

*Toric Resolution of Singularities in a Certain Class
of C^∞ Functions and Asymptotic Analysis
of Oscillatory Integrals*

By Joe KAMIMOTO and Toshihiro NOSE

Abstract. In a seminal work of A. N. Varchenko, the behavior at infinity of oscillatory integrals with real analytic phase is precisely investigated by using the theory of toric varieties based on the geometry of the Newton polyhedron of the phase. The purpose of this paper is to generalize his results to the case that the phase is contained in a certain class of C^∞ functions. The key in our analysis is a toric resolution of singularities in the above class of C^∞ functions. The properties of poles of local zeta functions, which are closely related to the behavior of oscillatory integrals, are also studied under the associated situation.

Contents

1. Introduction	426
2. Newton Polyhedra and the Classes $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$	431
2.1. Polyhedra	431
2.2. Newton polyhedra	432
2.3. The γ -part	432
2.4. The classes $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$	434
2.5. Examples	435
3. Main Results	436
3.1. Known results	437

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3.2. Our results	438
4. Lemmas on Polyhedra	440
5. Remarks on the γ -Part	442
6. Properties of $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$	446
6.1. Elementary properties	446
6.2. Equivalent conditions	447
6.3. Denjoy-Carleman classes	451
6.4. Remarks on the definition of the γ -part	452
7. Toric Varieties Constructed from Polyhedra	453
7.1. Cones and fans	453
7.2. The fan associated with P and its simplicial subdivision	454
7.3. Construction of toric varieties	455
8. Toric Resolution of Singularities in the Class $\hat{\mathcal{E}}(U)$	456
8.1. Preliminaries	456
8.2. Resolution of singularities	461
9. Poles of Local Zeta Functions	463
9.1. Candidate poles	464
9.2. Poles of $J_{\pm}(s)$ on negative integers	470
9.3. The first coefficients	471
10. Proofs of the Theorems in Section 3	477
10.1. Relationship between $I(t; \varphi)$ and $Z_{\pm}(s; \varphi)$	477
10.2. Proofs of Theorems 3.3, 3.5 and 3.7	478
10.3. The first coefficient in the asymptotics (3.1)	478
11. Examples	479
11.1. Example 1	479
11.2. Example 2	480
11.3. Example 3	481
11.4. Example 4	482
References	482

1. Introduction

In this paper, we investigate the asymptotic behavior of oscillatory integrals, that is, integrals of the form

$$(1.1) \quad I(t; \varphi) = \int_{\mathbb{R}^n} e^{itf(x)} \varphi(x) dx,$$

for large values of the real parameter t , where f and φ are real-valued C^∞ smooth functions defined near the origin and the support of φ is contained in a small neighborhood of the origin in \mathbb{R}^n . Here f and φ are called the *phase* and the *amplitude*, respectively.

By the principle of stationary phase, the main contribution in the behavior of the integral (1.1) as $t \rightarrow +\infty$ is given by the local properties of the phase on neighborhoods of its critical points. When the phase has a nondegenerate critical point, i.e., the $n \times n$ matrix $\nabla^2 f$ is invertible, the Morse lemma implies that there exists a coordinate where f is locally expressed as $x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2$ with some k . This fact easily gives the asymptotic expansion of $I(t; \varphi)$ through the computation of Fresnel integrals. On the other hand, the situation at degenerate critical points is quite different. There are very few cases that direct computations are available for the analysis of $I(t; \varphi)$ by using a smooth change of coordinates only. Up to now, there have been many studies about the degenerate case, which develop more intrinsic and ingenious methods to see the behavior of $I(t; \varphi)$ (see [41], [37], [38], [34], [12], [35], [7], [14], [15], [16], [17], [22], [23], [24], [5], [4], [32], etc.). Analogous studies about oscillatory integral operators are seen in [38], [33], [36], [19], [13], [18], etc.

The following classical results need the hypothesis of the real analyticity of the phase. By using Hironaka's resolution of singularities, it is known (c.f. [26], [30]) that $I(t; \varphi)$ admits an asymptotic expansion (see (3.1) in Section 3). More precisely, Varchenko [41] investigates the leading term of this asymptotic expansion by using the theory of toric varieties based on the geometry of the Newton polyhedron of the phase under a certain nondegeneracy condition on the phase (see Theorem 3.1 in Section 3). Since his study, the investigation of the behavior of oscillatory integrals has been more closely linked with the theory of singularities. Refer to the excellent exposition [2] for studies in this direction. The investigation under the nondegeneracy hypothesis has been developed in [8], [9], [6], [7].

In the same paper [41], Varchenko investigates the two-dimensional case in more detail. When the phase is real analytic, he proves the existence of a good coordinate system, which is called an *adapted coordinate*, and gives analogous results without the nondegeneracy hypothesis by using this coordinate. His proof is based on a two-dimensional resolution of singularities result. Notice that the adapted coordinate may not exist in dimensions

higher than two. Later, his two-dimensional results have been improved in [34], [12], [35], [15], [22], [23], [24], which are inspired by the work of Phong and Stein in their seminal paper [33].

In higher dimensions, recent interesting studies [14], [16], [17], [5] also emphasize the importance of the relationship between behavior of oscillatory integrals and resolution of singularities for the phase. Observing these studies and the two-dimensional works mentioned above, we see that explicit and elementary approaches to the resolution of singularities are useful for quantitative investigation of the decay rate of oscillatory integrals.

In this paper, we generalize the above results of Varchenko [41], under the same nondegeneracy hypothesis, to the case that the phase is contained in a certain class of C^∞ functions including real analytic functions. This class is denoted by $\hat{\mathcal{E}}(U)$, where U is an open neighborhood of the origin in \mathbb{R}^n . Under the nondegeneracy condition, we construct a toric resolution of singularities in the class $\hat{\mathcal{E}}(U)$. Using this resolution, we show that $I(t; \varphi)$ has an asymptotic expansion of the same form as in the real analytic phase case and succeed to generalize the above results of Varchenko. Moreover, we give an explicit formula of the coefficient of the leading term of the above asymptotic expansion.

Let us explain the properties of the class $\hat{\mathcal{E}}(U)$ in more detail. In the above earlier many investigations, the function γ -part, which corresponds to each face γ of the Newton polyhedron of the phase, plays an important role. By using summation, the γ -part is simply defined as a function for every face γ in the real analytic case. From the viewpoint of this definition, the γ -part is considered as a formal power series when γ is noncompact and the phase is only smooth. This γ -part may not become a function, so it is not useful for our analysis. From convex geometrical points of view (c.f. [42]), we give another definition of the γ -part, which always becomes a function defined near the origin (see Section 2.3). This definition is a natural generalization of that in the real analytic case. We remark that not all smooth functions admit the γ -part for every face γ of their Newton polyhedra in our sense. The class $\hat{\mathcal{E}}(U)$ is defined to be the set of C^∞ functions admitting the γ -part for every face γ of its Newton polyhedron (see Section 2.4). Many kinds of C^∞ functions are contained in this class. In particular, it contains the Denjoy-Carleman quasianalytic classes, which are interesting classes of C^∞ functions and have been studied from various points of view (c.f. [3],

[39]). The most important property of the class $\hat{\mathcal{E}}(U)$ is that its element is generated by finite monomials whose powers are contained in its Newton polyhedron. This property plays a crucial role in the construction of a toric resolution of singularities in the class $\hat{\mathcal{E}}(U)$.

There have been many attempts to understand the behavior of oscillatory integrals with *smooth* phases. Explicit asymptotic expansions of $I(t; \varphi)$ are computed in the case of one-dimensional nonflat phases (see the monograph [38]) and in the case of finite line type convex phases (see [37]). In the two-dimensional case, strong results are also obtained by using an adapted coordinate, which exists even in the smooth case, in [15], [22], [23], [24]. (As for analogous studies about oscillatory integral operators, the one-dimensional case has been completely understood when the phase is nonflat in [36], [13].) On the other hand, a simple example given by Iosevich and Sawyer [25] in two dimensions shows that some kind of restriction like the class $\hat{\mathcal{E}}(U)$ is necessary to generalize the results of Varchenko directly (see Section 11.4). The smooth case is difficult to deal with because analytical information of functions around critical points does not always appear in the geometry of their Newton polyhedra.

It is known (see, for instance, [21], [2] and Section 10.1 in this paper) that the asymptotic analysis of the oscillatory integral (1.1) can be reduced to an investigation of the poles of the functions $Z_{\pm}(s; \varphi)$ (see (9.1) in Section 9), which are similar to the local zeta function

$$Z(s; \varphi) = \int_{\mathbb{R}^n} |f(x)|^s \varphi(x) dx,$$

where f, φ are as in (1.1) and f vanishes at a critical point. The substantial analysis in this paper is to investigate the properties of poles of the local zeta function $Z(s; \varphi)$ and the above functions $Z_{\pm}(s; \varphi)$ by using the geometrical properties of the Newton polyhedron of the function f . We also give new results relating to the poles of these functions.

This paper is organized as follows. In Section 2, after explaining some important notions in convex geometry, we give the definition of Newton polyhedra and explain related important words in our analysis. Moreover, after generalizing the concept of the γ -part, we introduce the classes $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$ of C^{∞} functions. In Section 3, we state main results relating to oscillatory integrals. Some parts of the results are new even when the phase is real analytic. In Section 4, we consider elementary convex geometrical

properties of polyhedra, which are useful in this paper. In Section 5, basic properties of the generalized γ -part are investigated. In Section 6, more detailed properties of the class $\hat{\mathcal{E}}(U)$ are investigated, which play important roles in the resolution of singularities. In Section 7, we overview the method to construct toric varieties from a given polyhedron. In Section 8, we construct a resolution of singularities in the class $\hat{\mathcal{E}}(U)$ under the nondegeneracy condition in [41]. In Section 9, we investigate the properties of poles of the local zeta function $Z(s; \varphi)$ and the functions $Z_{\pm}(s; \varphi)$ by using the resolution of singularities constructed in Section 8. In Section 10, we give proofs of theorems on the behavior of oscillatory integrals stated in Section 3. Furthermore, we give explicit formulae for the leading term of the asymptotic expansion of $I(t; \varphi)$. In Section 11, we give concrete computations for some examples, which are not directly covered in earlier investigations.

Some of the results in this paper have been announced in [28].

Notation and Symbols.

- (i) We denote by $\mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$ the subsets consisting of all nonnegative numbers in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, respectively.
- (ii) We use the multi-index as follows. For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, define

$$\begin{aligned} |x| &= \sqrt{|x_1|^2 + \dots + |x_n|^2}, & \langle x, y \rangle &= x_1 y_1 + \dots + x_n y_n, \\ x^\alpha &= x_1^{\alpha_1} \dots x_n^{\alpha_n}, & \partial^\alpha &= \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n}, \\ \langle \alpha \rangle &= \alpha_1 + \dots + \alpha_n, & \alpha! &= \alpha_1! \dots \alpha_n! \quad (0! = 1! = 1). \end{aligned}$$

- (iii) For $A, B \subset \mathbb{R}^n$ and $c \in \mathbb{R}$, we set

$$A + B = \{a + b \in \mathbb{R}^n : a \in A \text{ and } b \in B\}, \quad c \cdot A = \{ca \in \mathbb{R}^n : a \in A\}.$$

- (iv) For a finite set A , $\#A$ means the cardinality of A .

- (v) For a nonnegative real number r and a subset I in $\{1, \dots, n\}$, the map $T_I^r : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by

$$(1.2) \quad (z_1, \dots, z_n) = T_I^r(x_1, \dots, x_n) \quad \text{with} \quad z_j := \begin{cases} r & \text{for } j \in I, \\ x_j & \text{otherwise.} \end{cases}$$

We define $T_I := T_I^0$. For a set A in \mathbb{R}^n , the image of A by T_I is denoted by $T_I(A)$. When $A = \mathbb{R}^n$ or \mathbb{Z}_+^n , its image is expressed as

$$(1.3) \quad T_I(A) = \{x \in A : x_j = 0 \text{ for } j \in I\}.$$

(vi) For a C^∞ function f , we denote by $\text{Supp}(f)$ the support of f , i.e., $\text{Supp}(f) = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$.

(vii) For $x \in \mathbb{R}$, $\alpha > 0$, the value of $e^{-1/|x|^\alpha}$ at the origin is defined by 0. Then $e^{-1/|x|^\alpha}$ is a C^∞ function defined on \mathbb{R} .

2. Newton Polyhedra and the Classes $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$

2.1. Polyhedra

Let us explain fundamental notions in the theory of convex polyhedra, which are necessary for our study. Refer to [42] for general theory of convex polyhedra.

For $(a, l) \in \mathbb{R}^n \times \mathbb{R}$, let $H(a, l)$ and $H^+(a, l)$ be a hyperplane and a closed halfspace in \mathbb{R}^n defined by

$$(2.1) \quad \begin{aligned} H(a, l) &:= \{x \in \mathbb{R}^n : \langle a, x \rangle = l\}, \\ H^+(a, l) &:= \{x \in \mathbb{R}^n : \langle a, x \rangle \geq l\}, \end{aligned}$$

respectively. A (*convex rational*) *polyhedron* is an intersection of closed halfspaces: a set $P \subset \mathbb{R}^n$ presented in the form $P = \bigcap_{j=1}^N H^+(a^j, l_j)$ for some $a^1, \dots, a^N \in \mathbb{Z}^n$ and $l_1, \dots, l_N \in \mathbb{Z}$.

Let P be a polyhedron in \mathbb{R}^n . A pair $(a, l) \in \mathbb{Z}^n \times \mathbb{Z}$ is said to be *valid* for P if P is contained in $H^+(a, l)$. A *face* of P is any set of the form $F = P \cap H(a, l)$, where (a, l) is valid for P . Since $(0, 0)$ is always valid, we consider P itself as a trivial face of P ; the other faces are called *proper faces*. Conversely, it is easy to see that any face is a polyhedron. Considering the valid pair $(0, -1)$, we see that the empty set is always a face of P . Indeed, $H^+(0, -1) = \mathbb{R}^n$, but $H(0, -1) = \emptyset$. The *dimension* of a face F is the dimension of its affine hull of F (i.e., the intersection of all affine flats that contain F), which is denoted by $\dim(F)$. The faces of dimensions 0, 1 and $\dim(P) - 1$ are called *vertices*, *edges* and *facets*, respectively. The *boundary* of a polyhedron P , denoted by ∂P , is the union of all proper faces of P . For a face F , ∂F is similarly defined.

2.2. Newton polyhedra

Let f be a real-valued C^∞ function defined on an open neighborhood of the origin in \mathbb{R}^n . Denote by $\hat{f}(x)$ the Taylor series of f at the origin, i.e.,

$$(2.2) \quad \hat{f}(x) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha \quad \text{with } c_\alpha = \frac{\partial^\alpha f(0)}{\alpha!}.$$

The *Newton polyhedron* of f is the integral polyhedron:

$$\Gamma_+(f) = \text{the convex hull of the set } \bigcup \{ \alpha + \mathbb{R}_+^n : c_\alpha \neq 0 \} \text{ in } \mathbb{R}_+^n$$

(i.e., the intersection of all convex sets which contain $\bigcup \{ \alpha + \mathbb{R}_+^n : c_\alpha \neq 0 \}$). It is known (cf. [42]) that the Newton polyhedron $\Gamma_+(f)$ is a polyhedron. The union of the compact faces of the Newton polyhedron $\Gamma_+(f)$ is called the *Newton diagram* $\Gamma(f)$ of f , while the boundary of $\Gamma_+(f)$ is denoted by $\partial\Gamma_+(f)$. The *principal part* of f is defined by $f_*(x) = \sum_{\alpha \in \Gamma(f) \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$. Note that $\Gamma_+(f) = \Gamma_+(f_*)$.

A C^∞ function f is said to be *convenient* if the Newton polyhedron $\Gamma_+(f)$ intersects all the coordinate axes.

We assume that f is *nonflat*, i.e., $\Gamma_+(f) \neq \emptyset$. Let q_* be the point at which the line $\alpha_1 = \dots = \alpha_n$ in \mathbb{R}^n intersects the boundary of $\Gamma_+(f)$. The coordinate of q_* is called the *Newton distance* of $\Gamma_+(f)$, which is denoted by $d(f)$, i.e., $q_* = (d(f), \dots, d(f))$. The face whose relative interior contains q_* is called the *principal face* of $\Gamma_+(f)$, which is denoted by τ_* . The codimension of τ_* is called the *Newton multiplicity* of $\Gamma_+(f)$, which is denoted by $m(f)$. Here, when q_* is a vertex of $\Gamma_+(f)$, τ_* is the point q_* and $m(f) = n$.

2.3. The γ -part

Let f be a real-valued C^∞ function defined on an open neighborhood V of the origin in \mathbb{R}^n with the Taylor series (2.2), $P \subset \mathbb{R}_+^n$ a nonempty polyhedron containing $\Gamma_+(f)$ and γ a face of P . Note that this polyhedron P satisfies the condition $P + \mathbb{R}_+^n \subset P$, which will be shown in Lemma 4.1, below. We say that f *admits the γ -part* on an open neighborhood $U \subset V$ of the origin if for any $x \in U$ the limit:

$$(2.3) \quad \lim_{t \rightarrow 0} \frac{f(t^{\alpha_1} x_1, \dots, t^{\alpha_n} x_n)}{t^l}$$

exists for *all* valid pairs $(a, l) = (a_1, \dots, a_n, l) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ defining γ (i.e., $H(a, l) \cap P = \gamma$). Proposition 5.2 (iii), below, implies that when f admits the γ -part, the above limits take the same value for any (a, l) , which is denoted by $f_\gamma(x)$. We consider f_γ as the function on U , which is called the γ -part of f on U .

REMARK 2.1. We give many remarks on the γ -part. Some of them are not trivial and they will be shown later.

- (i) The readers might feel that “all” is too strict in the above definition of the admission of the γ -part. Actually, even if “all” is replaced by “some” in the definition, this exchange does not affect the analysis in this paper. This subtle issue will be discussed in Section 6.4.
- (ii) If f admits the γ -part f_γ on U , then f_γ has the quasihomogeneous property:

$$(2.4) \quad f_\gamma(t^{a_1}x_1, \dots, t^{a_n}x_n) = t^l f_\gamma(x) \quad \text{for } t \in (0, 1] \text{ and } x \in U,$$

where (a, l) is a valid pair defining γ (see Lemma 5.4 (i)).

- (iii) The above γ -part f_γ is a C^∞ function defined on U (see Proposition 5.2 (iv)). Moreover, by using the above quasihomogeneity (2.4), f_γ can be uniquely extended to a C^∞ function with the property (2.4) (with $t \in \mathbb{R}$) defined on much wider regions (see Lemma 5.4 (ii)). This function is also denoted by f_γ .
- (iv) When $\gamma = P$, f always admits the γ -part on V and $f_P = f$. In fact, consider the case when $(a, l) = (0, 0)$.
- (v) For a compact face γ of $\Gamma_+(f)$, f always admits the γ -part near the origin and $f_\gamma(x)$ equals the polynomial $\sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$, which is the same as the well-known γ -part of f in [41], [2]. Note that γ is a compact face if and only if every valid pair $(a, l) = (a_1, \dots, a_n)$ defining γ satisfies $a_j > 0$ for any j (see Proposition 5.2 (iii)).
- (vi) Let f be a smooth function and γ a noncompact face of $\Gamma_+(f)$. Then, f does not admit the γ -part in general (see Section 2.5). If f admits the γ -part, then the Taylor series of $f_\gamma(x)$ at the origin is $\sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$, where the Taylor series of f is as in (2.2).

- (vii) Let f be a smooth function and γ a face defined by the intersection of $\Gamma_+(f)$ and some coordinate hyperplane. More exactly, there exists a nonempty subset $I \subset \{1, \dots, n\}$ such that $\gamma = \Gamma_+(f) \cap T_I(\mathbb{R}_+^n)$ (see (1.3)). Although γ is a noncompact face if $\gamma \neq \emptyset$, f always admits the γ -part. Indeed, $T_I(\mathbb{R}_+^n)$ can be regarded as a face of \mathbb{R}_+^n , which is expressed by $\mathbb{R}_+^n \cap H(a_I, 0)$, where $a_I = (a_1, \dots, a_n) \in \mathbb{R}_+^n$ satisfies that $a_j > 0$ iff $j \in I$. Thus every valid pair defining γ takes the form $(a, l) = (a_I, 0)$, so $l = 0$ implies the existence of the limit (2.3) (see Section 2.5.2, below).
- (viii) If f is real analytic on V and γ is a face of $\Gamma_+(f)$, then f admits the γ -part f_γ on U and, moreover, f_γ is equal to a convergent power series $\sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$ on U (see Lemma 5.3).
- (ix) An example, which shows the case of non-admission of the γ -part, will be given in Section 2.5. This example also indicates that for a face γ of P , the condition $\gamma \cap \Gamma_+(f) = \emptyset$ may not imply that f admits the γ -part: $f_\gamma \equiv 0$.

2.4. The classes $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$

Let $P \subset \mathbb{R}_+^n$ be a polyhedron (possibly an empty set) satisfying $P + \mathbb{R}_+^n \subset P$ if $P \neq \emptyset$ and U an open neighborhood of the origin. Denote by $\mathcal{E}[P](U)$ the set of C^∞ functions defined on U whose Newton polyhedra are contained in P . Moreover, when $P \neq \emptyset$, we denote by $\hat{\mathcal{E}}[P](U)$ the set of the elements f of $\mathcal{E}[P](U)$ admitting the γ -part on U for any face γ of P . We set $\hat{\mathcal{E}}[\emptyset](U) = \{0\}$, i.e., the set consisting of only the function identically equaling zero on U . We define

$$\hat{\mathcal{E}}(U) = \{f \in C^\infty(U) : f \in \hat{\mathcal{E}}[\Gamma_+(f)](U)\}.$$

REMARK 2.2. In the definition of $\hat{\mathcal{E}}[P](U)$, “any face” can be replaced by “any noncompact facet” (see Section 6.4).

REMARK 2.3. The class $\hat{\mathcal{E}}(U)$ contains many kinds of C^∞ functions. Here U is a small open neighborhood of the origin in \mathbb{R}^n .

- (i) The function identically equaling zero on U is contained in $\hat{\mathcal{E}}(U)$. This easily follows from the definition.

- (ii) Every real analytic function defined on U belongs to $\hat{\mathcal{E}}(U)$. This follows from Remark 2.1 (viii).
- (iii) Every convenient C^∞ function defined on U belongs to $\hat{\mathcal{E}}(U)$. This follows from Remark 2.1 (v), (vii).
- (iv) In the one-dimensional case, every nonflat C^∞ function defined on U belongs to $\hat{\mathcal{E}}(U)$. This is a particular case of the above (iii).
- (v) The Denjoy-Carleman classes $\mathcal{E}_M(U)$ are contained in $\hat{\mathcal{E}}(U)$. See Proposition 6.10.

REMARK 2.4. Tougeron [40] shows that if a C^∞ function f has a critical point of “finite multiplicity” (see [1], p.121), then f can be expressed as a polynomial around the critical point by using smooth coordinate changes. But, there are many elements in $\hat{\mathcal{E}}(U)$ or $\hat{\mathcal{E}}[P](U)$ which do not satisfy this hypothesis. (Our classes contain all real analytic functions.)

REMARK 2.5. The classes $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$ are useful for the investigation of the behavior of *weighted* oscillatory integrals:

$$\tilde{I}(t; \varphi) = \int_{\mathbb{R}^n} e^{itf(x)} g(x) \varphi(x) dx,$$

where f, φ are the same as in (1.1) and g is a *weight* function satisfying some conditions (see [27], [32], [29]).

More detailed properties of the classes $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$ are investigated in Section 6 below.

2.5. Examples

2.5.1 Example 1

Let us consider the following two-dimensional example.

$$(2.5) \quad \begin{aligned} f_k(x) &= f_k(x_1, x_2) = x_1^2 x_2^2 + x_1^k e^{-1/x_2^2}, \quad k \in \mathbb{Z}_+; \\ P &= \{(\alpha_1, \alpha_2) \in \mathbb{R}_+^2 : \alpha_1 \geq 1, \alpha_2 \geq 1\}. \end{aligned}$$

Of course, f_k is not real analytic around the origin. The set of the proper faces of $\Gamma_+(f_k)$ and P consists of $\gamma_1, \gamma_2, \gamma_3$ and τ_1, τ_2, τ_3 , where

$$(2.6) \quad \begin{aligned} \gamma_1 &= \{(2, \alpha_2) : \alpha_2 \geq 2\}, \quad \gamma_2 = \{(2, 2)\}, \quad \gamma_3 = \{(\alpha_1, 2) : \alpha_1 \geq 2\}, \\ \tau_1 &= \{(1, \alpha_2) : \alpha_2 \geq 1\}, \quad \tau_2 = \{(1, 1)\}, \quad \tau_3 = \{(\alpha_1, 1) : \alpha_1 \geq 1\}. \end{aligned}$$

It is easy to see that if $j = 2, 3$, then f_k admits the γ_j -part and τ_j -part near the origin for all $k \in \mathbb{Z}_+$ and they are written as $(f_k)_{\gamma_2}(x) = (f_k)_{\gamma_3}(x) = x_1^2 x_2^2$ and $(f_k)_{\tau_2}(x) = (f_k)_{\tau_3}(x) \equiv 0$. Consider the γ_1 -part and τ_1 -part of f_k for $k \in \mathbb{Z}_+$. The situation depends on the parameter k as follows.

- $(f_0)_{\gamma_1}$ and $(f_0)_{\tau_1}$ cannot be defined.
- $(f_1)_{\gamma_1}$ cannot be defined but $(f_1)_{\tau_1}(x) = x_1 e^{-1/x_2^2}$.
- $(f_2)_{\gamma_1}(x) = f_2(x)$ and $(f_2)_{\tau_1}(x) \equiv 0$.
- If $k \geq 3$, then $(f_k)_{\gamma_1}(x) = x_1^2 x_2^2$ and $(f_k)_{\tau_1}(x) \equiv 0$.

From the above, we see that $f_k \in \hat{\mathcal{E}}(U)$ if and only if $k \geq 2$; $f_k \in \hat{\mathcal{E}}[P](U)$ if and only if $k \geq 1$. Notice that $\tau_1 \cap \Gamma_+(f_1) = \emptyset$ but $(f_1)_{\tau_1}(x) = x_1 e^{-1/x_2^2} \neq 0$ (see Remark 2.1 (ix)).

2.5.2 Example 2

Let f be a smooth function defined near the origin in \mathbb{R}^3 . As mentioned in Remark 2.1 (vii), when γ is a face defined by the intersection of $\Gamma_+(f)$ and some coordinate hyperplane, f always admits the γ -part. But, the condition that γ is a noncompact face of $\Gamma_+(f)$ contained in some coordinate hyperplane is not sufficient for the admission of the γ -part. Indeed, the following three-dimensional example shows this subtle situation:

$$f(x_1, x_2, x_3) = x_1^2 + e^{-1/x_2^2} \text{ and } \gamma = \{(2, \alpha_2, 0) : \alpha_2 \geq 0\}.$$

It is easy to check that the valid pair $((1, 0, 1), 2)$ defines the face γ , but the limit (2.3) does not exist when $x_2 \neq 0$. This implies that the above f does not belong to the class $\hat{\mathcal{E}}(U)$.

3. Main Results

Let us explain our results relating to the behavior of the oscillatory integral $I(t; \varphi)$ in (1.1) as $t \rightarrow +\infty$.

Throughout this section, the functions f, φ satisfy the following conditions. Let U be an open neighborhood of the origin in \mathbb{R}^n .

- (A) $f : U \rightarrow \mathbb{R}$ is a C^∞ smooth function satisfying that $f(0) = 0, \nabla f(0) = (0, \dots, 0)$ and $\Gamma_+(f) \neq \emptyset$;
- (B) $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^∞ smooth function whose support is contained in U .

3.1. Known results

As mentioned in the Introduction, by using Hironaka’s resolution of singularities [20], an asymptotic expansion for $I(t; \varphi)$ is obtained (c.f. [26], [30]). To be more specific, if f is real analytic on U and the support of φ is contained in a sufficiently small neighborhood of the origin, then the integral $I(t; \varphi)$ has an asymptotic expansion of the form

$$(3.1) \quad I(t; \varphi) \sim \sum_{\alpha} \sum_{k=1}^n C_{\alpha k}(\varphi) t^{\alpha} (\log t)^{k-1} \quad \text{as } t \rightarrow +\infty,$$

where α runs through a finite number of arithmetic progressions, not depending on the amplitude φ , which consist of negative rational numbers. We are interested in the largest α occurring in the asymptotic expansion (3.1). Let $S(f)$ be the set of pairs (α, k) such that for each neighborhood of the origin in \mathbb{R}^n , there exists a C^∞ function φ with support in this neighborhood for which $C_{\alpha k}(\varphi) \neq 0$ in the asymptotic expansion (3.1). We denote by $(\beta(f), \eta(f))$ the maximum of the set $S(f)$ under the lexicographic ordering, i.e., $\beta(f)$ is the maximum of values α for which we can find k so that (α, k) belongs to $S(f)$; $\eta(f)$ is the maximum of integers k satisfying that $(\beta(f), k)$ belongs to $S(f)$. We call $\beta(f)$ the *oscillation index* of f and $\eta(f)$ the *multiplicity* of its index. (This multiplicity, less one, is equal to the corresponding multiplicity in [2], p.183.)

The oscillation index and its multiplicity are precisely estimated or determined by Varchenko in [41] and Arnold, Gusein-Zade and Varchenko [2]. Their investigations need the following condition. A C^∞ function f is said to be *nondegenerate* over \mathbb{R} with respect to the Newton polyhedron $\Gamma_+(f)$ if for every compact face γ of $\Gamma_+(f)$, the polynomial f_γ satisfies

$$(3.2) \quad \nabla f_\gamma = \left(\frac{\partial f_\gamma}{\partial x_1}, \dots, \frac{\partial f_\gamma}{\partial x_n} \right) \neq (0, \dots, 0) \quad \text{on the set } U \cap (\mathbb{R} \setminus \{0\})^n.$$

THEOREM 3.1 ([41], [2]). *Suppose that f is real analytic on U and is nondegenerate over \mathbb{R} with respect to its Newton polyhedron. Then one has the following:*

- (i) *The progression $\{\alpha\}$ in (3.1) belongs to finitely many arithmetic progressions, which are obtained by using the theory of toric varieties*

based on the geometry of the Newton polyhedron $\Gamma_+(f)$. (See Remark 3.4, below.)

(ii) $\beta(f) \leq -1/d(f)$.

(iii) If at least one of the following three conditions is satisfied:

(a) $d(f) > 1$;

(b) f is nonnegative or nonpositive on U ;

(c) $1/d(f)$ is not an odd integer and f_{τ_*} does not vanish on $U \cap (\mathbb{R} \setminus \{0\})^n$,

then $\beta(f) = -1/d(f)$ and $\eta(f) = m(f)$.

REMARK 3.2. In the assertion (iii), more precise situation for amplitudes is seen as follows. If $\varphi(0) > 0$ (resp. $\varphi(0) < 0$) and φ is nonnegative (resp. φ is nonpositive) on U and the support of φ is contained in a sufficiently small neighborhood of the origin, then we have $\lim_{t \rightarrow \infty} t^{1/d(f)} \cdot (\log t)^{-m(f)+1} \cdot I(t; \varphi) \neq 0$.

3.2. Our results

Let us explain our results. They need the following condition:

(C) f belongs to the class $\hat{\mathcal{E}}(U)$ and is nondegenerate over \mathbb{R} with respect to its Newton polyhedron.

Since Hironaka's resolution theorem requires the hypothesis of the real analyticity, the existence of the asymptotic expansion of $I(t; \varphi)$ is not trivial in the smooth phase case.

THEOREM 3.3. *If f satisfies the condition (C) and the support of φ is contained in a sufficiently small neighborhood of the origin, then $I(t; \varphi)$ admits an asymptotic expansion of the form (3.1), where $\{\alpha\}$ belongs to the same progressions as in the case that the phase is f_* , which is the principal part of f . (Since f_* is a polynomial, the progressions can be exactly constructed as in [41].)*

REMARK 3.4. To be more specific, the above set $\{\alpha\}$ belongs to the following set:

$$(3.3) \quad \left\{ -\frac{\langle a \rangle + \nu}{l(a)} : \nu \in \mathbb{Z}_+, a \in \tilde{\Sigma}^{(1)} \right\} \cup (-\mathbb{N}),$$

where $l(a)$ and $\tilde{\Sigma}^{(1)}$ are as in Theorem 9.1 in Section 9. We remark that $l(a)$ and $\tilde{\Sigma}^{(1)}$ are determined by the geometry of $\Gamma_+(f) = \Gamma_+(f_*)$ only.

Since the existence of the asymptotic expansion of the form (3.1) has been shown in the above theorem, the oscillation index $\beta(f)$ and its multiplicity $\eta(f)$ for a given f satisfying the condition (C) are defined in a similar fashion to the real analytic case.

The following theorem, which generalizes the assertion (ii) in Theorem 3.1, gives an accurate decay estimate for $I(t; \varphi)$ by using the Newton distance $d(f)$ and the Newton multiplicity $m(f)$.

THEOREM 3.5. *If f satisfies the condition (C) and the support of φ is contained in a sufficiently small neighborhood of the origin, then there exists a positive constant $C(\varphi)$ depending on φ but being independent of t such that*

$$(3.4) \quad |I(t; \varphi)| \leq C(\varphi)t^{-1/d(f)}(\log t)^{m(f)-1} \quad \text{for } t \geq 2.$$

This implies $\beta(f) \leq -1/d(f)$.

REMARK 3.6. The above theorem is not only a generalization to (ii) in Theorem 3.1 but also is a slightly stronger result even if f is real analytic. Indeed, from the argument in [41], [2], the estimate $|I(t; \varphi)| \leq \tilde{C}(\varphi)t^{-1/d(f)}(\log t)^{m(f)}$ for $t \geq 2$ with $\tilde{C}(\varphi) > 0$ is obtained, but more delicate computation of coefficients in the asymptotic expansion (3.1) can improve this estimate.

Next, let us consider the case that the equations $\beta(f) = -1/d(f)$ and $\eta(f) = m(f)$ hold. The following theorem generalizes the assertion (iii) in Theorem 3.1.

THEOREM 3.7. *If f satisfies the condition (C) and at least one of the following three conditions is satisfied:*

- (a) $d(f) > 1$;
 - (b) f is nonnegative or nonpositive on U ;
 - (c) $1/d(f)$ is not an odd integer and f_{τ_*} does not vanish on $U \cap (\mathbb{R} \setminus \{0\})^n$,
- then the equations $\beta(f) = -1/d(f)$ and $\eta(f) = m(f)$ hold.

REMARK 3.8. Even if the principal face τ_* is not compact, the τ_* -part of f is realized as a smooth function from the condition (C).

REMARK 3.9. The condition of amplitudes, which attain the above equalities, is the same as in Remark 3.2.

REMARK 3.10. Under the hypotheses in the above theorem, we will give explicit formulae for the coefficient of the leading term of the asymptotic expansion (3.1) (see Theorem 10.1 in Section 10). Related results have been obtained in [37] for convex finite line type phases and in [6], [7], [15] for real analytic phases.

REMARK 3.11. Some analogous results about Laplace integrals:

$$L(t; \varphi) = \int_{\mathbb{R}^n} e^{-tf(x)} \varphi(x) dx$$

are obtained in [31], where f, φ are the same as in (1.1) and, moreover, f satisfies $f \geq 0$ on U .

4. Lemmas on Polyhedra

Every polyhedron treated in this paper satisfies a condition in the following lemma.

LEMMA 4.1. *Let $P \subset \mathbb{R}_+^n$ be a polyhedron. Then the following conditions are equivalent.*

- (i) $P + \mathbb{R}_+^n \subset P$;
- (ii) *There exists a finite set of pairs $\{(a^j, l_j)\}_{j=1}^N \subset \mathbb{Z}_+^n \times \mathbb{Z}_+$ such that $P = \bigcap_{j=1}^N H^+(a^j, l_j)$;*

(iii) *There exists a finite set of pairs $\{(b^j, m_j)\}_{j=1}^M \subset \mathbb{Z}_+^n \times \mathbb{Z}_+$ such that $P = \bigcap_{j=1}^M H^+(b^j, m_j)$ and $P \cap H(b^j, m_j)$ is a facet of P for all j .*

PROOF. (i) \implies (ii). Suppose that (ii) does not hold. From the definition of the polyhedron, P is expressed as $P = \bigcap_{j=1}^N H^+(a^j, l_j)$ with $(a^j, l_j) \in \mathbb{Z}^n \times \mathbb{Z}$. Here, it may be assumed that the set $A := \{(a^j, l_j)\}_{j=1}^N$ satisfies that $P \cap H(a^j, l_j) \neq \emptyset$ for all j . If $(a, l) \in A$ belongs to $\mathbb{Z}_+^n \times (-\mathbb{N})$, then $P \cap H(a, l) = \emptyset$. On the other hand, if there exists $(a, l) \in A$ with $a \in \mathbb{Z}^n \setminus \mathbb{Z}_+^n$, then the nonempty face $\gamma := P \cap H(a, l)$ satisfies $\gamma + \mathbb{R}_+^n \not\subset H^+(a, l)$, which implies $P + \mathbb{R}_+^n \not\subset P$.

(ii) \implies (i). This implication easily follows from the following: For any $(a, l) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$, if $\alpha \in H^+(a, l)$, then $\alpha + \mathbb{R}_+^n \subset H^+(a, l)$.

(iii) \implies (ii). Obvious.

(ii) \implies (iii). This can be shown by using the *Representation theorem for polytopes* in [42] (Theorem 2.15, p. 65), which can be easily generalized to the case of polyhedra. \square

Hereafter in this section, we assume that $P + \mathbb{R}_+^n \subset P \subset \mathbb{R}_+^n$. For a face γ of P , we define the subsets in $\{1, \dots, n\}$ as

$$(4.1) \quad V(\gamma) := \{k : \gamma + \mathbb{R}_+ e_k \subset \gamma\} \quad \text{and} \quad W(\gamma) := \{1, \dots, n\} \setminus V(\gamma),$$

where $e_k := (0, \dots, \overset{(k)}{1}, \dots, 0)$.

LEMMA 4.2. *Let k be in $\{1, \dots, n\}$. Then the following conditions are equivalent.*

(i) $k \in V(\gamma)$;

(ii) *There exists a point $\alpha \in \gamma$ such that $\alpha + \mathbb{R}_+ e_k \subset \gamma$;*

(iii) *For any valid pair $(a, l) = (a_1, \dots, a_n, l)$ defining γ , $a_k = 0$.*

PROOF. (i) \implies (ii). Obvious.

(ii) \implies (iii). Let (a, l) be an arbitrary valid pair defining γ . Considering that $\alpha \in H(a, l)$ means that α is a solution of the equation $\langle a, \alpha \rangle = l$, we easily see that the following conditions are equivalent:

- (i') $H(a, l) + \mathbb{R}e_k \subset H(a, l)$;
- (ii') There exists a point $\alpha \in H(a, l)$ such that $\alpha + \mathbb{R}_+e_k \subset H(a, l)$;
- (iii') $a_k = 0$.

Since (ii) implies (ii'), the desired implication is shown.

(iii) \implies (i). By using the above equivalences and the condition $P + \mathbb{R}_+^n \subset P$, this implication is shown as follows.

$$\begin{aligned} \gamma + \mathbb{R}_+e_k &\subset (P \cap H(a, l)) + \mathbb{R}_+e_k \\ &\subset (P + \mathbb{R}_+^n) \cap (H(a, l) + \mathbb{R}e_k) \subset P \cap H(a, l) = \gamma. \quad \square \end{aligned}$$

As a corollary of the above lemma, we easily obtain the following. The proofs are omitted.

LEMMA 4.3. *A face γ of P is compact if and only if $V(\gamma) = \emptyset$.*

LEMMA 4.4. *Let γ be a nonempty face of P . For any valid pair $(a, l) = (a_1, \dots, a_n, l)$ defining γ , the following equations hold:*

$$(4.2) \quad V(\gamma) = \{k : a_k = 0\}, \quad W(\gamma) = \{k : a_k \neq 0\}.$$

(These equations mean that the set $\{k : a_k = 0\}$ is independent of the chosen valid pair defining γ .)

5. Remarks on the γ -Part

Throughout this section, we assume that f is a C^∞ function defined on an open neighborhood U of the origin in \mathbb{R}^n , whose Taylor series is $\sum_\alpha c_\alpha x^\alpha$, and $P \subset \mathbb{R}_+^n$ is a polyhedron containing the Newton polyhedron $\Gamma_+(f)$.

The following lemma is ‘‘Taylor’s formula’’, which is useful for the analysis below.

LEMMA 5.1. *Let V, W be subsets in $\{1, \dots, n\}$ such that the disjoint union of V and W is $\{1, \dots, n\}$. Then, f can be expressed as follows: For any $N \in \mathbb{N}$,*

$$(5.1) \quad f(x) = \sum_{\alpha \in A_V(N)} \frac{1}{\alpha!} (\partial^\alpha f)(T_W(x)) x^\alpha + \sum_{\alpha \in B_V(N)} R_\alpha(x) x^\alpha \quad \text{for } x \in U,$$

where

$$R_\alpha(x) = \frac{N}{\alpha!} \int_0^1 (1-t)^{N-1} (\partial^\alpha f)(tT_V(x) + T_W(x)) dt$$

and

$$(5.2) \quad \begin{aligned} A_V(N) &:= \{\alpha \in T_V(\mathbb{Z}_+^n) : \langle \alpha \rangle < N\}, \\ B_V(N) &:= \{\alpha \in T_V(\mathbb{Z}_+^n) : \langle \alpha \rangle = N\}. \end{aligned}$$

Here $T_V(\mathbb{Z}_+^n) = \{\alpha \in \mathbb{Z}_+^n : \alpha_j = 0 \text{ for } j \in V\}$ as in (1.3).

PROOF. For $\varphi \in C^\infty((-\delta, 1 + \delta))$ with $\delta > 0$, the integration by part gives

$$\varphi(1) = \sum_{k=0}^l \frac{\varphi^{(k)}(0)}{k!} + \frac{1}{l!} \int_0^1 (1-t)^l \varphi^{(l+1)}(t) dt.$$

Applying $\varphi(t) = f(tT_V(x) + T_W(x))$ to the above formula, we can easily obtain the lemma. \square

Hereafter, let $V := V(\gamma)$ and $W := W(\gamma)$, where $V(\gamma)$ and $W(\gamma)$ are as in (4.2). We use the following symbols:

$$(5.3) \quad \begin{aligned} H_V(a, l) &:= T_V(H(a, l)) \cap \mathbb{Z}_+^n (= H(a, l) \cap T_V(\mathbb{Z}_+^n)); \\ H_V^+(a, l) &:= T_V(H^+(a, l)) \cap \mathbb{Z}_+^n (= H^+(a, l) \cap T_V(\mathbb{Z}_+^n)); \\ \gamma_V &:= T_V(\gamma) \cap \mathbb{Z}_+^n. \end{aligned}$$

Note that the sets $H_V(a, l)$ and γ_V are finite.

Using Lemma 5.1, we easily see the following fundamental properties of the γ -part.

PROPOSITION 5.2.

- (i) *If γ is a compact face of P , then f admits the γ -part on U .*
- (ii) *If f is real analytic on U , then f admits the γ -part on U for any nonempty face γ of P .*
- (iii) *If f admits the γ -part on U , then the limits (2.3) take the same value for all valid pairs (a, l) defining γ . The value of these limits can be expressed as*

$$(5.4) \quad f_\gamma(x) = \sum_{\alpha \in \gamma_V} \frac{1}{\alpha!} (\partial^\alpha f)(T_W(x)) x^\alpha \quad \text{for } x \in U,$$

where $V = V(\gamma)$ and $W = W(\gamma)$. In particular, if γ is a compact face, then $f_\gamma(x) = \sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$ for $x \in U$.

(iv) The above f_γ is a C^∞ function defined on U . (If f is real analytic on U , then so is f_γ .)

PROOF. By using Lemma 5.1, we have (5.1), where $N \in \mathbb{N}$ satisfies the condition $H_V(a, l) \subset A_V(N)$. Noticing that the following equation holds:

$$(5.5) \quad T_W(t^{a_1}x_1, \dots, t^{a_n}x_n) = T_W(x) \quad \text{for } t \in [0, 1], x \in U,$$

we have

$$(5.6) \quad \begin{aligned} f(t^{a_1}x_1, \dots, t^{a_n}x_n) &= \sum_{\alpha \in A_V(N)} \frac{1}{\alpha!} (\partial^\alpha f)(T_W(x)) x^\alpha t^{\langle a, \alpha \rangle} \\ &+ \sum_{\alpha \in B_V(N)} R_\alpha(t^{a_1}x_1, \dots, t^{a_n}x_n) x^\alpha t^{\langle a, \alpha \rangle}. \end{aligned}$$

When γ is compact (equivalently $V(\gamma) = \emptyset$), the first summation of the right hand side of (5.6) is $\sum_{\alpha \in A(N)} c_\alpha x^\alpha t^{\langle a, \alpha \rangle}$, where $A(N) = \{\alpha \in \mathbb{Z}_+^n : \langle \alpha \rangle < N\}$. On the other hand, consider the case that f is real analytic on U and γ is a general nonempty face of P . Since $\partial^\alpha f$ is also real analytic on U for any $\alpha \in \mathbb{Z}_+^n$, it follows from the shape of the Newton polyhedron of f and the quasianalytic property (see (6.6) in Section 6) that $(\partial^\alpha f)(T_W(x))$ in (5.6) vanishes on U for $\alpha \in A_V(N)$ satisfying $\langle a, \alpha \rangle < l$. In these two cases, we see that the limits (2.3) always exist by using the condition $H_V(a, l) \subset A_V(N)$, which shows (i) and (ii).

Next, suppose that f admits the γ -part on U . Since $\langle a, \alpha \rangle > l$ holds for $\alpha \in B_V(N)$ from the condition $H_V(a, l) \subset A_V(N)$, the existence of the limit (2.3) implies that $(\partial^\alpha f)(T_W(x))$ in (5.6) must vanish for $\alpha \in A_V(N)$ satisfying $\langle a, \alpha \rangle < l$. Moreover, we see that the limit (2.3) is equal to (5.4), which implies (iii). From the form of (5.4), (iv) is easily obtained. \square

By using the expression of f_γ in (5.4), we see the following two properties of f_γ .

LEMMA 5.3. *Let f admit the γ -part f_γ on U and let γ be a face of P . Then the Taylor series of f_γ at the origin is $\sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$. In particular, if f is real analytic on U , then f_γ is equal to the convergent power series $\sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$ on U .*

PROOF. To prove the above, it suffices to show the following:

$$(\partial^\beta f_\gamma)(0) = \begin{cases} (\partial^\beta f)(0) & \text{if } \beta \in \gamma \cap \mathbb{Z}_+^n, \\ 0 & \text{if } \beta \in \mathbb{Z}_+^n \setminus \gamma. \end{cases}$$

If $\beta \in \gamma \cap \mathbb{Z}_+^n$, then

$$\begin{aligned} (\partial^\beta f_\gamma)(x) &= \sum_{\alpha \in \gamma_V} \frac{1}{\alpha!} \cdot \partial^{T_W(\beta)} [(\partial^\alpha f)(T_V(x))] \cdot (\partial^{T_W(\beta)} x^\alpha) \\ &= \frac{1}{\alpha!} \cdot \partial^{T_W(\beta)} [(\partial^{T_V(\beta)} f)(T_W(x))] \cdot \alpha! = (\partial^\beta f)(T_W(x)). \end{aligned}$$

Indeed, the value of $\partial^\beta x^\alpha$ at the origin is $\alpha!$ if $\alpha = \beta$ or it vanishes otherwise. If $\beta \in \mathbb{Z}_+^n \setminus \gamma$, then a similar computation gives $(\partial^\beta f_\gamma)(0) = 0$. \square

LEMMA 5.4. *Let f admit the γ -part f_γ on U . Let $(a, l) = (a_1, \dots, a_n, l) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ be a valid pair defining γ . Then we have the following:*

- (i) f_γ has the quasihomogeneous property:
- (5.7) $f_\gamma(t^{a_1} x_1, \dots, t^{a_n} x_n) = t^l f_\gamma(x)$ for $t \in (0, 1]$ and $x \in U$,
- (ii) f_γ can be uniquely extended to be a C^∞ function defined on the set $U \cup \left(\bigcup_{|r| < \delta} T_{V(\gamma)}^r(\mathbb{R}^n) \right)$ with the property (5.7), where δ is a positive number. This extended function is also denoted by f_γ .

PROOF. The expression (5.4) in Proposition 5.2 and the equation (5.5) give the above quasihomogeneous property in (i). The assertion (ii) easily follows from (i). \square

REMARK 5.5. Under the only assumption of the admission of the γ -part, (5.7) may not hold for all pairs (a, l) satisfying that $\gamma \subset H(a, l)$. Indeed, consider the two-dimensional example:

$$f(x) = x_1^2 x_2^2 + e^{-1/x_2^2}; \quad \gamma = \{(2, 2)\}.$$

Compare to Lemma 6.8, below.

6. Properties of $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$

Throughout this section, every polyhedron $P \subset \mathbb{R}_+^n$ always satisfies that $P + \mathbb{R}_+^n \subset P$ if $P \neq \emptyset$. Let U be an open neighborhood of the origin in \mathbb{R}^n .

6.1. Elementary properties

From the definitions of $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$, the following properties can be directly seen, so we omit the proofs.

PROPOSITION 6.1. *The classes $\mathcal{E}[P](U)$, $\hat{\mathcal{E}}[P](U)$ and $\hat{\mathcal{E}}(U)$ have the following properties:*

- (i) *When $n = 1$ and $P = [p, \infty)$ with $p \in \mathbb{Z}_+$,*
 - (a) $\mathcal{E}[P](U) = \hat{\mathcal{E}}[P](U) = \{x^\alpha \psi(x) : \alpha - p \in \mathbb{Z}_+, \psi \in C^\infty(U)\}$,
 - (b) $\hat{\mathcal{E}}(U) = \{x^\alpha \psi(x) : \alpha \in \mathbb{Z}_+, \psi \in C^\infty(U) \text{ with } \psi(0) \neq 0\} \cup \{0\}$
 $(= \{f \in C^\infty(U) : \Gamma_+(f) \neq \emptyset\} \cup \{0\})$.
- (ii) $\hat{\mathcal{E}}[\mathbb{R}_+^n](U) = \mathcal{E}[\mathbb{R}_+^n](U) = C^\infty(U)$.
- (iii) *If $P_1, P_2 \subset \mathbb{R}_+^n$ are polyhedra with $P_1 \subset P_2$, then $\mathcal{E}[P_1](U) \subset \mathcal{E}[P_2](U)$ and $\hat{\mathcal{E}}[P_1](U) \subset \hat{\mathcal{E}}[P_2](U)$.*
- (iv) $C^\omega(U) \cap \mathcal{E}[P](U) \subset \hat{\mathcal{E}}[P](U) \subset \mathcal{E}[P](U)$. *In particular, $C^\omega(U) \subset \hat{\mathcal{E}}(U) \subset C^\infty(U)$.*
- (v) $\mathcal{E}[P](U)$ and $\hat{\mathcal{E}}[P](U)$ are $C^\infty(U)$ -algebra and, moreover, they are ideals of $C^\infty(U)$.

REMARK 6.2. Unfortunately, the class $\hat{\mathcal{E}}(U)$ is not closed in the following sense.

- (i) (Addition.) Consider the following two-dimensional example: $f(x) = x_1 + x_1^2 + x_1 e^{-1/x_1^2}$; $g(x) = -x_1$. It is easy to see that $f, g \in \hat{\mathcal{E}}(U)$, but $f + g \notin \hat{\mathcal{E}}(U)$.
- (ii) (Change of coordinates.) Consider the following two-dimensional example: $f(x_1, x_2) = (x_1 - x_2)^2 + e^{-1/x_2^2}$. The diffeomorphism $x = \psi(y)$ defined around the origin is defined by $x_1 = y_1 + y_2$ and $x_2 = y_2$. It is easy to see that $f \in \hat{\mathcal{E}}(U)$, but $f \circ \psi \notin \hat{\mathcal{E}}(\psi^{-1}(U))$.

6.2. Equivalent conditions

The following is an important characterization of the class $\hat{\mathcal{E}}[P](U)$, which is considered as a generalization of the property (i-a) in Proposition 6.1. Denote by $\mathcal{S}[P]$ the set of finite sets in $P \cap \mathbb{Z}_+^n$.

PROPOSITION 6.3. *If P is a nonempty polyhedron, then the following conditions are equivalent.*

- (i) f belongs to the class $\hat{\mathcal{E}}[P](U)$;
- (ii) There exist $S \in \mathcal{S}[P]$ and $\psi_p \in C^\infty(U)$ for $p \in S$ such that

$$(6.1) \quad f(x) = \sum_{p \in S} x^p \psi_p(x).$$

Note that the above expression is not unique.

PROOF. It suffices to show the assertion in the proposition in the case of the polyhedron: $P \cap H^+(a, l)$ for any $(a, l) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ instead of P under the assumption that the assertion is satisfied in the case of P . Indeed, since every polyhedron is defined as an intersection of finitely many closed half spaces, an inductive argument gives the proof of the above proposition. Note that the case when $P = \mathbb{R}_+^n$ is obvious.

Since the implication (ii) \implies (i) is easy, we only show the implication (i) \implies (ii).

Now, let us assume that $f(x)$ can be expressed as in (6.1). Let a pair $(a, l) = (a_1, \dots, a_n, z) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ be fixed.

Using Lemma 5.1 with $V = V(\gamma)$ and $W = W(\gamma)$, we have

$$(6.2) \quad \psi_p(x) = \sum_{\alpha \in A_V(N)} C_{p\alpha}(T_{W(\gamma)}(x))x^\alpha + \sum_{\alpha \in B_V(N)} R_{p\alpha}(x)x^\alpha,$$

where $C_{p\alpha}, R_{p\alpha} \in C^\infty(U)$ and $N \in \mathbb{N}$ satisfies the condition: $H_V(a, l) \subset A_V(N)$. Substituting (6.2) into (6.1), we have

$$\begin{aligned} f(x) &= \sum_{p \in S, \alpha \in A_V(N)} C_{p\alpha}(T_{W(\gamma)}(x))x^{p+\alpha} + \sum_{p \in S, \alpha \in B_V(N)} R_{p\alpha}(x)x^{p+\alpha} \\ &=: f_1(x) + f_2(x). \end{aligned}$$

If $\alpha \in B_V(N)$, then the relationship $H_V(a, l) \subset A_V(N)$ implies $p + \alpha \in H_V^+(a, l)$. Therefore, it suffices to show the following: Under the assumption that the limit in (2.3) exists, the coefficients $C_{p\alpha}(T_{W(\gamma)}(x))$ in $f_1(x)$ vanish on U , if $p + \alpha \notin H_V^+(a, l)$.

First, let us give an estimate for the function $f_2(x)$. A simple computation gives

$$f_2(t^{a_1}y_1, \dots, t^{a_n}y_n) = \sum_{p \in S, \alpha \in B_V(N)} C_{p\alpha}(t^{a_1}y_1, \dots, t^{a_n}y_n) t^{\langle a, p+\alpha \rangle} y^{p+\alpha}$$

for $y \in U$ and $t \in (0, \delta)$. The condition: $H_V(a, l) \subset A_V(N)$ implies that $\langle a, T_{V(\gamma)}(\alpha) \rangle \geq l + \epsilon$ with some positive number ϵ for $\alpha \in B_V(N)$. Thus $\langle a, p + \alpha \rangle \geq \langle a, p \rangle + l + \epsilon \geq l + \epsilon$ hold for $p \in S$ and $\alpha \in B_V(N)$. From the above equation, there exist positive numbers C and δ such that

$$(6.3) \quad f_2(t^{a_1}y_1, \dots, t^{a_n}y_n) \leq Ct^{l+\epsilon}$$

for $y \in U$ and $t \in (0, \delta)$.

Next, let us consider the function $f_1(x)$. Noticing (5.5), we have

$$f_1(t^{a_1}y_1, \dots, t^{a_n}y_n) = \sum_{p \in S, \alpha \in A_V(N)} C_{p\alpha}(T_{W(\gamma)}(y)) t^{\langle a, p+\alpha \rangle} y^{p+\alpha}.$$

By using the estimate (6.3), the condition: $H_V(a, l) \subset A_V(N)$ implies that $C_{p\alpha}(T_{W(\gamma)}(y))$ must vanish on the set U if $p + \alpha \notin H^+(a, l)$. \square

Proposition 6.3 implies that the class $\hat{\mathcal{E}}[P](U)$ can be written in the form

$$\hat{\mathcal{E}}[P](U) = \left\{ \sum_{p \in S} x^p \psi_p(x) : S \in \mathcal{S}[P], \psi_p \in C^\infty(U) \text{ for } p \in S \right\}.$$

Next, let us consider an analogous problem in the case of $\hat{\mathcal{E}}(U)$. It seems difficult to express this class in such a simple form. For a polyhedron $P \subset \mathbb{R}_+^n$, denote by $\mathcal{V}(P)$ the set of vertices of P .

LEMMA 6.4. *If f belongs to the class $\hat{\mathcal{E}}(U)$, then f is expressed as $f(x) = \sum_{p \in S} x^p \psi_p(x)$, where $S \in \mathcal{S}[\Gamma_+(f)]$ and $\psi_p \in C^\infty(U)$. Moreover, S contains $\mathcal{V}(\Gamma_+(f))$ and $\psi_p(0) \neq 0$ if $p \in \mathcal{V}(\Gamma_+(f))$.*

PROOF. The expression is directly obtained from the Proposition 6.3 with the definition of $\hat{\mathcal{E}}(U)$. If S does not contain some vertices of $\Gamma_+(f)$ or $\psi_p(0) = 0$ for some vertex p , then $\Gamma_+(\sum_{p \in S} x^p \psi_p(x)) \subsetneq \Gamma_+(f)$, which is a contradiction. \square

The following lemma is a converse of the above lemma.

LEMMA 6.5. *Let P be a nonempty polyhedron. If f belongs to the class $\hat{\mathcal{E}}[P](U)$, which is expressed as in (6.1), where S contains $\mathcal{V}(P)$ and $\psi_p(0) \neq 0$ for $p \in \mathcal{V}(P)$, then f belongs to the class $\hat{\mathcal{E}}(U)$.*

PROOF. The assumption implies $P = \Gamma_+(f)$, which means $f \in \hat{\mathcal{E}}(U)$. \square

For a polyhedron $P \subset \mathbb{R}_+^n$, denote by $\check{\mathcal{E}}[P](U)$ the set of $f \in \hat{\mathcal{E}}[P](U)$ which is expressed as $f(x) = \sum_{p \in S} x^p \psi_p(x)$, where $S \in \mathcal{S}[P]$ satisfies $\mathcal{V}(P) \subset S$ and $\psi_p \in C^\infty(U)$ satisfies that $\psi_p(0) \neq 0$ if $p \in \mathcal{V}(P)$. Let $\tilde{\mathcal{E}}(U)$ be the subset in $C^\infty(U)$ defined by

$$\tilde{\mathcal{E}}(U) := \left\{ \sum_{p \in S} x^p \psi_p(x) : S \in \mathcal{S}[\mathbb{R}_+^n], \psi_p \in C^\infty(U) \right. \\ \left. \text{with } \psi_p(0) \neq 0 \text{ for } p \in S \right\}.$$

In order to understand the structure of the class $\hat{\mathcal{E}}(U)$, we express or compare this class by using the relatively simple classes $\check{\mathcal{E}}[P](U)$ and $\tilde{\mathcal{E}}(U)$.

PROPOSITION 6.6.

- (i) $\hat{\mathcal{E}}(U) = \bigcup_P \check{\mathcal{E}}[P](U)$, where the union is taken for all polyhedra P in \mathbb{R}_+^n .
- (ii) $\tilde{\mathcal{E}}(U) \subset \hat{\mathcal{E}}(U)$. More precisely,
 - (a) When $n = 1$ or 2 , $\tilde{\mathcal{E}}(U) = \hat{\mathcal{E}}(U)$;
 - (b) When $n \geq 3$, $\tilde{\mathcal{E}}(U) \subsetneq \hat{\mathcal{E}}(U)$.

PROOF. The equation in (i) and the inclusion $\tilde{\mathcal{E}}(U) \subset \hat{\mathcal{E}}(U)$ in (ii) easily follow from Lemmas 6.4 and 6.5. Let us show the properties (a), (b) in (ii).

(a) The case when $n = 1$ easily follows from the Proposition 6.1 (i-b).

Consider the case when $n = 2$. Let f belong to the class $\hat{\mathcal{E}}(U)$, which is expressed as in Lemma 6.4. When $p \in S \setminus \mathcal{V}(\Gamma_+(f))$, Taylor's formula implies that the term $x^p \psi_p(x)$ can be written in the form: $x^p \psi_p(x) =$ a polynomial $+$ $\sum_{\alpha \in \mathcal{V}(\Gamma_+(f))} x^\alpha \psi_{p\alpha}(x)$, where $\psi_{p\alpha} \in C^\infty(U)$ with $\psi_{p\alpha}(0) = 0$. Notice that $(\Gamma_+(f) \cap \mathbb{Z}_+^n) \setminus \bigcup_{p \in \mathcal{V}(\Gamma_+(f))} (p + \mathbb{R}_+^n)$ is a finite set in the two-dimensional case. By substituting the above into the expression in Lemma 6.4, f can be written in the form: $f(x) =$ a polynomial $+$ $\sum_{p \in \mathcal{V}(\Gamma_+(f))} x^p \tilde{\psi}_p(x)$, where $\tilde{\psi}_p \in C^\infty(U)$ with $\tilde{\psi}_p(0) \neq 0$. This means that f belongs to the class $\tilde{\mathcal{E}}(U)$.

(b) When $n \geq 3$, consider the example: $g(x) = x_1^2 + \dots + x_{n-1}^2 + x_1 x_2 e^{-1/x_n^2}$. It is easy to see that $g \in \hat{\mathcal{E}}(U)$, but $g \notin \tilde{\mathcal{E}}(U)$. \square

Using Proposition 6.3, we give another expression of f_γ . Compare to (5.4) in Proposition 5.2.

LEMMA 6.7. *Let f belong to $\hat{\mathcal{E}}(U)$, which is expressed as (6.1) in Proposition 6.3, and γ be a nonempty face of P . Then f_γ can be expressed as*

$$(6.4) \quad f_\gamma(x) = \sum_{p \in \gamma \cap S} x^p \psi_p(T_{W(\gamma)}(x)) \quad \text{for } x \in U.$$

PROOF. Let $(a, l) = (a_1, \dots, a_n, l) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ be a valid pair defining γ . Noticing that $a_k > 0$ if and only if $k \in W(\gamma)$, Proposition 6.3 gives

$$\begin{aligned} f_\gamma(x) &= \lim_{t \rightarrow 0} \frac{f(t^{a_1} x_1, \dots, t^{a_n} x_n)}{t^l} \\ &= \lim_{t \rightarrow 0} \sum_{p \in S} t^{\langle a, p \rangle - l} x^p \psi_p(t^{a_1} x_1, \dots, t^{a_n} x_n) = \sum_{p \in \gamma \cap S} x^p \psi_p(T_{W(\gamma)}(x)). \quad \square \end{aligned}$$

When $f \in \hat{\mathcal{E}}[P](U)$, a slightly stronger result for the quasihomogeneous property of f_γ is obtained. Compare to Lemma 5.4 (i).

LEMMA 6.8. *If $f \in \hat{\mathcal{E}}[P](U)$, then the identity (5.7) holds for all valid pairs (a, l) satisfying $\gamma \subset H(a, l)$.*

6.3. Denjoy-Carleman classes

Let us discuss the relationship between the classes $\hat{\mathcal{E}}[P](U)$, $\hat{\mathcal{E}}(U)$ and $\mathcal{E}_M(U)$. Here $\mathcal{E}_M(U)$ are the Denjoy-Carleman quasianalytic classes, which are interesting classes in $C^\infty(U)$ and have been studied from various viewpoints. These classes contain all real analytic functions but are strictly larger, so they also contain functions with non-convergent Taylor expansions. We briefly explain the Denjoy-Carleman quasianalytic classes and their properties. Refer to the paper [3] by Bierstone and Milman and the expository article [39] by Thilliez for more detailed properties and recent studies about these classes.

Let U be an open neighborhood of the origin in \mathbb{R}^n and $M = \{M_0, M_1, M_2, \dots\}$ an increasing sequence of positive real numbers, where $M_0 = 1$. Denote by $\mathcal{E}_M(U)$ the set consisting of all real-valued C^∞ functions satisfying that for every compact set $K \subset U$, there exist positive constants A, B such that

$$(6.5) \quad |\partial^\alpha f(x)| \leq AB^{|\alpha|} \alpha! M_{\langle \alpha \rangle} \quad \text{for any } x \in K \text{ and } \alpha \in \mathbb{Z}_+^n.$$

The class $\mathcal{E}_M(U)$ is said to be *quasianalytic*, if all its elements satisfy the following:

$$(6.6) \quad \text{If } \partial^\alpha f(0) = 0 \text{ for any } \alpha \in \mathbb{Z}_+^n, \text{ then } f \equiv 0 \text{ on } U.$$

Of course, the set of real analytic functions is quasianalytic. We assume that $M = \{M_k\}_{k \in \mathbb{Z}_+}$ satisfies the condition: M is logarithmically convex, i.e.,

$$(6.7) \quad \frac{M_{j+1}}{M_j} \leq \frac{M_{j+2}}{M_{j+1}} \quad \text{for all } j \in \mathbb{Z}_+.$$

This condition implies that $\mathcal{E}_M(U)$ is a ring and $\mathcal{E}_M(U)$ contains the ring $C^\omega(U)$ of real analytic functions on U . The Denjoy-Carleman theorem asserts that under the hypothesis (6.7), $\mathcal{E}_M(U)$ is quasianalytic if and only if

$$(6.8) \quad \sum_{j=0}^{\infty} \frac{M_j}{(j+1)M_{j+1}} = \infty.$$

If M satisfies the conditions (6.7) and (6.8), then $\mathcal{E}_M(U)$ is called a *Denjoy-Carleman class*.

Now, let us show that our classes $\hat{\mathcal{E}}[P](U), \hat{\mathcal{E}}(U)$ contain Denjoy-Carleman classes. For $f \in C^\infty(U)$ and $I \subset \{1, \dots, n\}$, $f \circ T_I$ can be regarded as the C^∞ function of $(n - \#I)$ -variables defined on $U_I := U \cap T_I(\mathbb{R}^n)$, which is denoted by f_I . For a sequence of positive numbers $M = \{M_j\}_{j \in \mathbb{Z}_+}$ and a nonnegative integer k , M^{+k} denotes the shifted sequence $M^{+k} = \{M_{j+k}\}_{j \in \mathbb{Z}_+}$.

LEMMA 6.9. *If f belongs to a Denjoy-Carleman class $\mathcal{E}_M(U)$, then*

- (i) $\partial^\alpha f$ belongs to $\mathcal{E}_{M^{+(\alpha)}}(U)$ for any $\alpha \in \mathbb{Z}_+^n$;
- (ii) f_I belongs to $\mathcal{E}_M(U_I)$ for any subset I in $\{1, \dots, n\}$.

PROOF. Easy. \square

PROPOSITION 6.10. *If $\mathcal{E}_M(U)$ is a Denjoy-Carleman class, then $\mathcal{E}_M(U) \cap \mathcal{E}[P](U)$ is contained in $\hat{\mathcal{E}}[P](U)$ and, in particular, $\mathcal{E}_M(U)$ is contained in $\hat{\mathcal{E}}(U)$.*

PROOF. Let f belong to $\mathcal{E}_M(U) \cap \mathcal{E}[P](U)$, let γ be an arbitrary proper face of P defined by a valid pair (a, l) and let $V = V(\gamma)$ and $W = W(\gamma)$. Then, from Lemma 5.1, f can be expressed as (5.1), where $N \in \mathbb{N}$ satisfies the condition $H_V(a, l) \subset A_V(N)$. It follows from the shape of the Newton polyhedron of f that if $\langle a, \alpha \rangle < l$, then $\partial^\beta(\partial^\alpha f)(0) = 0$ for any $\beta \in T_W(\mathbb{Z}_+^n)$. Since $(\partial^\alpha f)_W \in \mathcal{E}_{M^{+(\alpha)}}(U_W)$ from Lemma 6.9, the quasianalytic property (6.6) implies $(\partial^\alpha f)_W \equiv 0$ on U_W if $\langle a, \alpha \rangle < l$. In the same fashion as in the proof of Proposition 5.2, we can see that f admits the γ -part. \square

6.4. Remarks on the definition of the γ -part

We discuss delicate issues on the definitions of the γ -part and the class $\hat{\mathcal{E}}[P](U)$. Symbols are the same as in Sections 2 and 5. Let us consider the difference of the following two conditions:

- (a) For any $x \in U$, the limit (2.3) exists for *all* valid pairs defining γ .
- (b) For any $x \in U$, the limit (2.3) exists for *some* valid pair defining γ .

Recall that f is said to *admit the γ -part* on U if (a) holds. Here, (a) obviously implies (b), but (b) may not imply (a). Indeed, the following three-dimensional example shows this: $f(x) = x_1 + x_2^2 \exp(-1/x_3^2)$. In this case,

$\Gamma_+(f) = \{(1, 0, 0)\} + \mathbb{R}_+^3$. In the case of the face $\gamma = \{(1, 0, \alpha_3) : \alpha_3 \in \mathbb{R}_+\}$, the limit (2.3) exists for $a = (1, 1, 0), l = 1$, while it does not exist for $a = (3, 1, 0), l = 3$.

When γ is compact, both (a) and (b) always hold from the proof of Proposition 5.2. Moreover, if γ is a facet of P , then the above (a) and (b) are equivalent. Indeed, if (a, l) is some valid pair defining γ , then every valid pair defining γ is expressed as (ca, cl) with $c > 0$. These facts imply that the equivalence of (a) and (b) always holds in the two-dimensional case because every noncompact proper face is a facet.

Next, let us consider the definition of $\hat{\mathcal{E}}[P](U)$. From the proof of Proposition 6.3 and the above argument, the equivalence of (ii) and (iii) in Lemma 4.1 implies that “any face” can be replaced by “any noncompact proper facet” in the definition of $\hat{\mathcal{E}}[P](U)$ in Section 2.4. Therefore, even if (a) is replaced by (b), this exchange does not affect the definition of $\hat{\mathcal{E}}[P](U)$.

7. Toric Varieties Constructed from Polyhedra

Let $P \subset \mathbb{R}_+^n$ be a nonempty n -dimensional polyhedron satisfying $P + \mathbb{R}_+^n \subset P$. In this section, we recall the method to construct a toric variety from a given polyhedron P . Refer to [10], etc. for general theory of toric varieties.

7.1. Cones and fans

A *rational polyhedral cone* $\sigma \subset \mathbb{R}^n$ is a cone generated by finitely many elements of \mathbb{Z}^n . In other words, there are $u_1, \dots, u_k \in \mathbb{Z}^n$ such that

$$\sigma = \{\lambda_1 u_1 + \dots + \lambda_k u_k \in \mathbb{R}^n : \lambda_1, \dots, \lambda_k \geq 0\}.$$

We say that σ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$.

By regarding a cone as a polyhedron in \mathbb{R}^n , the definitions of *dimension*, *face*, *edge*, *facet* for the cone are given by the same way as in Section 2.

The *fan* is defined to be a finite collection Σ of cones in \mathbb{R}^n with the following properties:

- Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone;
- If $\sigma \in \Sigma$ and τ is a face of σ , then $\tau \in \Sigma$;

- If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each.

For a fan Σ , the union $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$ is called the *support* of Σ . For $k = 0, 1, \dots, n$, we denote by $\Sigma^{(k)}$ the set of k -dimensional cones in Σ . The *skeleton* of a cone $\sigma \in \Sigma$ is the set of all of its primitive integer vectors (i.e., with components relatively prime in \mathbb{Z}_+) in the edges of σ . It is clear that the skeleton of $\sigma \in \Sigma^{(k)}$ generates σ itself and that the number of the elements of skeleton is not less than k . Thus, the set of skeletons of the cones belonging to $\Sigma^{(k)}$ is also expressed by the same symbol $\Sigma^{(k)}$.

7.2. The fan associated with P and its simplicial subdivision

We denote by $(\mathbb{R}^n)^*$ the dual space of \mathbb{R}^n with respect to the standard inner product. For $a = (a_1, \dots, a_n) \in (\mathbb{R}^n)^*$, define

$$(7.1) \quad l(a) = \min \{ \langle a, \alpha \rangle : \alpha \in P \}$$

and $\gamma(a) = \{ \alpha \in P : \langle a, \alpha \rangle = l(a) \} (= H(a, l(a)) \cap P)$. We introduce an equivalence relation \sim in $(\mathbb{R}^n)^*$ by $a \sim a'$ if and only if $\gamma(a) = \gamma(a')$. For any k -dimensional face γ of P , there is an equivalence class γ^* which is defined by

$$(7.2) \quad \begin{aligned} \gamma^* &:= \{ a \in (\mathbb{R}^n)^* : \gamma(a) = \gamma \text{ and } a_j \geq 0 \text{ for } j = 1, \dots, n \} \\ & (= \{ a \in (\mathbb{R}^n)^* : \gamma = H(a, l(a)) \cap P \text{ and } a_j \geq 0 \text{ for } j = 1, \dots, n \}.) \end{aligned}$$

Here, $P^* := 0$. The closure of γ^* , denoted by $\overline{\gamma^*}$, is expressed as

$$(7.3) \quad \overline{\gamma^*} = \{ a \in (\mathbb{R}^n)^* : \gamma \subset H(a, l(a)) \cap P \text{ and } a_j \geq 0 \text{ for } j = 1, \dots, n \}.$$

It is easy to see that $\overline{\gamma^*}$ is an $(n - k)$ -dimensional strongly convex rational polyhedral cone in $(\mathbb{R}^n)^*$ and, moreover, the collection of $\overline{\gamma^*}$ gives a fan Σ_0 , which is called the *fan associated with a polyhedron P* . Note that $|\Sigma_0| = \mathbb{R}_+^n$.

It is known that there exists a *simplicial subdivision* Σ of Σ_0 , that is, Σ is a fan satisfying the following properties:

- The fans Σ_0 and Σ have the same support;
- Each cone of Σ lies in some cone of Σ_0 ;
- The skeleton of any cone belonging to Σ can be completed to a base of the lattice dual to \mathbb{Z}^n .

7.3. Construction of toric varieties

Let Σ_0 be the fan associated with P and fix a simplicial subdivision Σ of Σ_0 . For an n -dimensional cone $\sigma \in \Sigma$, let $a^1(\sigma), \dots, a^n(\sigma)$ be the skeleton of σ , ordered once and for all. Here, we set the coordinates of the vector $a^j(\sigma)$ as

$$a^j(\sigma) = (a_1^j(\sigma), \dots, a_n^j(\sigma)).$$

With every such cone σ , we associate a copy of \mathbb{R}^n which is denoted by $\mathbb{R}^n(\sigma)$. We denote by $\pi(\sigma) : \mathbb{R}^n(\sigma) \rightarrow \mathbb{R}^n$ the map defined by $(x_1, \dots, x_n) = \pi(\sigma)(y_1, \dots, y_n)$ with

$$(7.4) \quad x_k = \prod_{j=1}^n y_j^{a_k^j(\sigma)} = y_1^{a_k^1(\sigma)} \dots y_n^{a_k^n(\sigma)}, \quad k = 1, \dots, n.$$

Let Y_Σ be the union of $\mathbb{R}^n(\sigma)$ for σ which are glued along the images of $\pi(\sigma)$. Indeed, for any n -dimensional cones $\sigma, \sigma' \in \Sigma$, two copies $\mathbb{R}^n(\sigma)$ and $\mathbb{R}^n(\sigma')$ can be identified with respect to a rational mapping: $\pi^{-1}(\sigma') \circ \pi(\sigma) : \mathbb{R}^n(\sigma) \rightarrow \mathbb{R}^n(\sigma')$ (i.e., $x \in \mathbb{R}^n(\sigma)$ and $x' \in \mathbb{R}^n(\sigma')$ will coalesce if $\pi^{-1}(\sigma') \circ \pi(\sigma) : x \mapsto x'$). Then it is known that

- Y_Σ is an n -dimensional real algebraic manifold;
- The map $\pi : Y_\Sigma \rightarrow \mathbb{R}^n$ defined on each $\mathbb{R}^n(\sigma)$ as $\pi(\sigma) : \mathbb{R}^n(\sigma) \rightarrow \mathbb{R}^n$ is proper.

The manifold Y_Σ is called the (real) *toric variety* associated with Σ .

The following properties of $\pi(\sigma)$ are useful for the analysis in Section 9. They can be easily seen, so we omit their proofs.

LEMMA 7.1. *The set of the points in $\mathbb{R}^n(\sigma)$ in which $\pi(\sigma)$ is not an isomorphism is a union of coordinate planes.*

LEMMA 7.2. *The Jacobian of the mapping $\pi(\sigma)$ is equal to*

$$(7.5) \quad J_{\pi(\sigma)}(y) = \epsilon \prod_{j=1}^n y_j^{\langle a^j(\sigma) \rangle - 1},$$

where ϵ is 1 or -1 .

8. Toric Resolution of Singularities in the Class $\hat{\mathcal{E}}(U)$

8.1. Preliminaries

Let us show many lemmas which play important roles in the construction of toric resolutions of singularities in the class $\hat{\mathcal{E}}(U)$. Some of them will be useful for the analysis of local zeta functions in Section 9.

Let us explain symbols which will be used in this subsection.

- $P \subset \mathbb{R}_+^n$ is a polyhedron satisfying $P + \mathbb{R}_+^n \subset P$;
- Σ_0 is the fan associated with the polyhedron P ;
- Σ is a simplicial subdivision of Σ_0 ;
- $\Sigma^{(n)}$ consists of n -dimensional cones in Σ ;
- $a^1(\sigma), \dots, a^n(\sigma)$ is the skeleton of $\sigma \in \Sigma^{(n)}$, ordered once and for all;
- $\mathcal{P}(\{1, \dots, n\})$ is the set of all subsets in $\{1, \dots, n\}$;
- $\mathcal{F}(P)$ is the set of nonempty faces of P ;
- When $I \in \mathcal{P}(\{1, \dots, n\})$, we write $J := \{1, \dots, n\} \setminus I$;
- $H(\cdot, \cdot), l(\cdot)$ are as in (2.1), (7.1), respectively.

Let $\sigma \in \Sigma^{(n)}$, $\gamma \in \mathcal{F}(P)$ and $I \in \mathcal{P}(\{1, \dots, n\})$. Define

$$(8.1) \quad \gamma(I, \sigma) := \bigcap_{j \in I} H(a^j(\sigma), l(a^j(\sigma))) \cap P,$$

$$(8.2) \quad I(\gamma, \sigma) := \{j : \gamma \subset H(a^j(\sigma), l(a^j(\sigma)))\}.$$

Here set $\gamma(\emptyset, \sigma) := P$. It is easy to see that $\gamma(I, \sigma) \in \mathcal{F}(P)$ and $I(P, \sigma) = \emptyset$.

LEMMA 8.1. *For $\sigma \in \Sigma^{(n)}$, $\gamma \in \mathcal{F}(P)$, $I \in \mathcal{P}(\{1, \dots, n\})$, we have the following.*

- (i) $\gamma \subset \gamma(I(\gamma, \sigma), \sigma)$ and $\dim(\gamma) \leq n - \#I(\gamma, \sigma)$.
- (ii) $\gamma = \gamma(I, \sigma) \implies I \subset I(\gamma, \sigma) \implies \dim(\gamma) \leq n - \#I$.

PROOF. (i) is directly seen from the definitions of $\gamma(I, \sigma)$ and $I(\gamma, \sigma)$. The first implication in (ii) is shown as follows: $\gamma = \gamma(I, \sigma) \Rightarrow \gamma = \bigcap_{j \in I} H(a^j(\sigma), l(a^j(\sigma))) \cap P \Rightarrow \gamma \subset H(a^j(\sigma), l(a^j(\sigma)))$ for $j \in I \Rightarrow I \subset I(\gamma, \sigma)$. From the inequality in (i), the second implication in (ii) is obvious. \square

Next, consider the case when $\dim(\gamma) = n - \#I(\gamma, \sigma)$. Define

$$(8.3) \quad \Sigma^{(n)}(\gamma) := \{\sigma \in \Sigma^{(n)} : \dim(\gamma) = n - \#I(\gamma, \sigma)\}.$$

Note that $\Sigma^{(n)}(P) = \Sigma^{(n)}$.

LEMMA 8.2. For $\sigma \in \Sigma^{(n)}$, $\gamma \in \mathcal{F}(P)$, $I \in \mathcal{P}(\{1, \dots, n\})$, we have the following. Here γ^* is as in (7.2).

- (i) $\#I(\gamma, \sigma) = \dim(\gamma^* \cap \sigma)$.
- (ii) $\Sigma^{(n)}(\gamma) = \{\sigma \in \Sigma^{(n)} : \dim(\gamma^* \cap \sigma) = \dim(\gamma^*)\} \neq \emptyset$.
- (iii) If $\sigma \in \Sigma^{(n)}(\gamma)$, then $\gamma = \gamma(I(\gamma, \sigma), \sigma)$.

PROOF. (i) This equation follows from the following equivalences.

$$j \in I(\gamma, \sigma) \Leftrightarrow \gamma \subset H(a^j(\sigma), l(a^j(\sigma))) \Leftrightarrow a^j(\sigma) \in \overline{\gamma^*} \Leftrightarrow a^j(\sigma) \in \overline{\gamma^*} \cap \sigma,$$

where $\overline{\gamma^*}$ denotes the closure of γ^* . Note that the second equivalence follows from (7.3).

(ii) Putting the equation in (i) and $\dim(\gamma^*) = n - \dim(\gamma)$ together, we see the equality of the sets. Since the support of the fan Σ is \mathbb{R}_+^n , there exists σ such that $\dim(\gamma^* \cap \sigma) = \dim(\gamma^*)$, which implies $\Sigma^{(n)}(\gamma) \neq \emptyset$.

(iii) Lemma 8.1 (i),(ii) and the assumption imply $\dim(\gamma) = \dim(\gamma(I(\gamma, \sigma), \sigma))$. In fact, $\dim(\gamma) \leq \dim(\gamma(I(\gamma, \sigma), \sigma)) \leq n - \#I(\gamma, \sigma) = \dim(\gamma)$. Since $\gamma \subset \gamma(I(\gamma, \sigma), \sigma)$ from Lemma 8.1 (i), the above dimensional equation yields $\gamma = \gamma(I(\gamma, \sigma), \sigma)$. \square

REMARK 8.3. It follows from Lemma 8.2 (iii) that the map $\gamma : \mathcal{P}(\{1, \dots, n\}) \times \Sigma^{(n)} \rightarrow \mathcal{F}(P)$ is surjective.

Hereafter in this subsection, we always assume that $\sigma \in \Sigma^{(n)}$, $\gamma \in \mathcal{F}(P)$, $I \in \mathcal{P}(\{1, \dots, n\})$ have the relationship $\gamma(I, \sigma) = \gamma \in \mathcal{F}(P)$.

LEMMA 8.4. *The pair $(\sum_{j \in I} a^j(\sigma), \sum_{j \in I} l(a^j(\sigma)))$ is valid for P and defines the face γ .*

PROOF. It follows from the following equivalences that the pair in the lemma is valid for P .

$$\begin{aligned} & P \subset H^+(a^j(\sigma), l(a^j(\sigma))) \text{ for any } j \in I \\ \iff & \text{ If } \alpha \in P, \text{ then } \langle a^j(\sigma), \alpha \rangle \geq l(a^j(\sigma)) \text{ for any } j \in I \\ \iff & \text{ If } \alpha \in P, \text{ then } \langle \sum_{j \in I} a^j(\sigma), \alpha \rangle \geq \sum_{j \in I} l(a^j(\sigma)) \\ \iff & P \subset H^+(\sum_{j \in I} a^j(\sigma), \sum_{j \in I} l(a^j(\sigma))). \end{aligned}$$

Note that the second equivalence follows from the definition of $l(\cdot)$. Moreover, it is similarly shown that the above pair in the lemma defines the face γ , so we omit its proof. \square

LEMMA 8.5. *For any $(I, \sigma) \in \mathcal{P}(\{1, \dots, n\}) \times \Sigma^{(n)}$ (satisfying $\gamma(I, \sigma) = \gamma$), the subset $\{k : a_k^j(\sigma) = 0 \text{ for any } j \in I\}$ in $\{1, \dots, n\}$ is equal to $V(\gamma)$ defined as in (4.1). (This means that the above subset is independent of the chosen pair (I, σ) satisfying $\gamma(I, \sigma) = \gamma$.)*

PROOF. This follows from Lemma 4.4 and Lemma 8.4. \square

Let us consider the following two subsets in \mathbb{R}^n .

$$\begin{aligned} (8.4) \quad & T_I(\mathbb{R}^n) = \{y \in \mathbb{R}^n : y_j = 0 \text{ if } j \in I\} \text{ (as in (1.3)),} \\ & T_I^*(\mathbb{R}^n) := \{y \in \mathbb{R}^n : y_j = 0 \text{ if and only if } j \in I\} \\ & \quad (= \{y \in T_I(\mathbb{R}^n) : y_j \neq 0 \text{ if } j \notin I\}). \end{aligned}$$

The following are important equivalent conditions of the compactness of a face.

PROPOSITION 8.6. *The following conditions are equivalent.*

- (i) γ is compact;
- (ii) $\sum_{j \in I} a_k^j(\sigma) > 0$ for $k = 1, \dots, n$;
- (iii) $V(\gamma) = \emptyset$;

(iv) $\pi(\sigma)(T_I(\mathbb{R}^n)) = 0;$

(v) $\pi(\sigma)(T_I^*(\mathbb{R}^n)) = 0.$

PROOF. The equivalence of three conditions (i),(ii),(iii) follows from Lemmas 4.2, 4.3 and 8.4. An easy computation implies that $(x_1, \dots, x_n) = (\pi(\sigma) \circ T_I)(y_1, \dots, y_n)$, where

$$(8.5) \quad x_k := \begin{cases} \prod_{j=1}^n y_j^{a_k^j(\sigma)} = \prod_{j \in J} y_j^{a_k^j(\sigma)} & \text{for } k \in V(\gamma), \\ 0 & \text{for } k \in W(\gamma). \end{cases}$$

The equivalence of three conditions (iii), (iv), (v) follows from the equations in (8.5). \square

LEMMA 8.7. *The following equality as the map from \mathbb{R}^n to \mathbb{R}^n holds:*

$$(8.6) \quad \pi(\sigma) \circ T_I = T_{W(\gamma)} \circ \pi(\sigma).$$

PROOF. This follows from (8.5) and a computation of $T_{W(\gamma)} \circ \pi(\sigma)$. \square

Hereafter we assume that f belongs to the class $\hat{\mathcal{E}}(U)$ and set $P = \Gamma_+(f)$.

LEMMA 8.8. *For any $\sigma \in \Sigma^{(n)}$, there exists a C^∞ function f_σ defined on the set $\pi(\sigma)^{-1}(U)$ such that $f_\sigma(0) \neq 0$ and*

$$(8.7) \quad f(\pi(\sigma)(y)) = \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))} \right) f_\sigma(y) \quad \text{for } y \in \pi(\sigma)^{-1}(U).$$

PROOF. Let y be in $\pi(\sigma)^{-1}(U)$. Since f belongs to the class $\hat{\mathcal{E}}(U)$, f can be expressed as in (6.1) in Proposition 6.3. Substituting $x = \pi(\sigma)(y)$ into (6.1), we have

$$f(\pi(\sigma)(y)) = \sum_{p \in S} \left(\prod_{j=1}^n y_j^{\langle a^j(\sigma), p \rangle} \right) \psi_p(\pi(\sigma)(y)).$$

Now, define

$$(8.8) \quad f_\sigma(y) := \sum_{p \in S} \left(\prod_{j=1}^n y_j^{\langle a^j(\sigma), p \rangle - l(a^j(\sigma))} \right) \psi_p(\pi(\sigma)(y)).$$

Then we obtain the equation of the form (8.7). Noticing $\langle a^j(\sigma), p \rangle - l(a^j(\sigma)) \in \mathbb{Z}_+$ for all j , we see that f_σ is smooth on $\pi(\sigma)^{-1}(U)$. On the other hand, the face $\gamma(\{1, \dots, n\}, \sigma)$ becomes a vertex of $\Gamma_+(f)$, which is denoted by $p(\sigma)$. Lemma 6.4 and (8.8) imply that $p(\sigma) \in S$ and $f_\sigma(0) = \psi_{p(\sigma)}(0) \neq 0$. \square

The following equation plays an important role in the resolution of singularities and the analysis in Section 9.

LEMMA 8.9.

$$(8.9) \quad f_\gamma(\pi(\sigma)(y)) = \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))} \right) f_\sigma(T_I(y)) \quad \text{for } y \in \pi(\sigma)^{-1}(U).$$

PROOF. Let y be in $\pi(\sigma)^{-1}(U)$. From Lemma 6.7, we have

$$(8.10) \quad \begin{aligned} f_\gamma(\pi(\sigma)(y)) &= \sum_{p \in \gamma \cap S} \left(\prod_{j=1}^n y_j^{\langle a^j(\sigma), p \rangle} \right) \psi_p((T_{W(\gamma)} \circ \pi(\sigma))(y)) \\ &= \left(\prod_{j \in I} y_j^{l(a^j(\sigma))} \right) \sum_{p \in \gamma \cap S} \left(\prod_{j \in J} y_j^{\langle a^j(\sigma), p \rangle} \right) \psi_p((T_{W(\gamma)} \circ \pi(\sigma))(y)). \end{aligned}$$

On the other hand, the definition of f_σ in (8.8) gives

$$(8.11) \quad \begin{aligned} f_\sigma(T_I(y)) &= \sum_{p \in \gamma \cap S} \left(\prod_{j \in J} y_j^{\langle a^j(\sigma), p \rangle - l(a^j(\sigma))} \right) \psi_p((\pi(\sigma) \circ T_I)(y)) \\ &= \left(\prod_{j \in J} y_j^{l(a^j(\sigma))} \right)^{-1} \sum_{p \in \gamma \cap S} \left(\prod_{j \in J} y_j^{\langle a^j(\sigma), p \rangle} \right) \psi_p((\pi(\sigma) \circ T_I)(y)). \end{aligned}$$

Putting (8.6), (8.10), (8.11) together, we get the equation in the lemma. \square

8.2. Resolution of singularities

The purpose of this subsection is to show the following theorem.

THEOREM 8.10. *Let f belong to the class $\hat{\mathcal{E}}(U)$, where U is an open neighborhood of the origin in \mathbb{R}^n , let Σ be a simplicial subdivision of the fan Σ_0 associated with the Newton polyhedron $\Gamma_+(f)$ and let σ be an n -dimensional cone in Σ , whose skeleton is $a^1(\sigma), \dots, a^n(\sigma) \in \mathbb{Z}_+^n$. Then there exists a C^∞ function f_σ defined on the set $\pi(\sigma)^{-1}(U)$ such that $f_\sigma(0) \neq 0$ and*

$$(8.12) \quad (f \circ \pi(\sigma))(y) = \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))} \right) f_\sigma(y) \quad \text{for } y \in \pi(\sigma)^{-1}(U).$$

Furthermore, if f is nondegenerate over \mathbb{R} with respect to $\Gamma_+(f)$ and a subset $I \subset \{1, \dots, n\}$ satisfies $\pi(\sigma)(T_I^*(\mathbb{R}^n)) = 0$, then the set $\{y \in T_I^*(\mathbb{R}^n) : f_\sigma(y) = 0\}$ is nonsingular (the definition of $T_I^*(\mathbb{R}^n)$ was given in (8.4)), i.e., the gradient of the restriction of the function f_σ to $T_I^*(\mathbb{R}^n)$ does not vanish at the points of the set $\{y \in T_I^*(\mathbb{R}^n) : f_\sigma(y) = 0\}$.

Consider a toric variety Y_Σ and the map $\pi : Y_\Sigma \rightarrow \mathbb{R}^n$, which are constructed as in Section 7 when $P = \Gamma_+(f)$. The above theorem shows that if $f \in \hat{\mathcal{E}}(U)$ satisfies the nondegeneracy condition, then this map $\pi : Y_\Sigma \rightarrow \mathbb{R}^n$ is a real resolution of singularities. Indeed, the set $\pi(\sigma)^{-1}(0)$ is expressed as a disjoint union of $T_I^*(\mathbb{R}^n)$ for some subsets I in $\{1, \dots, n\}$.

REMARK 8.11. Let $b = (b_1, \dots, b_n)$ be a point on $T_I^*(\mathbb{R}^n)$ satisfying $f_\sigma(b) = 0$. By the implicit function theorem, there exist local coordinates around b in which $f \circ \pi(\sigma)$ can be expressed in a normal crossing form. To be more specific, there exists a local diffeomorphism Φ defined around b such that $y = \Phi(u)$ with $b = \Phi(b)$ and

$$(8.13) \quad (f \circ \pi(\sigma) \circ \Phi)(u) = (u_p - b_p) \prod_{j \in I} u_j^{l(a_j(\sigma))},$$

where $y_j = u_j$ for $j \in I$ and $p \in \{1, \dots, n\} \setminus I$.

PROOF OF THEOREM 8.10. Lemma 8.8 implies the existence of a C^∞ function f_σ satisfying (8.12) with $f_\sigma(0) \neq 0$. Let us show the rest of the theorem.

Let σ be as in the theorem and I a subset in $\{1, \dots, n\}$ satisfying $\pi(\sigma)(T_I^*(\mathbb{R}^n)) = 0$. Note that $\gamma = \gamma(I, \sigma)$ is a compact face from Proposition 8.6.

Since $\gamma = \gamma(I, \sigma)$, we have $\gamma \subset H(a^j(\sigma), l(a^j(\sigma)))$ for $j \in I$ from Lemma 8.1. Thus, Lemma 6.8 implies

$$f_\gamma(t^{a_1^j(\sigma)}x_1, \dots, t^{a_n^j(\sigma)}x_n) = t^{l(a^j(\sigma))}f_\gamma(x) \quad \text{for } j \in I.$$

Taking the derivative in (8.9) with respect to t and putting $t = 1$, we obtain Euler's identities:

$$(8.14) \quad \sum_{k=1}^n a_k^j(\sigma)x_k \frac{\partial f_\gamma}{\partial x_k}(x) = l(a^j(\sigma))f_\gamma(x) \quad \text{for } j \in I.$$

On the other hand, taking the partial derivative with respect to y_j for $j \in J$ and putting $x = \pi(\sigma)(y)$, we have

$$(8.15) \quad \sum_{k=1}^n a_k^j(\sigma)x_k \frac{\partial f_\gamma}{\partial x_k}(x) = \left(\prod_{i=1}^n y_i^{l(a^i(\sigma))} \right) \left[l(a^j(\sigma))(f_\sigma \circ T_I)(y) + y_j \frac{\partial}{\partial y_j}(f_\sigma \circ T_I)(y) \right]$$

for $j \in J$.

Now, let us assume that there exists a point $b \in T_I^*(\mathbb{R}^n)$ such that

$$f_\sigma(b) = \frac{\partial f_\sigma}{\partial y_j}(b) = 0 \quad \text{for } j \in J.$$

Then the set $U_I(b) = \{x \in U : x = \pi(\sigma)(T_I^r(b)) \text{ for } r \in \mathbb{R} \setminus \{0\}\}$ is contained in $(\mathbb{R} \setminus \{0\})^n$. Since f_γ vanishes on the set $U_I(b)$ from (8.9), the equations (8.14),(8.15) give

$$(8.16) \quad \sum_{k=1}^n a_k^j(\sigma)x_k \frac{\partial f_\gamma}{\partial x_k}(x) = 0 \quad \text{for } x \in U_I(b), \quad j = 1, \dots, n.$$

Since the determinant of the $n \times n$ matrix $(a_k^j(\sigma))_{1 \leq j, k \leq n}$ is equal to 1 or -1 , this matrix is invertible. Therefore, we have

$$\frac{\partial f_\gamma}{\partial x_k}(x) = 0 \quad \text{for } x \in U_I(b), \quad k = 1, \dots, n,$$

which is a contradiction to the nondegeneracy condition of f in (3.2). \square

9. Poles of Local Zeta Functions

Let U be an open neighborhood of the origin. Throughout this section, the functions f, φ always satisfy the conditions (A), (B) in the beginning of Section 3.

We investigate the properties of poles of the functions:

$$(9.1) \quad Z_{\pm}(s; \varphi) := \int_{\mathbb{R}^n} f(x)_{\pm}^s \varphi(x) dx,$$

where $f(x)_{+} = \max\{f(x), 0\}$, $f(x)_{-} = \max\{-f(x), 0\}$ and the *local zeta function*:

$$(9.2) \quad Z(s; \varphi) = \int_{\mathbb{R}^n} |f(x)|^s \varphi(x) dx.$$

Note that the above functions have a simple relationship: $Z(s; \varphi) = Z_{+}(s; \varphi) + Z_{-}(s; \varphi)$. Since $Z_{\pm}(s; \varphi)$ can be expressed as

$$(9.3) \quad Z_{\pm}(s; \varphi) = \sum_{\theta \in \{-1, 1\}^n} \int_{\mathbb{R}_+^n} f(\theta x)_{\pm}^s \varphi(\theta x) dx,$$

where $\theta x = (\theta_1 x_1, \dots, \theta_n x_n)$, we substantially investigate the properties of the functions:

$$(9.4) \quad \tilde{Z}_{\pm}(s; \varphi) := \int_{\mathbb{R}_+^n} f(x)_{\pm}^s \varphi(x) dx.$$

It is easy to see that the above functions are holomorphic functions in the region $\text{Re}(s) > 0$. For the moment, suppose that f is real analytic near the origin. It is known (c.f. [26], [30]) that if the support of φ is sufficiently small, then the functions $Z_{\pm}(s; \varphi)$ and $Z(s; \varphi)$ can be analytically continued to the complex plane as meromorphic functions and their poles belong to finitely many arithmetic progressions constructed from negative rational numbers. (In this section, this kind of process on analytic extension often appears. We denote by the same symbols these extended meromorphic functions defined on the complex plane.) More precisely, Varchenko [41] describes the positions of candidate poles of these functions and their orders by using the theory of toric varieties based on the geometry of Newton polyhedra. His works have been deeply developed in [8], [9], [6], [7].

The purpose of this section is to generalize the above Varchenko's results to the case that the function f belongs to the class $\hat{\mathcal{E}}(U)$. The results in this section need the following assumption stated as in Section 3.

- (C) f belongs to the class $\hat{\mathcal{E}}(U)$ and is nondegenerate over \mathbb{R} with respect to its Newton polyhedron.

In this section, we use the following notation.

- Σ_0 is the fan associated with $\Gamma_+(f)$;
- Σ is a simplicial subdivision of Σ_0 ;
- (Y_Σ, π) is the real resolution associated with Σ ;
- $a^1(\sigma), \dots, a^n(\sigma)$ is the skeleton of $\sigma \in \Sigma^{(n)}$, ordered once and for all;
- $J_\pi(y)$ is the Jacobian of the mapping of π .

9.1. Candidate poles

First, let us state our results on the positions and the orders of candidate poles of the functions $Z_\pm(s; \varphi)$, $Z(s; \varphi)$.

THEOREM 9.1. *Suppose that f satisfies the condition (C). If the support of φ is contained in a sufficiently small neighborhood of the origin, then the functions $Z_\pm(s; \varphi)$ and $Z(s; \varphi)$ can be analytically continued to the complex plane as meromorphic functions, which are also denoted by the same symbols, and their poles are contained in the set*

$$(9.5) \quad \left\{ -\frac{\langle a \rangle + \nu}{l(a)} : \nu \in \mathbb{Z}_+, a \in \tilde{\Sigma}^{(1)} \right\} \cup (-\mathbb{N}),$$

where $l(a)$ is as in (7.1) with $P = \Gamma_+(f)$ and $\tilde{\Sigma}^{(1)} = \{a \in \Sigma^{(1)} : l(a) \neq 0\}$. Moreover, the largest element of the first set in (9.5) is $-1/d(f)$. When $Z_\pm(s; \varphi)$ and $Z(s; \varphi)$ have poles at $s = -1/d(f)$, their orders are at most

$$\begin{cases} m(f) & \text{if } 1/d(f, \varphi) \text{ is not an integer,} \\ \min\{m(f) + 1, n\} & \text{otherwise.} \end{cases}$$

PROOF. First, let us show that the above assertions also hold in the case of the functions $\tilde{Z}_\pm(s; \varphi)$ in (9.4).

Step 1. (Decompositions of $\tilde{Z}_\pm(s; \varphi)$.)

For the moment, we assume that $s \in \mathbb{C}$ satisfies $\text{Re}(s) > 0$. By using the mapping $x = \pi(y)$, $\tilde{Z}_\pm(s; \varphi)$ are expressed as

$$\begin{aligned} \tilde{Z}_\pm(s; \varphi) &= \int_{\mathbb{R}_+^n} f(x)_\pm^s \varphi(x) dx \\ &= \int_{\tilde{Y}_\Sigma} ((f \circ \pi)(y))_\pm^s (\varphi \circ \pi)(y) |J_\pi(y)| dy, \end{aligned}$$

where $\tilde{Y}_\Sigma := Y_\Sigma \cap \pi^{-1}(\mathbb{R}_+^n)$ and dy is a volume element in Y_Σ . It is easy to see that there exists a set of C_0^∞ functions $\{\chi_\sigma : Y_\Sigma \rightarrow \mathbb{R}_+ : \sigma \in \Sigma^{(n)}\}$ satisfying the following properties:

- For each $\sigma \in \Sigma^{(n)}$, the support of the function χ_σ is contained in $\mathbb{R}^n(\sigma)$ and χ_σ identically equals one in some neighborhood of the origin.
- $\sum_{\sigma \in \Sigma^{(n)}} \chi_\sigma \equiv 1$ on the support of $\varphi \circ \pi$.

Applying Theorem 8.10 and Lemmas 7.1 and 7.2, we have

$$\tilde{Z}_\pm(s; \varphi) = \sum_{\sigma \in \Sigma^{(n)}} Z_\pm^{(\sigma)}(s)$$

with

$$\begin{aligned} Z_\pm^{(\sigma)}(s) &= \int_{\mathbb{R}_+^n} ((f \circ \pi(\sigma))(y))_\pm^s (\varphi \circ \pi(\sigma))(y) \chi_\sigma(y) |J_{\pi(\sigma)}(y)| dy \\ (9.6) \quad &= \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))} f_\sigma(y) \right)_\pm^s \left| \prod_{j=1}^n y_j^{(a^j(\sigma)) - 1} \right| \varphi_\sigma(y) dy, \end{aligned}$$

where $\varphi_\sigma(y) = (\varphi \circ \pi(\sigma))(y) \chi_\sigma(y)$.

Consider each function $Z_\pm^{(\sigma)}(s)$ for $\sigma \in \Sigma^{(n)}$. We easily see the existence of finite sets of C_0^∞ functions $\{\psi_k : \mathbb{R}^n \rightarrow \mathbb{R}_+\}$ and $\{\eta_l : \mathbb{R}^n \rightarrow \mathbb{R}_+\}$ satisfying the following conditions.

- The supports of ψ_k and η_l are sufficiently small and $\sum_k \psi_k + \sum_l \eta_l \equiv 1$ on the support of φ_σ .

- For each k , f_σ is always positive or negative on the support of ψ_k .
- For each l , the support of η_l intersects the set $\{y \in \text{Supp}(\varphi_\sigma) : f_\sigma(y) = 0\}$.
- The union of the support of η_l for all l contains the set $\{y \in \text{Supp}(\varphi_\sigma) : f_\sigma(y) = 0\}$.

By using the functions ψ_k and η_l , we have

$$(9.7) \quad Z_\pm^{(\sigma)}(s) = \sum_k I_{\sigma,\pm}^{(k)}(s) + \sum_l J_{\sigma,\pm}^{(l)}(s),$$

with

$$(9.8) \quad \begin{aligned} I_{\sigma,\pm}^{(k)}(s) &= \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))} f_\sigma(y) \right)_\pm^s \left| \prod_{j=1}^n y_j^{\langle a^j(\sigma) \rangle - 1} \right| \tilde{\psi}_k(y) dy, \\ J_{\sigma,\pm}^{(l)}(s) &= \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))} f_\sigma(y) \right)_\pm^s \left| \prod_{j=1}^n y_j^{\langle a^j(\sigma) \rangle - 1} \right| \tilde{\eta}_l(y) dy, \end{aligned}$$

where $\tilde{\psi}_k(y) = \varphi_\sigma(y)\psi_k(y)$ and $\tilde{\eta}_l(y) = \varphi_\sigma(y)\eta_l(y)$. If the set $\{y \in \text{Supp}(\varphi_\sigma) : f_\sigma(y) = 0\}$ is empty, then the functions $J_{\sigma,\pm}^{(l)}(s)$ do not appear.

From the viewpoint of the properties of singularities, we divide the functions $\tilde{Z}_\pm(s; \varphi)$ as $\tilde{Z}_\pm(s; \varphi) = I_\pm(s) + J_\pm(s)$, with

$$(9.9) \quad I_\pm(s) = \sum_{\sigma \in \Sigma^{(n)}} \sum_k I_{\sigma,\pm}^{(k)}(s), \quad J_\pm(s) = \sum_{\sigma \in \Sigma^{(n)}} \sum_l J_{\sigma,\pm}^{(l)}(s).$$

Step 2. (Poles of $I_\pm(s)$.)

Let us consider the functions $I_{\sigma,\pm}^{(k)}(s)$. An easy computation gives

$$(9.10) \quad I_{\sigma,\pm}^{(k)}(s) = \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^n y_j^{l(a^j(\sigma))s + \langle a^j(\sigma) \rangle - 1} \right) f_\sigma(y)_\pm^s \tilde{\psi}_k(y) dy.$$

The following lemma is useful for analyzing the poles of integrals of the above form.

LEMMA 9.2 ([11], [2]). *Let $\psi(y_1, \dots, y_n; \mu)$ be a C_0^∞ function of y on \mathbb{R}^n that is an entire function of the parameter $\mu \in \mathbb{C}$. Then the function*

$$L(\tau_1, \dots, \tau_n; \mu) = \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^n y_j^{\tau_j} \right) \psi(y_1, \dots, y_n; \mu) dy_1 \cdots dy_n$$

can be analytically continued at all the complex values of τ_1, \dots, τ_n and μ as a meromorphic function. Moreover all its poles are simple and lie on $\tau_j = -1, -2, \dots$ for $j = 1, \dots, n$.

PROOF OF LEMMA 9.2. The lemma is easily obtained by the integration by parts (see [11], [2]). \square

By applying Lemma 9.2 to (9.10), each $I_{\sigma, \pm}^{(k)}(s)$ can be analytically continued to the complex plane as a meromorphic function and their poles are contained in the set

$$(9.11) \quad \left\{ -\frac{\langle a^j(\sigma) \rangle + \nu}{l(a^j(\sigma))} : \nu \in \mathbb{Z}_+, j \in B(\sigma) \right\},$$

where

$$(9.12) \quad B(\sigma) := \{j : l(a^j(\sigma)) \neq 0\} \subset \{1, \dots, n\}.$$

From (9.9), $I_{\pm}(s)$ also become meromorphic functions on \mathbb{C} and their poles are contained in the union of the sets (9.11) for all $\sigma \in \Sigma^{(n)}$.

Step 3. (Poles of $J_{\pm}(s)$.)

Let us consider the functions $J_{\sigma, \pm}^{(l)}(s)$. By applying Theorem 8.10 and changing the integral variables as in Remark 8.11, $J_{\sigma, \pm}^{(l)}(s)$ can be expressed as follows.

$$J_{\sigma, \pm}^{(l)}(s) = \int_{\mathbb{R}_+^n} \left((y_p - b_p) \prod_{j \in B_l(\sigma)} y_j^{l(a^j(\sigma))} \right)_{\pm}^s \times \left| \prod_{j \in B_l(\sigma)} y_j^{\langle a^j(\sigma) \rangle - 1} \right| \hat{\eta}_l(y_1, \dots, y_p - b_p, \dots, y_n) dy,$$

where $b_p > 0$, $B_l(\sigma) \subsetneq \{1, \dots, n\}$, $p \in \{1, \dots, n\} \setminus B_l(\sigma)$ and $\hat{\eta}_l \in C_0^\infty(\mathbb{R}^n)$. In a similar fashion to the case of $I_{\sigma, \pm}^{(k)}(s)$, we have

$$(9.13) \quad J_{\sigma, \pm}^{(l)}(s) = \int_{\mathbb{R}_+^n} \left(y_p^s \prod_{j \in B_l(\sigma)} y_j^{l(a^j(\sigma))s + (a^j(\sigma)) - 1} \right) \hat{\eta}_l(y_1, \dots, \pm y_p, \dots, y_n) dy.$$

By applying Lemma 9.2 to (9.13), each $J_{\sigma, \pm}^{(l)}(s)$ can be analytically continued to the complex plane as a meromorphic function and their poles are contained in the set

$$(9.14) \quad \left\{ -\frac{\langle a^j(\sigma) \rangle + \nu}{l(a^j(\sigma))} : \nu \in \mathbb{Z}_+, j \in \tilde{B}_l(\sigma) \right\} \cup (-\mathbb{N}),$$

where $\tilde{B}_l(\sigma) = \{j \in B_l(\sigma) : l(a^j(\sigma)) \neq 0\}$. The necessity of the set $(-\mathbb{N})$ in (9.13) follows from the existence of y_p^s in (9.13). We remark that $y_j^{l(a^j(\sigma))s + (a^j(\sigma)) - 1}$ may also induce the poles on $(-\mathbb{N})$. From (9.9), $J_\pm(s)$ also become meromorphic functions on \mathbb{C} and their poles are contained in the union of the sets (9.14) for all $\sigma \in \Sigma^{(n)}$.

Now, in order to investigate properties of the *first* poles of $Z_\pm(s)$, we define

$$(9.15) \quad \tilde{\beta}(f) = \max \left\{ -\frac{\langle a \rangle}{l(a)} : a \in \tilde{\Sigma}^{(1)} \right\}.$$

Step 4. (Geometrical meanings of $\tilde{\beta}(f)$.)

Let us consider geometrical meanings of the quantity $\tilde{\beta}(f)$. For $a \in \Sigma^{(1)}$, we denote by $q(a)$ the point of the intersection of the hyperplane $H(a, l(a))$ with the line $\{(t, \dots, t) \in \mathbb{R}_+^n : t > 0\}$, where $H(\cdot, \cdot)$ is as in (2.1). Then it is easy to see $q(a) = (l(a)/\langle a \rangle, \dots, l(a)/\langle a \rangle)$. Roughly speaking, the fact that the value of $-\langle a \rangle/l(a)$ is large means that the point $q(a)$ is far from the origin. To be more specific, we have the following equivalences: For $a \in \tilde{\Sigma}^{(1)}$,

$$(9.16) \quad \tilde{\beta}(f) = -\frac{\langle a \rangle}{l(a)} \iff q_* = q(a) \iff q_* \in H(a, l(a)).$$

(The definition of the point q_* was given in Section 2.2.) Thus, it easily follows from the definition of $d(\cdot)$ that $\tilde{\beta}(f) = -1/d(f)$.

Step 5. (Orders of the poles at $\tilde{\beta}(f)$.)

For $\sigma \in \Sigma^{(n)}$, let

$$A(\sigma) = \left\{ j \in B(\sigma) : \tilde{\beta}(f) = -\frac{\langle a^j(\sigma) \rangle}{l(a^j(\sigma))} \right\} \subset \{1, \dots, n\},$$

where $B(\sigma)$ is as in (9.12). From (9.9), it suffices to analyze the poles of $I_{\sigma,\pm}^{(k)}(s)$ and $J_{\sigma,\pm}^{(l)}(s)$. When these functions have poles at $s = \tilde{\beta}(f)$, we see the upper bounds of orders of their poles at $s = \tilde{\beta}(f)$ as follows by applying Lemma 9.2 to the integrals (9.10),(9.13).

$I_{\sigma,\pm}^{(k)}(s)$	$\#A(\sigma)$
$J_{\sigma,\pm}^{(l)}(s)$	$\min\{\#A(\sigma), n - 1\}$ if $\tilde{\beta}(f) \notin (-\mathbb{N})$ $\min\{\#A(\sigma) + 1, n\}$ if $\tilde{\beta}(f) \in (-\mathbb{N})$

From the above table, in order to obtain the estimates of the orders of poles in the theorem, it suffices to show the following. (Here, we need the inequality “ \leq ” only in the lemma below. The equality will be needed in Section 9.3.)

LEMMA 9.3. $m(f) = \max \{ \#A(\sigma) : \sigma \in \Sigma^{(n)} \}.$

PROOF OF LEMMA 9.3. Recall $m(f) := n - \dim(\tau_*)$. From the definition of $A(\sigma)$ and (9.15), we have

$$\begin{aligned} A(\sigma) &= \{ j : q_* \in H(a^j(\sigma), l(a^j(\sigma))) \} \\ &= \{ j : \tau_* \subset H(a^j(\sigma), l(a^j(\sigma))) \} = I(\tau_*, \sigma). \end{aligned}$$

Here τ_* is the principal face of $\Gamma_+(f)$, i.e., its relative interior contains the point q_* , and $I(\cdot, \cdot)$ is as in (8.2). Lemma 8.1 implies that $\dim(\tau_*) \leq n - \#I(\tau_*, \sigma) = n - \#A(\sigma)$ for any $\sigma \in \Sigma^{(n)}$. On the other hand, Lemma 8.2 implies that there exists $\sigma \in \Sigma^{(n)}$ such that $\dim(\tau_*) = n - \#A(\sigma)$. Since the codimension of τ_* is $m(f)$, we obtain the equality in the lemma. \square

Since $\tilde{Z}_\pm(s) = I_\pm(s) + J_\pm(s)$, we see that the poles of $\tilde{Z}_\pm(s)$ have the same properties as in the theorem. Finally, considering the relationships: (9.3) and $Z(s) = Z_+(s) + Z_-(s)$, we obtain the theorem. \square

9.2. Poles of $J_{\pm}(s)$ on negative integers

We consider the poles of the functions $J_{\pm}(s)$ at negative integers in more detail.

The following lemma is useful for computing the coefficients of the Laurent expansion explicitly.

LEMMA 9.4. *Let ψ be a C_0^∞ function on \mathbb{R} and $k \in \mathbb{N}$. Then*

$$\lim_{s \rightarrow -k} (s + k) \int_0^\infty y^s \psi(y) dy = \frac{1}{(k - 1)!} \psi^{(k-1)}(0).$$

In particular,

$$\lim_{s \rightarrow -1} (s + 1) \int_0^\infty y^s \psi(y) dy = \psi(0).$$

PROOF. The above formula is easily obtained by the integration by parts. \square

For $\lambda \in \mathbb{N}$, define

$$A_\lambda(\sigma) := \{j \in B(\sigma) : l(a^j(\sigma))\lambda - \langle a^j(\sigma) \rangle \in \mathbb{Z}_+\},$$

$$\rho_\lambda := \min\{\max\{\#A_\lambda(\sigma) : \sigma \in \Sigma^{(n)}\}, n - 1\}.$$

The following proposition will be used in the computation of the coefficients of the asymptotic expansion (3.1) of $I(t; \varphi)$.

PROPOSITION 9.5. *Suppose that f satisfies the condition (C). If the support of φ is contained in a sufficiently small neighborhood of the origin, then the orders of poles of $J_{\pm}(s)$ at $s = -\lambda \in (-\mathbb{N})$ are not higher than $\rho_\lambda + 1$. In particular, if $\lambda < 1/d(f)$, then these orders are not higher than 1. Moreover, let a_λ^\pm be the coefficients of $(s + \lambda)^{-\rho_\lambda - 1}$ in the Laurent expansions of $J_{\pm}(s)$ at $s = -\lambda$, respectively, then we have $a_\lambda^+ = (-1)^{\lambda-1} a_\lambda^-$ for $\lambda \in \mathbb{N}$.*

PROOF. Let $\lambda \in \mathbb{N}$, $l_j, m_j \in \mathbb{N}$ for $j = 1, \dots, n - 1$ and $\eta \in C_0^\infty(\mathbb{R}^n)$. Let B_λ be the subset in $\{1, \dots, n - 1\}$ defined by $B_\lambda = \{j : l_j \lambda - m_j + 1 \in \mathbb{N}\}$, and let $k_j \in \mathbb{N}$ be defined by $k_j = l_j \lambda - m_j + 1$ for $j \in B_\lambda$. We define

$$g_\lambda^\pm(s) = \int_{\mathbb{R}_+^n} \left(y_n^s \prod_{j \in B_\lambda} y_j^{l_j s + m_j - 1} \right) \eta(y_1, \dots, y_{n-1}, \pm y_n) dy,$$

respectively.

It easily follows from Lemma 9.2 that the functions $g_\lambda^\pm(s)$ can be analytically extended to \mathbb{C} as meromorphic functions and they have at $s = -\lambda$ poles of order not higher than $\#B_\lambda + 1$. Let b_λ^\pm be the coefficients of $(s + \lambda)^{-\#B_\lambda - 1}$ in the Laurent expansions of $g_\lambda^\pm(s)$ at $s = -\lambda$, respectively.

By carefully observing the analysis of $J_{\sigma, \pm}^{(l)}(s)$ in the proof of Theorem 9.1, it suffices to show the equation: $b_\lambda^+ = (-1)^{\lambda-1} b_\lambda^-$ for $\lambda \in \mathbb{N}$. By using Lemma 9.4, a direct computation gives $b_\lambda^\pm = (\pm 1)^{\lambda-1} C_\lambda$, with

$$C_\lambda = \frac{1}{(\lambda - 1)!} \prod_{j \in B_\lambda} \left(\frac{1}{l_j \prod_{\nu_j=1}^{k_j-1} (l_j \lambda - m_j + 1 - \nu_j)} \right) \times \begin{cases} (\partial^{\alpha-1} \eta)(0) & \text{if } B_\lambda = \{1, \dots, n-1\}, \\ \int_{\mathbb{R}_+^{n-\#B_\lambda-1}} (\partial^{\alpha-1} \eta) (T_{B_\lambda \cup \{n\}}(y)) \\ \quad \times \prod_{j \notin B_\lambda \cup \{n\}} dy_j & \text{otherwise,} \end{cases}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ satisfies that $\alpha_j = k_j$ if $j \in B_\lambda$, $\alpha_n = \lambda$ and $\alpha_j = 1$ otherwise. From the above equation, we see that $b_\lambda^+ = (-1)^{\lambda-1} b_\lambda^-$ for $\lambda \in \mathbb{N}$. \square

9.3. The first coefficients

We define the subset of important cones in $\Sigma^{(n)}$ as follows.

$$\Sigma_*^{(n)} := \{\sigma \in \Sigma^{(n)} : m(f) = \#A(\sigma)\}.$$

It follows from Lemma 9.3 that $\Sigma_*^{(n)}$ is nonempty. From the definition of $m(f)$, we can see the following implications:

$$(9.17) \quad \begin{aligned} \sigma \in \Sigma_*^{(n)} &\iff \dim(\tau_*) = n - \#A(\sigma) \\ &\implies \tau_* = \bigcap_{j \in A(\sigma)} H(a^j(\sigma), l(a^j(\sigma))) \cap \Gamma_+(f). \end{aligned}$$

Thus, when $\sigma \in \Sigma_*^{(n)}$, $I = A(\sigma)$, $\gamma = \tau_*$, the equation $\gamma(I, \sigma) = \gamma$ holds, which is an important condition in Section 8.

Now, let us compute the coefficients of $(s + 1/d(f))^{-m(f)}$ in the Laurent expansions of $\tilde{Z}_\pm(s; \varphi)$. Let

$$(9.18) \quad \tilde{C}_\pm (= \tilde{C}_\pm(f, \varphi)) := \lim_{s \rightarrow -1/d(f)} (s + 1/d(f))^{m(f)} \tilde{Z}_\pm(s; \varphi).$$

PROPOSITION 9.6. *Suppose that f satisfies the condition (C) and that at least one of the following conditions is satisfied.*

- (i) $d(f) > 1$;
- (ii) $f_\sigma \circ T_{A(\sigma)}$ does not vanish on $\mathbb{R}_+^n \cap \pi(\sigma)^{-1}(U)$ for any $\sigma \in \Sigma_*^{(n)}$.

Here the above f_{τ_*} is considered as an extended smooth function defined on a wider region as in Lemma 5.4 (ii). Then we give explicit formulae for coefficients \tilde{C}_\pm : (9.23), (9.25), (9.26), (9.27) in the proof of this proposition.

PROOF. In this proof, we use the following notation and symbols to decrease the complexity.

- $\prod_{j \notin A(\sigma)} y_j^{a_j} dy_j$ means $\prod_{j \notin A(\sigma)} y_j^{a_j} \cdot \prod_{j \in A(\sigma)} dy_j$ with $a_j > 0$;
- $L_\sigma := \prod_{j \in A(\sigma)} l(a^j(\sigma))^{-1}$;
- $M_j(\sigma) := -l(a^j(\sigma))/d(f) + \langle a^j(\sigma) \rangle$.
- If $a = 0$, then the value of $a^{-1/d(f)}$ is defined by 0.

Note that $M_j(\sigma)$ is a nonnegative constant and, moreover, $M_j(\sigma) = 0$ if and only if $j \in A(\sigma)$.

Let us compute the limits \tilde{C}_\pm exactly. We divide the computation into the following two cases: $m(f) < n$ and $m(f) = n$. After obtaining the formulae (9.23),(9.25), (9.26),(9.27), below, we can easily see that $\tilde{C}_\pm \geq 0$ and $\tilde{C}_+ + \tilde{C}_- > 0$, which are as in the theorem.

The case: $m(f) < n$. First, we consider the case that the hypothesis (i) is satisfied. Let us explicitly compute the following limits:

$$\tilde{C}_\pm(\sigma) := \lim_{s \rightarrow -1/d(f)} (s + 1/d(f))^{m(f)} Z_\pm^{(\sigma)}(s).$$

Since $\tilde{C}_\pm(\sigma) = 0$ if $\sigma \notin \Sigma_*^{(n)}$, it suffices to consider the case that $\sigma \in \Sigma_*^{(n)}$. Considering the equations (9.7) and applying Lemma 9.4 to (9.10), (9.13) with respect to each y_j for $j \in A(\sigma)$, we have

$$(9.19) \quad \tilde{C}_\pm(\sigma) = \sum_k G_\pm^{(k)}(\sigma) + \sum_l H_\pm^{(l)}(\sigma),$$

with

$$(9.20) \quad G_\pm^{(k)}(\sigma) = L_\sigma \int_{\mathbb{R}_+^{n-m(f)}} \frac{\tilde{\psi}_k(T_{A(\sigma)}(y))}{f_\sigma(T_{A(\sigma)}(y))_\pm^{1/d(f)}} \prod_{j \notin A(\sigma)} y_j^{M_j(\sigma)-1} dy_j,$$

$$(9.21) \quad H_\pm^{(l)}(\sigma) = L_\sigma \int_{\mathbb{R}_+^{n-m(f)}} \frac{\hat{\eta}_l(T_{A(\sigma)}(y_1, \dots, \pm y_p, \dots, y_n))}{y_p^{1/d(f)}} \times \prod_{j \in B_l(\sigma) \setminus A(\sigma)} y_j^{M_j(\sigma)-1} \prod_{j \notin A(\sigma)} dy_j,$$

where $\tilde{\psi}_k$ and $\hat{\eta}_l$ are as in (9.8), (9.13), the summations in (9.19) are taken for all k, l satisfying $T_{A(\sigma)}(\mathbb{R}^n) \cap \text{Supp}(\psi_k) \neq \emptyset$ and $A(\sigma) \subset B_l(\sigma)$. We remark that the values of $G_\pm^{(k)}(\sigma)$ and $H_\pm^{(l)}(\sigma)$ may depend on the cut-off functions $\chi_\sigma, \psi_k, \eta_l$ in Section 9.1. We remark that if $f_\sigma(T_{A(\sigma)}(y)) < 0$, then $f_\sigma(T_{A(\sigma)}(y))_+^{-1/d(f)} = 0$ in (9.20). Since $d(f) > 1$, the integrals in (9.21) are convergent and they are interpreted as improper integrals.

In (9.20), (9.21), we deform the cut-off functions ψ_k and η_l as the volume of the support of η_l tends to zero for all l . Then, it is easy to see that the limit of $H_\pm^{(l)}(\sigma)$ is zero, while that of $\sum_k G_\pm^{(k)}(\sigma)$ can be computed explicitly. Considering the equation (9.19), we have

$$(9.22) \quad \tilde{C}_\pm(\sigma) = L_\sigma \int_{\mathbb{R}_+^{n-m(f)}} \frac{\varphi_\sigma(T_{A(\sigma)}(y))}{f_\sigma(T_{A(\sigma)}(y))_\pm^{1/d(f)}} \prod_{j \notin A(\sigma)} y_j^{M_j(\sigma)-1} dy_j,$$

where φ_σ is as in (9.6).

Now, let us compute the limits \tilde{C}_\pm explicitly. If the cut-off function χ_σ is deformed as the volume of the support of χ_σ tends to zero, then $\tilde{C}_\pm(\sigma)$ tends to zero. Notice that each \mathbb{R}^n (see Section 7.3) is densely embedded

in Y_Σ and that $\tilde{C}_\pm = \sum_{\sigma \in \Sigma_*^{(n)}} \tilde{C}_\pm(\sigma)$. Thus, for an arbitrary fixed cone $\sigma \in \Sigma_*^{(n)}$, we have

$$(9.23) \quad \tilde{C}_\pm = L_\sigma \int_{\mathbb{R}_+^{n-m(f)}} \frac{(\varphi \circ \pi(\sigma))(T_{A(\sigma)}(y))}{f_\sigma(T_{A(\sigma)}(y))_\pm^{1/d(f)}} \prod_{j \notin A(\sigma)} y_j^{M_j(\sigma)-1} dy_j.$$

Let us give the other formulae of \tilde{C}_\pm . From the condition (9.17), Lemma 8.9 implies

$$(9.24) \quad (f_{\tau_*} \circ \pi(\sigma))(T_{A(\sigma)}^1(y)) = \left(\prod_{j \notin A(\sigma)} y_j^{l(a^j(\sigma))} \right) f_\sigma(T_{A(\sigma)}(y)).$$

By using the above equation, (9.23) can be rewritten as

$$(9.25) \quad \tilde{C}_\pm = L_\sigma \int_{\mathbb{R}_+^{n-m(f)}} \frac{(\varphi \circ \pi(\sigma))(T_{A(\sigma)}(y))}{(f_{\tau_*} \circ \pi(\sigma))(T_{A(\sigma)}^1(y))_\pm^{1/d(f)}} \prod_{j \notin A(\sigma)} y_j^{\langle a^j(\sigma) \rangle - 1} dy_j.$$

Secondly, we consider the case that the hypothesis (ii) is satisfied. In this case, we immediately obtain the equations (9.22) for $\sigma \in \Sigma_*^{(n)}$ in the same argument as in the case of the hypothesis (i), because $J_{\sigma, \pm}^{(l)}(s)$ do not appear in the decompositions (9.7) and $f_\sigma \circ T_{A_p(\sigma)}$ does not vanish. But, in this case, we must remark that $\tilde{C}_\pm(\sigma)$ does not always vanish even in the case when $\sigma \notin \Sigma_*^{(n)}$. Indeed, some $J_{\sigma, \pm}^{(l)}(s)$ may have at $s = -1/d(f, x^p)$ a pole of order $m(f, x^p)$ even if $\sigma \notin \Sigma_*^{(n)}$. Here, the coefficients $\tilde{C}_\pm(\sigma)$ for $\sigma \notin \Sigma_*^{(n)}$ can be computed in a similar argument of the proof of Lemma 9.4, since $I_{\sigma, \pm}^{(k)}(s)$ do not have at $s = -1/d(f, x^p)$ poles of order $m(f, x^p)$. From the result of this computation, it is easy to see that these coefficients tend to zero if the volume of the support of χ_σ tends to zero. Therefore, the limits \tilde{C}_\pm can be computed as in (9.23) and (9.25) in the same way, where σ is as in the hypothesis (ii). We remark that \tilde{C}_+ or \tilde{C}_- is equal to zero in this case.

The case: $m(f) = n$. In this case, we see that $A(\sigma) = \{1, \dots, n\}$, $m(f) = n$ and the principal face τ_* is the point $q_* = (d(f), \dots, d(f))$. Similar computations give the following. The first expression, corresponding to

(9.23), is

$$(9.26) \quad \tilde{C}_\pm = L \frac{\varphi(0)}{f_\sigma(0)_\pm^{1/d(f)}},$$

where $\sigma \in \Sigma_*^{(n)}$ and $L := \sum_{\sigma \in \Sigma_*^{(n)}} L_\sigma$. We remark that $f_\sigma(0)$ takes the same value for any $\sigma \in \Sigma_*^{(n)}$. The second expression, corresponding to (9.25), is

$$(9.27) \quad \tilde{C}_\pm = L \frac{\varphi(0)}{f_{\tau_*}(1, \dots, 1)_\pm^{1/d(f)}} = L \frac{(d(f)!)^{n/d(f)} \varphi(0)}{(\partial^{q_*} f)(0)_\pm^{1/d(f)}},$$

where L is the same as in (9.26). \square

REMARK 9.7. From the proof of Proposition 9.6, we see the following.

- (i) The values of (9.23),(9.25),(9.26),(9.27) are independent of the chosen cone $\sigma \in \Sigma_*^{(n)}$.
- (ii) The integrals in (9.23),(9.25) are convergent.

REMARK 9.8. Let us consider the case that τ_* is compact. Then $\pi(\sigma) \circ T_{A(\sigma)}(\mathbb{R}^n) = 0$ from Lemma 8.6. Therefore, when $m(f) < n$, the coefficients \tilde{C}_\pm can be written in more simple forms:

$$(9.28) \quad \begin{aligned} \tilde{C}_\pm &= L_\sigma \varphi(0) \int_{\mathbb{R}_+^{n-m(f)}} \frac{1}{f_\sigma(T_{A(\sigma)}(y))_\pm^{1/d(f)}} \prod_{j \notin A(\sigma)} y_j^{M_j(\sigma)-1} dy_j \\ &= L_\sigma \varphi(0) \int_{\mathbb{R}_+^{n-m(f)}} \frac{1}{(f_{\tau_*} \circ \pi(\sigma))(T_{A(\sigma)}^1(y))_\pm^{1/d(f)}} \\ &\quad \times \prod_{j \notin A(\sigma)} y_j^{(a^j(\sigma))-1} dy_j, \end{aligned}$$

where $\sigma \in \Sigma_*^{(n)}$.

REMARK 9.9. In [6], [7], similar formulae of \tilde{C}_\pm are obtained in the real analytic phase case. Their results do not require the assumptions (i), (ii) in Proposition 9.6.

Next, let us compute the coefficients of $(s + 1/d(f))^{-m(f)}$ in the Laurent expansions of $Z_{\pm}(s; \varphi)$, $Z(s; \varphi)$. Let

$$C_{\pm} = \lim_{s \rightarrow -1/d(f)} (s + 1/d(f))^{m(f)} Z_{\pm}(s; \varphi),$$

$$C = \lim_{s \rightarrow -1/d(f)} (s + 1/d(f))^{m(f)} Z(s; \varphi),$$

respectively.

THEOREM 9.10. *Suppose that f satisfies the condition (C) and that at least one of the following conditions is satisfied.*

- (i) $d(f) > 1$;
- (ii) f is nonnegative or nonpositive on U ;
- (iii) f_{τ_*} does not vanish on $U \cap (\mathbb{R} \setminus \{0\})^n$.

If the support of φ is contained in a sufficiently small neighborhood of the origin, then we have

$$(9.29) \quad C_{\pm} = \sum_{\theta \in \{-1, 1\}^n} \tilde{C}_{\pm}(f_{\theta}, \varphi_{\theta}) \quad \text{and} \quad C = C_+ + C_-,$$

where $f_{\theta}(x) := f(\theta x)$, $\varphi_{\theta}(x) := \varphi(\theta x)$ and $\tilde{C}_{\pm}(f, \varphi)$ (defined as in (9.18)) are as in (9.23), (9.25), (9.26), (9.27). Furthermore, if $\varphi(0) > 0$ (resp. $\varphi(0) < 0$) and φ is nonnegative (resp. nonpositive) on U , then C_{\pm} are nonnegative and $C = C_+ + C_-$ is positive.

PROOF. From the relationship (9.3), it suffices to show that the above conditions (ii) and (iii) imply the condition (ii) in Proposition 9.6, when the support of φ is contained in a sufficiently small neighborhood of the origin.

(ii) Let us assume that f is nonnegative on U . (Needless to say, the nonpositive case is similarly proved.) Let σ be in $\Sigma_*^{(n)}$. From the equation (8.7), $l(a^j(\sigma))$ are even for all j and f_{σ} is nonnegative on $\pi(\sigma)^{-1}(U)$. Let us assume that there exists a point $b_0 \in T_I(\mathbb{R}^n) \cap \pi(\sigma)^{-1}(U)$ with nonempty $I \subset \{1, \dots, n\}$ such that $f_{\sigma}(b_0) = 0$. Since f is nondegenerate over \mathbb{R} with respect to $\Gamma_+(f)$, Theorem 8.10 implies that there is a point $b \in \pi(\sigma)^{-1}(U)$ close to b_0 such that $f_{\sigma}(b) < 0$. This contradicts the nonnegativity of

f on U , so we see that there exists an open neighborhood V such that $\{y \in \pi(\sigma)^{-1}(V) : f_\sigma(y) = 0\} \subset (\mathbb{R} \setminus \{0\})^n$ and φ is nonnegative on V . By replacing U by V , the condition (ii) in the above theorem implies the condition (ii) in Proposition 9.6.

(iii) We only consider the case that f_{τ_*} is positive on $U \cap (\mathbb{R} \setminus \{0\})^n$. It follows from the equation (8.9) that $f_\sigma \circ T_{A(\sigma)}$ is nonnegative on $\pi(\sigma)^{-1}(U)$. By the same argument as in the above case (ii), the nondegeneracy condition implies that $f_\sigma \circ T_{A(\sigma)}$ is positive on $\pi(\sigma)^{-1}(U)$ with a sufficiently small neighborhood U . \square

10. Proofs of the Theorems in Section 3

10.1. Relationship between $I(t; \varphi)$ and $Z_\pm(s; \varphi)$

It is known (see [21], [2], etc.) that the study of the asymptotic behavior of the oscillatory integral $I(t; \varphi)$ in (1.1) can be reduced to an investigation of the poles of the functions $Z_\pm(s; \varphi)$ in (9.1). Here, we overview this situation. Let f, φ satisfy the conditions (A),(B) in Section 3. Suppose that the support of φ is sufficiently small.

Define the Gelfand-Leray function $K : \mathbb{R} \rightarrow \mathbb{R}$ as

$$(10.1) \quad K(u) = \int_{W_u} \varphi(x)\omega,$$

where $W_u = \{x \in \mathbb{R}^n : f(x) = u\}$ and ω is the surface element on W_u which is determined by $df \wedge \omega = dx_1 \wedge \cdots \wedge dx_n$. By using $K(u)$, $I(t; \varphi)$ and $Z_\pm(s; \varphi)$ can be expressed as follows. Changing the integral variables in (1.1),(9.1), we have

$$(10.2) \quad I(t; \varphi) = \int_{-\infty}^{\infty} e^{itu} K(u) du = \int_0^{\infty} e^{itu} K(u) du + \int_0^{\infty} e^{-itu} K(-u) du,$$

$$(10.3) \quad Z_\pm(s; \varphi) = \int_0^{\infty} u^s K(\pm u) du,$$

respectively. Applying the inverse formula of the Mellin transform to (10.3), we have

$$(10.4) \quad K(\pm u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z_\pm(s; \varphi) u^{-s-1} ds,$$

where $c > 0$ and the integral contour follows the line $\operatorname{Re}(s) = c$ upwards. Let us consider the case that $Z_{\pm}(s; \varphi)$ are meromorphic functions on \mathbb{C} and their poles exist on the negative part of the real axis. By deforming the integral contour as c tends to $-\infty$ in (10.4), the residue formula gives the asymptotic expansions of $K(u)$ as $u \rightarrow \pm 0$. Substituting these expansions of $K(u)$ into (10.2), we can get an asymptotic expansion of $I(t; \varphi)$ as $t \rightarrow +\infty$.

Through the above calculation, we see a specific relationship for the coefficients. If $Z_{\pm}(s; \varphi)$ have the Laurent expansions at $s = -\lambda$:

$$Z_{\pm}(s; \varphi) = \frac{B_{\pm}}{(s + \lambda)^{\rho}} + O\left(\frac{1}{(s + \lambda)^{\rho-1}}\right),$$

then the corresponding part in the asymptotic expansion of $I(t; \varphi)$ has the form

$$B\tau^{-\lambda}(\log t)^{\rho-1} + O(\tau^{-\lambda}(\log t)^{\rho-2}).$$

Here a simple computation gives the following relationship:

$$(10.5) \quad B = \frac{\Gamma(\lambda)}{(\rho - 1)!} \left[e^{i\pi\lambda/2} B_+ + e^{-i\pi\lambda/2} B_- \right],$$

where Γ is the Gamma function.

10.2. Proofs of Theorems 3.3, 3.5 and 3.7

Applying the above argument to the results relating to $Z_{\pm}(s; \varphi)$ in Section 9, we obtain the theorems in Section 3.

PROOF OF THEOREMS 3.3 and 3.5. These theorems are shown by using Theorem 9.1 with Proposition 9.5. Notice that Proposition 9.5 and the relationship (10.5) induce the cancellation of the coefficients of the terms, whose orders are larger than $-1/d(f)$. \square

PROOF OF THEOREM 3.7. This theorem follows from Theorem 9.10. Notice that the relationship (10.5) gives the information about the coefficient of the first term of $I(t; \varphi)$. \square

10.3. The first coefficient in the asymptotics (3.1)

From the relationship (10.5) and the equations (9.29), we give explicit formulae for the coefficient of the leading term of the asymptotic expansion in (3.1) as follows.

THEOREM 10.1. *If f satisfies the same conditions in Theorem 3.7 and the support of φ is contained in a sufficiently small neighborhood of the origin, then we have*

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{1/d(f)} (\log t)^{-m(f)+1} \cdot I(t; \varphi) \\ = \frac{\Gamma(1/d(f))}{(m(f) - 1)!} [e^{i\pi/(2d(f))} C_+ + e^{-i\pi/(2d(f))} C_-], \end{aligned}$$

where C_{\pm} are as in (9.29).

11. Examples

In this section, we consider the oscillatory integrals (1.1) with specific phases f , which satisfy the condition (A) in Section 3 and have noncompact principal faces. Moreover, in the first three examples, the phases belong to the class $\hat{\mathcal{E}}(U)$ and satisfy the nondegeneracy condition in Section 3. These examples cannot be directly dealt with by earlier investigations. Note that, for each phase, the origin is not a critical point of finite multiplicity in Tougeron’s theorem (see Remark 2.4). The last example shows that Theorem 3.1 (iii) cannot be directly generalized to the smooth case. In this section, we assume that the amplitudes φ satisfy the condition (B) in Section 3.

11.1. Example 1

Consider the following two-dimensional example:

$$(11.1) \quad f(x_1, x_2) = x_1^8 + x_1^7 x_2^1 + x_1^6 x_2^2 (1 + e^{-1/x_2^2}).$$

It is easy to determine important quantities and functions as follows. Let σ be a cone whose skeleton is a^1, a^2 , where $a^1 = (1, 0)$, $a^2 = (1, 1)$.

- $d(f) = 6$ and $m(f) = 1$,
- $\tau_* = \{(6, \alpha_2) : \alpha_2 \geq 2\}$, $\Sigma_*^{(2)} = \{\sigma\}$ and $A(\sigma) = \{1\}$,
- $l(a^1) = 6$, $l(a^2) = 8$,
- $\pi(\sigma)(y_1, y_2) = (y_1 y_2, y_2)$,
- $f_{\tau_*}(x) = x_1^6 x_2^2 (1 + e^{-1/x_2^2})$ and $(f_{\tau_*} \circ \pi(\sigma))(y) = y_1^6 y_2^8 (1 + e^{-1/y_2^2})$,

- $f_\sigma(y) = y_1^2 + y_1 + 1 + e^{-1/y_2^2}$

Substituting the above into (9.23) or (9.25), we have

$$\tilde{C}_+(f, \varphi) = \frac{1}{6} \int_0^\infty \frac{\varphi(0, y)}{y^{1/3}(1 + e^{-1/y^2})^{1/6}} dy$$

and $\tilde{C}_-(f, \varphi) = 0$. Moreover, we have

$$\lim_{t \rightarrow \infty} t^{1/6} \cdot I(t; \varphi) = \frac{e^{\pi i/12}}{3} \int_{-\infty}^\infty \frac{\varphi(0, y)}{|y|^{1/3}(1 + e^{-1/y^2})^{1/6}} dy.$$

Note that strong results relating to this example have been obtained in [23], [24] in the case where the phase is smooth and the principal face is compact, and in [15] in the case where the principal face is noncompact but the phase needs the real analyticity.

11.2. Example 2

Consider the following three-dimensional example:

$$(11.2) \quad f(x_1, x_2, x_3) = x_1^6 + x_1^4 x_2^2 e^{-1/x_3^2} + x_1^2 x_2^4 e^{-1/x_3^4} + x_2^6.$$

It is easy to determine important quantities and functions as follows. Let σ be a cone whose skeleton is a^1, a^2, a^3 , where $a^1 = (1, 0, 0)$, $a^2 = (1, 1, 0)$, $a^3 = (0, 0, 1)$.

- $d(f) = 3$ and $m(f) = 1$,
- $\tau_* = \{\alpha \in \mathbb{R}_+^3 : \alpha_1 + \alpha_2 = 6\}$, $\sigma \in \Sigma_*^{(3)}$ and $A(\sigma) = \{2\}$,
- $l(a^1) = l(a^3) = 0$ and $l(a^2) = 6$,
- $\pi(\sigma)(y_1, y_2, y_3) = (y_1 y_2, y_2, y_3)$,
- $f_{\tau_*}(x) = f(x)$ and $(f_{\tau_*} \circ \pi(\sigma))(y) = y_2^6(y_1^6 + y_1^4 e^{-1/y_3^2} + y_1^2 e^{-1/y_3^4} + 1)$,
- $f_\sigma(y) = y_1^6 + y_1^4 e^{-1/y_3^2} + y_1^2 e^{-1/y_3^4} + 1$.

Substituting the above into (9.23) or (9.25), we have

$$\tilde{C}_+(f, \varphi) = \frac{1}{6} \int_{\mathbb{R}_+^2} \frac{\varphi(0, 0, y_3)}{(y_1^6 + y_1^4 e^{-1/y_3^2} + y_1^2 e^{-1/y_3^4} + 1)^{1/3}} dy_1 dy_3$$

and $\tilde{C}_-(f, \varphi) = 0$. Moreover, we have

$$\begin{aligned} & \lim_{t \rightarrow \infty} t^{1/3} \cdot I(t; \varphi) \\ &= \Gamma(4/3)e^{\pi i/6} \int_{\mathbb{R}^2} \frac{\varphi(0, 0, y_3)}{(y_1^6 + y_1^4 e^{-1/y_3^2} + y_1^2 e^{-1/y_3^4} + 1)^{1/3}} dy_1 dy_3. \end{aligned}$$

11.3. Example 3

In the case of the following three-dimensional example, the logarithmic factor appears in the leading term of asymptotics of $I(t; \varphi)$:

$$(11.3) \quad f(x_1, x_2, x_3) = x_1^6 + x_1^2 x_2^2 (1 + e^{-1/x_3^2}) + x_2^6.$$

It is easy to determine important quantities and functions as follows. Let σ be a cone whose skeleton is a^1, a^2, a^3 , where $a^1 = (2, 1, 0)$, $a^2 = (1, 1, 0)$, $a^3 = (0, 0, 1)$.

- $d(f) = 2$ and $m(f) = 2$,
- $\tau_* = \{(2, 2, \alpha_3) : \alpha_3 \geq 0\}$, $\sigma \in \Sigma_*^{(3)}$ and $A(\sigma) = \{1, 2\}$,
- $l(a^1) = 6$, $l(a^2) = 4$, $l(a^3) = 0$,
- $\pi(\sigma)(y_1, y_2, y_3) = (y_1^2 y_2, y_1 y_2, y_3)$,
- $f_{\tau_*}(x) = x_1^2 x_2^2 (1 + e^{-1/x_3^2})$ and $(f_{\tau_*} \circ \pi(\sigma))(y) = y_1^6 y_2^4 (1 + e^{-1/y_3^2})$,
- $f_\sigma(y) = y_1^6 y_2^2 + y_2^2 + 1 + e^{-1/y_3^2}$.

Substituting the above into (9.23) or (9.25), we have

$$(11.4) \quad \tilde{C}_+(f, \varphi) = \frac{1}{24} \int_0^\infty \frac{\varphi(0, 0, y)}{(1 + e^{-1/y^2})^{1/2}} dy$$

and $\tilde{C}_-(f, \varphi) = 0$. Moreover, we have

$$\lim_{t \rightarrow \infty} t^{1/2} (\log t)^{-1} \cdot I(t; \varphi) = \frac{\sqrt{\pi} e^{i\pi/4}}{6} \int_{-\infty}^\infty \frac{\varphi(0, 0, y)}{(1 + e^{-1/y^2})^{1/2}} dy.$$

11.4. Example 4

Consider the following two-dimensional example given by Iosevich and Sawyer in [25]:

$$(11.5) \quad f(x_1, x_2) = x_1^2 + e^{-1/|x_2|^\alpha}, \quad \alpha > 0.$$

Note that the above f satisfies the nondegeneracy condition as in Section 3 but it does not belong to $\hat{\mathcal{E}}(U)$. It is easy to see the following:

- $d(f) = 2$ and $m(f) = 1$,
- $\tau_* = \{\alpha \in \mathbb{R}_+^2 : \alpha_1 = 2\}$,
- $f_{\tau_*}(x_1, x_2) = x_1^2$.

Consider an amplitude of the form: $\varphi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$ where ψ_j are smooth nonnegative functions on \mathbb{R} satisfying $\psi_j(0) > 0$ and its support is small for $j = 1, 2$. In [25], Iosevich and Sawyer shows:

$$|I(t; \varphi)| \leq Ct^{-1/2}(\log t)^{-1/\alpha} \quad \text{for } t \geq 2.$$

In particular, we have $\lim_{t \rightarrow \infty} t^{1/2}I(t; \varphi) = 0$, which is a phenomenon different from that in Remark 3.2. This example shows that the assertion (iii) in Theorem 3.1 with Remark 3.2 cannot be directly generalized to the smooth case. Moreover, the pattern of the asymptotic expansion in this case might be different from that of (3.1).

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Joe Kamimoto
Faculty of Mathematics
Kyushu University
Motooka 744, Nishi-ku
Fukuoka 819-0395, Japan
E-mail: joe@math.kyushu-u.ac.jp

Toshihiro Nose
Faculty of Engineering
Kyushu Sangyo University
Matsukadai 2-3-1, Higashi-ku
Fukuoka 813-8503, Japan
E-mail: t-nose@ip.kyusan-u.ac.jp