On the Functional Equation of the Generalized L-Function.

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A. Weil [5] has generalized Artin's L-functions so as to include Hecke's L-functions "mit Grössencharackteren". He indicates that we have only to follow Artin [1] to develop the whole theory of these functions. In following this indication, we have established in a previous paper [3] the theory of conductors. In the present paper, we shall give an explicit form of functional equations for these functions and determine in particular the Γ-factors appearing in the functional equations. The situation is in fact quite parallel to Artin's case in the whole, but in detail we have to proceed more carefully especially in case where infinite primes are ramified.

§ 1. Remarks on Hecke's L-function.

Let k be a finite algebraic number field with degree n, and $\mathfrak{p}_{\infty,1}, \cdots, \mathfrak{p}_{\infty,r_1+r_2}$ the infinite primes of k of which $\mathfrak{p}_{\infty,1}, \cdots, \mathfrak{p}_{\infty,r_1}$ are real and $\mathfrak{p}_{\infty,r_1+1}, \cdots, \mathfrak{p}_{\infty,r_1+r_2}$ are complex ($r_1+2r_2=n$). We denote the idèle group and idèle class group of k with J_k and C_k respectively. The topology of J_k is introduced as usual, then the principal idèle group $\boldsymbol{P}_k$ of k is a discrete subgroup of J_k, and J_k and C_k are locally compact abelian groups. Let χ be a character of C_k, then χ may be considered as a character of J_k such that $\chi(\boldsymbol{P}_k)=1$. Let $\mathfrak{m}$ be an integral ideal of k, and $U_{\mathfrak{m},0}$ the group of all unit idèls of k such that $\mathfrak{u}\equiv 1$ mod. $\mathfrak{m}$ and whose components at the infinite primes are all equal to 1. Then we have $\chi(U_{\mathfrak{m},0})=1$ for a suitable $\mathfrak{m}$ in virtue of the continuity of χ. The g. c. m. $\mathfrak{f}_\chi$ of all such $\mathfrak{m}$ is called the conductor of χ. We define the function $L(s,\chi)$ as follows. Let J' be the group of all idèles of k whose components at the infinite primes and the finite primes contained in $\mathfrak{f}_\chi$ are all equal to 1. To each $J'\ni\mathfrak{a}$ corresponds an ideal $\tilde{\mathfrak{a}}=\prod_{\mathfrak{p}}'\mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{a})}$ where $\prod_{\mathfrak{p}}'$ means the products for all finite primes and $\nu_{\mathfrak{p}}(\mathfrak{a})$ the $\mathfrak{p}$-order of $\mathfrak{a}$. The mapping $\mathfrak{a}\to\tilde{\mathfrak{a}}$ maps J' homomorphically on the group G' of all ideals of k which are prime to $\mathfrak{f}_\chi$. The value $\chi(\mathfrak{a})$ depends only on the ideal $\tilde{\mathfrak{a}}$, and we have a character $\tilde{\chi}$ of G' in putting $\tilde{\chi}(\tilde{\mathfrak{a}})=\chi(\mathfrak{a})$, This character coincides essentially with Hecke's "Grössencharakter mit Führer $\mathfrak{f}_\chi$".

Put
$$L(s,\chi)=\sum_{\substack{\tilde{\mathfrak{a}}\in G' \\ \sim:\ \text{integral}}}\frac{\tilde{\chi}(\tilde{\mathfrak{a}})}{N(\tilde{\mathfrak{a}})^s}$$
where s is a complex variable and $N(\tilde{\mathfrak{a}})$ denotes the norm of the ideal $\tilde{\mathfrak{a}}$. The function $L(s,\chi)$ is extended analytically to the whole s-plane, and if χ is not trivial on the compact component[1] of C_k, then this function is integral. To giving the explicit form of the functional equation of $L(s,\chi)$, we must define several notations. Let J_∞ be the group of all idèles of k whose components at the finite primes are all equal to 1. The element $\mathfrak{u}'$ of J_∞ can be identified naturally with the vector $(\alpha_1,\cdots,\alpha_{r_1},\alpha_{r_1+1},\cdots,\alpha_{r_1+r_2})$ whose components are all $\neq 0$ where $\alpha_1\cdots\alpha_{r_1}$ are real and $\alpha_{r_1+1}\cdots\alpha_{r_1+r_2}$ complex numbers. Then we have
$$\chi(\mathfrak{u}')=\prod_{j=1}^{r_1+r_2}|\alpha_j|^{\lambda_j\sqrt{-1}}\prod_{j=1}^{r_1}(\operatorname{sgn}\alpha_j)^{e_j}\prod_{l=r_1+1}^{r_1+r_2}\left(\frac{\alpha_l}{|\alpha_l|}\right)^{e_l}$$
where $\lambda_j\ (1\leq j\leq r_1+r_2)$ are all real and e_j are all rational integers such that $e_j=0$ or 1 for $1\leq j\leq r_1$. The system $(e_1,\cdots,e_{r_1+r_2},\lambda_1,\cdots,\lambda_{r_1+r_2})$ is uniquely determined by χ. Put
$$\Gamma(s,\chi)=2^{-r_2s}\prod_{j=1}^{r_1}\Gamma\left(\frac{s+e_j+\lambda_j\sqrt{-1}}{2}\right)\prod_{l=r_1+1}^{r_1+r_2}\Gamma\left(s+\frac{|e_l|+\lambda_l\sqrt{-1}}{2}\right)$$
and
$$A(\chi)=\Delta N\mathfrak{f}_\chi/\pi^n$$
where Δ is the absolute value of the discriminant of k. Then we have the functional equation of $L(s,\chi)$ of the following form[2]
$$\xi(s,\chi)=W(\chi)\xi(1-s,\bar{\chi}),\quad |W(\chi)|=1$$
where $\xi(s,\chi)=A(\chi)^{s/2}\Gamma(s,\chi)L(s,\chi)$, and $W(\chi)$ is a certain constant which depens on the character χ.

§ 2. The definition of decomposition groups of infinite primes in the group $G_{K,k}$

Let K be a finite normal extension of k with the Galois group $\mathfrak{g}$, and $G_{K,k}$ the group attached to the pair K, k as defined by A. Weil [5]. Let A_K be the maximally abelian extension of K and κ an archimedian valuation of A_K. The infinite primes of K and k which are induced by κ on K and k are denoted with $\mathfrak{P}_\infty$ and $\mathfrak{p}_\infty$ respectively. If both $\mathfrak{P}_\infty$ and $\mathfrak{p}_\infty$ are real or complex, then the *decomposition group* $H(\kappa)$ in $G_{K,k}$ is defined as the natural map $V(\mathfrak{P}_\infty)$ in C_K of the multiplicative

1. [5], p. 3.
2. [2], p. 35.

group of non-zero elements of the completion of K with respect to $\mathfrak{P}_\infty$. Assume that $\mathfrak{P}_\infty$ is complex and $\mathfrak{p}_\infty$ real. Then there exists a uniquely deteremined automorphisms of A_K over k with the order 2 which leaves invariant κ. We denote this automorphism with s'_σ. s'_σ induces the automorphisms σ of K over k which belongs to the decomposition group of $\mathfrak{P}_\infty$ in $\mathfrak{g}$. Let $\mathfrak{c}$ be the idèle of k whose component at $\mathfrak{p}_\infty$ is equal to -1 and all other components are equal to 1, and γ the idèle class of $\mathfrak{c}$. It can easily be proved that the automorphism of the maximally abelian extension A_k of k induced by s'_σ coinsides with γ' where γ' is the automorphism of A_k defined naturally by γ. Hence we can take a representative s_σ of s'_σ in $G_{K;k}$ such that $s_\sigma^2=\gamma$. We put then $H_z(\kappa)=V(\mathfrak{P}_\infty)\smile s_\sigma V(\mathfrak{P}_\infty)$ and call this group in this case a *decomposition group of* κ *in* $G_{K;k}$. If $H_z(\kappa)$, $H_z(\kappa)'$ are any two decomposition groups of κ in $G_{K;k}$, then there exists an element d of the connected component D_k of 1 of C_k such that $H_z(\kappa)'=d^{-1}H_z(\kappa)d$. The following properties of $H_z(\kappa)$ can easily be proved from this definition.

1. If t' is an arbitrary automorphism of A_K over k and t is a representative of t' in $G_{K;k}$, then $t^{-1}H_z(\kappa)t$ is a decomposition group of $\kappa^{t'}$ in $G_{K;k}$.

2. If k' is an intermediate field between K and k, then the group $H_z'(\kappa)=H_z(\kappa)\frown G_{K;k'}$ is a decomposition group of κ in $G_{K;k'}$.

3. If $\bar{K}$ is a finite normal extension of k containing K, and $\bar{\kappa}$ an extension of κ to the maximally abelian extension of $\bar{K}$, then the natural map of a decomposition group of $\bar{\kappa}$ in $G_{\bar{K};k}$ into the group $G_{K;k}$ is a decomposition group of κ in $G_{K;k}$.

§ 3. Γ-facter of a character of $H_z(\kappa)$.

In the following, we shall have to consider the characters of $H_z(\kappa)$ and $G_{K,k}$, but all the characters, with which we have to deal, are solely those of unitary representations. So if we speak of characters of these groups, it should be always understood, that we mean by this term those of unitary representations.

Let χ be an irreducible character of $H_z(\kappa)$. We shall define the Γ-factor of χ as follows. We distinguish two cases.

Case I. $\mathfrak{p}_\infty$ and $\mathfrak{P}_\infty$ are both real or complex. First, let $\mathfrak{p}_\infty$, $\mathfrak{P}_\infty$ be real, then we denote with ι the natural isomorphism of the multiplicative group $\mathfrak{R}^*$ of all real numbers $\neq 0$ into the group $V(\mathfrak{P}_\infty)=H_z(\kappa)$.

If secondaly $\mathfrak{p}_\infty$ and $\mathfrak{P}_\infty$ are complex, then we denote with ι the natural isomorphism of the multiplicative group C^* of all complex numbers $\neq 0$ onto the group

$\Gamma(\mathfrak{P}_\infty)=H_z(\kappa)$. Put $\chi\circ\iota=\tilde{\chi}$. Since $\tilde{\chi}$ is irredubible, the degree of $\tilde{\chi}$ is equal to 1 and there exists a real number λ and a rational integer $e\geqq 0$ such that

$$\tilde{\chi}(\alpha)=\begin{cases}(\operatorname{sgn}\alpha)^e|\alpha|^{\lambda\sqrt{-1}}, & \mathfrak{p}_\infty:\text{ real},\\ \left(\frac{\alpha}{|\alpha|}\right)^{\pm e}\quad|\alpha|^{\lambda\sqrt{-1}}, & \mathfrak{p}_\infty:\text{ complex},\end{cases}$$

where $e=0$ or 1 if $\mathfrak{p}_\infty$ is real. The values of λ and e (resp. $\pm e$) are determind uniquely by χ. We shall call the system (e,λ) the *exponent system* of χ, and e the *first exponent* of χ. Put

$$\Gamma(s,\chi,\kappa:\ K/k)=\begin{cases}\Gamma\left(\frac{s+e+\lambda\sqrt{-1}}{2}\right) & ,\quad\mathfrak{p}_\infty:\text{ real},\\ 2^{-s}\Gamma\left(s+\frac{e+\lambda\sqrt{-1}}{2}\right), & \mathfrak{p}_\infty:\text{ complex}.\end{cases}$$

We shall call this the Γ-*factor* of χ.

Case II. $\mathfrak{p}_\infty$ is real and $\mathfrak{P}_\infty$ is complex. Let H be a group extension of C^* by a group of order 2 such that

$$H=C^*\smile sC^*,\quad s^2=-1,\quad s^{-1}\alpha s=\bar{\alpha}\quad(\alpha\in C^*).$$

Then we have a natural isomorphism ι of H onto $H_z(\kappa)$ naturally and s onto s_σ. Let t be the transfer of H into C^*, then t maps H onto $R^*\subset C^*$, $C^*\ni\alpha$ onto $N(\alpha)=|\alpha|^2$ and s onto -1. Let ι' be the natural isomorphism of R^* onto $V(\mathfrak{p}_\infty)\subset C_k$ and $\nu_{K/k}$ the natural homomorphism of $G_{K,k}$ onto C_k. Then we have

$$\nu_{K/k}\cdot\iota=\iota'\cdot t\qquad(*)$$

on the group H. This remark will be used later.

Let χ be an irreducible character of $H_z(\kappa)$ and put $\tilde{\chi}=\chi\cdot\iota$. Since $(H:C^*)=2$ and C^* is abelien, the degree of χ is equal to 1 or 2.

If the degree of χ is $=1$, we shall call χ a character of the first kind. In this case, we have $\tilde{\chi}(s^{-1}\alpha s)=\chi(\bar{\alpha})=\chi(\alpha)$ for all $\alpha\in C^*$, hence there exists a uniquely determined character η of R^* such that $\eta\cdot t=\tilde{\chi}$. Since η is irreducible, we have a system (e,λ) $(e=0$ or $1)$ as in the case I. We shall call (e,λ) the *exponent system* of χ, and e the *first exponent* of χ.

Next, we consider an irreducible character χ of degree 2. Such a character will be called a character of the second kind. Then we see by the well-known result of Clifford[3] that χ is induced by an irreducible character η of C^* such that $\eta(\alpha)\neq\eta(\bar{\alpha})$ for a some $\alpha\in C^*$. Hence we have a system (e,λ) such that

3. [4], pp. 159–163.

$$\eta(a)=\left(\frac{a}{|a|}\right)^{\pm e}|a|^{\lambda\sqrt{-1}}, \quad a\in C^*$$

as in the case I, where e is >0. We shall call (e, λ) the *exponent system* of χ and e the *first exponent* of χ.

$$\Gamma(s, \chi, \kappa;\ K/k)=\begin{cases}\Gamma\left(\dfrac{s+e+\lambda\sqrt{-1}}{2}\right), & \text{if } \chi \text{ is of the first kind,}\\ 2^{-s}\Gamma\left(s+\dfrac{e+\lambda\sqrt{-1}}{2}\right), & \text{if } \chi \text{ is of the second kind.}\end{cases}$$

We shall call this the *Γ-factor* of χ.

Thus we have defined the Γ-factor of an irreducible character χ. If a character χ of $H_z(\kappa)$ is not irreducible, then we decompose χ into irreducible parts $\chi_1, \cdots\cdots$, χ_t and obtain $\chi=\chi_1+\cdots\cdots+\chi_t$. We put then

$$\Gamma(s, \chi, \kappa;\ K/k)=\prod_{i=1}^{t}\Gamma(s, \chi_i, \kappa;\ K/k),$$

and call $\Gamma(s, \chi, \kappa; K/k)$ the *Γ-factor of χ for κ with respect to K/k.* The Γ-factor has the following properties.

i) Let χ, χ' be characters of $H_z(\kappa)$, then we have

$$\Gamma(s, \chi+\chi', \kappa;\ K/k)=\Gamma(s, \chi, \kappa;\ K/k)\ \Gamma(s, \chi'\ \kappa;\ K/k)$$

ii) Let k' and $H_z'(\kappa)$ have the meaning given in 2. of § 2. Let further ψ be a character of $H_z'(\kappa)$, and χ_ψ the character of $H_z(\kappa)$ induced by ψ, then we have

$$\Gamma(s, \chi_\psi, \kappa;\ K/k)=\Gamma(s, \psi, \kappa;\ K/k')$$

iii) Let $\bar{K}$ and $H_z(\bar{\kappa})$ have the meaning given in 3. of § 2. We may regard χ as a character of $H_z(\bar{\kappa})$, then we have

$$\Gamma(s, \chi, \kappa;\ K/k)=\Gamma(s, \chi, \kappa;\ \bar{K}/k).$$

The property i) is obvious from the definition.

Proof of ii). Let $\mathfrak{q}_\infty$ be the infinite prime of k' induced by κ. Obviously, we may consider only the following case; $\mathfrak{p}_\infty$ is real and $\mathfrak{q}_\infty$ complex and ψ irreducible. If the first exponent of ψ is >0, then χ_ψ is an irreducible character of $H_z(\kappa)$ of the 2-nd kind, and we have our assertion by the definition of the Γ-factor. Assume that the exponent system of ψ is $(0, \lambda)$. Then we have

$$M(a)=\begin{pmatrix}|a|^{\lambda\sqrt{-1}} & 0\\ 0 & |a|^{\lambda\sqrt{-1}}\end{pmatrix} \qquad (a\in C^*)$$

and

$$M(s)=\begin{pmatrix}0 & \psi(-1)\\ 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

where M is the representation of H which corresponds to the character $\tilde{\chi}_\psi$ of H. Since the eigen-value of $M(s)$ are ± 1, χ_ψ is decomposed into two irreducible characters χ, χ' of the first kind whose exponent systems are equal to $(0, \lambda/2)$ and $(1, \lambda/2)$ respectively. Then we have

$$\Gamma(s, \chi_\psi, \kappa;\ K/k) = \Gamma\left(\frac{s+\lambda\sqrt{-1}/2}{2}\right)\Gamma\left(\frac{s+1+\lambda\sqrt{-1}/2}{2}\right)$$
$$= 2^{-s}\Gamma(s+\lambda\sqrt{-1}/2) = \Gamma(s, \psi, \kappa;\ K/k)\,.$$

q. e. d.

Proof of iii) Let $\overline{\mathfrak{P}}_\infty$ be the infinite prime of $\overline{K}$ induced by $\bar{\kappa}$. If $\mathfrak{P}_\infty$ and $\overline{\mathfrak{P}}_\infty$ are both real or complex, then the natural homomorphism of $H_z(\kappa)$ onto $H_z(\kappa)$ is an isomorphism, hence our assertion is obvious from the definition of the Γ-factor. Assume that $\mathfrak{P}_\infty$ is real and $\overline{\mathfrak{P}}_\infty$ complex. We have only to consider the case when χ is irreducible. Then χ is a character of the first kind of $H_z(\bar{\kappa})$, and we have our assertion by the remark $(*)$ and the definition of the Γ-factor.

§ 4. Functional equation of the function $L(s, \chi; K/k)$.

Let χ be a character of $G_{K,k}$ then we can attach a function $L(s, \chi;\ K/k)$ of a complex variable to χ after A. Weil[4]. We shall define the Γ-factor of the function $L(s, \chi;\ K/k)$ as follows. Let $\mathfrak{p}_\infty$ be an infinite prime of k and κ an archimedian valuation of A_k which induces $\mathfrak{p}_\infty$ in k. We may regard χ as a character of $H_z(\kappa)$. Put

$$\Gamma(s, \chi, \mathfrak{p}_\infty;\ K/k) = \Gamma(s, \chi, \kappa;\ K/k)\,.$$

$\Gamma(s, \chi, \mathfrak{p}_\infty;\ K/k)$ depends only on χ and $\mathfrak{p}_\infty$ in virtue of the property 1, of § 2. It will be called the *Γ-factor of χ at $\mathfrak{p}_\infty$ with respect to K/k*. $\Gamma(s, \chi, \mathfrak{p}_\infty;\ K/k)$ has the following properties.

1. If χ, χ' are characters of $G_{K,k}$ then we have
$$\Gamma(s, \chi+\chi', \mathfrak{p}_\infty;\ K/k) = \Gamma(s, \chi, \mathfrak{p}_\infty;\ K/k)\Gamma(s, \chi', \mathfrak{p}_\infty;\ K/k)\,.$$

2. If ψ is a character of $G_{K,k'}$ and χ_ψ the chrracter of $G_{K,k}$ induced by ψ, then we have

$$\Gamma(s, \chi_\psi, \mathfrak{p}_\infty;\ K/k) = \prod_{i=1}^{g}\Gamma(s, \psi, \mathfrak{q}_{\infty,i};\ K/k)$$

where $\mathfrak{q}_{\infty,1}, \cdots, \mathfrak{q}_{\infty,g}$ are all the infinite primes of k' such that $\mathfrak{q}_{\infty,i} \mid \mathfrak{p}_\infty$.

3. $\Gamma(s, \chi, \mathfrak{p}_\infty;\ K/k) = \Gamma(s, \bar{\chi}, \mathfrak{p}_\infty;\ K/k)\,.$

The properties 1, and 3 are readily proved by i) and ii) of § 3. To prove 2.,

4. [5], p. 31.

we use the following general theorem on induced characters.

Theorem. Let G be a group, G' a subgroup of G with a finite index, and H an arbitrary subgroup of G; let ψ be a character of a unitary representation of G' and χ_ψ the character of G induced by ψ. Decompose G by G' and H as follows

$$G=G't_1^{-1}H+\cdots+G't_g^{-1}H,$$

and put $H_i=t_i^{-1}Ht_i$ and $H_i'=H_i\frown G'$. Let ψ_i be the character of H_i' obtained by the restriction of the domain of definition of ψ to H_i', and χ_{ψ_i} the character of H_i induced by ψ_i. Then we have

$$\chi_\psi(h)=\chi_{\psi_1}(t_1^{-1}ht_1)+\cdots+\chi_{\psi_g}(t_g^{-1}ht_g)$$

on H.

Put $G=G_{K,k}$, $G'=G_{K,k'}$ and $H=H_z(\kappa)$. Then the infinite primes $\mathfrak{q}_{\infty,1}\cdots\mathfrak{q}_{\infty,g}$ of k' induced by $\kappa^{t_1'}\cdots\kappa^{t_g'}$ are all the different primes of k' such that $\mathfrak{q}_{\infty,i}\mid\mathfrak{p}_\infty$. Hence we have

$$\begin{aligned}\Gamma(s,\chi_\psi,\mathfrak{p}_\infty;\ K/k)&=\prod_{i=1}^{g}\Gamma(s,\chi_{\psi_i},\kappa^{t_i'};\ K/k)=\prod_{i=1}^{g}\Gamma(s,\psi_i,\kappa^{t_i'};\ K/k)\\&=\prod_{i=1}^{g}\Gamma(s,\psi,\mathfrak{q}_{\infty,i};\ K/k')\end{aligned}$$

in virtue of ii) of § 3.

Thus we can define the *Γ-factor of* $L(s,\chi;\ K/k)$ as follows.

Put

$$\Gamma(s,\chi;\ K/k)=\prod_{\mathfrak{p}_\infty}\Gamma(s,\chi,\mathfrak{p}_\infty;\ K/k)\ .$$

(The product is extended over all the infinite primes of k.)

Then we have the following properties of $\Gamma(s,\chi;\ K/k)$.

1. $\Gamma(s,\chi+\chi';\ K/k)=\Gamma(s,\chi;\ K/k)\Gamma(s,\chi';\ K/k)$,
2. $\Gamma(s,\chi_\psi;\ K/\chi)=\Gamma(s,\psi;\ K/k)$,
3. $\Gamma(s,\chi;\ \bar{K}/k)=\Gamma(s,\chi;\ K/k)$.

Moreover, if $K=k$ and χ is irreducible, then $\Gamma(s,\chi;\ k/k)=\Gamma(s,\chi)$ is the Γ-factor of Hecke's L-function $L(s,\chi)$.

Now, let $\mathfrak{f}(\chi;\ K/k)$ be the conductor of χ and put

$$A(\chi;\ K/k)=\left(\frac{\varDelta}{\pi^n}\right)^{\chi(1)}N(\mathfrak{f}(\chi;\ K/k))\ ,$$

and

$$\xi(s,\chi;\ K/k)=[A(\chi;\ K/k)]^{s/2}\Gamma(s,\chi;\ K/k)L(s,\chi;K/k)\ .$$

Then the function $\xi(s,\chi;\ K/k)$ satisfies the following equations in virtue of the known properties of $L(s, \chi; K/k)$ and $\mathfrak{f}(\chi; K/k)$.

1. $\xi(s, \chi+\chi'; K/k)=\xi(s, \chi; K/k)\xi(s, \chi'; K/k)$,
2. $\xi(s, \chi_{\psi}; K/k)=\xi(s, \psi; K/k)$,
3. $\xi(s, \chi; \bar{K}/k)=\xi(s, \chi; K/k)$.

If $K=k$ and χ is irreducible, then the function $L(s, \chi; K/k)=L(s, \chi)$ is Hecke's L-function, so we have as remarked in §1, the following functional equation of $L(s, \chi)$:

$$\xi(s, \chi)=W(\chi)\xi(1-s, \bar{\chi}) .$$

In general cases, χ is a sum with integral coefficients of the characters induced by characters of degree 1 of $C_{K'}$ where K's are intermediate fields between k and A_K. So we obtain in the same way as Artin, the following functional equation

$$\xi(s, \chi; K/k)=W(\chi)\xi(1-s, \bar{\chi}; K/k) .$$

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