Note on the Holomorphy on an Analytic Subset.[*]

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Introduction.

It is a recent tendency to consider holomorphic functions not only on non-singular complex analytic manifolds, but also on "complex spaces". Though there are several different methods to define "complex spaces" (Cartan [3], Grauert and Remmert [5]), a "complex space" is, in the local considerations, essentially an analytic subset in the usual complex Euclidean space C^n. Here, an *analytic subset*[1] X means a relatively closed set in an open subset in C^n, such that around every point $p \in X$, X is defined locally by the common zeros of finite numbers of holomorphic functions in a neighborhood of p in C^n. Such sets can be represented in a standard fashion as described in Lemma 1. We can define a notion of *holomorphic function* over an analytic subset (see Def. 2). However, until now, the definition of holomorphy on an analytic subset seems to depend deeply upon the local representation of the analytic subsets, which is very difficult to prove rigorously.

In this brief note, I would like to introduce a definition of holomorphic functions over an arbitrary closed subset in an open set in C^n (§1), and show that if X is an irreducible analytic subset in C^n, this notion which I called D-holomorphy is equivalent to the usual definition of holomorphic functions on X (§2).

For a reducible X, a holomorphic function f on it is defined by a collection of functions holomorphic on each irreducible component, so that it is inevitable to introduce the decomposition into irreducible components. In §3, I give a remark on a function holomorphic on a reducible analytic subset X.

However, our notion of D-holomorphy seems to have rather strange character, if X is not an analytic subset, as will be seen in the examples in §1.

The author wishes to express his hearty gratitude to Prof. K. de Leeuw for his instructive discussions during the preparation of this note at Stanford.

§1. D-holomorphy and examples.

DEFINITION 1. Let X be a relatively closed subset in an open set in C^n (X need not be closed in C^n itself). Suppose that a complex-valued function

*) This work has been supported by a grant of National Science Foundation.

1) This word has no connection with the "analytic set" studied by Polish School.

f is defined on X. f is called D-*holomorphic*[2] at a point $p \in X$, if there exists a neighborhood U of p satisfying the following two conditions:

1) f is continuous in $U \cap X$, with respect to the relative topology induced from C^n to $U \cap X$.

2) For every analytic mapping $\varphi(t)=(\varphi_1(t), \cdots, \varphi_n(t))$ from the unit disc on t-plane $\{|t|<1\}$ into $U \cap X$, the composite function $f(\varphi_1(t), \cdots, \varphi_n(t))$ is holomorphic in $\{|t|<1\}$.

f is called D-holomorphic on X, if it is D-holomorphic at every point p on X.

EXAMPLES **1°**. If X has inner points, then at every inner point p, D-holomorphy coincides with the usual definition of holomorphy at p, because of the Hartogs' theorem on holomorphy (Bochner-Martin [1], p. 33).

2°. If X is a relatively open subset on a non-singular complex analytic manifold regularly imbedded in C^n, D-holomorphy coincides with the usual definition of holomorphy for the functions on the manifold defined by means of the local coordinates on X.

3°. Let X be a relatively open set in the hyperplane

$$P=\{\Im z_n=0\}.$$

Then a function on X is a function of $n-1$ complex variables $z_1, \cdots, z_{n-1}$ and a real variable $x_n=\Re z_n$. Now, a function $f(z_1, \cdots, z_{n-1}, x_n)$ is D-holomorphic on X if and only if it is continuous in all variables and holomorphic in $z_1, \cdots, z_{n-1}$. Because, an analytic mapping $(\varphi_1(t), \cdots, \varphi_n(t))$ from $\{|t|<1\}$ into P must have the property $\varphi_n(t)=\text{const.}$, though other components $\varphi_1(t), \cdots, \varphi_{n-1}(t)$ may be quite arbitrary.

Such kinds of functions usually appear in the holomorphically homotopy deformation.

4°. When there is no analytic mapping from $\{|t|<1\}$ into X other than constant mappings, every continuous function on X should be D-holomorphic.

One of the examples of such kind of sets is the *distinguished boundary surface of a polycylinder* $\{|z_1|<1, \cdots, |z_n|<1\}$; viz.,

$$X=\{|z_1|=1, \cdots, |z_n|=1\}.$$

When $n=2$, this X is of real dimension 2, but is quite different in the analytic nature from the analytic subsets which are also of real dimension 2.

Another example is the hypersphere

$$S=\{|z_1|^2+\cdots+|z_n|^2=1\}. \tag{1}$$

In fact, we have

THEOREM 1. *Let* $\varphi(t)=(\varphi_1(t), \cdots, \varphi_n(t))$ *be an analytic mapping from*

2) D means the capital letter of "disc". This notion is a slight modification of "discwise holomorphy" due to Prof. K. de Leeuw [4].

$\{|t|<1\}$ into a closed bowl $B=\{|z_1|^2+\cdots+|z_n|^2\leq 1\}$. If there exists an inner point t_0 in $\{|t|<1\}$ whose image $\varphi(t_0)$ is on the boundary S of B, then $\varphi(t)$ must be a constant mapping.

COROLLARY. *A function on the closed bowl $B=\{|z_1|^2+\cdots+|z_n|^2\leq 1\}$ is D-holomorphic on B if and only if it is holomorphic in the interior of B and continuous on the boundary.*

The hypersurface (1) is of real dimension $2n-1$, but it has quite different character in the analytic nature from the hypersurface given in example 3°.

PROOF OF THEOREM 1. Let $\varphi(t_0)$ $(|t_0|<1)$ be on S. We have a suitable linear transformation

$$w_j'=\sum_{k=1}^{n} a_{jk}w_k \qquad (j=1,\cdots,n) \tag{2}$$

which maps the hypersphere $\{|w_1|^2+\cdots+|w_n|^2=1\}$ onto itself, and brings the point $(\varphi_1(t_0),\cdots,\varphi_n(t_0))$ to $(1,0,\cdots,0)$. The function

$$\sum_{k=1}^{n} a_{1k}\varphi_k(t)$$

must be constant with absolute value 1, because it is holomorphic in $\{|t|<1\}$ and assumes its maximum in the absolute value at an inner point t_0. Hence, all other functions

$$\sum_{k=1}^{n} a_{jk}\varphi_k(t) \qquad (j=2,\cdots,n)$$

must be 0. Since (2) is a non-singular transformation, all functions $\varphi_j(t)$ $(j=1,\cdots,n)$ must be constants.

§2. Holomorphic functions on an analytic subset.

Let X be an analytic subset as defined in Introduction. Suppose that it is not an isolated point at every point p on it. The following result is well-known (Bochner-Martin [1], Chap. X, or Cartan [2], XIV). We omit the proof, because the full demonstration is too long to describe here.

LEMMA 1. An analytic subset is decomposed *uniquely into a sum of finite number of irreducible components* at every point p on it. *An irreducible analytic subset X* at a point $p\in X$ has the following type of local representation in a neighborhood U of p.

After suitable linear transformation of coordinates, whose origin is at p, we have a uniquely determined integer l $(1\leq l\leq n)$ called the *dimension* of X and a neighborhood U satisfying the conditions:

1) We may assume that U is a polycylinder in this new coordinates. We denote by U' the projection of U into the $(z_1,\cdots,z_l)$ space. Now, $z_1,\cdots,z_l$ *have no relation on X*, i.e., a holomorphic function depending only on $z_1,\cdots,z_l$ and vanishing all over $U\cap X$ must be identically 0.

2) There is an irreducible *distinguished polynomial*

$$(3)\qquad P(w; z_1, \cdots, z_l)=w^s+a_1(z)w^{s-1}+\cdots+a_s(z)$$

whose coefficients $a_1, \cdots, a_s$ are holomorphic functions of $z_1, \cdots, z_l$ in U' vanishing at the origin, such that

$$(4)\qquad P(z_{l+1}; z_1, \cdots, z_l)=0$$

holds on X.

We denote by $D(z_1, \cdots, z_l)$ the *discriminant* of (3). Since P is irreducible, D is not identically 0.

3) There are $n-l-1$ polynomials $Q_j(w; z_1, \cdots, z_l)$ $(j=1, 2, \cdots, n-l-1)$ in w with degree $\leqq s-1$, whose coefficients are functions of $z_1, \cdots, z_l$ holomorphic in U', such that

$$(5)\qquad \frac{\partial P}{\partial w}(z_{l+1}; z_1, \cdots, z_l)\cdot z_{l+1+j}=Q_j(z_{l+1}; z_1, \cdots, z_l) \qquad (j=1, 2, \cdots, n-l-1)$$

holds on X.

4) For every *ordinary point* $(z_1, \cdots, z_n)$ in $U\cap X$, i.e., a point satisfying $D(z_1, \cdots, z_l)\neq 0$, *the coordinates* $z_{l+1}, z_{l+2}, \cdots, z_n$ *are given by the equations* (4) *and* (5). Therefore, in a neighborhood U of p, X is the closure of the set of points satisfying (4), (5) and $D(z_1, \cdots, z_l)\neq 0$, $(z_1, \cdots, z_l)\in U'$. Hence, *the set of ordinary points is dense on* X.

Now, in a neighborhood of an ordinary point p, X can be considered as an l-dimensional analytic manifold regularly imbedded in C^n with local coordinates $z_1, \cdots, z_l$. Hence we can define a holomorphic function at an *ordinary point of* X.

DEFINITION 2. *A holomorphic function on an irreducible analytic subset* X is a function f defined on X satisfying the following two conditions:

1) f is continuous with respect to the relative topology induced from C^n into X.

2) f is holomorphic at every ordinary point of X in the above sense.

REMARK. A function holomorphic in an open set in C^n containing X gives a function holomorphic on X. But a function f holomorphic on X in the above sense need not be given by a trace of holomorphic function in C^n. For example, on an analytic subset

$$z_1^2-z_2^3=0$$

in C^2, the function $f=z_1/z_2=\sqrt{z_2}$ is holomorphic in the above sense, but it is not a trace of holomorphic function in C^2.

As is well-known, an analytic subset with the property that every holomorphic function on it is always given by a trace of holomorphic function in C^n is called *normal* (Cartan [3], VI, VII).

Now, we shall prove the following result which is the main purpose of this paper.

THEOREM 2. *On an open set of an irreducible analytic subset X, a function is holomorphic in the sense of Definition 2, if and only if it is* D-*holomorphic on X.*

PROOF. This is a local property. Take a point $p \in X$. We have the representation of X around p as in Lemma 1, and we use the notations in Lemma 1. It is evident that a D-holomorphic function is holomorphic in the sense of Definition 2, because at every ordinary point of X, D-holomorphy coincides with the usual definition of holomorphy (see example 2°, in §1).

Conversely, suppose that f is holomorphic in a neighborhood of $p \in X$ in the sense of Definition 2. We have a small neighborhood U of p in which f is holomorphic and the representation in Lemma 1 is valid. Since f is continuous, we have only to show the second condition in Definition 1, in order to prove the D-holomorphy. Suppose that $\varphi(t)=(\varphi_1(t), \cdots, \varphi_n(t))$ is an analytic mapping from $\{|t|<1\}$ into C^n, whose image is in $U \cap X$.

We put $d(t)=D(\varphi_1(t), \cdots, \varphi_n(t))$, and $\tilde{f}(t)=f(\varphi_1(t), \cdots, \varphi_n(t))$. All we have to show is that $\tilde{f}(t)$ is holomorphic in $\{|t|<1\}$. For a value t_0 such that $d(t_0)\neq 0$, the function $\tilde{f}(t)$ is holomorphic at t_0, because $f(z_1, \cdots, z_n)$ is a holomorphic function of $z_1, \cdots, z_l$ around the point $(\varphi_1(t_0), \cdots, \varphi_n(t_0)) \in X$. Hence, if $d(t)$ is not identically 0, the function $\tilde{f}(t)$ is certainly holomorphic in $\{|t|<1\}$, because it is continuous and holomorphic except at isolated points given by $d(t)=0$.

However, $d(t)\equiv 0$ may occur. In this case, we need more intensive considerations, because we have only the continuity of f where D vanishes.

For every point $(\zeta_1, \cdots, \zeta_l) \in U'$ such that $D(\zeta_1, \cdots, \zeta_l)\neq 0$, there exist exactly s distinct points $q_1, \cdots, q_s$ on X whose projection to the $(z_1, \cdots, z_l)$ space is $(\zeta_1, \cdots, \zeta_l)$. Let f_j be the value of the function f at q_j $(j=1, \cdots, s)$. The elementary symmetric expressions of $f_1, \cdots, f_s$ are all one-valued holomorphic and bounded functions of $z_1, \cdots, z_l$ in U' except on the analytic subset $D(z_1, \cdots, z_l)=0$. By a theorem on removable singularity (Osgood [6], Chap. 3, or Bochner-Martin, [1], p. 173), they must be holomorphic in U' without exception. Then we have a polynomial in w

$$\Pi(w; z_1, \cdots, z_l)=w^s+b_1(z)w^{s-1}+\cdots+b_s(z) \tag{6}$$

whose coefficients are holomorphic functions of $z_1, \cdots, z_l$ in U', such that

$$\Pi(f(z_1, \cdots, z_n); z_1, \cdots, z_l)=0 \tag{7}$$

holds if $(z_1, \cdots, z_n) \in U \cap X$. Now, we substitute $z_j=\varphi_j(t)$ into (7). We have a non-trivial algebraic equation

$$(\tilde{f}(t))^s+\tilde{b}_1(t)(\tilde{f}(t))^{s-1}+\cdots+\tilde{b}_s(t)=0 \tag{8}$$

regardless whether $d(t)\not\equiv 0$ or $\equiv 0$. Here we put

$$\tilde{b}_j(t)=b_j(\varphi_1(t), \cdots, \varphi_l(t)) \qquad (j=1, \cdots, s),$$

which are holomorphic in $\{|t|<1\}$. Therefore $\tilde{f}(t)$ is an algebraic integral element over the ring consisting of all holomorphic functions in $\{|t|<1\}$. On the other hand, since $\tilde{f}(t)$ is one-valued and continuous, it must be holomorphic all over $\{|t|<1\}$, which proves our assertion.

§3. Reducible analytic subset.

An analytic subset may be reducible at a point with irreducible components of different dimensions (cf. example 6° below). If X is reducible, *a function f holomorphic on X is defined by a collection of functions holomorphic on each irreducible component* in the sense of Definition 2, and when two distinct irreducible components X_1 and X_2 meet with each other, a function *f holomorphic on X may have different values along X_1 and X_2* at the intersection.

For example, on an analytic subset X defined by

$$z_1^2-z_2^2=0$$

in C^2 a function f given by

$$f=\begin{cases} +1 & \text{on } z_1-z_2=0 \\ 0 & \text{on } z_1+z_2=0 \end{cases} \tag{9}$$

is a holomorphic function on X. This is not continuous on X at $z_1=z_2=0$, so that it is not D-holomorphic in the sense of Definition 1 because of the continuity.

In fact, a holomorphic function f on a *reducible* analytic subset X should not be considered as a function on the set X in C^n. It is inevitable to introduce the "parameter space" $\tilde{X}$ as H. Cartan did ([2], [3]). Here, $\tilde{X}$ is a space consisting of the collection of germs of irreducible components of X with suitable topology. f must be considered as a function over $\tilde{X}$.

In this last section, I would like to remark the following fact:

THEOREM 3. *Let f be a holomorphic function on an analytic subset X (irreducible or reducible). At every point $p\in X$, we have a neighborhood U in C^n and a function ψ meromorphic in U whose trace on X is f, i.e., $\psi=f$ holds on a dense subset in $U\cap X$.*

As we have remarked above, f is not always a trace of holomorphic function. First we show some of the examples.

EXAMPLES 5°. The function (9) is given by the trace of meromorphic function

$$\psi=(z_1+z_2)/2z_1 .$$

The intersection $z_1=z_2=0$ of the distinct irreducible components is exactly the *singularity of indeterminacy*[3] of φ. In general, φ may have the singularity of indeterminacy even on the points lying only one of the irreducible components of X, so that we required that $\varphi=f$ holds only *on a dense subset in* X.

6°. (Osgood [6], Chap. 2, §20). Let X be defined by

$$z_1z_3-z_2z_4=0\,, \qquad z_1z_3+z_2z_4=0\,, \qquad z_1-z_2=0$$

in C^4. X is the sum of two irreducible components X_1 and X_2, say:

$$X_1=\{z_1=z_2=0\}\,, \qquad X_2=\{z_3=z_4=0,\ z_1-z_2=0\}\,.$$

A function f on X defined by

$$f=\begin{cases} +1 & \text{on } X_1 \\ 0 & \text{on } X_2 \end{cases}$$

is a holomorphic function on X which is the trace of a meromorphic function

$$\varphi=\frac{z_3-z_4}{z_1+z_2+z_3-z_4}\ .$$

The set of points $(0, 0, \alpha, \alpha)$ on X_1 is the singularity of indeterminacy of φ, and we have $\varphi=1=f$ on X_1 except this set.

PROOF OF THEOREM 3. Let $X=X_1\cup\cdots\cup X_m$, where X_k $(k=1, \cdots, m)$ be the irreducible components of X at p.

i) First we consider the case where *all* X_k *are of same dimension*. This implies the case where X itself is irreducible, i.e., $m=1$.

In fact, Theorem 3 has been proved already in Osgood [6], (Chap. 2, §12), if X is an analytic subset of co-dimension 1 (principal analytic subset), and similar method is applicable in this case (Cartan [3], VII). Therefore, the assertion in this case is a known result, but I would like to give the demonstration in order to make the proof self-contained.

Under the assumption, we can again represent X into the form given in Lemma 1, though (3) is not necessarily irreducible. However, the discriminant $D(z_1,\cdots, z_l)$ is still not identically 0. We have polynomials $T_j(\omega; z_1, \cdots, z_l)$ $(j=1, \cdots, s-1)$ in ω of degree $\leqq s-1$, whose coefficients are holomorphic functions of $z_1, \cdots, z_l$ in U', satisfying

$$\text{(10)}\qquad \begin{aligned} &P(w; z_1, \cdots, z_l)-P(\omega; z_1, \cdots, z_l) \\ &\quad=(w-\omega)[w^{s-1}+T_1w^{s-2}+\cdots+T_{s-1}]\,. \end{aligned}$$

At any point $(\zeta_1, \cdots, \zeta_l)$ where $D(\zeta_1, \cdots, \zeta_l)\neq 0$, there are exactly s distinct points $q_1, \cdots, q_s$ on X whose projection in $(z_1, \cdots, z_l)$ space is $(\zeta_1, \cdots, \zeta_l)$. Each

3) This means the set of points at which both of the local denominator and numerator vanish simultaneously. In the terminology of Osgood [6], it is called the "non-essential singularity of the second kind".

q_j is situated on only one irreducible component of X. We denote by $z_{l+1}^{(j)}$ and f_j $(j=1, \cdots, s)$ the z_{l+1}-coordinate of q_j and the value of f at q_j respectively.

From (10), we have easily

$$(11) \qquad f_j \cdot \frac{\partial P}{\partial w}(z_{l+1}^{(j)}; z_1, \cdots, z_l) = \sum_{k=0}^{s-1} B_k(z_1, \cdots, z_l) T_{s-1-k}(z_{l+1}^{(j)}; z_1, \cdots, z_l),$$

where we put

$$B_k(z_1, \cdots, z_l) = \sum_{i=1}^{s} f_i \cdot (z_{l+1}^{(i)})^k \qquad (k=0, 1, \cdots, s-1),$$

and $T_0=1$.

The functions B_k are one-valued, bounded and holomorphic in U' without exception due to theorem of removable singularity again. Hence the function

$$\psi(z_1, \cdots, z_l, z_{l+1}) = \sum_{k=0}^{s-1} B_k(z_1, \cdots, z_l) T_{s-1-k}(z_{l+1}; z_1, \cdots, z_l) \Big/ \frac{\partial P}{\partial w}(z_{l+1}; z_1, \cdots, z_l)$$

is meromorphic in U, and according to (11), $\psi=f$ holds in a dense subset in $U \cap X$, say, at every ordinary point.

ii) General Case. Let $X = X_1 \cup \cdots \cup X_m$ be the decomposition of X into irreducible components. By the considerations in i), for each k, we have a meromorphic function ψ_k, such that $\psi_k = f$ holds on $U \cap X_k$, where U is a suitable neighborhood of p. Of course, we have nothing on the behavior of ψ_k on other components X_j $(j \neq k)$.

Now, on each X_k, we can take an ordinary point q_k, which lies only on X_k but on no other components. We have a function g_k holomorphic in U which vanishes on X_k, but $g_k(q_j) \neq 0$ holds for all $j=1, \cdots, m$ except k. Then, if $j \neq k$, g_k vanishes only on a nowhere dense set on X_j, due to the "Nullstellensatz" (Cartan [2], XIV, Theorem 2). We put

$$(12) \qquad h_k = \frac{1}{g_k} \Big/ \left[\frac{1}{g_1} + \cdots + \frac{1}{g_m} \right] = \frac{g_1 \cdots \hat{g}_k \cdots g_m}{g_2 \cdots g_m + g_1 g_3 \cdots g_m + \cdots + g_1 \cdots g_{m-1}}.$$

The denominator on the right hand side of (12) does not vanish identically on each X_k, because it is not 0 at q_k, so that we have

$$h_k = \begin{cases} +1 & \text{on } X_k \\ 0 & \text{on all other components } X_j \quad (j \neq k). \end{cases}$$

Hence, the function $\psi = \sum_{k=1}^{m} h_k \psi_k$, meromorphic in U, satisfies the conditions of our theorem.

Nov. 13, 1957.

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