

$$(4) \qquad \varphi(z)-f(z)=\varepsilon(z)p_1(z)+\bar{\varepsilon}(z)p_2(z),$$

$p_1(z)$ and $p_2(z)$ being suitable polynomials of degree $<n$. Therefore

$$(5) \qquad g(x)=[\varepsilon(z)p_1(z)+\bar{\varepsilon}(z)p_2(z)]_{z=x}=p_1(x)-p_2(x). \qquad \text{(q. e. d.)}$$

Proposition 8.2. Eor each hyperfunction $g \in \mathfrak{B}(I)$, $I=(a,b)\subset \boldsymbol{R}$, *there exists a hyperfunction* $G_n \in \mathfrak{B}(I)$ *such that*

$$\frac{d^n G_n(x)}{dx^n} = g(x). \tag{6}$$

We shall call $G_n(x)$ the *n-fold indefinite integral* of $g(x)$ and denote:

$$G_n(x) = \underbrace{\int^x dx \cdots}_{n\text{-fold}} \int^x g(x)dx. \tag{7}$$

$G_n(x)$ *is determined to within an additional term of a polynomial of degree* $<n$ by Proposition 8.2.

Proof. Let $g=[\varphi, D]$, $D=D^+ \cup I \cup D^- \in \mathfrak{D}(I)$. We can further assume that D^+ and D^- are both connected and simply connected (by replacing D by a suitable $D' \in \mathfrak{D}(I)$, $D' \subset D$, if necessary). By the monodromy theorem we can find a $\psi_n(z) \in \mathfrak{A}(D-I)$ for which $\frac{d^n \psi_n(z)}{dz^n} = \varphi(z)$ holds. The hyperfunction $G_n(x) = [\psi_n(z), D]_{z \to x} \in \mathfrak{B}(I)$ clearly satisfies the equation (6). The remaining part of the proposition is obvious. (q. e. d.)

If $g(x)$ is $=0$ on a real neighborhood of $a \in I$, then the n-fold indefinite integral $G_n(x)$ of $g(x)$ is uniquely determined by imposing on it an additional condition that $G_n(x)=0$ on a real neighborhood of $a \in I$. We shall denote this particular $G_n(x) \in \mathfrak{B}(I)$ with

$$G_n(x) = \underbrace{\int_a^x dx \cdots}_{n\text{-fold}} \int_a^x g(x)dx. \tag{8}$$

The lower bound a in the expression (8) may be replaced by $-\infty$ (or ∞) if car $g(x)$ is contained in (a, ∞) (or in $(-\infty, a)$).

Propositions 8.2 and 8.1 are particular cases of proposition 26.1 and the corollary thereof, respectively.

Simple Examples of Hyperfunctions.

We shall now give in §§9–13 simplest examples of hyperfunctions: Dirac's δ-function, Heaviside's Y-function and power functions. Other examples will be found in [3].

§9. *Dirac δ-function.*[*)]

We define the Dirac δ-function $\delta(x)$ as a hyperfunction on $\boldsymbol{R}$ as follows

*) P. M. A. Dirac, The Principles of Quantum Mechanics, (Clarendon Press, Oxford, 1957), third edition, §15; W. Heitler, The Quantum Theory of Radiation, (Clarendon Press, Oxford, 1954), third edition, §8, especially (8.7.).

$$(1)\qquad \delta(x)=-\frac{1}{2\pi i}\left[\frac{1}{z}\right]_{z=x}\in\mathfrak{B}(\boldsymbol{R}).$$

By (4.4), we have

$$(2)\qquad \delta(x)=-\frac{1}{2\pi i}\left(\frac{1}{x+i0}-\frac{1}{x-i0}\right).$$

By (8.1), we have

$$(3)\qquad \delta^{(n)}(x)=(-1)^{n+1}\frac{n}{2\pi i}\left[\frac{1}{z^{n+1}}\right]_{z=x}=(-1)^{n+1}\frac{n!}{2\pi i}\left(\frac{1}{(x+i0)^{n+1}}-\frac{1}{(x-i0)^{n+1}}\right).$$

Clearly, these are all real-valued hyperfunctions with carrier $\{0\}$. Moreover, we have, by definition,

$$(4)\qquad x\cdot\delta(x)=0,\qquad x\cdot\delta^{(n)}(x)=-n\delta^{(n-1)}(x),$$

and hence, for any $f(x)\in\mathfrak{A}(\{0\})$,

$$(4)'\qquad \begin{aligned}&f(x)\delta(x)=f(0)\delta(x),\\ &f(x)\delta^{(n)}(x)=\sum_{\nu=0}^{n}c_{n\nu}\delta^{(n-\nu)}(x)\ \text{ with }\ c_{n\nu}=(-1)^{\nu}\binom{n}{\nu}f^{(\nu)}(0),\end{aligned}$$

which are familiar properties of $\delta(x)$ as introduced by P.M.A. Dirac. The general solution of the equation

$$(5)\qquad x^n\cdot g(x)=0\ \text{ with }\ g(x)\in\mathfrak{B}(S),\qquad (S\ni 0),$$

is given by the linear form (with constant coefficients) of $\delta^{(\nu)}(x)$, $\nu=0,1,\cdots,n-1$. Again, for any given $h(x)=[\varphi(x)]_{z=x}\in\mathfrak{B}(S)$, the general solution of the equation

$$(6)\qquad x^n\cdot g(x)=h(x)\ \text{ with }\ g(x)\in\mathfrak{B}(S)$$

is given by the sum of a particular solution $g_0(x)=[z^{-n}\varphi(z)]_{z=x}\in\mathfrak{B}(S)$ and the general solution of (5) described above.

For any $c>0$, we have

$$(7)\qquad \delta(\pm cx)=\frac{1}{c}\delta(x),\qquad \delta^{(n)}(\pm cx)=(\pm 1)^n\frac{1}{c^{n+1}}\delta^{(n)}(x).$$

Thus, $\delta(x)$ is an even hyperfunction of "homogeneous degree -1", and $\delta^{(n)}(x)$ is an even or odd hyperfunction of "homogeneous degree $-(n+1)$" according as n is even or odd.

§10. *General δ-functions and meromorphic hyperfunctions.*

Let $\varphi(x)$ be a mormorphic function on the open set $I\subset\boldsymbol{R}$. We can put $\varphi\in\mathfrak{A}(D-I)$ with suitable $D\in\mathfrak{D}(I)$, $D=D^+\cup I\cup D^-$. Therefore a hyperfunction on I is defined by $g=[\varphi, D]$. We shall call a hyperfunction of this type a *general δ-function* on I. Any general δ-function $g(x)$ on I possesses the discrete carrier (which is at the same time the carrier of singularity) on I, and is expressible, in the neighborhood I_a of each (isolated) point $a \in \operatorname{car} g$, in the following form:

$$(1) \qquad g(x)|I_a = c_0\delta(x-a) + \cdots + c_{n-1}\delta^{(n-1)}(x-a).$$

The coefficients c_ν are easily determined from the principal part of the Laurent expansion of $\varphi(z)$ at $z=a$. (See what follows the corollary of proposition 23.1.)

More generally, we shall define the *meromorphic hyperfunction* on I as follows. A hyperfunction $g \in \mathfrak{B}(I)$ is called a meromorphic hyperfunction if it is expressible in the form:

$$(2) \qquad g(x) = [\varepsilon(z)\varphi_1(z) + \bar{\varepsilon}(z)\varphi_2(z), D]_{z=x}, \quad D \in \mathfrak{D}(I), \ \varphi_1, \varphi_2 \in \mathfrak{A}(D-I),$$

where φ_1 and φ_2 are meromorphic funcfions on I. For example, for each meromorphic function $\varphi(x)$ on I, $\varphi(x+i0)$, $\varphi(x-i0)$, and hence their arithmetic mean

$$(3) \qquad P\varphi(x) = (\varphi(x+i0) + \varphi(x-i0))/2$$

are all meromorphic hyperfunctions on I. We call $P\varphi(x)$ the *principal value* of $\varphi(x)$. (Note that $\varphi(x)$ itself is not a hyperfunction). Every meromorphic hyperfunction $g(x)$ on I, as expressed by (2), possesses a discrete carrier of singularity, and is uniquely decomposed into the form

$$(4) \qquad g(x) = P\varphi(x) + (\text{general } \delta\text{-function})$$

with $\varphi(x) = \varphi_1(x) - \varphi_2(x)$.

For example, we have the Lippmann-Schwinger formula[*)]

$$(5) \qquad \begin{aligned} \frac{1}{x \pm i0} &= P\frac{1}{x} \mp i\pi\delta(x) \\ \frac{1}{(x \pm i0)^{n+1}} &= P\frac{1}{x^{n+1}} \mp \frac{(-1)^n}{n} i\pi\delta^{(n)}(x) \end{aligned}$$

by (9.2), (9.3) and (3). Since $P\frac{1}{x^{n+1}}$ is of the opposite parity to $\delta^{(n)}(x)$, the formulae (5) just furnish simple examples of the decomposition (7.4) (as well as of the decomposition (3.2)).

§ 11. *Heaviside Y-function.*

We define as follows the Heaviside Y-function $Y(x)$ as a hyperfunction $\in \mathfrak{B}(\boldsymbol{R})$:

$$(1) \qquad Y(x) = \frac{-1}{2\pi i}[\log(-z)]_{z=x}$$

in which we consider $\log(-z)$ a single-valued function of $z \in \boldsymbol{C} - [0, \infty)$:

*) This is a formula frequently used in the theory of dispersion of waves, especially in quantum theory of scattering processes. See, e. g., B. A. Lippmann-J. Schwinger, Variational Principles for Scattering Processes, I. Phys. Rev. **79** (1950), 469, formula (1.57); W. Heitler, loc. cit.; see also P. M. A. Dirac, loc. cit., formula (50.35).

$$(2) \qquad \log(-z)\in\mathfrak{A}(C-[0,\infty))$$

which assumes real values for $z\in(-\infty, 0)$. We have, by (7.2),

$$(3) \qquad Y(-x)=\frac{1}{2\pi i}[\log z]_{z=x}, \quad \text{with} \quad \log z\in\mathfrak{A}(C-(-\infty, 0]),$$

hence

$$(4) \qquad Y(x)+Y(-x)=\frac{1}{2}[\varepsilon(z)-\bar{\varepsilon}(z)]_{z=x}=1.$$

Now let us define $\operatorname{sgn} x\in\mathfrak{B}(R)$ by

$$(5) \qquad \operatorname{sgn} x=Y(x)-Y(-x)=-\frac{1}{\pi i}[\varepsilon(z)\log(-iz)+\bar{\varepsilon}(z)\log(iz)]_{z=x},$$

hence

$$(6) \qquad Y(x)=(1+\operatorname{sgn} x)/2, \qquad Y(-x)=(1-\operatorname{sgn} x)/2.$$

All three hyperfunctions $Y(x)$, $Y(-x)$, and $\operatorname{sgn} x$ have the isolated point $\{0\}\subset R$ as their carrier of singularity, while they are expressed as follows in $R-\{0\}$:

$$(7) \qquad Y(x)=\begin{cases}0\\ 1,\end{cases} \qquad Y(-x)=\begin{cases}1\\ 0,\end{cases} \qquad \operatorname{sgn} x=\begin{cases}-1 & (x\in(-\infty, 0))\\ 1 & (x\in(0,\infty)).\end{cases}$$

We have furthermore

$$(8) \qquad \frac{dY(x)}{dx}=\delta(x), \qquad \frac{dY(-x)}{dx}=-\delta(x)$$

and

$$(9) \qquad \underbrace{\int_{-\infty}^{x}dx\cdots\int_{-\infty}^{x}}_{n\text{-fold}}\delta(x)dx=\frac{x^{n-1}}{(n-1)!}Y(x), \quad \int_{\infty}^{x}dx\cdots\int_{\infty}^{x}\delta(x)dx=-\frac{x^{n-1}}{(n-1)!}Y(-x).$$

The unit step function of a interval $[a, b]\subset R$ is defined as a hyperfunction $\in\mathfrak{B}(R)$ as follows:

$$(10) \qquad Y(x\,;\,[a,b])=Y(x-a)-Y(x-b)=\frac{-1}{2\pi i}\left[\log\frac{a-z}{b-z}\right]_{z=x}\in\mathfrak{B}(R) \quad \text{with}$$

$$\log\frac{a-z}{b-z}\in\mathfrak{A}(C-[a,b]),$$

for which we have

$$(11) \qquad Y(x\,;\,[a,b])=\begin{cases}0 & x\in(-\infty, a)\\ 1 & x\in(a,b)\\ 0 & x\in(b,\infty)\end{cases}$$

§12. *The hyperfunction with holomorphic parameters.*

Let Δ be an open set of the product space $C^s=\underbrace{C\times\cdots\times C}_{s\text{-uple}}$ of the complex plane C. For any locally closed subset W of $C\times\Delta$ and any $\alpha=(\alpha_1,\cdots,\alpha_s)\in\Delta$, we shall write $W_\alpha=\{z\,;\,(z,\alpha)\in W\}$. W_α is a locally closed set of C. Now let S

be a locally closed subset of $\boldsymbol{R}\times\Delta$, D an open set of $\boldsymbol{C}\times\Delta$ which contains S as a closed subset (the 'complex neighborhood of S'), $\varphi(z\,;\,\alpha)$ an holomorphic function (of $(s+1)$-complex variables $(z, \alpha_1, \cdots, \alpha_s)$) defined on $D-S$. Then, for each $\alpha\in\Delta$, D_α is a complex neighborhood of S_α, and $\varphi(z\,;\,\alpha)\in\mathfrak{A}(D_\alpha-S_\alpha)$. Hence a hyperfunction $g(x\,;\,\alpha)\in\mathfrak{B}(S_\alpha)$ is defined by

$$(1)\qquad g(x\,;\,\alpha)=[\varphi(z\,;\,\alpha),\,D_\alpha]_{z\to x}.$$

We shall call such $g(x\,;\,\alpha)$ a *hyperfunction with (complex) holomorphic parameters* $\alpha\in\Delta$.*)

§13. *Power function.*

Let α be a complex number ($\neq$rational integer). We define $g_\alpha(x)\in\mathfrak{B}(\boldsymbol{R})$ by:

$$(1)\qquad g_\alpha(x)=\frac{-1}{2i\sin\pi\alpha}[(-z)^\alpha]_{z\to x},\ \text{with}\ (-z)^\alpha\in\mathfrak{A}(\boldsymbol{C}-[0,\infty))$$

in which we consider $(-z)^\alpha$ a single-valued function of $z\in\boldsymbol{C}-[0,\infty)$ which assumes real values for $z\in(-\infty,0)$. We have by (7.2)

$$(2)\qquad g_\alpha(-x)=\frac{1}{2i\sin\pi\alpha}[z^\alpha]_{z\to x},\ \text{with}\ z^\alpha\in\mathfrak{A}(\boldsymbol{C}-(-\infty,0]).$$

$g_\alpha(x)$ (and hence $g_\alpha(-x)$) is a real valued hyperfunction, and has the isolated point $\{0\}$ as the carrier of singularity, while they are expressed as follows in $\boldsymbol{R}-\{0\}$:

$$(3)\qquad g_\alpha(x)=\begin{cases}0\\ x^\alpha,\end{cases}\quad g_\alpha(-x)=\begin{cases}(-x)^\alpha & (x\in(-\infty,0))\\ 0 & (x\in(0,\infty)).\end{cases}$$

On the other hand, we have by (1) and (2)

$$(4)\qquad \begin{aligned}h_\alpha(x)&=e^{-i\pi\alpha/2}g_\alpha(x)+e^{i\pi\alpha/2}g_\alpha(-x)=[\varepsilon(z)(-iz)^\alpha]_{z\to x},\\ \bar{h}_\alpha(x)&=e^{i\pi\alpha/2}g_\alpha(x)+e^{-i\pi\alpha/2}g_\alpha(-x)=-[\bar{\varepsilon}(z)(iz)^\alpha]_{z\to x},\end{aligned}$$

$$(5)\qquad \bar{h}_\alpha(x)=h_\alpha(-x)$$

and conversely,

$$(6)\qquad \begin{aligned}g_\alpha(x)&=\frac{-1}{2i\sin\pi\alpha}(e^{-i\pi\alpha/2}h_\alpha(x)-e^{i\pi\alpha/2}\bar{h}_\alpha(x)),\\ g_\alpha(-x)&=\frac{1}{2i\sin\pi\alpha}(e^{i\pi\alpha/2}h_\alpha(x)-e^{-i\pi\alpha/2}\bar{h}_\alpha(x)).\end{aligned}$$

By (4), $h_\alpha(x)$ and $\bar{h}_\alpha(x)$ are upper and lower semi-holomorphic hyperfunctions respectively. By definition (1), the hyperfunction $g_\alpha(\pm x)$ is a hyperfunction with a holomorphic parameter $\alpha\in\boldsymbol{C}-\{0,\pm1,\pm2,\cdots\}$, while the expression (4)

*) We shall introduce the notion of *hyperfunction with real holomorphic parameters* in II.

shows that $h_\alpha(\pm x)$ is a hyperfunction with a holomorphic parameter $\alpha\in\boldsymbol{C}$. However, we can obtain a more precise result as follows. Define $\varphi(z,\alpha)\in\mathfrak{A}(\boldsymbol{C}-[0,\infty))$, $\alpha\in\boldsymbol{C}$, by

$$\varphi(z,\alpha)=(-z)^{\alpha}\int_{-\infty}^{z}e^{t}(-t)^{-\alpha-1}dt. \tag{7}$$

Then we have, dividing the intergal into $\int_{-\infty}^{0}+\int_{0}^{z}$,

$$\varphi(z,\alpha)=\Gamma(-\alpha)\cdot(-z)^{\alpha}+f(z,\alpha)^{*)} \tag{8}$$

where

$$f(z,\alpha)=(-z)^{\alpha}\int_{0}^{z}(-t)^{-\alpha-1}e^{t}\,dt=\sum_{n=0}^{\infty}\frac{1}{\alpha-n}\cdot\frac{z^n}{n!}\in\mathfrak{A}(\boldsymbol{C}). \tag{9}$$

Therefore we have for $\alpha\in\boldsymbol{C}-\{0,\pm1,\pm2,\cdots\}$:

$$\Gamma(\alpha+1)\varphi(z,\alpha)\equiv-\frac{\pi}{\sin\pi\alpha}\cdot(-z)^{\alpha}\qquad(\mathrm{mod}\,\mathfrak{A}(\boldsymbol{C})), \tag{10}$$

hence

$$\begin{aligned} g_\alpha(x)&=\frac{\Gamma(\alpha+1)}{2\pi i}[\varphi(z,\alpha)]_{z=x},^{**)}\\ g_\alpha(-x)&=\frac{\Gamma(\alpha+1)}{2\pi i}[\varphi(-z,\alpha)]_{z=x}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} h_\alpha(\pm x)&=\frac{\Gamma(\alpha+1)}{2\pi i}[e^{\mp i\pi\alpha/2}\varphi(z,\alpha)-e^{\pm i\pi\alpha/2}\varphi(-z,\alpha)]_{z=x}\\ &=\pm[\varepsilon(\pm z)(\mp iz)^{\alpha}-\sin\frac{\pi\alpha}{2}\Gamma(\alpha+1)f(z,\alpha)]_{z=x}. \end{aligned} \tag{12}$$

Moreover the right hand sides of (1) and (2) give effective definitions of $g_\alpha(\pm x)$ and $h_\alpha(\pm x)$ even for $\alpha=n=0,1,2,\cdots$. We have by (7)

$$\varphi(z,n)=-z^{n}\int^{z}\frac{e^{t}}{t^{n+1}}dt\equiv-\frac{z^n}{n!}\log(-z)\qquad(\mathrm{mod}\,\mathfrak{A}(\boldsymbol{C})), \tag{13}$$

whence we obtain

$$g_n(\pm x)=(\pm x)^{n}Y(\pm x). \tag{14}$$

The corresponding expression for $h_n(\pm x)$:

$$h_n(\pm x)=(\mp ix)^{n}\in\mathfrak{A}(\boldsymbol{R}) \tag{15}$$

---

*) Actually, the identity (8) is derived here only for $\Re\alpha<0$, but the result is true for any $\alpha\in\boldsymbol{C}-\{0,1,2,\cdots\}$ on account of the *principle of analytic continuation*.

**) The defining functions $\pm\dfrac{\Gamma(\alpha+1)}{2\pi i}\varphi(\pm z,\alpha)$ which appear in (11) are the standard defining functions of $e^{\mp x}g_\alpha(\pm x)$ multiplied by $e^{\pm z}$. (See [3], p. 26, comment (17), and forth coming note II).

is a particular case of the expression (4), while the expression (6) cease to be effective for $\alpha=n=0, 1, 2, \cdots$. To sum up, $g_\alpha(\pm x)$ is a hyperfunction with a holomorphic parameter $\alpha\in C-\{-1, -2, \cdots\}$. Moreover, the expressions (11) shows that $g_\alpha(\pm x)/\Gamma(\alpha+1)$ is a hyperfunction with a holomorphic parameter $\alpha\in \boldsymbol{C}$, which, as is seen from (7), reduces to a δ-function for $\alpha=-n-1=-1, -2, \cdots$:

$$\varphi(z,-n-1)=-\frac{1}{z^{n+1}}\int_{-\infty}^{z}e^t t^n dt=-\frac{1}{z^{n+1}}\left(\frac{d^n}{d\lambda^n}\int_{-\infty}^{z}e^{\lambda t}\,dt\right)_{\lambda=1}$$

$$=-\frac{1}{z^{n+1}}\left(\frac{d^n}{d\lambda^n}\frac{e^{\lambda z}}{\lambda}\right)_{\lambda=1}\equiv(-1)^{n+1}\frac{n!}{z^{n+1}}\ (\mathrm{mod}\,\mathfrak{A}(\boldsymbol{C})),$$

(16) $$(g_\alpha(\pm x)/\Gamma(\alpha+1))_{\alpha=-n-1}=\delta^{(n)}(\pm x).$$

Representation of Hyperfunctions by Harmonic Functions.

§14. *Representation of hyperfunctions by harmonic functions.*

It is possible to define the notion of hyperfunctions in terms of harmonic functions instead of analytic functions ([3] §1, p. 5). Let us denote with $\mathfrak{H}(D)$ the module of (single-valued) harmonic functions defined on an open set D of $\boldsymbol{C}$. For $D\supset D'$ (D and D' denoting open sets of $\boldsymbol{C}$), we have a natural homomorphism by restriction of the domains

(1) $$\rho_{D'D}:\ \mathfrak{H}(D)\longrightarrow\mathfrak{H}(D').$$

For an open set I of $\boldsymbol{R}$, we shall denote with $\mathfrak{H}^+(I)$ the inductive limit of $\{\mathfrak{H}(D^+)\,;\ D\in\mathfrak{D}^*(I)\}$ by $\rho_{D'D}$, while we denote with $\mathfrak{H}_0^+(I)$ the inductive limit of $\{\mathfrak{H}_0(D^+)\,;\ D\in\mathfrak{D}^*(I)\}$ where $\mathfrak{H}_0(D^+)$ denotes the submodule of $\mathfrak{H}(D^+)$ cosisting of harmonic functions whose boundary values on I are identically 0. $\mathfrak{H}_0^+(I)$ consistute a submodule of $\mathfrak{H}^+(I)$ in a natural sense. Now let $\varphi(z)\in\mathfrak{A}(D)$, then we have

(2) $$u(x,y)=\varphi(x+iy)-\varphi(x-iy)\in\mathfrak{H}(D^+).$$

The correspondence $\varphi(z)\to u(x, y)$ induces a canonical homomorphism

(3) $$\tilde{\mathfrak{A}}(I)\longrightarrow\mathfrak{H}^+(I)$$

which turns out to be *surjective* because there are arbitrarily small $D\in\mathfrak{D}^*(I)$ such that each connected component of D^+ is simply connected. Besides, the inverse image of $\mathfrak{H}_0^+(I)\subset\mathfrak{H}^+(I)$ coincidens with $\mathfrak{A}(I)$ in accordance with the principle of reflection for harmonic prolongation. Thus we have

(4) $$\mathfrak{B}(I)\simeq\mathfrak{H}^+(I)\ \mathrm{mod}\,\mathfrak{H}_0^+(I).$$

For $g(x)=[\varphi(z), D]_{z=x}\in\mathfrak{B}(I)$ and $u(x, y)=\varphi(x+iy)-\varphi(x-iy)\in\mathfrak{H}(D^+)$ we say that

$u(x, y)$ is a defining harmonic function of $g(x)$, and denote

(5) $$g(x)=[u(x, y), D]_{y=+0}=u(x, +0).$$

For instance, we have, instead of (3.1) and (8.1) respectively,

(6) $$\overline{g}(x)=[\overline{u(x, y)}]_{y=+0}=\overline{u(x, +0)},$$

(7) $$g^{(n)}(x)=\left[\frac{\partial^n u(x, y)}{\partial x^n}\right]_{y=+0}=\frac{\partial^n(x, +0)}{\partial x^n}.$$

Example 1.

(8) $$\begin{cases} \delta(x)=\dfrac{1}{\pi}\left[\dfrac{y}{x^2+y^2}\right]_{y=+0}, \qquad P\dfrac{1}{x}=\left[\dfrac{x}{x^2+y^2}\right]_{y=+0}, \\ \delta^{(n)}(x)=\dfrac{1}{\pi}\left[\left(\dfrac{\partial}{\partial x}\right)^n \dfrac{y}{x^2+y^2}\right]_{y=+0}=(-1)^n\dfrac{n!}{\pi}\left[\dfrac{\Im(x+iy)^{n+1}}{(x^2+y^2)^{n+1}}\right]_{y=+0}, \\ P\dfrac{1}{x^{n+1}}=\dfrac{(-1)^n}{n!}\left[\left(\dfrac{\partial}{\partial x}\right)^n \dfrac{x}{x^2+y^2}\right]_{y=+0}=\left[\dfrac{\Re(x+iy)^{n+1}}{(x^2+y^2)^{n+1}}\right]_{y=+0}. \end{cases}$$

Example 2.

(9) $$\begin{cases} Y(\pm x)=\dfrac{1}{\pi}[\arg(\mp x+iy)]_{y=+0}, \\ \text{with } \arg z\in(0, \pi) \text{ for } z\in \boldsymbol{C} \text{ (the principal value of } \arg z), \end{cases}$$

(10) $$\begin{cases} \operatorname{sgn} x=\dfrac{2}{\pi}\left[\operatorname{Tan}^{-1}\dfrac{x}{y}\right]_{y=+0}, \\ \text{where } \operatorname{Tan}^{-1}\dfrac{x}{y}\in\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \quad \text{for} \quad z\in \boldsymbol{C}, \end{cases}$$

(11) $$\begin{cases} Y(x\,;\,[a, b])=\dfrac{1}{\pi}[\theta(x, y)]_{y=+0}, \\ \text{where } \theta(x, y)=\arg(z-b)-\arg(z-a)=\widehat{azb} \\ \qquad \text{(the angle between } \overline{za} \text{ and } \overline{zb}), \text{ with } z=x+iy. \end{cases}$$

Integration.

One of the most important operations on hyperfunctions is *definite integral.* Not only it extends the usual notion of integral but it brings about great improvement for handling integral formulae of practical use in applied analysis. Various examples will be found in [3] §2.

§15. *Integration.*

We now define as follows the *definite integral* of a hyperfunction on a *compact* $K\subset \boldsymbol{R}$.

Definition. The definite integral of $g(x)=[\varphi(z), D]_{z=x}\in\mathfrak{B}(K)$ *is given by*

(1) $$\int_K g(x)dx=-\oint_\Gamma \varphi(z)dz$$

*) W. Heitler, loc. cit. (8.17, b, c).

where Γ is a rectifiable path in $D\in\mathfrak{D}(K)$ going round K in the positive sense.[*)] By the integral theorem of Cauchy, the value of the integral (1) does not depend on the choice of the defining function (φ, D) and the path Γ. Taking this fact into consideration, we shall hereafter often denote the right hand side of (1) as follows:

$$-\oint_{\Gamma}\varphi(z)\,dz=-\oint_{\partial D}\varphi(z)dz. \tag{2}$$

For each particular hyperfunction $g\in\mathfrak{B}(K)$, we can consider $g\in\mathfrak{B}(I)$ with some real neighborhood I of K (proposition 4.1). In this case, the region K of integration in (1) may be replaced by I:

$$\int_I g(x)\,dx=\int_K g(x)\,dx.$$

For instance, for $K=[a, b]$ we write

$$\int_{a-\delta}^{b+\delta} g(x)\,dx=\int_{[a,b]} g(x)dx$$

with some δ, $0<\delta\leqslant\infty$.

Example 1. Let $f=[f, D]\in\mathfrak{A}(\{0\})$. We have

$$f(x)\delta(x)\in\mathfrak{B}(\{0\}),\quad f(x)\delta^{(n)}(x)\in\mathfrak{B}(\{0\})$$

and

$$\int_{-a}^{a} f(x)\,\delta(x)\,dx=\int_{\{0\}} f(x)\,\delta(x)\,dx=f(0), \tag{3}$$

$$\int_{-a}^{a} f(x)\,\delta^{(n)}\,dx=\int_{\{0\}} f(x)\,\delta^{(n)}(x)\,dx=(-1)^n f^{(n)}(0), \tag{4}$$

which follows from the definition (9.1) of $\delta(x)$ and the integral formula of Cauchy.

Example 2. Let $[a, b]\subset \boldsymbol{R}$ be a closed interval, and $f(x)\in\mathfrak{A}([a, b])$. Then we have

$$f(x)Y(x\,;\,[a, b])\in\mathfrak{B}([a, b])$$

and

$$\int_{[a,b]} f(x)Y(x\,;\,[a, b])dx=\int_a^b f(x)dx \tag{5}$$

[*)] Such Γ can be always found e.g. if we cover K with a finite number of discs D_ν:

$$K\subset D'=\bigcup_{\nu=1}^{n} D_\nu,\quad\text{with}\quad \overline{D}_\nu\subset D,$$

and set $\Gamma=\partial D'$. This Γ consists of a finite number of arc polygons in D.

where the expression in the right hand side denotes the integral of $f(x)$ in the usual sense (i.e. the Riemann or Lebesgue integral of $f(x)$).

Proof of (5). By definition, we have

$$\begin{aligned}(\text{The left hand side}) &= -\oint_{\Gamma} f(z)\left(\frac{1}{2\pi i}\int_a^b \frac{dx}{x-z}dx\right)dz \\ &= -\frac{1}{2\pi i}\int_a^b\left(\oint_{\Gamma}\frac{f(z)}{x-z}dz\right)dx = \int_a^b f(x)dx.\end{aligned}$$

(q. e. d.)

§16. *The standard representation.*

Proposition 16.1. Let $g(x)\in\mathfrak{B}(K)$ *be a hyperfunction on a compact* $K\subset \boldsymbol{R}$. *Then* $\varphi_0(z)\in\mathfrak{A}(\boldsymbol{C}-K)$ *defined by*

$$(1) \qquad \varphi_0(z)=\frac{1}{2\pi i}\int_K \frac{g(x)}{x-z}\,dx, \qquad z\in \boldsymbol{C}-K,$$

is a defining function of $g(x)$:

$$(2) \qquad g(x)=[\varphi_0(z), \boldsymbol{C}]_{z\to x}.$$

This $\varphi_0(z)$, which is uniquely determined by $g(x)$, will be called the *standard defining function* of $g(x)$.

Proof. Let $g(x)=[\varphi(z), D]_{z\to x}$, $D\in\mathfrak{D}(K)$. By definition (15.1),

$$(3) \qquad \varphi_0(z)=-\frac{1}{2\pi i}\int_{\Gamma} \frac{\varphi(\zeta)}{\zeta-z}\,d\zeta, \qquad z\in \boldsymbol{C}-K.$$

where Γ is a rectifiable path in D going round K in the positive sense such that z is outside Γ. We can write, in agreement with (15.2),

$$(4) \qquad \varphi_0(z)=\frac{1}{2\pi i}\oint_{\partial(\boldsymbol{C}-K)} \frac{\varphi(\zeta)}{\zeta-z}\,d\zeta, \qquad z\in \boldsymbol{C}-K.$$

On the other hand, define $f(z)\in\mathfrak{A}(D)$ by

$$(5) \qquad f(z)=\frac{1}{2\pi i}\oint_{\partial D} \frac{\varphi(\zeta)}{\zeta-z}\,d\zeta, \qquad z\in D.$$

Combining (4) and (5), we have for $z\in D-K$,

$$(6) \qquad \varphi_0(z)+f(z)=\frac{1}{2\pi i}\int_{\partial(D-K)} \frac{\varphi(\zeta)}{\zeta-z}\,d\zeta=\varphi(z)$$

by the integral formula of Cauchy. Thus we have

$$(7) \qquad \varphi(z)\equiv\varphi_0(z) \qquad (\mathrm{mod}\,\mathfrak{A}(D))$$

and hence the desired result (2). (q. e. d.)

Among defining functions of $g(x)\in\mathfrak{B}(K)$, the standard defining function $(\varphi_0(z), C)$ is characterized by

$$\varphi_0(z)\in\mathfrak{A}(C-K) \tag{8}$$

and

$$\varphi_0(\infty)=\lim_{|z|\to\infty}\varphi_0(z)=0, \tag{9}$$

by the Liouville theorem. (From (8), (9) follows in particullar that $\varphi_0(z)$ is holomorphic at $z=\infty$. We shall write $\varphi_0(z)\in\mathfrak{A}(\overline{C})$ for it, where $\overline{C}$ denotes the Riemann sphere $C\smile\{\infty\}$.)

Proposition 16.2. Let $g=[\varphi_0, C]\in\mathfrak{B}(K)$, *and let* φ_0 *be the standard defining function of* g. *We have then*

$$\int_K g(x)dx=2\pi i\alpha \tag{10}$$

where α *denotes the residue of the differential form* $\varphi_0(z)dz$ *at* $z=\infty$:

$$\alpha=-(z\varphi_0(z))_{z=\infty}. \tag{11}$$

Proof. By definition,

$$\int_K g_0(x)dx=-\oint_{\partial C}\varphi_0(z)\,dz=2\pi i\alpha. \qquad \text{(q. e. d.)}$$

Though the notion of standard defining function is useful for various purposes, it must be remembered that it is *not an invariant notion* under the analytic transformation of the domain K.

Concerning the representation by defining harmonic function, we shall define the *standard defining harmonic function* of $g\in\mathfrak{B}(K)$ by

$$u_0(x, y)=\frac{1}{\pi}\int_K g(t)\frac{y}{(t-x)^2+y^2}dt, \qquad (y>0). \tag{12}$$

Then we have $u_0(x, y)\in\mathfrak{H}(C^+)$ and

$$g(x)=u_0(x, +0) \tag{13}$$

by (4).

Example 1. The expressions of $g(x)$, $\delta^{(n)}(x)$, and $Y(x, [a, b])$ in the definitions (9.1), (9.3), and (11.10) are all representations by the standard defining functions. We have, by (10)

$$\int_{[0]}\delta(x)dx=1,$$

$$\int_{[0]}\delta^{(n)}(x)dx=0 \qquad (n>0),$$

and, combining this result with (9.4′) we obtain the result (16.3) and (16.4) again.

The expressions in (14.8) and (14.11) are representations by the standard defining harmonic functions.

Example 2. Let $P_n(z)$, $Q_n(z)$ be the Legendre coefficients of degree n of the first and the second kind. Then we have a formula of C. Neumann:

$$Q_n(z)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(x)}{x-z}dx, \qquad z\in \boldsymbol{C}-[-1,1].$$

By (15.5), the standard defining function of $P_n(x)Y(x,[-1,1])\in\mathfrak{B}([-1,1])$ is given by $\frac{1}{\pi i}Q_n(z)\in\mathfrak{A}(\boldsymbol{C}-[-1,1])$:

(14) $$P_n(x)Y(x,[-1,1])=\frac{1}{\pi i}[Q_n(z),\boldsymbol{C}]_{z=x}.$$

§17. *The perfect hyperfunction.*

Now let K be a compact sebset of a locally closed $S\subset\boldsymbol{R}$. Let $D_0\in\mathfrak{D}(S)$ be given (e.g. $D_0=\boldsymbol{C}-(\bar{S}-S)$).

Proposition 17.1 guarantees that each $g\in\mathfrak{B}(K)$ possesses a defining function $\varphi\in\mathfrak{A}(D_0-K)$:

(1) $$g=[\varphi,D_0],$$

e.g.

(2) $$\varphi=\varphi_0|(D_0-K),$$
$$\varphi_0:\ \text{standard defining function of } g.$$

Define now a hyperfunction $g_0\in\mathfrak{B}(S)$ by

(3) $$g_0=[\varphi|(D_0-S),D_0].$$

Clearly this g_0 is determined only by $g\in\mathfrak{B}(K)$ and is independent of the choice of D_0 and φ. The correspondence $g\to g_0$ induces a canonical homomorphism

(4) $$\mathfrak{B}(K)\longrightarrow\mathfrak{B}(S)$$

which is *injective* because $g_0=0$ clearly implies $g=0$. Consequently, we can agree hereafther to set

(5) $$\mathfrak{B}(K)\subset\mathfrak{B}(S)$$

in a natural sense. A hyperfunction $g\in\mathfrak{B}(S)$ belongs to $\mathfrak{B}(K)$ if and only if $g|(S-K)=0$, i.e. if and only if

(6) $$\mathrm{car}\, g\subset K.$$

*Definition. A hyperfunction $g\in\mathfrak{B}(S)$ is called a perfect hyperfunction**)

*) In [1], we named it *particular hyperfunction* (*p.h.f.*).

if $g\in\mathfrak{B}(K)$ *for some compact* $K\subset S$, *i.e. if* car g *is compact.*

More generally, we define a *perfect hyperfunction* g *on any given set* $E\subset \boldsymbol{R}$ to mean a hyperfunction on some compact subset $K\subset E$, and define the *integral of* g *over* E by

$$(7)\qquad \int_E g(x)\,dx=\int_K g(x)dx.$$

The family $\mathfrak{B}^*(E)$ of all perfect hyperfunctions on E is given by the inductive limit of $\{\mathfrak{B}(K)\,;\ K\subset E\}$ by the (injective) canonical homomorphisms:

$$(8)\qquad \mathfrak{B}(K)\longrightarrow\mathfrak{B}(K'),\qquad K\subset K',$$

K, K' denoting compact subsets of E. By (5), we can set

$$(9)\qquad \mathfrak{B}^*(E)=\bigcup_{K\subset E}\mathfrak{B}(K).$$

Each $\mathfrak{B}(K)$ is an $\mathfrak{A}(K)$- and hence $\mathfrak{A}(E)$-module, and each homomorphism (8) is an $\mathfrak{A}(E)$-homomorphism (i.e. $\mathfrak{B}(K)$ is an $\mathfrak{A}(E)$-submodule of $\mathfrak{B}(K')$ for $K\subset K'$). Hence $\mathfrak{B}^*(E)$, defined as the inductive limit of these $\mathfrak{B}(K)$, is also an $\mathfrak{A}(E)$-module. In particular, $\mathfrak{B}^*(S)$ is an $\mathfrak{A}(S)$-submodule of $\mathfrak{B}(S)$ for any locally closed S. On the other hand, let $\mathfrak{A}^*(E)$ denote the inductive limit of $\{\mathfrak{A}(D-K)\,;\ D:$ open set $\supset E$, $K:$ compact set $\subset E\}$ by the restrictions $\mathfrak{A}(D-K)\to\mathfrak{A}(D'-K')$ with $D\supset D'$, $K\subset K'$. We can consider $\mathfrak{A}^*(E)$ as an extension ring of $\mathfrak{A}(E)$, and hence as an $\mathfrak{A}(E)$-module, in a natural manner. Moreover, we can easily derive the relation

$$(10)\qquad \mathfrak{B}^*(E)\simeq\mathfrak{A}^*(E)\bmod\mathfrak{A}(E)$$

by a natural correspondence.

Decomposition of Hyperfunctions.

In the following §§18, 19, the "decomposability" of hyperfunction is studied. As we shall observe in §25, we can prove fundamental properties of hyperfunction such as localizability (proposition 24.1) and completeness theorem (proposition 22.1) by means of the results of §§18, 19[*], but we shall derive them (including decomposition theorem) in a more unified manner in §§20–24.

§18. *Decomposition theorem for perfect hyperfunction.*

Let $-\infty<a=c_0\leqslant c_1\leqslant\cdots\leqslant c_n=b<\infty$ and let $g_\nu\in\mathfrak{B}([c_{\nu-1},c_\nu])$, $\nu=1,\cdots,n$, be given. Then, by (17.5), we can consider g_ν as belonging to $\mathfrak{B}([a,b])$, and hence we have their sum

$$(1)\qquad g=g_1+\cdots+g_n\in\mathfrak{B}([a,b])$$

as a hyperfunction *on* $[a,b]$. The purpose of this paragraph is to derive the

[*] This is the way in which the author was first led to these results in autumn, 1957.

converse of this fact (proposition 2).

Lemma 1. Let $k(x)$ be a (complex valued) continuous function[*] *on $[0, \infty)$. Then there exists an entire function $\chi(z)\in\mathfrak{A}(\boldsymbol{C})$ such that*

$$\chi(x)\geqslant|k(x)| \quad \text{for } x\in[0, \infty) \tag{2}$$

Proof. Replacing $k(x)$ by $\max\limits_{0\leqslant t\leqslant x}|k(t)|$ if necessary, we can assume that $k(x)$ is positive-valued ($k(x)\geqslant 0$) and monotone increasing on $[0, \infty)$. Choose a pair of sequences of numbers: $\{a_n;\ n=1, 2, \cdots\}$, $\{b_n;\ n=1, 2, \cdots\}$ such that

$$0<a_1\leqslant a_2\leqslant\cdots, \qquad 0<b_1\leqslant b_2\leqslant\cdots \tag{3}$$
$$\text{and} \qquad b_n<a_n, \quad (n=1, 2, \cdots).$$

(E. g. $a_n=2n$, $b_n=n$). Then, for every n, we have

$$\left(\frac{a_n}{b_n}\right)^{m_n}\geqslant k(a_{n+1}) \tag{4}$$

for a sufficiently large natural number m_n. Clearly we can assume moreover

$$m_1<m_2<\cdots.$$

Now define $\chi(z)$ by

$$\chi(z)=k(a_1)+\sum_{n=1}^{\infty}\left(\frac{z}{b_n}\right)^{m_n}, \tag{5}$$

a power series whose radius of convergence is ∞. Therefore, $\chi(z)$ is an entire function:

$$\chi(z)\in\mathfrak{A}(\boldsymbol{C})$$

which is, by definition (5), positive-valued and monotone increasing on $[0, \infty)$. Now we have, by (5) and (4),

$$\chi(a_n)\geqslant\left(\frac{a_n}{b_n}\right)^{m_n}\geqslant k(a_{n+1})$$

and hence, we have for any $x\in[a_n, a_{n+1}]$,

$$\chi(x)\geqslant k(x). \tag{6}$$

As n is arbitrary, we see that (6) is valid for any $x\in[a_1, \infty)$. (6) is valid also for $x\in[0, a_1]$, as we have by (5)

$$\chi(x)\geqslant k(a_1)\geqslant k(x).$$

Hence $\chi(z)$ satisfies all the required conditions. (q. e. d.)

[*] The condition '$k(x)$ is continuous' can be replaced by '$k(x)$ is bounded on any finite interval $[0, a]$, $(0\leqslant a<\infty)$'. The proof which follows is valid for this case, too, if we replace $\max\limits_{0\leqslant t\leqslant x}|k(t)|$ therein by $\sup\limits_{0\leqslant t\leqslant x}|k(t)|$.

Lemma 2. In the preceding lemma 1, $\chi(z)$ can be chosen as an even entire function without zeros (i.e. $\chi(-z)=\chi(z)$, $\frac{1}{\chi(z)}\in\mathfrak{A}(\boldsymbol{C})$).

Proof. Apply lemma 1 to $k_1(x)=\log^+|k(\sqrt{x})|$. Then we have a $\chi_1(z)\in\mathfrak{A}(\boldsymbol{C})$ such that

$$\chi_1(x)\geqslant k_1(x) \qquad (0\leqslant x<\infty).$$

Put $\chi(z)=\exp\chi_1(z^2)$. Then $\chi(z)$ is clearly an even entire function without zeros for which we have

$$\chi(x)\geqslant \exp k_1(x^2)\geqslant k(x). \qquad \text{(q. e. d.)}$$

Proposition 18.1. Let $-\infty<a\leqslant c\leqslant b<\infty$, and let $g\in\mathfrak{B}([a,b])$ be given. Then we have a decomposition of $g(x)$:

$$g(x)=g_1(x)+g_2(x) \tag{7}$$

with some $g_1(x)\in\mathfrak{B}([a,c])$ and $g_2(x)\in\mathfrak{B}([c,b])$.

Proof. Let $\varphi_0(z)$ be the standard defining function of $g(x)$.

$z=c$ and $\Re z=c$ are transformed to $z'=\infty$ and $\Im z'=0$ (real axis on z'-plane) respectively, by a Möbius transformation: $z=c+iz'^{-1}$, $z'=\frac{i}{z-c}$. Hence there exists, by lemma 2, a $\chi(z)\in\mathfrak{A}(\bar{\boldsymbol{C}}-\{c\})$ such that

$$\frac{1}{\chi(z)}\in\mathfrak{A}(\bar{\boldsymbol{C}}-\{c\}), \quad \text{and} \tag{8}$$
$$\chi(z)=\overline{\chi(\bar{z})}\geqslant\max(|\varphi(z)|,\ |\varphi(\bar{z})|) \quad \text{for} \quad \Re z=c.$$

Consequently we have

$$\varphi_0(z)=\chi(z)\psi_0(z) \tag{9}$$

with $\psi_0(z)\in\mathfrak{A}(\bar{\boldsymbol{C}}-\{c\})$ such that

$$|\psi_0(z)|\leqslant 1 \ \text{for}\ \Re z=c,\ z\neq c. \tag{10}$$

Now let us consider two paths Γ_1, Γ_2 defined by

$$\Gamma_1:\quad c\to c+i\varepsilon\to a-\varepsilon+i\varepsilon\to a-\varepsilon-i\varepsilon\to c-i\varepsilon\to c$$
$$\Gamma_2:\quad c\to c-i\varepsilon\to b+\varepsilon-i\varepsilon\to b+\varepsilon+i\varepsilon\to c+i\varepsilon\to c$$

with sufficiently small $\varepsilon>0$ (Fig. 1).

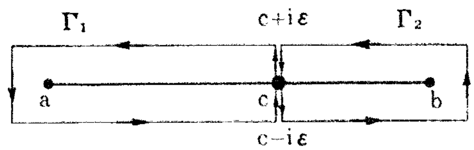


Fig. 1

By (10), we can define $\psi_\nu(z)$, $\nu=1,2$, by

$$\psi_\nu(z)=\frac{-1}{2\pi i}\int_{\Gamma_\nu}\frac{\psi_0(\zeta)}{\zeta-z}\,d\zeta \qquad \begin{cases} z\in\boldsymbol{C}-[a,c] & \text{for } \nu=1,\\ z\in\boldsymbol{C}-[c,b] & \text{for } \nu=2.\end{cases} \tag{11}$$

As ε can be taken arbitrarily small, we get

$$(12)\qquad \psi_1(z)\in\mathfrak{A}(\bar{C}-[a,c]),\qquad \psi_2(z)\in\mathfrak{A}(\bar{C}-[c,b]).$$

Moreover we have for $z\in C-[a,b]$

$$(13)\qquad \psi_1(z)+\psi_2(z)=\frac{-1}{2\pi i}\int_{\Gamma_1+\Gamma_2}\frac{\psi_0(\zeta)}{\zeta-z}d\zeta=\psi_0(z).$$

Defining $g_\nu(x)$, $\nu=1,2$, by

$$(14)\qquad g_\nu(x)=[\varphi_\nu(z),C]_{z=x}\ \text{with}\ \varphi_\nu(z)=\chi(z)\psi_\nu(z),$$

we have, by (9), (12), and (13),

$$g(x)=g_1(x)+g_2(x)$$
$$g_1(x)\in\mathfrak{B}([a,c]),\qquad g_2(x)\in\mathfrak{B}([c,b]).\qquad\text{(q. e. d.)}$$

By repeated application of the above proposition 1, we obtain

Proposition 18.2. Let $-\infty<a=c_0\leqslant c_1\leqslant\cdots\leqslant c_n=b<\infty$, *and let* $g\in\mathfrak{B}([a,b])$ *be given. Then we have a decomposition of* $g(x)$:

$$(15)\qquad g(x)=g_1(x)+\cdots+g_n(x)$$

with some $g_\nu(x)\in\mathfrak{B}([c_{\nu-1},c_\nu])$, $\nu=1,\cdots,n$.

Remark. Let $h_\nu\in\mathfrak{B}(\{c_\nu\})$, $\nu=1,\cdots,n-1$, be any hyperfunctions on a point $x=c_\nu$, and put $h_0\equiv h_n\equiv0$. Then we have from (15)

$$(16)\qquad \begin{aligned} &g=g_1'+\cdots+g_n'\\ \text{with}\quad &g_\nu'=-h_{\nu-1}+g_\nu+h_\nu\in\mathfrak{B}([c_{\nu-1},c_\nu]),\end{aligned}$$

a decomposition of g similar to (15). It is easy to see that (16) gives the *general form* of decomposition of g as required in proposition 2.

§19. *Decomposition of a hyperfunction into perfect hyperfunctions.*

The purpose of this paragraph is to generalize the decomposition theorem 18.2 to a hyperfunction which is not necessarily *perfect.*

Proposition 19.1. Let $I=(a,b)\subset\boldsymbol{R}$, $-\infty\leqslant a<b\leqslant\infty$, *be any open interval,* $\{c_n;\ n=0,\pm1,\pm2,\cdots\}$ *be any sequence of points on I such that*

$$(1)\qquad c_n\downarrow a\ (\textit{for}\ n\to-\infty),\qquad c_n\uparrow b\ (\textit{for}\ n\to\infty).$$

Then, for any $g\in\mathfrak{B}(I)$, *we have a decomposition*

$$(2)\qquad g=\sum_{\nu=-\infty}^{\infty}g_\nu$$

with some $g_\nu\in\mathfrak{B}([c_{\nu-1},c_\nu])$, $\nu=0,\pm1,\pm2,\cdots$. (2) *means that, for any* $I'=(a',b')$, $a<a'<b'<b$, *we have*

$$(3)\qquad g|I'=\sum_{n=-N}^{N}g_n|I'$$

for sufficiently large N. Each g_ν isu niquely determined mod. $(\mathfrak{A}\{c_{\nu-1}\}), \mathfrak{A}(\{c_\nu\})$.

Proof. Let $g=[\varphi, D]$, $\varphi\in\mathfrak{A}(D-I)$, $D\in\mathfrak{D}(I)$. Let $d(x)$ be a continuous function on I such that

$$0<d(x)<\infty, \text{ and } \{x+iy\,;\ |y|\leqslant d(x), x\in I\}\subset D$$

and define $D'\in D(I)$, $D'\subset D$, by

$$D'=\{x+iy\,;\ |y|<d(x), x\in I\}.$$

On the other hand, we can find, for each ν, $\chi_\nu(z)\in\mathfrak{A}(C-\{c_\nu\})$ such that we have

$$\varphi(z)=\chi_\nu(z)\psi_\nu(z), \qquad \psi_\nu(z)\in\mathfrak{A}(D-I),$$
$$|\psi_\nu(z)|\leqslant 1 \text{ for } \Re z=c_\nu,\ z\in D'-I,$$

by lemma 2. Define $\xi_\nu(z)$ and $\eta_\nu(z)$ by

$$(4) \qquad \frac{-\chi_\nu(z)}{2\pi i}\int_{-id(c_\nu)}^{id(c_\nu)}\frac{\psi_\nu(\zeta)}{\zeta-z}\,d\zeta=\begin{cases}\xi_\nu(z) & (\text{for } \Re z>c_\nu,\ z\in D')\\ \eta_\nu(z) & (\text{for } \Re z<c_\nu,\ z\in D').\end{cases}$$

Then we have, by analytic continuation,

$$(5) \qquad \xi_\nu(z)\in\mathfrak{A}(D'-(a, c_\nu]), \qquad \eta_\nu(z)\in\mathfrak{A}(D'-[c_\nu, b))$$

and

$$(6) \qquad \xi_\nu(z)+\eta_\nu(z)=\varphi(z) \qquad (\text{for } z\in D'-I).$$

Now put

$$(7) \qquad \varphi_\nu(z)=\xi_\nu(z)-\xi_{\nu-1}(z)=-\eta_\nu(z)+\eta_{\nu-1}(z),$$

then we have

$$(8) \qquad \varphi_\nu(z)\in\mathfrak{A}(D'-[c_{\nu-1}, c_\nu]).$$

Therefore we can define $g_\nu\in\mathfrak{B}([c_{\nu-1}, c_\nu])$ by

$$(9) \qquad g_\nu=[\varphi_\nu, D'].$$

We have, by (7), (6), and (5),

$$\varphi(z)-\sum_{\nu=-N}^{N}\varphi_\nu(z)=\xi_{-N-1}(z)+\eta_N(z)\in\mathfrak{A}(D'-(a, c_{-N-1}]\smile[c_N, b)).$$

By (1), we have $c_{-N-1}\leqslant a'$ and $b'\leqslant c_N$ for sufficiently large N, whence

$$(\varphi(z)-\sum_{\nu=-N}^{N}\varphi_\nu(z))|I'=0$$

viz.

$$g|I'=\sum_{\nu=-N}^{N}g_\nu|I'.$$

The uniqueness of each g_ν mod. $(\mathfrak{A}(\{c_{\nu-1}\}), \mathfrak{A}(\{c_\nu\}))$ is easily verified. (q. e. d.)
Conversely we have

Proposition 19.2. Let $I=(a, b)$ *and* $\{c_n;\ n=0, \pm 1, \pm 2, \cdots\}$ *be as in proposition 1, and let* $g_\nu(x)\in\mathfrak{B}([c_{\nu-1}, c_\nu])$, $\nu=0, \pm 1, \pm 2, \cdots$, *be any given hyperfunction on each* $[c_{\nu-1}, c_\nu]$. *Then there exists a uniquely determined* $g(x)\in\mathfrak{B}(I)$ *such that*

$$g(x)=\sum_{\nu=-\infty}^{\infty}{}' g_\nu(x). \tag{10}$$

This proposition is included in the following

Proposition 19.3. Under the same assumptions as in proposition 19.2, there exists a $h(x)\in\mathfrak{B}([a, b])$*) *such that*

$$h(x)|I=\sum_{\nu=-\infty}^{\infty}{}' g_\nu(x). \tag{11}$$

Proof. Let $\varphi_\nu(z)$ denote the standard defining function of $g_\nu(x)$. For each $\nu\leqslant 0$, we shall define $f_\nu(z)\in\mathfrak{A}(\bar{C}-\{a\})$ as follows. Let

$$\varphi_\nu(z)=\sum_{n=1}^{\infty}{}' c_{\nu n}(z-a)^{-n} \qquad (|z-a|>c_\nu-a)^{**)} \tag{12}$$

be the Laurent expansion of $\varphi_\nu(z)$. Then $f_\nu(z)$ is a partial sum of (12):

$$f_\nu(z)=\sum_{n=1}^{N_\nu}{}' c_{\nu n}(z-a)^{-n} \qquad (z\neq a) \tag{13}$$

where N_ν is a sufficiently large natural number such that

$$|\varphi_\nu(z)-f_\nu(z)|\leqslant 2^\nu \qquad \text{for } |z-a|\geqslant 2(c_\nu-a).^{***)} \tag{14}$$

Similarily, for $\nu>0$ we can define $f_\nu(z)\in\mathfrak{A}(\bar{C}-\{b\})$ so that

$$|\varphi_\nu(z)-f_\nu(z)|\leqslant 2^{-\nu} \qquad \text{for } |z-b|\geqslant 2(b-c_{\nu-1}). \tag{15}$$

By (14) and (15), the infinite series

$$\psi_n(z)=\Big(\sum_{\nu\leqslant -n}{}'+\sum_{\nu>n}{}'\Big)(\varphi_\nu(\nu)-f_\nu(z))$$

is uniformly convergent in the interior of $\bar{C}-[a, c_{-n}]\smile[c_n, b]$. Hence we have

$$\psi_n(z)\in\mathfrak{A}(\bar{C}-[a, c_{-n}]\smile[c_n, b]).$$

Now define $h_n(x)\in\mathfrak{B}([a, c_{-n}]\smile[c_n, b])$ and $h(x)\in\mathfrak{B}([a, b])$ by

*) For $a=-\infty$ and/or $b=+\infty$, $[a, b]$ should mean the closure of (a, b) *in* C, e.g. $[-\infty, \infty]=\bar{\boldsymbol{R}}=\boldsymbol{R}\smile\{\infty\}$.

**) For $a=-\infty$, (12) and (13) should be replaced by $\varphi_\nu(z)=\sum_{n=0}^{\infty}{}' c_{\nu n}z^n$ and $f_\nu(z)=\sum_{n=0}^{N_\nu}{}' c_{\nu n}z^n$ respectively.

***) If we use the approximation theorem of Runge, it becomes unnecessary to invoke to the Laurent expansion (12) to prove the existence of $f_\nu(z)$ which satisfies (14).

$$h_n(x) = [\varphi'_n(z), \boldsymbol{C}]_{z=x}$$
$$h(x) = h_0(x)$$

Then we have, for any $I' = (a', b')$, $a < a' < b' < b$,

$$(h(x) - \sum_{\nu=-N+1}^{N} g_\nu(x))|I' = h_N(x)|I' = 0$$

with sufficiently large N. Hence we have (11). (q. e. d.)

Now we have

Proposition 19.4. Let $I = (a, b)$ be an open interval, and let $g \in \mathfrak{B}(I)$ be given. Then there exists a perfect hyperfunction $h(x) \in \mathfrak{B}([a, b])$ such that

$$(16) \qquad h|I = g.$$

$h(x)$ is uniquely determined mod. $\mathfrak{B}(\{a\} \smile \{b\})$.

Proof. Take a sequence of points on I: $\{c_n\,;\ n = 0, \pm 1, \pm 2, \cdots\}$ such that (1) holds. Then, by proposition 1, we have decompostion (2) with some $g_\nu \in \mathfrak{B}([c_{\nu-1}, c_\nu])$. By proposition 3, there exists a $h(x) \in \mathfrak{B}([a, b])$ for which (11) holds. Hence we have (16). (q. e. d.)

Localizablity of Hyperfunctions.

Now we derive "localizability" and other basic properties of hyperfunctions along the line of [3], §3.*)

§20. *An existence theorem for defining functions.*

We have proved in §16 the existence of the standard defining function for any perfect hyperfunction. We shall prove in the following a corresponding result for an arbitrary (not necessarily perfect) hyperfunction.

Proposition 20.1. Let $S \subset \boldsymbol{R}$ be any locally closed set. Let $g \in \mathfrak{B}(S)$ and $D_0 \in \mathfrak{D}(S)$ be given. Then there exists a $\varphi_0(z) \in \mathfrak{A}(D_0 - S)$ such that

$$(1) \qquad g(x) = [\varphi_0(x), D_0]_{z=x}$$

*Proof.***) We need prove the proposition only for a non-compact S.

First step. Let $g = [\varphi, D]$, $D \in \mathfrak{D}(S)$, where we shall assume $D \subset D_0$ without loss of generality. Take $D' \in \mathfrak{D}(S)$, $D' \subset D$ such that

*) The meaning of these results becomes most clear if we look at them from the standpoint of sheaf theory (see [3], p. 11, and comment (19), (20)).

**) An open set $D_0 \subset \boldsymbol{C}$ is a complex neighborhood of S if and only if $S \subset D_0 \subset \boldsymbol{C} - K$ where K is a closed set of $\boldsymbol{C}$ defined by $K = \bar{S} - S$. In particular, $\boldsymbol{C} - K$ is the greatest complex neighborhood of S. Therefore it suffices to prove proposition 1 only for the case $D_0 = \boldsymbol{C} - K$, though in the following we have described the proof for general D_0.

(2)*) $\left\{\begin{array}{l}\text{the closure of } D' \text{ relative to } D_0 \text{ is contained in } D: \\ \qquad\qquad\qquad D'^{\frown} D_0 \subset D,\end{array}\right.$

in other words, closure $\bar{D}'^{\frown} D$ of D' relative to D is (relatively) closed in D_0, and

(3) $\left\{\begin{array}{l}\text{the boundary of } D' \text{ relative to } D_0 \text{ is a locally rectifiable path} \\ \Gamma \text{ in } D_0: \\ \qquad \partial_{D_0} D' = \Gamma \qquad (|\Gamma| = \bar{D}'^{\frown} D_0 - D').\end{array}\right.$

The carrier $|\Gamma|$ of Γ is not compact: some of the connected components of $|\Gamma|$ may be (rectifiable) closed Jordan curves, but some must be (locally rectifiable) open Jordan curves.**) We can, in any case, decompose Γ into chains with compact carriers (e. g. into closed arcs), i. e. we have

(4) $$\Gamma = \gamma_1 + \gamma_2 + \cdots \quad \text{(sum of chains)}$$

where $|\gamma_n|$ are compact subsets of $|\Gamma|$ such that $\{|\gamma_n|;\ n=1, 2, \cdots\}$ constitutes a locally finite covering of $|\Gamma|$.

Now let us suppose that there exist $\psi_n(z) \in \mathfrak{A}(D_0 - |\gamma_n|)$, $n=1,2,\cdots$ such that

(5) $$\psi_n(z) \equiv \frac{-1}{2\pi i} \int_{\gamma_n} \frac{\varphi(\zeta)}{\zeta - z} d\zeta \qquad (\mathrm{mod}\ \mathfrak{A}(D_0)),$$

and for any $m=0, 1, 2, \cdots$, the series

(6) $$\xi_m(z) = \sum_{n>m} \psi_n(z).$$

*) Such D' is obtained e.g. if we cover S with a locally finite family $\{D_\nu; \nu=1, 2, \cdots\}$ of open discs D_ν in D such that $\bar{D}_\nu \subset D$, and set $D' = \bigcup_\nu D_\nu$.

**) Thus, each component Γ_n of Γ is expressible in the following form:

i) for closed Γ_n:

$$\Gamma_n = (z_n(t_n); 0 \leqslant t_n \leqslant 1)$$

where $z_n(t_n)$ is a continuous map from $[0, 1]$ into D such that $z_n(0) = z_n(1)$;

ii) for open Γ_n:

$$\Gamma_n = (z_n(t_n); 0 < t_n < 1)$$

where $z_n(t_n)$ is a continuous map from $(0, 1)$ into D which is at the same time *closed* as a map from $(0, 1)$ into D_0. The carrier $|\Gamma_n|$ of Γ_n is the image of the map z_n, i. e. the point set $\{z_n(t_n);\ 0 \leqslant t_n \leqslant 1 \text{ (or } 0 < t_n < 1)\}$, while $|\Gamma|$, the carrier of Γ, is given by $\bigcup_n |\Gamma_n|$. The condition that Γ is locally rectifiable is equivalent to say that each $z_n(t_n)$ is of (locally) bounded variation.

In abbreviating, we shall write as follows for the full path $\Gamma = \Gamma_1 + \Gamma_2 + \cdots$:

$$\Gamma = (z(t);\ t \in T),$$

where T stands for a direct union of some circles and some copies of interval $(0, 1)$.

Now divide T into closed intervals τ_ν, $\nu = 1, 2, \cdots$, by taking points relatively discrete on $|T|$. Then Γ is decomposed as follows:

$$\Gamma = \gamma_1 + \gamma_2 + \cdots$$

with

$$\gamma_\nu = (z(t);\ t \in \tau_\nu),$$

where each $|\gamma_\nu|$ is a compact set of D, and $\{|\gamma_\nu|; \nu = 1, 2, \cdots\}$ is locally finite in D_0. In other words, Γ is a locally finite singular chain in D_0 with carrier in D.

is locally uniformly convergent in $D-|\sum_{n>m}\gamma_n|$.*) By (4), we can define $\varphi_0(z)\in\mathfrak{A}(D_0-S\smile|\Gamma|)$ and $f(z)\in\mathfrak{A}(D-|\Gamma|)$ by

$$\varphi_0(z)=\begin{cases}\xi_0(z) & (z\in D_0-\bar{D}')\\ \varphi(z)-\xi_0(z) & (z\in D'-S)\end{cases} \tag{7}$$

$$f(z)=\begin{cases}\varphi(z)-\xi_0(z) & (z\in D-\bar{D}')\\ \xi_0(z) & (z\in D')\end{cases} \tag{8}$$

By definition, $|\gamma_n|$ converges to the boundary of D_0:

$$\limsup_{n\to\infty}|\gamma_n|\subset\bar{D}-D_0\subset|\partial D_0|, \tag{9}$$

where $\bar{D}$ and ∂D_0 denote the closure of D *in* $\bar{C}$ and the boundary of D_0 *in* C respectively. Hence, for any compact subset $K\subset D_0$, we have

$$|\gamma_n|\frown K=\phi \qquad \text{for } n>m \tag{10}$$

with sufficiently large m. By (10) and (6),

$$\xi_m(z)\equiv 0 \qquad (\mathrm{mod}\,\mathfrak{A}(K)),$$

$$\xi_0(z)=\psi_1(z)+\cdots+\psi_m(z)+\xi_m(z)\equiv\psi_1(z)+\cdots+\psi_m(z) \qquad (\mathrm{mod}\,\mathfrak{A}(K)),$$

hence we have by (5)

$$\xi_0(z)\equiv\frac{-1}{2\pi i}\int_{\gamma_1+\cdots+\gamma_m}\frac{\varphi(\zeta)}{\zeta-z}d\zeta \qquad (\mathrm{mod}\,\mathfrak{A}(K)). \tag{12}$$

The formula (12), combined with the integral formula of Cauchy, shows that $\varphi_0(z)$ and $f(z)$ defined by (7), (8) are holomorphic in the neighborhood of $K\frown|\Gamma|$ $(\subset|\gamma_1+\cdots+\gamma_m|)$. As K is *any* compact subset of D_0, we have now

$$\varphi_0(z)\in\mathfrak{A}(D_0-S), \qquad f(z)\in\mathfrak{A}(D). \tag{13}$$

Consequently,

$$\varphi(z)=\varphi_0(z)+f(z)\equiv\varphi_0(z) \qquad (\mathrm{mod}\,\mathfrak{A}(D)),$$

i.e. $\varphi_0(z)$ is the defining function of $g(x)$ as required in proposition 1.

Second step. Now we shall show the existence of $\psi_n(z)\in\mathfrak{A}(D_0-|\gamma_n|)$ which satisfies (5), (6). For this purpose, let Δ_n, $n=1,2,\cdots$, be relatively compact open subsets of D_0 such that

$$\Delta_n\frown|\gamma_n|=\phi, \tag{14}$$

*) A sequence of continuous functions $\{f_n(p),\ n=1,2,\cdots\}$ on a topological space X is called *locally uniformly convergent on* X if, for each $p_0\in X$, $\{f_n(p)\}$ is uniformly convegent on some neighborhood U_{p_0} of p_0. If X is locally compact (which is the case for $X=D$, an open set of C), this is equivalent to the condition that $\{f_n(p)\}$ is *uniformly convergent on* X *in the wider sense* (or *uniformly convergent in the interior of* X) i.e. $\{f_n(p)\}$ is uniformly convergent on any compact subset $K\subset X$.

(15) Δ_n is simply connected relatively to D_0, and

(16) $$\Delta_n \uparrow D_0 \quad \text{for} \quad n\to\infty.$$

(Such Δ_n are furnished e.g. by

(17) $$\Delta_n=\Delta(2d_n),$$

where

(18) $$\Delta(r)=\{z\,;\ D_s(z,r)\subset D_0\} \ \text{ for } r>0,$$
$$D_s(z,r)=\left\{z'\,;\ \frac{|z'-z|}{\sqrt{1+|z'|^2}\sqrt{1+|z|^2}}\leqslant r\right\}$$
=the disc with center z and radius r in the spherical distance, and
$$d_n=\max\{r\,;\ \Delta(r)\subset D_0-\bigcup_{\nu\geqslant n}|\gamma_\nu|\}.$$

It is clear that Δ_n defined by (17) are open and relatively compact in D_0, and satisfy (14). (15) is also satisfied because no connected component of $D_0-\Delta_n$ is disjoint to the ideal boundary of D_0. (16) is clear because we have $d_n\downarrow 0$ for $n\to\infty$ by definition.)

According to the generalized Runge's theorem[*], we see that, for each $h(z)\in\mathfrak{A}(\bar{\Delta}_n)$ and $\varepsilon>0$, there exists a $f(z)\in\mathfrak{A}(D_0)$ such that
$$|h(z)-f(z)|\leqslant\varepsilon \qquad \text{for } z\in\bar{\Delta}_n.$$
In particular we have, for $h(z)=\dfrac{-1}{2\pi i}\displaystyle\int_{\gamma_n}\frac{\varphi(\zeta)}{\zeta-z}d\zeta$ $(\in\mathfrak{A}(\bar{\Delta}_n))$ and $\varepsilon=2^{-n}$,

(19) $$\begin{cases} |\psi_n(z)|\leqslant 2^{-n} \quad (z\in\bar{\Delta}_n) \\ \text{where } \ \psi_n(z)=\dfrac{-1}{2\pi i}\displaystyle\int_{\gamma_n}\frac{\varphi(\zeta)}{\zeta-z}d\zeta-f_n(z) \qquad (z\in D_0-|\gamma_n|)\end{cases}$$

with some $f_n(z)\in\mathfrak{A}(D_0)$. It is then clear that $\psi_n(z)$, $n=1,2,\cdots$, given by (19) satisfy conditions (5) and (6). (q.e.d.)

Corollary. For any $D_0\in\mathfrak{D}(S)$, we have a canonical isomorphism

(20) $$\mathfrak{B}(S)\simeq\mathfrak{A}(D_0-S) \bmod \mathfrak{A}(D_0).$$

Furthermore, for $D_0\supset D_1\supset\cdots\supset S$ $(D_0, D_1,\cdots\in\mathfrak{D}(S))$, we have a commutative diagram of the form:

(21) $$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{A}(D_0) & \longrightarrow & \mathfrak{A}(D_1) & \longrightarrow\cdots\longrightarrow & \mathfrak{A}(S) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{A}(D_0-S) & \longrightarrow & \mathfrak{A}(D_1-S) & \longrightarrow\cdots\longrightarrow & \tilde{\mathfrak{A}}(S) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{B}(S) & = & \mathfrak{B}(S) & =\ \cdots\ = & \mathfrak{B}(S) \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

[*] See M. H. Behnke: Généralisation du théorème de Runge pour des fonctions multiformes de variables complexes, Colloque sur les fonctions de plusieurs variables, Bruxelles (1953), p. 81.

where each sequence in vertical direction is exact.

§21. *Alternative proofs of the existence theorem.*

We shall next give an alternative proof of the proposition 20.1. by constructing $\psi_n(z)$ in (20.5) directly, without employming Runge's theorem.

Let $p_n(z)\in\mathfrak{A}(D_0)$, $n=1, 2, \cdots$ and suppose $p_n(z)$ has no zero on $|\gamma_n|$. Define $\psi_n(z)\in\mathfrak{A}(D_0-|\gamma_n|)$ by

$$(1)\qquad \psi_n(z)=\frac{-1}{2\pi i}p_n(z)\int_{\gamma_n}\frac{1}{p_n(\zeta)}\frac{\varphi(\zeta)}{\zeta-z}d\zeta, \qquad (z\in D_0-|\gamma_n|).$$

Clearly, we can write

$$(2)\qquad p_n(\zeta)-p_n(z)=(\zeta-z)q_n(\zeta, z)$$

with a holomorphic function $g_n(\zeta, z)$ of $(\zeta, z)\in D_0\times D_0$. Consequently

$$\psi_n(z)=\frac{-1}{2\pi i}\int_{\gamma_n}\left(\frac{1}{\zeta-z}-\frac{g_n(\zeta, z)}{p_n(\zeta)}\right)\varphi(\zeta)\,d\zeta\equiv\frac{1}{2\pi i}\int_{\gamma_n}\frac{\varphi(\zeta)}{\zeta-z}d\zeta \quad (\mathrm{mod}\ \mathfrak{A}(D_0)).$$

Namely, $\psi_n(z)$ defined by (1) satisfies (20, 5). We shall now show that, by a suitable choice of $p_n(z)$, (20, 6) is also satisfied by these $\psi_n(z)$. First, dividing each γ_n into smaller arcs if necessary, we can assume that the diameter of $|\gamma_n|$ tends to zero for $n\to\infty$. Consequently we can choose $\alpha_n\in\hat{C}-D$, $n=1, 2, \cdots$, such that

$$(3)\qquad D(\alpha_n, d_n)\supset|\gamma_n|$$

with some $d_n>0$ tending to zero:

$$(4)\qquad d_n\downarrow 0 \quad \text{for } n\to\infty,$$

where

$$(5)\qquad \begin{cases} D(\alpha, r)=\{z\,;\ |z-\alpha|\leqslant r\} \\ \qquad =\text{the disc with center } \alpha \text{ and radius } r \text{ (for } \alpha\neq\infty), \\ D(\infty, r)=\{z\,;\ |z-\beta|\geqslant\frac{1}{r} \text{ with a fixed } \beta\in C-\bigcup_{n=1}^{\infty}|\gamma_n|. \end{cases}$$

By (4), there exist natural numbers N_n, $n=1, 2, \cdots$, such that $N_n\uparrow\infty$ and

$$(6)\qquad d_n\cdot\sqrt[N_n]{M_n}\longrightarrow 0 \text{ for } n\to\infty$$

with

$$M_n=\frac{1}{2\pi}\int_{\gamma_n}|\varphi(z)dz|.$$

We shall now put

$$(7)\qquad \begin{cases} p_n(z)=(z-\alpha_n)^{-N_n} & \text{for } n \text{ for which } \alpha_n\neq\infty, \\ p_n(z)=(z-\beta)^{N_n} & \text{for } n \text{ for which } \alpha_n=\infty. \end{cases}$$

From (1) we have, for $\alpha_n \neq \infty$,

$$|\varphi_n(z)| \leqslant |z-\alpha_n|^{-\lambda_n} \cdot \frac{1}{2\pi} \int_{\gamma_n} |\zeta-\alpha_n|^{\lambda_n} \frac{|\varphi(\zeta)d\zeta|}{|\zeta-z|}$$

$$\leqslant \frac{|z-\alpha_n|^{-\lambda_n}}{\mathrm{dist}(z, |\gamma_n|)} \cdot d_n^{\lambda_n} M_n = \frac{1}{\mathrm{dist}(z, |\gamma_n|)} \left| \frac{\varepsilon_n}{z-\alpha_n} \right|^{\lambda_n},$$

and for $\alpha_n = \infty$,

$$|\varphi_n(z)| \leqslant \frac{|z-\beta|^{\lambda_n}}{\mathrm{dist}(z, |\gamma_n|)} \cdot d_n^{\lambda_n} M_n = \frac{1}{\mathrm{dist}(z, |\gamma_n|)} |\varepsilon_n(z-\beta)|^{\lambda_n},$$

where we set $\varepsilon_n = d_n \cdot \sqrt[\lambda_n]{M_n}$ (for which we have $\varepsilon_n \to 0$ $(n \to \infty)$ by (6)). Consequently we have, for any compact subset $K \subset D_0$,

(8) $$|\varphi_n(z)| \leqslant A \cdot (B\varepsilon_n)^{\lambda_n} \qquad \text{for } z \in K$$

with some $A>0$, $B>0$ and sufficiently large n. It is now clear that $\varphi_n(z)$ satisfies the condition (20, 6) in question. (q. e. d.)

We remark now that the proof can be subsumed in the following proof-scheme:

Let $\Gamma = \{\zeta(t)\,;\, t \in T\}$ be a parametric representation of $\Gamma = \partial_{D_0} D'$ and let $\mathfrak{F}$ be a submodule of $\mathfrak{A}(D-S)$ consisting of $\varphi(z) \in \mathfrak{A}(D-S)$ such that $\frac{\varphi(\zeta(t))}{m(t)}$ is bounded for $t \in T$ with some (fixed) $m(t)>0$, a continuous function of $t \in T$. Let us consider a differential form of $\zeta = \zeta(t)$ of the form

$$\omega(t, z) = \frac{d\zeta(t)}{\zeta(t)-z} + f(t, z)dt$$

such that

(9) $m(t)\omega(t, z)$ is integrable in $t \in T$ locally uniformly in $z \in D_0 - |\Gamma|$, i. e. for any compact subset K of $D_0 - |\Gamma|$, there exists an integrable function $k(t, K)$ of $t \in T$ such that

$$\left| \frac{\omega(t, z)}{dt} \right| \leqslant \frac{k(t, K)}{m(t)} \qquad \text{for } z \in K,$$

$$\int_T k(t, K)dt < \infty,$$

(10) $f(t, z)$ is locally integrable in $t \in T$ locally uniformly in $z \in D_0$, i. e. for any compact subset K_0 of D_0, there exists a locally integrable function $k_0(t, K_0)$ of $t \in T$ such that

$$|f(t, z)| \leqslant k_0(t, K_0) \text{ for } z \in K_0, \text{ and}$$

(11) for any continuous function $h(t)$ of $t \in T$ with compact carrier,

$$\int_T h(t) f(t, z) dt$$

is a holomorphic function of $z \in D_0$. (We shall say "$f(t, z)$ is holomorphic in z" in this case).

The existence of such a differential form $\omega(t, z)$ being assumed, define for each $\varphi(z) \in \mathfrak{F}$, $\xi_0(z) \in \mathfrak{A}(D_0 - |\Gamma|)$ by

$$\xi_0(z) = \frac{-1}{2\pi i} \int_{t \in T} \varphi(\zeta(t)) \omega(t, z). \tag{12}$$

We have, for any compact subset K of D_0,

$$\frac{-1}{2\pi i} \int_{t \in T - T_1} \varphi(\zeta(t)) \omega(t, z) \equiv 0 \qquad (\mathrm{mod}\ \mathfrak{A}(K))$$

with some compact subset T_1 of T. Hence

$$\xi_0(z) \equiv \frac{-1}{2\pi i} \int_{t \in T_1} \varphi(\zeta(t)) \omega(t, z) \equiv \frac{-1}{2\pi i} \int_{\zeta(T_1)} \varphi(\zeta) \frac{d\zeta}{\zeta - z} \ (\mathrm{mod}\ \mathfrak{A}(K)),$$

whence we see, just as in the first step of the proof of the proposition 20.1, that $\varphi_0(z)$ and $f(z)$ defined from $\xi_0(z)$ by (20, 7) and (20, 8) respectively, satisfy (20, 13), and thus we again arrive at the proposition 20.1.

It is clear from the above considerations that the map

$$\varphi(z) \longrightarrow \varphi_0(z) \tag{13}$$

is a linear map from $\mathfrak{F}$ into $\mathfrak{A}(D_0 - S)$.

Previously we have constructed $\varphi_n(z)$ by (1), and obtained $\xi_0(z)$ (and hence $\varphi_0(z)$) by (20, 6). This can be regarded as a special case of (12). In fact, (12) reduces to

$$\xi_0(z) = \frac{-1}{2\pi i} \sum_{n=1}^{\infty} p_n(z) \int_{\gamma_n} \frac{1}{p_n(\zeta)} \frac{\varphi(\zeta)}{\zeta - z} d\zeta = \sum_{n=1}^{\infty} \varphi_n(z) \tag{14}$$

if we take

$$\left\{\begin{array}{l} \omega(t, z) = \dfrac{p_n(z)}{p_n(\zeta(t))} \dfrac{d\zeta(t)}{\zeta(t) - z} \qquad \text{for } t \in T_n,\ n = 1, 2, \cdots, \\ \quad \text{i. e.} \\ \omega(t, z) = \displaystyle\sum_{n=1}^{\infty} \chi(t, T_n) \cdot \frac{p_n(z)}{p_n(\zeta(t))} \frac{d\zeta(t)}{\zeta(t) - z} \\ \quad \text{with} \qquad \chi(t, T_n) = \begin{cases} 1 & \text{for } t \in T_n \\ 0 & \text{for } t \in T - T_n, \end{cases} \end{array}\right. \tag{15}$$

where γ_n and $p_n(z)$ are as described above and $\gamma_n = \{\zeta(t) : t \in T_n\}$, $T = T_1 \smile T_2 \smile \cdots$.

On the other hand, decompositions of $\varphi_0(z) \in \mathfrak{A}(C - [a, b])$ and $\varphi(z) \in \mathfrak{A}(D - I)$ given in the proofs of proposition 18.1 and proposition 19.1 respectively, are also special cases of (12). In fact, we have, as to the proposition 18.1,

(16) $$\varphi_\nu(z)=\frac{-1}{2\pi i}\int_{\Gamma_\nu}\varphi_0(z)\omega(\zeta, z), \qquad \nu=1, 2,$$

with

(17) $$\omega(\zeta, z)=\frac{k(z)}{k(\zeta)}\frac{d\zeta}{\zeta-z}$$

and as to the proposition 19.1,

(18) $$\frac{-1}{2\pi i}\int_{\Gamma_\nu}\varphi(\zeta)\omega_\nu(\zeta, z)=\begin{cases}\xi_\nu(z) & (\Re z>c_\nu)\\ \eta_\nu(z) & (\Re z<c_\nu)\end{cases}$$

with

(19) $$\omega_\nu(\zeta, z)=\frac{k_\nu(z)}{k_\nu(\zeta)}\frac{d\zeta}{\zeta-z}.$$

To sum up, we see that the proof of the existence theorem (proposition 20.1) as well as of the decomposition theorems (propositions 18.1 and 19.1) are all reduced to the construction of $\omega(t, z)$ for which (9), (10) and (11) hold,—and it is easy to see that $\omega(t, z)$, $\omega(\zeta(t), z)$, $\omega_\nu(\zeta(t), z)$ given by (15), (17), (19) respectively, satisfy (9), (10), (11). We shall prove however the following proposition which assures the existence of such $\omega(t, z)$ satisfying even stronger conditions than (9), (10), (11). Thus this proposition will furnish still another proof of propositions 18.1, 19.1, and 20.1. The method of the following proof will be also used later on.

Proposition 21.1. Let $D_0\subset C$ be an open set and let $k(z)dz$ be a given differential form where $k(z)$ is a continuous function defined on a closed subset F of D_0 with positive value:

(20) $$k(z)>0 \qquad (z\in F).$$

Then there exists a differential form of $\zeta\in D_0$ of the form

(21) $$\omega(\zeta, z)=\frac{d\zeta}{\zeta-z}+f(\zeta, z)d\zeta$$

such that

(22) *$f(\zeta, z)$ is a C^∞-function of $(\zeta, z)\in D_0\times D_0$, and is a holomorphic function of $z\in D_0$ for each fixed $\zeta\in D_0$, and*

(23) *$\frac{1}{k(\zeta)}\left|\frac{\omega(\zeta, z)}{d\zeta}\right|$ is bounded for $(\zeta, z)\in F\times K$, K being any compact subset of D_0-F.*

Proof. Let U_n, $n=1, 2,\cdots$, be relatively compact open subsets of D_0 such that $\{U_n;\ n=1, 2, \cdots\}$ constitute a locally finite open covering of D_0, and assume at first that there exist $p_n(z)\in\mathfrak{A}(D_0)$, $n=1, 2, \cdots$, such that $p_n(z)\neq 0$ for $z\in U_n$ and

(24) $$\left|\frac{p_n(z)}{p_n(\zeta)}\right|\leqslant h(z)k(z) \quad \text{for} \quad \zeta\in U_n,\ z\in D_0,\ n=1, 2, \cdots$$

with some $h(z)>0$, a continuous function of $z\in D_0$ independent of n. On the other hand, it is well-known as the decomposition-of-unity-theorem for C^∞-function that there are C^∞-functions $\varepsilon_n(z)$ of $z\in D_0$, $n=1,2,\cdots$, such that

$$(25)\qquad \begin{cases} 0\leqslant\varepsilon_n(z)\leqslant 1,\quad \sum_{n=1}^{\infty}\varepsilon_n(z)=1, \\ \text{and the carrier of } \varepsilon_n(z) \text{ is contained in } U_n \text{ (i. e. } \varepsilon_n(z)=0 \text{ for } z\in D-F_n \\ \text{with some } F_n\subset U_n, \text{ a closed set of } D_0). \end{cases}$$

We shall define $\omega(\zeta, z)$ by

$$(26)\qquad \omega(\zeta,z)=\sum_{n=1}^{\infty}\varepsilon_n(\zeta)\frac{p_n(z)}{p_n(\zeta)}\frac{d\zeta}{\zeta-z}\qquad ((\zeta,z)\in D_0\times D_0,\ \zeta\neq z).$$

we have

$$(27)\qquad \frac{p_n(z)}{p_n(\zeta)}=1+(\zeta-z)\frac{q_n(\zeta,z)}{p_n(\zeta)},$$

$$q_n(\zeta,z):\ \text{holomorphic function of } (\zeta,z)\in D_0\times D_0,$$

an identity parallel to (2). Consequently

$$(28)\qquad \begin{cases} \omega(\zeta,z)=\dfrac{d\zeta}{\zeta-z}+f(\zeta,z)d\zeta \\ \text{with}\qquad f(\zeta,z)=\sum_{n=1}^{\infty}\dfrac{\varepsilon_n(\zeta)}{p_n(\zeta)}q_n(\zeta,z). \end{cases}$$

Clearly $f(\zeta, z)$ satisfies the condition (22). We have moreover, by (24),

$$\frac{1}{k(\zeta)}\left|\frac{\omega(\zeta,z)}{d\zeta}\right|\leqslant\frac{1}{k(\zeta)}\sum_{n=1}^{\infty}\varepsilon_n(\zeta)\cdot h(z)k(z)\cdot\frac{1}{|\zeta-z|}=\frac{h(z)}{|\zeta-z|}$$

Hence the condition (23) is also satisfied.

To complete the proof, we shall verify (24) by constructing suitable $p_n(z)$. Replacing the covering $\{U_n;\ n=1,2,\cdots\}$ by a suitable refinement of it, we can assume from the beginning that the diameter of U_n tends to zero for $n\to\infty$. Consequently we can choose $\alpha_n\in\dot{C}-D$, $n=1,2,\cdots$, such that, in the notation of (5),

$$(29)\qquad D(\alpha_n, d_n)\supset U_n$$

with some $d_n>0$ tending to zero:

$$(30)\qquad d_n\downarrow 0\qquad \text{for } n\to\infty.$$

We can furthermore choose natural numbers N_n, $n=1,2,\cdots$, such that $N_n\uparrow\infty$ and

$$(31)\qquad \varepsilon_n=\frac{d_n}{\sqrt[N_n]{\mu_n}}\longrightarrow 0\qquad \text{for } n\to\infty$$

with

$$\mu_n = \min_{\zeta \in U_n} k(\zeta).$$

We shall now put

$$(32) \qquad \begin{cases} p_n(z) = (z-\alpha_n)^{-N_n} & \text{for } n \text{ for which } \alpha_n \neq \infty, \\ p_n(z) = (z-\beta)^{N_n} & \text{for } n \text{ for which } \alpha_n = \infty, \end{cases}$$

and define $h(z)$ by

$$(33) \qquad h(z) = \max_n h_n(z),$$

with

$$\begin{cases} h_n(z) = \left(\dfrac{\varepsilon_n}{|z-\alpha_n|}\right)^{N_n} & \text{for } \alpha_n \neq \infty, \\ h_n(z) = (\varepsilon_n |z-\beta|)^{N_n} & \text{for } \alpha_n = \infty. \end{cases}$$

As $h_n(z)$ tends to zero locally uniformly in $z \in D_0$, a continuous function $h(z)$ of $z \in D_0$ is defined by (33). We have, for $\zeta \in U_n$, $z \in D_0$,

$$\left|\frac{p_n(z)}{p_n(\zeta)}\right| = \left|\frac{\zeta-\alpha_n}{z-\alpha_n}\right|^{N_n} \leqslant \left(\frac{d_n}{|z-\alpha_n|}\right)^{N_n} = \left(\frac{\varepsilon_n}{|z-\alpha_n|}\right)^{N_n} \cdot \mu_n \leqslant h(z) \cdot k(\zeta)$$

if $\alpha_n \neq \infty$. It is easy to see that the same result is also valid for $\alpha_n = \infty$. Thus (24) is fully verified. (q. e. d.)

§22. *The completeness of* $\mathfrak{B}(S)$.

The existence theorem (proposision 20.1, or equivalently, the formula (20,20)) proved in §§20 and 21, plays an essential rôle in the theory of hyperfunctions, as will be explained now.

First of all: let F be any closed subset of a locally closed $S \subset \boldsymbol{R}$. Then it is clear that any complex neighborhood D *of* S is always a complex neighborhood *of* F:

$$D \in \mathfrak{D}(S) \Rightarrow D \in \mathfrak{D}(F).$$

Now let D be any one of these complex neighborhoods of S. By proposition 1, each $g \in \mathfrak{B}(F)$ possesses a defining function $\varphi(z) \in \mathfrak{A}(D-F)$, and hence determines a hyperfunction *on* S, as explained in proposition 4.1. It is clear that this correspondence $\mathfrak{B}(F) \to \mathfrak{B}(S)$ (the canonical embedding) does not depend on the choice of $D \in \mathfrak{D}(S)$; each element of $\mathfrak{B}(F)$ corresponds in a 1-1 manner to each element of $\mathfrak{B}(S)$ whose restriction onto the open subset $S-F$ of S vanishes. In other words, we have an exact sequence:

$$0 \longrightarrow \mathfrak{B}(F) \longrightarrow \mathfrak{B}(S) \xrightarrow{\text{rest.}} \mathfrak{B}(S-F),$$
$$(\text{rest.} = \text{restriction}).$$

On the other hand, we have $D-F \in \mathfrak{D}(S-F)$, whence we see by proposition 20.1 that each $g_1(x) \in \mathfrak{B}(S-F)$ possesses a defining function $\varphi_0(z) \in \mathfrak{A}(D-S)$:

$$g_1(x)=[\varphi_0(z), D-F]_{z=x}.$$

Hence, defining $g_0(x)\in\mathfrak{B}(S)$ by $g_0(x)=[\varphi_0(z), D]_{z=x}$, we have

$$(1) \qquad g_1=g_0|(S-F).$$

Thus, a hyperfunction *on* $S-F$ is always the restriction of some hyperfunction *on* S, or equivalently: the restriction $\mathfrak{B}(S)\to\mathfrak{B}(S-F)$ is a *surjective* mapping.

To sum up, we obtain the following result, one of the most basic properties of hyperfunctions, which we shall call the *completeness* of $\mathfrak{B}(S)$. (If we replace "hyperfunction" by "Schwartz's distribution," the following proposition ceases to be valid.)

Proposition 22.1. For any closed subset F of locally closed $S\subset\boldsymbol{R}$, we have an exact sequence:

$$(2) \qquad 0 \longrightarrow \mathfrak{B}(F) \longrightarrow \mathfrak{B}(S) \xrightarrow{\text{rest.}} \mathfrak{B}(S-F) \longrightarrow 0,$$

or equivalently,

$$(3) \qquad \mathfrak{B}(S-F)\simeq\mathfrak{B}(S) \bmod \mathfrak{B}(F)$$

where we consider $\mathfrak{B}(F)$ as a submodule of $\mathfrak{B}(S)$ by canonical embedding.

For any locally closed $S\subset\boldsymbol{R}$, the closure $\bar{S}$ of S in $\bar{\boldsymbol{R}}$ is always a compact set containing S as an open subset. Hence we have

Corollary. Any hyperfunction $g\in\mathfrak{B}(S)$ can be "extended to" a suitable perfect hyperfunction $g_0(x)\in\mathfrak{B}(\bar{S})$ (and hence $g_0(x)\in\mathfrak{B}(\bar{\boldsymbol{R}})$):

$$(4) \qquad g=g_0|S.$$

§23. *Decomposition of a hyperfunction, general case.*

Proposition 23.1. Let $\{F_n,\ n=1,2,\cdots\}$, $F_n\subset S$, be a locally finite closed covering of S, and let $g_n(x)\in\mathfrak{B}(F_n)$, $n=1,2,\cdots$, be given. Then there exists a $g(x)\in\mathfrak{B}(S)$ such that

$$(1) \qquad g|S'=\sum_{n=1}^{\infty} g_n|S'$$

for any relatively compact open subset S' of S.[*]

The proof below is similar to, but somewhat simpler than, that of proposition 20.1.

Proof. Let $D_0\in\mathfrak{D}(S)$ be a (fixed) complex neighborhood of S. By proposition 20.1, each $g_n(x)$ possesses a defining function $\varphi_n(z)\in\mathfrak{A}(D_0-F_n)$. Let Δ_n, $n=1,2,\cdots$, be a relatively compact open subset of $D_0-\bigcup_{\nu\geqq n}F_\nu$ such that Δ_n is simply connected relatively to D_0, and $\Delta_n\uparrow D_0$ for $n\to\infty$. (Such Δ_n are furnished e.g.

[*] Note that the number of non-vanishing terms in the right-hand side of (1) is finite.

by (20.17) where d_n is defined by

$$d_n = \max\{r\,;\, \Delta(r) \subset D_0 - \bigcup_{\nu \geq n} F_\nu\}$$

instead of the definition of d_n in (20.18).)

According to the generalized Runge's theorem, there exist $f_n(z) \in \mathfrak{A}(D_0)$, $n = 1, 2, \cdots$, such that

$$|\varphi_n(z) - f_n(z)| \leqslant 2^{-n} \text{ for } z \in \bar{\Delta}_n.$$

Consequently

$$(2) \qquad \xi_m(z) = \sum_{n>m} (\varphi_n(z) - f_n(z))$$

is convergent locally uniformly in $D_0 - \bigcup_{n>m} F_n$, and hence a $\xi_m(z) \in \mathfrak{A}(D_0 - \bigcup_{n>m} F_n)$ is determined by (2). In particular we have

$$(3) \qquad \xi_0(z) = \sum_{n=1}^{m} (\varphi_n(z) - f_n(z)) + \xi_m(z) \equiv \sum_{n=1}^{m} \varphi_n(z) \qquad (\operatorname{mod} \mathfrak{A}(\Delta_n)).$$

Define $g(x) \in \mathfrak{B}(S)$ by

$$(4) \qquad g(x) = [\xi_0(z)\,;\, D_0]_{z=x}.$$

We have, by (3),

$$(5) \qquad g|S_m = \sum_{n=1}^{m} g_n|S_m \qquad (= \sum_{n=1}^{\infty} g_n|S_m)$$

with $S_n = \boldsymbol{R} \frown \Delta_n$. As any relatively compact subset S' of S is contained in some S_n, (5) yields the general relation (1). (q. e. d.)

Clearly $g(x) \in \mathfrak{B}(S)$ is uniquely determined by the condition (1). We shall therefore call $g(x)$ the sum of $\{g_n(x)\,;\, n = 1, 2, \cdots\}$ and denote

$$(6) \qquad g(x) = \sum_{n=1}^{\infty} g_n(x).$$

Remark. In the above proof of proposition 23.1, proposition 20.1 is used only to assure the existence of the defining functions $\varphi_n \in \mathfrak{A}(D_0 - F_n)$ of $g_n \in \mathfrak{B}(F_n)$. Consequently, the proof goes well without proposition 20.1, if each g_n is *perfect* (by proposition 16.1). In other words, following corollary is derived *without* proposition 20.1. (This is a generalization of propositions 19.2–3.)

Corollary. Let I be any open set of $\boldsymbol{R}$, $\{K_n\,;\, n = 1, 2, \cdots\}$ a locally finite family of compact subsets of I, g_n a hyperfunction on each K_n. Then, for any $D_0 \in \mathfrak{D}(I)$, there exists a $\varphi_0 \in \mathfrak{A}(D_0 - I)$ such that

$$\sum_{n=1}^{\infty} g_n = [\varphi_0, D_0] \qquad (\in \mathfrak{B}(I)).$$

Now let $g(x)$ be a general δ-function on an open set $I\subset \boldsymbol{R}$ as defined in §10, with carrier $\{a_j\,;\ j=1, 2, \cdots\}$, and let

$$g(x)|I_j=g_j(x)=\sum_{\nu=0}^{m_j}{}' c_{j\nu}\delta^{(\nu)}(x-a_j)$$

be the local expression of $g(x)$ in the real neighborhood I_j of $x=a_j$. Then we have, by the definition (6) of the infinite sum,

$$(7)\qquad g(x)=\sum_j{}' g_j(x) \quad \text{on} \quad I.$$

Conversely, *if a discrete point set* $\{a_j\,;\ j=1, 2, \cdots\}\subset I$ *and general δ-functions* $g_j(x)$ *on* $x=a_j$, $j=1, 2, \cdots$, *are arbitrarily given, then by corollary of proposition 23.1, a general δ-function* $g(x)\in\mathfrak{B}(I)$ *is determined by* (7).

This is essentially the well-known Mittag-Leffler theorem.

Example. Let $\xi(x)$ be a real valued holomorphic function on $I\subset\boldsymbol{R}$ whose zeros $x=a_1, a_2, \cdots$ $(\in I)$ are all simple. Then, by (7.1), $\delta(\xi(x))$ is well-defined in the neighborhood of each $x=a_\nu$. Accordingly, we have*)

$$\delta(\xi(x))=\sum_j{}' |f'(a_j)|^{-1}\cdot\delta(x-a_j),$$

and this is a general δ-function on I. For example,

$$\delta(x^2-a^2)=\frac{1}{|2a|}(\delta(x+a)+\delta(x-a)), \ (a\neq 0).$$

Utilizing the proposition 22.1, we can now prove the converse of proposition 23.1, the *general decomposition theorem* for a hyperfunction which contains the decomposition theorems of §§18, 19 as special cases.

Proposition 23.2. Let S and $\{F_n\,;\ n=1, 2, \cdots\}$ have the same meanings as in proposition 23.1, and let $g(x)\in\mathfrak{B}(S)$ be given. Then there exist $g_n(x)\in\mathfrak{B}(F_n)$, $n=1, 2, \cdots$, for which (6) holds.

Proof. Put $S_n=S-\bigcup_{\nu>n} F_\nu$, $n=1, 2, \cdots$. Clearly S_n is an open subset of $F_1\smile\cdots\smile F_n$ as well as of S. Hence, by proposition 22.1, the restriction $g|S_n\in\mathfrak{B}(S_n)$ can be extended to a hyperfunction $h_n(x)$ on $F_1\smile\cdots\smile F_n$:

$$(8)\qquad h_n|S_n=g|S_n \quad \text{with } h_n\in\mathfrak{B}(F_1\smile\cdots\smile F_n)\subset\mathfrak{B}(S).$$

Define $g_n(x)\in\mathfrak{B}(F_1\smile\cdots\smile F_n)$ by

$$(9)\qquad g_1(x)=h_1(x), \qquad g_n(x)=h_n(x)-h_{n-1}(x).$$

We have $g_n|S_{n-1}=0$ by (7). Hence $g_n(x)$ is a hyperfunction on $F_1\smile\cdots\smile F_n-S_{n-1}$ $=F_n$:

*) The definition of $\delta(f(x))$ in footnote (3), [1] should be abandoned (cf. [3] comment 16). Accordingly, the expression in the line next to (4) on p. 127 [1] should be replaced by (9.3) of the present paper.

$$g_n(x) \in \mathfrak{B}(F_n), \quad n=1, 2, \cdots.$$

Moreover we have from (7) and (8)

$$g|S_n = \sum_{\nu=1}^{n} g_\nu|S_n,$$

and hence the identity (6). (q. e. d.)

In particular we have, as a generalization of proposition 18.2,

Corollary. Let $K_1, \cdots, K_n$ be compact subsets of $\boldsymbol{R}$. Then we have

$$\mathfrak{B}(K_1 \smile \cdots \smile K_n) = (\mathfrak{B}(K_1), \cdots, \mathfrak{B}(K_n)).$$

Now let D_0 be a complex neighborhood of S, and let $g(x) = [\varphi(z), D_0]_{z=x}$, $h_n(x) = [\psi_n(x), D_0]_{z=x}$. Then we have

$$g_n(x) = [\varphi_n(z), D_0]_{z=x} \text{ with } \varphi_n(z) = \psi_n(z) - \psi_{n-1}(z) \in \mathfrak{A}(D_0 - F_n)$$

and

(10) $$\varphi(z) = \sum_{n=1}^{\infty} \varphi_n(z) \qquad \text{for } z \in D_0 - S.$$

Hence we have

Proposition 23.3. Let $g(x) \in \mathfrak{B}(S)$, $g_n(x) \in \mathfrak{B}(F_n)$ be as described above, and let $f(x) \in \mathfrak{A}(S)$ be given. Then the identity (6) implies

(11) $$f(x)g(x) = \sum_{n=1}^{\infty} f(x)g_n(x),$$

(12) $$\bar{g}(x) = \sum_{n=1}^{\infty} \bar{g}_n(x),$$

(13) $$g^{(\nu)}(x) = \sum_{n=1}^{\infty} g_n^{(\nu)}(x).$$

§ 24. *The localization theorem.*

Definition. Let $\{S_\alpha ; \alpha \in N\}$, $S_\alpha \subset S$, be any open covering of S, and let $g_\alpha(x) \in \mathfrak{B}(S_\alpha)$ be given on each S_α. We say that $\{(S_\alpha, g_\alpha); \alpha \in N\}$ constitutes a localized hyperfunction or a hyperfunction in the wider sense, if every pair g_α, g_β has a common restriction on $S_\alpha \frown S_\beta$:

(1) $$g_\alpha|S_\alpha \frown S_\beta = g_\beta|S_\alpha \frown S_\beta{}^{*)}, \qquad \alpha, \beta \in N,$$
$$\text{i. e.} \quad g_\alpha(x) = g_\beta(x) \qquad \text{for } x \in S_\alpha \frown S_\beta.$$

For example, $\{(S_\alpha, g|S_\alpha); \alpha \in N\}$ with a $g \in \mathfrak{B}(S)$ always constitutes a localized hyperfunction. Each localized hyperfunction of this type will be called "equivalent to a hyperfunction $g(x)$".

Now we can state as follows the *localization theorem* the hyperfunctions, a basic result assuring localizable nature of hyperfunctions.

*) If $S_\alpha \frown S_\beta = \phi$, each side of (1) stands for $0 \in \mathfrak{B}(\phi)$.

Proposition 24.1. Every localized hyperfunction is equivalent to a hyperfunction, i.e. for any localized hyperfunction $\{(S_\alpha, g|S_\alpha); \alpha\in N\}$ on S there exists a hyperfunction $g(x)\in\mathfrak{B}(S)$ such that $g_\alpha=g|S_\alpha$ for every $\alpha\in N$.

Proof. There exists a refinement of $\{S_\alpha; \alpha\in N\}$ by a locally finite closed covering of S, say by $\{F_n; n=1, 2, \cdots\}$. By hypothesis, $F^{(n)}=\bigcup_{\nu\geq n} F_\nu$ is a closed subset of S. We shall now define hyperfunctions $h_n(x)\in\mathfrak{B}(F_n)$ $(\subset\mathfrak{B}(S))$, $n=1, 2, \cdots$, such that

$$(2)\qquad g_\alpha\equiv\sum_{\nu=1}^{n} h_\nu|S_\alpha \qquad (\mathrm{mod}\,\mathfrak{B}(S_\alpha\frown F^{(n+1)}))$$

as follows, by induction on n.

Let $h_1(x),\cdots, h_{n-1}(x)$ be already defined, and set

$$(3)\qquad g_\alpha=\sum_{\nu=1}^{n-1} h_\nu|S_\alpha+g_{n\alpha}.$$

By hypothesis, we have $g_\alpha(x)=g_\beta(x)$ for $x\in S_\alpha\frown S_\beta$, and hence

$$(4)\qquad g_{n\alpha}(x)=g_{n\beta}(x) \qquad \text{for } x\in S_\alpha\frown S_\beta.$$

Moreover, we have $g_{n\alpha}\in\mathfrak{B}(S\frown F^{(n)})$ by hypothesis of induction. Hence $\{(S_\alpha, g_{n\alpha}(x)); \alpha\in N\}$ constitute a localized hyperfunction *on* $F^{(n)}$. Consequently, defining $g^*_{n\alpha}\in\mathfrak{B}(S_\alpha-F^{(n+1)})$ by $g^*_{n\alpha}=g_{n\alpha}|(S_\alpha-F^{(n+1)})$, we have a localized hyperfunction $\{(S_\alpha-F^{(n+1)}, g^*_{n\alpha}); \alpha\in N\}$ *on* $F^{(n)}-F^{(n+1)}$. By hypothesis, there exists a $\alpha(n)\in N$ such that $F_n\subset S_{\alpha(n)}$. Clearly we have

$$g^*_{n\alpha(n)}\in\mathfrak{B}(F^{(n)}-F^{(n+1)}),$$

whence the localized hyperfunction $\{(S_\alpha-F^{(n+1)}, g^*_{n\alpha}); \alpha\in N\}$ introduced above is equivalent to a hyperfunction $g^*_n=g^*_{n\alpha(n)}$ *on* $F^{(n)}-F^{(n+1)}$:

$$g^*_{n\alpha}=g^*_n|(S_\alpha-F^{(n+1)}) \qquad (\text{for every } \alpha\in N).$$

By proposition 22.1, there exists a $h_n(x)\in\mathfrak{B}(F_n)$ such that

$$h_n|(F^{(n)}-F^{(n+1)})=g^*_n$$

or equivalently,

$$h_n|(S_\alpha-F^{(n+1)})=g^*_{n\alpha} \qquad (\text{for every } \alpha\in N),$$

$$\text{i.e.}\qquad h_n|S_\alpha\equiv g_n \qquad \mathrm{mod}\,\mathfrak{B}(S_\alpha\frown F^{(n+1)}).$$

We have now

$$g_\alpha-\sum_{\nu=1}^{n} h_\nu|S_\alpha=g_{n\alpha}-h_n|S_\alpha\in\mathfrak{B}(S_\alpha\frown F^{(n+1)}),$$

and hence the condition (2) is verified also for $h_n(x)$.

To complete the proof, we shall invoke to the proposition 23.1 and define

$g(x)\in\mathfrak{B}(S)$ as follows by means of $h_n(x)\in\mathfrak{B}(F_n)$, $n=1,2,\cdots$:

$$(5)\qquad g(x)=\sum_{\nu=1}^{\infty} h_\nu(x).$$

Then we have, by (2),

$$g|S_\alpha=g_\alpha.$$

Thus the proposition is completely proved. (q. e. d.)

Let $\{(S_\alpha, g_\alpha);\ \alpha\in N\}$ be a localized hyperfunction on S, and let

$$(6)\qquad g_\alpha=[\varphi_\alpha, U_\alpha],\quad U_\alpha\in\mathfrak{D}(S_\alpha).$$

Now set $D=\bigcup_{\alpha\in N} U_\alpha$. Then clearly D is a complex neighborhood of S, while $\{U_\alpha\ ;\ \alpha\in N\}$ constitutes an open covering of D. Clearly we have

$$(7)\qquad \varphi_\alpha\equiv\varphi_\beta \quad (\mathrm{mod}\ \mathfrak{A}(U_\alpha\frown U_\beta))$$

for any $\alpha, \beta\in N$. Conversely, if a complex neighborhood D of S, an open covering $\{U_\alpha\ ;\ \alpha\in N\}$ of D, and a family of holomorphic functions $\varphi_\alpha\in\mathfrak{A}(U_\alpha-S)$, $\alpha\in N$, satisfying the condition (6), are given, then a localized hyperfunction $\{(S_\alpha, g_\alpha);\ \alpha\in N\}$ on S is determined by (5), and hence a hyperfunction $g(x)$ on S.

Definition. We call $\{(\varphi_\alpha, U_\alpha);\ \alpha\in N\}$ *a family of localized defining functions of* $g(x)$, *and denote:*

$$(8)\qquad \begin{cases} g=[(\varphi_\alpha, U_\alpha)\ ;\ \alpha\in N]=[\varphi_\alpha\ ;\ \alpha\in N], & or \\ g(x)=[(\varphi_\alpha(z), U_\alpha)\ ;\ \alpha\in N]_{z=x}=[\varphi_\alpha(z)\ ;\ \alpha\in N]_{z=x}. \end{cases}$$

If the suffix set N *is the finite set* $\{1, 2, \cdots, n\}$, *we shall write, in particular,*

$$(9)\qquad g=[(\varphi_1, U_1),\cdots,(\varphi_n, U_n)]=[\varphi_1,\cdots,\varphi_n].$$

As to this paragraph, see further [3] §3, and the forthcoming note II.

§25. *A remark.*

In §§20–24 above, the decomposition theorem and the localization theorem are derived from proposition 20.1 (existence theorem) and proposition 23.1. The results of §18 and §19 are special cases of the decomposition theorem thus obtained. Conversely, we shall show in the following that we can also prove the proposition 20.1 (and hence all the results of §§20–24) with the help of the results of §§18, 19 (and corollary of proposition 23.1).

In fact: let I be an open set of $\boldsymbol{R}$, $g(x)$ a hyperfunction on I. By applying proposition 19.1 to each $g(x)|I_j$, $j=1,2,\cdots$, I_j denoting connected components of I, we see that $g(x)$ can be decomposed into a locally finite sum of perfect hyperfunctions:

$$\begin{cases} g=\sum_{\nu=1}^{\infty} g_\nu, \qquad g_\nu \in \mathfrak{B}(K_\nu), \\ \{K_\nu\,;\ \nu=1,2,\cdots\}\text{: locally finite family of compact subsets of } I. \end{cases}$$

Hence, by corollary 1 of proposition 23.1, $g(x)$ has a defining function $\varphi_0 \in \mathfrak{A}(D_0 - I)$ for any $D_0 \in \mathfrak{D}(I)$.

Analytic Hyperfunctions.

§26. *Linear differential equations.*

Let $L=L_x$ be a holomorphic (linear) differential operator on an open set $I \subset \boldsymbol{R}$:

$$L_x=\sum_{\nu=0}^{n} f_\nu(x)\frac{d^{n-\nu}}{dx^{n-\nu}}, \quad f_\nu \in \mathfrak{A}(I),\ \nu=0,1,\cdots,n, \tag{1}$$

and let $h=[\psi, D] \in \mathfrak{B}(I)$ be a hyperfunction on I.

We shall consider linear differential equations of the homogeneous and inhomogeneous type:

$$L[g]=0, \tag{2}$$

$$L[g]=h, \tag{3}$$

where the unknown function is a hyperfunction $g \in \mathfrak{B}(I)$.

We can and shall restrict ourselves, with no loss of generality, to the case where I is connected, i. e.

$$I=(a,b) \subset \boldsymbol{R}, \quad -\infty \leqslant a < b \leqslant \infty. \tag{4}$$

We shall also assume that $f_0(x)$ is not identically zero on (a, b). By replacing D by a suitable $D' \in \mathfrak{D}(I)$, $D' \subset D$, if necessary, we can assume furthermore following conditions from the beginning:

$$\begin{cases} D=D^+ \cup I \cup D^- \qquad (\text{i. e. } D \cup \boldsymbol{R}=I), \\ D^+ \text{ and } D^- \text{ are both connected and simply connected,} \end{cases} \tag{5}$$

and

$$\begin{cases} f_\nu \in \mathfrak{A}(D), \qquad \nu=0,1,\cdots,n, \\ f_0(z) \neq 0 \qquad \text{for } z \in D-I. \end{cases} \tag{6}$$

Under these assumptions, the following equation clearly admits a solution $\varphi=\varphi_0 \in \mathfrak{A}(D-I)$ (by the monodromy theorem):

$$L_z(\varphi(z))=\psi(z) \qquad (z \in D-I). \tag{7}$$

We get consequently

Proposition 26.1. Equation (3) always admits a solution $\in\mathfrak{B}(I)$. *Let* $g=g_0$ *be one of the solution of* (3), *and* $g=g'$ $(\in\mathfrak{B}(I))$ *the general solution of* (2), *then the general solution of* (3) *is given by*

$$g=g_0+g'. \tag{3}$$

If, in addition, we have $h\in\mathfrak{A}(I)$ *and* $f_0(x)\neq 0$ *(for* $x\in I$*), then we have* $g\in\mathfrak{A}(I)$ *for* g *in* (8).

In particular, we have

Corollary. Any hyperfunction $g\in\mathfrak{B}(I)$ *which satisfies the equation* (2) *is everywhere holomorphic on* I *except on zero points of* f_0. Consequently, car g is (relatively) discrete in I.

Moreover we have, by an analogous consideration,

Proposition 26.2. Let $g'(x)$ *be a solution of equation (3) on* I', I' *being a open sub-interval of* I *(i. e.* $I'=(a', b')$ *with* $a\leqslant a'<b'\leqslant b'$*). Then there exists a solution* $g(x)$ *of (3) on* I *such that*

$$g|I=g'. \tag{9}$$

§27. *analytic hyperfunctions.*

Definition. Let $g\in\mathfrak{B}(I)$, $I\subset \boldsymbol{R}$ *be open,* $a\in I$ *be an isolated singularity of* g *(i. e. an isolated point of* sing.car g*). We say that* $x=a$ is a *threshold of* g *if there exists a real neighborhood* $I_a\subset I$ *of* $x=a$, *and a differential operator* L *holomorphic on* I_a, *such that*

$$L_x[g(x)]=0 \qquad (x\in I_a). \tag{1}$$

Definition. We say that $g\in\mathfrak{B}(I)$ *is an analytic hyperfunction if* $g(x)$ *is everywhere holomorphic except on thresholds (i. e. if* sing.car g *is a discrete subset of* I *and each* $a\in$sing·car g *is a threshold of* $g(x)$*).*

Proposition 27.1. Let $f_j\in\mathfrak{A}(I)$, $g_j\in\mathfrak{B}(I)$ $(j=1, \cdots, n)$, *and let* g_j *be all analytic hyperfunctions. Then the linear form* $g=\sum_{j=1}^{n} f_j g_j$ *is also an anlytic hyperfunction. More generally, let* $A^{(j)}$ *be holomorphic linear differential operators defined on* I, *then*

$$g=\sum_{j=1}^{n} A^{(j)}[g_j] \tag{2}$$

is also an analytic hyperfunction on I.

This proposition follows from the fact that, if each g_j satisfies the differential equation

$$L^{(j)}[g_j]=0 \tag{3}$$

of rank $\leqslant m_j$ on a neighborhood I_a of $a\in I$, then we can construct an equation of rank $\leqq \sum_{j=1}^{n}{}' m_j$ satisfied by g from these equations, by eliminating all g_j from the equations (2) and (3).

We can easily derive from the theory of monodromy groups of holomorphic linear differential equations, that, if $g=[\varphi, D]\in\mathfrak{B}(I)$ has $x=a$ as the threshold, then $\varphi(z)$ is uniquely decomposed into the following form:

$$\varphi(z)=\sum_{j=0}^{r}{}'\varphi_j(z) \tag{4}$$

where

$$\begin{cases} \varphi_j(z)=(z-a)^{\alpha_j}(\varepsilon(z)P_j(z, \log(z-a))+\bar{\varepsilon}(z)Q_j(z, \log(z-a))), \\ P_j(z, w) \text{ and } Q_j(z, w) \text{ are polynomials of } w \text{ whose coefficients are} \\ \text{holomorphic functions of } z \text{ in } 0<|z-a|<\delta \text{ with some } \delta>0, \end{cases} \tag{5}$$

and, in addition,

$$\begin{cases} \alpha_j \not\equiv \alpha_k \quad (\text{mod } 1) \text{ for } j\neq k, \\ \alpha_0=0. \end{cases} \tag{6}$$

Each α_j are determined uniquely mod 1.

In particular, $x=a$ is an isolated point of car g if and only if $r=0$ in (4), i. e.

$$\varphi(z)=\varphi_0(z)=\varepsilon(z)P_0(z, \log(z-a))+\bar{\varepsilon}(z)Q_0(z, \log(z-a)). \tag{7}$$

Now we shall consider a special kind of thresholds for which every coefficient appearing in the representation of P_j and Q_j as polynomials of w in (5) is meromorphic at $z=a$. We shall call such a threshold a *regular threshold*, because it corresponds to the *regular singularity* of an analytic linear differential equation.

Now it is easy to see that g satisfies an equation of the following form:

$$L_x[g(x)]=0 \tag{8}$$

$$\text{with} \qquad L_x=\sum_{\nu=0}^{n}{}' f_\nu(x)\frac{d^{n-\nu}}{dx^{n-\nu}},$$

where

$$f_\nu\in\mathfrak{A}((a-\delta, a+\delta)),$$

$$\frac{(x-a)^\nu\cdot f_\nu(x)}{f_0(x)}\in\mathfrak{A}((a-\delta, a+\delta)), \text{ for some } \delta>0, \tag{9}$$

and conversely, any hyperfunction g which satisfies an equation of a form such as (8)-(9), has $x=a$ as a regular thershold.

Again, let us call the regular threshold $x=a$ *non-degenerate*, if each coefficient appearing in the representation of P_0 and Q_0 in (7) as polynomials of w is, not merely meromorphic but holomorphic at $z=a$. Then we have

Proposition 27.2. Let $g\in\mathfrak{B}(I)$ be an analytic hyperfunction whose thresholds are all non-degenerate regular thresholds. Then, for any discrete subset $\Delta\subset I$ (e.g. $\Delta=\text{sing.car}\, g$), $g(x)$ is uniquely determined by its restriction onto $I-\Delta$:

$$g|(I-\Delta)\in\mathfrak{B}(I-\Delta).$$

Proof. Let $g_1\in\mathfrak{B}(I)$ be another analytic hyperfunction whose thresholds are all non-degenerate regular thresholds, and assume

$$g_1|(I-\Delta)\in g|(I-\Delta).$$

Let $g_0=g_1-g=[\varphi, D]$, $D\in\mathfrak{D}(I)$. Then, for each $a\in\Delta$, $\varphi(z)$ is a holomorphic function in $0<|z|<\delta$ with some $\delta>0$. On the other hand, $\varphi(z)$ is expressible in the form (25.3). By the uniqueness of the expression, we know that the expression is of the form

$$\varphi(z)=\varphi_0(z)=\varepsilon(z)\varphi_0(z)+\tilde{\varepsilon}(z)\varphi_0(z),$$

where $\varphi_0(z)$ is a holomorphic function in $|z|<\delta$.

As a result, we have

$$g_0=0 \text{ on } I. \qquad \text{(q. e. d.)}$$

If we take $\Delta\supset$ sing.car g, then $f=g|(I-\Delta)$ is a holomorphic function on $I-\Delta$, and, for each $a\in\Delta$, $f(x)$ is expressed in the form

$$\text{(10)}\qquad f(x)=\begin{cases}\sum_{j=0}^{r}(x-a)^{\alpha_j}P_j(x, \log(x-a)) & (a-\delta<x<a)\\ \sum_{j=0}^{r}(a-x)^{\alpha_j}Q_j(x, \log(a-x)) & (a<x<a+\delta)\end{cases}$$

for some $\delta>0$, where $\alpha_0, \cdots, \alpha_r$ satisfy the condition (6), while $P_j(z,w)$ and $Q_j(z,w)$ are polynomials of w whose coefficients are holomorphic functions of z in $|z-a|<\delta$. Conversely, let Δ be any discrete subset of I, and $f(x)$, any analytic function on $I-\Delta$ which is expressed in the form (10) for each $a\in\Delta$. Then there exists a uniquely determined analytic hyperfunction $g(x)$ on I whose thresholds are all non-degenerate regular thresholds.

Example. Power functions $g_\alpha(x)$, $g_\alpha(-x)$ ($\alpha\in\boldsymbol{C}-\{-1, -2, \cdots\}$) in §13 are analytic hyperfunctions on $\boldsymbol{R}$ and satisfies the equation

$$\left(x\frac{d}{dx}-\alpha\right)g_\alpha(\pm x)=0.^{*)}$$

*) Conversely, the general solution of the equation $\left(x\dfrac{d}{dx}-\alpha\right)w=0$ with

(i) $\qquad w\in\mathfrak{B}(\boldsymbol{R})$

is given by $w=c_1g_\alpha(x)+c_2g_\alpha(-x)$, c_1, c_2 being arbitrary constants. The same result is true even if we replace (i) by a weaker condition: $w\in\mathfrak{B}(I)$, I being an open interval on $\boldsymbol{R}$, according to the proposition 26.2. In case $\alpha=-(n+1)=-1,-2,\cdots$, which was excluded above, the general solution of the equation with $w\in\mathfrak{B}(\boldsymbol{R})$ or $w\in\mathfrak{B}(I)$ is given by $w=c_1P\dfrac{1}{x^{n+1}}+c_2\delta^{(n)}(x)$.

$x=0$ is the only threshold of $g_a(\pm x)$. Clearly it is a non-degenerate regular threshold. Consequently, $g_a(\pm x)$ is completely determined by (13.3), in the sense of proposition 27.2.

Various examples of analytic hyperfunctions (of practical use in the applied analysis) and their integrals will be found in [3], §§2-3. (Cf. also forthcoming paper II.)

§28. *Irregular thresholds.*

Thresholds of $g\in\mathfrak{B}(I)$ other than regular thresholds will be called *irregular thresholds*. They correspond to irregular singularities of holomorphic linear differential equations.

Let $z=a$ $(\in \boldsymbol{R})$ be a singularity of the holomorphic linear differential equation

$$(1) \qquad L_z[\psi(z)]=0,$$

where coefficients of operator L_z are holomorphic functions in $|z-a|<\rho$ with some $\rho>0$, then the solution $\psi(z)$ of equation (1) is a holomorphic function with (possibly) a logarithmic branch point on $z=a$, i.e. a holomorphic function on the covering surface of the z-plane:

$$(2) \qquad \begin{cases} 0<|z-a|<\rho \\ -\infty<\arg(z-a)<\infty. \end{cases}$$

If $z=a$ is an irregular singularity of (1), it is generally difficult to determine $\psi(z)$ exactly. According to H. Poincaré,[*] however, the equation (1) has then a solution $\psi(z)$ which has an asymptotic expansion of the following form for $|z-a|\to 0$ in a domain $\{z\,;\ \theta_1<\arg(z-a)<\theta_2\}$:

$$(3) \qquad \psi(z)\sim(z-a)^{\alpha}\cdot\exp P((z-a)^{-1/r})\cdot Q((z-a)^{1/r}),$$

where α is a complex number, r a natural number, $P(w)$ a polynomial in w, and $Q(w)$ a formal power series (not necessarily convergent) in w. The expansion (3) holds uniformly in $\theta_1+\delta\leqslant\arg\cdot(z-a)\leqslant\theta_2-\delta$ $(\delta>0)$. Now let Aw^n be the term of the polynomial $P(w)$ with the highest exponent. The domain $\{z\,;\ \Re A(z-a)^{-\frac{n}{r}}<0\}$ consists of the angular domains $\Delta_m=\{z\,;\ \omega_{2m}<\arg(z-a)<\omega_{2m+1}\}$, $m=0, \pm1, \pm2, \cdots$, where

$$\omega_\mu=\arg A+\left(\mu+\frac{1}{2}\right)\frac{r\pi}{n}.$$

In each Δ_m which lies in $\theta_1<\arg|z-a|<\theta_2$, $\psi(z)$ converges to zero exponentially, and hence faster than any power of $|z-a|$, when $|z-a|$ tends to zero.

For this reason, we shall now consider a solution $\psi(z)$ of (1) defined in the

[*] H. Poincaré, Arta Math. **8** (1886), 295.

domain (2) which satisfies for every $N<\infty$:

$$(4)\quad \begin{cases} \psi(z)=O(|z-a|^N) \text{ as } |z-a|\to 0, \\ \text{uniformly in } \theta_1+\delta\leqslant \arg(z-a)\leqslant\theta_2-\delta,\ (\delta>0) \end{cases}$$

with some θ_1, θ_2 $(\theta_1<\theta_2)$.

Lemma 1. Let $A=A_z$ be a holomorphic linear differential operator defined in $|z-a|<\rho$. Then $A_z[\psi(z)]$ is again a holomorphic function in the domain (2) satisfying the same condition as that of (4) for $\psi(z)$.

Proof. By the integral formula of Cauchy, we have

$$(5)\quad A_z[\psi(z)]=\frac{1}{2\pi i}\oint_\gamma \psi(\zeta)A_z\left(\frac{1}{\zeta-z}\right)d\zeta,$$

where we shall take $\gamma=(z+ce^{i\kappa}(z-a)\,;\,0\leqslant\kappa\leqslant 2\pi)$ with a suitable $c>0$ so that we have in $|z-a|\leqslant\rho_1<\rho$

$$(6)\quad \begin{cases} |\psi(z+ce^{i\kappa}(z-a))|\leqslant A|z-a|^{N+n} \text{ for } \theta_1+\delta\leqslant\arg(z-a)\leqslant\theta_2-\delta \\ \text{with some } A>0. \end{cases}$$

As $A_z\left(\frac{1}{\zeta-z}\right)$ is a polynomial in $\frac{1}{\zeta-z}$ of degree $n+1$, n being the rank of A, we have

$$(7)\quad \left|A_z\left(\frac{1}{\zeta-z}\right)\right|\leqslant B\left|\frac{1}{\zeta-z}\right|^{n+1} \text{ for } |\zeta-z|\leqslant\rho_2$$

with some $\rho_2>0$ and $B<\infty$. Consequently we have from (5)

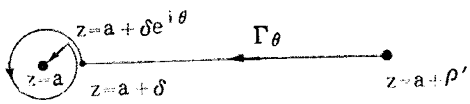
$$|A_z(\psi(z))|\leqslant\frac{B}{(c|z-a|)^n}\cdot A|z-a|^{N+n}=A'\cdot|z-a|^N$$
$$\text{for } |\zeta-z|\leqslant\min(\rho_1,\rho_2)$$

with $A'=c^{-n}\cdot BA$. (q. e. d.)

Now let $0<\rho_0<\rho$, and let $\Gamma_\theta=\Gamma_\theta(\rho')$ $(\rho_0\leqslant\rho'<\rho)$ be a path in the domain (2) defined by

$$(8)\quad \Gamma_\theta:\ a+\rho'\longrightarrow a+\delta\longrightarrow a+\delta e^{i\theta}\longrightarrow a+0\cdot e^{i\theta}$$

δ being a sufficiently small positive number (Fig. 2). Define $\varphi_{\rho'}(z)\in\mathfrak{A}(C-[a,a+\rho'])$ by



(Fig. 2)

$$(9)\quad \varphi_{\rho'}(z)=-\frac{1}{2\pi i}\int_{\Gamma_\theta(\rho')}\frac{\psi(\zeta)}{\zeta-z}d\zeta$$

with a θ between θ_1 and θ_2. Clearly $\varphi_{\rho'}(z)$ does not depend on the choice of θ. Now let

$$(10) \qquad g=[\varphi_{\rho'}, D_{\rho_0}] \quad (D_{\rho_0}=\{z\,;\ |z-a|<\rho_0\})$$

be a hyperfunction on $(a-\rho_0, a+\rho_0)$ with $\varphi_{\rho'}(z)$ in (6) as the defining function, then we have

$$(11) \qquad g(x)=\begin{cases} 0 & (a-\rho_0<x<0) \\ \psi(x) & (0<x<a+\rho_0). \end{cases}$$

On the other hand, lemma 1 yields

$$(12) \qquad A_z[\varphi_{\rho'}(z)]\equiv -\frac{1}{2\pi i}\int_{\Gamma_\theta(\rho')} \frac{A_\zeta[\psi(\zeta)]}{\zeta-z}d\zeta \qquad (\mathrm{mod}\, \mathfrak{A}(D_{\rho_0}))$$

for any analytic linear differential operator A defined in D_ρ. In particular, we have

$$(13) \qquad L_z[\varphi_{\rho'}(z)]\equiv 0 \qquad (\mathrm{mod}\, \mathfrak{A}(D_{\rho_0}))$$

and hence

$$(14) \qquad L_x[g(x)]=0 \qquad (x\in(a-\rho_0,\ a+\rho_0)).$$

It is clear that $g(x)$ does not depend on the choice of ρ'.

To sum up, we shall state as follows:

Proposition 28.1. Let $\psi(z)$ be a solution of the equation (1) defined in the domain (2) satisfying (4) for every $N<\infty$. Then an analytic hyperfunction $g(x)\in\mathfrak{B}((a-\rho_0,\ a+\rho_0))$ is determined by (9)-(10), and satisfies (11), (14).

Definition. We shall call this $g(x)$ the analytic hyperfunction determined by $\psi(z)$ and Γ_θ (or any path Γ which is equivalent to Γ_θ as an asymptotic path for $\psi(z)$).

Futhermore, by lemma 1 and proposition 26.1 we have easily

Proposition 28.2. Let $L^{(j)}$, $A^{(j)}$, $j=1, \cdots, n$, be all analytic linear differential operators in D_ρ, and for each j, let $g_j(x)$ be an analytic hyperfunction determined by $\psi_j(z)$ and Γ_θ where $\psi_j(z)$ is a solution of the equation $L^{(j)}(\psi_j(z))=0$ satisfying the same condition as (4):

$$\begin{cases} \psi_j(z)=O(|z-a|^N) \text{ as } |z-a|\to 0, \text{ for every } N<\infty, \\ \text{uniformly in } \theta_1+\delta\leqslant\arg(z-a)\leqslant\theta_2-\delta \ (\delta>0). \end{cases}$$

Then $g(x)=\sum_{j=1}^{n} A_x^{(j)}[g_j(x)]$ is also an analytic hyperfunction determined by $\psi(z)=\sum_{j=1}^{n} A_z^{(j)}[\psi_j(z)]$ and Γ_θ.

Hyperfunctions with Diffrent Types.

Thus far, we have considered hyperfunctions on a locally closed subset of $\boldsymbol{R}$. Hyperfunctions, however, can be defined on any C^{ω}-manifold as we have stated in the foreword, and at this juncture, we are led to introduce the notion of *analytic distributions* as a natural generalization of that of hyperfunctions ([2] §2, [3] §7, [4]). In the following, we shall explain a generalization of this sort in case of dimension 1.

§29. *Analytic distributions.*

Let X be a Riemann surface. (We do not assume the *connectedness* of X, i.e. we admit that X consists of an arbitrary (not necessarily countable) number of connected components.) For any open set D of X, we denote with $\mathfrak{A}(D)$ the ring of all holomorphic functions in D (which is a generalization of the definition of $\mathfrak{A}(D)$ in §1), and for any subset E of X, we define a ring $\mathfrak{A}(E)$ as the inductive limit of $\{\mathfrak{A}(D);\ D\supset E\}$ by restrictions $\rho_{D'D}:\mathfrak{A}(D)\to\mathfrak{A}(D')$ (D, D' being open sets of X).

Now let S be any nowhere dense*) locally closed set of X. We shall, in accordance with §1, denote with $\mathfrak{D}(S)=\mathfrak{D}(S, X)$ the family of all the "complex neighborhoods", i.e. the open sets of X containing S as a closed subset. For locally closed S, $\mathfrak{A}(S)$ is clearly defined as the inductive limit of $\{\mathfrak{A}(D);\ D\in\mathfrak{D}(S, X)\}$ in accordance with §1. Similarly, $\tilde{\mathfrak{A}}(S)$ and $\mathfrak{B}(S)$ are now introduced as $\mathfrak{A}(S)$-moduli by

$$(1)\qquad \tilde{\mathfrak{A}}(S)=\text{inductive limit of } \{\mathfrak{A}(D-S);\ D\in\mathfrak{D}(S, X)\},$$

$$(2)\qquad \mathfrak{B}(S)=\tilde{\mathfrak{A}}(S) \bmod \mathfrak{A}(S),$$

where we consider $\mathfrak{A}(S)\subset\tilde{\mathfrak{A}}(S)$ in a natural manner. We call each element of $\mathfrak{B}(S)$ an *analytic distribution on S*. An analytic distribution $g\in\mathfrak{B}(S)$ is determined by a couple (φ, D) such that $\varphi\in\mathfrak{A}(D-S)$, $D\in\mathfrak{D}(S, X)$. We call (φ, D) (or simply φ) a defining function of g, and write:

$$(3)\qquad g=[\varphi, D]=[\varphi].$$

Now, let (X', S') be another couple of a Riemann surface X' and a nowhere dense locally closed set S' of X'. Let σ be an *analytic homeomorphism* from S onto S', i.e. a 1-1 analytic map from some $D_0\in\mathfrak{D}(S)$ onto some $D_0'\in\mathfrak{D}(S')$ such that $\sigma(S)=S'$. (Hence σ^{-1} is an analytic homemomorphism from S' onto

*) Although analytic distributions can be defined on any locally closed S, we confine our considerations on *nowhere dense S*. (This implies $Dist^0(S, \mathfrak{A})=0$ in notation of [2], $\mathfrak{A}$ denoting the sheaf of holomorphic functions of some type (§31). Hence only $Dist^1(S, \mathfrak{A})$ is to be considered.

S.) For any $g'=[\varphi', D']\in\mathfrak{B}(K')$, we define $g=g'\circ\sigma\in\mathfrak{B}(K)$ by

$$(4)\qquad g=[\varphi,\sigma^{-1}(D_0'\frown D')]$$
$$\text{with}\quad \varphi(p)=\varphi'(\sigma(p))\quad (p\in\sigma^{-1}(D_0'\frown D')-S).$$

The definition of $g'\sigma$ is clearly independent of the choice of (φ', D'). Moreover, we have from (4)

$$g\circ\sigma^{-1}=g'.$$

Therefore, the carrespondence $g\leftrightarrow g'$ furnishes a canonical isomorphism

$$(5)\qquad \mathfrak{B}(S)\simeq\mathfrak{B}(S').$$

The transformation $g=g'\circ\sigma$ defined by (4) is a generalization of the transformation of variable defined in (7.1) for hyperfunctions.

The corresponding relation between $\mathfrak{A}(S)$ and $\mathfrak{A}'(S)$:

$$(6)\qquad \mathfrak{A}(S)\simeq\mathfrak{A}(S')$$

is also easily derived. We have, for $f_j'\in\mathfrak{A}(S')$, $g_j'\in\mathfrak{A}(S')$, $j=1,\cdots,n$,

$$(7)\qquad \left(\sum_{j=1}^{n}f_j'g_j'\right)\circ\sigma=\sum_{j=1}^{n}f_jg_j$$

with $f_j'\circ\sigma=f_j$, $g_j'\circ\sigma=g_j$.

§30. *Hyperfunctions on an analytic curve.*

Let M be an *oriented* simple analytic curve on a Riemann surface X. Thus, for a suitable $D\in\mathfrak{D}(M,X)$, we have a decomposition analogous to (1.1)

$$(1)\qquad D=D^{+}+M+D^{-}$$

such that for each $p\in M$ there exists a complex neighborhood $D_p\in\mathfrak{D}(\{p\})$ and an *univalent* holomorphic function $z_p\in\mathfrak{A}(D_p)$ (i. e. a local parameter in D_p) with which

$$(2)\qquad \begin{aligned} z_p^{-1}(\boldsymbol{R})&=D_p\frown M\\ z_p^{-1}(\boldsymbol{C}^{\pm})&=D_p\frown D^{\pm}(=D_p^{\pm})\end{aligned}$$

holds. Replacing X by some $X'\in\mathfrak{D}(M,X)$ (e. g. $X'=X-(\bar{M}-M)$, $\bar{M}$ denoting the closure of M in X) if necessary, we may assume, with no loss of generality, that M is (relatively) closed in X:

$$(3)\qquad \bar{M}=M.$$

The Riemann surface X may be called a "complex analytic prolongation" of the "1-dimensional C^{ω}-manifold M". The relation (29.5) assures that these hyperfunctions on $S\subset M$ are well defined if C^{ω}-*manifold* M is given, and do not depend essentially on the choice of X in which M is embedded.

Now we shall generalize as follows the notion of hyperfunction defined in §2.

*Definition. Let S be a locally closed subset of M, M being an (oriented) simple analytic curve on a Riemann surface X.**) *We call a hyperfunction on S each element of $\mathfrak{B}(S)$ (i.e. each analytic distribution on S).*

Needless to say, the properties described thus far for hyperfunctions on a locally closed S *in* $\boldsymbol{R}$ also holds, mutatis mutandis, for hyperfunctions on a locally closed set S *in* M. For instance, for $g=[\varphi]\in\mathfrak{B}(S)$ we can define the complex conjugate $\bar{g}\in\mathfrak{B}(S)$ of g by

$$\bar{g}=-[\bar{\varphi}] \quad \text{with } \bar{\varphi}(p)=\overline{\varphi(p^*)} \tag{4}$$

where p^* denotes the reflection of $p\in D$ (with some $D\in\mathfrak{D}(S,X)$) with respect to M, and for any analytic linear differential operator L on S, we can define $L[g]\in\mathfrak{B}(S)$ as a generalization of (8.1) by

$$L[g]=[L[\varphi]]. \tag{5}$$

Each $g\in\mathfrak{B}(S)$ can be considered as an element of $\mathfrak{B}(I)$, I being a suitable *real neighborhood* of S (i.e. an open set $I\subset M$ which contains S as a closed subset), and we have, by a canonical identification similar to (5.3),

$$\mathfrak{A}(I)\subset\mathfrak{B}(I). \tag{6}$$

§31. *Hyperfunctions with different types.*

Through handling *manifolds*, we are naturally led to introduce *hyperfunctions with various transformation properties* as follows.

In the first place, let $\boldsymbol{B}$ be a (complex) analytic vector bundle over the Riemann surface X. For any open set D of X, we denote with $\mathfrak{A}_B(D)$ the $\mathfrak{A}(D)$-module consisting of all the sections of $\boldsymbol{B}$ over D (i.e. of all the analytic mapping φ from D into $\boldsymbol{B}$ such that $\pi\circ\varphi=1_D=$idential map of D, π denoting the projection of $\boldsymbol{B}$ onto D). Replacing every $\mathfrak{A}(D)$ in §29 by $\mathfrak{A}_B(D)$, we obtain the definition of $\mathfrak{A}_B(E)$ (E: subset of X), $\tilde{\mathfrak{A}}_B(S)$, $\mathfrak{B}_B(S)$ (S: locally closed set of X), in place of $\mathfrak{A}(E)$, $\tilde{\mathfrak{A}}(S)$, $\mathfrak{B}(S)$ respectively, and expressions and relations corresponding to (29.3)~(29.7). Clearly $\mathfrak{A}_B(E)$ constitutes an $\mathfrak{A}(E)$-module, and $\tilde{\mathfrak{A}}_B(S)$ and $\mathfrak{B}_B(S)$ constitute $\mathfrak{A}(S)$-moduli. Each element of $\mathfrak{B}_B(S)$ will be called an *analytic distribution of type* $\boldsymbol{B}$ on S.

In the next place, let M be an (oriented) analytic curve in X, and let $\boldsymbol{B}$ be

*) Clearly M (and hence S) is a nowhere dense locally closed set of X.

an analytic vector bundle *over the 1-dimensional C^ω-manifold M.* Then there exists an $X_1 \in \mathfrak{D}(M, X)$ and a complex analytic vector bundle $\boldsymbol{B}_1$ *over* X_1 such that $\boldsymbol{B}$ is (canonically isomorphic with) the restriction of $\boldsymbol{B}_1$ onto M (i. e. the analytic vector bundle on M induced from $\boldsymbol{B}_1$ by the injection $M \to X_1$). If $(X_2, \boldsymbol{B}_2)$ is another such couple of $X_2 \in \mathfrak{D}(M, X)$ and a complex analytic vector bundle $\boldsymbol{B}_2$ over X_2, then we have a canonical isomorphism

$$(6) \qquad \begin{aligned} \mathfrak{A}_{B_1}(S) &\simeq \mathfrak{A}_{B_2}(S) \\ \mathfrak{B}_{B_1}(S) &\simeq \mathfrak{B}_{B_2}(S) \end{aligned}$$

in a natural manner. Therefore, we can define $\mathfrak{A}(S)$-moduli $\mathfrak{A}_B(S)$ and $\mathfrak{B}_B(S)$ by

$$(2) \qquad \begin{aligned} \mathfrak{A}_B(S) &= \mathfrak{A}_{B_1}(S) \\ \mathfrak{B}_B(S) &= \mathfrak{B}_{B_1}(S) \end{aligned}$$

independent of the choice of $(X_1, \boldsymbol{B}_1)$.

Definition. Each element of $\mathfrak{A}_B(S)$ and $\mathfrak{B}_B(S)$, $S \subset M$, is called holomorphic function of type $\boldsymbol{B}$ and hyperfunction of type $\boldsymbol{B}$, respectively.

If $\boldsymbol{B}$ is e. g. the analytic vector bundle of differential forms (i.e. convariant vectors), or of (linear) differential operators, then analytic functions of the corresponding type $\boldsymbol{B}$ are *holomorphic differential forms* or *holomorphic differential operators* on M, respectively, while the hyperfunctions of the corresponding type will be called *hyperfunctions of differential form* or *of differential operator*, respectively. If $\boldsymbol{B} = X \times \boldsymbol{C}$ (the product bundle), then $\mathfrak{A}_B(S)$ and $\mathfrak{B}_B(S)$ reduce to $\mathfrak{A}(S)$ and $\mathfrak{B}(S)$ respectively (so that the qualifying phrase "of type $\boldsymbol{B}$" becomes unnecessary).

Corresponding to (30.6), we have in a natural manner

$$\mathfrak{A}_B(I) \subset \mathfrak{B}_B(I)$$

for any open set $I \subset M$ and any type $\boldsymbol{B}$, and consequently, we can legitimately call each $f \in \mathfrak{A}_B(I)$ a holomorphic hyperfunction of type $\boldsymbol{B}$ (cf. [2] §2, [3] §7).

In II, we shall give an equivalent definition of analytic distributions from the local stand-point, and a direct proof of the 'localization theorem' and 'completeness theorem' for analytic distributions.

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Errata.

The author profits by this opportunity to correct some errors in his former papers [1], [2].

[1] p. 127, expression (2): '$\varphi^-(z)$' should be replaced by '$-\varphi^-(z)$'.

[2] p. 607, line 9: 'completely separable' should be read 'perfectly separable'.

p. 607, line 5 from the bottom: 'T' in the bracket should be replaced by '$\boldsymbol{T}$'.