J. Math. Sci. Univ. Tokyo
23 (2016), 499–528.

On the Initial Value Problem for the Navier-Stokes Equations with the Initial Datum in Critical Sobolev and Besov Spaces

By D. Q. Khai and N. M. Tri

Abstract. The existence of local unique mild solutions to the Navier-Stokes equations in the whole space with an initial tempered distribution datum in critical homogeneous or inhomogeneous Sobolev spaces is shown. Especially, the case when the integral-exponent is less than 2 is investigated. The global existence is also obtained for the initial datum in critical homogeneous Sobolev spaces with a norm small enough in suitable critical Besov spaces. The key lemma is to establish the bilinear estimates in these spaces, due to the point-wise decay of the kernel of the heat semigroup.

§1. Introduction

We consider the Navier-Stokes equations (NSE) in d dimensions in special setting of a viscous, homogeneous, incompressible fluid which fills the entire space and is not submitted to external forces. Thus, the equations we consider are the system:

$$\begin{cases} \partial_t u = \Delta u - \nabla.(u \otimes u) - \nabla p, \\ \operatorname{div}(u) = 0, \\ u(0, x) = u_0, \end{cases}$$

which is a condensed writing for

$$\begin{cases} 1 \leq k \leq d, \quad \partial_t u_k = \Delta u_k - \sum_{l=1}^{d} \partial_l (u_l u_k) - \partial_k p, \\ \sum_{l=1}^{d} \partial_l u_l = 0, \\ 1 \leq k \leq d, \quad u_k(0, x) = u_{0k}. \end{cases}$$

2010 *Mathematics Subject Classification.* Primary 35Q30; Secondary 76D05, 76N10.
Key words: Navier-Stokes equations, existence and uniqueness of local and global mild solutions, critical Sobolev and Besov spaces.

The unknown quantities are the velocity $u(t,x) = (u_1(t,x), \dots, u_d(t,x))$ of the fluid element at time t and position x and the pressure $p(t,x)$.

A translation invariant Banach space of tempered distributions $\mathcal{E}$ is called a critical space for NSE if its norm is invariant under the action of the scaling $f(.) \longrightarrow \lambda f(\lambda.)$. One can take, for example, $\mathcal{E} = L^d(\mathbb{R}^d)$ or the smaller space $\mathcal{E} = \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$. In fact, one has the chain of critical spaces given by the continuous embeddings

$$\begin{aligned} (1) \qquad \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d) \hookrightarrow \dot{B}_q^{\frac{d}{q}-1,\infty}(\mathbb{R}^d)_{(d\leq q<\infty)} \\ \hookrightarrow BMO^{-1}(\mathbb{R}^d) \hookrightarrow \dot{B}_\infty^{-1,\infty}(\mathbb{R}^d). \end{aligned}$$

It is remarkable feature that NSE are well-posed in the sense of Hadarmard (existence, uniqueness and continuous dependence on data) when the initial datum is divergence-free and belong to the critical function spaces (except $\dot{B}_\infty^{-1,\infty}$) listed in (1) (see [7] for $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$, $L^d(\mathbb{R}^d)$, and $\dot{B}_q^{\frac{d}{q}-1,\infty}(\mathbb{R}^d)$, see [28] for $BMO^{-1}(\mathbb{R}^d)$. The recent ill-posedness result for $\dot{B}_\infty^{-1,\infty}(\mathbb{R}^d)$) with $d \geq 3$ was established in [3]. However, the ill-posedness in $\dot{B}_\infty^{-1,\infty}(\mathbb{R}^d)$ is still open when $d = 2$.

In the 1960s, mild solutions were first constructed by Kato and Fujita ([20], [16]) that are continuous in time and take values in the Sobolev space $H^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$, say $u \in C([0,T]; H^s(\mathbb{R}^d))$. In 1992, a modern treatment for mild solutions in $H^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$ was given by Chemin [11]. In 1995, using the simplified version of the bilinear operator, Cannone proved the existence of mild solutions in $\dot{H}^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$, see [7]. Results on the existence of mild solutions with value in $L^q(\mathbb{R}^d), (q > d)$ were established in the papers of Fabes, Jones and Rivière [14] and of Giga [17]. Concerning the initial datum in the space $L^\infty(\mathbb{R}^d)$, the existence of a mild solution was obtained by Cannone and Meyer in ([7], [10]). Moreover, in ([7], [10]), they also obtained theorems on the existence of mild solutions with value in Morrey-Campanato space $M_2^q(\mathbb{R}^d), (q > d)$ and Sobolev space $H_q^s(\mathbb{R}^d), (q < d, \frac{1}{q} - \frac{s}{d} < \frac{1}{d})$, and in general in the case of a so-called well-suited space $\mathcal{W}$ for NSE. NSE in the Morrey-Campanato spaces were also treated by Kato [22], Taylor [33], Kozono and Yamazaki [24].

In 1981, Weissler [34] gave the first existence result of mild solutions in the half space $L^3(\mathbb{R}^3_+)$. Then Giga and Miyakawa [18] generalized the result to $L^3(\Omega)$, where Ω is an open bounded domain in $\mathbb{R}^3$. Finally, in

1984, Kato [21] obtained, by means of a purely analytical tool (involving only the Hölder and Young inequalities and without using any estimate of fractional powers of the Stokes operator), an existence theorem in the whole space $L^3(\mathbb{R}^3)$. In ([7], [8], [9]), Cannone showed how to simplify Kato's proof. The idea is to take advantage of the structure of the bilinear operator in its scalar form. In particular, the divergence ∇ and heat $e^{t\Delta}$ operators can be treated as a single convolution operator. In 1994, Kato and Ponce [23] showed that NSE are well-posed when the initial datum belongs to the homogeneous Sobolev spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d), (d \le q < \infty)$. Recently, the authors of this article have considered NSE in mixed-norm Sobolev-Lorentz spaces and Sobolev-Fourier-Lorentz spaces, see [25] and [26] respectively. In [27], we showed that the bilinear operator

$$B(u,v)(t) = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau,.)\otimes v(\tau,.)\big)\mathrm{d}\tau \tag{2}$$

is bicontinuous in $L^\infty([0,T];\dot{H}_q^s(\mathbb{R}^d))$ with super-critical, non-negative-regular indexes $(0 \le s \le d, q > 1$, and $\frac{s}{d} < \frac{1}{q} < \min\Big\{\frac{s+1}{d}, \frac{s+d}{2d}\Big\})$, and we obtain the inequality

$$\big\|B(u,v)\big\|_{L^\infty([0,T];\dot{H}_q^s)} \le C_{s,q,d}T^{\frac{1}{2}(1+s-\frac{d}{q})}\big\|u\big\|_{L^\infty([0,T];\dot{H}_q^s)}\big\|v\big\|_{L^\infty([0,T];\dot{H}_q^s)}.$$

In this case existence and uniqueness theorems of local mild solutions can therefore be easily deduced.

In this paper, first, for $d \ge 3, s \ge 0, p > 1,$ and $r > 2$ be such that $\frac{s}{d} < \frac{1}{p} < \frac{1}{2} + \frac{s}{2d}$ and $\frac{2}{r} + \frac{d}{p} - s \le 1$, we investigate mild solutions to NSE in the spaces $L^r\big([0,T];\dot{H}_p^s(\mathbb{R}^d)\big)$. We obtain the existence of local mild solutions with arbitrary initial tempered distribution datum in the Besov spaces $B_p^{s-\frac{2}{r},r}$. In the case of critical indexes $\frac{2}{r} - s + \frac{d}{p} = 1$, we obtain the existence of global mild solutions when the norm of the initial tempered distribution datum in the Besov space $\dot{B}_p^{s-\frac{2}{r},r}$ is small enough. The particular case of the above result, when $s = 0$, was presented in the book by Lemarie-Rieusset [29]. We also note that the Cauchy problem for an incompressible magneto-hydrodynamics system with positive viscosity and magnetic resistivity, in the framework of the Besov spaces was considered in [30].

Next, we present two different algorithms for constructing mild solutions in $C([0,T];\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d))$ or $C([0,T];H_q^{\frac{d}{q}-1}(\mathbb{R}^d))$ to the Cauchy problem for the Navier-Stokes equations when the initial datum belongs to the Sobolev spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ $\big($or $H_q^{\frac{d}{q}-1}(\mathbb{R}^d)\big)$. We use the first algorithm to consider the case when the initial datum belongs to $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ or $H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ with $3 \leq d \leq 4$ and $2 \leq q \leq d$. Our results, when $q = d$, are a generalization the ones obtained in [29]. With the second algorithm, we can treat the case when the initial datum belongs to the critical spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ with $d \geq 3$ and $1 < q \leq d$. The cases $q = 2$ and $q = d$ were considered by many authors, see ([7], [9], [11], [12], [16], [20], [21], [29], [31]). A part of our results in the case when $2 < q < d$ can also be obtained by using the interpolation method of the results between the spaces $\dot{H}^{\frac{d}{2}}$ and L^d. So we will concentrate our efforts on the case $1 < q < 2$. To obtain the existence theorem in $C([0,T];\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d))$, we need to establish the continuity of the bilinear operator B from

$$L^{2q}\Big([0,T];\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\Big) \times L^{2q}\Big([0,T];\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\Big) \text{ to } C([0,T];\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)),$$

and establishes the continuity of the bilinear operator B from $L^r([0,T];H_p^s) \times L^r([0,T];H_p^s)$ into $L^r([0,T];H_p^s)$. In order to evaluate the norm of the bilinear operator B in these spaces we use Lemma 7 which estimates the point-wise product of two functions in $\dot{H}_q^s(\mathbb{R}^d)$.

The paper is organized as follows. In Section 2 we recall some embedding theorems in the Triebel and Besov spaces and auxiliary lemmas. In Section 3 we present the main results of the paper.

In the sequence, for a space of functions defined on $\mathbb{R}^d$, say $E(\mathbb{R}^d)$, we will abbreviate it as E.

§2. Some Imbedding Theorems

In this paper we use the definition of the Besov space $B_q^{s,p}$, the Triebel space $F_q^{s,p}$, and their homogeneous space $\dot{B}_q^{s,p}$ and $\dot{F}_q^{s,p}$ in [5, 6, 13, 32]. A known property of these spaces is the Riesz potential $\dot{\Lambda}^s = (-\Delta)^{s/2}$ which is an isomorphism from $\dot{B}_q^{s_0,p}$ onto $\dot{B}_q^{s_0-s,p}$ and from $\dot{F}_q^{s_0,p}$ to $\dot{F}_q^{s_0-s,p}$, see [4].

Let $1 < q < \infty$ and $s < d/q$, we define the homogeneous Sobolev space $\dot{H}^s_q$ as the closure of the space $S_0 = \{f \in \mathcal{S} : \ 0 \notin \mathrm{Supp}\hat{f}\}$ in the norm $\|f\|_{\dot{H}^s_q} = \|\dot{\Lambda}^s f\|_q$. Let us recall the following lemmas.

LEMMA 1. *Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$.*
(a) *If $s < 1$ then the two quantities*

$$\Big(\int_0^\infty \big(t^{-\frac{s}{2}}\big\|e^{t\Delta}t^{\frac{1}{2}}\dot{\Lambda}f\big\|_q\big)^p\frac{\mathrm{d}t}{t}\Big)^{1/p} \text{ and } \big\|f\big\|_{\dot{B}^{s,p}_q} \text{ are equivalent.}$$

(b) *If $s < 0$ then the two quantities*

$$\Big(\int_0^\infty \big(t^{-\frac{s}{2}}\big\|e^{t\Delta}f\big\|_q\big)^p\frac{\mathrm{d}t}{t}\Big)^{1/p} \text{ and } \big\|f\big\|_{\dot{B}^{s,p}_q} \text{ are equivalent.}$$

PROOF. See ([15], Proposition 1, p. 181 and Proposition 3, p. 182), or see ([29], Theorem 5.4, p. 45). □

The following lemma is a generalization of the above lemma.

LEMMA 2. *Let $1 \leq p, q \leq \infty$, $\alpha \geq 0$, and $s < \alpha$. Then the two quantities*

$$\Big(\int_0^\infty (t^{-\frac{s}{2}}\big\|e^{t\Delta}t^{\frac{\alpha}{2}}\dot{\Lambda}^\alpha f\big\|_{L^q})^p\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}} \text{ and } \big\|f\big\|_{\dot{B}^{s,p}_q} \text{ are equivalent.}$$

PROOF. Note that $\dot{\Lambda}^{s_0}$ is an isomorphism from $\dot{B}^{s,p}_q$ to $\dot{B}^{s-s_0,p}_q$, then we can easily prove the lemma. □

LEMMA 3. *For $1 \leq p, q, r \leq \infty$ and $s \in \mathbb{R}$, we have the following embedding mappings.*
(a) *If $1 < q \leq 2$ then*

$$\dot{B}^{s,q}_q \hookrightarrow \dot{H}^s_q \hookrightarrow \dot{B}^{s,2}_q,\ B^{s,q}_q \hookrightarrow H^s_q \hookrightarrow B^{s,2}_q.$$

(b) *If $2 \leq q < \infty$ then*

$$\dot{B}^{s,2}_q \hookrightarrow \dot{H}^s_q \hookrightarrow \dot{B}^{s,q}_q,\ B^{s,2}_q \hookrightarrow H^s_q \hookrightarrow B^{s,q}_q.$$

(c) *If* $1 \le p_1 < p_2 \le \infty$ *then*

$$\dot{B}_q^{s,p_1} \hookrightarrow \dot{B}_q^{s,p_2}, B_q^{s,p_1} \hookrightarrow B_q^{s,p_2},\ \dot{F}_q^{s,p_1} \hookrightarrow \dot{F}_q^{s,p_2}, F_q^{s,p_1} \hookrightarrow F_q^{s,p_2}.$$

(d) *If* $s_1 > s_2,\ 1 \le q_1,\ q_2 \le \infty$, *and* $s_1 - \frac{d}{q_1} = s_2 - \frac{d}{q_2}$ *then*

$$\dot{B}_{q_1}^{s_1,p} \hookrightarrow \dot{B}_{q_2}^{s_2,p}, B_{q_1}^{s_1,p} \hookrightarrow B_{q_2}^{s_2,p}, \dot{F}_{q_1}^{s_1,p} \hookrightarrow \dot{F}_{q_2}^{s_2,r},\ F_{q_1}^{s_1,p} \hookrightarrow F_{q_2}^{s_2,r}.$$

(e) *If* $p \le q$ *then*

$$B_q^{s,p} \hookrightarrow F_q^{s,p},\ \dot{B}_q^{s,p} \hookrightarrow \dot{F}_q^{s,p}.$$

(f) *If* $q \le p$ *then*

$$F_q^{s,p} \hookrightarrow B_q^{s,p},\ \dot{F}_q^{s,p} \hookrightarrow \dot{B}_q^{s,p}.$$

(g)

$$F_q^{s,q} = B_q^{s,q},\ \dot{F}_q^{s,q} = \dot{B}_q^{s,q}.$$

(h) *If* $1 < q < \infty$

$$H_q^s = F_q^{s,2},\ \dot{H}_q^s = \dot{F}_q^{s,2}.$$

PROOF. For the proof of (a) and (b) see Theorem 6.4.4 ([2], p. 152). For the proof of (c) see [1] and [2]. For the proof of (d) see Theorem 6.5.1 ([2], p. 153) and [4]. For the proof of (e), (f), (g), and (h) see [1] and [4]. □

LEMMA 4. *Let* $p \ge 1$ *and* $s \in \mathbb{R}$. *Then the following statements hold*
(1) *Assume that* $u_0 \in H_p^s$. *Then*

$$e^{t\Delta}u_0 \in L^\infty([0,\infty); H_p^s) \text{ and } \big\|e^{t\Delta}u_0\big\|_{L^\infty([0,\infty);H_p^s)} \le \big\|u_0\big\|_{H_p^s}.$$

(2) *Assume that* $u_0 \in \dot{H}_p^s$. *Then*

$$e^{t\Delta}u_0 \in L^\infty([0,\infty); \dot{H}_p^s) \text{ and } \big\|e^{t\Delta}u_0\big\|_{L^\infty([0,\infty);\dot{H}_p^s)} \le \big\|u_0\big\|_{\dot{H}_p^s}.$$

PROOF. (1) We have

$$\begin{aligned}&\big\|e^{t\Delta}u_0\big\|_{H^s_p} = \big\|e^{t\Delta}(Id-\Delta)^{s/2}u_0\big\|_{L^p} = \\ &\frac{1}{(4\pi t)^{d/2}}\Big\|\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\big((Id-\Delta)^{s/2}u_0\big)(\,.-\xi)\mathrm{d}\xi\Big\|_{L^p} \\ &\leq \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\big\|\big((Id-\Delta)^{s/2}u_0\big)(\,.-\xi)\big\|_{L^p}\mathrm{d}\xi \\ &= \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\big\|u_0\big\|_{H^s_p}\mathrm{d}\xi = \big\|u_0\big\|_{H^s_p},\ t\geq 0.\end{aligned}$$

(2) The proof of (2) is similar to the proof of (1). □

THEOREM 1. *Let E be an Banach space, and let $B : E\times E \to E$ be a continuous bilinear form such that there exists $\eta > 0$ so that*

$$\|B(x,y)\| \leq \eta\|x\|\|y\|,$$

for all x and y in E. Then for any fixed $y\in E$ such that $\|y\| \leq \frac{1}{4\eta}$, the equation $x = y - B(x,x)$ has a unique solution $\overline{x}\in E$ satisfying $\|\overline{x}\| \leq \frac{1}{2\eta}$.

PROOF. See Theorem 22.4 ([29], p. 227). □

The following lemmas, in which we estimate the point-wise product of two functions in $\dot{H}^s_p(\mathbb{R}^d)$ is more general than the Hölder inequality. In the case when $s = 0, p \geq 2$, we get back the usual Hölder inequality.

LEMMA 5. *Assume that*

$$1 < p,q < d \text{ and } \frac{1}{p}+\frac{1}{q} < 1+\frac{1}{d}.$$

Then there exists a constant C independent of u,v such that the following inequality holds

$$\big\|uv\big\|_{\dot{H}^1_r} \leq C\big\|u\big\|_{\dot{H}^1_p}\big\|v\big\|_{\dot{H}^1_q},\ \forall u\in\dot{H}^1_p, v\in\dot{H}^1_q,$$

where $\frac{1}{r} = \frac{1}{p}+\frac{1}{q}-\frac{1}{d}$. In the subsequence the above kinds of conclusions will be shorten as

$$\big\|uv\big\|_{\dot{H}^1_r} \lesssim \big\|u\big\|_{\dot{H}^1_p}\big\|v\big\|_{\dot{H}^1_q}.$$

Proof. By applying the Leibniz formula for the derivatives of a product of two functions, we have

$$\|uv\|_{\dot{H}^1_r} \simeq \sum_{|\alpha|=1} \|\partial^\alpha(uv)\|_{L^r} \leq \sum_{|\alpha|=1} \|(\partial^\alpha u)v\|_{L^r} + \sum_{|\alpha|=1} \|u(\partial^\alpha v)\|_{L^r}.$$

From the Hölder and Sobolev inequalities it follows that

$$\sum_{|\alpha|=1} \|(\partial^\alpha u)v\|_{L^r} \leq \sum_{|\alpha|=1} \|\partial^\alpha u\|_{L^p} \|v\|_{L^{q_1}} \lesssim \|u\|_{\dot{H}^1_p} \|v\|_{\dot{H}^1_q},$$

where

$$\frac{1}{q_1} = \frac{1}{q} - \frac{1}{d}.$$

Similar to the above proof, we have

$$\sum_{|\alpha|=1} \|u(\partial^\alpha v)\|_{L^r} \lesssim \|u\|_{\dot{H}^1_p} \|v\|_{\dot{H}^1_q}.$$

This gives the desired result

$$\|uv\|_{\dot{H}^1_r} \lesssim \|u\|_{\dot{H}^1_p} \|v\|_{\dot{H}^1_q}. \square$$

Lemma 6. *Assume that*

$$0 \leq s \leq 1, \frac{1}{p} > \frac{s}{d}, \frac{1}{q} > \frac{s}{d}, \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}. \tag{3}$$

Then the following inequality holds

$$\|uv\|_{\dot{H}^s_r} \lesssim \|u\|_{\dot{H}^s_p} \|v\|_{\dot{H}^s_q}, \ \forall u \in \dot{H}^s_p, v \in \dot{H}^s_q,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{s}{d}$.

Proof. It is not difficult to show that if $p, q,$ and s satisfy (3) then there exist numbers $p_1, p_2, q_1, q_2 \in (1, +\infty)$ (may be many of them) such that

$$\frac{1}{p} = \frac{1-s}{p_1} + \frac{s}{p_2}, \frac{1}{q} = \frac{1-s}{q_1} + \frac{s}{q_2}, \frac{1}{p_1} + \frac{1}{q_1} < 1,$$
$$p_2 < d, q_2 < d, \text{ and } \frac{1}{p_2} + \frac{1}{q_2} < 1 + \frac{1}{d}.$$

Setting

$$\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{q_1}, \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{d},$$

we have

$$\frac{1}{r} = \frac{1-s}{r_1} + \frac{s}{r_2}.$$

Therefore, applying Theorem 6.4.5 (p. 152) of [2] (see also [19] for $\dot{H}^s_p$), we get

$$\dot{H}^s_p = [L^{p_1}, \dot{H}^1_{p_2}]_s, \dot{H}^s_q = [L^{q_1}, \dot{H}^1_{q_2}]_s, \dot{H}^s_r = [L^{r_1}, \dot{H}^1_{r_2}]_s.$$

Applying the Hölder inequality and Lemma 5 in order to obtain

$$\big\|uv\big\|_{L^{r_1}} \lesssim \big\|u\big\|_{L^{p_1}} \big\|v\big\|_{L^{q_1}}, \ \forall u \in L^{p_1}, v \in L^{q_1},$$
$$\big\|uv\big\|_{\dot{H}^1_{r_2}} \lesssim \big\|u\big\|_{\dot{H}^1_{p_2}} \big\|v\big\|_{\dot{H}^1_{q_2}}, \ \forall u \in \dot{H}^1_{p_2}, v \in \dot{H}^1_{q_2}.$$

From Theorem 4.4.1 (p. 96) of [2] we get

$$\big\|uv\big\|_{\dot{H}^s_r} \lesssim \big\|u\big\|_{\dot{H}^s_p}\big\|v\big\|_{\dot{H}^s_q}. \ \square$$

Lemma 7. *Assume that*

$$0 \le s < d, \frac{s}{d} < \frac{1}{p}, \frac{s}{d} < \frac{1}{q}, \ \text{and} \ \frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}. \tag{4}$$

Then we have the inequality

$$\big\|uv\big\|_{\dot{H}^s_r} \lesssim \big\|u\big\|_{\dot{H}^s_p}\big\|v\big\|_{\dot{H}^s_q}, \ \forall u \in \dot{H}^s_p, v \in \dot{H}^s_q,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{s}{d}$.

Proof. Denote by $[s]$ the integer part of s and by $\{s\}$ the fraction part of s. Using formula for the derivatives of a product of two functions, we have

$$\big\|uv\big\|_{\dot{H}^s_r} = \big\|\dot{\Lambda}^s(uv)\big\|_{L^r} = \big\|\dot{\Lambda}^{\{s\}}(uv)\big\|_{\dot{H}^{[s]}_r} \simeq$$
$$\sum_{|\alpha|=[s]} \big\|\partial^\alpha \dot{\Lambda}^{\{s\}}(uv)\big\|_{L^r} = \sum_{|\alpha|=[s]} \big\|\dot{\Lambda}^{\{s\}}\partial^\alpha(uv)\big\|_{L^r}$$
$$= \sum_{|\alpha|=[s]} \big\|\partial^\alpha(uv)\big\|_{\dot{H}^{\{s\}}_r} \lesssim \sum_{|\gamma|+|\beta|=[s]} \big\|\partial^\gamma u \partial^\beta v\big\|_{\dot{H}^{\{s\}}_r}.$$

Set
$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{s - |\gamma| - \{s\}}{d}, \frac{1}{\tilde{q}} = \frac{1}{q} - \frac{s - |\beta| - \{s\}}{d}.$$
Applying Lemma 6 and the Sobolev inequality in order to obtain
$$\begin{aligned}\big\|\partial^\gamma u \partial^\beta v\big\|_{\dot{H}^{\{s\}}_r} &\lesssim \big\|\partial^\gamma u\big\|_{\dot{H}^{\{s\}}_{\tilde{p}}} \big\|\partial^\beta v\big\|_{\dot{H}^{\{s\}}_{\tilde{q}}} \\ &\lesssim \big\|u\big\|_{\dot{H}^{|\gamma|+\{s\}}_{\tilde{p}}} \big\|v\big\|_{\dot{H}^{|\beta|+\{s\}}_{\tilde{q}}} \lesssim \big\|u\big\|_{\dot{H}^s_p} \big\|v\big\|_{\dot{H}^s_q}.\end{aligned}$$
This gives the desired result
$$\big\|uv\big\|_{\dot{H}^s_r} \lesssim \big\|u\big\|_{\dot{H}^s_p} \big\|v\big\|_{\dot{H}^s_q}. \ \square$$

Remark 1. Lemmas 5, 6, and 7 are still valid when the homogeneous space $\dot{H}^s_p$ is replaced by the inhomogeneous space H^s_p.

§3. The Main Results

For $T > 0$, we say that u is a mild solution of NSE on $[0, T]$ corresponding to a divergence-free initial data u_0 when u satisfies the integral equation
$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau, .) \otimes u(\tau, .)\big)\mathrm{d}\tau.$$
Above we have used the following notation: For a tensor $F = (F_{ij})$ we define the vector $\nabla.F$ by $(\nabla.F)_i = \sum_{i=1}^d \partial_j F_{ij}$ and for vectors u and v, we define their tensor product $(u \otimes v)_{ij} = u_i v_j$. The operator $\mathbb{P}$ is the Helmholtz-Leray projection onto the divergence-free fields
$$(\mathbb{P}f)_j = f_j + \sum_{1 \le k \le d} R_j R_k f_k, \tag{5}$$
where R_j is the Riesz transforms defined on a scalar function g as
$$\widehat{R_j g}(\xi) = \frac{i\xi_j}{|\xi|}\hat{g}(\xi).$$
The heat kernel $e^{t\Delta}$ is defined as
$$e^{t\Delta}u(x) = ((4\pi t)^{-d/2}e^{-|.|^2/4t} * u)(x).$$

If X is a normed space and $u = (u_1, u_2, ..., u_d), u_i \in X, 1 \le i \le d$, then we write

$$u \in X, \|u\|_X = \Big(\sum_{i=1}^{d} \|u_i\|_X^2\Big)^{1/2}.$$

3.1. On the continuity and regularity of the bilinear operator

In this subsection a particular attention will be devoted to the study of the bilinear operator $B(u, v)(t)$ defined by (2).

LEMMA 8. *Let*

$$d \ge 3,\ \ s \ge 0,\ \ p > 1,\ \ r > 2, \textit{and } T > 0 \tag{6}$$

be such that

$$\frac{s}{d} < \frac{1}{p} < \frac{1}{2} + \frac{s}{2d} \textit{ and } \frac{2}{r} + \frac{d}{p} - s \le 1. \tag{7}$$

Then the bilinear operator $B(u, v)(t)$ is continuous from

$$L^r([0, T]; H_p^s) \times L^r([0, T]; H_p^s)$$

into

$$L^r([0, T]; H_p^s),$$

and the following inequality holds

$$\big\|B(u, v)\big\|_{L^r([0,T];H_p^s)} \le CT^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})}\big\|u\big\|_{L^r([0,T];H_p^s)}\big\|v\big\|_{L^r([0,T];H_p^s)}, \tag{8}$$

where C is a positive constant independent of T.

PROOF. We have

$$\big\|B(u, v)(t)\big\|_{H_p^s} \le \int_0^t \Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau, .) \otimes v(\tau, .)\big)\Big\|_{H_p^s} \mathrm{d}\tau = \tag{9}$$
$$\int_0^t \Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id - \Delta)^{s/2}\big(u(\tau, .) \otimes v(\tau, .)\big)\Big\|_{L^p} \mathrm{d}\tau,$$

where the operator $(Id - \Delta)^{\frac{s}{2}}$ is defined via the Fourier transform as

$$\big((Id - \Delta)^{\frac{s}{2}} g\big)^{\wedge}(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \hat{g}(\xi).$$

We have

$$\Big(e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big)_j =$$
$$e^{(t-\tau)\Delta}\sum_{l,k=1}^{d}\Big(\delta_{jk}-\frac{\partial_j\partial_k}{\Delta}\Big)\partial_l(Id-\Delta)^{s/2}\big(u_l(\tau,.)v_k(\tau,.)\big).$$

From the property of the Fourier transform we have

$$\Big(e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big)_j^{\wedge}(\xi) =$$
$$e^{-(t-\tau)|\xi|^2}\sum_{l,k=1}^{d}\Big(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l)\Big((Id-\Delta)^{s/2}\big(u_l(\tau,.)v_k(\tau,.)\big)\Big)^{\wedge}(\xi),$$

and therefore

$$\Big(e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big)_j = \tag{10}$$
$$\frac{1}{(t-\tau)^{\frac{d+1}{2}}}\sum_{l,k=1}^{d}K_{l,k,j}\Big(\frac{.}{\sqrt{t-\tau}}\Big)*\Big((Id-\Delta)^{s/2}\big(u_l(\tau,.)v_k(\tau,.)\big)\Big),$$

where

$$\widehat{K_{l,k,j}}(\xi)=\frac{1}{(2\pi)^{d/2}}.e^{-|\xi|^2}\Big(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l).$$

Applying Proposition 11.1 ([29], p. 107) with $|\alpha|=1$ we obtain

$$|K_{l,k,j}(x)|\lesssim\frac{1}{(1+|x|)^{d+1}}.$$

Thus, the tensor $K(x)=\{K_{l,k,j}(x)\}$ satisfies

$$|K(x)|\lesssim\frac{1}{(1+|x|)^{d+1}}. \tag{11}$$

So, we can rewrite the equality (10) in the tensor form

$$e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big) =$$
$$\frac{1}{(t-\tau)^{\frac{d+1}{2}}}K\Big(\frac{.}{\sqrt{t-\tau}}\Big)*\Big((Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big).$$

Set

$$\frac{1}{\tilde{p}} = \frac{2}{p} - \frac{s}{d},\ \frac{1}{h} = \frac{s}{d} - \frac{1}{p} + 1. \tag{12}$$

Note that from the inequalities (6) and (7), we can check that the following relations are satisfied

$$1 < h, \tilde{p} < \infty \text{ and } \frac{1}{p} + 1 = \frac{1}{h} + \frac{1}{\tilde{p}}.$$

Applying the Young inequality for convolution we obtain

$$\begin{aligned} &\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big\|_{L^p} \lesssim \\ &\frac{1}{(t-\tau)^{\frac{d+1}{2}}}\Big\|K\Big(\frac{.}{\sqrt{t-\tau}}\Big)\Big\|_{L^h}\Big\|(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big\|_{L^{\tilde{p}}}. \end{aligned} \tag{13}$$

Applying Lemma 7

$$\begin{aligned} \Big\|(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big\|_{L^{\tilde{p}}} &= \big\|u(\tau,.)\otimes v(\tau,.)\big\|_{H^s_{\tilde{p}}} \\ &\lesssim \big\|u(\tau,.)\big\|_{H^s_p}\big\|v(\tau,.)\big\|_{H^s_p}. \end{aligned} \tag{14}$$

From the estimate (11) and the equality (12), we have

$$\Big\|K\Big(\frac{.}{\sqrt{t-\tau}}\Big)\Big\|_{L^h} = (t-\tau)^{\frac{d}{2h}}\big\|K\big\|_{L^h} \simeq (t-\tau)^{\frac{s}{2}-\frac{d}{2p}+\frac{d}{2}}. \tag{15}$$

The inequalities (13), (14), and (15) imply that

$$\begin{aligned} &\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big\|_{L^p} \lesssim \\ &(t-\tau)^{\frac{s}{2}-\frac{d}{2p}-\frac{1}{2}}\big\|u(\tau,.)\big\|_{H^s_p}\big\|v(\tau,.)\big\|_{H^s_p}. \end{aligned} \tag{16}$$

From the inequalities (9) and (16), we get

$$\big\|B(u,v)(t)\big\|_{H^s_p} \lesssim \int_0^t (t-\tau)^{\frac{s}{2}-\frac{d}{2p}-\frac{1}{2}}\big\|u(\tau,.)\big\|_{H^s_p}\big\|v(\tau,.)\big\|_{H^s_p}\mathrm{d}\tau.$$

Applying of Proposition 2.4 (c) in ([29], p. 20) for the convolution in the Lorentz spaces, we have the following estimates

$$\begin{aligned}(17)\qquad \Big\|\big\|B(u,v)(t)\big\|_{H^s_p}\Big\|_{L^r_t(0,T)} &= \Big\|\big\|B(u,v)(t)\big\|_{H^s_p}\Big\|_{L^{r,r}_t(0,T)} \\ &\le \Big\|\big\|B(u,v)(t)\big\|_{H^s_p}\Big\|_{L^{r,\frac{r}{2}}_t(0,T)} \lesssim \\ \big\|1_{[0,T]}t^{\frac{s}{2}-\frac{d}{2p}-\frac{1}{2}}\big\|_{L^{r',\infty}} &\Big\|\big\|u(t,.)\big\|_{H^s_p}\big\|v(t,.)\big\|_{H^s_p}\Big\|_{L^{\frac{r}{2},\frac{r}{2}}_t(0,T)},\end{aligned}$$

where $\frac{1}{r'}+\frac{1}{r}=1$ and $1_{[0,T]}$ is the indicator function of set $[0,T]$ on $\mathbb{R}$.

By applying the Hölder inequality we get

$$\begin{aligned}(18)\qquad \Big\|\big\|u(t,.)\big\|_{H^s_p}\big\|v(t,.)\big\|_{H^s_p}\Big\|_{L^{\frac{r}{2},\frac{r}{2}}_t(0,T)} &= \Big\|\big\|u(t,.)\big\|_{H^s_p}\big\|v(t,.)\big\|_{H^s_p}\Big\|_{L^{\frac{r}{2}}_t(0,T)} \\ &\le \Big\|\big\|u(t,.)\big\|_{H^s_p}\Big\|_{L^r_t(0,T)}\Big\|\big\|v(t,.)\big\|_{H^s_p}\Big\|_{L^r_t(0,T)}.\end{aligned}$$

Note that

$$(19)\qquad \Big\|1_{[0,T]}t^{\frac{s}{2}-\frac{d}{2p}-\frac{1}{2}}\Big\|_{L^{r',\infty}} \simeq T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})}.$$

Therefore the inequality (8) can be deduced from the inequalities (17), (18), and (19). □

Remark 2. Lemma 8 is still valid when the inhomogeneous space H^s_p is replaced by the homogeneous space $\dot{H}^s_p$.

Lemma 9. *Let*

$$d\ge 3,\ 0\le s<d,\ p>1,\ r>2, \textit{and } T>0$$

be such that

$$\frac{1}{p}<\frac{1}{2}+\frac{s}{2d}, \frac{2}{p}\ge\frac{s+1}{d},\ \textit{and } \frac{2}{r}+\frac{d}{p}-s=1.$$

Then the bilinear operator $B(u,v)(t)$ is continuous from

$$L^r\big([0,T];\dot{H}^s_p\big)\times L^r\big([0,T];\dot{H}^s_p\big)$$

into

$$L^\infty\Big([0,T];\dot{B}_{\tilde{p}}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}\Big),$$

where

$$\frac{1}{\tilde{p}}=\frac{2}{p}-\frac{s}{d},$$

and we have the inequality

$$\|B(u,v)\|_{L^\infty\big([0,T];\dot{B}_{\tilde{p}}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}\big)}\leq C\|u\|_{L^r([0,T];\dot{H}^s_p)}\|v\|_{L^r([0,T];\dot{H}^s_p)}, \tag{20}$$

where C is a positive constant independent of T.

Proof. To prove this lemma by duality (in the x-variable), (see Proposition 3.9 in ([29], p. 29)), let us consider an arbitrary test function $h(x)\in\mathcal{S}(\mathbb{R}^d)$ and evaluate the quantity

$$I_t=\big\langle B(u,v)(t),h\big\rangle=\int_{\mathbb{R}^d}\big(B(u,v)(t)\big)(x)h(x)\mathrm{d}x. \tag{21}$$

We have

$$\begin{aligned}\big\langle B(u,v)(t),h\big\rangle&=\int_0^t\big\langle e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau,.)\otimes v(\tau,.)\big),h\big\rangle\mathrm{d}\tau=\\&\int_0^t\Big\langle e^{(t-\tau)\Delta}\dot{\Lambda}\mathbb{P}\frac{\nabla}{\dot{\Lambda}}.\big(u(\tau,.)\otimes v(\tau,.)\big),h\Big\rangle\mathrm{d}\tau=\\&\int_0^t\Big\langle\mathbb{P}\frac{\nabla}{\dot{\Lambda}}.\big(u(\tau,.)\otimes v(\tau,.)\big),e^{(t-\tau)\Delta}\dot{\Lambda}h\Big\rangle\mathrm{d}\tau=\\&\int_0^t\Big\langle\mathbb{P}\frac{\nabla}{\dot{\Lambda}}.\dot{\Lambda}^s\big(u(\tau,.)\otimes v(\tau,.)\big),e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\Big\rangle\mathrm{d}\tau.\end{aligned} \tag{22}$$

By applying the Hölder inequality in the x-variable, from the equality (22) and the fact that (see [29])

$$\mathbb{P}\text{ and }\frac{\nabla}{\dot{\Lambda}}\text{ are continuous from }L^p\text{ into }L^p, 1<p<\infty,$$

we get

$$\begin{aligned}|I_t|&\leq\int_0^t\Big\|\mathbb{P}\frac{\nabla}{\dot{\Lambda}}.\dot{\Lambda}^s\big(u(\tau,.)\otimes v(\tau,.)\big)\Big\|_{L^{\tilde{p}}}\big\|e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\ \mathrm{d}\tau\\&\lesssim\int_0^t\big\|\dot{\Lambda}^s\big(u(\tau,.)\otimes v(\tau,.)\big)\big\|_{L^{\tilde{p}}}\big\|e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\ \mathrm{d}\tau,\end{aligned} \tag{23}$$

where

$$\frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1.$$

Applying Lemma 7, we have

$$\begin{aligned} (24) \qquad \big\|\dot{\Lambda}^s\big(u(\tau,.)\otimes v(\tau,.)\big)\big\|_{L^{\tilde{p}}} &= \big\|u(\tau,.)\otimes v(\tau,.)\big\|_{\dot{H}^s_{\tilde{p}}} \\ &\lesssim \big\|u(\tau,.)\big\|_{\dot{H}^s_p}\big\|v(\tau,.)\big\|_{\dot{H}^s_p}. \end{aligned}$$

From the inequalities (23) and (24), applying the Hölder inequality in the t-variable, we deduce that

$$\begin{aligned} (25) \qquad |I_t| &\lesssim \int_0^t \big\|u(\tau,.)\big\|_{\dot{H}^s_p}\big\|v(\tau,.)\big\|_{\dot{H}^s_p}\big\|e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\,\mathrm{d}\tau \leq \\ &\Big(\int_0^t \big(\big\|u(\tau,.)\big\|_{\dot{H}^s_p}\big\|v(\tau,.)\big\|_{\dot{H}^s_p}\big)^{\frac{r}{2}}\mathrm{d}\tau\Big)^{\frac{2}{r}}\Big(\int_0^t \big(\big\|e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}d\tau\Big)^{\frac{r-2}{r}} \\ &\leq \big\|u\big\|_{L^r([0,T];\dot{H}^s_p)}\big\|v\big\|_{L^r([0,T];\dot{H}^s_p)}\Big(\int_0^t \big(\big\|e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\mathrm{d}\tau\Big)^{\frac{r-2}{r}}. \end{aligned}$$

From Lemma 1 and note that $\dot{\Lambda}^{s_0}$ is an isomorphism from $\dot{B}^{s,p}_q$ to $\dot{B}^{s-s_0,p}_q$ (see [4]), we have the following estimates

$$\begin{aligned} (26) \qquad &\Big(\int_0^t \big(\big\|e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\mathrm{d}\tau\Big)^{\frac{r-2}{r}} \\ &\leq \Big(\int_0^\infty \big(\big\|e^{t\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\mathrm{d}t\Big)^{\frac{r-2}{r}} \\ &= \Big(\int_0^\infty \big(t^{\frac{r-4}{2r}}\big\|e^{t\Delta}t^{\frac{1}{2}}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\frac{\mathrm{d}t}{t}\Big)^{\frac{r-2}{r}} \simeq \big\|\dot{\Lambda}^{-s}h\big\|_{\dot{B}^{\frac{4-r}{r},\frac{r}{r-2}}_{\tilde{p}'}} \\ &\simeq \big\|h\big\|_{\dot{B}^{\frac{4-r}{r}-s,\frac{r}{r-2}}_{\tilde{p}'}} = \big\|h\big\|_{\dot{B}^{1-\frac{d}{\tilde{p}},\frac{r}{r-2}}_{\tilde{p}'}}. \end{aligned}$$

From the equality (21) and the inequalities (25) and (26), we get

$$\big|\big\langle B(u,v)(t),h\big\rangle\big| \lesssim \big\|u\big\|_{L^r([0,T];\dot{H}^s_p)}\big\|v\big\|_{L^r([0,T];\dot{H}^s_p)}\Big\|h\Big\|_{\dot{B}^{1-\frac{d}{\tilde{p}},\frac{r}{r-2}}_{\tilde{p}'}}.$$

However, $\dot{B}^{1-\frac{d}{\tilde{p}},\frac{r}{r-2}}_{\tilde{p}'}$ is exactly the dual of $\dot{B}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}_{\tilde{p}}$, (the restriction $\frac{2}{p} \geq \frac{s+1}{d}$ is mainly because we are interested in non-negative indexes), therefore we conclude that

$$\big\|B(u,v)(t)\big\|_{\dot{B}_{\tilde{p}}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}} \lesssim \|u\|_{L^r([0,T];\dot{H}^s_p)}\|v\|_{L^r([0,T];\dot{H}^s_p)},\ 0\le t\le T. \tag{27}$$

Finally, the estimate (20) can be deduced from the inequality (27). □

Combining Theorem 1 with Lemma 8, we get the following existence results, the particular case of which, when $s=0$, was obtained in [29].

Theorem 2. *Let*

$$d\ge 3, s\ge 0,\ p>1,\ \textit{and}\ r>2,$$

be such that

$$\frac{s}{d}<\frac{1}{p}<\frac{1}{2}+\frac{s}{2d}\ \textit{and}\ \frac{2}{r}+\frac{d}{p}-s\le 1.$$

(a) *There exists a positive constant* $\delta_{s,p,r,d}$ *such that for all* $T>0$ *and for all* $u_0\in\mathcal{S}'(\mathbb{R}^d)$ *with* $\operatorname{div}(u)=0$, *satisfying*

$$T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})}\big\|e^{t\Delta}u_0\big\|_{L^r([0,T];\dot{H}^s_p)}\le\delta_{s,p,r,d}, \tag{28}$$

there is a unique mild solution $u\in L^r\big([0,T];\dot{H}^s_p)\big)$ *for NSE.*

If

$$e^{t\Delta}u_0\in L^r\big([0,1];\dot{H}^s_p\big),$$

then the inequality (28) *holds when* $T(u_0)$ *is small enough.*
(b) *If* $\frac{2}{r}+\frac{d}{p}-s=1$ *then there exists a positive constant* $\delta_{s,p,d}$ *such that we can take* $T=\infty$ *whenever* $\big\|e^{t\Delta}u_0\big\|_{L^r([0,\infty];\dot{H}^s_p)}\le\delta_{s,p,d}$.

Proof. (a) From Lemma 8, we use the estimate

$$\big\|B\big\|_{L^r([0,T];\dot{H}^s_p)}\le C_{s,p,r,d}T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})},$$

where $C_{s,p,r,d}$ is a positive constant independent of T. From Theorem 1 and the above inequality, we deduce the existence of a solution to the Navier-Stokes equations on the interval $(0,T)$ with

$$4C_{s,p,r,d}T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})}\big\|e^{t\Delta}u_0\big\|_{L^r([0,T];\dot{H}^s_p)}\le 1.$$

If $e^{t\Delta}u_0 \in L^r([0,1];\dot{H}^s_p)$ then this condition is fulfilled for $T = T(u_0)$ small enough, this is obvious for the case when $\frac{2}{r} + \frac{d}{p} - s < 1$ since $\lim_{T\to 0} T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})} = 0$. For the case when $\frac{2}{r} + \frac{d}{p} - s = 1$, the condition is fulfilled since we have $\lim_{T\to 0}\big\|e^{t\Delta}u_0\big\|_{L^r([0,T];\dot{H}^s_p)} = 0$.
(b) This is obvious. □

Remark 3. From Theorem 5.3 ([29], p. 44), if $u_0 \in B_p^{s-\frac{2}{r},r} \cap \mathcal{S}'(\mathbb{R}^d)$ then $e^{t\Delta}u_0 \in L^r([0,1];\dot{H}^s_p)$. From Lemma 2, if $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ the two quantities $\big\|u_0\big\|_{\dot{B}_p^{s-\frac{2}{r},r}}$ and $\big\|e^{t\Delta}u_0\big\|_{L^r([0,\infty];\dot{H}^s_p)}$ are equivalent.

3.2. Solutions to the Navier-Stokes equations with initial value in the critical spaces $H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ and $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $3 \leq d \leq 4,\ 2 \leq q \leq d$

Lemma 10. *Let $d \geq 3$ and $2 \leq q \leq d$. Then the bilinear operator $B(u,v)(t)$ is continuous from*

$$L^4\Big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\Big) \times L^4\Big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\Big)$$

into

$$L^\infty\Big([0,T];\dot{B}_q^{\frac{d}{q}-1,2}\Big),$$

and we have the inequality

$$\big\|B(u,v)\big\|_{L^\infty([0,T];\dot{H}_q^{\frac{d}{q}-1})} \lesssim \big\|B(u,v)\big\|_{L^\infty\big([0,T];\dot{B}_q^{\frac{d}{q}-1,2}\big)} \leq C\big\|u\big\|_{L^4\big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\big)}\big\|v\big\|_{L^4\big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\big)}, \tag{29}$$

where C is a positive constant and independent of T.

Proof. Applying Lemma 9 with $r = 4, p = \frac{2dq}{2d-q}$, and $s = \frac{d}{q} - 1$, we get

$$\frac{1}{\tilde{p}} = \frac{2}{p} - \frac{s}{d} = \frac{2d-q}{dq} - \frac{\frac{d}{q}-1}{d} = \frac{1}{q},$$

$$\big\|B(u,v)\big\|_{L^\infty\big([0,T];\dot{B}_q^{\frac{d}{q}-1,2}\big)} \lesssim \big\|u\big\|_{L^4\big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\big)}\big\|v\big\|_{L^4\big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\big)}. \tag{30}$$

From (b) of Lemma 3, we have

$$\dot{B}_q^{\frac{d}{q}-1,2} \hookrightarrow \dot{H}_q^{\frac{d}{q}-1}. \tag{31}$$

Finally, the estimate (29) can be deduced from the inequality (30) and the imbedding (31). □

LEMMA 11. *Let $d \geq 3$ and $2 \leq q \leq d$. Then the bilinear operator $B(u,v)(t)$ is continuous from*

$$L^4\Big([0,T]; H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\Big) \times L^4\Big([0,T]; H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\Big)$$

into

$$L^\infty\Big([0,T]; H_q^{\frac{d}{q}-1}\Big),$$

and we have the inequality

$$\big\|B(u,v)\big\|_{L^\infty\big([0,T];H_q^{\frac{d}{q}-1}\big)} \leq C\big\|u\big\|_{L^4\big([0,T];H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\big)}\big\|v\big\|_{L^4\big([0,T];H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\big)}, \tag{32}$$

where C is a positive constant and independent of T.

PROOF. To prove this lemma by duality (in the x-variable), let us consider an arbitrary test function $h(x) \in \mathcal{S}(\mathbb{R}^d)$. Similar to the proof of Lemma 9, we have

$$\begin{aligned}&\Big|\Big\langle(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t), h\Big\rangle\Big|\\ &\lesssim \big\|u\big\|_{L^4\big([0,T];H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\big)}\big\|v\big\|_{L^4\big([0,T];H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\big)}\big\|h\big\|_{\dot{B}_{q'}^{0,2}},\end{aligned}$$

where

$$\frac{1}{q}+\frac{1}{q'}=1.$$

However the dual space of $\dot{B}_{q'}^{0,2}$ is $\dot{B}_q^{0,2}$, therefore we get

$$\begin{aligned}&\Big\|(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t)\Big\|_{\dot{B}_q^{0,2}}\\ &\lesssim \big\|u\big\|_{L^4\big([0,T];H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\big)}\big\|v\big\|_{L^4\big([0,T];H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\big)}.\end{aligned} \tag{33}$$

From (b) of Lemma 3 and the estimate (33), we have

$$\Big\|B(u,v)(t)\Big\|_{H_q^{\frac{d}{q}-1}} = \Big\|(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t)\Big\|_{L^q} = \Big\|(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t)\Big\|_{\dot H_q^0} \lesssim \Big\|(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t)\Big\|_{\dot B_q^{0,2}} \lesssim \big\|u\big\|_{L^4\left([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\right)}\big\|v\big\|_{L^4\left([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\right)},\ 0\le t\le T. \tag{34}$$

Finally, the estimate (32) can be deduced from the inequality (34). □

LEMMA 12. *Let $d \geq 3$ and $2 \leq q \leq 4$.*
(a) *If $u_0 \in H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ then*

$$\big\|e^{t\Delta}u_0\big\|_{L^4\left([0,\infty);H^{d/q-1}_{2dq/(2d-q)}\right)} \lesssim \big\|u_0\big\|_{H_q^{d/q-1}}.$$

(b) *If $u_0 \in \dot H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ then*

$$\Big\|e^{t\Delta}u_0\Big\|_{L^4\left([0,\infty);\dot H^{d/q-1}_{2dq/(2d-q)}\right)} \simeq \big\|u_0\big\|_{\dot B^{d/q-3/2,4}_{2dq/(2d-q)}} \lesssim \big\|u_0\big\|_{\dot H_q^{d/q-1}}.$$

PROOF. (a) From Lemma 1, we have the estimates

$$\begin{aligned}\Big\|e^{t\Delta}u_0\Big\|_{L^4\left([0,\infty);H^{d/q-1}_{2dq/(2d-q)}\right)} &= \Big(\int_0^\infty \Big\|e^{t\Delta}(\sqrt{Id-\Delta})^{d/q-1}u_0\Big\|^4_{L^{2dq/(2d-q)}}\mathrm{d}t\Big)^{1/4}\\ &= \Big(\int_0^\infty \Big(t^{\frac14}\Big\|e^{t\Delta}(\sqrt{Id-\Delta})^{d/q-1}u_0\Big\|_{L^{2dq/(2d-q)}}\Big)^4\frac{\mathrm{d}t}{t}\Big)^{1/4}\\ &\simeq \Big\|(\sqrt{Id-\Delta})^{d/q-1}u_0\Big\|_{\dot B^{-1/2,4}_{2dq/(2d-q)}}.\end{aligned} \tag{35}$$

Applying (b), (c), and (d) of Lemma 3 in order to obtain

$$L^q = \dot H_q^0 \hookrightarrow \dot B_q^{0,q} \hookrightarrow \dot B_q^{0,4} \hookrightarrow \dot B^{-1/2,4}_{2dq/(2d-q)}. \tag{36}$$

From the inequality (35) and the imbedding (36), we get

$$\left\|e^{t\Delta}u_0\right\|_{L^4\left([0,\infty);H^{d/q-1}_{2dq/(2d-q)}\right)} \simeq \left\|(\sqrt{Id-\Delta})^{d/q-1}u_0\right\|_{\dot{B}^{-1/2,4}_{2dq/(2d-q)}}$$
$$\lesssim \left\|(\sqrt{Id-\Delta})^{d/q-1}u_0\right\|_{L^q} = \left\|u_0\right\|_{H^{d/q-1}_q}.$$

(b) Similar to the proof of (a) we have

$$\left\|e^{t\Delta}u_0\right\|_{L^4\left([0,\infty);\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \simeq \left\|\dot{\Lambda}^{\frac{d}{q}-1}u_0\right\|_{\dot{B}^{-1/2,4}_{2dq/(2d-q)}}$$
$$\lesssim \left\|\dot{\Lambda}^{\frac{d}{q}-1}u_0\right\|_{L^q} = \left\|u_0\right\|_{\dot{H}^{d/q-1}_q},$$
$$\text{and} \quad \left\|\dot{\Lambda}^{\frac{d}{q}-1}u_0\right\|_{\dot{B}^{-1/2,4}_{2dq/(2d-q)}} \simeq \left\|u_0\right\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}}. \square$$

Combining Theorem 1 with Lemmas 4, 8, 10, and 12 we obtain the following existence result.

THEOREM 3. *Let $3 \le d \le 4$ and $2 \le q \le d$. There exists a positive constant $\delta_{q,d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^{d/q-1}_q(\mathbb{R}^d)$ with* $\operatorname{div}(u_0) = 0$ *satisfying*

$$\left\|e^{t\Delta}u_0\right\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \le \delta_{q,d}, \tag{37}$$

NSE has a unique mild solution $u \in L^4\big([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\big) \cap C\big([0,T];\dot{H}^{d/q-1}_q\big)$. Denoting $w = u - e^{t\Delta}u_0$, then we have

$$w \in L^4\big([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\big) \cap L^\infty\big([0,T];\dot{B}^{d/q-1,2}_q\big).$$

Finally, we have

$$\left\|e^{t\Delta}u_0\right\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \lesssim \left\|u_0\right\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \lesssim \left\|u_0\right\|_{\dot{H}^{d/q-1}_q},$$

in particular, for arbitrary $u_0 \in \dot{H}^{d/q-1}_q(\mathbb{R}^d)$ the inequality (37) holds when $T(u_0)$ is small enough; and there exists a positive constant $\sigma_{q,d}$ such that for all $\left\|u_0\right\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \le \sigma_{q,d}$ we can take $T = \infty$.

Proof. By applying Lemma 8 with $r = 4,\ p = \frac{2dq}{2d-q},\ s = \frac{d}{q} - 1$, and notice that $1 + s - \frac{2}{r} - \frac{d}{p} = 0$ we have

$$\big\|B\big\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \leq C_{q,d},$$

where $C_{q,d}$ is a positive constant independent of T. From Theorem 1 and the above inequality, we deduce that for any $u_0 \in \dot{H}_q^{\frac{d}{q}-1}$ such that

$$\mathrm{div}(u_0) = 0, \quad \big\|e^{t\Delta}u_0\big\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \leq \frac{1}{4C_{q,d}},$$

NSE has a mild solution u on the interval $(0, T)$ so that

$$u \in L^4\big([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\big). \tag{38}$$

From Lemma 10 and (38), we have $B(u,u) \in L^\infty\big([0,T];\dot{H}_q^{d/q-1}\big)$. From (2) of Lemma 4, we have $e^{t\Delta}u_0 \in L^\infty\big([0,T];\dot{H}_q^{d/q-1}\big)$. Therefore

$$u = e^{t\Delta}u_0 - B(u,u) \in L^\infty([0,T];\dot{H}_q^{d/q-1}).$$

In the space $H^{d/2-1}$ or L^d (see [29]), the solutions can also be constructed by a successive approximation via the integral equation and therefore they are continuous in time up to the initial time. Since $e^{t\Delta}$ is a (C_0)-semigroup in H_q^s and $\dot{H}_q^s$ with finite integral-exponent ($q < \infty$), by the same way as, we can easily show that the obtained mild solution $u \in C\big([0,T];\dot{H}_q^{d/q-1}\big)$.

From (b) of Lemma 12, we have

$$\begin{aligned}\big\|e^{t\Delta}u_0\big\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} &\lesssim \big\|e^{t\Delta}u_0\big\|_{L^4\left([0,\infty);\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \\ &\simeq \Big\|u_0\Big\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \lesssim \big\|u_0\big\|_{\dot{H}_q^{d/q-1}} < \infty.\end{aligned}$$

Hence, the left-hand side of the inequality (37) converges to 0 when T tends to 0. Therefore, for arbitrary $u_0 \in \dot{H}_q^{\frac{d}{q}-1}$ there is $T(u_0)$ small enough such that the inequality (37) holds. Also, there exists a positive constants $\sigma_{q,d}$ such that for all $\Big\|u_0\Big\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \leq \sigma_{q,d}$ and $T = \infty$ the inequality (37) holds. □

REMARK 4. Theorem 3 in the particular case $q = d$ is Proposition 20.1 in [29].

THEOREM 4. *Let $3 \leq d \leq 4$ and $2 \leq q \leq d$. There exists a positive constant $\delta_{q,d}$ such that for all $T > 0$ and for all $u_0 \in H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ with* $\text{div}(u_0) = 0$ *satisfying*

$$\|e^{t\Delta}u_0\|_{L^4\left([0,T];H^{d/q-1}_{2dq/(2d-q)}\right)} \leq \delta_{q,d}, \tag{39}$$

NSE has a unique mild solution $u \in L^4\big([0,T];H^{d/q-1}_{2dq/(2d-q)}\big) \cap C\big([0,T];H_q^{d/q-1}\big)$. Finally, we have

$$\|e^{t\Delta}u_0\|_{L^4\left([0,T];H^{d/q-1}_{2dq/(2d-q)}\right)} \leq \|u_0\|_{H_q^{d/q-1}},$$

in particular, for arbitrary $u_0 \in H_q^{\frac{d}{q}-1}$ the inequality (39) *holds when $T(u_0)$ is small enough;*

PROOF. The proof of Theorem 4 is similar to the one of Theorem 3, by combining Theorem 1 with Lemmas 4, 8, 11, and 12. □

3.3. Solutions to the Navier-Stokes equations with initial value in the critical spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $d \geq 3$ and $1 < q \leq d$

We consider two cases $2 < q \leq d$ and $1 < q \leq 2$ separately.

3.3.1 Solutions to the Navier-Stokes equations with initial value in the critical spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $d \geq 3$ and $2 < q \leq d$

LEMMA 13. *Let $d \geq 3$ and $2 < q \leq d$. Then for all p such that*

$$2 < p < \min\Big\{\frac{(d-2)q}{d-q},\ d+2\Big\}, (if\ q = d\ then\ \frac{(d-2)q}{d-q} = +\infty),$$

the bilinear operator $B(u,v)(t)$ is continuous from

$$L^p([0,T];\dot{H}_p^{\frac{2+d-p}{p}}) \times L^p([0,T];\dot{H}_p^{\frac{2+d-p}{p}})$$

into

$$L^\infty\big([0,T];\dot{B}^{\frac{d+p-2}{p}-1,\frac{p}{2}}_{\frac{dp}{d+p-2}}\big),$$

and we have the inequality

$$\big\|B(u,v)\big\|_{L^\infty\left([0,T];\dot{H}_q^{\frac{d}{q}-1}\right)} \lesssim \big\|B(u,v)\big\|_{L^\infty\left([0,T];\dot{B}_{\frac{dp}{d+p-2}}^{\frac{d+p-2}{p}-1,\frac{p}{2}}\right)} \tag{40}$$
$$\leq C\big\|u\big\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)}\big\|v\big\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)},$$

where C is a positive constant independent of T.

PROOF. Applying Lemma 9 with $r=p$ and $s=\frac{2+d-p}{p}$, we get

$$\frac{1}{\tilde{p}}=\frac{2}{p}-\frac{s}{d}=\frac{d+p-2}{dp},$$
$$\Big\|B(u,v)\Big\|_{L^\infty\left([0,T];\dot{B}_{\frac{dp}{d+p-2}}^{\frac{d+p-2}{p}-1,\frac{p}{2}}\right)} \tag{41}$$
$$\lesssim \big\|u\big\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)}\big\|v\big\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)}.$$

Applying (e), (d), and (h) of Lemma 3 in order to obtain

$$\dot{B}_{\frac{dp}{d+p-2}}^{\frac{d+p-2}{p}-1,\frac{p}{2}} \hookrightarrow \dot{F}_{\frac{dp}{d+p-2}}^{\frac{d+p-2}{p}-1,\frac{p}{2}} \hookrightarrow \dot{F}_q^{\frac{d}{q}-1,2}=\dot{H}_q^{\frac{d}{q}-1}. \tag{42}$$

Therefore the estimate (40) is deduced from the inequality (41) and the imbedding (42).

LEMMA 14. *Let $2<q<p<+\infty$. Then for all $u_0\in\dot{H}_q^{\frac{d}{q}-1}$ we have the estimates*

$$\big\|e^{t\Delta}u_0\big\|_{L^p\left([0,\infty);\dot{H}_p^{\frac{2+d-p}{p}}\right)} \simeq \big\|u_0\big\|_{\dot{B}_p^{\frac{d}{p}-1,p}} \lesssim \big\|u_0\big\|_{\dot{H}_q^{\frac{d}{q}-1}}.$$

PROOF. From Lemma 1, we have the estimates

$$\big\|e^{t\Delta}u_0\big\|_{L^p\left([0,\infty);\dot{H}_p^{\frac{2+d-p}{p}}\right)} \simeq \big\|u_0\big\|_{\dot{B}_p^{\frac{d}{p}-1,p}}. \tag{43}$$

Applying (b), (d), and (c) of Lemma 3 in order to obtain

$$\dot{H}_q^{\frac{d}{q}-1} \hookrightarrow \dot{B}_q^{\frac{d}{q}-1,q} \hookrightarrow \dot{B}_p^{\frac{d}{p}-1,q} \hookrightarrow \dot{B}_p^{\frac{d}{p}-1,p}. \tag{44}$$

From the estimate (43) and the imbedding (44), we have

$$\left\|e^{t\Delta}u_0\right\|_{L^p\left([0,\infty);\dot{H}_p^{\frac{2+d-p}{p}}\right)} \simeq \left\|u_0\right\|_{\dot{B}_p^{\frac{d}{p}-1,p}} \lesssim \left\|u_0\right\|_{\dot{H}_q^{\frac{d}{q}-1}}. \square$$

THEOREM 5. *Let $d \geq 3$ and $2 < q \leq d$. Then for any p be such that*

$$q < p < \min\Big\{\frac{(d-2)q}{d-q},\ d+2\Big\},$$

there exists a constant $\delta_{q,p,d} > 0$ such that for all $T > 0$ and for all $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$ satisfying

$$\left\|e^{t\Delta}u_0\right\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)} \leq \delta_{q,p,d}, \tag{45}$$

NSE has a unique mild solution $u \in L^p\big([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\big) \cap C\big([0,T];\dot{H}_q^{d/q-1}\big)$. Denoting $w = u - e^{t\Delta}u_0$, then we have

$$w \in L^p\big([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\big) \cap L^\infty\Big([0,T];\dot{B}_{\frac{dp}{d+p-2}}^{\frac{d+p-2}{p}-1,\frac{p}{2}}\Big).$$

Finally, we have

$$\left\|e^{t\Delta}u_0\right\|_{L^p\left([0,T];\dot{H}_p^{\frac{2+d-p}{p}}\right)} \leq \left\|u_0\right\|_{\dot{B}_p^{\frac{d}{p}-1,p}} \lesssim \left\|u_0\right\|_{\dot{H}_q^{\frac{d}{q}-1}},$$

in particular, for arbitrary $u_0 \in \dot{H}_q^{d/q-1}$ the inequality (45) holds when $T(u_0)$ is small enough; and there exists a positive constant $\sigma_{q,p,d}$ such that for all $\left\|u_0\right\|_{\dot{B}_p^{\frac{d}{p}-1,p}} \leq \sigma_{q,p,d}$ we can take $T = \infty$.

PROOF. The proof of Theorem 5 is similar to the one of Theorem 3, by combining Theorem 1 with Lemmas 4, 8 (for $r = p,\ s = \frac{2+d-p}{p}$), 13, and 14. $\square$

REMARK 5. The case $q = d$ was treated by several authors, see for example ([7], [12], [21]). However their results are different from ours.

3.3.2 Solutions to the Navier-Stokes equations with initial value in the critical spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $d \geq 3$ and $1 < q \leq 2$

Lemma 15. *Let $d \geq 3$ and $1 < q \leq 2$. Then the bilinear operator $B(u,v)(t)$ is continuous from*

$$L^{2q}\Big([0,T]; \dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\Big) \times L^{2q}\Big([0,T]; \dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\Big)$$

into

$$L^{\infty}\big([0,T]; \dot{B}_q^{\frac{d}{q}-1,q}\big),$$

and we have the inequality

$$\big\|B(u,v)\big\|_{L^{\infty}\big([0,T];\dot{H}_q^{\frac{d}{q}-1}\big)} \lesssim \big\|B(u,v)\big\|_{L^{\infty}\big([0,T];\dot{B}_q^{\frac{d}{q}-1,q}\big)}$$
$$\leq C\Big\|u\Big\|_{L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)}\Big\|v\Big\|_{L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)},$$

where C is a positive constant independent of T.

Proof. Applying Lemma 9 with $r = 2q$, $p = \frac{dq}{d+1-q}$, and $s = \frac{d+2-2q}{q}$, we get

$$\frac{1}{\tilde{p}} = \frac{2}{p} - \frac{s}{d} = \frac{1}{q},$$

and from (a) of Lemma 3, we have

$$\big\|B(u,v)\big\|_{L^{\infty}\big([0,T];\dot{H}_q^{\frac{d}{q}-1}\big)} \lesssim \big\|B(u,v)\big\|_{L^{\infty}\big([0,T];\dot{B}_q^{\frac{d}{q}-1,q}\big)}$$
$$\lesssim \Big\|u\Big\|_{L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)}\Big\|v\Big\|_{L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)}. \quad \square$$

Lemma 16. *Assume that $u_0 \in \dot{H}_q^{\frac{d}{q}-1}$ with $d \geq 3$ and $1 < q \leq 2$. Then*

$$\Big\|e^{t\Delta}u_0\Big\|_{L^{2q}\big([0,\infty);\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)} \simeq \big\|u_0\big\|_{\dot{B}^{(d+1)/q-2,2q}_{dq/(d+1-q)}} \lesssim \big\|u_0\big\|_{\dot{H}_q^{d/q-1}}.$$

PROOF. By using (a), (c), and (d) of Lemma 3 in order to obtain

$$\dot{H}_q^{\frac{d}{q}-1} \hookrightarrow \dot{B}_q^{\frac{d}{q}-1,2} \hookrightarrow \dot{B}_q^{\frac{d}{q}-1,2q} \hookrightarrow \dot{B}_{dq/(d+1-q)}^{(d+1)/q-2,2q}. \tag{46}$$

Applying Lemma 1 and from the imbedding (46) we have the estimates

$$\begin{aligned}\left\|e^{t\Delta}u_0\right\|_{L^{2q}\left([0,\infty);\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\right)} &\simeq \left\|\dot{\Lambda}^{\frac{d+2-2q}{q}}u_0\right\|_{\dot{B}_{dq/(d+1-q)}^{-1/q,2q}} \\ &\simeq \left\|u_0\right\|_{\dot{B}_{dq/(d+1-q)}^{(d+1)/q-2,2q}} \lesssim \left\|u_0\right\|_{\dot{H}_q^{d/q-1}}. \ \square\end{aligned}$$

THEOREM 6. *Let $d \geq 3$ and $1 < q \leq 2$. There exists a positive constant $\delta_{q,d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$ satisfying*

$$\left\|e^{t\Delta}u_0\right\|_{L^{2q}\left([0,T];\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\right)} \leq \delta_{q,d}, \tag{47}$$

NSE has a unique mild solution $u \in L^{2q}\big([0,T];\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\big) \cap C\big([0,T];\dot{H}_q^{d/q-1}\big)$. Denoting $w = u - e^{t\Delta}u_0$, then we have

$$w \in L^{2q}\big([0,T];\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\big) \cap L^{\infty}\big([0,T];\dot{B}_q^{\frac{d}{q}-1,q}\big).$$

Finally, we have

$$\left\|e^{t\Delta}u_0\right\|_{L^{2q}\left([0,T];\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\right)} \leq \left\|u_0\right\|_{\dot{B}_{dq/(d+1-q)}^{(d+1)/q-2,2q}} \lesssim \left\|u_0\right\|_{\dot{H}_q^{d/q-1}},$$

in particular, for arbitrary $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ the inequality (47) holds when $T(u_0)$ is small enough; and there exists a positive constant $\sigma_{q,d}$ such that for all $\|u_0\|_{\dot{B}_{dq/(d+1-q)}^{(d+1)/q-2,2q}} \leq \sigma_{q,d}$ we can take $T = \infty$.

PROOF. The proof of Theorem 6 is similar to the one of Theorem 3, by combining Theorem 1 with Lemmas 4, 8 (for $r = 2q, p = \frac{dq}{d+1-q}, s = \frac{d+2-2q}{q}$), 15, and 16. $\square$

Remark 6. The case $q = 2$ was treated by several authors, see for example ([7],[16], [29]). However their results are different from ours.

Acknowledgments. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2014.50.

References

[1] Adams, A. R., *Sobolev Spaces*, Academic press, Boston, MA, 1975, 268 p.

[2] Bergh, J. and J. Lofstrom, *Interpolation Spaces*, Springer-Verlag, 1976, 264 p.

[3] Bourgain, J. and N. Pavloviéc, Ill-posedness of the Navier-Stokes equations in a critical space in 3D, J. Funct. Anal. **255** (9) (2008), 2233–2247.

[4] Jawerth, B., Some observations on Besov and Lizorkin-Triebel space, Math. Scand. **40** (1977), 94–104.

[5] Bourdaud, G., Réalisation des espaces de Besov homogènes, Ark. Mat. **26** (1) (1988), 41–54.

[6] Bourdaud, G., Ce qu'il faut savoir sur les espaces de Besov, Prépublication de l'Universitéde Paris 7 (janvier 1993).

[7] Cannone, M., *Ondelettes, Paraproduits et Navier-Stokes*, Diderot Editeur, Paris, 1995, 191 p.

[8] Cannone, M., A generalization of a theorem by Kato on Navier-Stokes equations, Rev. Mat. Iberoamericana **13** (3) (1997), 515–541.

[9] Cannone, M. and F. Planchon, On the nonstationary Navier-Stokes equations with an external force, Adv. in Diff. Eq. **4** (5) (1999), 697–730.

[10] Cannone, M. and Y. Meyer, Littlewood-Paley decomposition and the Navier-Stokes equations, Meth. and Appl. of Anal. **2** (1995), 307–319.

[11] Chemin, J. M., Remarques sur l'existence globale pour le système de Navier-Stokes incompressible, SIAM J. Math. Anal. **23** (1992), 20–28.

[12] Dong, H. and D. Du, On the local smoothness of solutions of the Navier-Stokes equations, J. Math. Fluid Mech. **9** (2) (2007), 139–152.

[13] Frazier, M., Jawerth, B. and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, CBMS Regional Conference Series in Mathematics, 79, AMS, Providence (1991).

[14] Fabes, E., Jones, B. and N. Riviere, The initial value problem for the Navier-Stokes equations with data in L^p, Arch. Rat. Mech. Anal. **45** (1972), 222–240.

[15] Friedlander, S. and D. Serre, *Handbook of Mathematical Fluid Dynamics*, Volume 3, Elsevier, 2004.

[16] Fujita, H. and T. Kato, On the Navier-Stokes initial value problem I, Arch. Rat. Mech. Anal. **16** (1964), 269–315.

[17] Giga, Y., Solutions of semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, J. Differ. Eq. **62** (1986), 186–212.

[18] Giga, Y. and T. Miyakawa, Solutions in L^r of the Navier-Stokes initial value problem, Arch. Rat. Mech. Anal. **89** (1985), 267–281.

[19] Kalton, N., Mayboroda, S. and M. Mitrea, *Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin Spaces and Applications to Problems in Partial Differential Equations. Interpolation Theory and Applications*, Contemp. Math., **445**, Amer. Math. Soc., Providence, RI, 2007, 121–177.

[20] Kato, T. and H. Fujita, On the non-stationary Navier-Stokes system, Rend. Sem. Mat. Univ. Padova **32** (1962), 243–260.

[21] Kato, T., Strong L^p-solutions of the Navier-Stokes equation in R^m, with applications to weak solutions, Math. Z. **187** (4) (1984), 471–480.

[22] Kato, T., Strong solutions of the Navier-Stokes equations in Morrey spaces, Bol. Soc. Brasil. Math. **22** (1992), 127–155.

[23] Kato, T. and G. Ponce, The Navier-Stokes equation with weak initial data, Int. Math. Res. Notes **10** (1994), 435–444.

[24] Kozono, H. and M. Yamazaki, Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data, Comm. Partial Differential Equations **19** (1994), 959–1014.

[25] Khai, D. Q. and N. M. Tri, Solutions in mixed-norm Sobolev-Lorentz spaces to the initial value problem for the Navier-Stokes equations, Journal of Mathematical Analysis and Applications **417** (2014), 819–833.

[26] Khai, D. Q. and N. M. Tri, Well-posedness for the Navier-Stokes equations with datum in Sobolev-Fourier-Lorentz spaces, Preprint arXiv:1601.01441.

[27] Khai, D. Q. and N. M. Tri, Well-posedness for the Navier-Stokes equations with data f in homogeneous Sobolev-Lorentz spaces, Preprint arXiv: 1601.01742.

[28] Koch, H. and D. Tataru, Well-posedness for the Navier-Stokes equations, Adv. Math. **157** (2001), 22–35.

[29] Lemarie-Rieusset, P. G., *Recent Developments in the Navier-Stokes Problem*, Chapman and Hall/CRC Research Notes in Mathematics, vol. 431, Chapman and Hall/CRC, Boca Raton, FL, 2002.

[30] Miao, C. and B. Yuan, On the well-posedness of the Cauchy problem for an MHD system in Besov spaces, Math. Methods Appl. Sci. **32** (2009), 53–76.

[31] Planchon, F., *Solutions Globales et Comportement Asymptotique pour les Equations de Navier-Stokes*, Doctoral Thesis, Ecole Polytechnique, France (1996).

[32] Peetre, J., *New Thoughts on Besov Spaces*, Duke Univ. Math. Series, 1976.

[33] Taylor, M. E., Analysis on Morrey spaces and applications to Navier-Stokes equations and other evolution equations, Comm. Partial Differential Equations **17** (1992), 1407–1456.

[34] Weissler, F. B., The Navier-Stokes initial value problem in L^p, Arch. Rat. Mech. Anal. **74** (1981), 219–230.

(Received October 27, 2014)
(Revised October 23, 2015)

Institute of Mathematics
Vietnam Academy of Science and Technology
18 Hoang Quoc Viet, 10307 Cau Giay
Hanoi, Vietnam
E-mail: triminh@math.ac.vn
Khaitoantin@gmail.com