

Renormalization Group Analysis of Multi-Band Many-Electron Systems at Half-Filling

By Yohei KASHIMA

Abstract. Renormalization group analysis for multi-band many-electron systems at half-filling at positive temperature is presented. The analysis includes the Matsubara ultra-violet integration and the infrared integration around the zero set of the dispersion relation. The multi-scale integration schemes are implemented in a finite-dimensional Grassmann algebra indexed by discrete position-time variables. In order that the multi-scale integrations are justified inductively, various scale-dependent estimates on Grassmann polynomials are established. We apply these theories in practice to prove that for the half-filled Hubbard model with nearest-neighbor hopping on a square lattice the infinite-volume, zero-temperature limit of the free energy density exists as an analytic function of the coupling constant in a neighborhood of the origin if the system contains the magnetic flux $\pi \pmod{2\pi}$ per plaquette and $0 \pmod{2\pi}$ through the large circles around the periodic lattice. Combined with Lieb’s result on the flux phase problem ([Lieb, E. H., Phys. Rev. Lett. **73** (1994), 2158]), this theorem implies that the minimum free energy density of the flux phase problem converges to an analytic function of the coupling constant in the infinite-volume, zero-temperature limit. The proof of the theorem is based on a four-band formulation of the model Hamiltonian and an extension of Giuliani-Mastropietro’s renormalization designed for the half-filled Hubbard model on the honeycomb lattice ([Giuliani, A. and V. Mastropietro, Commun. Math. Phys. **293** (2010), 301–346]).

Contents

1. Introduction	3
1.1. Introduction	3
1.2. The model and the main results	10
2. Formulation	18

2010 *Mathematics Subject Classification.* Primary 81T17; Secondary 81T28.

Key words: Many-electron system, renormalization group, the Hubbard model, zero-temperature limit.

2.1. The multi-band Hamiltonian	18
2.2. The finite-dimensional Grassmann integrals	20
2.3. The full covariance	23
2.4. The Grassmann Gaussian integral formulation	25
2.5. A symmetric formulation	30
3. General Estimation	46
3.1. Estimation of the free integration	48
3.2. Estimation of the tree expansion	49
3.3. Invariance of Grassmann polynomials	72
4. General Estimation at Different Temperatures	77
4.1. Estimation of the free integration at different temperatures	80
4.2. Estimation of the tree expansion at different temperatures	84
5. Generalized Multi-Scale Integrations	107
5.1. The generalized ultra-violet integration	108
5.2. The generalized ultra-violet integration at different temperatures	118
5.3. The generalized infrared integration	130
5.4. The generalized infrared integration at different temperatures	139
6. The Matsubara Ultra-Violet Integration	146
6.1. The covariance matrices with the Matsubara ultra-violet cut-off	146
6.2. Application of the generalized ultra-violet integration	158
7. The Infrared Integration of the Model	163
7.1. The four-band formulation	164
7.2. The cut-off function for the infrared integration	167
7.3. The covariance matrices in the infrared integration	174
7.4. Application of the generalized infrared integration	208
Appendix A. The Flux Phase Problem	242
Appendix B. L^1 -Estimates of Kernels of Grassmann Polynomials	254
Appendix C. Estimation of Gevrey-Class Functions	262
Appendix D. The Time-Continuum, Infinite-Volume Limit of the Truncated Grassmann Integral Formulation	269
Appendix E. Direct Treatment of the Free Energy Density	278
Acknowledgments	282
Notation	282
Parameters and constants	282
Sets and spaces	283

Functions and maps	285
Inner products, norms and semi-norms	286
Other notations	287
References	287

1. Introduction

1.1. Introduction

It is becoming clear that many-electron lattice systems at positive temperature can be constructed rigorously within the framework of the finite-dimensional Grassmann integrals and various physical quantities defined in the system can be analyzed by solid calculus on the finite-dimensional Grassmann algebra. One analytical technique at the core of this research field is the multi-scale integration. Since its iterative operation with decomposed covariances formally obeys a semi-group property, the multi-scale integration is also called the renormalization group (RG) method. For instance, the existence of infinite-volume limit of thermodynamic physical quantities in many-electron systems and their analyticity with respect to the coupling constant can be proved by carrying out a multi-scale integration over the Matsubara frequency. This type of multi-scale integration is called the Matsubara ultra-violet (UV) integration. Nowadays, however, it is known that a wide class of many-electron systems can be controlled independently of the volume factor by a simple single-scale analysis thanks to the development of volume-independent determinant bounds on the covariances by Pedra and Salmhofer ([19]). Though the Matsubara UV integration or the single-scale integration based on Pedra-Salmhofer's determinant bound proves the analyticity of physical quantities in the infinite-volume limit with the coupling constant, these methods do not improve the temperature-dependency of the domain in which such analytic statements can be made. Without any further treatment, the allowed magnitude of the coupling typically shrinks in a power order of temperature. As a consequence, the theory gives little insight into physics caused by interacting electrons in low temperatures. A multi-scale integration designed to ease the temperature-dependency of the maximal magnitude of interaction is called the infrared (IR) integration. Proper implementation of the IR integration is believed to guarantee the analyticity of physical quantities down to exponentially small temperatures or even to the absolute zero-temperature. Since there are demands for rigor-

ous tools which enable us to treat many-electron models in wide parameter regions, the RG methods need to be systematically investigated from various view points as a hopeful candidate for such anticipated mathematical methods.

This paper has two purposes. One is to construct necessary estimates for the multi-scale integrations on a finite-dimensional Grassmann algebra to ensure the convergence of infinite-volume, zero-temperature limit of thermodynamic physical quantities in half-filled multi-band many-electron systems. The other is to apply these general estimates in practice to a specific many-electron model and reach rigorous conclusions in low temperatures. More precise explanation of the second purpose is the following. We prove that for the half-filled Hubbard model on a square lattice there exists an analytic function of the complex coupling constants on a multi-disk around the origin such that the free energy density is equal to the restriction of the analytic function on the real axis and the analytic function uniformly converges in the infinite-volume, zero-temperature limit, if the nearest-neighbor hopping parameter of the Hubbard model contains the magnetic flux $\pi \pmod{2\pi}$ per plaquette and $0 \pmod{2\pi}$ through the large circles around the periodic lattice. The Hubbard model with this constraint on the magnetic flux is rarely seen in the study of mathematical RG so far. However, it is not irrelevant in mathematical physics. In fact this model defines the minimum free energy in the flux phase problem, which seeks a configuration of the arguments of the complex-valued hopping parameter in the half-filled Hubbard model in order that the free energy of the system is minimum. Lieb ([15]) essentially gave a sufficient condition for the arguments to attain the minimum, which is the above condition on the magnetic flux. The sufficiency of this condition was emphasized by Macris and Nachtergaele in [17]. Since our model is the minimizer, the same analytic and convergent properties hold for the minimum free energy density in the flux phase problem. These results are officially stated in Subsection 1.2.

Let us explain the motive for this work by reviewing recent developments in the multi-scale analysis concerning the 2-dimensional Hubbard models at positive temperature, especially by focusing on the temperature-dependency of the possible magnitude of the coupling constant. In the series [20], [1], [2] the half-filled Hubbard model on a square lattice was studied. These multi-scale analysis suggest that the correlation functions in the sys-

tem are analytic with respect to the coupling constant U in the domain $|U| < c|\log T|^{-2}$, where T is the temperature and c is a generic positive constant. In the doctoral thesis [18] Pedra characterized the 2-point correlation function in the Hubbard model away from half-filling on a square lattice under the constraint $|U| \leq c|\log T|^{-1}$ and concluded that the system in this domain of the coupling is a Fermi liquid. The RG analysis by Benfatto, Giuliani and Mastropietro [3] also showed that the behavior of the 2-point correlation function in the Hubbard model away from half-filling on a square lattice corresponds to a Fermi liquid if the coupling constant U obeys the condition $|U| \leq c|\log T|^{-1}$. One remarkable achievement was made by Giuliani and Mastropietro in [9]. They developed an infrared integration technique for the half-filled Hubbard model on the honeycomb lattice and concluded that the free energy density and the correlation functions in the infinite-volume limit are analytic in the temperature-independent domain $|U| < c$. Giuliani, Mastropietro and Porta continued their RG analysis for the same model in the following article [10]. Despite the conceptual importance of the 2-d Hubbard models in condensed matter physics, complete implementation of RG methods leading to rigorous conclusions on the model in low temperatures is still scarce. It is necessary to clarify the applicability of rigorous versions of RG to the 2-d Hubbard models. This paper is aimed at achieving this goal by presenting another example of analytic control of the 2-d Hubbard model down to the absolute zero-temperature together with a general framework constructed in a self-contained style.

Let us look into more details of related research articles to understand new aspects of this paper from a technical view point. It was shown in [12] that the partition function in many-electron systems can be formulated into a time-continuum limit of the finite-dimensional Grassmann Gaussian integral, whose derivation is based on a discretization of the Riemann integral with respect to the time variable inside the perturbative expansion of the partition function. In this formulation the basis of Grassmann algebra is indexed by the discrete space-time points. The following papers [13], [14] adopted the same formulation and proved exponential decay properties of the finite-temperature correlation functions in the Hubbard models by a single-scale analysis based on Pedra-Salmhofer's determinant bound and a multi-scale integration over the Matsubara frequency respectively. The Matsubara UV integration in [14] was inductively constructed as a transform

on the finite-dimensional Grassmann algebra. Since no infrared integration is performed in [12], [13], [14], the results in these papers are restricted within a domain of the coupling constant depending on temperature as significantly as $|U| < cT^n$ with some $n \in \mathbb{N}$. So the next step is to analyze a many-electron system in low temperatures by means of an IR integration in the same finite-dimensional Grassmann algebra as in [12], [13], [14] and to justify the IR integration process by the mathematical induction with the scale index in the same manner as in the Matsubara UV integration of [14].

A key idea of the IR analysis in [18], [3], [9] is the modification of the covariance at each integration step by the insertion of the kernel of the quadratic term produced by the previous integration. Because of the symmetries of Grassmann polynomials preserved during the multi-scale integration process, this modification does not qualitatively change the shape of the zero set of the denominator of the covariance, and thus the IR integration approaching the zero points of the denominator is guaranteed to continue. The IR integration in this paper uses this adaptive modification method introduced in [18], [3], [9]. Though this renormalization technique explicitly plays a role only when we solve the model problem in Section 7, keeping it in mind, we prepare general estimates in Subsection 5.3 and Subsection 5.4 by giving Grassmann polynomials whose degrees are at least 4 as the input to the integrations.

The main reasons why the physical quantities in the half-filled Hubbard model on the honeycomb lattice are proved to be analytic independently of temperature in [9] are the following. The zero set of the free dispersion relation degenerates into 2 distinct points, which remain to be the zero points of the denominator of the effective covariance, and consequently the integral of each effective interaction term of order ≥ 4 is bounded from above by a negative power of the support size of IR cut-off at the scale, in other words, effective interaction terms of order ≥ 4 are irrelevant under the iterative IR integrations. In fact the invariance of the 2 Fermi-points is a remarkable discovery made by Giuliani and Mastropietro in [9]. In this paper we formulate the half-filled Hubbard model with the flux π condition on a square lattice into a 4-band many-electron model, in which the zero set of the free dispersion relation consists of a single point. Then, we prove that the zero point of the free dispersion relation essentially continues to be a zero point of the denominator of the effective covariance during RG process

by extending Giuliani-Mastropietro's renormalization method originally developed for the 2-band half-filled Hubbard model on the honeycomb lattice. More precisely speaking, our effective covariance in momentum space is the inverse of a 4×4 matrix. What we prove is that each element of the effective 4×4 matrix becomes negligibly small when either the momentum variable is close to the zero point of the free dispersion relation or the Matsubara frequency is close to zero and thus the point where the effective matrix is not invertible is the same as in the free case. The non-corresponding property of the free covariance at equal space-time points erases the quadratic term of the interaction in the Grassmann integral formulation adopted in [9], while the quadratic term remains if we formulate the model by using the Grassmann Gaussian integral proposed in [12], [13], [14]. The quadratic term in the interaction breaks one of the invariances called 'inversion' in [9, Lemma 1], which was used especially to prove that the diagonal elements of the effective 2×2 matrix vanish as the Matsubara frequency approaches zero in [9]. Because of a lack of necessary invariances, the argument of [9] to confirm that certain elements including diagonal ones of the effective matrix vanish in the IR limit does not immediately fit in our formulation. In this paper, therefore, we start from reforming the formulation built in the same manner as in [12], [13], [14] into a more convenient form having desirable symmetries for the IR integration.

The proof of validity of RG in this paper is based on the mathematical induction with respect to the integration scale, which assumes a scale-dependent norm bound on the input as the induction hypothesis and shows the succeeding norm bound on the output of the single-scale integration, while the Gallavotti-Nicolò tree spreading over all the scales is the main tool to organize the multi-scale integration process in [9] as well as in [3]. For this reason the major part of this paper is devoted to establish norm bounds on Grassmann polynomials produced by the single-scale integrations, especially by the tree expansions for derivatives of logarithm of the Grassmann Gaussian integral. Norm estimations on finite-dimensional Grassmann algebra were rigorously summarized with the aim of validating RG by induction by Feldman, Knörrer and Trubowitz in the book [5], in which, however, a representation theorem for the Schwinger functional developed in [4], rather than the tree formula, underlay the Fermionic expansion. The concepts of [5] were extended into the RG analysis on infinite-dimensional Grassmann

algebra for interacting Fermions in [6], [7]. This paper intends to keep the finite-dimensionality of Grassmann algebra and shows the existence of infinite-volume, zero-temperature limit as a result of calculus on the finite-dimensional vector space. In summary what this paper newly presents apart from the statements of the main theorem and its corollary in Subsection 1.2 are

- (i) Inductive construction of the multi-scale integrations, which lead to the zero-temperature limit of the free energy density, on the finite-dimensional Grassmann algebra indexed by discrete space-time points.
- (ii) An extension of Giuliani-Mastropietro's renormalization to a 4-band many-electron system.

Before closing the introductory remarks we should also argue possible limitations of our framework for IR analysis. Our IR multi-scale integration procedure is based on a general proposition, namely Proposition 5.6, which concerns scale-dependent bound properties of the output of a single-scale integration generalizing a real IR integration step. The validity of the proposition is due to the structure that with respect to the scale-dependent norm and semi-norm set in the proposition, any Grassmann monomial of order ≥ 4 with the bound of scale $l + 1$ automatically admits the bound of scale l . In more details the norm bound on a monomial of order ≥ 4 at scale $l + 1$ amounts to requiring the integral of the monomial to be bounded from above by a negative power of the factor M^{l+1} , where $M(> 1)$ is a parameter to control the support size of IR cut-off. Since the negative power of M^{l+1} is smaller than that of M^l , the monomial satisfies the norm bound of scale l as well. When we solve the model problem in Section 7, for example, the power of the factor M^l for a monomial of order m is $-m + 7/2$, which is negative for $m \geq 4$. As long as we go through Proposition 5.6, therefore, our constructive theory is such that effective interaction terms of order ≥ 4 are irrelevant at every step of IR integrations. In this paper we do not have a rigorous a priori criterion of to which model the proposition does or does not apply. The proposition is built upon an assumption, namely (5.57), determined by exponents in the determinant and L^1 bounds on an effective covariance. A heuristic argument in Remark 5.7 suggests that the assumption of the proposition is unlikely to be realized in a d -dimensional many-electron model where the $d - 1$ -dimensional Hausdorff measure of the zero

set of the free dispersion relation is non-zero such as in the 1-dimensional Hubbard models with free Fermi points or the 2-dimensional Hubbard models whose free Fermi curve does not degenerate into finite points. For this reason we expect that these usual many-electron models cannot be analyzed at zero-temperature by an immediate application of our framework.

The contents of this paper after this section are outlined as follows. In Section 2 we introduce the Hamiltonian operator in a generalized setting and formulate the free energy density as a time-continuum limit of logarithm of the finite-dimensional Grassmann Gaussian integral. In Section 3 we present norm estimates on single-scale integrations without assuming quantitative upper bounds on the covariances. In Section 4 we establish norm estimates on the difference between single-scale integrations at 2 different temperatures without assuming quantitative upper bounds on the covariances. In Section 5 we apply the general norm estimates developed in Section 3 and Section 4 to construct both the UV integration process and the IR integration process as well as to measure the difference between Grassmann polynomials produced by these integrations at 2 different temperatures in a model-independent general setting. In Section 6 we complete the UV integration over the Matsubara frequency by showing that the covariance with UV cut-off actually satisfies the bound properties assumed in Section 5. In Section 7 we apply the estimations prepared in Section 5 for the IR integration to the model Hamiltonian and prove the main theorem of this paper. In Appendix A we restate Lieb's result on the flux phase problem with some supplementary arguments concerning the repeated reflection. In Appendix B we establish L^1 -norm bounds on kernels of Grassmann polynomials, which are necessary for the proof of the convergence of the symmetric Grassmann integral formulation to the free energy density in Section 2. In Appendix C we summarize basic estimates on functions of Gevrey-class, to which our cut-off functions belong. In Appendix D we prove that the time-continuum, infinite-volume limit of derivatives of logarithm of the Grassmann Gaussian integral exists at the origin. These convergence properties are used in the proof of the existence of infinite-volume limit of the free energy density in Section 7. Finally in Appendix E some lemmas concerning the free energy density are directly proved without going through the Grassmann integral formulation. The flow chart of our construction is shown in Figure 1.

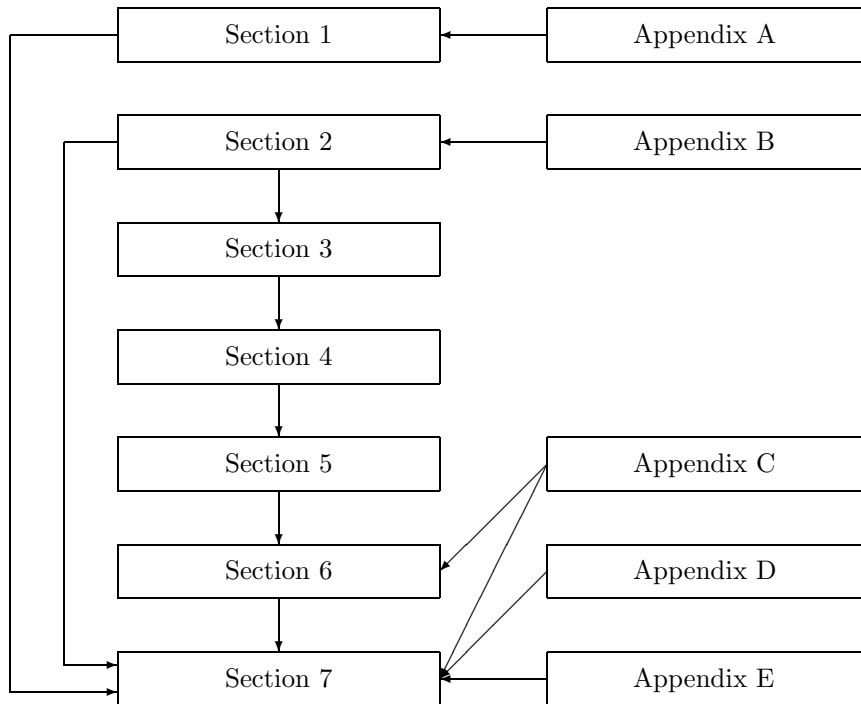


Fig. 1. Flow chart of our construction, where the arrows mean major dependency.

1.2. The model and the main results

Here we introduce the model Hamiltonian and state the main results of this paper. For $L \in \mathbb{N}$ we define the spatial lattice $\Gamma(2L)$ by $\Gamma(2L) := \{0, 1, \dots, 2L - 1\}^2$. For $(\mathbf{x}, \sigma) \in \Gamma(2L) \times \{\uparrow, \downarrow\}$ let $\psi_{\mathbf{x}\sigma}$ denote the annihilation operator of the Fermionic Fock space $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$ and $\psi_{\mathbf{x}\sigma}^*$ denote its adjoint operator, which is called the creation operator. For $\mathbf{x} \in \mathbb{Z}^2$ we define $\psi_{\mathbf{x}\sigma}, \psi_{\mathbf{x}\sigma}^*$ by identifying \mathbf{x} with the corresponding site of $\Gamma(2L)$ which is equal to \mathbf{x} in $(\mathbb{Z}/2L\mathbb{Z})^2$.

Let $\mathbf{e}_1 := (1, 0)$, $\mathbf{e}_2 := (0, 1) \in \mathbb{Z}^2$. We define the amplitude $t(\cdot, \cdot) : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}_{\geq 0}$ of the hopping matrix elements as follows. With parameters

$t_{h,e}, t_{h,o}, t_{v,e}, t_{v,o} \in \mathbb{R}_{>0}$,

$$t(\mathbf{x}, \mathbf{y}) := \begin{cases} t_{h,e} & \text{if } \mathbf{x} - \mathbf{y} = \mathbf{e}_1, -\mathbf{e}_1 \text{ in } (\mathbb{Z}/2L\mathbb{Z})^2 \text{ and } x_2 = 0 \text{ in } \mathbb{Z}/2\mathbb{Z}, \\ t_{h,o} & \text{if } \mathbf{x} - \mathbf{y} = \mathbf{e}_1, -\mathbf{e}_1 \text{ in } (\mathbb{Z}/2L\mathbb{Z})^2 \text{ and } x_2 = 1 \text{ in } \mathbb{Z}/2\mathbb{Z}, \\ t_{v,e} & \text{if } \mathbf{x} - \mathbf{y} = \mathbf{e}_2, -\mathbf{e}_2 \text{ in } (\mathbb{Z}/2L\mathbb{Z})^2 \text{ and } x_1 = 0 \text{ in } \mathbb{Z}/2\mathbb{Z}, \\ t_{v,o} & \text{if } \mathbf{x} - \mathbf{y} = \mathbf{e}_2, -\mathbf{e}_2 \text{ in } (\mathbb{Z}/2L\mathbb{Z})^2 \text{ and } x_1 = 1 \text{ in } \mathbb{Z}/2\mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

$(\forall \mathbf{x} = (x_1, x_2), \mathbf{y} \in \mathbb{Z}^2)$.

We allow the hopping matrix elements to be complex. Assume that the argument $\theta_L(\cdot, \cdot) : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfies

$$(1.1) \quad \begin{aligned} \theta_L(\mathbf{x}, \mathbf{y}) &= -\theta_L(\mathbf{y}, \mathbf{x}) \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \\ \theta_L(\mathbf{x} + 2mL\mathbf{e}_1 + 2nL\mathbf{e}_2, \mathbf{y}) &= \theta_L(\mathbf{x}, \mathbf{y}) \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{Z}^2, m, n \in \mathbb{Z}) \end{aligned}$$

and

$$(1.2) \quad \begin{aligned} &\theta_L(\mathbf{x} + \mathbf{e}_1, \mathbf{x}) + \theta_L(\mathbf{x} + \mathbf{e}_1 + \mathbf{e}_2, \mathbf{x} + \mathbf{e}_1) \\ &\quad + \theta_L(\mathbf{x} + \mathbf{e}_2, \mathbf{x} + \mathbf{e}_1 + \mathbf{e}_2) + \theta_L(\mathbf{x}, \mathbf{x} + \mathbf{e}_2) \\ &= \pi \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall \mathbf{x} \in \mathbb{Z}^2), \\ &\sum_{j=0}^{2L-1} \theta_L((j+1, x), (j, x)) = \sum_{j=0}^{2L-1} \theta_L((x, j+1), (x, j)) \\ &= 0 \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall x \in \mathbb{Z}). \end{aligned}$$

The kinetic part \mathbf{H}_0 of the Hamiltonian is defined by

$$(1.3) \quad \begin{aligned} \mathbf{H}_0 := &\sum_{\mathbf{x} \in \Gamma(2L)} \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{j=1}^2 (t(\mathbf{x}, \mathbf{x} + \mathbf{e}_j) e^{i\theta_L(\mathbf{x}, \mathbf{x} + \mathbf{e}_j)} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x} + \mathbf{e}_j \sigma} \\ &+ t(\mathbf{x}, \mathbf{x} - \mathbf{e}_j) e^{i\theta_L(\mathbf{x}, \mathbf{x} - \mathbf{e}_j)} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x} - \mathbf{e}_j \sigma}). \end{aligned}$$

One can see that $\mathbf{H}_0^* = \mathbf{H}_0$.

The condition (1.2) is interpreted as having the magnetic flux $\pi \pmod{2\pi}$ per plaquette and $0 \pmod{2\pi}$ through the circles winding around the periodic lattice, because the sum in (1.2) is the value of the line integral

of the magnetic vector potential around the corresponding contour, if we adopt the Peierls substitution. One simple example of such θ_L is that

$$(1.4) \quad \theta_L(\mathbf{x}, \mathbf{y}) = \begin{cases} \pi & \text{if } \mathbf{x} - \mathbf{y} = \mathbf{e}_2, -\mathbf{e}_2 \text{ in } (\mathbb{Z}/2L\mathbb{Z})^2 \text{ and} \\ & x_1 = 1 \text{ in } \mathbb{Z}/2\mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\forall \mathbf{x} = (x_1, x_2), \mathbf{y} \in \mathbb{Z}^2).$$

In this case the nearest-neighbor hopping is pictured as in Figure 2.

To define the interacting part of the Hamiltonian, we assume that the magnitude of the on-site interaction may depend on sites. More specifically, with parameters $U_{e,e}, U_{o,e}, U_{e,o}, U_{o,o} \in \mathbb{R}$ we define $U(\cdot) : \mathbb{Z}^2 \rightarrow \mathbb{R}$ by

$$U(\mathbf{x}) := \begin{cases} U_{e,e} & \text{if } \mathbf{x} = (0, 0) \text{ in } (\mathbb{Z}/2\mathbb{Z})^2, \\ U_{o,e} & \text{if } \mathbf{x} = (1, 0) \text{ in } (\mathbb{Z}/2\mathbb{Z})^2, \\ U_{e,o} & \text{if } \mathbf{x} = (0, 1) \text{ in } (\mathbb{Z}/2\mathbb{Z})^2, \\ U_{o,o} & \text{if } \mathbf{x} = (1, 1) \text{ in } (\mathbb{Z}/2\mathbb{Z})^2, \end{cases} \quad (\forall \mathbf{x} \in \mathbb{Z}^2).$$

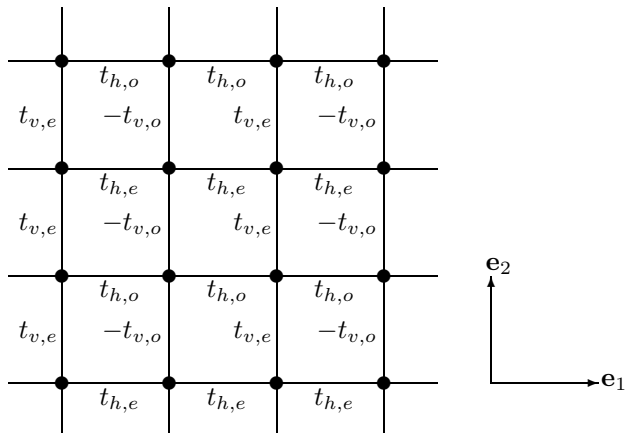


Fig. 2. The nearest-neighbor hopping with the phase θ_L defined by (1.4).

With this $U(\cdot)$, define the interacting part \mathbf{V} by

$$(1.5) \quad \mathbf{V} := \sum_{\mathbf{x} \in \Gamma(2L)} U(\mathbf{x}) \left(\psi_{\mathbf{x}\uparrow}^* \psi_{\mathbf{x}\downarrow}^* \psi_{\mathbf{x}\downarrow} \psi_{\mathbf{x}\uparrow} - \frac{1}{2} \sum_{\sigma \in \{\uparrow, \downarrow\}} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x}\sigma} \right).$$

The Hamiltonian \mathbf{H} is defined by $\mathbf{H} := \mathbf{H}_0 + \mathbf{V}$, which is a self-adjoint operator on $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$. Including the quadratic term in the interacting part as above makes the system half-filled. This fact can be confirmed by a well-known argument. We provide the proof in Remark 1.4 below for completeness. With the inverse temperature $\beta \in \mathbb{R}_{>0}$, the free energy density of the system is given by

$$-\frac{1}{\beta(2L)^2} \log(\text{Tr } e^{-\beta \mathbf{H}}).$$

To shorten formulas, we set $\mathbf{t} := (t_{h,e}, t_{h,o}, t_{v,e}, t_{v,o}) \in (\mathbb{R}_{>0}^4)$,

$$(1.6) \quad f_{\mathbf{t}} := \frac{\min\{t_{h,e}t_{h,o}, t_{v,e}t_{v,o}\}}{(\max\{t_{h,e}, t_{h,o}, t_{v,e}, t_{v,o}\})^{\frac{3}{2}}} \cdot \min \left\{ \frac{t_{h,o}}{t_{h,e}}, \frac{t_{h,e}}{t_{h,o}}, \frac{t_{v,o}}{t_{v,e}}, \frac{t_{v,e}}{t_{v,o}} \right\}$$

and

$$D_{\mathbf{t}}(c) := \{z \in \mathbb{C} \mid |z| < cf_{\mathbf{t}}^2\}$$

for $c \in \mathbb{R}_{>0}$. The goal of this paper is to prove the following theorem.

THEOREM 1.1. *Set $\mathbf{U} := (U_{e,e}, U_{o,e}, U_{e,o}, U_{o,o})$. There exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that the following statements hold true.*

- (1) *There exists a function $F_{\beta,L}(\cdot) : \overline{D_{\mathbf{t}}(c)}^4 \rightarrow \mathbb{C}$ parameterized by $\beta \in \mathbb{R}_{>0}$ and $L \in \mathbb{N}$ satisfying $L \geq \max\{t_{h,e}, t_{h,o}, t_{v,e}, t_{v,o}\}\beta$ such that $F_{\beta,L}(\cdot)$ is continuous in $\overline{D_{\mathbf{t}}(c)}^4$, analytic in $D_{\mathbf{t}}(c)^4$ and*

$$\begin{aligned} F_{\beta,L}(\mathbf{U}) &= -\frac{1}{\beta(2L)^2} \log(\text{Tr } e^{-\beta \mathbf{H}}), \\ (\forall \mathbf{U} \in \overline{D_{\mathbf{t}}(c)}^4 \cap \mathbb{R}^4, \beta \in \mathbb{R}_{>0}, \\ &L \in \mathbb{N} \text{ with } L \geq \max\{t_{h,e}, t_{h,o}, t_{v,e}, t_{v,o}\}\beta). \end{aligned}$$

- (2) *There exists a function $F_\beta(\cdot) : \overline{D_t(c)}^4 \rightarrow \mathbb{C}$ parameterized by $\beta \in \mathbb{R}_{>0}$, independent of $L \in \mathbb{N}$ such that*

$$\lim_{\substack{L \rightarrow \infty \\ L \in \mathbb{N}}} \sup_{\mathbf{z} \in \overline{D_t(c)}^4} |F_{\beta,L}(\mathbf{z}) - F_\beta(\mathbf{z})| = 0, \quad (\forall \beta \in \mathbb{R}_{>0}).$$

- (3) *There exists a function $F(\cdot) : \overline{D_t(c)}^4 \rightarrow \mathbb{C}$ independent of $\beta \in \mathbb{R}_{>0}$ such that*

$$\lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{\mathbf{z} \in \overline{D_t(c)}^4} |F_\beta(\mathbf{z}) - F(\mathbf{z})| = 0.$$

If we impose additional conditions on $t_{h,e}, t_{h,o}, t_{v,e}, t_{v,o}, U_{e,e}, U_{o,e}, U_{e,o}, U_{o,o}$ and L , we can relate the free energy density considered in Theorem 1.1 to the minimum free energy in the flux phase problem, which seeks a phase $\phi : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ of the hopping parameter minimizing the free energy. Lieb ([15]) essentially gave a sufficient condition for a phase to be a minimizer of the flux phase problem. The sufficient condition was also claimed by Macris and Nachtergaele in [17]. That is the condition (1.2) if $L \in 2\mathbb{N} + 1$. For readers who are not familiar with the flux phase problem, we restate Lieb's result in Appendix A with some supplementary arguments which were not explicit in the letter [15]. In mathematical terms, the flux phase problem is to find $\tilde{\phi} : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying (1.1) such that

$$(1.7) \quad \begin{aligned} & -\frac{1}{\beta(2L)^2} \log(\text{Tr } e^{-\beta \mathbf{H}(\tilde{\phi})}) \\ & = \min \left\{ -\frac{1}{\beta(2L)^2} \log(\text{Tr } e^{-\beta \mathbf{H}(\phi)}) \mid \phi : \right. \\ & \quad \left. \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R} \text{ satisfying (1.1)} \right\}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{H}(\phi) := & \sum_{(\mathbf{x}, \sigma) \in \Gamma(2L) \times \{\uparrow, \downarrow\}} \sum_{j=1}^2 (t(\mathbf{x}, \mathbf{x} + \mathbf{e}_j) e^{i\phi(\mathbf{x}, \mathbf{x} + \mathbf{e}_j)} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x} + \mathbf{e}_j \sigma} \\ & + t(\mathbf{x}, \mathbf{x} - \mathbf{e}_j) e^{i\phi(\mathbf{x}, \mathbf{x} - \mathbf{e}_j)} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x} - \mathbf{e}_j \sigma}) + \mathbf{V}. \end{aligned}$$

Since this is equivalent to a minimization problem of a continuous function defined on the compact set $[0, 2\pi]^{2(2L)^2}$, a minimizer exists. Under the additional conditions that $t_{h,e} = t_{h,o}$, $t_{v,e} = t_{v,o}$, $U_{e,e} = U_{o,e} = U_{e,o} = U_{o,o}$ and $L \in 2\mathbb{N} + 1$, Theorem A.5 in Appendix A ensures that any phase θ_L satisfying (1.1) and (1.2) is a minimizer of the flux phase problem. Thus, we have the following corollary.

COROLLARY 1.2. *Assume that $t_{h,e} = t_{h,o}$, $t_{v,e} = t_{v,o}$, $U_{e,e} = U_{o,e} = U_{e,o} = U_{o,o} = U \in \mathbb{R}$. Let $G_{\beta,L}(U)$ denote the right-hand side of (1.7). Then, there exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that the following statements hold true.*

- (1) *There exists a function $F_{\beta,L}(\cdot) : \overline{D_{\mathbf{t}}(c)} \rightarrow \mathbb{C}$ parameterized by $\beta \in \mathbb{R}_{>0}$ and $L \in 2\mathbb{N} + 1$ satisfying $L \geq \max\{t_{h,e}, t_{v,e}\}\beta$ such that $F_{\beta,L}(\cdot)$ is continuous in $\overline{D_{\mathbf{t}}(c)}$, analytic in $D_{\mathbf{t}}(c)$ and*

$$\begin{aligned} F_{\beta,L}(U) &= G_{\beta,L}(U), \\ (\forall U \in \overline{D_{\mathbf{t}}(c)} \cap \mathbb{R}, \beta \in \mathbb{R}_{>0}, \\ L \in 2\mathbb{N} + 1 \text{ with } L &\geq \max\{t_{h,e}, t_{v,e}\}\beta). \end{aligned}$$

- (2) *There exists a function $F_{\beta}(\cdot) : \overline{D_{\mathbf{t}}(c)} \rightarrow \mathbb{C}$ parameterized by $\beta \in \mathbb{R}_{>0}$, independent of $L \in 2\mathbb{N} + 1$ such that*

$$\lim_{\substack{L \rightarrow \infty \\ L \in 2\mathbb{N} + 1}} \sup_{z \in \overline{D_{\mathbf{t}}(c)}} |F_{\beta,L}(z) - F_{\beta}(z)| = 0, \quad (\forall \beta \in \mathbb{R}_{>0}).$$

- (3) *There exists a function $F(\cdot) : \overline{D_{\mathbf{t}}(c)} \rightarrow \mathbb{C}$ independent of $\beta \in \mathbb{R}_{>0}$ such that*

$$\lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{z \in \overline{D_{\mathbf{t}}(c)}} |F_{\beta}(z) - F(z)| = 0.$$

REMARK 1.3. Theorem 1.1 implies the analyticity of the infinite-volume, zero-temperature limit of the free energy density in the following sense. There exists a function $F(\cdot) : \overline{D_{\mathbf{t}}(c)}^4 \rightarrow \mathbb{C}$ independent of $\beta \in \mathbb{R}_{>0}$, $L \in \mathbb{N}$ such that $F(\cdot)$ is continuous in $\overline{D_{\mathbf{t}}(c)}^4$, analytic in $D_{\mathbf{t}}(c)^4$ and

$$\lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \lim_{\substack{L \rightarrow \infty \\ L \in \mathbb{N}}} \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c)}^4 \cap \mathbb{R}^4} \left| -\frac{1}{\beta(2L)^2} \log(\text{Tr } e^{-\beta \mathbf{H}}) - F(\mathbf{U}) \right| = 0.$$

REMARK 1.4. The system is half-filled. To confirm this, let us define the operator A on $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$ by

$$\begin{aligned} A(\alpha\Omega_{2L}) &:= \bar{\alpha} \prod_{\mathbf{x} \in \Gamma(2L)} (\psi_{\mathbf{x}\uparrow}^* \psi_{\mathbf{x}\downarrow}^*) \Omega_{2L}, \\ A(\alpha\psi_{(x_{1,1}, x_{1,2})\sigma_1}^* \psi_{(x_{2,1}, x_{2,2})\sigma_2}^* \cdots \psi_{(x_{n,1}, x_{n,2})\sigma_n}^* \Omega_{2L}) \\ &:= (-1)^{\sum_{j=1}^n (x_{j,1} + x_{j,2})} \bar{\alpha} \psi_{(x_{1,1}, x_{1,2})\sigma_1} \psi_{(x_{2,1}, x_{2,2})\sigma_2} \cdots \psi_{(x_{n,1}, x_{n,2})\sigma_n} \\ &\quad \cdot \prod_{\mathbf{x} \in \Gamma(2L)} (\psi_{\mathbf{x}\uparrow}^* \psi_{\mathbf{x}\downarrow}^*) \Omega_{2L}, \\ &(\forall \alpha \in \mathbb{C}, (x_{j,1}, x_{j,2}) \in \Gamma(2L), \sigma_j \in \{\uparrow, \downarrow\} (j = 1, 2, \dots, n)) \end{aligned}$$

and by linearity, where Ω_{2L} denotes the vacuum of $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$. One can check that A is unitary, $AHA^* = H$ and

$$A\psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x}\sigma} A^* = id_{2L} - \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x}\sigma}, \quad (\forall (\mathbf{x}, \sigma) \in \Gamma(2L) \times \{\uparrow, \downarrow\}),$$

where id_{2L} denotes the identity map on $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$. Thus,

$$\begin{aligned} \mathrm{Tr}(e^{-\beta H} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x}\sigma}) &= \mathrm{Tr}(e^{-\beta AHA^*} A\psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x}\sigma} A^*) \\ &= \mathrm{Tr} e^{-\beta H} - \mathrm{Tr}(e^{-\beta H} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x}\sigma}), \end{aligned}$$

which implies that

$$\frac{\mathrm{Tr}(e^{-\beta H} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x}\sigma})}{\mathrm{Tr} e^{-\beta H}} = \frac{1}{2}, \quad (\forall (\mathbf{x}, \sigma) \in \Gamma(2L) \times \{\uparrow, \downarrow\}).$$

REMARK 1.5. In Theorem 1.1 we have freedom to choose a phase θ_L satisfying (1.1) and (1.2). However, the free energy density is independent of the choice of θ_L . Let θ_L, θ'_L be phases satisfying (1.1) and (1.2) and $H(\theta_L), H(\theta'_L)$ be the Hamiltonian having the phase θ_L, θ'_L respectively. Then, Lemma A.4 given in Appendix A implies that $\mathrm{Tr} e^{-\beta H(\theta_L)} = \mathrm{Tr} e^{-\beta H(\theta'_L)}$. In brief, this equality is due to the fact that the flux of θ_L through any circuit in the periodic lattice $(\mathbb{Z}/2L\mathbb{Z})^2$ is the same as that of θ'_L .

REMARK 1.6. The proof of Theorem A.5 requires that the hopping amplitude and the magnitude of on-site interaction are invariant under vertical

and horizontal reflections. To meet this requirement, we need to assume that $t_{h,e} = t_{h,o}$, $t_{v,e} = t_{v,o}$, $U_{e,e} = U_{o,e} = U_{e,o} = U_{o,o}$. Moreover, on the assumption $L \in 2\mathbb{N} + 1$, having the flux $\pi(L - 1) \pmod{2\pi}$ through the circles around the periodic lattice, another requirement of Theorem A.5, is equal to having the flux $0 \pmod{2\pi}$, which is satisfied by our model Hamiltonian. In the case $L \in 2\mathbb{N}$ we do not have the equivalence between the free energy governed by our model Hamiltonian and the minimum free energy in the flux phase problem.

REMARK 1.7. Consider the Hamiltonian H with the phase defined by (1.4). If $t_{h,o} = t_{v,o} = U_{o,o} = 0$, the Hamiltonian H becomes the half-filled Hubbard model on the copper-oxide (CuO) lattice. Since the condition $t_{h,o}, t_{v,o} \neq 0$ is indispensable for our analysis, we cannot treat the half-filled CuO Hubbard model itself in this paper. As an operator on the finite-dimensional space our Hamiltonian can be arbitrarily close to the half-filled CuO Hubbard model as $t_{h,o}, t_{v,o} \searrow 0$. For such an approximate model with small but non-zero $t_{h,o}, t_{v,o}$, Theorem 1.1 guarantees the existence of infinite-volume, zero-temperature limit of the free energy density and its analyticity with the coupling constants $U_{e,e}, U_{o,e}, U_{e,o}$. However, since the domain $D_{\mathbf{t}}(c)$ shrinks as $t_{h,o}, t_{v,o} \searrow 0$, we cannot extract any information on the free energy density defined in the half-filled CuO Hubbard model from Theorem 1.1.

REMARK 1.8. Later in Lemma 7.15 in Section 7 we will see that the integral of modulus of the free covariance is bounded by a constant times β from above and below if L is sufficiently large. This also implies that the free covariance with the Matsubara UV cut-off has the same bound property in low temperature, since the integral of modulus of the free covariance with the large Matsubara frequency, the difference between the free covariance and that with the Matsubara UV cut-off, is bounded from above independently of β . These facts tell us that we cannot prove the analyticity of the free energy density in the infinite-volume limit in a domain larger than $\{U \in \mathbb{C} \mid |U| < c\beta^{-1}\}^4$ by means of a single-scale analysis based on Pedra-Salmhofer's determinant bound as in [12] or a multi-scale analysis over the Matsubara frequency as in [14], since the inverse of the L^1 -bound of the free covariance with or without UV cut-off determines the maximal magnitude of the coupling in these theories. Therefore, we are led to perform an IR

analysis in order to reach the infinite-volume, zero-temperature limit in this model of interacting electrons.

2. Formulation

In this section, first we define the multi-band Hamiltonian H consisting of the free part H_0 and the interacting part V in a generalized setting. In Subsection 1.2 we introduced the single-band Hamiltonian \mathbf{H} . We will prove Theorem 1.1 by formulating the Hamiltonian \mathbf{H} into a 4-band Hamiltonian in Section 7. The Hamiltonian H should be considered as a generalization of the 4-band model Hamiltonian. Then we introduce the finite-dimensional Grassmann integral formulation of the normalized free energy density $-\frac{1}{\beta L^d} \log(\text{Tr } e^{-\beta H} / \text{Tr } e^{-\beta H_0})$. Though the main theorem of this paper concerns the free energy density of the form $-\frac{1}{\beta L^d} \log(\text{Tr } e^{-\beta H})$, it is more convenient to deal with the normalized one, since it fits in the framework of Grassmann Gaussian integration. We can reach the conclusions on the free energy density from the analysis of the normalized free energy density, since the non-interacting free energy density $-\frac{1}{\beta L^d} \log(\text{Tr } e^{-\beta H_0})$, the difference between them, is exactly computable.

2.1. The multi-band Hamiltonian

Let us set up a system which we focus on until we analyze the specific model in Section 7. Let $d \in \mathbb{N}$ denote the spatial dimension. Take a basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d$ of \mathbb{R}^d . Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d$ be another basis of \mathbb{R}^d satisfying $\langle \mathbf{u}_l, \mathbf{v}_m \rangle = \delta_{l,m}$ ($\forall l, m \in \{1, 2, \dots, d\}$), where $\langle \cdot, \cdot \rangle$ is the standard inner product of \mathbb{R}^d . The spatial lattice Γ is defined by

$$\Gamma := \left\{ \sum_{j=1}^d m_j \mathbf{u}_j \mid m_j \in \{0, 1, \dots, L-1\} \ (j = 1, 2, \dots, d) \right\}.$$

The momentum lattice Γ^* dual to Γ is given by

$$\Gamma^* := \left\{ \frac{2\pi}{L} \sum_{j=1}^d m_j \mathbf{v}_j \mid m_j \in \{0, 1, \dots, L-1\} \ (j = 1, 2, \dots, d) \right\}.$$

With a number $b \in \mathbb{N}$ we assume that the crystal lattice is modeled by the lattice Γ with a b -point basis. The integer b stands for the number of atomic

sites in a primitive unit cell of the lattice Γ . Each site of the crystal lattice is identified with an element of the set $\{1, 2, \dots, b\} \times \Gamma$. For conciseness we set $\mathcal{B} := \{1, 2, \dots, b\}$.

The Hamiltonian H is defined as a self-adjoint operator on the Fermionic Fock space $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$. To define the free part of the Hamiltonian H , we assume that in the momentum space the hopping matrix is represented by $E(\mathbf{k}) \in \text{Mat}(b, \mathbb{C})$ ($\mathbf{k} \in \Gamma^*$). Moreover we assume that the domain of $E(\cdot)$ can be extended to \mathbb{R}^d and

$$(2.1) \quad \begin{aligned} E &\in C(\mathbb{R}^d; \text{Mat}(b, \mathbb{C})), \\ E(\mathbf{k})^* &= E(\mathbf{k}), \quad (\forall \mathbf{k} \in \mathbb{R}^d), \end{aligned}$$

$$(2.2) \quad E(\mathbf{k} + 2\pi\mathbf{v}_j) = E(\mathbf{k}), \quad (\forall \mathbf{k} \in \mathbb{R}^d, j \in \{1, 2, \dots, d\}).$$

We consider $\text{Mat}(b, \mathbb{C})$ as a b^2 -dimensional complex Banach space with the norm $\|\cdot\|_{b \times b}$ defined by

$$\|A\|_{b \times b} := \sup_{\substack{\mathbf{v} \in \mathbb{C}^b \text{ with} \\ \|\mathbf{v}\|_{\mathbb{C}^b} = 1}} \|A\mathbf{v}\|_{\mathbb{C}^b}, \quad (A \in \text{Mat}(b, \mathbb{C})),$$

where $\|\cdot\|_{\mathbb{C}^b}$ denotes the norm of \mathbb{C}^b induced by the standard inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}^b}$. With $E(\cdot)$ we define the free part H_0 by

$$(2.3) \quad H_0 := \frac{1}{L^d} \sum_{\rho, \eta \in \mathcal{B}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma} \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{k} \in \Gamma^*} e^{i\langle \mathbf{k}, \mathbf{x} - \mathbf{y} \rangle} E(\mathbf{k})(\rho, \eta) \psi_{\rho\mathbf{x}\sigma}^* \psi_{\eta\mathbf{y}\sigma},$$

where $\psi_{\rho\mathbf{x}\sigma}$ is the annihilation operator destroying an electron with the spin σ on the site (ρ, \mathbf{x}) and $\psi_{\rho\mathbf{x}\sigma}^*$ is its adjoint operator called the creation operator.

The interacting part V is defined by

$$(2.4) \quad V := \sum_{\rho \in \mathcal{B}} \sum_{\mathbf{x} \in \Gamma} U_\rho \left(\psi_{\rho\mathbf{x}\uparrow}^* \psi_{\rho\mathbf{x}\downarrow}^* \psi_{\rho\mathbf{x}\downarrow} \psi_{\rho\mathbf{x}\uparrow} - \frac{1}{2} \sum_{\sigma \in \{\uparrow, \downarrow\}} \psi_{\rho\mathbf{x}\sigma}^* \psi_{\rho\mathbf{x}\sigma} \right),$$

with the coupling constants $U_\rho \in \mathbb{R}$ ($\rho \in \mathcal{B}$). To be more precise, the second term of V should be considered as a part representing the on-site energy minus the chemical potential. Since we are going to construct a theory for the half-filled systems, the on-site quadratic term of this form needs

to be included in V . The Hamiltonian governing the multi-band many-electron system is defined by $H := H_0 + V: F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})) \rightarrow F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$. By the condition (2.1), H is self-adjoint. In the rest of this section we will introduce the Grassmann integral formulation of the normalized free energy density $-\frac{1}{\beta L^d} \log(\text{Tr } e^{-\beta H} / \text{Tr } e^{-\beta H_0})$.

2.2. The finite-dimensional Grassmann integrals

Let us summarize the notions of Grassmann integration over a finite-dimensional Grassmann algebra. Take a parameter $h \in (2/\beta)\mathbb{N}$ and introduce the discrete analogue of the interval $[0, \beta)$ by

$$[0, \beta)_h := \left\{ 0, \frac{1}{h}, \frac{2}{h}, \dots, \beta - \frac{1}{h} \right\}.$$

We take the parameter h from $(2/\beta)\mathbb{N}$ rather than from $(1/\beta)\mathbb{N}$ in order to refer to the basic results of [12, Appendix C] constructed with h belonging to $(2/\beta)\mathbb{N}$. The index sets I_0, I are defined by

$$\begin{aligned} I_0 &:= \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)_h, \\ I &:= I_0 \times \{1, -1\}. \end{aligned}$$

Let N stand for the number $4b\beta hL^d$, the cardinality of I . Let \mathcal{V} denote the complex vector space spanned by the abstract basis $\{\psi_X\}_{X \in I}$. Similarly for $p \in \mathbb{N}$ let \mathcal{V}_p be the complex vector space spanned by the basis $\{\psi_X^p\}_{X \in I}$. For $X \in I_0$ we sometimes write $\bar{\psi}_X, \psi_X$ in place of $\psi_{(X,1)}, \psi_{(X,-1)}$ respectively.

For a finite-dimensional complex vector space W and $n \in \mathbb{N}$, set $\bigwedge^0 W := \mathbb{C}$ and let $\bigwedge^n W$ denote the n -fold anti-symmetric tensor product of W . Moreover, set

$$\bigwedge W := \bigoplus_{n=0}^{\dim W} \bigwedge^n W.$$

For $n \in \{0, 1, \dots, \dim W\}$ let $\mathcal{P}_n : \bigwedge W \rightarrow \bigwedge^n W$ denote the standard projection.

We call $\bigwedge \mathcal{V}$ Grassmann algebra generated by $\{\psi_X\}_{X \in I}$. We write an element f of $\bigwedge \mathcal{V}$ as $f(\psi)$ when we want to show its Grassmann variable explicitly. We can define $f(\psi + \psi^p) \in \bigwedge(\mathcal{V} \oplus \mathcal{V}_p)$ from $f(\psi) \in \bigwedge \mathcal{V}$ by replacing each ψ_X by $\psi_X + \psi_X^p$ inside $f(\psi)$. For $\mathbf{X} = (X_1, X_2, \dots, X_m) \in I^m$ we simply write $\psi_{\mathbf{X}}$ in place of $\psi_{X_1} \psi_{X_2} \cdots \psi_{X_m}$ and \mathbf{X}_σ in place of

$(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(m)})$ for any $\sigma \in \mathbb{S}_m$, where \mathbb{S}_m denotes the set of all permutations over $\{1, 2, \dots, m\}$. We call a function $f_m : I^m \rightarrow \mathbb{C}$ anti-symmetric if $f_m(\mathbf{X}) = \text{sgn}(\sigma)f_m(\mathbf{X}_\sigma)$ for any $\mathbf{X} \in I^m$, $\sigma \in \mathbb{S}_m$.

For any $f(\psi) \in \bigwedge \mathcal{V}$ there uniquely exist $f_0 \in \mathbb{C}$ and anti-symmetric functions $f_m(\cdot) : I^m \rightarrow \mathbb{C}$ ($m \in \{1, 2, \dots, N\}$) such that

$$f(\psi) = f_0 + \sum_{m=1}^N \left(\frac{1}{h}\right)^m \sum_{\mathbf{X} \in I^m} f_m(\mathbf{X})\psi_{\mathbf{X}}.$$

Throughout the paper we follow the notational convention that for $f(\psi) \in \bigwedge \mathcal{V}$, $f_m(\psi)$ denotes $\mathcal{P}_m f(\psi)$ and $f_m(\cdot) : I^m \rightarrow \mathbb{C}$ denotes the anti-symmetric kernel of $f_m(\psi)$. For example, we write as follows.

$$f(\psi) = \sum_{m=0}^N f_m(\psi), \quad f_m(\psi) = \left(\frac{1}{h}\right)^m \sum_{\mathbf{X} \in I^m} f_m(\mathbf{X})\psi_{\mathbf{X}}.$$

We can construct a norm in the complex vector space $\bigwedge \mathcal{V}$ by defining a norm in the space of anti-symmetric functions on I^m for all $m \in \{1, 2, \dots, N\}$. In this paper we will introduce various norms in the space of anti-symmetric functions. We will define the norms one by one when necessary rather than by listing them all together at this stage.

Let $a \in \mathbb{N}$ and D be a domain of \mathbb{C}^a . Assume that $f(\mathbf{z})(\psi) \in \bigwedge \mathcal{V}$ is parameterized by $\mathbf{z} \in \overline{D}$. We say that $f(\mathbf{z})(\psi)$ is continuous with \mathbf{z} in \overline{D} if so is $f(\mathbf{z})_m(\mathbf{X})$ ($\forall m \in \{0, 1, \dots, N\}$, $\mathbf{X} \in I^m$). Similarly we say that $f(\mathbf{z})(\psi)$ is analytic with \mathbf{z} in D if so is $f(\mathbf{z})_m(\mathbf{X})$ ($\forall m \in \{0, 1, \dots, N\}$, $\mathbf{X} \in I^m$). In this case we define the Grassmann polynomials

$$\prod_{j=1}^a \left(\frac{\partial}{\partial z_j}\right)^{n_j} f(\mathbf{z})(\psi) \in \bigwedge \mathcal{V},$$

$$(\mathbf{z} = (z_1, z_2, \dots, z_a) \in D, n_j \in \mathbb{N} \cup \{0\} \ (j = 1, 2, \dots, a))$$

by

$$\prod_{j=1}^a \left(\frac{\partial}{\partial z_j}\right)^{n_j} f(\mathbf{z})(\psi) := \sum_{m=0}^N \left(\frac{1}{h}\right)^m \sum_{\mathbf{X} \in I^m} \prod_{j=1}^a \left(\frac{\partial}{\partial z_j}\right)^{n_j} f(\mathbf{z})_m(\mathbf{X})\psi_{\mathbf{X}}.$$

Consider a sequence $(f^n(\psi))_{n=1}^\infty$ of $\bigwedge \mathcal{V}$. We say that $f^n(\psi)$ converges as $n \rightarrow \infty$ if so does $f_m^n(\mathbf{X})$ ($\forall m \in \{0, 1, \dots, N\}$, $\mathbf{X} \in I^m$). Consider

a sequence $(f^n(\mathbf{z})(\psi))_{n=1}^\infty$ of $\bigwedge \mathcal{V}$ parameterized by $\mathbf{z} \in \overline{D}$. We say that $f^n(\mathbf{z})(\psi)$ uniformly converges with $\mathbf{z} \in \overline{D}$ as $n \rightarrow \infty$ if so does $f^n(\mathbf{z})_m(\mathbf{X})$ ($\forall m \in \{0, 1, \dots, N\}, \mathbf{X} \in I^m$). If a norm is defined in $\bigwedge \mathcal{V}$, the normed space $\bigwedge \mathcal{V}$ is complete, since $\dim \bigwedge \mathcal{V} < \infty$. These definitions of continuity, analyticity, derivative, convergence and uniform convergence are equivalent to those defined in the Banach space $\bigwedge \mathcal{V}$.

For $p_1, p_2, \dots, p_n, p \in \mathbb{N}$ with $p_j \neq p$ ($\forall j \in \{1, 2, \dots, n\}$) the Grassmann Gaussian integral $\int \cdot d\mu_{C_o}(\psi^p)$ with a covariance matrix $C_o : I_0^2 \rightarrow \mathbb{C}$ is a linear map from $\bigwedge(\mathcal{V}_{p_1} \oplus \dots \oplus \mathcal{V}_{p_n} \oplus \mathcal{V}_p)$ to $\bigwedge(\mathcal{V}_{p_1} \oplus \dots \oplus \mathcal{V}_{p_n})$ defined as follows. For any $f \in \bigwedge(\mathcal{V}_{p_1} \oplus \dots \oplus \mathcal{V}_{p_n})$,

$$\begin{aligned} & \int f d\mu_{C_o}(\psi^p) := f, \\ & \int f \overline{\psi}_{X_1}^p \cdots \overline{\psi}_{X_l}^p \psi_{Y_m}^p \cdots \psi_{Y_1}^p d\mu_{C_o}(\psi^p) \\ & := \begin{cases} \det(C_o(X_i, Y_j))_{1 \leq i, j \leq l} f & \text{if } l = m, \\ 0 & \text{if } l \neq m. \end{cases} \end{aligned}$$

Then, for any $g \in \bigwedge(\mathcal{V}_{p_1} \oplus \dots \oplus \mathcal{V}_{p_n} \oplus \mathcal{V}_p)$ the value of $\int g d\mu_{C_o}(\psi^p)$ can be uniquely determined by linearity and anti-symmetry. For $Y \in I$ the left derivative $\partial/\partial\psi_Y^p$ is a linear transform on $\bigwedge(\mathcal{V}_{p_1} \oplus \dots \oplus \mathcal{V}_{p_n} \oplus \mathcal{V}_p)$ defined as follows.

$$\begin{aligned} & \frac{\partial}{\partial\psi_Y^p}(\alpha\psi_{X_1}^{q_1} \cdots \psi_{X_l}^{q_l} \psi_Y^p \psi_{X_{l+1}}^{q_{l+1}} \cdots \psi_{X_m}^{q_m}) := (-1)^l \alpha \psi_{X_1}^{q_1} \cdots \psi_{X_l}^{q_l} \psi_{X_{l+1}}^{q_{l+1}} \cdots \psi_{X_m}^{q_m}, \\ & \frac{\partial}{\partial\psi_Y^p}(\alpha\psi_{X_1}^{q_1} \cdots \psi_{X_l}^{q_l} \psi_{X_{l+1}}^{q_{l+1}} \cdots \psi_{X_m}^{q_m}) := 0, \\ & (\forall \alpha \in \mathbb{C}, \psi_{X_1}^{q_1}, \dots, \psi_{X_m}^{q_m} \in \{\psi_X^{p_1}, \dots, \psi_X^{p_n}, \psi_X^p\}_{X \in I} \setminus \{\psi_Y^p\}). \end{aligned}$$

Then, the value of $\partial/\partial\psi_Y^p$ on any element of $\bigwedge(\mathcal{V}_{p_1} \oplus \dots \oplus \mathcal{V}_{p_n} \oplus \mathcal{V}_p)$ can be uniquely determined by linearity and anti-symmetry. For $\mathbf{X} = (X_1, X_2, \dots, X_m) \in I^m$ we sometimes write $\partial/\partial\psi_{\mathbf{X}}$ in place of $\partial/\partial\psi_{X_1} \cdot \partial/\partial\psi_{X_2} \cdots \partial/\partial\psi_{X_m}$ for simplicity.

We will frequently deal with the exponential and the logarithm of a Grassmann polynomial. Let us recall their definitions. For $f \in \bigwedge \mathcal{V}$ the polynomial e^f ($\in \bigwedge \mathcal{V}$) is defined by

$$e^f := e^{f_0} \sum_{n=0}^N \frac{1}{n!} (f - f_0)^n.$$

Additionally, assume that $f_0 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. The logarithm of f is defined by

$$\log f := \log f_0 + \sum_{n=1}^N \frac{(-1)^{n-1}}{n} \left(\frac{f - f_0}{f_0} \right)^n.$$

For any $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ we define $\log z$ by the principal value $\log |z| + i\theta$, where $\theta \in (-\pi, \pi)$ satisfies $z = |z|e^{i\theta}$.

2.3. The full covariance

The covariance in our Grassmann Gaussian integral formulation of the free energy density is equal to the non-interacting 2-point correlation function. For $(\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)$,

$$(2.5) \quad C(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) := \frac{\text{Tr}(e^{-\beta H_0} T(\psi_{\rho \mathbf{x} \sigma}^*(x) \psi_{\eta \mathbf{y} \tau}(y)))}{\text{Tr} e^{-\beta H_0}},$$

where $\psi_{\rho \mathbf{x} \sigma}^*(x) := e^{x H_0} \psi_{\rho \mathbf{x} \sigma}^* e^{-x H_0}$, $\psi_{\eta \mathbf{y} \tau}(y) := e^{y H_0} \psi_{\eta \mathbf{y} \tau} e^{-y H_0}$,

$$T(\psi_{\rho \mathbf{x} \sigma}^*(x) \psi_{\eta \mathbf{y} \tau}(y)) := 1_{x \geq y} \psi_{\rho \mathbf{x} \sigma}^*(x) \psi_{\eta \mathbf{y} \tau}(y) - 1_{x < y} \psi_{\eta \mathbf{y} \tau}(y) \psi_{\rho \mathbf{x} \sigma}^*(x).$$

For a proposition P the value of 1_P is defined as follows. $1_P := 1$ if P is true, $1_P := 0$ otherwise. We use the same symbol C even when its variables are restricted to be in the finite subset I_0^2 .

Let \mathcal{M} denote the set of the Matsubara frequency $(\pi/\beta)(2\mathbb{Z} + 1)$. We define the h -dependent finite subset \mathcal{M}_h of the Matsubara frequency by

$$\mathcal{M}_h := \left\{ \omega \in \frac{\pi}{\beta}(2\mathbb{Z} + 1) \mid |\omega| < \pi h \right\}.$$

Let I_b denote the $b \times b$ unit matrix. The covariance matrix $C : I_0^2 \rightarrow \mathbb{C}$ is characterized as follows.

LEMMA 2.1. *For any $(\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in I_0$,*

$$(2.6) \quad C(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) = \frac{\delta_{\sigma, \tau}}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} e^{-i(\mathbf{x}-\mathbf{y}, \mathbf{k})} e^{i(x-y)\omega} h^{-1} (I_b - e^{-i\frac{\omega}{h} I_b + \frac{1}{h} \overline{E(\mathbf{k})}})^{-1}(\rho, \eta).$$

PROOF. One can complete the characterization in the same way as in [14, Appendix A]. For readers' convenience we provide a sketch of the proof.

For any $\mathbf{k} \in \mathbb{R}^d$ let $\alpha_1(\mathbf{k}), \dots, \alpha_b(\mathbf{k}) \in \mathbb{R}$ be the eigen values of $E(\mathbf{k})$. There exists a unitary matrix $U(\mathbf{k}) \in \text{Mat}(b, \mathbb{C})$ such that

$$(2.7) \quad (U(\mathbf{k})^* E(\mathbf{k}) U(\mathbf{k}))(\rho, \eta) = \alpha_\rho(\mathbf{k}) \delta_{\rho, \eta}, \quad (\forall \rho, \eta \in \mathcal{B}).$$

Define the matrix $W : (\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})^2 \rightarrow \mathbb{C}$ by

$$W(\rho \mathbf{x} \sigma, \eta \mathbf{y} \tau) := \frac{\delta_{\sigma, \tau}}{L^d} \sum_{\mathbf{k} \in \Gamma^*} e^{-i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} \overline{U(\mathbf{k})(\rho, \eta)}.$$

Set

$$(W\psi^*)_{\rho \mathbf{x} \sigma} := \sum_{(\eta, \mathbf{y}, \tau) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}} W(\rho \mathbf{x} \sigma, \eta \mathbf{y} \tau) \psi_{\eta \mathbf{y} \tau}^*,$$

$$(\forall (\rho, \mathbf{x}, \sigma) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}).$$

Let us define the linear transform F on $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$ by

$$F\Omega := \Omega,$$

$$F\psi_{X_1}^* \psi_{X_2}^* \cdots \psi_{X_n}^* \Omega := (W\psi^*)_{X_1} (W\psi^*)_{X_2} \cdots (W\psi^*)_{X_n} \Omega,$$

$$(\forall n \in \mathbb{N}, X_1, X_2, \dots, X_n \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})$$

and by linearity, where Ω denotes the vacuum of $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$. By using the unitary property of $U(\mathbf{k})$ and the equality (2.7) one can check that the transform F is unitary and

$$FH_0F^* = \sum_{\rho \in \mathcal{B}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma} \sum_{\sigma \in \{\uparrow, \downarrow\}} \frac{1}{L^d} \sum_{\mathbf{k} \in \Gamma^*} e^{i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} \alpha_\rho(\mathbf{k}) \psi_{\rho \mathbf{x} \sigma}^* \psi_{\rho \mathbf{y} \sigma}.$$

For any $(\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)$ set

$$\tilde{\psi}_{\rho \mathbf{x} \sigma}^*(x) := e^{xFH_0F^*} \psi_{\rho \mathbf{x} \sigma}^* e^{-xFH_0F^*},$$

$$\tilde{\psi}_{\rho \mathbf{x} \sigma}(x) := e^{xFH_0F^*} \psi_{\rho \mathbf{x} \sigma} e^{-xFH_0F^*},$$

$$T(\tilde{\psi}_{\rho \mathbf{x} \sigma}^*(x) \tilde{\psi}_{\eta \mathbf{y} \tau}(y)) := 1_{x \geq y} \tilde{\psi}_{\rho \mathbf{x} \sigma}^*(x) \tilde{\psi}_{\eta \mathbf{y} \tau}(y) - 1_{x < y} \tilde{\psi}_{\eta \mathbf{y} \tau}(y) \tilde{\psi}_{\rho \mathbf{x} \sigma}^*(x).$$

Since F is unitary,

$$(2.8) \quad C(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y)$$

$$\begin{aligned}
 &= \sum_{X, Y \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}} W(\rho \mathbf{x} \sigma, X) \overline{W(\eta \mathbf{y} \tau, Y)} \\
 &\quad \cdot \frac{\text{Tr}(e^{-\beta F H_0 F^*} T(\tilde{\psi}_X^*(x) \tilde{\psi}_Y(y)))}{\text{Tr} e^{-\beta F H_0 F^*}}.
 \end{aligned}$$

Since $F H_0 F^*$ is diagonalized with respect to the band index, the 2-point function $\text{Tr}(e^{-\beta F H_0 F^*} T(\tilde{\psi}_X^*(x) \tilde{\psi}_Y(y))) / \text{Tr} e^{-\beta F H_0 F^*}$ can be computed by a standard procedure (see, e.g., [12, Lemma B.10]). The result is that

$$\begin{aligned}
 (2.9) \quad &\frac{\text{Tr}(e^{-\beta F H_0 F^*} T(\tilde{\psi}_{\rho' \mathbf{x}' \sigma'}^*(x) \tilde{\psi}_{\eta' \mathbf{y}' \tau'}(y)))}{\text{Tr} e^{-\beta F H_0 F^*}} \\
 &= \frac{\delta_{\rho', \eta'} \delta_{\sigma', \tau'}}{L^d} \sum_{\mathbf{k} \in \Gamma^*} e^{-i \langle \mathbf{x}' - \mathbf{y}', \mathbf{k} \rangle} \\
 &\quad \cdot e^{(x-y) \alpha_{\rho'}(\mathbf{k})} \left(\frac{1_{x \geq y}}{1 + e^{\beta \alpha_{\rho'}(\mathbf{k})}} - \frac{1_{x < y}}{1 + e^{-\beta \alpha_{\rho'}(\mathbf{k})}} \right), \\
 &(\forall (\rho', \mathbf{x}', \sigma'), (\eta', \mathbf{y}', \tau') \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}).
 \end{aligned}$$

By substituting (2.9) into (2.8),

$$\begin{aligned}
 (2.10) \quad C(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) &= \frac{\delta_{\sigma, \tau}}{L^d} \sum_{\mathbf{k} \in \Gamma^*} e^{-i \langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} \sum_{\gamma \in \mathcal{B}} \overline{U(\mathbf{k})(\rho, \gamma)} U(\mathbf{k})(\eta, \gamma) \\
 &\quad \cdot e^{(x-y) \alpha_{\gamma}(\mathbf{k})} \left(\frac{1_{x \geq y}}{1 + e^{\beta \alpha_{\gamma}(\mathbf{k})}} - \frac{1_{x < y}}{1 + e^{-\beta \alpha_{\gamma}(\mathbf{k})}} \right), \\
 &(\forall (\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)).
 \end{aligned}$$

Since $h \in (2/\beta)\mathbb{N}$, we can apply [12, Lemma C.3] to obtain that for any $x, y \in [0, \beta)_h$,

$$\begin{aligned}
 (2.11) \quad &e^{(x-y) \alpha_{\gamma}(\mathbf{k})} \left(\frac{1_{x \geq y}}{1 + e^{\beta \alpha_{\gamma}(\mathbf{k})}} - \frac{1_{x < y}}{1 + e^{-\beta \alpha_{\gamma}(\mathbf{k})}} \right) \\
 &= \frac{1}{\beta} \sum_{\omega \in \mathcal{M}_h} \frac{e^{i(x-y)\omega}}{h(1 - e^{-i\frac{\omega}{h} + \frac{1}{h} \alpha_{\gamma}(\mathbf{k})})}.
 \end{aligned}$$

By combining (2.11) with (2.10) and using (2.7) we can derive (2.6). \square

2.4. The Grassmann Gaussian integral formulation

Here we formulate $\log(\text{Tr} e^{-\beta H} / \text{Tr} e^{-\beta H_0})$ into the Grassmann Gaussian integral with the covariance C . Let us introduce a counterpart of the

interaction V in the Grassmann algebra $\wedge \mathcal{V}$.

$$(2.12) \quad V(\psi) := \frac{1}{h} \sum_{(\rho, \mathbf{x}, x) \in \mathcal{B} \times \Gamma \times [0, \beta)_h} \\ \cdot U_\rho \left(\bar{\psi}_{\rho \mathbf{x} \uparrow x} \bar{\psi}_{\rho \mathbf{x} \downarrow x} \psi_{\rho \mathbf{x} \downarrow x} \psi_{\rho \mathbf{x} \uparrow x} - \frac{1}{2} \sum_{\sigma \in \{\uparrow, \downarrow\}} \bar{\psi}_{\rho \mathbf{x} \sigma x} \psi_{\rho \mathbf{x} \sigma x} \right).$$

From now we simply write \mathbf{U} in place of (U_1, U_2, \dots, U_b) .

LEMMA 2.2. *The following statements hold.*

(1) *For any $U_{max} \in \mathbb{R}_{>0}$ there exists $h_0 \in \mathbb{R}_{>0}$ such that*

$$\operatorname{Re} \int e^{-V(\psi)} d\mu_C(\psi) > 0, \\ (\forall \mathbf{U} \in \mathbb{R}^b \text{ with } |U_\rho| \leq U_{max} \ (\forall \rho \in \mathcal{B}), \ \forall h \in (2/\beta)\mathbb{N} \text{ with } h \geq h_0).$$

(2) *For any $U_{max} \in \mathbb{R}_{>0}$,*

$$\lim_{\substack{h \rightarrow \infty \\ h \in (2/\beta)\mathbb{N}}} \sup_{\substack{\mathbf{U} \in \mathbb{R}^b \text{ with} \\ |U_\rho| \leq U_{max} \ (\forall \rho \in \mathcal{B})}} \\ \cdot \left| \log \left(\frac{\operatorname{Tr} e^{-\beta H}}{\operatorname{Tr} e^{-\beta H_0}} \right) - \log \left(\int e^{-V(\psi)} d\mu_C(\psi) \right) \right| = 0.$$

PROOF. These claims can be proved in the same way as in the proof of [13, Lemma 3.4], [14, Appendix B]. We outline the proof for self-containedness. We can rewrite the interacting part V of the Hamiltonian H as follows.

$$V = \sum_{\substack{(\rho, \mathbf{x}, \sigma), (\eta, \mathbf{y}, \tau) \\ \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}}} w_1(\rho \mathbf{x} \sigma, \eta \mathbf{y} \tau) \psi_{\rho \mathbf{x} \sigma}^* \psi_{\eta \mathbf{y} \tau} \\ + \prod_{j=1}^2 \left(\sum_{\substack{(\rho_j, \mathbf{x}_j, \sigma_j), (\eta_j, \mathbf{y}_j, \tau_j) \\ \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}}} \right) w_2(\rho_1 \mathbf{x}_1 \sigma_1, \rho_2 \mathbf{x}_2 \sigma_2, \eta_1 \mathbf{y}_1 \tau_1, \eta_2 \mathbf{y}_2 \tau_2) \\ \cdot \psi_{\rho_1 \mathbf{x}_1 \sigma_1}^* \psi_{\rho_2 \mathbf{x}_2 \sigma_2}^* \psi_{\eta_2 \mathbf{y}_2 \tau_2} \psi_{\eta_1 \mathbf{y}_1 \tau_1},$$

where

$$\begin{aligned} w_1(\rho \mathbf{x} \sigma, \eta \mathbf{y} \tau) &:= -\frac{1_{(\rho, \mathbf{x}, \sigma) = (\eta, \mathbf{y}, \tau)}}{2} U_\rho, \\ w_2(\rho_1 \mathbf{x}_1 \sigma_1, \rho_2 \mathbf{x}_2 \sigma_2, \eta_1 \mathbf{y}_1 \tau_1, \eta_2 \mathbf{y}_2 \tau_2) \\ &:= 1_{(\rho_1, \mathbf{x}_1) = (\rho_2, \mathbf{x}_2) = (\eta_1, \mathbf{y}_1) = (\eta_2, \mathbf{y}_2)} 1_{(\sigma_1, \sigma_2, \tau_1, \tau_2) = (\uparrow, \downarrow, \uparrow, \downarrow)} U_{\rho_1}. \end{aligned}$$

By repeating the same argument as in [13, Proposition 3.2] we can derive the following series.

$$(2.13) \quad \frac{\text{Tr } e^{-\beta H}}{\text{Tr } e^{-\beta H_0}} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \prod_{k=1}^m \left(\sum_{l_k=1}^2 \sum_{\substack{\mathbf{x}_k, \mathbf{y}_k \\ \in (\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})^{l_k}}} \int_0^\beta dx_k w_{l_k}(\mathbf{X}_k, \mathbf{Y}_k) \right) \cdot \det(C(X_p, Y_q))_{1 \leq p, q \leq \sum_{k=1}^m l_k},$$

where the variables are defined by the following rule.

$$(2.14) \quad \begin{aligned} \mathbf{X}_k &:= ((\rho_{k,1}, \mathbf{x}_{k,1}, \sigma_{k,1}), \dots, (\rho_{k,l_k}, \mathbf{x}_{k,l_k}, \sigma_{k,l_k})), \\ \mathbf{Y}_k &:= ((\eta_{k,1}, \mathbf{y}_{k,1}, \tau_{k,1}), \dots, (\eta_{k,l_k}, \mathbf{y}_{k,l_k}, \tau_{k,l_k})), \\ X_p &:= (\rho_{u+1,v}, \mathbf{x}_{u+1,v}, \sigma_{u+1,v}, x_{u+1}), \\ Y_p &:= (\eta_{u+1,v}, \mathbf{y}_{u+1,v}, \tau_{u+1,v}, x_{u+1}), \end{aligned}$$

for $p = \sum_{k=1}^u l_k + v$, $u \in \{0, 1, \dots, m-1\}$, $v \in \{1, \dots, l_{u+1}\}$.

We define the function $\mathbf{U} \mapsto P(\mathbf{U}) : \mathbb{C}^b \rightarrow \mathbb{C}$ by the right-hand side of (2.13). By replacing the integral $\int_0^\beta dx_k$ by the Riemann sum $\frac{1}{h} \sum_{x \in [0, \beta)_h}$ we introduce the discrete analogue P_h of P as follows.

$$\begin{aligned} P_h(\mathbf{U}) &:= 1 + \sum_{m=1}^{N/2} \frac{(-1)^m}{m!} \prod_{k=1}^m \left(\sum_{l_k=1}^2 \sum_{\substack{\mathbf{x}_k, \mathbf{y}_k \\ \in (\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})^{l_k}}} \frac{1}{h} \sum_{x_k \in [0, \beta)_h} w_{l_k}(\mathbf{X}_k, \mathbf{Y}_k) \right) \\ &\quad \cdot \det(C(X_p, Y_q))_{1 \leq p, q \leq \sum_{k=1}^m l_k}, \end{aligned}$$

where the variables are defined by the rule (2.14).

Define the function $\tilde{C} : (\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})^2 \times [0, \beta)^2 \rightarrow \mathbb{C}$ by $\tilde{C}(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) := C(\rho \mathbf{x} \sigma \hat{x}, \eta \mathbf{y} \tau \hat{y})$ with $\hat{x}, \hat{y} \in [0, \beta)_h$ satisfying $x \in [\hat{x}, \hat{x} + 1/h)$, $y \in [\hat{y}, \hat{y} + 1/h)$.

The function P_h can be rewritten as follows.

$$P_h(\mathbf{U}) = 1 + \sum_{m=1}^{N/2} \frac{(-1)^m}{m!} \prod_{k=1}^m \left(\sum_{l_k=1}^2 \sum_{\substack{\mathbf{X}_k, \mathbf{Y}_k \\ \in (\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})^{l_k}}} \int_0^\beta dx_k w_{l_k}(\mathbf{X}_k, \mathbf{Y}_k) \right) \\ \cdot \det(\tilde{C}(X_p, Y_q))_{1 \leq p, q \leq \sum_{k=1}^m l_k},$$

Then, we have for any $U_{max} \in \mathbb{R}_{>0}$ that

$$(2.15) \quad \sup_{\substack{\mathbf{U} \in \mathcal{C}^b \text{ with} \\ |U_\rho| \leq U_{max} (\forall \rho \in \mathcal{B})}} |P(\mathbf{U}) - P_h(\mathbf{U})| \\ \leq \sum_{m=1}^{\infty} \frac{1}{m!} \prod_{k=1}^m \\ \cdot \left(\sum_{l_k=1}^2 \sum_{\substack{\mathbf{X}_k, \mathbf{Y}_k \\ \in (\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})^{l_k}}} \int_0^\beta dx_k \sup_{\substack{\mathbf{U} \in \mathcal{C}^b \text{ with} \\ |U_\rho| \leq U_{max} (\forall \rho \in \mathcal{B})}} |w_{l_k}(\mathbf{X}_k, \mathbf{Y}_k)| \right) \\ \cdot |\det(C(X_p, Y_q))_{1 \leq p, q \leq \sum_{k=1}^m l_k} - \det(\tilde{C}(X_p, Y_q))_{1 \leq p, q \leq \sum_{k=1}^m l_k}|.$$

Now let us confirm the fact that C, \tilde{C} have β, L -dependent determinant bounds. For any $(\rho_j, \mathbf{x}_j, \sigma_j, x_j), (\eta_j, \mathbf{y}_j, \tau_j, y_j) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)$ ($j = 1, 2, \dots, n$) we can choose operators A_j ($j = 1, 2, \dots, 2n$) from $\{e^{x_j H_0} \psi_{\rho_j \mathbf{x}_j \sigma_j}^* e^{-x_j H_0}, e^{y_j H_0} \psi_{\eta_j \mathbf{y}_j \tau_j} e^{-y_j H_0}\}_{j=1}^n$ so that

$$|\det(C(\rho_p \mathbf{x}_p \sigma_p x_p, \eta_q \mathbf{y}_q \tau_q y_q))_{1 \leq p, q \leq n}| = \left| \frac{\text{Tr}(e^{-\beta H_0} A_1 A_2 \cdots A_{2n})}{\text{Tr} e^{-\beta H_0}} \right|.$$

Let $\langle \cdot, \cdot \rangle_{F_f}$ denote the inner product of the Fermionic Fock space $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$ and $\|\cdot\|_{F_f}$ denote the norm induced by $\langle \cdot, \cdot \rangle_{F_f}$. For any linear transform ζ on $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$ let $\|\zeta\|_{\mathfrak{B}(F_f)}$ denote its operator norm defined by

$$\|\zeta\|_{\mathfrak{B}(F_f)} := \sup_{\substack{g \in F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})) \\ \text{with } \|g\|_{F_f} = 1}} \|\zeta g\|_{F_f}.$$

Since $\|e^{x H_0} \psi_{\rho \mathbf{x} \sigma}^* e^{-x H_0}\|_{\mathfrak{B}(F_f)} \leq e^{2\beta \|H_0\|_{\mathfrak{B}(F_f)}}$ ($(\forall (\rho, \mathbf{x}, \sigma, x) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)$), we have that

$$(2.16) \quad |\det(C(X_p, Y_q))_{1 \leq p, q \leq n}| \leq D_1 \cdot D_2^n,$$

$$(\forall X_j, Y_j \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta) \quad (j = 1, 2, \dots, n)),$$

where

$$D_1 := \frac{2^{2bL^d} e^{\beta \|H_0\| \mathfrak{A}_{F_f}}}{\text{Tr } e^{-\beta H_0}}, \quad D_2 := e^{4\beta \|H_0\| \mathfrak{A}_{F_f}}.$$

By using the determinant bound (2.16) we obtain the inequality

$$(2.17) \quad \frac{1}{m!} \prod_{k=1}^m \left(\sum_{l_k=1}^2 \sum_{\substack{\mathbf{X}_k, \mathbf{Y}_k \\ \in (\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})^{l_k}}} \int_0^\beta dx_k \sup_{\substack{\mathbf{U} \in \mathcal{C}^b \text{ with} \\ |U_\rho| \leq U_{\max}(\forall \rho \in \mathcal{B})}} |w_{l_k}(\mathbf{X}_k, \mathbf{Y}_k)| \right) \\ \cdot |\det(C(X_p, Y_q))_{1 \leq p, q \leq \sum_{k=1}^m l_k} - \det(\tilde{C}(X_p, Y_q))_{1 \leq p, q \leq \sum_{k=1}^m l_k}| \\ \leq \frac{2D_1}{m!} (U_{\max} b L^b \beta (D_2 + D_2^2))^m.$$

Since $(x, y) \mapsto C(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y)$ is continuous a.e. in $[0, \beta)^2$, so is $(x_1, x_2, \dots, x_m) \mapsto \det(C(X_p, Y_q))_{1 \leq p, q \leq \sum_{k=1}^m l_k}$ in $[0, \beta)^m$. Thus,

$$\lim_{\substack{h \rightarrow \infty \\ h \in (2/\beta)\mathbb{N}}} |\det(C(X_p, Y_q))_{1 \leq p, q \leq \sum_{k=1}^m l_k} - \det(\tilde{C}(X_p, Y_q))_{1 \leq p, q \leq \sum_{k=1}^m l_k}| = 0,$$

for a.e. $(x_1, x_2, \dots, x_n) \in [0, \beta)^m$. Therefore, the dominated convergence theorem for $L^1([0, \beta)^m)$ ensures that

$$(2.18) \quad \lim_{\substack{h \rightarrow \infty \\ h \in (2/\beta)\mathbb{N}}} \frac{1}{m!} \prod_{k=1}^m \left(\sum_{l_k=1}^2 \sum_{\substack{\mathbf{X}_k, \mathbf{Y}_k \\ \in (\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})^{l_k}}} \int_0^\beta dx_k \sup_{\substack{\mathbf{U} \in \mathcal{C}^b \text{ with} \\ |U_\rho| \leq U_{\max}(\forall \rho \in \mathcal{B})}} |w_{l_k}(\mathbf{X}_k, \mathbf{Y}_k)| \right) \\ \cdot |\det(C(X_p, Y_q))_{1 \leq p, q \leq \sum_{k=1}^m l_k} - \det(\tilde{C}(X_p, Y_q))_{1 \leq p, q \leq \sum_{k=1}^m l_k}| = 0.$$

By (2.17) and (2.18) we can apply the dominated convergence theorem for $l^1(\mathbb{N})$ to deduce from (2.15) that

$$(2.19) \quad \lim_{\substack{h \rightarrow \infty \\ h \in (2/\beta)\mathbb{N}}} \sup_{\substack{\mathbf{U} \in \mathcal{C}^b \text{ with} \\ |U_\rho| \leq U_{\max}(\forall \rho \in \mathcal{B})}} |P(\mathbf{U}) - P_h(\mathbf{U})| = 0.$$

By replacing the determinant in P_h by the Grassmann Gaussian integral we can derive that

$$(2.20) \quad P_h(\mathbf{U}) = \int e^{-V(\psi)} d\mu_C(\psi).$$

By (2.19) and (2.20),

$$(2.21) \quad \lim_{\substack{h \rightarrow \infty \\ h \in (2/\beta)\mathbb{N}}} \sup_{\substack{\mathbf{U} \in \mathbb{R}^b \text{ with} \\ |U_\rho| \leq U_{\max}(\forall \rho \in \mathcal{B})}} \left| \frac{\text{Tr } e^{-\beta H}}{\text{Tr } e^{-\beta H_0}} - \int e^{-V(\psi)} d\mu_C(\psi) \right| = 0,$$

which implies the claim (1). The claim (2) follows from the claim (1) and (2.21). \square

2.5. A symmetric formulation

It is vital for the validity of the forthcoming IR integration that any Grassmann polynomial produced by the single-scale IR integration is invariant under certain transforms. The Grassmann integral formulation constructed in the previous subsection does not satisfy one of the necessary invariant properties by itself if we connect it to the IR integration process. We need to modify the formulation into a more suitable form for the forthcoming IR analysis. For this purpose we introduce a few more covariances. Then, we propose another formulation, which will be shown to have desired symmetries in Section 7, by using the newly introduced covariances. For any $(\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in I_0$ set

$$(2.22) \quad C_{\leq 0}^+(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) := \frac{\delta_{\sigma, \tau}}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} e^{-i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \\ \cdot \chi(h|1 - e^{i\frac{\omega}{h}}|) h^{-1} (I_b - e^{-i\frac{\omega}{h}} I_b + \frac{1}{h} \overline{E(\mathbf{k})})^{-1}(\rho, \eta),$$

where $\chi(\cdot) : \mathbb{R} \rightarrow [0, 1]$ is a smooth function. We assume that the support of $\chi(\cdot)$ is contained in the interval $[-c_\chi, c_\chi]$, where $c_\chi \in \mathbb{R}_{>0}$ is a constant. In this section we do not need more detailed information on the cut-off function $\chi(\cdot)$. For any $(\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in I_0$ set

$$(2.23) \quad C_{> 0}^+(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) \\ := \frac{\delta_{\sigma, \tau}}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} e^{-i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega}$$

$$\cdot (1 - \chi(h|1 - e^{i\frac{\omega}{h}}|))h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1}(\rho, \eta),$$

so that $C(X, Y) = C_{\leq 0}^+(X, Y) + C_{> 0}^+(X, Y)$, ($\forall X, Y \in I_0$). Define $C_{\leq 0}^\infty : I_0^2 \rightarrow \mathbb{C}$ by

$$(2.24) \quad C_{\leq 0}^\infty(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) \\ := \frac{\delta_{\sigma, \tau}}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} e^{-i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \chi(|\omega|) (i\omega I_b - \overline{E(\mathbf{k})})^{-1}(\rho, \eta).$$

Moreover, we define the covariance $C_{> 0}^- : I_0^2 \rightarrow \mathbb{C}$ as follows. For $(\rho, \mathbf{x}, \sigma, x)$, $(\eta, \mathbf{y}, \tau, y) \in I_0$,

$$(2.25) \quad C_{> 0}^-(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) \\ := \frac{\delta_{\sigma, \tau}}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} e^{-i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \\ \cdot (1 - \chi(h|1 - e^{i\frac{\omega}{h}}|))h^{-1}(e^{i\frac{\omega}{h}I_b - \frac{1}{h}\overline{E(\mathbf{k})}} - I_b)^{-1}(\rho, \eta).$$

We can derive the following equality from the definitions.

LEMMA 2.3. *For any $(\rho, \mathbf{x}, \sigma, x)$, $(\eta, \mathbf{y}, \tau, y) \in I_0$,*

$$C_{> 0}^+(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) + \frac{1_{(\rho, \mathbf{x}, \sigma) = (\eta, \mathbf{y}, \tau)}}{\beta h} \sum_{\omega \in \mathcal{M}_h} e^{i(x-y)\omega} \chi(h|1 - e^{i\frac{\omega}{h}}|) \\ = C_{> 0}^-(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) + 1_{(\rho, \mathbf{x}, \sigma, x) = (\eta, \mathbf{y}, \tau, y)}.$$

Finally we define the covariances $C_{> 0}^{+(h)}$, $\mathcal{I} : I_0^2 \rightarrow \mathbb{C}$ by

$$C_{> 0}^{+(h)}(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) \\ := C_{> 0}^+(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) + \frac{1_{(\rho, \mathbf{x}, \sigma) = (\eta, \mathbf{y}, \tau)}}{\beta h} \sum_{\omega \in \mathcal{M}_h} e^{i(x-y)\omega} \chi(h|1 - e^{i\frac{\omega}{h}}|),$$

$$\mathcal{I}(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) := 1_{(\rho, \mathbf{x}, \sigma, x) = (\eta, \mathbf{y}, \tau, y)}.$$

We will make another Grassmann integral formulation out of these covariances. In order to prove that the Grassmann integral formulation converges to the normalized free energy density as $h \rightarrow \infty$, we must know that these covariances have suitable determinant bounds.

LEMMA 2.4. *There exist (β, L^d, b, χ, E) -dependent, h -independent constants $h_0, c_1 \in \mathbb{R}_{>0}$ such that the following inequalities hold true for any $h \in (2/\beta)\mathbb{N}$ with $h \geq h_0$.*

$$\begin{aligned} |\det(C_o(X_i, Y_j))_{1 \leq i, j \leq n}| &\leq c_1^n, \\ |\det(C_{>0}^{+(h)}(X_i, Y_j) - C_{>0}^+(X_i, Y_j))_{1 \leq i, j \leq n}| &\leq \frac{1}{h} c_1^n, \\ |\det(C_{\leq 0}^+(X_i, Y_j) - C_{\leq 0}^\infty(X_i, Y_j))_{1 \leq i, j \leq n}| &\leq \frac{1}{h} c_1^n, \\ (\forall n \in \mathbb{N}, X_j, Y_j \in I_0 \ (j = 1, 2, \dots, n)) \end{aligned}$$

for $C_o = C, C_{\leq 0}^+, C_{>0}^+, C_{\leq 0}^\infty, C_{>0}^-, C_{>0}^{+(h)}$ respectively.

PROOF. The determinant bound on C has been given in (2.16). The determinant bounds on the other covariances can be proved by applying Gram's inequality in the complex Hilbert space $\mathcal{H} = L^2(\mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\} \times \mathcal{M}_h)$, which consists of all complex-valued functions on $\mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\} \times \mathcal{M}_h$ and is equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ defined by

$$\langle f, g \rangle_{\mathcal{H}} := \frac{1}{\beta L^d} \sum_{K \in \mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\} \times \mathcal{M}_h} \overline{f(K)} g(K), \quad (\forall f, g \in \mathcal{H}).$$

Define the vectors $f_{\rho\mathbf{x}\sigma x}, g_{\rho\mathbf{x}\sigma x} \in \mathcal{H}$ ($(\rho, \mathbf{x}, \sigma, x) \in I_0$) by

$$\begin{aligned} f_{\rho\mathbf{x}\sigma x}(\eta, \mathbf{k}, \tau, \omega) &:= \delta_{\rho, \eta} \delta_{\sigma, \tau} e^{i\langle \mathbf{x}, \mathbf{k} \rangle} e^{-ix\omega} \chi(h|1 - e^{i\frac{\omega}{h}}|)^{\frac{1}{2}}, \\ g_{\rho\mathbf{x}\sigma x}(\eta, \mathbf{k}, \tau, \omega) &:= \delta_{\sigma, \tau} e^{i\langle \mathbf{x}, \mathbf{k} \rangle} e^{-ix\omega} \chi(h|1 - e^{i\frac{\omega}{h}}|)^{\frac{1}{2}} \\ &\quad \cdot h^{-1} (I_b - e^{-i\frac{\omega}{h}} I_b + \frac{1}{h} \overline{E(\mathbf{k})})^{-1}(\eta, \rho), \\ (\forall (\eta, \mathbf{k}, \tau, \omega) \in \mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\} \times \mathcal{M}_h). \end{aligned}$$

It follows that $C_{\leq 0}^+(X, Y) = \langle f_X, g_Y \rangle_{\mathcal{H}}$, ($\forall X, Y \in I_0$). Moreover, by using the inequality $h|1 - e^{i\frac{\omega}{h}}| \geq (2/\pi)|\omega|$ ($\forall \omega \in \mathcal{M}_h$) we have

$$\|f_X\|_{\mathcal{H}}^2 \leq \frac{1}{\beta} \sum_{\omega \in \mathcal{M}_h} \chi(h|1 - e^{i\frac{\omega}{h}}|) \leq \frac{1}{\beta} \sum_{\omega \in \mathcal{M}} 1_{|\omega| \leq \frac{\pi}{2} c_X},$$

where $\|\cdot\|_{\mathcal{H}}$ denotes the norm of \mathcal{H} induced by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Since

$$(2.26) \quad h^{-1} (I_b - e^{-i\frac{\omega}{h}} I_b + \frac{1}{h} \overline{E(\mathbf{k})})^{-1}$$

$$\begin{aligned}
 &= (i\omega I_b - \overline{E(\mathbf{k})})^{-1} \sum_{m=0}^{\infty} \left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \frac{1}{h^{n-1}} (i\omega I_b - \overline{E(\mathbf{k})})^{n-1} \right)^m, \\
 (2.27) \quad &\|h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1}\|_{b \times b} \\
 &\leq \|(i\omega I_b - \overline{E(\mathbf{k})})^{-1}\|_{b \times b} \sum_{m=0}^{\infty} \left(\frac{1}{h} \sum_{n=2}^{\infty} \frac{1}{n!} \|i\omega I_b - \overline{E(\mathbf{k})}\|_{b \times b}^{n-1} \right)^m \\
 &\leq \frac{\beta}{\pi} \sum_{m=0}^{\infty} \left(\frac{1}{h} e^{\frac{\pi}{2}c_\chi + \sup_{\mathbf{k} \in \mathbb{R}^d} \|E(\mathbf{k})\|_{b \times b}} \right)^m \\
 &\leq \beta, \quad (\forall \mathbf{k} \in \Gamma^*, \omega \in \mathcal{M} \text{ with } |\omega| \leq (\pi/2)c_\chi),
 \end{aligned}$$

if h is large enough. Therefore,

$$\begin{aligned}
 \|g_X\|_{\mathcal{H}}^2 &\leq \frac{1}{\beta L^d} \sum_{(\mathbf{k}, \omega) \in \Gamma^* \times \mathcal{M}} 1_{|\omega| \leq \frac{\pi}{2}c_\chi} \|h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1}\|_{b \times b}^2 \\
 &\leq \beta \sum_{\omega \in \mathcal{M}} 1_{|\omega| \leq \frac{\pi}{2}c_\chi}.
 \end{aligned}$$

We can, thus, apply Gram's inequality to deduce that

$$\begin{aligned}
 (2.28) \quad |\det(C_{\leq 0}^+(X_i, Y_j))_{1 \leq i, j \leq n}| &\leq \prod_{j=1}^n \|f_{X_j}\|_{\mathcal{H}} \|g_{Y_j}\|_{\mathcal{H}} \\
 &\leq \left(\sum_{\omega \in \mathcal{M}} 1_{|\omega| \leq \frac{\pi}{2}c_\chi} \right)^n.
 \end{aligned}$$

Take any $X_1, \dots, X_n, Y_1, \dots, Y_n \in I_0$. Define $C_{(n)}, C_{>0, (n)}^+, C_{\leq 0, (n)}^+ \in \text{Mat}(n, \mathbb{C})$ by

$$\begin{aligned}
 C_{(n)}(i, j) &:= C(X_i, Y_j), \quad C_{>0, (n)}^+(i, j) := C_{>0}^+(X_i, Y_j), \\
 C_{\leq 0, (n)}^+(i, j) &:= C_{\leq 0}^+(X_i, Y_j), \quad (\forall i, j \in \{1, 2, \dots, n\}).
 \end{aligned}$$

Since

$$C_{>0, (n)}^+ = C_{(n)} - C_{\leq 0, (n)}^+ = (C_{(n)}, I_n) \begin{pmatrix} I_n \\ -C_{\leq 0, (n)}^+ \end{pmatrix},$$

the Cauchy-Binet formula gives that

$$(2.29) \quad \det(C_{>0, (n)}^+)$$

$$\begin{aligned}
&= \sum_{\substack{\phi: \{1,2,\dots,n\} \rightarrow \{1,2,\dots,2n\} \\ \text{with } \phi(1) < \phi(2) < \dots < \phi(n)}} \det((C_{(n)}, I_n)(i, \phi(j)))_{1 \leq i, j \leq n} \\
&\quad \cdot \det \left(\begin{pmatrix} I_n \\ -C_{\leq 0, (n)}^+ \end{pmatrix} (\phi(i), j) \right)_{1 \leq i, j \leq n}.
\end{aligned}$$

By using (2.16), (2.28) and admitting that $\phi(0) = n$, $\phi(n+1) = n+1$ we can derive from (2.29) that

$$\begin{aligned}
&|\det(C_{>0, (n)}^+)| \\
&\leq \sum_{m=0}^n \sum_{\substack{\phi: \{1,2,\dots,n\} \rightarrow \{1,2,\dots,2n\} \\ \text{with } \phi(1) < \phi(2) < \dots < \phi(n)}} 1_{\phi(m) \leq n < \phi(m+1)} \\
&\quad \cdot \left(1_{m=0} + 1_{m>0} \sup_{\substack{X'_j, Y'_j \in I_0 \\ (j=1,2,\dots,m)}} |\det(C(X'_i, Y'_j))_{1 \leq i, j \leq m}| \right) \\
&\quad \cdot \left(1_{m=n} + 1_{m<n} \sup_{\substack{X'_j, Y'_j \in I_0 \\ (j=1,2,\dots,n-m)}} |\det(C_{\leq 0}^+(X'_i, Y'_j))_{1 \leq i, j \leq n-m}| \right) \\
&\leq \sum_{m=0}^n \binom{n}{m} \binom{n}{n-m} \max\{1, D_1\} D_2^m \left(\sum_{\omega \in \mathcal{M}} 1_{|\omega| \leq \frac{\pi}{2} c_X} \right)^{n-m} \\
&\leq \max\{1, D_1\} \sum_{m=0}^n \binom{2n}{2m} D_2^m \left(\sum_{\omega \in \mathcal{M}} 1_{|\omega| \leq \frac{\pi}{2} c_X} \right)^{n-m} \\
&\leq \max\{1, D_1\} \left(D_2^{\frac{1}{2}} + \left(\sum_{\omega \in \mathcal{M}} 1_{|\omega| \leq \frac{\pi}{2} c_X} \right)^{\frac{1}{2}} \right)^{2n}.
\end{aligned}$$

The determinant bounds on the covariances $C_{\leq 0}^\infty$, $C_{>0}^-$, $C_{>0}^{+(h)}$ can be derived in the same way as above.

Since $(C_{>0}^{+(h)} - C_{>0}^+)(X, Y) = \langle \frac{1}{h} f_X, f_Y \rangle_{\mathcal{H}}$ ($\forall X, Y \in I_0$), we have

$$\begin{aligned}
|\det((C_{>0}^{+(h)} - C_{>0}^+)(X_i, Y_j))_{1 \leq i, j \leq n}| &\leq \frac{1}{h} \prod_{j=1}^n \|f_{X_j}\|_{\mathcal{H}} \|f_{Y_j}\|_{\mathcal{H}} \\
&\leq \frac{1}{h} \left(\frac{1}{\beta} \sum_{\omega \in \mathcal{M}} 1_{|\omega| \leq \frac{\pi}{2} c_X} \right)^n.
\end{aligned}$$

To prove the determinant bound on $C_{\leq 0}^+ - C_{\leq 0}^\infty$, let us define the vectors $f'_{\rho\mathbf{x}\sigma x}, g'_{\rho\mathbf{x}\sigma x} \in \mathcal{H}$ ($(\rho, \mathbf{x}, \sigma, x) \in I_0$) by

$$\begin{aligned} f'_{\rho\mathbf{x}\sigma x}(\eta, \mathbf{k}, \tau, \omega) &:= \delta_{\rho, \eta} \delta_{\sigma, \tau} e^{i\langle \mathbf{x}, \mathbf{k} \rangle} e^{-ix\omega} 1_{|\omega| \leq \frac{\pi}{2} c_\chi}, \\ g'_{\rho\mathbf{x}\sigma x}(\eta, \mathbf{k}, \tau, \omega) &:= \delta_{\sigma, \tau} e^{i\langle \mathbf{x}, \mathbf{k} \rangle} e^{-ix\omega} 1_{|\omega| \leq \frac{\pi}{2} c_\chi} (\chi(h|1 - e^{i\frac{\omega}{h}}|)h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1}(\eta, \rho) \\ &\quad - \chi(|\omega|)(i\omega I_b - \overline{E(\mathbf{k})})^{-1}(\eta, \rho)). \end{aligned}$$

Since $\chi(h|1 - e^{i\frac{\omega}{h}}|) = 1_{|\omega| \leq \frac{\pi}{2} c_\chi} \chi(h|1 - e^{i\frac{\omega}{h}}|)$, $\chi(|\omega|) = 1_{|\omega| \leq \frac{\pi}{2} c_\chi} \chi(|\omega|)$ ($\forall \omega \in \mathcal{M}_h$), $(C_{\leq 0}^+ - C_{\leq 0}^\infty)(X, Y) = \langle f'_X, g'_Y \rangle_{\mathcal{H}}$ ($\forall X, Y \in I_0$). Note that

$$\|f'_X\|_{\mathcal{H}}^2 \leq \frac{1}{\beta} \sum_{\omega \in \mathcal{M}} 1_{|\omega| \leq \frac{\pi}{2} c_\chi}.$$

By assuming that h is sufficiently large and using (2.26), (2.27) we deduce that for any $\omega \in \mathcal{M}_h$, $\mathbf{k} \in \Gamma^*$,

$$\begin{aligned} &\|\chi(h|1 - e^{i\frac{\omega}{h}}|)h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1} - \chi(|\omega|)(i\omega I_b - \overline{E(\mathbf{k})})^{-1}\|_{b \times b} \\ &\leq |\chi(h|1 - e^{i\frac{\omega}{h}}|) - \chi(|\omega|)| \|h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1}\|_{b \times b} \\ &\quad + \chi(|\omega|) \|h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1} - (i\omega I_b - \overline{E(\mathbf{k})})^{-1}\|_{b \times b} \\ &\leq |h|1 - e^{i\frac{\omega}{h}}| - |\omega| \sup_{x \in \mathbb{R}} |\chi'(x)| 1_{|\omega| \leq \frac{\pi}{2} c_\chi} \beta \\ &\quad + 1_{|\omega| \leq \frac{\pi}{2} c_\chi} \|(i\omega I_b - \overline{E(\mathbf{k})})^{-1}\|_{b \times b} \sum_{m=1}^{\infty} \left(\frac{1}{h} \sum_{n=2}^{\infty} \frac{1}{n!} \|i\omega I_b - \overline{E(\mathbf{k})}\|_{b \times b}^{n-1} \right)^m \\ &\leq \frac{\beta}{h} e^{\frac{\pi}{2} c_\chi} \sup_{x \in \mathbb{R}} |\chi'(x)| + \frac{\beta}{h} e^{\frac{\pi}{2} c_\chi + \sup_{\mathbf{k} \in \mathbb{R}^d} \|E(\mathbf{k})\|_{b \times b}}, \end{aligned}$$

which implies that

$$\|g'_X\|_{\mathcal{H}}^2 \leq \left(\frac{\beta}{h} e^{\frac{\pi}{2} c_\chi + \sup_{\mathbf{k} \in \mathbb{R}^d} \|E(\mathbf{k})\|_{b \times b}} (1 + \sup_{x \in \mathbb{R}} |\chi'(x)|) \right)^2 \frac{1}{\beta} \sum_{\omega \in \mathcal{M}} 1_{|\omega| \leq \frac{\pi}{2} c_\chi}.$$

Again by Gram's inequality,

$$|\det(C_{\leq 0}^+ - C_{\leq 0}^\infty)(X_i, Y_j)_{1 \leq i, j \leq n}| \leq \prod_{j=1}^n \|f'_{X_j}\|_{\mathcal{H}} \|g'_{X_j}\|_{\mathcal{H}}$$

$$\leq \frac{1}{h} \left(e^{\frac{\pi}{2}c_\chi + \sup_{\mathbf{k} \in \mathbb{R}^d} \|E(\mathbf{k})\|_{b \times b}} \left(1 + \sup_{x \in \mathbb{R}} |\chi'(x)| \right) \sum_{\omega \in \mathcal{M}} 1_{|\omega| \leq \frac{\pi}{2}c_\chi} \right)^n.$$

The claimed determinant bounds have been obtained. \square

To continue our analysis, we introduce an L^1 -norm on functions on I^m . For $m \in \{1, 2, \dots, N\}$ and a function $f_m : I^m \rightarrow \mathbb{C}$, let $\|f_m\|_{L^1}$ be defined by

$$\|f_m\|_{L^1} := \left(\frac{1}{h} \right)^m \sum_{\mathbf{X} \in I^m} |f_m(\mathbf{X})|.$$

To simplify arguments, we let $\|f_0\|_{L^1}$ denote $|f_0|$ for $f_0 \in \mathbb{C}$ as well. Necessary basic estimations with this norm are separately prepared in Appendix B.

Let us introduce the Grassmann polynomials $V^+(\psi)$, $V^-(\psi)$, $S^+(\psi)$, $S^-(\psi)$, $S^0(\psi)$ ($\in \wedge \mathcal{V}$) as follows.

$$(2.30) \quad V^+(\psi) := V(\psi), \quad V^-(\psi) := V(\psi) + \frac{1}{h} \sum_{(\rho, \mathbf{x}, \sigma, x) \in I_0} U_\rho \bar{\psi}_{\rho \mathbf{x} \sigma x} \psi_{\rho \mathbf{x} \sigma x},$$

$$S^\delta(\psi) := \int e^{-V^\delta(\psi + \psi^1)} d\mu_{C_{>0}^\delta}(\psi^1), \quad (\delta \in \{+, -\}),$$

$$S^0(\psi) := \int e^{-V^+(\psi + \psi^1)} d\mu_{C_{>0}^{+(h)}}(\psi^1),$$

where $V(\psi)$ is the Grassmann polynomial defined in (2.12). In the following ‘ c_1 ’ denotes the h -independent constant appearing in Lemma 2.4 and the parameter $h \in (2/\beta)\mathbb{N}$ is assumed to be larger than h_0 appearing in the same lemma. Also, let us assume that $\mathbf{U} \in \mathbb{C}^b$ satisfies $|U_\rho| \leq U_{max}$ ($\forall \rho \in \mathcal{B}$) for some $U_{max} (\in \mathbb{R}_{\geq 0})$.

LEMMA 2.5. *The following inequalities hold.*

$$(1) \quad |S_0^\delta - 1| \leq e^{b\beta L^d U_{max}(c_1 + c_1^2)} - 1, \quad (\forall \delta \in \{+, -, 0\}).$$

(2) For any $\alpha \in \mathbb{R}_{\geq 0}$,

$$\sum_{m=0}^N \alpha^m c_1^{\frac{m}{2}} \|S_m^\delta\|_{L^1} \leq e^{b\beta L^d U_{max}((\alpha+1)^2 c_1 + (\alpha+1)^4 c_1^2)}, \quad (\forall \delta \in \{+, -, 0\}).$$

(3) For any $\alpha \in \mathbb{R}_{\geq 0}$,

$$\sum_{m=0}^N \alpha^m c_1^{\frac{m}{2}} \|S_m^+ - S_m^0\|_{L^1} \leq \frac{1}{h} (e^{b\beta L^d U_{max}((\alpha+2)^2 c_1 + (\alpha+2)^4 c_1^2)} - 1).$$

PROOF. The anti-symmetric kernels $V_m^\delta(\cdot) : I^m \rightarrow \mathbb{C}$ ($m \in \{2, 4\}$) of $V^\delta(\psi)$ are characterized as follows. For $\delta \in \{+, -\}$,

(2.31)

$$\begin{aligned} & V_2^\delta((\rho_1, \mathbf{x}_1, \sigma_1, x_1, \theta_1), (\rho_2, \mathbf{x}_2, \sigma_2, x_2, \theta_2)) \\ &= -\frac{\delta h}{4} U_{\rho_1} 1_{(\rho_1, \mathbf{x}_1, \sigma_1, x_1) = (\rho_2, \mathbf{x}_2, \sigma_2, x_2)} (1_{(\theta_1, \theta_2) = (1, -1)} - 1_{(\theta_1, \theta_2) = (-1, 1)}), \\ & V_4^\delta((\rho_1, \mathbf{x}_1, \sigma_1, x_1, \theta_1), (\rho_2, \mathbf{x}_2, \sigma_2, x_2, \theta_2), \\ & \quad (\rho_3, \mathbf{x}_3, \sigma_3, x_3, \theta_3), (\rho_4, \mathbf{x}_4, \sigma_4, x_4, \theta_4)) \\ &= \frac{h^3}{4!} U_{\rho_1} 1_{(\rho_1, \mathbf{x}_1, x_1) = (\rho_2, \mathbf{x}_2, x_2) = (\rho_3, \mathbf{x}_3, x_3) = (\rho_4, \mathbf{x}_4, x_4)} \\ & \quad \cdot \sum_{\zeta \in \mathbb{S}_4} \text{sgn}(\zeta) 1_{((\sigma_{\zeta(1)}, \theta_{\zeta(1)}), (\sigma_{\zeta(2)}, \theta_{\zeta(2)}), (\sigma_{\zeta(3)}, \theta_{\zeta(3)}), (\sigma_{\zeta(4)}, \theta_{\zeta(4)})) = ((\uparrow, 1), (\downarrow, 1), (\downarrow, -1), (\uparrow, -1))}. \end{aligned}$$

From this we can see that

$$\|V_m^\delta\|_{L^1} \leq b\beta L^d U_{max}, \quad (\forall m \in \{2, 4\}, \delta \in \{+, -\}).$$

By these inequalities and Lemma 2.4 we can readily apply Lemma B.2 (1),(2),(4) proved in Appendix B to verify the statements (1),(2),(3) respectively. \square

LEMMA 2.6. For any $\alpha \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} & \sum_{m=0}^N \alpha^m c_1^{\frac{m}{2}} \|S_m^- - S_m^0\|_{L^1} \\ & \leq \frac{1}{\beta h} (b\beta L^d U_{max}((\alpha+1)^2 c_1 + (\alpha+1)^4 c_1^2))^2 e^{b\beta L^d U_{max}((\alpha+1)^2 c_1 + (\alpha+1)^4 c_1^2)}. \end{aligned}$$

PROOF. For any $(\rho, \mathbf{x}, x) \in \mathcal{B} \times \Gamma \times [0, \beta)_h$ set

$$(2.32) \quad V_{\rho \mathbf{x}}^+(\psi) := \bar{\psi}_{\rho \mathbf{x} \uparrow x} \bar{\psi}_{\rho \mathbf{x} \downarrow x} \psi_{\rho \mathbf{x} \downarrow x} \psi_{\rho \mathbf{x} \uparrow x} - \frac{1}{2} \sum_{\sigma \in \{\uparrow, \downarrow\}} \bar{\psi}_{\rho \mathbf{x} \sigma x} \psi_{\rho \mathbf{x} \sigma x},$$

$$V_{\rho\mathbf{x}x}^-(\psi) := \bar{\psi}_{\rho\mathbf{x}\uparrow x} \bar{\psi}_{\rho\mathbf{x}\downarrow x} \psi_{\rho\mathbf{x}\downarrow x} \psi_{\rho\mathbf{x}\uparrow x} + \frac{1}{2} \sum_{\sigma \in \{\uparrow, \downarrow\}} \bar{\psi}_{\rho\mathbf{x}\sigma x} \psi_{\rho\mathbf{x}\sigma x}.$$

Recall the general property of Grassmann Gaussian integral that for any covariances $A, B : I_0^2 \rightarrow \mathbb{C}$ and $f \in \wedge \mathcal{V}$,

$$(2.33) \quad \int f(\psi + \psi^1) d\mu_{A+B}(\psi^1) = \int \int f(\psi + \psi^1 + \psi^2) d\mu_A(\psi^2) d\mu_B(\psi^1)$$

(see [5, Proposition I.21]). Assume that $x_1, x_2, \dots, x_m \in [0, \beta)_h$ satisfy $x_i \neq x_j$ if $i \neq j$. Using Lemma 2.3 and (2.33), we observe that

$$(2.34) \quad \begin{aligned} & \int \prod_{j=1}^m V_{\rho_j \mathbf{x}_j x_j}^+(\psi + \psi^1) d\mu_{C_{>0}^+(h)}(\psi^1) \\ &= \int \int \prod_{j=1}^m V_{\rho_j \mathbf{x}_j x_j}^+(\psi + \psi^1 + \psi^2) d\mu_{\mathcal{I}}(\psi^2) d\mu_{C_{>0}^-}(\psi^1) \\ &= \int \int \prod_{j=1}^m (V_{\rho_j \mathbf{x}_j x_j}^+(\psi + \psi^1) + V_{\rho_j \mathbf{x}_j x_j}^+(\psi^2) \\ &\quad + (\bar{\psi} + \bar{\psi}^1)_{\rho_j \mathbf{x}_j \uparrow x_j} (\psi + \psi^1)_{\rho_j \mathbf{x}_j \uparrow x_j} \bar{\psi}_{\rho_j \mathbf{x}_j \downarrow x_j}^2 \psi_{\rho_j \mathbf{x}_j \downarrow x_j}^2 \\ &\quad + (\bar{\psi} + \bar{\psi}^1)_{\rho_j \mathbf{x}_j \downarrow x_j} (\psi + \psi^1)_{\rho_j \mathbf{x}_j \downarrow x_j} \bar{\psi}_{\rho_j \mathbf{x}_j \uparrow x_j}^2 \psi_{\rho_j \mathbf{x}_j \uparrow x_j}^2 \\ &\quad \cdot d\mu_{\mathcal{I}}(\psi^2) d\mu_{C_{>0}^-}(\psi^1) \\ &= \int \prod_{j=1}^m \left(V_{\rho_j \mathbf{x}_j x_j}^+(\psi + \psi^1) \right. \\ &\quad \left. + \sum_{\sigma \in \{\uparrow, \downarrow\}} (\bar{\psi} + \bar{\psi}^1)_{\rho_j \mathbf{x}_j \sigma x_j} (\psi + \psi^1)_{\rho_j \mathbf{x}_j \sigma x_j} \right) d\mu_{C_{>0}^-}(\psi^1) \\ &= \int \prod_{j=1}^m V_{\rho_j \mathbf{x}_j x_j}^-(\psi + \psi^1) d\mu_{C_{>0}^-}(\psi^1). \end{aligned}$$

Let us define $f_{V^+}(\psi), f_{V^-}(\psi) \in \wedge \mathcal{V}$ by

$$f_{V^\delta}(\psi) := 1 + \sum_{n=1}^N \frac{(-1)^n}{n!} \prod_{j=1}^n \left(\frac{1}{h} \sum_{(\rho_j, \mathbf{x}_j, x_j) \in \mathcal{B} \times \Gamma \times [0, \beta)_h} U_{\rho_j} \right)$$

$$\cdot \mathbf{1}_{\forall i \forall j (i \neq j \rightarrow x_i \neq x_j)} \prod_{j=1}^n V_{\rho_j \mathbf{x}_j x_j}^\delta(\psi), \quad (\delta \in \{+, -\}).$$

The equality (2.34) implies that

$$\int f_{V^+}(\psi + \psi^1) d\mu_{C_{>0}^{+(h)}}(\psi^1) = \int f_{V^-}(\psi + \psi^1) d\mu_{C_{>0}^-}(\psi^1),$$

and thus,

$$\begin{aligned} S^0(\psi) - S^-(\psi) &= \int (e^{-V^+(\psi+\psi^1)} - f_{V^+}(\psi + \psi^1)) d\mu_{C_{>0}^{+(h)}}(\psi^1) \\ &\quad - \int (e^{-V^-(\psi+\psi^1)} - f_{V^-}(\psi + \psi^1)) d\mu_{C_{>0}^-}(\psi^1). \end{aligned}$$

Set

$$\tilde{S}^0(\psi) := \int (e^{-V^+(\psi+\psi^1)} - f_{V^+}(\psi + \psi^1)) d\mu_{C_{>0}^{+(h)}}(\psi^1).$$

We can see that

$$\begin{aligned} &\tilde{S}_m^0(\psi) \\ &= \sum_{n=2}^N \frac{(-1)^n}{n!} \prod_{j=1}^n \left(\frac{1}{h} \sum_{(\rho_j, \mathbf{x}_j, x_j) \in \mathcal{B} \times \Gamma \times [0, \beta)_h} U_{\rho_j} \right) \mathbf{1}_{\exists i \exists j (i \neq j \wedge x_i = x_j)} \\ &\quad \cdot \mathcal{P}_m \int \prod_{l=1}^n V_{\rho_l \mathbf{x}_l x_l}^+(\psi + \psi^1) d\mu_{C_{>0}^{+(h)}}(\psi^1) \\ &= \sum_{n=2}^N \frac{(-1)^n}{n!} \prod_{j=1}^n \left(\frac{1}{h} \sum_{(\rho_j, \mathbf{x}_j, x_j) \in \mathcal{B} \times \Gamma \times [0, \beta)_h} U_{\rho_j} \right) \mathbf{1}_{\exists i \exists j (i \neq j \wedge x_i = x_j)} \\ &\quad \cdot \prod_{l=1}^n \left(\sum_{m_l \in \{2, 4\}} \left(\frac{1}{h} \right)^{m_l} \sum_{k_l=0}^{m_l} \binom{m_l}{k_l} \sum_{\mathbf{X}_l \in I^{m_l - k_l}} \sum_{\mathbf{Y}_l \in I^{k_l}} V_{\rho_l \mathbf{x}_l x_l, m_l}^+(\mathbf{X}_l, \mathbf{Y}_l) \right) \\ &\quad \cdot \varepsilon_{\pm} \mathbf{1}_{\sum_{l=1}^n k_l = m} \int \psi_{\mathbf{X}_1}^1 \psi_{\mathbf{X}_2}^1 \cdots \psi_{\mathbf{X}_n}^1 d\mu_{C_{>0}^{+(h)}}(\psi^1) \psi_{\mathbf{Y}_1} \psi_{\mathbf{Y}_2} \cdots \psi_{\mathbf{Y}_n}, \end{aligned}$$

where the factor $\varepsilon_{\pm} \in \{1, -1\}$ depends only on $(m_l)_{l=1}^n, (k_l)_{l=1}^n$. The anti-symmetric kernels $V_{\rho \mathbf{x}, m}^+(\cdot)$ ($m = 2, 4$) are characterized as follows.

$$V_{\rho \mathbf{x}, 2}^+((\eta_1, \mathbf{y}_1, \tau_1, y_1, \xi_1), (\eta_2, \mathbf{y}_2, \tau_2, y_2, \xi_2))$$

$$\begin{aligned}
&= -\frac{h^2}{4} \mathbf{1}_{(\eta_1, \mathbf{Y}_1, y_1) = (\eta_2, \mathbf{Y}_2, y_2) = (\rho, \mathbf{x}, x)} \mathbf{1}_{\tau_1 = \tau_2} \left(\mathbf{1}_{(\xi_1, \xi_2) = (1, -1)} - \mathbf{1}_{(\xi_1, \xi_2) = (-1, 1)} \right), \\
&V_{\rho \mathbf{x} x, 4}^+ \left((\eta_1, \mathbf{Y}_1, \tau_1, y_1, \xi_1), (\eta_2, \mathbf{Y}_2, \tau_2, y_2, \xi_2), \right. \\
&\quad \left. (\eta_3, \mathbf{Y}_3, \tau_3, y_3, \xi_3), (\eta_4, \mathbf{Y}_4, \tau_4, y_4, \xi_4) \right) \\
&= \frac{h^4}{4!} \mathbf{1}_{(\eta_i, \mathbf{Y}_i, y_i) = (\rho, \mathbf{x}, x), (\forall i \in \{1, 2, 3, 4\})} \\
&\quad \cdot \sum_{\zeta \in \mathbb{S}_4} \operatorname{sgn}(\zeta) \mathbf{1}_{((\tau_{\zeta(1)}, \xi_{\zeta(1)}), (\tau_{\zeta(2)}, \xi_{\zeta(2)}), (\tau_{\zeta(3)}, \xi_{\zeta(3)}), (\tau_{\zeta(4)}, \xi_{\zeta(4)})) = ((\uparrow, 1), (\downarrow, 1), (\downarrow, -1), (\uparrow, -1))},
\end{aligned}$$

which imply that

$$(2.35) \quad \|V_{\rho \mathbf{x} x, m}^+\|_{L^1} \leq 1, \quad (\forall (\rho, \mathbf{x}, x) \in \mathcal{B} \times \Gamma \times [0, \beta)_h, m \in \{2, 4\}).$$

By using Lemma 2.4, (2.35) and Lemma B.1 we can estimate $\|\tilde{S}_m^0\|_{L^1}$ as follows.

$$\begin{aligned}
\|\tilde{S}_m^0\|_{L^1} &\leq \sum_{n=2}^N \frac{1}{n!} \prod_{j=1}^n \left(\frac{U_{max}}{h} \sum_{(\rho_j, \mathbf{x}_j, x_j) \in \mathcal{B} \times \Gamma \times [0, \beta)_h} \right) \mathbf{1}_{\exists i \exists j (i \neq j \wedge x_i = x_j)} \\
&\quad \cdot \prod_{l=1}^n \left(\sum_{m_l \in \{2, 4\}} \sum_{k_l=0}^{m_l} \binom{m_l}{k_l} \right) \mathbf{1}_{\sum_{l=1}^n k_l = m} c_1^{\frac{1}{2} \sum_{l=1}^n (m_l - k_l)} \\
&\leq \frac{c_1^{-\frac{m}{2}}}{\beta h} \sum_{n=2}^N \frac{1}{n!} \binom{n}{2} (b\beta L^d U_{max})^n \\
&\quad \cdot \prod_{l=1}^n \left(\sum_{m_l \in \{2, 4\}} c_1^{\frac{m_l}{2}} \sum_{k_l=0}^{m_l} \binom{m_l}{k_l} \right) \mathbf{1}_{\sum_{l=1}^n k_l = m}.
\end{aligned}$$

Thus, for any $\alpha \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned}
\sum_{m=0}^N \alpha^m c_1^{\frac{m}{2}} \|\tilde{S}_m^0\|_{L^1} &\leq \frac{1}{2\beta h} \sum_{n=2}^N \frac{1}{(n-2)!} \left(b\beta L^d U_{max} \sum_{m \in \{2, 4\}} (\alpha+1)^m c_1^{\frac{m}{2}} \right)^n \\
&\leq \frac{1}{2\beta h} \left(b\beta L^d U_{max} \sum_{m \in \{2, 4\}} (\alpha+1)^m c_1^{\frac{m}{2}} \right)^2 e^{b\beta L^d U_{max} \sum_{m \in \{2, 4\}} (\alpha+1)^m c_1^{\frac{m}{2}}}.
\end{aligned}$$

We can estimate $\int (e^{-V^-(\psi+\psi^1)} - f_{V^-}(\psi+\psi^1)) d\mu_{C_{>0}^-}(\psi^1)$ in the same way as above and obtain the claimed estimation of $S^-(\psi) - S^0(\psi)$. \square

LEMMA 2.7. *Take any $\alpha \in \mathbb{R}_{\geq 0}$, $\varepsilon \in (0, 1)$. Assume that*

$$U_{max} \leq (b\beta L^d((\alpha + 1)^2 c_1 + (\alpha + 1)^4 c_1^2))^{-1} \log \left(\frac{2(\varepsilon + 1)}{\varepsilon + 2} \right).$$

Then, the following inequalities hold.

(1)

$$|S_0^\delta - 1| \leq \frac{\varepsilon}{\varepsilon + 2}, \quad (\forall \delta \in \{+, -, 0\}).$$

(2)

$$\sup_{\delta \in \{+, -, 0\}} \sum_{m=1}^N \alpha^m c_1^{\frac{m}{2}} \|S_m^\delta\|_{L^1} \leq \varepsilon \inf_{\delta \in \{+, -, 0\}} |S_0^\delta|.$$

PROOF. (1): The inequalities follow from the assumption and Lemma 2.5 (1).

(2): By the assumption,

$$e^{b\beta L^d U_{max}((\alpha+1)^2 c_1 + (\alpha+1)^4 c_1^2)} \leq (\varepsilon + 1)(2 - e^{b\beta L^d U_{max}(c_1 + c_1^2)}).$$

Thus, by Lemma 2.5 (1),(2),

$$\begin{aligned} \sup_{\delta \in \{+, -, 0\}} \sum_{m=1}^N \alpha^m c_1^{\frac{m}{2}} \|S_m^\delta\|_{L^1} &\leq \sup_{\delta \in \{+, -, 0\}} \sum_{m=0}^N \alpha^m c_1^{\frac{m}{2}} \|S_m^\delta\|_{L^1} - \inf_{\delta \in \{+, -, 0\}} |S_0^\delta| \\ &\leq (\varepsilon + 1)(2 - e^{b\beta L^d U_{max}(c_1 + c_1^2)}) - \inf_{\delta \in \{+, -, 0\}} |S_0^\delta| \\ &\leq \varepsilon \inf_{\delta \in \{+, -, 0\}} |S_0^\delta|. \quad \square \end{aligned}$$

LEMMA 2.8. *Take any $\alpha \in \mathbb{R}_{\geq 0}$, $\varepsilon \in (0, 1)$. Assume that*

$$U_{max} \leq (b\beta L^d((\alpha + 2)^2 c_1 + (\alpha + 2)^4 c_1^2))^{-1} \log \left(\frac{2(\varepsilon + 1)}{\varepsilon + 2} \right).$$

Set $R^\delta(\psi) := \log S^\delta(\psi)$, ($\delta \in \{+, -, 0\}$). Then, the following inequalities hold for any $h \in (2/\beta)\mathbb{N}$ satisfying $h > (1/2) \max\{1, 4/\beta\}$.

(1)

$$|R_0^\delta| \leq \log \left(\frac{\varepsilon + 2}{2} \right), \quad (\forall \delta \in \{+, -, 0\}).$$

(2)

$$\sum_{m=1}^N \alpha^m c_1^{\frac{m}{2}} \|R_m^\delta\|_{L^1} \leq -\log(1 - \varepsilon), \quad (\forall \delta \in \{+, -, 0\}).$$

(3)

$$|R_0^\delta - R_0^0| \leq -\log \left(1 - \max \left\{ 1, \frac{4}{\beta} \right\} \frac{1}{2h} \right), \quad (\forall \delta \in \{+, -\}).$$

(4)

$$\sum_{m=1}^N \alpha^m c_1^{\frac{m}{2}} \|R_m^\delta - R_m^0\|_{L^1} \leq \max \left\{ 1, \frac{4}{\beta} \right\} \frac{1}{2(1 - \varepsilon)h}, \quad (\forall \delta \in \{+, -\}).$$

REMARK 2.9. By Lemma 2.7 (1),

$$\operatorname{Re} S_0^\delta \geq \frac{2}{\varepsilon + 2} > 0, \quad (\forall \delta \in \{+, -, 0\}).$$

Thus, $R^\delta(\psi)$ ($\delta \in \{+, -, 0\}$) are well-defined.

PROOF OF LEMMA 2.8. (1): The result follows from Lemma 2.7 (1) and Lemma B.3 (1).

(2): By the assumption and Lemma 2.7 (2) we can apply Lemma B.3 (2) to obtain the result.

(3): By the assumption, Lemma 2.5 (3), Lemma 2.6, Lemma 2.7 (1) and Lemma B.3 (3) we have that

$$\begin{aligned} |R_0^+ - R_0^0| &\leq \sum_{n=1}^{\infty} \frac{1}{n} \left| \frac{S_0^+ - S_0^0}{S_0^0} \right|^n \leq \sum_{n=1}^{\infty} \frac{1}{n} \left| \frac{\frac{1}{h} \left(\frac{2(\varepsilon+1)}{\varepsilon+2} - 1 \right)}{\frac{2}{\varepsilon+2}} \right|^n \\ &\leq -\log \left(1 - \frac{1}{2h} \right), \end{aligned}$$

$$|R_0^- - R_0^0| \leq \sum_{n=1}^{\infty} \frac{1}{n} \left| \frac{S_0^- - S_0^0}{S_0^0} \right|^n \leq \sum_{n=1}^{\infty} \frac{1}{n} \left| \frac{\frac{2(\varepsilon+1)}{\beta h(\varepsilon+2)}}{\frac{2}{\varepsilon+2}} \right|^n \leq -\log \left(1 - \frac{2}{\beta h} \right).$$

These imply the result.

(4): The assumption, Lemma 2.5 (3), Lemma 2.6, Lemma 2.7 (1),(2) and Lemma B.3 (4) ensure that

$$\begin{aligned} & \sum_{m=1}^N \alpha^m c_1^{\frac{m}{2}} \|R_m^+ - R_m^0\|_{L^1} \\ & \leq \frac{1}{1-\varepsilon} |S_0^0|^{-1} |S_0^+|^{-1} \sum_{m=0}^N \alpha^m c_1^{\frac{m}{2}} \|S_m^+\|_{L^1} \sum_{n=0}^N \alpha^n c_1^{\frac{n}{2}} \|S_n^+ - S_n^0\|_{L^1} \\ & \leq \frac{1}{1-\varepsilon} \cdot \frac{\varepsilon+2}{2} \cdot \varepsilon \cdot \frac{1}{h} \left(\frac{2(\varepsilon+1)}{\varepsilon+2} - 1 \right) \\ & \leq \frac{1}{2h(1-\varepsilon)}, \\ & \sum_{m=1}^N \alpha^m c_1^{\frac{m}{2}} \|R_m^- - R_m^0\|_{L^1} \\ & \leq \frac{1}{1-\varepsilon} |S_0^0|^{-1} |S_0^-|^{-1} \sum_{m=0}^N \alpha^m c_1^{\frac{m}{2}} \|S_m^-\|_{L^1} \sum_{n=0}^N \alpha^n c_1^{\frac{n}{2}} \|S_n^- - S_n^0\|_{L^1} \\ & \leq \frac{1}{1-\varepsilon} \cdot \frac{\varepsilon+2}{2} \cdot \varepsilon \cdot \frac{1}{\beta h} \cdot \frac{2(\varepsilon+1)}{\varepsilon+2} \\ & \leq \frac{2}{\beta h(1-\varepsilon)}. \end{aligned}$$

The claimed inequality follows from these inequalities. \square

We conclude this section by proving the following lemma, which enables us to adopt

$$-\frac{1}{\beta L^d} \log \left(\int e^{\frac{1}{2}(R^+(\psi) + R^-(\psi))} d\mu_{C_{\leq 0}^\infty}(\psi) \right)$$

as a formulation of the normalized free energy density. Later in Section 7 we will see that this Grassmann integral formulation is suited to the IR analysis since it has desirable symmetries.

LEMMA 2.10. *There exist (β, L^d, b, χ, E) -dependent, h -independent constants $h_0, c_2, c_3 \in \mathbb{R}_{>0}$ such that the following statements hold for any $h \in (2/\beta)\mathbb{N}$ satisfying $h \geq h_0$.*

(1)

$$\begin{aligned} \operatorname{Re} \int e^{-V(\psi)} d\mu_C(\psi) &> 0, \\ \operatorname{Re} \int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C_{\leq 0}^\infty}(\psi) &> 0, \\ (\forall \mathbf{U} \in \mathbb{C}^b \text{ satisfying } |U_\rho| \leq c_2 \ (\forall \rho \in \mathcal{B})). \end{aligned}$$

(2)

$$\begin{aligned} \left| \log \left(\int e^{-V(\psi)} d\mu_C(\psi) \right) - \log \left(\int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C_{\leq 0}^\infty}(\psi) \right) \right| &\leq \frac{1}{h} c_3, \\ (\forall \mathbf{U} \in \mathbb{C}^b \text{ satisfying } |U_\rho| \leq c_2 \ (\forall \rho \in \mathcal{B})). \end{aligned}$$

PROOF. Take any $\varepsilon \in (0, 2/5)$ and assume that

$$(2.36) \quad U_{max} \leq (b\beta L^d (4^2 c_1 + 4^4 c_1^2))^{-1} \log \left(\frac{2(\varepsilon + 1)}{\varepsilon + 2} \right).$$

(1): By Lemma B.2 (1) and Lemma 2.8 (1),(2),

$$\begin{aligned} (2.37) \quad &\left| \int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C_{\leq 0}^+}(\psi) - 1 \right| \\ &\leq \left| \int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C_{\leq 0}^+}(\psi) - e^{\frac{1}{2}(R_0^+ + R_0^-)} \right| + |e^{\frac{1}{2}(R_0^+ + R_0^-)} - 1| \\ &\leq e^{\frac{1}{2}(|R_0^+| + |R_0^-|)} \left(e^{\frac{1}{2} \sum_{m=1}^N c_1^{\frac{m}{2}} (\|R_m^+\|_{L^1} + \|R_m^-\|_{L^1})} - 1 \right) \\ &\quad + e^{\frac{1}{2}(|R_0^+| + |R_0^-|)} - 1 \\ &\leq \frac{\varepsilon + 2}{2(1 - \varepsilon)} - 1. \end{aligned}$$

Since $\varepsilon \in (0, 2/5)$,

$$\operatorname{Re} \int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C_{\leq 0}^+}(\psi) \geq \frac{2 - 5\varepsilon}{2(1 - \varepsilon)} > 0.$$

It follows from (2.36) and the same argument as in the proof of Lemma 2.5 (1) that

$$\left| \int e^{-V(\psi)} d\mu_C(\psi) - 1 \right| \leq e^{b\beta L^d U_{max}(c_1+c_1^2)} - 1 \leq \frac{2(\varepsilon+1)}{\varepsilon+2} - 1.$$

Therefore,

$$\operatorname{Re} \int e^{-V(\psi)} d\mu_C(\psi) \geq \frac{2}{\varepsilon+2} > 0.$$

(2): The same calculation as in (2.37) yields that

$$(2.38) \quad \left| \int e^{R^+(\psi)} d\mu_{C_{\leq 0}^+}(\psi) \right| \geq \frac{2-5\varepsilon}{2(1-\varepsilon)}.$$

By Lemma 2.4 and Lemma B.2 (3),(4),

$$\begin{aligned} & \left| \int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C_{\leq 0}^\infty}(\psi) - \int e^{R^+(\psi)} d\mu_{C_{\leq 0}^+}(\psi) \right| \\ & \leq \left| \int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C_{\leq 0}^\infty}(\psi) - \int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C_{\leq 0}^+}(\psi) \right| \\ & \quad + \left| \int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C_{\leq 0}^+}(\psi) - \int e^{R^+(\psi)} d\mu_{C_{\leq 0}^+}(\psi) \right| \\ & \leq \frac{1}{h} e^{\frac{1}{2}(|R_0^+|+|R_0^-|)} \left(e^{\frac{1}{2} \sum_{m=1}^N 2^m c_1^{\frac{m}{2}} (\|R_m^+\|_{L^1} + \|R_m^-\|_{L^1})} - 1 \right) \\ & \quad + \left((e^{\frac{1}{2}|R_0^+ - R_0^-|} - 1) e^{|R_0^+|} + \frac{1}{2} e^{|R_0^+|} \sum_{m=1}^N c_1^{\frac{m}{2}} \|R_m^+ - R_m^-\|_{L^1} \right) \\ & \quad \cdot e^{\sup_{\delta \in \{+, -\}} \sum_{m=1}^N c_1^{\frac{m}{2}} \|R_m^\delta\|_{L^1}} \\ & \leq \frac{1}{h} e^{\frac{1}{2}(|R_0^+|+|R_0^-|)} \left(e^{\frac{1}{2} \sum_{m=1}^N 2^m c_1^{\frac{m}{2}} (\|R_m^+\|_{L^1} + \|R_m^-\|_{L^1})} - 1 \right) \\ & \quad + \left((e^{\frac{1}{2} \sum_{\delta \in \{+, -\}} |R_0^\delta - R_0^0|} - 1) e^{|R_0^+|} \right. \\ & \quad \left. + \frac{1}{2} e^{|R_0^+|} \sum_{\delta \in \{+, -\}} \sum_{m=1}^N c_1^{\frac{m}{2}} \|R_m^\delta - R_m^0\|_{L^1} \right) \\ & \quad \cdot e^{\sup_{\delta \in \{+, -\}} \sum_{m=1}^N c_1^{\frac{m}{2}} \|R_m^\delta\|_{L^1}}. \end{aligned}$$

Set $c' := \max\{1, 4/\beta\}$. By the assumption (2.36) we can substitute the inequalities proved in Lemma 2.8 for $\alpha = 2$ to derive that

$$\begin{aligned} & \left| \int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C_{\leq 0}^{\infty}}(\psi) - \int e^{R^+(\psi)} d\mu_{C_{\leq 0}^+}(\psi) \right| \\ & \leq \frac{\varepsilon + 2}{2h} \left(\frac{1}{1-\varepsilon} - 1 \right) \\ & \quad + \left(\left(\frac{1}{1-c'/(2h)} - 1 \right) \frac{\varepsilon + 2}{2} + \frac{\varepsilon + 2}{4} \cdot \frac{c'}{(1-\varepsilon)h} \right) \frac{1}{1-\varepsilon}. \end{aligned}$$

This inequality implies that there exist h -independent constants $c'', h_0 \in \mathbb{R}_{>0}$ such that for any $h \in (2/\beta)\mathbb{N}$ with $h \geq h_0$,

$$(2.39) \quad \left| \int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C_{\leq 0}^{\infty}}(\psi) - \int e^{R^+(\psi)} d\mu_{C_{\leq 0}^+}(\psi) \right| \leq \frac{1}{h} c''.$$

By using the inequalities (2.38), (2.39) and taking a larger h if necessary we have that

$$(2.40) \quad \left| \log \left(\int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C_{\leq 0}^{\infty}}(\psi) \right) - \log \left(\int e^{R^+(\psi)} d\mu_{C_{\leq 0}^+}(\psi) \right) \right| \\ \leq \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2(1-\varepsilon)}{2-5\varepsilon} \cdot \frac{c''}{h} \right)^n \leq \frac{2(1-\varepsilon)c''}{2-5\varepsilon-2(1-\varepsilon)c''/h} \cdot \frac{1}{h}.$$

We saw in Remark 2.9 that $\operatorname{Re} S_0^+ > 0$. Therefore, we can apply [14, Lemma C.2] to justify that

$$\begin{aligned} \int e^{R^+(\psi)} d\mu_{C_{\leq 0}^+}(\psi) &= \int \int e^{-V(\psi+\psi^1)} d\mu_{C_{>0}^+}(\psi^1) d\mu_{C_{\leq 0}^+}(\psi) \\ &= \int e^{-V(\psi)} d\mu_C(\psi). \end{aligned}$$

By combining this equality with (2.40) we obtain the inequality claimed in (2). \square

3. General Estimation

In this section we establish various inequalities which will form the basis of both the Matsubara UV integration and the IR integration around the

zero set of the free dispersion relation. We also show that a Grassmann polynomial produced by a single-scale integration inherits symmetric properties which the covariance and the input polynomial originally have. Here we assume that a covariance $C_o : I_0^2 \rightarrow \mathbb{C}$ is given and satisfies

$$(3.1) \quad \begin{aligned} |\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_o(X_i, Y_j))_{1 \leq i, j \leq n}| &\leq D_{et}^n, \\ (\forall r, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} &\leq 1, \\ X_i, Y_i \in I_0 \text{ (} i = 1, 2, \dots, n)), & \end{aligned}$$

with a constant $D_{et} \in \mathbb{R}_{>0}$. It is sometimes more convenient to deal with the anti-symmetric extension $\widetilde{C}_o : I^2 \rightarrow \mathbb{C}$ of C_o than C_o itself. The definition of \widetilde{C}_o is that

$$(3.2) \quad \begin{aligned} &\widetilde{C}_o((X, \theta), (Y, \xi)) \\ &:= \frac{1}{2}(1_{(\theta, \xi) = (1, -1)} C_o(X, Y) - 1_{(\theta, \xi) = (-1, 1)} C_o(Y, X)), \\ &(\forall X, Y \in I_0, \theta, \xi \in \{1, -1\}). \end{aligned}$$

In order to measure sizes of Grassmann polynomials during the multi-scale integrations, we need to define a family of norms and semi-norms on the linear space of anti-symmetric functions on I^m . For any $(\rho, \mathbf{x}, \sigma, x, \theta)$, $(\eta, \mathbf{y}, \tau, y, \xi) \in I$ and $j \in \{0, 1, \dots, d\}$ set

$$\begin{aligned} &d_j((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)) \\ &:= \begin{cases} \frac{\beta}{2\pi} |e^{i\frac{2\pi}{\beta}x} - e^{i\frac{2\pi}{\beta}y}| & \text{if } j = 0, \\ \frac{L}{2\pi} |e^{i\frac{2\pi}{L}\langle \mathbf{x}, \mathbf{v}_j \rangle} - e^{i\frac{2\pi}{L}\langle \mathbf{y}, \mathbf{v}_j \rangle}| & \text{if } j \in \{1, 2, \dots, d\}. \end{cases} \end{aligned}$$

Assume that a set of positive numbers $\{w(l)\}_{l \in \mathbb{Z}}$ is given. Fix $r \in (0, 1]$. For any $m \in \{2, 3, \dots, N\}$, anti-symmetric function $f : I^m \rightarrow \mathbb{C}$ and $l \in \mathbb{Z}$, let

$$(3.3) \quad \begin{aligned} \|f\|_{l,0} &:= \sup_{X \in I} \left(\frac{1}{h}\right)^{m-1} \sum_{\mathbf{Y} \in I^{m-1}} e^{\sum_{j=0}^d (w(l)d_j(X, Y_1))^r} |f(X, \mathbf{Y})|, \\ \|f\|_{l,1} &:= \sup_{j' \in \{0, 1, \dots, d\}} \sup_{q \in \{1, 2, \dots, m-1\}} \sup_{X \in I} \\ &\quad \cdot \left(\frac{1}{h}\right)^{m-1} \sum_{\mathbf{Y} \in I^{m-1}} d_{j'}(X, Y_q) e^{\sum_{j=0}^d (w(l)d_j(X, Y_1))^r} |f(X, \mathbf{Y})|. \end{aligned}$$

To understand these definitions clearly, recall the notational rule that $\mathbf{Y} = (Y_1, Y_2, \dots, Y_{m-1})$ for $\mathbf{Y} \in I^{m-1}$. Since no Grassmann polynomial of degree 1 appears in our multi-scale analysis, there is no need to newly introduce a norm in the space of functions on I . To organize formulas we sometimes write $\|f_0\|_{l,0}$ in place of $|f_0|$ for $f_0 \in \mathbb{C}$ as well.

When we practically use the norm $\|\cdot\|_{l,0}$ and the semi-norm $\|\cdot\|_{l,1}$ in Section 6 and Section 7, the integer l will represent an integration scale in the multi-scale integration procedure. Moreover, in these sections the weight $w(l)$ and the exponent r will be specifically defined. However, the general theory in this section can be completed without more detailed information on these parameters.

3.1. Estimation of the free integration

Here we estimate a Grassmann polynomial produced by the free integration. With $J(\psi) \in \bigwedge \mathcal{V}$ satisfying $J_m(\psi) = 0$ if $m \notin 2\mathbb{N} \cup \{0\}$, set

$$(3.4) \quad F(\psi) := \int J(\psi + \psi^1) d\mu_{C_o}(\psi^1).$$

By the definition of the Grassmann Gaussian integral, $F_m(\psi) = 0$ if $m \notin 2\mathbb{N} \cup \{0\}$.

LEMMA 3.1. *The following inequalities hold.*

$$\begin{aligned} |F_0| &\leq |J_0| + \frac{N}{h} \sum_{n=1}^N D_{et}^{\frac{n}{2}} \|J_n\|_{l,0}, \\ \|F_m\|_{l,r} &\leq \|J_m\|_{l,r} + \sum_{n=m+1}^N 2^n D_{et}^{\frac{n-m}{2}} \|J_n\|_{l,r}, \\ &(\forall r \in \{0, 1\}, l \in \mathbb{Z}, m \in \mathbb{N}_{\geq 2}). \end{aligned}$$

PROOF. By anti-symmetry,

$$\begin{aligned} F_m(\psi) &= J_m(\psi) + \left(\frac{1}{h}\right)^m \sum_{\mathbf{X} \in I^m} \left(\sum_{n=m+1}^N \left(\frac{1}{h}\right)^{n-m} \sum_{\mathbf{Y} \in I^{n-m}} \binom{n}{m} \right. \\ &\quad \left. \cdot J_n(\mathbf{X}, \mathbf{Y}) \int \psi_{\mathbf{Y}}^1 d\mu_{C_o}(\psi^1) \right) \psi_{\mathbf{X}}, \end{aligned}$$

which implies that for any $\mathbf{X} \in I^m$,

$$(3.5) \quad F_m(\mathbf{X}) = J_m(\mathbf{X}) + \sum_{n=m+1}^N \left(\frac{1}{h}\right)^{n-m} \cdot \sum_{\mathbf{Y} \in I^{n-m}} \binom{n}{m} J_n(\mathbf{X}, \mathbf{Y}) \int \psi_{\mathbf{Y}}^1 d\mu_{C_o}(\psi^1).$$

Since $\|J_n\|_{L^1} \leq 1_{n=0}|J_0| + 1_{n \geq 1} \frac{N}{h} \|J_n\|_{l,0}$,

$$|F_0| \leq \sum_{n=0}^N D_{et}^{\frac{n}{2}} \|J_n\|_{L^1} \leq |J_0| + \frac{N}{h} \sum_{n=1}^N D_{et}^{\frac{n}{2}} \|J_n\|_{l,0}.$$

By using the inequality

$$\binom{n}{m} \leq 2^n,$$

we can derive the claimed upper bound on $\|F_m\|_{l,r}$ from (3.5). \square

3.2. Estimation of the tree expansion

Take $J(\psi) \in \bigwedge \mathcal{V}$ satisfying $J_m(\psi) = 0$ if $m \notin 2\mathbb{N} \cup \{0\}$. We can see from definition that there exists a domain O of \mathbb{C} containing 0 such that

$$\log \left(\int e^{zJ(\psi+\psi^1)} d\mu_{C_o}(\psi^1) \right)$$

is analytic with z in O . It is known that for any $n \in \mathbb{N}_{\geq 2}$,

$$\left(\frac{d}{dz} \right)^n \log \left(\int e^{zJ(\psi+\psi^1)} d\mu_{C_o}(\psi^1) \right) \Big|_{z=0}$$

can be characterized as a sum over trees with n vertices. We adopt one version of such formulas clearly proved in [22]. The formula [22, Theorem 3] states that for any $n \in \mathbb{N}_{\geq 2}$ and $J(\psi) \in \bigwedge \mathcal{V}$ satisfying $J_m(\psi) = 0$ if $m \notin 2\mathbb{N} \cup \{0\}$,

$$(3.6) \quad \begin{aligned} & \frac{1}{n!} \left(\frac{d}{dz} \right)^n \log \left(\int e^{zJ(\psi+\psi^1)} d\mu_{C_o}(\psi^1) \right) \Big|_{z=0} \\ &= \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \prod_{\{p,q\} \in T} (\Delta_{p,q}(C_o) + \Delta_{q,p}(C_o)) \int_{[0,1]^{n-1}} ds \sum_{\xi \in \mathbb{S}_n(T)} \varphi(T, \xi, \mathbf{s}) \end{aligned}$$

$$\cdot e^{\sum_{r,s=1}^n M_{at}(T,\xi,\mathbf{s})(r,s)\Delta_{r,s}(C_o)} \prod_{j=1}^n J(\psi^j + \psi) \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}},$$

where \mathbb{T}_n is the set of all trees over the vertices $\{1, 2, \dots, n\}$,

$$(3.7) \quad \Delta_{r,s}(C_o) := - \sum_{X,Y \in I_0} C_o(X,Y) \frac{\partial}{\partial \psi_X^r} \frac{\partial}{\partial \psi_Y^s} \\ : \bigwedge \left(\bigoplus_{j=1}^n \mathcal{V}_j \right) \rightarrow \bigwedge \left(\bigoplus_{j=1}^n \mathcal{V}_j \right), \quad (\forall r, s \in \{1, 2, \dots, n\}),$$

$\mathbb{S}_n(T)$ is a T -dependent subset of \mathbb{S}_n , $\varphi(T, \xi, \cdot)$ is a (T, ξ) -dependent continuous function from $[0, 1]^{n-1}$ to $\mathbb{R}_{\geq 0}$ satisfying that

$$(3.8) \quad \int_{[0,1]^{n-1}} ds \sum_{\xi \in \mathbb{S}_n(T)} \varphi(T, \xi, \mathbf{s}) = 1, \quad (\forall T \in \mathbb{T}_n),$$

$(M_{at}(T, \xi, \mathbf{s})(r, s))_{1 \leq r, s \leq n}$ is a (T, ξ, \mathbf{s}) -dependent real symmetric non-negative matrix satisfying that $M_{at}(T, \xi, \mathbf{s})(r, r) = 1$ ($\forall r \in \{1, \dots, n\}$). Moreover, $\mathbf{s} \mapsto M_{at}(T, \xi, \mathbf{s})(r, s)$ is continuous in $[0, 1]^{n-1}$ ($\forall r, s \in \{1, \dots, n\}$).

For a given polynomial $J(\psi) \in \bigwedge \mathcal{V}$ satisfying $J_m(\psi) = 0$ if $m \notin 2\mathbb{N} \cup \{0\}$, let us define $T^{(n)}(\psi) \in \bigwedge \mathcal{V}$ by the right-hand side of (3.6). The goal of this subsection is to establish norm estimates on the anti-symmetric kernels of $T^{(n)}(\psi)$.

To facilitate our analysis, let us fix some notational conventions. For $\mathbf{X} \in I^m$, $\mathbf{Y} \in I^n$ we write $\mathbf{X} \subset \mathbf{Y}$ if $m \leq n$ and there exist $j_1, j_2, \dots, j_m \in \{1, 2, \dots, n\}$ such that $j_1 < j_2 < \dots < j_m$ and $\mathbf{X} = (Y_{j_1}, Y_{j_2}, \dots, Y_{j_m})$. In this case we also define $\mathbf{Y} \setminus \mathbf{X} \in I^{n-m}$ by $\mathbf{Y} \setminus \mathbf{X} := (Y_{k_1}, Y_{k_2}, \dots, Y_{k_{n-m}})$, where $1 \leq k_1 < k_2 < \dots < k_{n-m} \leq n$ and $\{k_1, k_2, \dots, k_{n-m}\} \cap \{j_1, j_2, \dots, j_m\} = \emptyset$. The following abbreviation will be often used. For any object $f(\mathbf{X})$ parameterized by the variable $\mathbf{X} \in I^m$,

$$\sum_{\substack{\mathbf{X} \subset \mathbf{Y} \\ \mathbf{X} \in I^m}} f(\mathbf{X})$$

denotes

$$\sum_{1 \leq i_1 < i_2 < \dots < i_m \leq n} f((Y_{i_1}, Y_{i_2}, \dots, Y_{i_m})).$$

For any $T \in \mathbb{T}_n$, $p, q \in \{1, 2, \dots, n\}$ let $n_p(T) (\in \{1, 2, \dots, n-1\})$ be the incidence number of the vertex p and $\text{dis}_T(p, q) (\in \{0, 1, \dots, n\})$ denote the distance between the vertex p and the vertex q along the unique path connecting p to q in T . Moreover, set

$$d_T(p) := \max_{r \in \{1, 2, \dots, n\}} \text{dis}_T(p, r).$$

Define $L_q^p(T) (\subset T)$ by

$$L_q^p(T) := \{\{q, r\} \in T \mid \text{dis}_T(p, r) = \text{dis}_T(p, q) + 1\}.$$

Note that

$$\#L_p^p(T) = n_p(T), \quad \#L_q^p(T) = n_q(T) - 1, \quad (\forall q \in \{1, 2, \dots, n\} \setminus \{p\}).$$

For $q \in \{1, 2, \dots, n\}$ with $\#L_q^p(T) \neq 0$ let ζ_q denote the bijective map from $L_q^p(T)$ to $\{1, 2, \dots, \#L_q^p(T)\}$ satisfying that

$$\zeta_q(\{q, r\}) < \zeta_q(\{q, s\}), \quad (\forall \{q, r\}, \{q, s\} \in L_q^p(T) \text{ with } r < s).$$

For non-commutative mathematical objects f_r, f_{r+1}, \dots, f_s ($r, s \in \mathbb{Z}$, $r < s$) we set

$$\prod_{\substack{n=r \\ \text{order}}}^s f_n := f_r f_{r+1} \cdots f_s.$$

This notation will help us to shorten formulas on various occasions. Also for conciseness, let us set

$$\begin{aligned} ope(T, C_o) &:= \int_{[0,1]^{n-1}} ds \sum_{\xi \in \mathbb{S}_n(T)} \varphi(T, \xi, \mathbf{s}) e^{\sum_{r,s=1}^n Mat(T, \xi, \mathbf{s})(r,s) \Delta_{r,s}(C_o)}, \\ Ope(T, C_o) &:= ope(T, C_o) \prod_{\{p,q\} \in T} (\Delta_{p,q}(C_o) + \Delta_{q,p}(C_o)), \quad (\forall T \in \mathbb{T}_n). \end{aligned}$$

We construct necessary estimates step by step. By using the assumption (3.1), the properties of $Mat(T, \xi, \mathbf{s})$ and by repeating the same argument as in [12, Lemma 4.5] one can prove the next lemma. The reason why the covariance C_o needs to be multiplied by $\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^n}$ in (3.1) is that $Mat(T, \xi, \mathbf{s})(r, s)$ can be rewritten as $\langle \mathbf{p}_r, \mathbf{q}_s \rangle_{\mathbb{C}^n}$ with some $\mathbf{p}_r, \mathbf{q}_s \in \mathbb{R}^n$ satisfying $\|\mathbf{p}_r\|_{\mathbb{C}^n} = \|\mathbf{q}_s\|_{\mathbb{C}^n} = 1$ during the proof of the next lemma.

LEMMA 3.2. *For any $T \in \mathbb{T}_n$, $\xi \in \mathbb{S}_n(T)$, $\mathbf{s} \in [0, 1]^{n-1}$, $\mathbf{X}_j \in I^{m_j}$ ($j = 1, 2, \dots, n$),*

$$\left| e^{\sum_{r,s=1}^n \text{Mat}(T, \xi, \mathbf{s})(r, s) \Delta_{r, s}(C_o)} \prod_{\substack{j=1 \\ \text{order}}}^n \psi_{\mathbf{X}_j}^j \right|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \leq D_{\text{et}}^{\frac{1}{2} \sum_{j=1}^n m_j}.$$

Lemma 3.2 will be used in the proof of the following lemma.

LEMMA 3.3. *Take any $T \in \mathbb{T}_n$, $m_j \in \mathbb{N}$ and $\mathbf{X}_j \in I^{m_j}$ ($j = 1, 2, \dots, n$). The following inequality holds.*

$$\begin{aligned} & \left| \text{Ope}(T, C_o) \prod_{\substack{j=1 \\ \text{order}}}^n \psi_{\mathbf{X}_j}^j \right|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\ & \leq 1_{n_j(T) \leq m_j (\forall j \in \{1, 2, \dots, n\})} 2^{n-1} D_{\text{et}}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} \\ & \quad \cdot \sum_{\substack{\mathbf{w}_1 \subset \mathbf{X}_1 \\ \mathbf{w}_1 \in I^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \prod_{\{1, s\} \in L_1^1(T)} \left(\sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{1, \sigma_1 \circ \zeta_1}(\{1, s\}), Z_s)| \right) \\ & \quad \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2, 3, \dots, n\} \\ \text{dis}_T(1, j) = u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{w}_j \subset \mathbf{X}_j \setminus Z_j \\ \mathbf{w}_j \in I^{n_j(T)-1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \right) \right) \\ & \quad \cdot \prod_{\{j, s\} \in L_j^1(T)} \left(\sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{j, \sigma_j \circ \zeta_j}(\{j, s\}), Z_s)| \right) \Bigg). \end{aligned}$$

PROOF. For $\{p, q\} \in T$ set $\Delta_{\{p, q\}} := \Delta_{p, q}(C_o) + \Delta_{q, p}(C_o)$. Note that

$$(3.9) \quad \Delta_{\{p, q\}} = -2 \sum_{\mathbf{X} \in I^2} \widetilde{C}_o(\mathbf{X}) \frac{\partial}{\partial \psi_{X_1}^p} \frac{\partial}{\partial \psi_{X_2}^q}.$$

The operator $\prod_{l \in T} \Delta_l$ erases $n_j(T)$ elements from the Grassmann monomial $\psi_{\mathbf{X}_j}^j$ for every $j \in \{1, 2, \dots, n\}$. Hence, we need the constraint

$1_{n_j(T) \leq m_j} (\forall j \in \{1, 2, \dots, n\})$. The operator $\prod_{l \in T} \Delta_l$ can be decomposed as follows.

$$\prod_{l \in T} \Delta_l = \prod_{l \in L_1^1(T)} \Delta_l \prod_{u=1}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2, 3, \dots, n\} \\ \text{dis}_T(1, j) = u, n_j(T) \neq 1}} \prod_{l \in L_j^1(T)} \Delta_l \right).$$

We apply Δ_l ($l \in L_1^1(T)$) to the input Grassmann monomial first. Then, from $u = 1$ up to $u = d_T(1) - 1$ we let the operator

$$\prod_{\substack{j \in \{2, 3, \dots, n\} \\ \text{dis}_T(1, j) = u, n_j(T) \neq 1}} \prod_{l \in L_j^1(T)} \Delta_l$$

act on the remaining polynomial by turns. This procedure yields that

$$\begin{aligned} & \left| \text{Ope}(T, C_o) \prod_{\substack{j=1 \\ \text{order}}}^n \psi_{\mathbf{X}_j}^j \right|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\ & \leq 1_{n_j(T) \leq m_j} (\forall j \in \{1, 2, \dots, n\}) \\ & \cdot \sum_{\substack{\mathbf{w}_1 \subset \mathbf{X}_1 \\ \mathbf{w}_1 \in I^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \prod_{\{1, s\} \in L_1^1(T)} \left(2 \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{1, \sigma_1 \circ \zeta_1}(\{1, s\}), Z_s)| \right) \\ & \cdot \left| \text{ope}(T, C_o) \prod_{u=1}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2, 3, \dots, n\} \\ \text{dis}_T(1, j) = u, n_j(T) \neq 1}} \prod_{l \in L_j^1(T)} \Delta_l \right) \right. \\ & \cdot \left. \psi_{\mathbf{X}_1 \setminus \mathbf{w}_1}^1 \prod_{\substack{j=2 \\ \text{order}}}^n (1_{\text{dis}_T(1, j) \leq 1} \psi_{\mathbf{X}_j \setminus Z_j}^j + 1_{\text{dis}_T(1, j) > 1} \psi_{\mathbf{X}_j}^j) \right|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\ & \leq 1_{n_j(T) \leq m_j} (\forall j \in \{1, 2, \dots, n\}) \\ & \cdot \sum_{\substack{\mathbf{w}_1 \subset \mathbf{X}_1 \\ \mathbf{w}_1 \in I^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \prod_{\{1, s\} \in L_1^1(T)} \left(2 \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{1, \sigma_1 \circ \zeta_1}(\{1, s\}), Z_s)| \right) \\ & \cdot \prod_{\substack{j \in \{2, 3, \dots, n\} \\ \text{dis}_T(1, j) = 1, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{w}_j \subset \mathbf{X}_j \setminus Z_j \\ \mathbf{w}_j \in I^{n_j(T)-1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \right) \end{aligned}$$

$$\begin{aligned}
& \cdot \prod_{\{j,s\} \in L_j^1(T)} \left(2 \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{j,\sigma_j \circ \zeta_j}(\{j,s\}), Z_s)| \right) \\
& \cdot \left| \text{ope}(T, C_o) \prod_{u=2}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j)=u, n_j(T) \neq 1}} \prod_{l \in L_j^1(T)} \Delta_l \right) \right. \\
& \cdot \psi_{\mathbf{X}_1 \setminus \mathbf{w}_1}^1 \prod_{\substack{j=2 \\ \text{order}}}^n (1_{\text{dis}_T(1,j) \leq 1 \text{ and } n_j(T) \neq 1} \psi_{(\mathbf{X}_j \setminus Z_j) \setminus \mathbf{w}_j}^j \\
& \quad + 1_{\text{dis}_T(1,j) \leq 1 \text{ and } n_j(T)=1} \psi_{\mathbf{X}_j \setminus Z_j}^j + 1_{\text{dis}_T(1,j)=2} \psi_{\mathbf{X}_j \setminus Z_j}^j \\
& \quad + 1_{\text{dis}_T(1,j) > 2} \psi_{\mathbf{X}_j}^j) \left| \begin{array}{c} \psi^j=0 \\ (\forall j \in \{1,2,\dots,n\}) \end{array} \right| \\
& \leq 1_{n_j(T) \leq m_j (\forall j \in \{1,2,\dots,n\})} \\
& \cdot \sum_{\substack{\mathbf{w}_1 \subset \mathbf{X}_1 \\ \mathbf{w}_1 \in I^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \prod_{\{1,s\} \in L_1^1(T)} \left(2 \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{1,\sigma_1 \circ \zeta_1}(\{1,s\}), Z_s)| \right) \\
& \cdot \prod_{\substack{u=1 \\ \text{order}}}^v \left(\prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j)=u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{w}_j \subset \mathbf{X}_j \setminus Z_j \\ \mathbf{w}_j \in I^{n_j(T)-1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \right) \right) \\
& \cdot \prod_{\{j,s\} \in L_j^1(T)} \left(2 \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{j,\sigma_j \circ \zeta_j}(\{j,s\}), Z_s)| \right) \\
& \cdot \left| \text{ope}(T, C_o) \prod_{u=v+1}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j)=u, n_j(T) \neq 1}} \prod_{l \in L_j^1(T)} \Delta_l \right) \right. \\
& \cdot \psi_{\mathbf{X}_1 \setminus \mathbf{w}_1}^1 \prod_{\substack{j=2 \\ \text{order}}}^n (1_{\text{dis}_T(1,j) \leq v \text{ and } n_j(T) \neq 1} \psi_{(\mathbf{X}_j \setminus Z_j) \setminus \mathbf{w}_j}^j \\
& \quad + 1_{\text{dis}_T(1,j) \leq v \text{ and } n_j(T)=1} \psi_{\mathbf{X}_j \setminus Z_j}^j + 1_{\text{dis}_T(1,j)=v+1} \psi_{\mathbf{X}_j \setminus Z_j}^j \\
& \quad + 1_{\text{dis}_T(1,j) > v+1} \psi_{\mathbf{X}_j}^j) \left| \begin{array}{c} \psi^j=0 \\ (\forall j \in \{1,2,\dots,n\}) \end{array} \right|
\end{aligned}$$

$$\begin{aligned}
 &\leq 1_{n_j(T) \leq m_j (\forall j \in \{1, 2, \dots, n\})} \\
 &\cdot \sum_{\substack{\mathbf{w}_1 \subset \mathbf{x}_1 \\ \mathbf{w}_1 \in I^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \prod_{\{1, s\} \in L_1^1(T)} \left(2 \sum_{\substack{Z_s \subset \mathbf{x}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{1, \sigma_1 \circ \zeta_1(\{1, s\}), Z_s})| \right) \\
 &\cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2, 3, \dots, n\} \text{ with} \\ \text{dis}_T(1, j) = u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{w}_j \subset \mathbf{x}_j \setminus Z_j \\ \mathbf{w}_j \in I^{n_j(T)-1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \right. \right. \\
 &\cdot \left. \left. \prod_{\{j, s\} \in L_j^1(T)} \left(2 \sum_{\substack{Z_s \subset \mathbf{x}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{j, \sigma_j \circ \zeta_j(\{j, s\}), Z_s})| \right) \right) \right) \\
 &\cdot \left| \text{ope}(T, C_o) \psi_{\mathbf{x}_1 \setminus \mathbf{w}_1}^1 \right. \\
 &\cdot \left. \prod_{\substack{j=2 \\ \text{order}}}^n (1_{n_j(T) \neq 1} \psi_{(\mathbf{x}_j \setminus Z_j) \setminus \mathbf{w}_j}^j + 1_{n_j(T) = 1} \psi_{\mathbf{x}_j \setminus Z_j}^j) \right|_{\substack{\psi^j = 0 \\ (\forall j \in \{1, 2, \dots, n\})}}.
 \end{aligned}$$

Collecting the factor 2 gives 2^{n-1} , since the tree T has $n-1$ lines. Then, by using (3.8), Lemma 3.2 and the fact that $\sum_{j=1}^n n_j(T) = 2(n-1)$ we obtain the claimed inequality. \square

LEMMA 3.4. *Take any $T \in \mathbb{T}_n$ and $m_j \in \mathbb{N}_{\geq 2}$ ($j = 1, 2, \dots, n$). Let $J_{m_j} : I^{m_j} \rightarrow \mathbb{C}$ ($j = 1, 2, \dots, n$) be anti-symmetric functions. Then, the following inequalities hold.*

(1) *For any $X_{1,1} \in I$,*

$$\begin{aligned}
 (3.10) \quad &\left| \text{Ope}(T, C_o) \left(\frac{1}{h} \right)^{m_1-1} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1, m_1}) \in I^{m_1-1}} J_{m_1}(\mathbf{X}_1) \right. \\
 &\cdot \prod_{j=2}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in I^{m_j}} J_{m_j}(\mathbf{X}_j) \right) \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{\mathbf{x}_k}^k \left| \right|_{\substack{\psi^j = 0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\
 &\leq 1_{n_j(T) \leq m_j (\forall j \in \{1, 2, \dots, n\})} \\
 &\cdot 2^{n-1} \prod_{j=1}^n \left(m_j \binom{m_j-1}{n_j(T)-1} (n_j(T)-1)! \right)
 \end{aligned}$$

$$\cdot D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} \|\widetilde{C}_o\|_{l,0}^{n-1} \prod_{k=1}^n \|J_{m_k}\|_{l,0}.$$

(2) *In addition, assume that $k_j \in \{0, 1, \dots, m_j - 1\}$ ($\forall j \in \{1, 2, \dots, n\}$), $p, q \in \{1, 2, \dots, n\}$, $k_1, k_p, k_q \geq 1$, $r \in \{1, 2, \dots, k_q\}$, $j' \in \{0, 1, \dots, d\}$ and $a \in \{0, 1\}$. Then, for any $Y_{1,1} \in I$,*

$$\begin{aligned}
(3.11) \quad & \left| \text{Ope}(T, C_o) \left(\frac{1}{h} \right)^{m_1 - 1} \sum_{\mathbf{X}_1 \in I^{m_1 - k_1}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1, k_1}) \in I^{k_1 - 1}} J_{m_1}(\mathbf{X}_1, \mathbf{Y}_1) \right. \\
& \cdot \prod_{j=2}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in I^{m_j - k_j}} \sum_{\mathbf{Y}_j \in I^{k_j}} J_{m_j}(\mathbf{X}_j, \mathbf{Y}_j) \right) \\
& \cdot d_{j'}(Y_{1,1}, Y_{q,r})^a e^{\sum_{j=0}^d (w(l)d_j(Y_{1,1}, Y_{p,1}))^F} \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{\mathbf{X}_k}^k \left| \begin{array}{c} \psi^j = 0 \\ (\forall j \in \{1, 2, \dots, n\}) \end{array} \right| \\
& \leq 1_{n_j(T) \leq m_j - k_j (\forall j \in \{1, 2, \dots, n\})} 2^{n-1} \\
& \cdot \prod_{i=1}^n \left((m_i - k_i) \binom{m_i - k_i - 1}{n_i(T) - 1} (n_i(T) - 1)! \right) \\
& \cdot D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\
& \cdot \prod_{j=1}^n \left(\sum_{q_j=0}^1 \|J_{m_j}\|_{l, q_j} \right) \prod_{k=2}^n \left(\sum_{r_k=0}^1 \|\widetilde{C}_o\|_{l, r_k} \right) 1_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = a}.
\end{aligned}$$

PROOF. (1): By Lemma 3.3 we have that

(the left-hand side of (3.10))

$$\begin{aligned}
& \leq 1_{n_j(T) \leq m_j (\forall j \in \{1, 2, \dots, n\})} 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} \\
& \cdot \left(\frac{1}{h} \right)^{m_1 - 1} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1, m_1}) \in I^{m_1 - 1}} |J_{m_1}(\mathbf{X}_1)| \sum_{\substack{\mathbf{w}_1 \subset \mathbf{X}_1 \\ \mathbf{w}_1 \in I^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \\
& \cdot \prod_{\{1, s\} \in L_1^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s}} |J_{m_s}(\mathbf{X}_s)| \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{1, \sigma_1 \circ \zeta_1}(\{1, s\}), Z_s)| \right)
\end{aligned}$$

$$\begin{aligned}
 & \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j)=u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{w}_j \subset \mathbf{X}_j \setminus Z_j \\ \mathbf{w}_j \in I^{n_j(T)-1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \right. \right. \\
 & \cdot \prod_{\{j,s\} \in L_j^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s}} |J_{m_s}(\mathbf{X}_s)| \right. \\
 & \cdot \left. \left. \left. \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{j,\sigma_j \circ \zeta_j}(\{j,s\}), Z_s)| \right) \right) \right).
 \end{aligned}$$

Then, estimating recursively from $u = d_T(1) - 1$ to $u = 1$, we observe that

(the left-hand side of (3.10))

$$\begin{aligned}
 & \leq \mathbf{1}_{n_j(T) \leq m_j (\forall j \in \{1,2,\dots,n\})} 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} \\
 & \cdot \left(\frac{1}{h} \right)^{m_1 - 1} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1,m_1}) \in I^{m_1 - 1}} |J_{m_1}(\mathbf{X}_1)| \sum_{\substack{\mathbf{w}_1 \subset \mathbf{X}_1 \\ \mathbf{w}_1 \in I^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \\
 & \cdot \prod_{\{1,s\} \in L_1^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s}} |J_{m_s}(\mathbf{X}_s)| \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{1,\sigma_1 \circ \zeta_1}(\{1,s\}), Z_s)| \right) \\
 & \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-2} \left(\prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j)=u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{w}_j \subset \mathbf{X}_j \setminus Z_j \\ \mathbf{w}_j \in I^{n_j(T)-1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \right. \right. \\
 & \cdot \prod_{\{j,s\} \in L_j^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s}} |J_{m_s}(\mathbf{X}_s)| \right. \\
 & \cdot \left. \left. \left. \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{j,\sigma_j \circ \zeta_j}(\{j,s\}), Z_s)| \right) \right) \right) \\
 & \cdot \prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j)=d_T(1)-1, n_j(T) \neq 1}} \left(\binom{m_j - 1}{n_j(T) - 1} (n_j(T) - 1)! \right)
 \end{aligned}$$

$$\begin{aligned}
& \cdot \prod_{\{j,s\} \in L_j^1(T)} (m_s \|J_{m_s}\|_{l,0} \|\widetilde{C}_o\|_{l,0}) \\
& \leq 1_{n_j(T) \leq m_j (\forall j \in \{1,2,\dots,n\})} 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} \\
& \cdot \left(\frac{1}{h} \right)^{m_1 - 1} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1,m_1}) \in I^{m_1 - 1}} |J_{m_1}(\mathbf{X}_1)| \sum_{\substack{\mathbf{w}_1 \subset \mathbf{X}_1 \\ \mathbf{w}_1 \in I^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \\
& \cdot \prod_{\{1,s\} \in L_1^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s}} |J_{m_s}(\mathbf{X}_s)| \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{1,\sigma_1 \circ \zeta_1}(\{1,s\}), Z_s)| \right) \\
& \cdot \prod_{\substack{u=1 \\ \text{order}}}^v \left(\prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j) = u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{w}_j \subset \mathbf{X}_j \setminus Z_j \\ \mathbf{w}_j \in I^{n_j(T) - 1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T) - 1}} \right. \right. \\
& \cdot \prod_{\{j,s\} \in L_j^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s}} |J_{m_s}(\mathbf{X}_s)| \right. \\
& \cdot \left. \left. \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{j,\sigma_j \circ \zeta_j}(\{j,s\}), Z_s)| \right) \right) \\
& \cdot \prod_{w=v+1}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j) = w, n_j(T) \neq 1}} \left(\binom{m_j - 1}{n_j(T) - 1} (n_j(T) - 1)! \right. \right. \\
& \cdot \left. \left. \prod_{\{j,s\} \in L_j^1(T)} (m_s \|J_{m_s}\|_{l,0} \|\widetilde{C}_o\|_{l,0}) \right) \right) \\
& \leq 1_{n_j(T) \leq m_j (\forall j \in \{1,2,\dots,n\})} 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} \\
& \cdot \binom{m_1}{n_1(T)} n_1(T)! \|J_{m_1}\|_{l,0} \prod_{\{1,s\} \in L_1^1(T)} (m_s \|J_{m_s}\|_{l,0} \|\widetilde{C}_o\|_{l,0}) \\
& \cdot \prod_{u=1}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j) = u, n_j(T) \neq 1}} \left(\binom{m_j - 1}{n_j(T) - 1} (n_j(T) - 1)! \right. \right.
\end{aligned}$$

$$\cdot \prod_{\{j,s\} \in L_j^1(T)} \left(m_s \|J_{m_s}\|_{l,0} \|\widetilde{C}_o\|_{l,0} \right) \Bigg),$$

which is equal to the right-hand side of (3.10).

(2): Using Lemma 3.3 and the anti-symmetric property of the functions $J_{m_j}(\cdot)$ ($j = 1, 2, \dots, n$), we see that

$$\begin{aligned} (3.12) \quad & \text{(the left-hand side of (3.11))} \\ & \leq 1_{n_j(T) \leq m_j - k_j (\forall j \in \{1, 2, \dots, n\})} 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\ & \cdot \left(\frac{1}{h} \right)^{m_1 - 1} \sum_{\mathbf{X}_1 \in I^{m_1 - k_1}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1,k_1}) \in I^{k_1 - 1}} |J_{m_1}(\mathbf{X}_1, \mathbf{Y}_1)| \\ & \cdot \sum_{\substack{\mathbf{w}_1 \subset \mathbf{X}_1 \\ \mathbf{w}_1 \in I^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \\ & \cdot \prod_{\{1,s\} \in L_1^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s - k_s}} \sum_{\mathbf{Y}_s \in I^{k_s}} |J_{m_s}(\mathbf{X}_s, \mathbf{Y}_s)| \right. \\ & \quad \cdot \left. \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{1,\sigma_1 \circ \zeta_1}(\{1,s\}), Z_s)| \right) \\ & \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j) = u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{w}_j \subset \mathbf{X}_j \setminus Z_j \\ \mathbf{w}_j \in I^{n_j(T)-1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \right. \right. \\ & \quad \cdot \prod_{\{j,s\} \in L_j^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s - k_s}} \sum_{\mathbf{Y}_s \in I^{k_s}} |J_{m_s}(\mathbf{X}_s, \mathbf{Y}_s)| \right. \\ & \quad \cdot \left. \left. \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I}} |\widetilde{C}_o(W_{j,\sigma_j \circ \zeta_j}(\{j,s\}), Z_s)| \right) \right) \Bigg) \\ & \cdot d_{j'}(Y_{1,1}, Y_{q,r})^a e^{\sum_{j=0}^d (w(l) d_j(Y_{1,1}, Y_{p,1}))^r} \\ & = 1_{n_j(T) \leq m_j - k_j (\forall j \in \{1, 2, \dots, n\})} 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \end{aligned}$$

$$\begin{aligned}
& \cdot \left(\frac{1}{h} \right)^{m_1-1} \sum_{\mathbf{X}_1 \in I^{m_1-k_1-n_1}(T)} \sum_{\mathbf{W}_1 \in I^{n_1}(T)} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1,k_1}) \in I^{k_1-1}} \\
& \cdot |J_{m_1}(\mathbf{X}_1, \mathbf{W}_1, \mathbf{Y}_1)| \left(\binom{m_1-k_1}{n_1(T)} n_1(T)! \right) \\
& \cdot \prod_{\{1,s\} \in L_1^1(T)} \left((m_s - k_s) \left(\frac{1}{h} \right)^{m_s} \right. \\
& \quad \cdot \sum_{\mathbf{X}_s \in I^{m_s-k_s-n_s}(T)} \sum_{\mathbf{W}_s \in I^{n_s(T)-1}} \sum_{Z_s \in I} \sum_{\mathbf{Y}_s \in I^{k_s}} \\
& \quad \cdot |J_{m_s}(\mathbf{X}_s, \mathbf{W}_s, Z_s, \mathbf{Y}_s)| |\widetilde{\mathcal{C}}_o(W_{1,\zeta_1(\{1,s\})}, Z_s)| \left. \right) \\
& \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j)=u, n_j(T) \neq 1}} \left(\binom{m_j-k_j-1}{n_j(T)-1} (n_j(T)-1)! \right) \right. \\
& \quad \cdot \prod_{\{j,s\} \in L_j^1(T)} \left((m_s - k_s) \left(\frac{1}{h} \right)^{m_s} \right) \\
& \quad \cdot \sum_{\mathbf{X}_s \in I^{m_s-k_s-n_s}(T)} \sum_{\mathbf{W}_s \in I^{n_s(T)-1}} \sum_{Z_s \in I} \sum_{\mathbf{Y}_s \in I^{k_s}} \\
& \quad \cdot |J_{m_s}(\mathbf{X}_s, \mathbf{W}_s, Z_s, \mathbf{Y}_s)| |\widetilde{\mathcal{C}}_o(W_{j,\zeta_j(\{j,s\})}, Z_s)| \left. \right) \left. \right) \\
& \cdot d_{j'}(Y_{1,1}, Y_{q,r})^a e^{\sum_{j=0}^d (w(l)d_j(Y_{1,1}, Y_{p,1}))^r} \\
& = 1_{n_j(T) \leq m_j-k_j (\forall j \in \{1,2,\dots,n\})} 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j-k_j)-n+1} \\
& \cdot \prod_{j=1}^n \left((m_j - k_j) \binom{m_j - k_j - 1}{n_j(T) - 1} (n_j(T) - 1)! \right) \\
& \cdot \left(\frac{1}{h} \right)^{m_1-1} \sum_{\mathbf{X}_1 \in I^{m_1-k_1-n_1}(T)} \sum_{\mathbf{W}_1 \in I^{n_1}(T)} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1,k_1}) \in I^{k_1-1}} \\
& \cdot |J_{m_1}(\mathbf{X}_1, \mathbf{W}_1, \mathbf{Y}_1)|
\end{aligned}$$

$$\begin{aligned}
 & \cdot \prod_{\{1,s\} \in L_1^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s - k_s - n_s(T)}} \sum_{\mathbf{W}_s \in I^{n_s(T) - 1}} \sum_{Z_s \in I} \sum_{\mathbf{Y}_s \in I^{k_s}} \right. \\
 & \quad \cdot |J_{m_s}(\mathbf{X}_s, \mathbf{W}_s, Z_s, \mathbf{Y}_s)| |\widetilde{\mathcal{C}}_o(W_{1, \zeta_1}(\{1, s\}), Z_s)| \Big) \\
 & \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2, 3, \dots, n\} \text{ with} \\ \text{dis}_T(1, j) = u, n_j(T) \neq 1}} \right. \\
 & \cdot \prod_{\{j, s\} \in L_j^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s - k_s - n_s(T)}} \sum_{\mathbf{W}_s \in I^{n_s(T) - 1}} \sum_{Z_s \in I} \sum_{\mathbf{Y}_s \in I^{k_s}} \right. \\
 & \quad \cdot |J_{m_s}(\mathbf{X}_s, \mathbf{W}_s, Z_s, \mathbf{Y}_s)| |\widetilde{\mathcal{C}}_o(W_{j, \zeta_j}(\{j, s\}), Z_s)| \Big) \Big) \\
 & \cdot d_{j'}(Y_{1,1}, Y_{q,r})^a e^{\sum_{j=0}^d (w(l) d_j(Y_{1,1}, Y_{p,1}))^r}.
 \end{aligned}$$

Then, we can apply Lemma 3.5, which will be proved next, to derive the claimed inequality. \square

LEMMA 3.5. *On the same assumption as in Lemma 3.4 (2) plus that $n_j(T) \leq m_j - k_j$ ($\forall j \in \{1, 2, \dots, n\}$), the following inequality holds.*

$$\begin{aligned}
 (3.13) \quad & \left(\frac{1}{h} \right)^{m_1 - 1} \sum_{\mathbf{X}_1 \in I^{m_1 - k_1 - n_1(T)}} \sum_{\mathbf{W}_1 \in I^{n_1(T)}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1, k_1}) \in I^{k_1 - 1}} \\
 & \cdot |J_{m_1}(\mathbf{X}_1, \mathbf{W}_1, \mathbf{Y}_1)| \\
 & \cdot \prod_{\{1, s\} \in L_1^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s - k_s - n_s(T)}} \sum_{\mathbf{W}_s \in I^{n_s(T) - 1}} \sum_{Z_s \in I} \sum_{\mathbf{Y}_s \in I^{k_s}} \right. \\
 & \quad \cdot |J_{m_s}(\mathbf{X}_s, \mathbf{W}_s, Z_s, \mathbf{Y}_s)| |\widetilde{\mathcal{C}}_o(W_{1, \zeta_1}(\{1, s\}), Z_s)| \Big) \\
 & \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2, 3, \dots, n\} \text{ with} \\ \text{dis}_T(1, j) = u, n_j(T) \neq 1}} \right.
 \end{aligned}$$

$$\begin{aligned}
& \cdot \prod_{\{j,s\} \in L_j^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s - k_s - n_s(T)}} \sum_{\mathbf{W}_s \in I^{n_s(T) - 1}} \sum_{Z_s \in I} \sum_{\mathbf{Y}_s \in I^{k_s}} \right. \\
& \quad \left. \cdot |J_{m_s}(\mathbf{X}_s, \mathbf{W}_s, Z_s, \mathbf{Y}_s)| |\widetilde{C}_o(W_{j, \zeta_j(\{j, s\}), Z_s})| \right) \\
& \cdot d_{j'}(Y_{1,1}, Y_{q,r})^a e^{\sum_{j=0}^d (w(l) d_j(Y_{1,1}, Y_{p,1}))^r} \\
& \leq \prod_{j=1}^n \left(\sum_{q_j=0}^1 \|J_{m_j}\|_{l, q_j} \right) \prod_{k=2}^n \left(\sum_{r_k=0}^1 \|\widetilde{C}_o\|_{l, r_k} \right) 1_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = a}.
\end{aligned}$$

PROOF. We prove the claimed inequality by induction with n . In the following we will repeatedly use the inequality $(x + y)^r \leq x^r + y^r$, ($\forall x, y \in \mathbb{R}_{\geq 0}$). If $n = 2$,

$$\begin{aligned}
& \text{(the left-hand side of (3.13))} \\
& = \left(\frac{1}{h} \right)^{m_1 - 1} \sum_{\mathbf{X}_1 \in I^{m_1 - k_1 - 1}} \sum_{W_1 \in I} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1, k_1}) \in I^{k_1 - 1}} |J_{m_1}(\mathbf{X}_1, W_1, \mathbf{Y}_1)| \\
& \cdot \left(\frac{1}{h} \right)^{m_2} \sum_{\mathbf{X}_2 \in I^{m_2 - k_2 - 1}} \sum_{Z_2 \in I} \sum_{\mathbf{Y}_2 \in I^{k_2}} |J_{m_2}(\mathbf{X}_2, Z_2, \mathbf{Y}_2)| |\widetilde{C}_o(W_1, Z_2)| \\
& \cdot d_{j'}(Y_{1,1}, Y_{q,r})^a e^{\sum_{j=0}^d (w(l) d_j(Y_{1,1}, Y_{p,1}))^r} \\
& \leq \left(\frac{1}{h} \right)^{m_1 - 1} \sum_{\mathbf{X}_1 \in I^{m_1 - k_1 - 1}} \sum_{W_1 \in I} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1, k_1}) \in I^{k_1 - 1}} |J_{m_1}(\mathbf{X}_1, W_1, \mathbf{Y}_1)| \\
& \cdot \left(\frac{1}{h} \right)^{m_2} \sum_{\mathbf{X}_2 \in I^{m_2 - k_2 - 1}} \sum_{Z_2 \in I} \sum_{\mathbf{Y}_2 \in I^{k_2}} |J_{m_2}(\mathbf{X}_2, Z_2, \mathbf{Y}_2)| |\widetilde{C}_o(W_1, Z_2)| \\
& \cdot \left(1_{q=1} \sum_{q_1=0}^1 d_{j'}(Y_{1,1}, Y_{1,r})^{q_1} 1_{q_1=a} \right. \\
& \quad + 1_{q=2} \sum_{q_1=0}^1 d_{j'}(Y_{1,1}, W_1)^{q_1} \sum_{r_2=0}^1 d_{j'}(W_1, Z_2)^{r_2} \\
& \quad \left. \cdot \sum_{q_2=0}^1 d_{j'}(Z_2, Y_{2,r})^{q_2} 1_{q_1 + q_2 + r_2 = a} \right)
\end{aligned}$$

$$\begin{aligned}
 & \cdot \left(1_{p=1} + 1_{p=2} e^{\sum_{j=0}^d (\mathbf{w}(l) d_j(Y_{1,1}, W_1))^r} \right. \\
 & \quad \left. \cdot e^{\sum_{j=0}^d (\mathbf{w}(l) d_j(W_1, Z_2))^r} e^{\sum_{j=0}^d (\mathbf{w}(l) d_j(Z_2, Y_{2,1}))^r} \right) \\
 & \leq 1_{q=1} \sum_{q_1=0}^1 \|J_{m_1}\|_{l, q_1} \|\widetilde{C}_0\|_{l, 0} \|J_{m_2}\|_{l, 0} 1_{q_1=a} \\
 & \quad + 1_{q=2} \sum_{q_1=0}^1 \|J_{m_1}\|_{l, q_1} \sum_{r_2=0}^1 \|\widetilde{C}_0\|_{l, r_2} \sum_{q_2=0}^1 \|J_{m_2}\|_{l, q_2} 1_{q_1+q_2+r_2=a},
 \end{aligned}$$

which is less than or equal to the right-hand side of (3.13) for $n = 2$.

Assume that the claim holds for some $n \in \mathbb{N}_{\geq 2}$. Let us estimate the left-hand side of (3.13) for $n + 1$. Take a vertex $\hat{s} \in \{1, 2, \dots, n + 1\}$ satisfying $\text{dis}_T(1, \hat{s}) = d_T(1)$. Take $\sigma \in \mathbb{S}_{n+1}$ satisfying $\sigma(1) = 1$, $\sigma(\hat{s}) = n + 1$. Then, define the tree $T' \in \mathbb{T}_{n+1}$ by $T' := \{\{\sigma(j), \sigma(s)\} \mid \{j, s\} \in T\}$. Note that $\text{dis}_{T'}(1, n + 1) = d_{T'}(1)$. Setting $m'_j := m_{\sigma^{-1}(j)}$, $k'_j := k_{\sigma^{-1}(j)}$ ($j = 1, 2, \dots, n + 1$), we see that $n_j(T') \leq m'_j - k'_j$ ($\forall j \in \{1, 2, \dots, n + 1\}$) and

(the left-hand side of (3.13))

$$\begin{aligned}
 & = \left(\frac{1}{h}\right)^{m'_1-1} \sum_{\mathbf{X}_1 \in I^{m'_1-k'_1-n_1(T')}} \sum_{\mathbf{W}_1 \in I^{n_1(T')}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1, k'_1}) \in I^{k'_1-1}} \\
 & \quad \cdot |J_{m'_1}(\mathbf{X}_1, \mathbf{W}_1, \mathbf{Y}_1)| \\
 & \quad \cdot \prod_{\{1, s\} \in L_1^1(T')} \left(\left(\frac{1}{h}\right)^{m'_s} \sum_{\mathbf{X}_s \in I^{m'_s-k'_s-n_s(T')}} \sum_{\mathbf{W}_s \in I^{n_s(T')-1}} \sum_{Z_s \in I} \sum_{\mathbf{Y}_s \in I^{k'_s}} \right. \\
 & \quad \left. \cdot |J_{m'_s}(\mathbf{X}_s, \mathbf{W}_s, Z_s, \mathbf{Y}_s)| \|\widetilde{C}_o(W_{1, \zeta_{\sigma^{-1}(1)}(\{\sigma^{-1}(1), \sigma^{-1}(s)\}), Z_s})\| \right) \\
 & \quad \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_{T'}(1)-1} \left(\prod_{\substack{j \in \{2, 3, \dots, n\} \\ \text{dis}_{T'}(1, j) = u, n_j(T') \neq 1}} \right)
 \end{aligned}$$

$$\begin{aligned}
& \cdot \prod_{\{j,s\} \in L_j^1(T')} \left(\left(\frac{1}{h} \right)^{m'_s} \sum_{\mathbf{X}_s \in I^{m'_s - k'_s - n_s(T')}} \sum_{\mathbf{W}_s \in I^{n_s(T') - 1}} \sum_{Z_s \in I} \sum_{\mathbf{Y}_s \in I^{k'_s}} \right. \\
& \quad \cdot \left. |J_{m'_s}(\mathbf{X}_s, \mathbf{W}_s, Z_s, \mathbf{Y}_s)| |\widetilde{C}_o(W_{j, \zeta_{\sigma^{-1}(j)}(\{\sigma^{-1}(j), \sigma^{-1}(s)\})}, Z_s)| \right) \\
& \cdot d_{j'}(Y_{1,1}, Y_{\sigma(q),r})^a e^{\sum_{j=0}^d (\mathbf{w}(l) d_j(Y_{1,1}, Y_{\sigma(p),1}))^r}.
\end{aligned}$$

Let ζ'_j denote the bijective map from $L_j^1(T')$ to $\{1, 2, \dots, \#L_j^1(T')\}$ satisfying that $\zeta'_j(\{j, r\}) < \zeta'_j(\{j, s\})$, ($\forall \{j, r\}, \{j, s\} \in L_j^1(T')$ with $r < s$). By the anti-symmetry of $J_{m'_j}(\mathbf{X}_j, \mathbf{W}_j, Z_j, \mathbf{Y}_j)$ with respect to the variable \mathbf{W}_j we can replace $\zeta_{\sigma^{-1}(j)}(\{\sigma^{-1}(j), \sigma^{-1}(s)\})$ by $\zeta'_j(\{j, s\})$ in the right-hand side of the above equality. This argument implies that we may assume $\text{dis}_T(1, n+1) = d_T(1)$ in the left-hand side of (3.13) without losing generality. Thus, we assume so in the following.

If $\{\tilde{j}, n+1\} \in T$,

$$\begin{aligned}
(3.14) \quad & d_{j'}(Y_{1,1}, Y_{q,r})^a e^{\sum_{j=0}^d (\mathbf{w}(l) d_j(Y_{1,1}, Y_{p,1}))^r} \\
& \leq \left(\mathbf{1}_{q \leq n} d_{j'}(Y_{1,1}, Y_{q,r})^a \right. \\
& \quad + \mathbf{1}_{q=n+1} \sum_{q_{\tilde{j}}=0}^1 d_{j'}(Y_{1,1}, W_{\tilde{j}, \zeta_{\tilde{j}}(\{\tilde{j}, n+1\})})^{q_{\tilde{j}}} \\
& \quad \cdot \sum_{r_{n+1}=0}^1 d_{j'}(W_{\tilde{j}, \zeta_{\tilde{j}}(\{\tilde{j}, n+1\})}, Z_{n+1})^{r_{n+1}} \\
& \quad \cdot \sum_{q_{n+1}=0}^1 d_{j'}(Z_{n+1}, Y_{n+1,r})^{q_{n+1}} \mathbf{1}_{q_{\tilde{j}} + q_{n+1} + r_{n+1} = a} \left. \right) \\
& \cdot \left(\mathbf{1}_{p \leq n} e^{\sum_{j=0}^d (\mathbf{w}(l) d_j(Y_{1,1}, Y_{p,1}))^r} \right. \\
& \quad + \mathbf{1}_{p=n+1} e^{\sum_{j=0}^d (\mathbf{w}(l) d_j(Y_{1,1}, W_{\tilde{j}, \zeta_{\tilde{j}}(\{\tilde{j}, n+1\})}))^r} \\
& \quad \cdot e^{\sum_{j=0}^d (\mathbf{w}(l) d_j(W_{\tilde{j}, \zeta_{\tilde{j}}(\{\tilde{j}, n+1\})}, Z_{n+1}))^r} \\
& \quad \cdot \left. e^{\sum_{j=0}^d (\mathbf{w}(l) d_j(Z_{n+1}, Y_{n+1,1}))^r} \right).
\end{aligned}$$

Set $\tilde{T} := T \setminus \{\{\tilde{j}, n+1\}\}$, $\tilde{k}_j := k_j$ ($\forall j \in \{1, 2, \dots, n\} \setminus \{\tilde{j}\}$), $\tilde{k}_{\tilde{j}} := k_{\tilde{j}} + 1$. It follows that $\tilde{k}_j \in \{0, 1, \dots, m_j - 1\}$, $n_j(\tilde{T}) \leq m_j - \tilde{k}_j$, ($\forall j \in \{1, 2, \dots, n\}$). Substitution of (3.14) yields that

$$\begin{aligned}
 (3.15) \quad & \text{(the left-hand side of (3.13))} \\
 & \leq \left(\frac{1}{\hbar} \right)^{m_1-1} \sum_{\mathbf{X}_1 \in I^{m_1-k_1-n_1(T)}} \sum_{\mathbf{W}_1 \in I^{n_1(T)}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1,k_1}) \in I^{k_1-1}} \\
 & \cdot |J_{m_1}(\mathbf{X}_1, \mathbf{W}_1, \mathbf{Y}_1)| \\
 & \cdot \prod_{\{1,s\} \in L_1^1(T)} \left(\left(\frac{1}{\hbar} \right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s-k_s-n_s(T)}} \sum_{\mathbf{W}_s \in I^{n_s(T)-1}} \sum_{Z_s \in I} \sum_{\mathbf{Y}_s \in I^{k_s}} \right. \\
 & \cdot |J_{m_s}(\mathbf{X}_s, \mathbf{W}_s, Z_s, \mathbf{Y}_s)| |\widetilde{C}_o(W_{1,\zeta_1(\{1,s\})}, Z_s)| \Big) \\
 & \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j)=u, n_j(T) \neq 1}} \right. \\
 & \cdot \prod_{\{j,s\} \in L_j^1(T) \setminus \{\{\tilde{j}, n+1\}\}} \left(\left(\frac{1}{\hbar} \right)^{m_s} \right. \\
 & \cdot \sum_{\mathbf{X}_s \in I^{m_s-k_s-n_s(T)}} \sum_{\mathbf{W}_s \in I^{n_s(T)-1}} \sum_{Z_s \in I} \sum_{\mathbf{Y}_s \in I^{k_s}} \\
 & \cdot |J_{m_s}(\mathbf{X}_s, \mathbf{W}_s, Z_s, \mathbf{Y}_s)| |\widetilde{C}_o(W_{j,\zeta_j(\{j,s\})}, Z_s)| \Big) \Big) \\
 & \cdot \left(1_{q \leq n} 1_{p \leq n} d_{j'}(Y_{1,1}, Y_{q,r})^a \right. \\
 & \cdot e^{\sum_{j=0}^d (w(l) d_j(Y_{1,1}, Y_{p,1}))^r} \|\widetilde{C}_o\|_{l,0} \|J_{m_{n+1}}\|_{l,0} \\
 & + 1_{q \leq n} 1_{p=n+1} d_{j'}(Y_{1,1}, Y_{q,r})^a \\
 & \cdot e^{\sum_{j=0}^d (w(l) d_j(Y_{1,1}, W_{\tilde{j}, \zeta_{\tilde{j}}(\{\tilde{j}, n+1\}))})^r} \|\widetilde{C}_o\|_{l,0} \|J_{m_{n+1}}\|_{l,0} \\
 & + 1_{q=n+1} 1_{p \leq n} \sum_{q_{\tilde{j}}=0}^1 d_{j'}(Y_{1,1}, W_{\tilde{j}, \zeta_{\tilde{j}}(\{\tilde{j}, n+1\})})^{q_{\tilde{j}}}
 \end{aligned}$$

$$\begin{aligned}
& \cdot e^{\sum_{j=0}^d (w(l)d_j(Y_{1,1}, Y_{p,1}))^r} \\
& \cdot \sum_{r_{n+1}=0}^1 \|\widetilde{C}_o\|_{l, r_{n+1}} \sum_{q_{n+1}=0}^1 \|J_{m_{n+1}}\|_{l, q_{n+1}} \mathbf{1}_{q_j+q_{n+1}+r_{n+1}=a} \\
& + \mathbf{1}_{q=n+1} \mathbf{1}_{p=n+1} \sum_{q_j=0}^1 d_{j'}(Y_{1,1}, W_{\tilde{j}, \zeta_{\tilde{j}}(\{\tilde{j}, n+1\})})^{q_j} \\
& \cdot e^{\sum_{j=0}^d (w(l)d_j(Y_{1,1}, W_{\tilde{j}, \zeta_{\tilde{j}}(\{\tilde{j}, n+1\}))^r} \\
& \cdot \sum_{r_{n+1}=0}^1 \|\widetilde{C}_o\|_{l, r_{n+1}} \sum_{q_{n+1}=0}^1 \|J_{m_{n+1}}\|_{l, q_{n+1}} \mathbf{1}_{q_j+q_{n+1}+r_{n+1}=a} \Big) \\
= & \left(\frac{1}{h}\right)^{m_1-1} \sum_{\mathbf{X}_1 \in I^{m_1 - \tilde{k}_1 - n_1}(\tilde{T})} \sum_{\mathbf{W}_1 \in I^{n_1}(\tilde{T})} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1, \tilde{k}_1}) \in I^{\tilde{k}_1-1}} \\
& \cdot |J_{m_1}(\mathbf{X}_1, \mathbf{W}_1, \mathbf{Y}_1)| \\
& \cdot \prod_{\{1,s\} \in L_1^1(\tilde{T})} \left(\left(\frac{1}{h}\right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s - \tilde{k}_s - n_s}(\tilde{T})} \sum_{\mathbf{W}_s \in I^{n_s}(\tilde{T})-1} \sum_{Z_s \in I} \sum_{\mathbf{Y}_s \in I^{\tilde{k}_s}} \right. \\
& \cdot |J_{m_s}(\mathbf{X}_s, \mathbf{W}_s, Z_s, \mathbf{Y}_s)| \|\widetilde{C}_o(W_{1, \zeta_1(\{1,s\}), Z_s})\| \Big) \\
& \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_{\tilde{T}}(1)-1} \left(\prod_{\substack{j \in \{2,3, \dots, n\} \text{ with} \\ \text{dis}_{\tilde{T}}(1,j)=u, n_j(\tilde{T}) \neq 1}} \right. \\
& \cdot \prod_{\{j,s\} \in L_j^1(\tilde{T})} \left(\left(\frac{1}{h}\right)^{m_s} \sum_{\mathbf{X}_s \in I^{m_s - \tilde{k}_s - n_s}(\tilde{T})} \sum_{\mathbf{W}_s \in I^{n_s}(\tilde{T})-1} \sum_{Z_s \in I} \sum_{\mathbf{Y}_s \in I^{\tilde{k}_s}} \right. \\
& \cdot |J_{m_s}(\mathbf{X}_s, \mathbf{W}_s, Z_s, \mathbf{Y}_s)| \|\widetilde{C}_o(W_{j, \zeta_j(\{j,s\}), Z_s})\| \Big) \Big) \\
& \cdot \left(\mathbf{1}_{q \leq n} \mathbf{1}_{p \leq n} d_{j'}(Y_{1,1}, Y_{q,r})^a \right. \\
& \cdot e^{\sum_{j=0}^d (w(l)d_j(Y_{1,1}, Y_{p,1}))^r} \|\widetilde{C}_o\|_{l,0} \|J_{m_{n+1}}\|_{l,0} \\
& \left. + \mathbf{1}_{q \leq n} \mathbf{1}_{p=n+1} d_{j'}(Y_{1,1}, Y_{q,r'})^a \right)
\end{aligned}$$

$$\begin{aligned}
 & \cdot e^{\sum_{j=0}^d (w(l)d_j(Y_{1,1}, Y_{\tilde{j},1}))^r} \|\widetilde{C}_o\|_{l,0} \|J_{m_{n+1}}\|_{l,0} \\
 & + \mathbf{1}_{q=n+1} \mathbf{1}_{p \leq n} \sum_{q_{\tilde{j}}=0}^1 d_{j'}(Y_{1,1}, Y_{\tilde{j},r''})^{q_{\tilde{j}}} e^{\sum_{j=0}^d (w(l)d_j(Y_{1,1}, Y_{p,1}))^r} \\
 & \cdot \sum_{r_{n+1}=0}^1 \|\widetilde{C}_o\|_{l,r_{n+1}} \sum_{q_{n+1}=0}^1 \|J_{m_{n+1}}\|_{l,q_{n+1}} \mathbf{1}_{q_{\tilde{j}}+q_{n+1}+r_{n+1}=a} \\
 & + \mathbf{1}_{q=n+1} \mathbf{1}_{p=n+1} \sum_{q_{\tilde{j}}=0}^1 d_{j'}(Y_{1,1}, Y_{\tilde{j},1})^{q_{\tilde{j}}} e^{\sum_{j=0}^d (w(l)d_j(Y_{1,1}, Y_{\tilde{j},1}))^r} \\
 & \cdot \sum_{r_{n+1}=0}^1 \|\widetilde{C}_o\|_{l,r_{n+1}} \sum_{q_{n+1}=0}^1 \|J_{m_{n+1}}\|_{l,q_{n+1}} \mathbf{1}_{q_{\tilde{j}}+q_{n+1}+r_{n+1}=a} \Big),
 \end{aligned}$$

where we used the anti-symmetry of $J_{m_{\tilde{j}}}(\cdot)$ to shift the variable $W_{\tilde{j},\zeta_{\tilde{j}}}(\{\tilde{j},n+1\})$ to be in front of $\mathbf{Y}_{\tilde{j}}$ (or behind $\mathbf{Y}_{\tilde{j}}$) and replaced $(W_{\tilde{j},\zeta_{\tilde{j}}}(\{\tilde{j},n+1\}), \mathbf{Y}_{\tilde{j}})$ (or $(\mathbf{Y}_{\tilde{j}}, W_{\tilde{j},\zeta_{\tilde{j}}}(\{\tilde{j},n+1\}))$) by $\mathbf{Y}_{\tilde{j}} \in I^{\tilde{k}_{\tilde{j}}}$. Because of this change of the variable, the component inside $d_{j'}(\cdot, \cdot)$ may be changed from the original one. We used the numbers $r' \in \{1, 2, \dots, \tilde{k}_q\}$, $r'' \in \{1, 2, \dots, \tilde{k}_{\tilde{j}}\}$ to represent the possible new components. Now, we can apply the hypothesis of induction to derive from (3.15) that

$$\begin{aligned}
 & \text{(the left-hand side of (3.13))} \\
 & \leq \prod_{j=1}^n \left(\sum_{q_j=0}^1 \|J_{m_j}\|_{l,q_j} \right) \prod_{k=2}^n \left(\sum_{r_k=0}^1 \|\widetilde{C}_o\|_{l,r_k} \right) \\
 & \cdot \left(\mathbf{1}_{q \leq n} \mathbf{1}_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = a} \|\widetilde{C}_o\|_{l,0} \|J_{m_{n+1}}\|_{l,0} \right. \\
 & \quad + \mathbf{1}_{q=n+1} \sum_{q_{\tilde{j}}=0}^1 \mathbf{1}_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = q_{\tilde{j}}} \\
 & \quad \left. \cdot \sum_{r_{n+1}=0}^1 \|\widetilde{C}_o\|_{l,r_{n+1}} \sum_{q_{n+1}=0}^1 \|J_{m_{n+1}}\|_{l,q_{n+1}} \mathbf{1}_{q_{\tilde{j}}+q_{n+1}+r_{n+1}=a} \right),
 \end{aligned}$$

which is less than or equal to the right-hand side of (3.13) for $n+1$. Thus, the induction concludes that the inequality (3.13) holds for all $n \in \mathbb{N}_{\geq 2}$. \square

In order to deal with combinatorial factors in the tree expansion, we use the following concise estimate, though it is not quantitatively optimal.

LEMMA 3.6. *For any $m_j \in \mathbb{N}$ ($j = 1, 2, \dots, n$) the following inequality holds.*

$$(3.16) \quad \begin{aligned} & \frac{2^{n-1}}{n!} \sum_{T \in \mathbb{T}_n} 1_{n_j(T) \leq m_j (\forall j \in \{1, 2, \dots, n\})} \\ & \quad \cdot \prod_{j=1}^n \left(m_j \binom{m_j - 1}{n_j(T) - 1} (n_j(T) - 1)! \right) \\ & \leq 2^{2 \sum_{j=1}^n m_j}. \end{aligned}$$

PROOF. By Cayley's theorem on the number of trees with fixed incidence numbers we can replace the sum over $T \in \mathbb{T}_n$ by the sum over possible incidence numbers. As the result, we have that

$$\begin{aligned} & \text{(the left-hand side of (3.16))} \\ & = \frac{2^{n-1}}{n!} \prod_{i=1}^n \left(\sum_{l_i=1}^{m_i} \right) 1_{\sum_{i=1}^n l_i = 2(n-1)} \frac{(n-2)!}{\prod_{k=1}^n (l_k - 1)!} \\ & \quad \cdot \prod_{j=1}^n \left(m_j \binom{m_j - 1}{l_j - 1} (l_j - 1)! \right) \\ & \leq \frac{2^{n-1}}{n(n-1)} \prod_{j=1}^n \left(m_j \sum_{l_j=1}^{m_j} \binom{m_j - 1}{l_j - 1} \right) \leq 2^{2 \sum_{j=1}^n m_j}. \quad \square \end{aligned}$$

LEMMA 3.7. *Take any $m_j \in \mathbb{N}_{\geq 2}$ ($j = 1, 2, \dots, n$). Let $J_{m_j} : I^{m_j} \rightarrow \mathbb{C}$ ($j = 1, 2, \dots, n$) be anti-symmetric functions. Then, the following inequalities hold.*

(1) *For any $X_{1,1} \in I$,*

$$\left| \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o) \left(\frac{1}{h} \right)^{m_1 - 1} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1, m_1}) \in I^{m_1 - 1}} J_{m_1}(\mathbf{X}_1) \right|$$

$$\begin{aligned}
 & \cdot \prod_{j=2}^n \left(\left(\frac{1}{\hbar} \right)^{m_j} \sum_{\mathbf{X}_j \in I^{m_j}} J_{m_j}(\mathbf{X}_j) \right) \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{\mathbf{X}_k}^k \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}} \\
 & \leq 2^{2 \sum_{j=1}^n m_j} D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} \|\widetilde{C}_o\|_{l,0}^{n-1} \prod_{j=1}^n \|J_{m_j}\|_{l,0}.
 \end{aligned}$$

(2) In addition, assume that $k_j \in \{0, 1, \dots, m_j - 1\}$ ($\forall j \in \{1, 2, \dots, n\}$), $p, q \in \{1, 2, \dots, n\}$, $k_1, k_p, k_q \geq 1$, $r \in \{1, 2, \dots, k_q\}$, $j' \in \{0, 1, \dots, d\}$ and $a \in \{0, 1\}$. Then, for any $Y_{1,1} \in I$,

$$\begin{aligned}
 & \left| \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o) \left(\frac{1}{\hbar} \right)^{m_1 - 1} \sum_{\mathbf{X}_1 \in I^{m_1 - k_1}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1,k_1}) \in I^{k_1 - 1}} \right. \\
 & \quad \cdot J_{m_1}(\mathbf{X}_1, \mathbf{Y}_1) \\
 & \quad \cdot \prod_{j=2}^n \left(\left(\frac{1}{\hbar} \right)^{m_j} \sum_{\mathbf{X}_j \in I^{m_j - k_j}} \sum_{\mathbf{Y}_j \in I^{k_j}} J_{m_j}(\mathbf{X}_j, \mathbf{Y}_j) \right) \\
 & \quad \cdot d_{j'}(Y_{1,1}, Y_{q,r})^a e^{\sum_{j=0}^d (w(l)d_j(Y_{1,1}, Y_{p,1}))^r} \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{\mathbf{X}_k}^k \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}} \\
 & \leq 2^{2 \sum_{j=1}^n (m_j - k_j)} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\
 & \quad \cdot \prod_{j=1}^n \left(\sum_{q_j=0}^1 \|J_{m_j}\|_{l,q_j} \right) \prod_{k=2}^n \left(\sum_{r_k=0}^1 \|\widetilde{C}_o\|_{l,r_k} \right) 1_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = a}.
 \end{aligned}$$

PROOF. We only need to sum the right-hand sides of (3.10) and (3.11) over trees. The claimed upper bounds follow from Lemma 3.6. \square

Here let us recall the definition of $T^{(n)}(\psi)$ ($n \in \mathbb{N}_{\geq 2}$). With $J(\psi) \in \bigwedge \mathcal{V}$ satisfying $J_m(\psi) = 0$ if $m \notin 2\mathbb{N} \cup \{0\}$,

$$T^{(n)}(\psi) = \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o) \prod_{j=1}^n J(\psi^j + \psi) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}}.$$

We conclude this subsection by proving the next lemma.

LEMMA 3.8. *The following inequalities hold for any $n \in \mathbb{N}_{\geq 2}$.*

(1)

$$|T_0^{(n)}| \leq \frac{N}{h} D_{et}^{-n+1} \|\widetilde{C}_o\|_{l,0}^{n-1} \left(\sum_{m=2}^N 2^{2m} D_{et}^{\frac{m}{2}} \|J_m\|_{l,0} \right)^n.$$

(2) For any $m \in \{2, 3, \dots, N\}$ and $a \in \{0, 1\}$,

$$\begin{aligned} & \|T_m^{(n)}\|_{l,a} \\ & \leq 2^{-2m} D_{et}^{-\frac{m}{2}-n+1} \prod_{i=1}^n \left(\sum_{q_i=0}^1 \right) \prod_{j=2}^n \left(\sum_{r_j=0}^1 \right) 1_{\sum_{i=1}^n q_i + \sum_{j=2}^n r_j = a} \\ & \cdot \prod_{k=2}^n \|\widetilde{C}_o\|_{l,r_k} \prod_{p=1}^n \left(\sum_{m_p=2}^N 2^{3m_p} D_{et}^{\frac{m_p}{2}} \|J_{m_p}\|_{l,q_p} \right) 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m}. \end{aligned}$$

PROOF. First note that the constant part J_0 of the input $J(\psi)$ does not affect the result since the operator $\prod_{l \in T} \Delta_l$ erases it. By using the anti-symmetric property of the kernels we have that

$$\begin{aligned} (3.17) \quad T_m^{(n)}(\psi) &= \mathcal{P}_m \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o) \\ & \cdot \prod_{j=1}^n \left(\sum_{m_j=2}^N \left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in I^{m_j}} \right. \\ & \quad \left. \cdot J_{m_j}(\mathbf{X}_j)(\psi^j + \psi)_{\mathbf{X}_j} \right) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\ &= \prod_{i=1}^n \left(\sum_{m_i=2}^N \sum_{k_i=0}^{m_i-1} \binom{m_i}{k_i} \left(\frac{1}{h} \right)^{k_i} \sum_{\mathbf{Y}_i \in I^{k_i}} \right) \\ & \cdot 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m} 1_{\sum_{j=1}^n k_j = m} \\ & \cdot \frac{\varepsilon_{\pm}}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o) \\ & \cdot \prod_{j=1}^n \left(\left(\frac{1}{h} \right)^{m_j - k_j} \sum_{\mathbf{X}_j \in I^{m_j - k_j}} J_{m_j}(\mathbf{X}_j, \mathbf{Y}_j) \right) \end{aligned}$$

$$\cdot \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{\mathbf{X}_k}^k \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}} \prod_{\substack{p=1 \\ \text{order}}}^n \psi_{\mathbf{Y}_p},$$

where the factor $\varepsilon_{\pm} \in \{1, -1\}$ depends only on $(m_i)_{i=1}^n$, $(k_i)_{i=1}^n$ and the constraint $\sum_{j=1}^n m_j - 2n + 2 \geq m$ is due to the fact that $\prod_{l \in T} \Delta_l$ erases $2n - 2$ Grassmann variables. By the uniqueness of anti-symmetric kernels we can characterize the kernel of $T_m^{(n)}(\psi)$ as follows. For any $\mathbf{Y} \in I^m$,

$$(3.18) \quad T_m^{(n)}(\mathbf{Y}) = \prod_{i=1}^n \left(\sum_{m_i=2}^N \sum_{k_i=0}^{m_i-1} \binom{m_i}{k_i} \sum_{\mathbf{Y}_i \in I^{k_i}} \right) \\ \cdot 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m} 1_{\sum_{j=1}^n k_j = m} \\ \cdot \frac{1}{m!} \sum_{\sigma \in \mathbb{S}_m} \text{sgn}(\sigma) 1_{\mathbf{Y}_\sigma = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n)} \frac{\varepsilon_{\pm}}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o) \\ \cdot \prod_{j=1}^n \left(\left(\frac{1}{\hbar} \right)^{m_j - k_j} \sum_{\mathbf{X}_j \in I^{m_j - k_j}} J_{m_j}(\mathbf{X}_j, \mathbf{Y}_j) \right) \\ \cdot \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{\mathbf{X}_k}^k \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}}.$$

If $m \geq 2$, by changing the numbering if necessary we can apply Lemma 3.7 (2) to (3.18) to deduce that

$$\|T_m^{(n)}\|_{l,a} \leq \prod_{i=1}^n \left(\sum_{m_i=2}^N \sum_{k_i=0}^{m_i-1} \binom{m_i}{k_i} \right) 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m} 1_{\sum_{j=1}^n k_j = m} \\ \cdot 2^{2 \sum_{j=1}^n m_j - 2m} D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j - \frac{m}{2} - n + 1} \\ \cdot \prod_{j=1}^n \left(\sum_{q_j=0}^1 \|J_{m_j}\|_{l,q_j} \right) \prod_{k=2}^n \left(\sum_{r_k=0}^1 \|\widetilde{C}_o\|_{l,r_k} \right) 1_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = a}.$$

Then, by substituting the inequality

$$\prod_{i=1}^n \left(\sum_{k_i=0}^{m_i-1} \binom{m_i}{k_i} \right) 1_{\sum_{j=1}^n k_j = m} \leq 2^{\sum_{j=1}^n m_j},$$

we obtain the inequality claimed in (2). By applying Lemma 3.7 (1) to (3.18) we can derive the inequality claimed in (1). \square

3.3. Invariance of Grassmann polynomials

Here we show that Grassmann polynomials produced by the free integration or the tree expansion inherit symmetric properties from the covariance C_o and the input polynomial $J(\psi)$. The general results summarized in this subsection will have practical applications in Section 7.

Let S be a bijective map from I to I and Q be a map from I to \mathbb{R} . For $m \in \mathbb{N}$, define $S_m : I^m \rightarrow I^m$, $Q_m : I^m \rightarrow \mathbb{R}$ by

$$\begin{aligned} S_m(X_1, X_2, \dots, X_m) &:= (S(X_1), S(X_2), \dots, S(X_m)), \\ Q_m(X_1, X_2, \dots, X_m) &:= \sum_{j=1}^m Q(X_j). \end{aligned}$$

For $X \in I$, set $(\mathcal{R}\psi)_X := e^{iQ(S(X))}\psi_{S(X)}$. For $f(\psi) \in \bigwedge \mathcal{V}$ we define $f(\mathcal{R}\psi) \in \bigwedge \mathcal{V}$ by replacing each ψ_X by $(\mathcal{R}\psi)_X$ ($X \in I$) inside $f(\psi)$. More precisely, for $f(\psi) = \sum_{m=0}^N \left(\frac{1}{\hbar}\right)^m \sum_{\mathbf{X} \in I^m} f_m(\mathbf{X})\psi_{\mathbf{X}}$,

$$f(\mathcal{R}\psi) := \sum_{m=0}^N \left(\frac{1}{\hbar}\right)^m \sum_{\mathbf{X} \in I^m} f_m(\mathbf{X}) e^{iQ_m(S_m(\mathbf{X}))} \psi_{S_m(\mathbf{X})}.$$

For $f(\psi) \in \bigwedge \mathcal{V}$ we define $\bar{f}(\psi) \in \bigwedge \mathcal{V}$ by

$$\bar{f}(\psi) := \sum_{m=0}^N \left(\frac{1}{\hbar}\right)^m \sum_{\mathbf{X} \in I^m} \overline{f_m(\mathbf{X})} \psi_{\mathbf{X}}.$$

LEMMA 3.9. *Let $F(\psi)$, $T^{(n)}(\psi) \in \bigwedge \mathcal{V}$ ($n \in \mathbb{N}_{\geq 2}$) be defined by (3.4) and the right-hand side of (3.6) respectively with the covariance $C_o : I_0^2 \rightarrow \mathbb{C}$ and the input $J(\psi)$ satisfying $J_m(\psi) = 0$ if $m \notin 2\mathbb{N} \cup \{0\}$. Let $\widetilde{C}_o : I^2 \rightarrow \mathbb{C}$ be the anti-symmetric extension of C_o defined by (3.2). Then, the following statements hold true.*

(1) *If*

$$J(\mathcal{R}\psi) = J(\psi), \quad \widetilde{C}_o(\mathbf{X}) = e^{iQ_2(S_2(\mathbf{X}))} \widetilde{C}_o(S_2(\mathbf{X})), \quad (\forall \mathbf{X} \in I^2),$$

then,

$$\begin{aligned} F(\mathcal{R}\psi) &= F(\psi), \\ T^{(n)}(\mathcal{R}\psi) &= T^{(n)}(\psi), \quad (\forall n \in \mathbb{N}_{\geq 2}). \end{aligned}$$

- (2) Let $a \in \mathbb{N}$ and D be a domain of \mathbb{C}^a satisfying that $\bar{\mathbf{z}} \in \bar{D}$ ($\forall \mathbf{z} \in \bar{D}$), where \bar{D} denotes the closure of D . Additionally assume that $J(\psi)$ and C_o are parameterized by $\mathbf{z} \in \bar{D}$ and write $J(\mathbf{z})(\psi)$, $C_o(\mathbf{z})$, $\widetilde{C}_o(\mathbf{z})$, $F(\mathbf{z})(\psi)$, $T^{(n)}(\mathbf{z})(\psi)$ in place of $J(\psi)$, C_o , \widetilde{C}_o , $F(\psi)$, $T^{(n)}(\psi)$ respectively. If

$$\begin{aligned} \overline{J(\bar{\mathbf{z}})}(\mathcal{R}\psi) &= J(\mathbf{z})(\psi), \quad \overline{\widetilde{C}_o(\mathbf{z})}(\mathbf{X}) = e^{-iQ_2(S_2(\mathbf{X}))} \overline{\widetilde{C}_o(\bar{\mathbf{z}})}(S_2(\mathbf{X})), \\ (\forall \mathbf{z} \in \bar{D}, \mathbf{X} \in I^2), \end{aligned}$$

then,

$$\begin{aligned} \overline{F(\bar{\mathbf{z}})}(\mathcal{R}\psi) &= F(\mathbf{z})(\psi), \\ \overline{T^{(n)}(\bar{\mathbf{z}})}(\mathcal{R}\psi) &= T^{(n)}(\mathbf{z})(\psi), \quad (\forall \mathbf{z} \in \bar{D}, n \in \mathbb{N}_{\geq 2}). \end{aligned}$$

PROOF. We provide the proof for the claim (2) first. The claim (1) can be proved similarly.

- (2): Let us show the invariance of $F(\mathbf{z})(\psi)$. Define $\widetilde{C}_o(\mathbf{z})' : I^2 \rightarrow \mathbb{C}$ by

$$\widetilde{C}_o(\mathbf{z})'(\mathbf{X}) := e^{iQ_2(\mathbf{X})} \widetilde{C}_o(\mathbf{z})(S_2^{-1}(\mathbf{X})).$$

It follows from the assumption that

$$(3.19) \quad \overline{\widetilde{C}_o(\bar{\mathbf{z}})'(\mathbf{X})} = \widetilde{C}_o(\mathbf{z})(\mathbf{X}), \quad (\forall \mathbf{z} \in \bar{D}, \mathbf{X} \in I^2).$$

Recalling the definition of the Grassmann Gaussian integral, we observe that for any $\mathbf{X} \in I^n$,

$$\begin{aligned} (3.20) \quad \int \psi_{\mathbf{X}}^1 d\mu_{C_o(\mathbf{z})}(\psi^1) &= e^{-\sum_{Y,Z \in I_0} C_o(\mathbf{z})(Y,Z) \frac{\partial}{\partial \psi_Y^1} \frac{\partial}{\partial \psi_Z^1} \psi_{\mathbf{X}}^1} \Big|_{\psi^1=0} \\ &= e^{-\sum_{\mathbf{Y} \in I^2} \widetilde{C}_o(\mathbf{z})(\mathbf{Y}) \frac{\partial}{\partial \psi_{\mathbf{Y}}^1} \psi_{\mathbf{X}}^1} \Big|_{\psi^1=0} \\ &= e^{-\sum_{\mathbf{Y} \in I^2} \widetilde{C}_o(\mathbf{z})'(\mathbf{Y}) \frac{\partial}{\partial \psi_{\mathbf{Y}}^1} e^{-iQ_n(S_n(\mathbf{X}))} \psi_{S_n(\mathbf{X})}^1} \Big|_{\psi^1=0}. \end{aligned}$$

Moreover, by (3.19),

$$(3.21) \quad \int \psi_{\mathbf{X}}^1 d\mu_{\overline{C_o(\bar{\mathbf{z}})}}(\psi^1) = e^{-\sum_{\mathbf{Y} \in I^2} \widetilde{C}_o(\mathbf{z})(\mathbf{Y}) \frac{\partial}{\partial \psi_{\mathbf{Y}}^1} e^{iQ_n(S_n(\mathbf{X}))} \psi_{S_n(\mathbf{X})}^1} \Big|_{\psi^1=0}$$

$$= \int e^{iQ_n(S_n(\mathbf{X}))} \psi_{S_n(\mathbf{X})}^1 d\mu_{C_o(\mathbf{z})}(\psi^1).$$

By using anti-symmetry we can characterize $F(\mathbf{z})(\psi)$ as follows.

$$(3.22) \quad \begin{aligned} F(\mathbf{z})(\psi) &= J(\mathbf{z})_0 + \sum_{m=1}^N \left(\frac{1}{h}\right)^m \sum_{\mathbf{X} \in I^m} \sum_{n=m}^N \binom{n}{m} \\ &\quad \cdot \left(\frac{1}{h}\right)^{n-m} \sum_{\mathbf{Y} \in I^{n-m}} J(\mathbf{z})_n(\mathbf{X}, \mathbf{Y}) \\ &\quad \cdot \int \psi_{\mathbf{Y}}^1 d\mu_{C_o(\mathbf{z})}(\psi^1) \psi_{\mathbf{X}}. \end{aligned}$$

By the invariance of $J(\mathbf{z})(\psi)$ and (3.21) we have that

$$\begin{aligned} &\overline{F(\bar{\mathbf{z}})}(\mathcal{R}\psi) \\ &= \overline{J(\bar{\mathbf{z}})}_0 + \sum_{m=1}^N \left(\frac{1}{h}\right)^m \sum_{\mathbf{X} \in I^m} \sum_{n=m}^N \binom{n}{m} \left(\frac{1}{h}\right)^{n-m} \sum_{\mathbf{Y} \in I^{n-m}} \overline{J(\bar{\mathbf{z}})}_n(\mathbf{X}, \mathbf{Y}) \\ &\quad \cdot \int e^{iQ_{n-m}(S_{n-m}(\mathbf{Y}))} \psi_{S_{n-m}(\mathbf{Y})}^1 d\mu_{C_o(\mathbf{z})}(\psi^1) e^{iQ_m(S_m(\mathbf{X}))} \psi_{S_m(\mathbf{X})} \\ &= \int \overline{J(\bar{\mathbf{z}})}(\mathcal{R}\psi + \mathcal{R}\psi^1) d\mu_{C_o(\mathbf{z})}(\psi^1) = \int J(\mathbf{z})(\psi + \psi^1) d\mu_{C_o(\mathbf{z})}(\psi^1) \\ &= F(\mathbf{z})(\psi). \end{aligned}$$

Next let us prove the invariance of $T^{(n)}(\mathbf{z})(\psi)$. Since $M_{at}(T, \xi, \mathbf{s})^t = M_{at}(T, \xi, \mathbf{s})$,

$$(3.23) \quad \begin{aligned} &\sum_{r,s=1}^n M_{at}(T, \xi, \mathbf{s})(r, s) \Delta_{r,s}(C_o(\mathbf{z})) \\ &= - \sum_{r,s=1}^n M_{at}(T, \xi, \mathbf{s})(r, s) \sum_{(X_1, X_2) \in I^2} \widetilde{C}_o(\mathbf{z})(X_1, X_2) \frac{\partial}{\partial \psi_{X_1}^r} \frac{\partial}{\partial \psi_{X_2}^s}. \end{aligned}$$

For an anti-symmetric function $A : I^2 \rightarrow \mathbb{C}$ and $T \in \mathbb{T}_n$ we define the operator $\widetilde{Ope}(T, A)$ by

$$\widetilde{Ope}(T, A) := \prod_{\{p,q\} \in T} \left(-2 \sum_{(X_1, X_2) \in I^2} A(X_1, X_2) \frac{\partial}{\partial \psi_{X_1}^p} \frac{\partial}{\partial \psi_{X_2}^q} \right)$$

$$\cdot \int_{[0,1]^{n-1}} d\mathbf{s} \sum_{\xi \in \mathbb{S}_n(T)} \varphi(T, \xi, \mathbf{s})$$

$$\cdot e^{-\sum_{r,s=1}^n \text{Mat}(T, \xi, \mathbf{s})(r, s) \sum_{(Y_1, Y_2) \in I^2} A(Y_1, Y_2) \frac{\partial}{\partial \psi_{Y_1}^r} \frac{\partial}{\partial \psi_{Y_2}^s}}.$$

The equalities (3.9), (3.23) ensure that

$$(3.24) \quad \widetilde{Ope}(T, \widetilde{C}_o(\mathbf{z})) = Ope(T, C_o(\mathbf{z})).$$

By the same argument as in (3.20) we have for any $m_j \in \{1, 2, \dots, N\}$, $\mathbf{X}_j \in I^{m_j}$ ($j = 1, 2, \dots, n$) that

$$\widetilde{Ope}(T, \widetilde{C}_o(\mathbf{z})) \prod_{\substack{j=1 \\ \text{order}}}^n \psi_{\mathbf{X}_j}^j \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}$$

$$= \widetilde{Ope}(T, \widetilde{C}_o(\mathbf{z})') \prod_{\substack{j=1 \\ \text{order}}}^n \left(e^{-iQ_{m_j}(S_{m_j}(\mathbf{X}_j))} \psi_{S_{m_j}(\mathbf{X}_j)}^j \right) \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}.$$

Thus, by (3.19),

$$(3.25) \quad \widetilde{Ope}(T, \overline{\widetilde{C}_o(\mathbf{z})}) \prod_{\substack{j=1 \\ \text{order}}}^n \psi_{\mathbf{X}_j}^j \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}$$

$$= \widetilde{Ope}(T, \widetilde{C}_o(\mathbf{z})) \prod_{\substack{j=1 \\ \text{order}}}^n \left(e^{iQ_{m_j}(S_{m_j}(\mathbf{X}_j))} \psi_{S_{m_j}(\mathbf{X}_j)}^j \right) \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}.$$

By using the invariance of $J(\mathbf{z})(\psi)$, (3.24) and (3.25) we can deduce from (3.17) that for any $m \in \{0, 1, \dots, N\}$,

$$\overline{T^{(n)}(\mathbf{z})_m}(\mathcal{R}\psi)$$

$$= \prod_{i=1}^n \left(\sum_{m_i=2}^N \sum_{k_i=0}^{m_i-1} \binom{m_i}{k_i} \left(\frac{1}{\hbar} \right)^{k_i} \sum_{\mathbf{Y}_i \in I^{k_i}} \right) \mathbf{1}_{\sum_{j=1}^n m_j - 2n + 2 \geq m} \mathbf{1}_{\sum_{j=1}^n k_j = m}$$

$$\cdot \frac{\varepsilon_{\pm}}{n!} \sum_{T \in \mathbb{T}_n} \widetilde{Ope}(T, \widetilde{C}_o(\mathbf{z})) \prod_{j=1}^n \left(\left(\frac{1}{\hbar} \right)^{m_j - k_j} \sum_{\mathbf{X}_j \in I^{m_j - k_j}} \overline{J(\mathbf{z})_{m_j}(\mathbf{X}_j, \mathbf{Y}_j)} \right)$$

$$\begin{aligned}
& \cdot e^{i \sum_{j=1}^n Q_{m_j - k_j}(S_{m_j - k_j}(\mathbf{X}_j))} \prod_{\substack{p=1 \\ \text{order}}}^n \psi_{S_{m_p - k_p}}^p(\mathbf{X}_p) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\
& \cdot e^{i \sum_{j=1}^n Q_{k_j}(S_{k_j}(\mathbf{Y}_j))} \prod_{\substack{q=1 \\ \text{order}}}^n \psi_{S_{k_q}}(\mathbf{Y}_q) \\
& = \mathcal{P}_m \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \widetilde{Ope}(T, \widetilde{C}_o(\mathbf{z})) \prod_{j=1}^n \overline{J(\bar{\mathbf{z}})}(\mathcal{R}\psi + \mathcal{R}\psi^j) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\
& = \mathcal{P}_m \frac{1}{n!} \sum_{T \in \mathbb{T}_n} Ope(T, C_o(\mathbf{z})) \prod_{j=1}^n J(\mathbf{z})(\psi + \psi^j) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\
& = T^{(n)}(\mathbf{z})_m(\psi),
\end{aligned}$$

which implies that $\overline{T^{(n)}(\bar{\mathbf{z}})}(\mathcal{R}\psi) = T^{(n)}(\mathbf{z})(\psi)$.

(1): It follows from the invariance of C_o that for any $\mathbf{X} \in I^n$,

$$\begin{aligned}
(3.26) \quad & e^{-\sum_{\mathbf{Y} \in I^2} \bar{C}_o(\mathbf{Y}) \frac{\partial}{\partial \psi_{\mathbf{Y}}^1}} \psi_{\mathbf{X}}^1 \Bigg|_{\psi^1=0} \\
& = e^{-\sum_{\mathbf{Y} \in I^2} \bar{C}_o(\mathbf{Y}) \frac{\partial}{\partial \psi_{\mathbf{Y}}^1}} e^{iQ_n(S_n(\mathbf{X}))} \psi_{S_n(\mathbf{X})}^1 \Bigg|_{\psi^1=0}.
\end{aligned}$$

We can see from the definition of the Grassmann Gaussian integral and (3.26) that for any $\mathbf{X} \in I^n$,

$$(3.27) \quad \int \psi_{\mathbf{X}}^1 d\mu_{C_o}(\psi^1) = \int e^{iQ_n(S_n(\mathbf{X}))} \psi_{S_n(\mathbf{X})}^1 d\mu_{C_o}(\psi^1).$$

By substituting (3.27) into (3.22) and using the invariance of $J(\psi)$ we obtain that $F(\psi) = F(\mathcal{R}\psi)$.

Using (3.24) and (3.26), we can prove that for any $m_j \in \{1, 2, \dots, N\}$, $\mathbf{X}_j \in I^{m_j}$ ($j = 1, 2, \dots, n$),

$$\begin{aligned}
(3.28) \quad & Ope(T, C_o) \prod_{\substack{j=1 \\ \text{order}}}^n \psi_{\mathbf{X}_j}^j \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\
& = Ope(T, C_o) \prod_{\substack{j=1 \\ \text{order}}}^n \left(e^{iQ_{m_j}(S_{m_j}(\mathbf{X}_j))} \psi_{S_{m_j}(\mathbf{X}_j)}^j \right) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}.
\end{aligned}$$

Then, by inserting (3.28) into (3.17) and using the invariance of $J(\psi)$ we can confirm that $T_m^{(n)}(\psi) = T_m^{(n)}(\mathcal{R}\psi)$ ($\forall m \in \{0, 1, \dots, N\}$), which implies that $T^{(n)}(\psi) = T^{(n)}(\mathcal{R}\psi)$. \square

4. General Estimation at Different Temperatures

In this section we estimate differences between 2 Grassmann polynomials produced by a single-scale integration at 2 different temperatures. One can prove that the free energy density is analytic with the coupling constants in a β -independent domain around the origin without measuring the differences between Grassmann polynomials created at different temperatures. However, in order to prove the existence of zero-temperature limit of the free energy density, we need the temperature-dependent estimates constructed in this section.

Let us set up notations which we start using from this section. Since we consider the problems at 2 different temperatures, we sometimes add the notation (β) to the right of a β -dependent object. For example, we write $I_0(\beta)$, $I(\beta)$ instead of the index sets I_0 , I when we want to indicate with which β these sets are defined.

Let us introduce the extended index sets $I_{0,\infty}$, I_∞ by

$$I_{0,\infty} := \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times \frac{1}{h}\mathbb{Z}, \quad I_\infty := I_{0,\infty} \times \{1, -1\}.$$

For any $x \in (1/h)\mathbb{Z}$ there uniquely exist $n_\beta(x) \in \mathbb{Z}$ and $r_\beta(x) \in [0, \beta)_h$ such that $x = n_\beta(x)\beta + r_\beta(x)$. For any

$$\mathbf{X} = ((\rho_1, \mathbf{x}_1, \sigma_1, x_1), (\rho_2, \mathbf{x}_2, \sigma_2, x_2), \dots, (\rho_m, \mathbf{x}_m, \sigma_m, x_m)) \in I_{0,\infty}^m$$

we define $R_\beta(\mathbf{X}) \in I_0^m$, $N_\beta(\mathbf{X}) \in \mathbb{Z}$ by

$$\begin{aligned} R_\beta(\mathbf{X}) &:= ((\rho_1, \mathbf{x}_1, \sigma_1, r_\beta(x_1)), (\rho_2, \mathbf{x}_2, \sigma_2, r_\beta(x_2)), \dots, \\ &\quad (\rho_m, \mathbf{x}_m, \sigma_m, r_\beta(x_m))), \\ N_\beta(\mathbf{X}) &:= \sum_{j=1}^m n_\beta(x_j). \end{aligned}$$

Moreover, for any $x \in (1/h)\mathbb{Z}$ let $\mathbf{X} + x \in I_{0,\infty}^m$ be defined by

$$\mathbf{X} + x := ((\rho_1, \mathbf{x}_1, \sigma_1, x_1 + x), (\rho_2, \mathbf{x}_2, \sigma_2, x_2 + x), \dots, (\rho_m, \mathbf{x}_m, \sigma_m, x_m + x)).$$

Similarly for any

$$\mathbf{X} = ((\rho_1, \mathbf{x}_1, \sigma_1, x_1, \theta_1), (\rho_2, \mathbf{x}_2, \sigma_2, x_2, \theta_2), \dots, (\rho_m, \mathbf{x}_m, \sigma_m, x_m, \theta_m)) \in I_\infty^m$$

we define $R_\beta(\mathbf{X}) \in I^m$, $N_\beta(\mathbf{X}) \in \mathbb{Z}$ by

$$\begin{aligned} R_\beta(\mathbf{X}) &:= ((\rho_1, \mathbf{x}_1, \sigma_1, r_\beta(x_1), \theta_1), (\rho_2, \mathbf{x}_2, \sigma_2, r_\beta(x_2), \theta_2), \dots, \\ &\quad (\rho_m, \mathbf{x}_m, \sigma_m, r_\beta(x_m), \theta_m)), \\ N_\beta(\mathbf{X}) &:= \sum_{j=1}^m n_\beta(x_j), \end{aligned}$$

though we must admit that these are abuse of notation. Also, for $x \in (1/h)\mathbb{Z}$ let

$$\begin{aligned} \mathbf{X} + x &:= ((\rho_1, \mathbf{x}_1, \sigma_1, x_1 + x, \theta_1), (\rho_2, \mathbf{x}_2, \sigma_2, x_2 + x, \theta_2), \dots, \\ &\quad (\rho_m, \mathbf{x}_m, \sigma_m, x_m + x, \theta_m)) \in I_\infty^m. \end{aligned}$$

Set

$$\Gamma_\infty := \left\{ \sum_{j=1}^d m_j \mathbf{u}_j \mid m_j \in \mathbb{Z} (j = 1, 2, \dots, d) \right\}.$$

Define the map r_L from Γ_∞ to Γ by

$$r_L \left(\sum_{j=1}^d m_j \mathbf{u}_j \right) := \sum_{j=1}^d m'_j \mathbf{u}_j,$$

where $m'_j \in \{0, 1, \dots, L-1\}$ and $m_j = m'_j$ in $\mathbb{Z}/L\mathbb{Z}$ ($\forall j \in \{1, 2, \dots, d\}$). In fact the notations Γ_∞ , $r_L(\cdot)$ are used only in Section 7 and Appendix D. However, it is systematic to introduce them at this stage together with other notations.

In this section we treat Grassmann polynomials whose anti-symmetric kernels $f_m : I(\beta)^m \rightarrow \mathbb{C}$ ($m \in \{1, 2, \dots, N\}$) satisfy that

$$(4.1) \quad f_m(\mathbf{X}) = (-1)^{N_\beta(\mathbf{X}+x)} f_m(R_\beta(\mathbf{X} + x)), \quad (\forall \mathbf{X} \in I(\beta)^m, x \in (1/h)\mathbb{Z}).$$

In our practical multi-scale integrations all relevant Grassmann polynomials will be proved to have the kernels satisfying (4.1).

From now until the end of this section we assume that

$$(4.2) \quad \beta_1, \beta_2 \in \mathbb{N}, \quad \beta_2 \geq \beta_1, \quad h \in 4\mathbb{N}.$$

It will eventually turn out that the condition (4.2) can be naturally imposed during the proof of the main theorem about the existence of zero-temperature limit of the free energy density in Section 7.

We introduce discrete versions of the intervals $[-\beta_1/4, \beta_1/4)$, $[\beta_1/4, \beta_a - \beta_1/4)$ ($a = 1, 2$) by

$$\begin{aligned} \left[-\frac{\beta_1}{4}, \frac{\beta_1}{4}\right)_h &:= \left\{-\frac{\beta_1}{4}, -\frac{\beta_1}{4} + \frac{1}{h}, \dots, \frac{\beta_1}{4} - \frac{1}{h}\right\}, \\ \left[\frac{\beta_1}{4}, \beta_a - \frac{\beta_1}{4}\right)_h &:= \left\{\frac{\beta_1}{4}, \frac{\beta_1}{4} + \frac{1}{h}, \dots, \beta_a - \frac{\beta_1}{4} - \frac{1}{h}\right\}, \quad (a = 1, 2). \end{aligned}$$

Note that $0 \in [-\beta_1/4, \beta_1/4)_h$ by the assumption $h \in 4\mathbb{N}$.

Define the index sets $\hat{I}_0, \hat{I}, I_0^0, I^0$ by

$$\begin{aligned} \hat{I}_0 &:= \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times \left[-\frac{\beta_1}{4}, \frac{\beta_1}{4}\right)_h, \quad \hat{I} := \hat{I}_0 \times \{1, -1\}, \\ I_0^0 &:= \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times \{0\}, \quad I^0 := I_0^0 \times \{1, -1\}. \end{aligned}$$

We assume that covariances $C_o(\beta_a) : I_0(\beta_a)^2 \rightarrow \mathbb{C}$ ($a = 1, 2$) are given and, as in Section 3, there exists a constant $D_{et} \in \mathbb{R}_{\geq 0}$ such that $C_o(\beta_a)$ ($a = 1, 2$) satisfy the determinant bound (3.1) with D_{et} . Moreover, assume that there is a β_1, β_2 -dependent constant $D(\beta_1, \beta_2) \in \mathbb{R}_{\geq 0}$ such that

$$(4.3) \quad \begin{aligned} &|\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_o(\beta_1)(R_{\beta_1}(X_i, Y_j)))_{1 \leq i, j \leq n} \\ &\quad - \det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_o(\beta_2)(R_{\beta_2}(X_i, Y_j)))_{1 \leq i, j \leq n}| \leq D(\beta_1, \beta_2) \cdot D_{et}^n, \\ &(\forall r, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} \leq 1, \\ &\quad X_i, Y_i \in \hat{I}_0 \ (i = 1, 2, \dots, n)). \end{aligned}$$

Furthermore, the covariances $C_o(\beta_a)$ ($a = 1, 2$) are assumed to satisfy that

$$(4.4) \quad \begin{aligned} C_o(\beta_a)(\mathbf{X}) &= (-1)^{N_{\beta_a}(\mathbf{X}+x)} C_o(\beta_a)(R_{\beta_a}(\mathbf{X}+x)), \\ &(\forall \mathbf{X} \in I_0(\beta_a)^2, x \in (1/h)\mathbb{Z}). \end{aligned}$$

For any $(\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi) \in \hat{I}, j \in \{0, 1, \dots, d\}$, set

$$\hat{d}_j((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi))$$

$$:= \begin{cases} |x - y| & \text{if } j = 0, \\ \frac{L}{2\pi} |e^{i\frac{2\pi}{L}\langle \mathbf{x}, \mathbf{v}_j \rangle} - e^{i\frac{2\pi}{L}\langle \mathbf{y}, \mathbf{v}_j \rangle}| & \text{if } j \in \{1, 2, \dots, d\}, \end{cases}$$

which is an analogue of $d_j(\cdot, \cdot)$ introduced in Section 3. Let $m \in \mathbb{N}_{\geq 2}$. For anti-symmetric functions $f_m(\beta_a) : I(\beta_a)^m \rightarrow \mathbb{C}$ ($a = 1, 2$) we estimate the difference between them by the quantities $|f_m(\beta_1) - f_m(\beta_2)|_l$ ($l \in \mathbb{Z}$) defined as follow.

$$\begin{aligned} |f_m(\beta_1) - f_m(\beta_2)|_l &:= \sup_{X \in I^0} \left(\frac{1}{h} \right)^{m-1} \sum_{\mathbf{Y} \in \hat{I}^{m-1}} e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(X, Y_1))^r} \\ &\quad \cdot |f_m(\beta_1)(R_{\beta_1}(X, \mathbf{Y})) - f_m(\beta_2)(R_{\beta_2}(X, \mathbf{Y}))|, \end{aligned}$$

where $\{w(l)\}_{l \in \mathbb{Z}} \subset \mathbb{R}_{>0}$ and $r \in (0, 1]$ are the same parameters as those used in the definitions of $\|\cdot\|_{l,0}$, $\|\cdot\|_{l,1}$ in Section 3.

By the assumption (4.2), $N(\beta_1)(= \sharp I(\beta_1)) \leq N(\beta_2)(= \sharp I(\beta_2))$. It will be convenient to write

$$f(\beta_1)(\psi) = \sum_{m=0}^{N(\beta_2)} f_m(\beta_1)(\psi)$$

for any $f(\beta_1)(\psi) \in \bigwedge \mathcal{V}(\beta_1)$ by admitting that $f_m(\beta_1)(\psi) = 0$ ($\forall m \in \{N(\beta_1) + 1, N(\beta_1) + 2, \dots, N(\beta_2)\}$).

4.1. Estimation of the free integration at different temperatures

As in Subsection 3.1 we set for $a = 1, 2$,

$$F(\beta_a)(\psi) := \int J(\beta_a)(\psi + \psi^1) d\mu_{C_o(\beta_a)}(\psi^1) \quad \left(\in \bigwedge \mathcal{V}(\beta_a) \right)$$

with $J(\beta_a)(\psi) \in \bigwedge \mathcal{V}(\beta_a)$ satisfying that $J_m(\beta_a)(\psi) = 0$ if $m \notin 2\mathbb{N} \cup \{0\}$ and having the anti-symmetric kernels satisfying (4.1). It follows from definition that $F_m(\beta_a)(\psi) = 0$ if $m \notin 2\mathbb{N} \cup \{0\}$. In this subsection we measure differences between $F(\beta_1)(\psi)$ and $F(\beta_2)(\psi)$. The result is the following.

LEMMA 4.1.

(1) For any $l \in \mathbb{Z}$,

$$\left| \frac{h}{N(\beta_1)} F_0(\beta_1) - \frac{h}{N(\beta_2)} F_0(\beta_2) \right|$$

$$\begin{aligned}
 &\leq \left| \frac{h}{N(\beta_1)} J_0(\beta_1) - \frac{h}{N(\beta_2)} J_0(\beta_2) \right| \\
 &\quad + \sum_{n=2}^{N(\beta_2)} 2^n D_{et}^{\frac{n}{2}} \left(|J_n(\beta_1) - J_n(\beta_2)|_l + D(\beta_1, \beta_2) \sum_{a=1}^2 \|J_n(\beta_a)\|_{l,0} \right. \\
 &\quad \quad \left. + \frac{2\pi}{\beta_1} \sum_{a=1}^2 \|J_n(\beta_a)\|_{l,1} \right).
 \end{aligned}$$

(2) For any $l \in \mathbb{Z}$, $m \in \mathbb{N}_{\geq 2}$,

$$\begin{aligned}
 &|F_m(\beta_1) - F_m(\beta_2)|_l \\
 &\leq |J_m(\beta_1) - J_m(\beta_2)|_l \\
 &\quad + \sum_{n=m+1}^{N(\beta_2)} 2^{2n} D_{et}^{\frac{n-m}{2}} \left(|J_n(\beta_1) - J_n(\beta_2)|_l + D(\beta_1, \beta_2) \sum_{a=1}^2 \|J_n(\beta_a)\|_{l,0} \right. \\
 &\quad \quad \left. + \frac{2\pi}{\beta_1} \sum_{a=1}^2 \|J_n(\beta_a)\|_{l,1} \right).
 \end{aligned}$$

PROOF. (1): By the property (4.4) and the definition of the Grassmann Gaussian integral we see that

$$\begin{aligned}
 (4.5) \quad &\int \psi_{\mathbf{X}} d\mu_{C_o(\beta_a)}(\psi) = (-1)^{N_{\beta_a}(\mathbf{X}+x)} \int \psi_{R_{\beta_a}(\mathbf{X}+x)} d\mu_{C_o(\beta_a)}(\psi), \\
 &(\forall \mathbf{X} \in I(\beta_a)^m, x \in (1/h)\mathbb{Z}, a \in \{1, 2\}).
 \end{aligned}$$

It follows from (3.5), (4.1) and (4.5) that

$$\begin{aligned}
 &F_0(\beta_a) \\
 &= J_0(\beta_a) + \sum_{n=2}^{N(\beta_2)} \left(\frac{1}{h}\right)^n \sum_{x \in [0, \beta_a)_h} \sum_{X \in I^0} \sum_{\mathbf{Y} \in I(\beta_a)^{n-1}} J_n(\beta_a)(X, R_{\beta_a}(\mathbf{Y} - x)) \\
 &\quad \cdot \int \psi_X^1 \psi_{R_{\beta_a}(\mathbf{Y}-x)}^1 d\mu_{C_o(\beta_a)}(\psi^1) \\
 &= J_0(\beta_a) + \beta_a \sum_{n=2}^{N(\beta_2)} \left(\frac{1}{h}\right)^{n-1} \sum_{X \in I^0} \sum_{\mathbf{Y} \in I(\beta_a)^{n-1}} J_n(\beta_a)(X, \mathbf{Y})
 \end{aligned}$$

$$\begin{aligned}
& \cdot \int \psi_{(X, \mathbf{Y})}^1 d\mu_{C_o(\beta_a)}(\psi^1) \\
= & J_0(\beta_a) + \beta_a \sum_{n=2}^{N(\beta_2)} \left(\frac{1}{h}\right)^{n-1} \sum_{X \in I^0} \sum_{\mathbf{Y} \in \hat{I}^{n-1}} J_n(\beta_a)(R_{\beta_a}(X, \mathbf{Y})) \\
& \cdot \int \psi_{R_{\beta_a}(X, \mathbf{Y})}^1 d\mu_{C_o(\beta_a)}(\psi^1) \\
+ & \beta_a \sum_{n=2}^{N(\beta_2)} \left(\frac{1}{h}\right)^{n-1} \sum_{X \in I^0} \sum_{\mathbf{Y} \in I(\beta_a)^{n-1}} \\
& \cdot \mathbf{1}_{\exists(\rho, \mathbf{x}, \sigma, x, \theta) \in I(\beta_a) \text{ s.t. } (\rho, \mathbf{x}, \sigma, x, \theta) \subset \mathbf{Y} \text{ and } x \in [\frac{\beta_1}{4}, \beta_a - \frac{\beta_1}{4}]_h} \\
& \cdot J_n(\beta_a)(X, \mathbf{Y}) \int \psi_{(X, \mathbf{Y})}^1 d\mu_{C_o(\beta_a)}(\psi^1).
\end{aligned}$$

Then, by using the determinant bounds (3.1), (4.3) we have

$$\begin{aligned}
& \left| \frac{h}{N(\beta_1)} F_0(\beta_1) - \frac{h}{N(\beta_2)} F_0(\beta_2) \right| \\
\leq & \left| \frac{h}{N(\beta_1)} J_0(\beta_1) - \frac{h}{N(\beta_2)} J_0(\beta_2) \right| \\
& + \sum_{n=2}^{N(\beta_2)} (|J_n(\beta_1) - J_n(\beta_2)|_l D_{et}^{\frac{n}{2}} + \|J_n(\beta_2)\|_{l,0} D(\beta_1, \beta_2) D_{et}^{\frac{n}{2}}) \\
& + \frac{2\pi}{\beta_1} \sum_{n=2}^{N(\beta_2)} (n-1) \sum_{a=1}^2 \|J_n(\beta_a)\|_{l,1} D_{et}^{\frac{n}{2}},
\end{aligned}$$

where we also used the inequalities that

$$(4.6) \quad \left| \frac{\beta_a}{2\pi} (e^{i\frac{2\pi}{\beta_a}x} - 1) \right| \geq \frac{\beta_1}{2\pi}, \quad (\forall x \in [\beta_1/4, \beta_a - \beta_1/4], a \in \{1, 2\})$$

and

$$1 \leq e^{\sum_{j=0}^d (w(l)d_j(X, Y_1))^r}, \quad 1 \leq e^{\sum_{j=0}^d (\frac{1}{\pi} w(l)\hat{d}_j(X, Y_1))^r}.$$

(2): We can see from (3.5) that for any $X \in I^0$, $\mathbf{Y} \in \hat{I}^{m-1}$,

$$\begin{aligned}
& F_m(\beta_a)(R_{\beta_a}(X, \mathbf{Y})) \\
& = J_m(\beta_a)(R_{\beta_a}(X, \mathbf{Y}))
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{n=m+1}^{N(\beta_2)} \left(\frac{1}{h}\right)^{n-m} \sum_{\mathbf{Z} \in \hat{I}^{n-m}} \binom{n}{m} J_n(\beta_a)(R_{\beta_a}(X, \mathbf{Y}, \mathbf{Z})) \\
 & \quad \cdot \int \psi_{R_{\beta_a}}^1(\mathbf{Z}) d\mu_{C_o(\beta_a)}(\psi^1) \\
 & + \sum_{n=m+1}^{N(\beta_2)} \left(\frac{1}{h}\right)^{n-m} \\
 & \quad \cdot \sum_{\mathbf{Z} \in I(\beta_a)^{n-m}} \mathbf{1}_{\exists(\rho, \mathbf{x}, \sigma, x, \theta) \in I(\beta_a) \text{ s.t. } (\rho, \mathbf{x}, \sigma, x, \theta) \subset \mathbf{Z} \text{ and } x \in [\frac{\beta_1}{4}, \beta_a - \frac{\beta_1}{4}]_h} \\
 & \quad \cdot \binom{n}{m} J_n(\beta_a)(R_{\beta_a}(X, \mathbf{Y}), \mathbf{Z}) \int \psi_{\mathbf{Z}}^1 d\mu_{C_o(\beta_a)}(\psi^1).
 \end{aligned}$$

By using (3.1), (4.3), (4.6) and the inequality that

$$\begin{aligned}
 (4.7) \quad & \frac{1}{\pi} |x - y| \leq \left| \frac{\beta_a}{2\pi} (e^{i\frac{2\pi}{\beta_a} r_{\beta_a}(x)} - e^{i\frac{2\pi}{\beta_a} r_{\beta_a}(y)}) \right|, \\
 & (\forall x, y \in [-\beta_1/4, \beta_1/4], a \in \{1, 2\}),
 \end{aligned}$$

we obtain

$$\begin{aligned}
 |F_m(\beta_1) - F_m(\beta_2)|_l & \leq |J_m(\beta_1) - J_m(\beta_2)|_l \\
 & + \sum_{n=m+1}^{N(\beta_2)} \binom{n}{m} \left(|J_n(\beta_1) - J_n(\beta_2)|_l D_{et}^{\frac{n-m}{2}} \right. \\
 & \quad \left. + \|J_n(\beta_2)\|_{l,0} D(\beta_1, \beta_2) D_{et}^{\frac{n-m}{2}} \right) \\
 & + \frac{2\pi}{\beta_1} \sum_{n=m+1}^{N(\beta_2)} (n-m) \binom{n}{m} \sum_{a=1}^2 \|J_n(\beta_a)\|_{l,1} D_{et}^{\frac{n-m}{2}}.
 \end{aligned}$$

Finally, by substituting the inequalities

$$\binom{n}{m}, (n-m) \binom{n}{m} \leq 2^{2n},$$

we reach the claimed inequality. \square

4.2. Estimation of the tree expansion at different temperatures

Here we estimate the differences between Grassmann polynomials produced by the tree expansion at 2 different temperatures. In the same style as in Subsection 3.2 we prepare necessary lemmas step by step. Our strategy is to decompose $T(\beta_a)(\psi)$ into 2 parts. One is a polynomial which integrates at least one time-variable away from 0 and β_a . The other is a polynomial which integrates only the time-variables close to 0 or β_a . We will find an upper bound on the first polynomial. We will measure the differences between the second polynomial at β_1 and that at β_2 . The next lemma is necessary to bound the first polynomials.

LEMMA 4.2. *Fix $a \in \{1, 2\}$. Take any $m_j \in \mathbb{N}_{\geq 2}$ ($j = 1, 2, \dots, n$). Let $J_{m_j}(\beta_a) : I(\beta_a)^{m_j} \rightarrow \mathbb{C}$ ($j = 1, 2, \dots, n$) be anti-symmetric functions. Then, the following inequalities hold.*

(1) *For any $X_{1,1} \in I^0$,*

$$\begin{aligned}
(4.8) \quad & \left| \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_a)) \right. \\
& \cdot \left(\frac{1}{h} \right)^{m_1-1} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1,m_1}) \in I(\beta_a)^{m_1-1}} J_{m_1}(\beta_a)(\mathbf{X}_1) \\
& \cdot \prod_{j=2}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in I(\beta_a)^{m_j}} J_{m_j}(\beta_a)(\mathbf{X}_j) \right) \\
& \cdot \mathbf{1}_{\exists(\rho, \mathbf{x}, \sigma, x, \theta) \in I(\beta_a) \text{ s.t. } (\rho, \mathbf{x}, \sigma, x, \theta) \subset (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) \text{ and } x \in [\frac{\beta_1}{4}, \beta_a - \frac{\beta_1}{4}]_h} \\
& \cdot \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{\mathbf{X}_k}^k \left| \begin{array}{c} \psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\}) \end{array} \right| \\
& \leq \frac{2\pi}{\beta_1} 2^{3 \sum_{j=1}^n m_j} D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} \\
& \cdot \prod_{j=1}^n \left(\sum_{q_j=0}^1 \|J_{m_j}(\beta_a)\|_{l, q_j} \right) \\
& \cdot \prod_{k=2}^n \left(\sum_{r_k=0}^1 \|\widetilde{C}_o(\beta_a)\|_{l, r_k} \right) \mathbf{1}_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = 1}.
\end{aligned}$$

(2) In addition, assume that $k_j \in \{0, 1, \dots, m_j - 1\}$ ($\forall j \in \{1, 2, \dots, n\}$), $p \in \{1, 2, \dots, n\}$ and $k_1, k_p \geq 1$. Then, for any $Y_{1,1} \in I^0$,

$$\begin{aligned}
 (4.9) \quad & \left| \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_a)) \right. \\
 & \cdot \left(\frac{1}{h} \right)^{m_1 - 1} \sum_{\mathbf{X}_1 \in I(\beta_a)^{m_1 - k_1}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1,k_1}) \in \hat{I}^{k_1 - 1}} \\
 & \cdot J_{m_1}(\beta_a)(\mathbf{X}_1, R_{\beta_a}(\mathbf{Y}_1)) \\
 & \cdot \prod_{j=2}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in I(\beta_a)^{m_j - k_j}} \sum_{\mathbf{Y}_j \in \hat{I}^{k_j}} J_{m_j}(\beta_a)(\mathbf{X}_j, R_{\beta_a}(\mathbf{Y}_j)) \right) \\
 & \cdot e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(Y_{1,1}, Y_{p,1}))^r} \\
 & \cdot \mathbf{1}_{\exists(\rho, \mathbf{x}, \sigma, x, \theta) \in I(\beta_a) \text{ s.t. } (\rho, \mathbf{x}, \sigma, x, \theta) \subset (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) \text{ and } x \in [\frac{\beta_1}{4}, \beta_a - \frac{\beta_1}{4}]_h} \\
 & \cdot \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{\mathbf{X}_k}^k \left| \begin{array}{c} \psi^{j=0} \\ (\forall j \in \{1, 2, \dots, n\}) \end{array} \right| \\
 & \leq \frac{2\pi}{\beta_1} 2^{3 \sum_{j=1}^n (m_j - k_j)} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\
 & \cdot \prod_{j=1}^n \left(\sum_{q_j=0}^1 \|J_{m_j}(\beta_a)\|_{l, q_j} \right) \\
 & \cdot \prod_{k=2}^n \left(\sum_{r_k=0}^1 \|\widetilde{C}_o(\beta_a)\|_{l, r_k} \right) \mathbf{1}_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = 1}.
 \end{aligned}$$

PROOF. We show the claim (2) first. The proof for the claim (1) is parallel to the proof of (2).

(2): Using (4.6), we can prove that

$$\begin{aligned}
 (4.10) \quad & \mathbf{1}_{\exists(\rho, \mathbf{x}, \sigma, x, \theta) \in I(\beta_a) \text{ s.t. } (\rho, \mathbf{x}, \sigma, x, \theta) \subset (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) \text{ and } x \in [\frac{\beta_1}{4}, \beta_a - \frac{\beta_1}{4}]_h} \\
 & \leq \frac{2\pi}{\beta_1} \sum_{q=1}^n \sum_{r=1}^{m_q - k_q} d_0(Y_{1,1}, X_{q,r}).
 \end{aligned}$$

We can deduce from Lemma 3.2, Lemma 3.3 and (4.10) that

$$\begin{aligned}
(4.11) \quad & \text{(the left-hand side of (4.9))} \\
& \leq \frac{2\pi}{\beta_1} \sum_{q=1}^n \sum_{r=1}^{m_q - k_q} \frac{1}{n!} \sum_{T \in \mathbb{T}_n} 1_{n_j(T) \leq m_j - k_j (\forall j \in \{1, 2, \dots, n\})} \\
& \quad \cdot 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\
& \quad \cdot \left(\frac{1}{h}\right)^{m_1 - 1} \sum_{\mathbf{X}_1 \in I(\beta_a)^{m_1 - k_1}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1,k_1}) \in \hat{I}^{k_1 - 1}} \\
& \quad \cdot |J_{m_1}(\beta_a)(\mathbf{X}_1, R_{\beta_a}(\mathbf{Y}_1))| \\
& \quad \cdot \sum_{\substack{\mathbf{w}_1 \subset \mathbf{X}_1 \\ \mathbf{w}_1 \in I(\beta_a)^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \\
& \quad \cdot \prod_{\{1,s\} \in L_1^1(T)} \left(\left(\frac{1}{h}\right)^{m_s} \sum_{\mathbf{X}_s \in I(\beta_a)^{m_s - k_s}} \sum_{\mathbf{Y}_s \in \hat{I}^{k_s}} \right. \\
& \quad \quad \cdot |J_{m_s}(\beta_a)(\mathbf{X}_s, R_{\beta_a}(\mathbf{Y}_s))| \\
& \quad \quad \cdot \left. \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I(\beta_a)}} |\widetilde{C}_o(\beta_a)(W_{1, \sigma_1 \circ \zeta_1(\{1,s\})}, Z_s)| \right) \\
& \quad \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2, 3, \dots, n\} \\ \text{dis}_T(1,j) = u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{w}_j \subset \mathbf{X}_j \setminus Z_j \\ \mathbf{w}_j \in I(\beta_a)^{n_j(T)-1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \right. \right. \\
& \quad \cdot \prod_{\{j,s\} \in L_j^1(T)} \left(\left(\frac{1}{h}\right)^{m_s} \sum_{\mathbf{X}_s \in I(\beta_a)^{m_s - k_s}} \sum_{\mathbf{Y}_s \in \hat{I}^{k_s}} \right. \\
& \quad \quad \cdot |J_{m_s}(\beta_a)(\mathbf{X}_s, R_{\beta_a}(\mathbf{Y}_s))| \\
& \quad \quad \cdot \left. \left. \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I(\beta_a)}} |\widetilde{C}_o(\beta_a)(W_{j, \sigma_j \circ \zeta_j(\{j,s\})}, Z_s)| \right) \right) \right) \\
& \quad \cdot d_0(Y_{1,1}, X_{q,r}) e^{\sum_{j=0}^d (w(l) d_j(Y_{1,1}, R_{\beta_a}(Y_{p,1})))^r},
\end{aligned}$$

where we also used (4.7) to justify the inequality

$$e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(Y, Y_{p,1}))^r} \leq e^{\sum_{j=0}^d (w(l) d_j(Y, R_{\beta_a}(Y_{p,1})))^r}.$$

We can estimate the right-hand side of (4.11) by straightforwardly following the argument in Subsection 3.2 leading to Lemma 3.7 (2). This procedure is summarized as follows.

$$\begin{aligned}
 (4.12) \quad & \text{(The right-hand side of (4.11))} \\
 & \leq \frac{2\pi}{\beta_1} \sum_{q=1}^n (m_q - k_q) \frac{1}{n!} \sum_{T \in \mathbb{T}_n} 1_{n_j(T) \leq m_j - k_j (\forall j \in \{1, 2, \dots, n\})} \\
 & \quad \cdot 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\
 & \quad \cdot \prod_{i=1}^n \left((m_i - k_i) \binom{m_i - k_i - 1}{n_i(T) - 1} (n_i(T) - 1)! \right) \\
 & \quad \cdot \prod_{j=1}^n \left(\sum_{q_j=0}^1 \|J_{m_j}(\beta_a)\|_{l, q_j} \right) \\
 & \quad \cdot \prod_{k=2}^n \left(\sum_{r_k=0}^1 \|\widetilde{C}_o(\beta_a)\|_{l, r_k} \right) 1_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = 1} \\
 & \leq \frac{2\pi}{\beta_1} 2^3 \sum_{j=1}^n (m_j - k_j) D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\
 & \quad \cdot \prod_{j=1}^n \left(\sum_{q_j=0}^1 \|J_{m_j}(\beta_a)\|_{l, q_j} \right) \\
 & \quad \cdot \prod_{k=2}^n \left(\sum_{r_k=0}^1 \|\widetilde{C}_o(\beta_a)\|_{l, r_k} \right) 1_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = 1}.
 \end{aligned}$$

To derive the first inequality we followed the proof of Lemma 3.4 (2). Then, we applied Lemma 3.6 to derive the second inequality.

(1): We let $X_{1,1} \in I^0$ play the same role as $Y_{1,1} \in I^0$ did in the proof of (2). Application of Lemma 3.3, (4.10) and repetition of the argument in Subsection 3.2 leading to Lemma 3.7 (2) ensure that

(the left-hand side of (4.8))

$$\begin{aligned}
&\leq \frac{2\pi}{\beta_1} \sum_{q=1}^n \sum_{r=1}^{m_q} \frac{1}{n!} \sum_{T \in \mathbb{T}_n} 1_{n_j(T) \leq m_j (\forall j \in \{1, 2, \dots, n\})} 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} \\
&\quad \cdot \left(\frac{1}{h} \right)^{m_1 - 1} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1,m_1}) \in I(\beta_a)^{m_1 - 1}} |J_{m_1}(\beta_a)(\mathbf{X}_1)| \\
&\quad \cdot \sum_{\substack{\mathbf{w}_1 \subset \mathbf{X}_1 \\ \mathbf{w}_1 \in I(\beta_a)^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \\
&\quad \cdot \prod_{\{1,s\} \in L_1^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I(\beta_a)^{m_s}} |J_{m_s}(\beta_a)(\mathbf{X}_s)| \right. \\
&\quad \quad \cdot \left. \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I(\beta_a)}} |\widetilde{C}_o(\beta_a)(W_{1, \sigma_1 \circ \zeta_1(\{1,s\})}, Z_s)| \right) \\
&\quad \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2, 3, \dots, n\} \text{ with} \\ \text{dis}_T(1,j) = u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{w}_j \subset \mathbf{X}_j \setminus Z_j \\ \mathbf{w}_j \in I(\beta_a)^{n_j(T)-1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \right. \right. \\
&\quad \cdot \prod_{\{j,s\} \in L_j^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in I(\beta_a)^{m_s}} |J_{m_s}(\beta_a)(\mathbf{X}_s)| \right. \\
&\quad \quad \cdot \left. \left. \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in I(\beta_a)}} |\widetilde{C}_o(\beta_a)(W_{j, \sigma_j \circ \zeta_j(\{j,s\})}, Z_s)| \right) \right) \\
&\quad \cdot d_0(X_{1,1}, X_{q,r}) \\
&\leq \frac{2\pi}{\beta_1} \sum_{q=1}^n m_q \frac{1}{n!} \sum_{T \in \mathbb{T}_n} 1_{n_j(T) \leq m_j (\forall j \in \{1, 2, \dots, n\})} 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} \\
&\quad \cdot \prod_{i=1}^n \left(m_i \binom{m_i - 1}{n_i(T) - 1} (n_i(T) - 1)! \right) \\
&\quad \cdot \prod_{j=1}^n \left(\sum_{q_j=0}^1 \|J_{m_j}(\beta_a)\|_{l, q_j} \right) \prod_{k=2}^n \left(\sum_{r_k=0}^1 \|\widetilde{C}_o(\beta_a)\|_{l, r_k} \right) 1_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = 1},
\end{aligned}$$

which is proved to be less than or equal to the right-hand side of (4.8) by Lemma 3.6. \square

Next we construct necessary lemmas to measure the differences between the polynomial containing only the time-integrals close to 0 or β_1 and that containing only the time-integrals close to 0 or β_2 .

LEMMA 4.3. *Take any $T \in \mathbb{T}_n$, $m_j \in \mathbb{N}$ and $\mathbf{X}_j \in \hat{I}^{m_j}$ ($j = 1, 2, \dots, n$). The following inequality holds.*

$$\begin{aligned} & \left| \text{ope}(T, C_o(\beta_1)) \prod_{\substack{j=1 \\ \text{order}}}^n \psi_{R_{\beta_1}}^j(\mathbf{X}_j) \right|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\ & - \text{ope}(T, C_o(\beta_2)) \prod_{\substack{j=1 \\ \text{order}}}^n \psi_{R_{\beta_2}}^j(\mathbf{X}_j) \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\ & \leq D(\beta_1, \beta_2) D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j}. \end{aligned}$$

PROOF. The result follows from the equality (3.8), the properties of the matrix $M_{at}(T, \xi, \mathbf{s})$ and the assumption (4.3). \square

Here let us introduce a couple of notations to organize formulas. Let $m \in \{2, 3, \dots, N(\beta_2)\}$. For anti-symmetric functions $J_m(\beta_a) : I(\beta_a)^m \rightarrow \mathbb{C}$ ($a = 1, 2$), $\mathbf{X} \in \hat{I}^m$, $i, j \in \mathbb{N}$ and $l \in \mathbb{Z}$, set

$$\begin{aligned} J_m^{(i,j)}(\mathbf{X}) & := \begin{cases} J_m(\beta_1)(R_{\beta_1}(\mathbf{X})) - J_m(\beta_2)(R_{\beta_2}(\mathbf{X})) & \text{if } i = j, \\ J_m(\beta_1)(R_{\beta_1}(\mathbf{X})) & \text{if } i < j, \\ J_m(\beta_2)(R_{\beta_2}(\mathbf{X})) & \text{if } i > j, \end{cases} \\ A(J_m(\beta_1), J_m(\beta_2))_l^{(i,j)} & := \begin{cases} |J_m(\beta_1) - J_m(\beta_2)|_l + \frac{2\pi}{\beta_1}(m-1) \sum_{a=1}^2 \|J_m(\beta_a)\|_{l,1} & \text{if } i = j, \\ \sum_{a=1}^2 \|J_m(\beta_a)\|_{l,0} & \text{if } i \neq j. \end{cases} \end{aligned}$$

LEMMA 4.4. *Take any $T \in \mathbb{T}_n$, $m_j \in \mathbb{N}$ and $\mathbf{X}_j \in \hat{I}^{m_j}$ ($j = 1, 2, \dots, n$). The following inequality holds.*

$$(4.13) \quad \left| \text{Ope}(T, C_o(\beta_1)) \prod_{\substack{j=1 \\ \text{order}}}^n \psi_{R_{\beta_1}}^j(\mathbf{X}_j) \right|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}$$

$$\begin{aligned}
 & \cdot \left| \text{ope}(T, C_o(\beta_1)) \psi_{R_{\beta_1}}^1(\mathbf{X}_1 \setminus \mathbf{W}_1) \right. \\
 & \cdot \prod_{\substack{k=2 \\ \text{order}}}^n \left(1_{n_k(T) \neq 1} \psi_{R_{\beta_1}}^k((\mathbf{X}_k \setminus Z_k) \setminus \mathbf{W}_k) + 1_{n_k(T)=1} \psi_{R_{\beta_1}}^k(\mathbf{X}_k \setminus Z_k) \right) \left. \right|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\
 & - \text{ope}(T, C_o(\beta_2)) \psi_{R_{\beta_2}}^1(\mathbf{X}_1 \setminus \mathbf{W}_1) \\
 & \cdot \prod_{\substack{k=2 \\ \text{order}}}^n \left(1_{n_k(T) \neq 1} \psi_{R_{\beta_2}}^k((\mathbf{X}_k \setminus Z_k) \setminus \mathbf{W}_k) \right. \\
 & \quad \left. + 1_{n_k(T)=1} \psi_{R_{\beta_2}}^k(\mathbf{X}_k \setminus Z_k) \right) \left. \right|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\
 & + 1_{n_j(T) \leq m_j (\forall j \in \{1, 2, \dots, n\})} 2^{n-1} \\
 & \cdot \sum_{\substack{\mathbf{W}_1 \subset \mathbf{X}_1 \\ \mathbf{W}_1 \in \tilde{I}^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \prod_{\{1, s\} \in L_1^1(T)} \left(\sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in \tilde{I}}} \right) \\
 & \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2, 3, \dots, n\} \\ \text{dis}_T(1, j) = u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{W}_j \subset \mathbf{X}_j \setminus Z_s \\ \mathbf{W}_j \in \tilde{I}^{n_j(T)-1}}} \right) \right. \\
 & \cdot \left. \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \prod_{\{j, s\} \in L_j^1(T)} \left(\sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in \tilde{I}}} \right) \right) \\
 & \cdot \left| \prod_{\substack{j \in \{1, 2, \dots, n\} \\ \text{with } n_j(T) \neq 1}} \prod_{\{j, s\} \in L_j^1(T)} \widetilde{C}_o(\beta_1)(R_{\beta_1}(W_{j, \sigma_j \circ \zeta_j(\{j, s\})}, Z_s)) \right. \\
 & - \prod_{\substack{j \in \{1, 2, \dots, n\} \\ \text{with } n_j(T) \neq 1}} \prod_{\{j, s\} \in L_j^1(T)} \widetilde{C}_o(\beta_2)(R_{\beta_2}(W_{j, \sigma_j \circ \zeta_j(\{j, s\})}, Z_s)) \left. \right| \\
 & \cdot \left| \text{ope}(T, C_o(\beta_2)) \psi_{R_{\beta_2}}^1(\mathbf{X}_1 \setminus \mathbf{W}_1) \right.
 \end{aligned}$$

$$\begin{aligned}
& \cdot \prod_{\substack{k=2 \\ \text{order}}}^n \left(1_{n_k(T) \neq 1} \psi_{R_{\beta_2}}^k((\mathbf{X}_k \setminus Z_k) \setminus \mathbf{W}_k) + 1_{n_k(T) = 1} \psi_{R_{\beta_2}}^k(\mathbf{X}_k \setminus Z_k) \right) \Bigg|_{\substack{\psi^j = 0 \\ (\forall j \in \{1, 2, \dots, n\})}} \Bigg| \\
& \leq 1_{n_j(T) \leq m_j (\forall j \in \{1, 2, \dots, n\})} 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} D(\beta_1, \beta_2) \\
& \cdot \sum_{\substack{\mathbf{W}_1 \subset \mathbf{X}_1 \\ \mathbf{W}_1 \in \tilde{I}^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \prod_{\{1, s\} \in L_1^1(T)} \left(\sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in \tilde{I}}} |\widetilde{C}_o(\beta_1)(R_{\beta_1}(W_{1, \sigma_1 \circ \zeta_1}(\{1, s\}), Z_s))| \right) \\
& \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2, 3, \dots, n\} \\ \text{dis}_T(1, j) = u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{W}_j \subset \mathbf{X}_j \setminus Z_j \\ \mathbf{W}_j \in \tilde{I}^{n_j(T)-1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \right) \right. \\
& \cdot \left. \prod_{\{j, s\} \in L_j^1(T)} \left(\sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in \tilde{I}}} |\widetilde{C}_o(\beta_1)(R_{\beta_1}(W_{j, \sigma_j \circ \zeta_j}(\{j, s\}), Z_s))| \right) \right) \\
& + 1_{n_j(T) \leq m_j (\forall j \in \{1, 2, \dots, n\})} 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} \sum_{v=2}^n \\
& \cdot \sum_{\substack{\mathbf{W}_1 \subset \mathbf{X}_1 \\ \mathbf{W}_1 \in \tilde{I}^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \prod_{\{1, s\} \in L_1^1(T)} \left(\sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in \tilde{I}}} |\widetilde{C}_o^{(v, s)}(W_{1, \sigma_1 \circ \zeta_1}(\{1, s\}), Z_s))| \right) \\
& \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2, 3, \dots, n\} \\ \text{dis}_T(1, j) = u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{W}_j \subset \mathbf{X}_j \setminus Z_j \\ \mathbf{W}_j \in \tilde{I}^{n_j(T)-1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \right) \right. \\
& \cdot \left. \prod_{\{j, s\} \in L_j^1(T)} \left(\sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in \tilde{I}}} |\widetilde{C}_o^{(v, s)}(W_{j, \sigma_j \circ \zeta_j}(\{j, s\}), Z_s))| \right) \right),
\end{aligned}$$

which is equal to the right-hand side of (4.13). \square

LEMMA 4.5. *Take any $m_j \in \{2, 3, \dots, N(\beta_2)\}$ ($j = 1, 2, \dots, n$). Let $J_{m_j}(\beta_a) : I(\beta_a)^{m_j} \rightarrow \mathbb{C}$ ($j = 1, 2, \dots, n$, $a = 1, 2$) be anti-symmetric functions. Then, the following inequalities hold.*

- (1) For any $X_{1,1} \in I^0$,

$$\begin{aligned}
 & \left| \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_1)) \right. \\
 & \quad \cdot \left(\frac{1}{h} \right)^{m_1-1} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1,m_1}) \in \hat{I}^{m_1-1}} J_{m_1}(\beta_1)(R_{\beta_1}(\mathbf{X}_1)) \\
 & \quad \cdot \prod_{j=2}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in \hat{I}^{m_j}} J_{m_j}(\beta_1)(R_{\beta_1}(\mathbf{X}_j)) \right) \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{R_{\beta_1}}^k(\mathbf{x}_k) \left. \right|_{(\forall j \in \{1, 2, \dots, n\})}^{\psi^j=0} \\
 & - \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_2)) \\
 & \quad \cdot \left(\frac{1}{h} \right)^{m_1-1} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1,m_1}) \in \hat{I}^{m_1-1}} J_{m_1}(\beta_2)(R_{\beta_2}(\mathbf{X}_1)) \\
 & \quad \cdot \prod_{j=2}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in \hat{I}^{m_j}} J_{m_j}(\beta_2)(R_{\beta_2}(\mathbf{X}_j)) \right) \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{R_{\beta_2}}^k(\mathbf{x}_k) \left. \right|_{(\forall j \in \{1, 2, \dots, n\})}^{\psi^j=0} \\
 & \leq 2^{2 \sum_{j=1}^n m_j} D_{\text{et}}^{\frac{1}{2} \sum_{j=1}^n m_j - n + 1} \left(\|\widetilde{C}_o(\beta_1)\|_{l,0}^{n-1} \sum_{v=1}^n \prod_{j=1}^n A(J_{m_j}(\beta_1), J_{m_j}(\beta_2))_l^{(v,j)} \right. \\
 & \quad + \sum_{v=2}^n \prod_{j=2}^n A(\widetilde{C}_o(\beta_1), \widetilde{C}_o(\beta_2))_l^{(v,j)} \prod_{k=1}^n \|J_{m_k}(\beta_2)\|_{l,0} \\
 & \quad \left. + D(\beta_1, \beta_2) \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0}^{n-1} \prod_{j=1}^n \|J_{m_j}(\beta_2)\|_{l,0} \right).
 \end{aligned}$$

(2) In addition, assume that $k_j \in \{0, 1, \dots, m_j - 1\}$ ($\forall j \in \{1, 2, \dots, n\}$), $p \in \{1, 2, \dots, n\}$ and $k_1, k_p \geq 1$. Then, for any $Y_{1,1} \in I^0$,

$$\begin{aligned}
 (4.14) \quad & \left| \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_1)) \right. \\
 & \quad \cdot \left(\frac{1}{h} \right)^{m_1-1} \sum_{\mathbf{X}_1 \in \hat{I}^{m_1-k_1}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1,k_1}) \in \hat{I}^{k_1-1}} \\
 & \quad \cdot J_{m_1}(\beta_1)(R_{\beta_1}(\mathbf{X}_1, \mathbf{Y}_1))
 \end{aligned}$$

$$\begin{aligned}
& \cdot \prod_{j=2}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in \hat{I}^{m_j - k_j}} \sum_{\mathbf{Y}_j \in \hat{I}^{k_j}} J_{m_j}(\beta_1)(R_{\beta_1}(\mathbf{X}_j, \mathbf{Y}_j)) \right) \\
& \cdot e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(Y_{1,1}, Y_{p,1}))^r} \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{R_{\beta_1}}^k(\mathbf{X}_k) \Bigg|_{(\forall j \in \{1, 2, \dots, n\})}^{\psi^j=0} \\
& - \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_2)) \\
& \cdot \left(\frac{1}{h} \right)^{m_1 - 1} \sum_{\mathbf{X}_1 \in \hat{I}^{m_1 - k_1}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1,k_1}) \in \hat{I}^{k_1 - 1}} \\
& \cdot J_{m_1}(\beta_2)(R_{\beta_2}(\mathbf{X}_1, \mathbf{Y}_1)) \\
& \cdot \prod_{j=2}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in \hat{I}^{m_j - k_j}} \sum_{\mathbf{Y}_j \in \hat{I}^{k_j}} J_{m_j}(\beta_2)(R_{\beta_2}(\mathbf{X}_j, \mathbf{Y}_j)) \right) \\
& \cdot e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(Y_{1,1}, Y_{p,1}))^r} \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{R_{\beta_2}}^k(\mathbf{X}_k) \Bigg|_{(\forall j \in \{1, 2, \dots, n\})}^{\psi^j=0} \\
& \leq 2^{2 \sum_{j=1}^n (m_j - k_j)} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\
& \cdot \left(\|\widetilde{C}_o(\beta_1)\|_{l,0}^{n-1} \sum_{v=1}^n \prod_{j=1}^n A(J_{m_j}(\beta_1), J_{m_j}(\beta_2))_l^{(v,j)} \right. \\
& \quad + \sum_{v=2}^n \prod_{j=2}^n A(\widetilde{C}_o(\beta_1), \widetilde{C}_o(\beta_2))_l^{(v,j)} \prod_{k=1}^n \|J_{m_k}(\beta_2)\|_{l,0} \\
& \quad \left. + D(\beta_1, \beta_2) \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0}^{n-1} \prod_{j=1}^n \|J_{m_j}(\beta_2)\|_{l,0} \right).
\end{aligned}$$

PROOF. We present the proof of (2) first. The claim (1) can be proved similarly.

(2): Note that

(4.15) (the left-hand side of (4.14))

$$\leq \sum_{v=1}^n \left| \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_1)) \right|$$

$$\begin{aligned}
 & \cdot \left(\frac{1}{h}\right)^{m_1-1} \sum_{\mathbf{X}_1 \in \hat{I}^{m_1-k_1}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1,k_1}) \in \hat{I}^{k_1-1}} J_{m_1}^{(v,1)}(\mathbf{X}_1, \mathbf{Y}_1) \\
 & \cdot \prod_{j=2}^n \left(\left(\frac{1}{h}\right)^{m_j} \sum_{\mathbf{X}_j \in \hat{I}^{m_j-k_j}} \sum_{\mathbf{Y}_j \in \hat{I}^{k_j}} J_{m_j}^{(v,j)}(\mathbf{X}_j, \mathbf{Y}_j) \right) \\
 & \cdot e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(Y_{1,1}, Y_{p,1}))^r} \\
 & \cdot \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{R_{\beta_1}}^k(\mathbf{x}_k) \left| \begin{array}{c} \psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\}) \end{array} \right| \\
 & + \left| \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \right. \\
 & \cdot \left(\frac{1}{h}\right)^{m_1-1} \sum_{\mathbf{X}_1 \in \hat{I}^{m_1-k_1}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1,k_1}) \in \hat{I}^{k_1-1}} \\
 & \cdot J_{m_1}(\beta_2)(R_{\beta_2}(\mathbf{X}_1, \mathbf{Y}_1)) \\
 & \cdot \prod_{j=2}^n \left(\left(\frac{1}{h}\right)^{m_j} \sum_{\mathbf{X}_j \in \hat{I}^{m_j-k_j}} \sum_{\mathbf{Y}_j \in \hat{I}^{k_j}} J_{m_j}(\beta_2)(R_{\beta_2}(\mathbf{X}_j, \mathbf{Y}_j)) \right) \\
 & \cdot e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(Y_{1,1}, Y_{p,1}))^r} \\
 & \cdot \left(\text{Ope}(T, C_o(\beta_1)) \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{R_{\beta_1}}^k(\mathbf{x}_k) \left| \begin{array}{c} \psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\}) \end{array} \right| \right. \\
 & \quad \left. - \text{Ope}(T, C_o(\beta_2)) \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{R_{\beta_2}}^k(\mathbf{x}_k) \left| \begin{array}{c} \psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\}) \end{array} \right| \right).
 \end{aligned}$$

By the same procedure leading to the proof of Lemma 3.7 (2) we have that

$$\begin{aligned}
 (4.16) \quad & \text{(the first term in the right-hand side of (4.15))} \\
 & \leq 2^{2 \sum_{j=1}^n (m_j - k_j)} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\
 & \cdot \left(\sup_{X \in \hat{I}} \frac{1}{h} \sum_{Y \in \hat{I}} e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(X, Y))^r} |\widetilde{C}_o(\beta_1)(R_{\beta_1}(X, Y))| \right)^{n-1}
 \end{aligned}$$

$$\cdot \sum_{v=1}^n \prod_{i=1}^n \left(\sup_{X \in \hat{I}} \left(\frac{1}{h} \right)^{m_i-1} \sum_{\mathbf{Y} \in \hat{I}^{m_i-1}} e^{\sum_{j=0}^d (\frac{1}{\pi} \mathbf{w}(l) \hat{d}_j(X, Y_1))^r} \cdot |J_{m_i}^{(v,i)}(X, \mathbf{Y})| \right).$$

It follows from (4.7) that

$$(4.17) \quad \sup_{X \in \hat{I}} \frac{1}{h} \sum_{Y \in \hat{I}} e^{\sum_{j=0}^d (\frac{1}{\pi} \mathbf{w}(l) \hat{d}_j(X, Y))^r} |\widetilde{C}_o(\beta_1)(R_{\beta_1}(X, Y))| \leq \|\widetilde{C}_o(\beta_1)\|_{l,0},$$

$$\sup_{X \in \hat{I}} \left(\frac{1}{h} \right)^{m_i-1} \sum_{\mathbf{Y} \in \hat{I}^{m_i-1}} e^{\sum_{j=0}^d (\frac{1}{\pi} \mathbf{w}(l) \hat{d}_j(X, Y_1))^r} |J_{m_i}^{(v,i)}(X, \mathbf{Y})|$$

$$\leq \sum_{a=1}^2 \|J_{m_i}(\beta_a)\|_{l,0} = A(J_{m_i}(\beta_1), J_{m_i}(\beta_2))_l^{(v,i)},$$

if $v \neq i$. Note that for any $x, y \in [-\beta_1/4, \beta_1/4]_h$,

$$r_{\beta_a}(r_{\beta_a}(y) - x) = r_{\beta_a}(y - x), \quad (\forall a \in \{1, 2\}),$$

$$n_{\beta_1}(r_{\beta_1}(y) - x) = n_{\beta_2}(r_{\beta_2}(y) - x).$$

By these equalities, (4.1), (4.6) and (4.7), for any $X = (\rho, \mathbf{x}, \sigma, x, \theta) \in \hat{I}$,

$$(4.18) \quad \left(\frac{1}{h} \right)^{m_v-1} \sum_{\mathbf{Y} \in \hat{I}^{m_v-1}} e^{\sum_{j=0}^d (\frac{1}{\pi} \mathbf{w}(l) \hat{d}_j(X, Y_1))^r} |J_{m_v}^{(v,v)}(X, \mathbf{Y})|$$

$$= \left(\frac{1}{h} \right)^{m_v-1} \sum_{Y=(\eta, \mathbf{y}, \tau, y, \xi) \in \hat{I}} \sum_{\mathbf{W} \in \hat{I}^{m_v-2}} e^{\sum_{j=0}^d (\frac{1}{\pi} \mathbf{w}(l) \hat{d}_j(X, Y))^r}$$

$$\cdot |J_{m_v}(\beta_1)(R_{\beta_1}(X, Y, \mathbf{W})) - J_{m_v}(\beta_2)(R_{\beta_2}(X, Y, \mathbf{W}))|$$

$$= \left(\frac{1}{h} \right)^{m_v-1} \sum_{Y=(\eta, \mathbf{y}, \tau, y, \xi) \in \hat{I}} \sum_{\mathbf{W} \in \hat{I}^{m_v-2}} e^{\sum_{j=0}^d (\frac{1}{\pi} \mathbf{w}(l) \hat{d}_j(X, Y))^r}$$

$$\cdot |J_{m_v}(\beta_1)(X - x, R_{\beta_1}((Y, \mathbf{W}) - x))$$

$$\quad - J_{m_v}(\beta_2)(X - x, R_{\beta_2}((Y, \mathbf{W}) - x))|$$

$$\cdot (1_{(Y, \mathbf{W}) - x \in \hat{I}^{m_v-1}} + 1_{(Y, \mathbf{W}) - x \notin \hat{I}^{m_v-1}})$$

$$\leq |J_{m_v}(\beta_1) - J_{m_v}(\beta_2)|_l$$

$$\begin{aligned}
 & + \sum_{a=1}^2 \left(\frac{1}{\hbar}\right)^{m_v-1} \sum_{Y=(\eta, \mathbf{y}, \tau, y, \xi) \in \hat{I}^{m_v-2}} \sum_{\mathbf{W} \in \hat{I}^{m_v-2}} \\
 & \cdot e^{\sum_{j=0}^d (w(l) d_j(\beta_a)(X-x, R_{\beta_a}(Y-x)))^r} \\
 & \cdot |J_{m_v}(\beta_a)(X-x, R_{\beta_a}((Y, \mathbf{W})-x))| 1_{(Y, \mathbf{W})-x \notin \hat{I}^{m_v-1}} \\
 & \leq |J_{m_v}(\beta_1) - J_{m_v}(\beta_2)|_l + \frac{2\pi}{\beta_1} (m_v - 1) \sum_{a=1}^2 \|J_{m_v}(\beta_a)\|_{l,1} \\
 & = A(J_{m_v}(\beta_1), J_{m_v}(\beta_2))_l^{(v,v)}.
 \end{aligned}$$

Substitution of (4.17), (4.18) into (4.16) gives

$$\begin{aligned}
 (4.19) \quad & \text{(the first term in the right-hand side of (4.15))} \\
 & \leq 2^{2 \sum_{j=1}^n (m_j - k_j)} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \|\widetilde{C}_o(\beta_1)\|_{l,0}^{n-1} \\
 & \cdot \sum_{v=1}^n \prod_{i=1}^n A(J_{m_i}(\beta_1), J_{m_i}(\beta_2))_l^{(v,i)}.
 \end{aligned}$$

On the other hand, by applying Lemma 4.4 we have that

$$\begin{aligned}
 & \text{(the second term in the right-hand side of (4.15))} \\
 & \leq \sum_{v=1}^n (1_{v=1} D(\beta_1, \beta_2) + 1_{v \geq 2}) \frac{1}{n!} \sum_{T \in \mathbb{T}_n} 1_{n_j(T) \leq m_j - k_j (\forall j \in \{1, 2, \dots, n\})} \\
 & \cdot 2^{n-1} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\
 & \cdot \left(\frac{1}{\hbar}\right)^{m_1-1} \sum_{\mathbf{X}_1 \in \hat{I}^{m_1-k_1}} \sum_{(Y_{1,2}, Y_{1,3}, \dots, Y_{1,k_1}) \in \hat{I}^{k_1-1}} |J_{m_1}(\beta_2)(R_{\beta_2}(\mathbf{X}_1, \mathbf{Y}_1))| \\
 & \cdot \sum_{\substack{\mathbf{W}_1 \subset \mathbf{X}_1 \\ \mathbf{W}_1 \in \hat{I}^{n_1(T)}}} \sum_{\sigma_1 \in \mathbb{S}_{n_1(T)}} \\
 & \cdot \prod_{\{1,s\} \in L_1^+(T)} \left(\left(\frac{1}{\hbar}\right)^{m_s} \sum_{\mathbf{X}_s \in \hat{I}^{m_s-k_s}} \sum_{\mathbf{Y}_s \in \hat{I}^{k_s}} |J_{m_s}(\beta_2)(R_{\beta_2}(\mathbf{X}_s, \mathbf{Y}_s))| \right. \\
 & \quad \left. \cdot \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in \hat{I}}} |\widetilde{C}_o^{(v,s)}(W_{1, \sigma_1 \circ \zeta_1}(\{1,s\}), Z_s)| \right)
 \end{aligned}$$

$$\begin{aligned}
& \cdot \prod_{\substack{u=1 \\ \text{order}}}^{d_T(1)-1} \left(\prod_{\substack{j \in \{2,3,\dots,n\} \text{ with} \\ \text{dis}_T(1,j)=u, n_j(T) \neq 1}} \left(\sum_{\substack{\mathbf{w}_j \subset \mathbf{X}_j \setminus Z_j \\ \mathbf{w}_j \in \hat{I}^{n_j(T)-1}}} \sum_{\sigma_j \in \mathbb{S}_{n_j(T)-1}} \right. \right. \\
& \cdot \prod_{\{j,s\} \in L_j^1(T)} \left(\left(\frac{1}{h} \right)^{m_s} \sum_{\mathbf{X}_s \in \hat{I}^{m_s-k_s}} \sum_{\mathbf{Y}_s \in \hat{I}^{k_s}} |J_{m_s}(\beta_2)(R_{\beta_2}(\mathbf{X}_s, \mathbf{Y}_s))| \right. \\
& \quad \cdot \left. \left. \sum_{\substack{Z_s \subset \mathbf{X}_s \\ Z_s \in \hat{I}}} |\widetilde{C}_o^{(v,s)}(W_{j,\sigma_j \circ \zeta_j(\{j,s\}), Z_s)| \right) \right) \Bigg) \\
& \cdot e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(Y_{1,1}, Y_{p,1}))^r}.
\end{aligned}$$

Moreover, by following the argument leading to Lemma 3.7 (2) we can deduce that

$$\begin{aligned}
(4.20) \quad & \text{(the second term in the right-hand side of (4.15))} \\
& \leq \sum_{v=1}^n (1_{v=1} D(\beta_1, \beta_2) + 1_{v \geq 2}) 2^{2 \sum_{j=1}^n (m_j - k_j)} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\
& \cdot \prod_{i=2}^n \left(\sup_{X \in \hat{I}} \frac{1}{h} \sum_{Y \in \hat{I}} e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(X, Y))^r} |\widetilde{C}_o^{(v,i)}(X, Y)| \right) \\
& \cdot \prod_{k=1}^n \left(\sup_{X \in \hat{I}} \left(\frac{1}{h} \right)^{m_k - 1} \sum_{\mathbf{Y} \in \hat{I}^{m_k - 1}} e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(X, Y_1))^r} \right. \\
& \quad \left. \cdot |J_{m_k}(\beta_2)(R_{\beta_2}(\mathbf{X}, \mathbf{Y}))| \right).
\end{aligned}$$

The inequality (4.7) guarantees that

$$\begin{aligned}
& \sup_{X \in \hat{I}} \left(\frac{1}{h} \right)^{m_k - 1} \sum_{\mathbf{Y} \in \hat{I}^{m_k - 1}} e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(X, Y_1))^r} |J_{m_k}(\beta_2)(R_{\beta_2}(\mathbf{X}, \mathbf{Y}))| \\
& \leq \|J_{m_k}(\beta_2)\|_{l,0}, \\
& \sup_{X \in \hat{I}} \frac{1}{h} \sum_{Y \in \hat{I}} e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(X, Y))^r} |\widetilde{C}_o^{(v,i)}(X, Y)| \\
& \leq \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} = A(\widetilde{C}_o(\beta_1), \widetilde{C}_o(\beta_2))_i^{(v,i)},
\end{aligned}$$

if $v \neq i$. It follows from (4.18) that

$$\sup_{X \in \hat{I}} \frac{1}{h} \sum_{Y \in \hat{I}} e^{\sum_{j=0}^d (\frac{1}{\pi} w(l) \hat{d}_j(X, Y))^r} |\widetilde{C}_o^{(v,v)}(X, Y)| \leq A(\widetilde{C}_o(\beta_1), \widetilde{C}_o(\beta_2))_l^{(v,v)}.$$

By inserting these inequalities into (4.20) we obtain

$$\begin{aligned} (4.21) \quad & \text{(the second term in the right-hand side of (4.15))} \\ & \leq 2^{2 \sum_{j=1}^n (m_j - k_j)} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\ & \quad \cdot \sum_{v=1}^n (1_{v=1} D(\beta_1, \beta_2) + 1_{v \geq 2}) \\ & \quad \cdot \prod_{i=2}^n A(\widetilde{C}_o(\beta_1), \widetilde{C}_o(\beta_2))_i^{(v,i)} \prod_{k=1}^n \|J_{m_k}(\beta_2)\|_{l,0}. \end{aligned}$$

By combining (4.19), (4.21) with (4.15) we reach the inequality claimed in (2).

(1): By considering the fixed variable $X_{1,1} \in I^0$ as $Y_{1,1}$ we can straightforwardly transform the proof of (2) to derive the claimed inequality. \square

For $a = 1, 2$ set

$$T^{(n)}(\beta_a)(\psi) := \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_a)) \prod_{j=1}^n J(\beta_a)(\psi + \psi^j) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}$$

with $J(\beta_a) (\in \bigwedge \mathcal{V}(\beta_a))$ satisfying that $J_m(\beta_a)(\psi) = 0$ if $m \notin 2\mathbb{N} \cup \{0\}$ and having the anti-symmetric kernels satisfying (4.1). By putting the preceding lemmas together we can prove the following lemma, which is the goal of this subsection.

LEMMA 4.6. *The following inequalities hold true.*

(1) *For any $n \in \mathbb{N}_{\geq 2}$ and $l \in \mathbb{Z}$,*

$$\left| \frac{h}{N(\beta_1)} T_0^{(n)}(\beta_1) - \frac{h}{N(\beta_2)} T_0^{(n)}(\beta_2) \right|$$

$$\begin{aligned}
&\leq 2n D_{et}^{-n+1} \left(\sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \right)^{n-2} \left(\sum_{m=2}^{N(\beta_2)} 2^{3m} D_{et}^{\frac{m}{2}} \sum_{a=1}^2 \|J_m(\beta_a)\|_{l,0} \right)^{n-1} \\
&\quad \cdot \sum_{m=2}^{N(\beta_2)} 2^{3m} D_{et}^{\frac{m}{2}} \left(\frac{2\pi}{\beta_1} \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \sum_{c=1}^2 \|J_m(\beta_c)\|_{l,1} \right. \\
&\quad + \frac{2\pi}{\beta_1} \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,1} \sum_{c=1}^2 \|J_m(\beta_c)\|_{l,0} \\
&\quad + \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} |J_m(\beta_1) - J_m(\beta_2)|_l \\
&\quad + |\widetilde{C}_o(\beta_1) - \widetilde{C}_o(\beta_2)|_l \sum_{c=1}^2 \|J_m(\beta_c)\|_{l,0} \\
&\quad \left. + D(\beta_1, \beta_2) \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \sum_{c=1}^2 \|J_m(\beta_c)\|_{l,0} \right).
\end{aligned}$$

(2) For any $n \in \mathbb{N}_{\geq 2}$, $l \in \mathbb{Z}$ and $m \in \{2, 3, \dots, N(\beta_2)\}$,

$$\begin{aligned}
&|T_m^{(n)}(\beta_1) - T_m^{(n)}(\beta_2)|_l \\
&\leq 2n \cdot 2^{-2m} D_{et}^{-\frac{m}{2}-n+1} \left(\sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \right)^{n-2} \\
&\quad \cdot \prod_{j=2}^n \left(\sum_{m_j=2}^{N(\beta_2)} 2^{4m_j} D_{et}^{\frac{m_j}{2}} \sum_{a=1}^2 \|J_{m_j}(\beta_a)\|_{l,0} \right) \\
&\quad \cdot \sum_{m_1=2}^{N(\beta_2)} 2^{4m_1} D_{et}^{\frac{m_1}{2}} \left(\frac{2\pi}{\beta_1} \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \sum_{c=1}^2 \|J_{m_1}(\beta_c)\|_{l,1} \right. \\
&\quad + \frac{2\pi}{\beta_1} \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,1} \sum_{c=1}^2 \|J_{m_1}(\beta_c)\|_{l,0} \\
&\quad + \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} |J_{m_1}(\beta_1) - J_{m_1}(\beta_2)|_l \\
&\quad \left. + |\widetilde{C}_o(\beta_1) - \widetilde{C}_o(\beta_2)|_l \sum_{c=1}^2 \|J_{m_1}(\beta_c)\|_{l,0} \right)
\end{aligned}$$

$$+ D(\beta_1, \beta_2) \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \sum_{c=1}^2 \|J_{m_1}(\beta_c)\|_{l,0} \Big) 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m}.$$

PROOF. (1): It follows from (4.4) that

$$(4.22) \quad \begin{aligned} & \text{Ope}(T, C_o(\beta_a)) \prod_{\substack{j=1 \\ \text{order}}}^n \psi_{\mathbf{X}_j}^j \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}} \\ &= \text{Ope}(T, C_o(\beta_a)) \prod_{\substack{j=1 \\ \text{order}}}^n \left((-1)^{N_{\beta_a}(\mathbf{X}_j+x)} \psi_{R_{\beta_a}(\mathbf{X}_j+x)}^j \right) \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}}, \\ & (\forall a \in \{1,2\}, \mathbf{X}_j \in I(\beta_a)^{m_j} \ (j = 1, 2, \dots, n), x \in (1/h)\mathbb{Z}). \end{aligned}$$

By using (4.1) and (4.22) we can transform $T_0^{(n)}(\beta_a)$ as follows.

$$\begin{aligned} & T_0^{(n)}(\beta_a) \\ &= \prod_{i=1}^n \left(\sum_{m_i=2}^{N(\beta_2)} \right) \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_a)) \\ & \quad \cdot \prod_{j=1}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in I(\beta_a)^{m_j}} J_{m_j}(\beta_a)(\mathbf{X}_j) \right) \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{\mathbf{X}_k}^k \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}} \\ &= \prod_{i=1}^n \left(\sum_{m_i=2}^{N(\beta_2)} \right) \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_a)) \\ & \quad \cdot \left(\frac{1}{h} \right)^{m_1} \sum_{X_{1,1} \in I^0} \sum_{x \in [0, \beta_a)_h} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1,m_1}) \in I(\beta_a)^{m_1-1}} \\ & \quad \cdot (-1)^{N_{\beta_a}((X_{1,2}, X_{1,3}, \dots, X_{1,m_1})-x)} \\ & \quad \cdot J_{m_1}(\beta_a)(X_{1,1}, R_{\beta_a}((X_{1,2}, X_{1,3}, \dots, X_{1,m_1})-x)) \\ & \quad \cdot \prod_{j=2}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in I(\beta_a)^{m_j}} (-1)^{N_{\beta_a}(\mathbf{X}_j-x)} J_{m_j}(\beta_a)(R_{\beta_a}(\mathbf{X}_j-x)) \right) \\ & \quad \cdot (-1)^{N_{\beta_1}((X_{1,2}, X_{1,3}, \dots, X_{1,m_1})-x)} \psi_{(X_{1,1}, R_{\beta_a}((X_{1,2}, X_{1,3}, \dots, X_{1,m_1})-x))}^1 \\ & \quad \cdot \prod_{\substack{k=2 \\ \text{order}}}^n \left((-1)^{N_{\beta_a}(\mathbf{X}_k-x)} \psi_{R_{\beta_a}(\mathbf{X}_k-x)}^k \right) \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}} \end{aligned}$$

$$\begin{aligned}
&= \beta_a \prod_{i=1}^n \left(\sum_{m_i=2}^{N(\beta_2)} \right) \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_a)) \\
&\quad \cdot \left(\frac{1}{h} \right)^{m_1-1} \sum_{X_{1,1} \in I^0} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1,m_1}) \in I(\beta_a)^{m_1-1}} J_{m_1}(\beta_a)(\mathbf{X}_1) \\
&\quad \cdot \prod_{j=2}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in I(\beta_a)^{m_j}} J_{m_j}(\beta_a)(\mathbf{X}_j) \right) \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{\mathbf{X}_k}^k \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}}.
\end{aligned}$$

Then, we decompose $T_0^{(n)}(\beta_a)$ as follows.

$$T_0^{(n)}(\beta_a) = S_0^{(n)}(\beta_a) + U_0^{(n)}(\beta_a),$$

where

$$\begin{aligned}
&S_0^{(n)}(\beta_a) \\
&:= \beta_a \prod_{i=1}^n \left(\sum_{m_i=2}^{N(\beta_2)} \right) \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_a)) \\
&\quad \cdot \left(\frac{1}{h} \right)^{m_1-1} \sum_{X_{1,1} \in I^0} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1,m_1}) \in I(\beta_a)^{m_1-1}} J_{m_1}(\beta_a)(\mathbf{X}_1) \\
&\quad \cdot \prod_{j=2}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in I(\beta_a)^{m_j}} J_{m_j}(\beta_a)(\mathbf{X}_j) \right) \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{\mathbf{X}_k}^k \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}} \\
&\quad \cdot \mathbf{1}_{\exists(\rho, \mathbf{x}, \sigma, x, \theta) \in I(\beta_a) \text{ s.t. } (\rho, \mathbf{x}, \sigma, x, \theta) \subset (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) \text{ and } x \in [\frac{\beta_1}{4}, \beta_a - \frac{\beta_1}{4}]_h}, \\
&U_0^{(n)}(\beta_a) \\
&:= \beta_a \prod_{i=1}^n \left(\sum_{m_i=2}^{N(\beta_2)} \right) \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_a)) \\
&\quad \cdot \left(\frac{1}{h} \right)^{m_1-1} \sum_{X_{1,1} \in I^0} \sum_{(X_{1,2}, X_{1,3}, \dots, X_{1,m_1}) \in \hat{I}^{m_1-1}} J_{m_1}(\beta_a)(R_{\beta_a}(\mathbf{X}_1)) \\
&\quad \cdot \prod_{j=2}^n \left(\left(\frac{1}{h} \right)^{m_j} \sum_{\mathbf{X}_j \in \hat{I}^{m_j}} J_{m_j}(\beta_a)(R_{\beta_a}(\mathbf{X}_j)) \right) \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{R_{\beta_a}(\mathbf{X}_k)}^k \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}}.
\end{aligned}$$

By applying Lemma 4.2 (1) we have that

$$\begin{aligned}
 (4.23) \quad & |S_0^{(n)}(\beta_a)| \\
 & \leq \frac{N(\beta_a)}{h} \frac{2\pi}{\beta_1} D_{et}^{-n+1} \prod_{j=1}^n \left(\sum_{q_j=0}^1 \sum_{m_j=2}^{N(\beta_2)} 2^{3m_j} D_{et}^{\frac{m_j}{2}} \|J_{m_j}(\beta_a)\|_{l,q_j} \right) \\
 & \quad \cdot \prod_{k=2}^n \left(\sum_{r_k=0}^1 \|\widetilde{C}_o(\beta_a)\|_{l,r_k} \right) 1_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = 1} \\
 & = \frac{N(\beta_a)}{h} \frac{2\pi}{\beta_1} D_{et}^{-n+1} \left(\sum_{m=2}^{N(\beta_2)} 2^{3m} D_{et}^{\frac{m}{2}} \|J_m(\beta_a)\|_{l,0} \right)^{n-1} \|\widetilde{C}_o(\beta_a)\|_{l,0}^{n-2} \\
 & \quad \cdot \sum_{m=2}^{N(\beta_2)} 2^{3m} D_{et}^{\frac{m}{2}} ((n-1)\|\widetilde{C}_o(\beta_a)\|_{l,1} \|J_m(\beta_a)\|_{l,0} \\
 & \quad + n\|\widetilde{C}_o(\beta_a)\|_{l,0} \|J_m(\beta_a)\|_{l,1}).
 \end{aligned}$$

On the other hand, Lemma 4.5 (1) ensures that

$$\begin{aligned}
 (4.24) \quad & \left| \frac{1}{\beta_1} U_0^{(n)}(\beta_1) - \frac{1}{\beta_2} U_0^{(n)}(\beta_2) \right| \\
 & \leq \#I^0 \prod_{i=1}^n \left(\sum_{m_i=2}^{N(\beta_2)} \right) 2^{2\sum_{j=1}^n m_j} D_{et}^{\frac{1}{2}\sum_{j=1}^n m_j - n+1} \\
 & \quad \cdot \left(\|\widetilde{C}_o(\beta_1)\|_{l,0}^{n-1} \sum_{v=1}^n \prod_{j=1}^n A(J_{m_j}(\beta_1), J_{m_j}(\beta_2))_l^{(v,j)} \right. \\
 & \quad + \sum_{v=2}^n \prod_{j=2}^n A(\widetilde{C}_o(\beta_1), \widetilde{C}_o(\beta_2))_l^{(v,j)} \prod_{k=1}^n \|J_{m_k}(\beta_2)\|_{l,0} \\
 & \quad \left. + D(\beta_1, \beta_2) \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0}^{n-1} \prod_{j=1}^n \|J_{m_j}(\beta_2)\|_{l,0} \right) \\
 & \leq \#I^0 D_{et}^{-n+1} \left(\sum_{m=2}^{N(\beta_2)} 2^{2m} D_{et}^{\frac{m}{2}} \sum_{a=1}^2 \|J_m(\beta_a)\|_{l,0} \right)^{n-1} \\
 & \quad \cdot \left(\sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \right)^{n-2}
 \end{aligned}$$

$$\begin{aligned}
& \cdot \sum_{m=2}^{N(\beta_2)} 2^{2m} D_{et}^{\frac{m}{2}} \left(n \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \right. \\
& \quad \cdot \left(|J_m(\beta_1) - J_m(\beta_2)|_l + \frac{2\pi}{\beta_1} (m-1) \sum_{c=1}^2 \|J_m(\beta_c)\|_{l,1} \right) \\
& \quad + (n-1) \left(|\widetilde{C}_o(\beta_1) - \widetilde{C}_o(\beta_2)|_l + \frac{2\pi}{\beta_1} \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,1} \right) \\
& \quad \cdot \sum_{c=1}^2 \|J_m(\beta_c)\|_{l,0} \\
& \quad \left. + D(\beta_1, \beta_2) \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \sum_{c=1}^2 \|J_m(\beta_c)\|_{l,0} \right).
\end{aligned}$$

Substitution of (4.23), (4.24) into the inequality

$$\begin{aligned}
& \left| \frac{h}{N(\beta_1)} T_0^{(n)}(\beta_1) - \frac{h}{N(\beta_2)} T_0^{(n)}(\beta_2) \right| \\
& \leq \sum_{a=1}^2 \frac{h}{N(\beta_a)} |S_0^{(n)}(\beta_a)| + \left| \frac{h}{N(\beta_1)} U_0^{(n)}(\beta_1) - \frac{h}{N(\beta_2)} U_0^{(n)}(\beta_2) \right|
\end{aligned}$$

gives the inequality claimed in (1).

(2): The anti-symmetric kernel $T_m^{(n)}(\beta_a)(\cdot)$ characterized in (3.18) can be decomposed as follows. For any $\mathbf{Y} \in I(\beta_a)^m$,

$$T_m^{(n)}(\beta_a)(\mathbf{Y}) = S_m^{(n)}(\beta_a)(\mathbf{Y}) + U_m^{(n)}(\beta_a)(\mathbf{Y}),$$

where

$$\begin{aligned}
& S_m^{(n)}(\beta_a)(\mathbf{Y}) \\
& := \prod_{i=1}^n \left(\sum_{m_i=2}^{N(\beta_2)} \sum_{k_i=0}^{m_i-1} \binom{m_i}{k_i} \sum_{\mathbf{Y}_i \in I(\beta_a)^{k_i}} \right) 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m} 1_{\sum_{j=1}^n k_j = m} \\
& \quad \cdot \frac{1}{m!} \sum_{\sigma \in \mathbb{S}_m} \text{sgn}(\sigma) 1_{\mathbf{Y}_\sigma = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n)} \frac{\varepsilon_\pm}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_a)) \\
& \quad \cdot \prod_{j=1}^n \left(\left(\frac{1}{h} \right)^{m_j - k_j} \sum_{\mathbf{X}_j \in I(\beta_a)^{m_j - k_j}} J_{m_j}(\beta_a)(\mathbf{X}_j, \mathbf{Y}_j) \right) \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{\mathbf{X}_k}^k \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}
\end{aligned}$$

$$\begin{aligned}
 & \cdot 1_{\exists(\rho, \mathbf{x}, \sigma, x, \theta) \in I(\beta_a) \text{ s.t. } (\rho, \mathbf{x}, \sigma, x, \theta) \subset (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) \text{ and } x \in [\frac{\beta_1}{4}, \beta_a - \frac{\beta_1}{4}]_h}, \\
 & U_m^{(n)}(\beta_a)(\mathbf{Y}) \\
 & := \prod_{i=1}^n \left(\sum_{m_i=2}^{N(\beta_2)} \sum_{k_i=0}^{m_i-1} \binom{m_i}{k_i} \sum_{\mathbf{Y}_i \in I(\beta_a)^{k_i}} \right) 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m} 1_{\sum_{j=1}^n k_j = m} \\
 & \cdot \frac{1}{m!} \sum_{\sigma \in \mathbb{S}_m} \text{sgn}(\sigma) 1_{\mathbf{Y}_\sigma = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n)} \frac{\varepsilon_\pm}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_o(\beta_a)) \\
 & \cdot \prod_{j=1}^n \left(\left(\frac{1}{h} \right)^{m_j - k_j} \sum_{\mathbf{X}_j \in \tilde{I}^{m_j - k_j}} J_{m_j}(\beta_a)(R_{\beta_a}(\mathbf{X}_j), \mathbf{Y}_j) \right) \\
 & \cdot \prod_{\substack{k=1 \\ \text{order}}}^n \psi_{R_{\beta_a}}^k(\mathbf{X}_k) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}.
 \end{aligned}$$

Application of Lemma 4.2 (2) yields that

$$\begin{aligned}
 (4.25) \quad & |S_m^{(n)}(\beta_1) - S_m^{(n)}(\beta_2)| \\
 & \leq \prod_{i=1}^n \left(\sum_{m_i=2}^{N(\beta_2)} \sum_{k_i=0}^{m_i-1} \binom{m_i}{k_i} \right) 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m} 1_{\sum_{j=1}^n k_j = m} \\
 & \cdot \frac{2\pi}{\beta_1} 2^{3 \sum_{j=1}^n (m_j - k_j)} D_{et}^{\frac{1}{2} \sum_{j=1}^n (m_j - k_j) - n + 1} \\
 & \cdot \sum_{a=1}^2 \prod_{j=1}^n \left(\sum_{q_j=0}^1 \|J_{m_j}(\beta_a)\|_{l, q_j} \right) \\
 & \cdot \prod_{k=2}^n \left(\sum_{r_k=0}^1 \|\tilde{C}_o(\beta_a)\|_{l, r_k} \right) 1_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = 1} \\
 & \leq \frac{2\pi}{\beta_1} 2^{-3m} D_{et}^{-\frac{m}{2} - n + 1} \\
 & \cdot \prod_{j=1}^n \left(\sum_{m_j=2}^{N(\beta_2)} 2^{4m_j} D_{et}^{\frac{m_j}{2}} \sum_{q_j=0}^1 \sum_{a=1}^2 \|J_{m_j}(\beta_a)\|_{l, q_j} \right) \\
 & \cdot \prod_{k=2}^n \left(\sum_{r_k=0}^1 \sum_{a=1}^2 \|\tilde{C}_o(\beta_a)\|_{l, r_k} \right)
 \end{aligned}$$

$$\begin{aligned}
& \cdot 1_{\sum_{j=1}^n q_j + \sum_{k=2}^n r_k = 1} 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m} \\
& \leq n \frac{2\pi}{\beta_1} 2^{-3m} D_{et}^{-\frac{m}{2} - n + 1} \left(\sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \right)^{n-2} \\
& \cdot \prod_{j=2}^n \left(\sum_{m_j=2}^{N(\beta_2)} 2^{4m_j} D_{et}^{\frac{m_j}{2}} \sum_{a=1}^2 \|J_{m_j}(\beta_a)\|_{l,0} \right) \\
& \cdot \sum_{m_1=2}^{N(\beta_2)} 2^{4m_1} D_{et}^{\frac{m_1}{2}} \left(\sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,1} \sum_{c=1}^2 \|J_{m_1}(\beta_c)\|_{l,0} \right. \\
& \quad \left. + \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \sum_{c=1}^2 \|J_{m_1}(\beta_c)\|_{l,1} \right) 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m}.
\end{aligned}$$

On the other hand, Lemma 4.5 (2) implies that

$$\begin{aligned}
(4.26) \quad & |U_m^{(n)}(\beta_1) - U_m^{(n)}(\beta_2)|_l \\
& \leq \prod_{i=1}^n \left(\sum_{m_i=2}^{N(\beta_2)} \sum_{k_i=0}^{m_i-1} \binom{m_i}{k_i} \right) 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m} 1_{\sum_{j=1}^n k_j = m} \\
& \cdot 2^{2\sum_{j=1}^n (m_j - k_j)} D_{et}^{\frac{1}{2}\sum_{j=1}^n (m_j - k_j) - n + 1} \\
& \cdot \left(\|\widetilde{C}_o(\beta_1)\|_{l,0}^{n-1} \sum_{v=1}^n \prod_{j=1}^n A(J_{m_j}(\beta_1), J_{m_j}(\beta_2))_l^{(v,j)} \right. \\
& \quad + \sum_{v=2}^n \prod_{j=2}^n A(\widetilde{C}_o(\beta_1), \widetilde{C}_o(\beta_2))_l^{(v,j)} \prod_{k=1}^n \|J_{m_k}(\beta_2)\|_{l,0} \\
& \quad \left. + D(\beta_1, \beta_2) \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0}^{n-1} \prod_{j=1}^n \|J_{m_j}(\beta_2)\|_{l,0} \right) \\
& \leq n 2^{-2m} D_{et}^{-\frac{1}{2}m - n + 1} \left(\sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \right)^{n-2} \\
& \cdot \prod_{j=2}^n \left(\sum_{m_j=2}^{N(\beta_2)} 2^{3m_j} D_{et}^{\frac{m_j}{2}} \sum_{a=1}^2 \|J_{m_j}(\beta_a)\|_{l,0} \right)
\end{aligned}$$

$$\begin{aligned}
 & \cdot \sum_{m_1=2}^{N(\beta_2)} 2^{4m_1} D_{et}^{\frac{m_1}{2}} \left(\sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} |J_{m_1}(\beta_1) - J_{m_1}(\beta_2)|_l \right. \\
 & + \frac{2\pi}{\beta_1} \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \sum_{c=1}^2 \|J_{m_1}(\beta_c)\|_{l,1} \\
 & + |\widetilde{C}_o(\beta_1) - \widetilde{C}_o(\beta_2)|_l \sum_{a=1}^2 \|J_{m_1}(\beta_a)\|_{l,0} \\
 & + \frac{2\pi}{\beta_1} \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,1} \sum_{c=1}^2 \|J_{m_1}(\beta_c)\|_{l,0} \\
 & \left. + D(\beta_1, \beta_2) \sum_{a=1}^2 \|\widetilde{C}_o(\beta_a)\|_{l,0} \sum_{c=1}^2 \|J_{m_1}(\beta_c)\|_{l,0} \right) \\
 & \cdot \mathbf{1}_{\sum_{j=1}^n m_j - 2n + 2 \geq m}.
 \end{aligned}$$

Finally, by combining the inequalities (4.25), (4.26) with the inequality

$$|T_m^{(n)}(\beta_1) - T_m^{(n)}(\beta_2)|_l \leq |S_m^{(n)}(\beta_1) - S_m^{(n)}(\beta_2)|_l + |U_m^{(n)}(\beta_1) - U_m^{(n)}(\beta_2)|_l,$$

we obtain the inequality claimed in (2). \square

5. Generalized Multi-Scale Integrations

In this section we present multi-scale integrations, assuming that a family of covariances is given and each covariance belonging to the family has certain scale-dependent upper bounds. We inductively define a family of Grassmann polynomials by means of the free integration and the tree expansion with the covariance at one scale. Then, we establish scale-dependent estimates on the Grassmann polynomials by applying the general lemmas prepared in Section 3 and Section 4. The analysis of this section can be seen as a generalization of the multi-scale integration over the Matsubara frequency and that around the zero set of the dispersion relation in the momentum space. The results obtained in this section will underlie more concrete, model-dependent analysis in Section 6 and Section 7.

From this section we use the symbol ‘ c ’ to represent a real positive constant independent of any parameter. When we construct inequalities, we will frequently replace the generic constant c by a larger constant with the

same symbol in the following lines without acknowledging the replacement. However, it must be clear from the context that such replacement does not change what the arguments conclude in the last line.

5.1. The generalized ultra-violet integration

Let $N_+ \in \mathbb{N}$ be a fixed number. Assume that a family of covariances $\{C_{o,l}\}_{l=1}^{N_+}$ is given and it satisfies the following properties with constants M , $c_0, c'_0 \in \mathbb{R}_{\geq 1}$, a weight $w(0) \in \mathbb{R}_{>0}$ and an exponent $r \in (0, 1]$.

$$(5.1) \quad \begin{aligned} C_{o,l}(\rho \mathbf{x} \uparrow x, \eta \mathbf{y} \uparrow y) &= C_{o,l}(\rho \mathbf{x} \downarrow x, \eta \mathbf{y} \downarrow y), \\ C_{o,l}(\rho \mathbf{x} \uparrow x, \eta \mathbf{y} \downarrow y) &= C_{o,l}(\rho \mathbf{x} \downarrow x, \eta \mathbf{y} \uparrow y) = 0, \\ (\forall (\rho, \mathbf{x}, x), (\eta, \mathbf{y}, y) \in \mathcal{B} \times \Gamma \times [0, \beta)_h, l \in \{1, 2, \dots, N_+\}), \end{aligned}$$

$$(5.2) \quad \begin{aligned} C_{o,l}(\rho \mathbf{x} \sigma x, \eta \mathbf{x} \tau x) &= C_{o,l}(\rho \mathbf{0} \sigma 0, \eta \mathbf{0} \tau 0), \\ (\forall (\rho, \mathbf{x}, \sigma, x) \in I_0, \eta \in \mathcal{B}, \tau \in \{\uparrow, \downarrow\}, l \in \{1, 2, \dots, N_+\}), \end{aligned}$$

$$(5.3) \quad \begin{aligned} |\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_{o,l}(X_i, Y_j))_{1 \leq i, j \leq n}| &\leq c_0^n, \\ (\forall r, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} \leq 1, \\ X_i, Y_i \in I_0 \ (i = 1, 2, \dots, n), l \in \{1, 2, \dots, N_+\}), \end{aligned}$$

$$(5.4) \quad \|\widetilde{C}_{o,l}\|_{0,j} \leq c_0 M^{-l}, \quad (\forall j \in \{0, 1\}, l \in \{1, 2, \dots, N_+\}),$$

where $\widetilde{C}_{o,l} : I^2 \rightarrow \mathbb{C}$ is the anti-symmetric extension of $C_{o,l}$ defined as in (3.2),

$$(5.5) \quad \sum_{l=1}^{N_+} \max_{\rho \in \mathcal{B}} |C_{o,l}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \leq c'_0.$$

These are the conditions typically satisfied by an actual covariance with the Matsubara UV cut-off. In fact the parameters $w(0)$, r do not play any explicit role here. We need these parameters only to introduce the norm $\|\cdot\|_{0,0}$ and the semi-norm $\|\cdot\|_{0,1}$.

Using the covariances $\{C_{o,l}\}_{l=1}^{N_+}$, we inductively define a family of Grassmann polynomials as follows. With parameters $U_\rho \in \mathbb{C}$ ($\rho \in \mathcal{B}$), $\delta \in \{1, -1\}$, define $F^{N_+}(\psi)$, $T^{N_+,(n)}(\psi)$ ($n \in \mathbb{N}_{\geq 2}$), $T^{N_+}(\psi)$, $J^{N_+}(\psi) \in \bigwedge \mathcal{V}$ by

$$(5.6) \quad F^{N_+}(\psi) := \frac{\delta}{2h} \sum_{(\rho, \mathbf{x}, \sigma, x) \in I_0} U_\rho \bar{\psi}_{\rho \mathbf{x} \sigma x} \psi_{\rho \mathbf{x} \sigma x}$$

$$\begin{aligned}
 & -\frac{1}{h} \sum_{(\rho, \mathbf{x}, x) \in \mathcal{B} \times \Gamma \times [0, \beta)_h} U_\rho \bar{\psi}_{\rho \mathbf{x} \uparrow x} \bar{\psi}_{\rho \mathbf{x} \downarrow x} \psi_{\rho \mathbf{x} \downarrow x} \psi_{\rho \mathbf{x} \uparrow x}, \\
 T^{N_+, (n)}(\psi) & := 0, \quad (\forall n \in \mathbb{N}_{\geq 2}), \quad T^{N_+}(\psi) := 0, \\
 J^{N_+}(\psi) & := F^{N_+}(\psi) + T^{N_+}(\psi).
 \end{aligned}$$

Assume that $l \in \{0, 1, \dots, N_+ - 1\}$ and $J^{l+1}(\psi) \in \bigwedge \mathcal{V}$ is given. Define $F^l(\psi)$, $T^{l, (n)}(\psi)$ ($n \in \mathbb{N}_{\geq 2}$), $T^l(\psi)$, $J^l(\psi) \in \bigwedge \mathcal{V}$ by

$$\begin{aligned}
 (5.7) \quad F^l(\psi) & := \int J^{l+1}(\psi + \psi^1) d\mu_{C_{o, l+1}}(\psi^1), \\
 T^{l, (n)}(\psi) & := \frac{1}{n!} \sum_{T \in \mathbb{T}_n} \text{Ope}(T, C_{o, l+1}) \prod_{\substack{j=1 \\ \text{order}}}^n J^{l+1}(\psi^j + \psi) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}, \\
 & (\forall n \in \mathbb{N}_{\geq 2}), \\
 T^l(\psi) & := \sum_{n=2}^{\infty} T^{l, (n)}(\psi), \\
 J^l(\psi) & := F^l(\psi) + T^l(\psi),
 \end{aligned}$$

on the assumption that $\sum_{n=2}^{\infty} T^{l, (n)}(\psi)$ converges.

Though one can directly see from definition, let us prove the following lemma by applying Lemma 3.9.

LEMMA 5.1. *Assume that $J^l(\psi)$ ($l = 0, 1, \dots, N_+$) are well-defined. Then, if $m \notin 2\mathbb{N} \cup \{0\}$,*

$$(5.8) \quad T_m^{l, (n)}(\psi) = F_m^l(\psi) = 0, \quad (\forall l \in \{0, 1, \dots, N_+\}, n \in \mathbb{N}_{\geq 2}).$$

PROOF. Apparently the equalities (5.8) hold for $l = N_+$. Assume that $J_m^l(\psi) = 0$ if $m \notin 2\mathbb{N} \cup \{0\}$ for some $l \in \{1, 2, \dots, N_+\}$.

Let $S : I \rightarrow I$, $Q : I \rightarrow \mathbb{R}$ be defined by $S(X) := X$, $Q(X) := \pi$, ($\forall X \in I$). Using the notations introduced in Subsection 3.3, we see that

$$J^l(\mathcal{R}\psi) = J^l(\psi), \quad \widetilde{C_{o, l}}(\mathbf{X}) = e^{iQ_2(S_2(\mathbf{X}))} \widetilde{C_{o, l}}(S_2(\mathbf{X})), \quad (\forall \mathbf{X} \in I^2).$$

Thus, we can apply Lemma 3.9 (1) to deduce that

$$F^{l-1}(\mathcal{R}\psi) = F^{l-1}(\psi),$$

$$T^{l-1,(n)}(\mathcal{R}\psi) = T^{l-1,(n)}(\psi), \quad (\forall n \in \mathbb{N}_{\geq 2}).$$

This implies (5.8) for $l - 1$. By induction the claim holds true. \square

The following proposition is a generalization of the multi-scale integration over the Matsubara frequency.

PROPOSITION 5.2. *There exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that if the parameters $M, \alpha \in \mathbb{R}_{\geq 1}, U_\rho \in \mathbb{C} (\rho \in \mathcal{B})$ satisfy*

$$(5.9) \quad M \geq c, \quad \alpha^2 \geq cM, \quad \sup_{\rho \in \mathcal{B}} |U_\rho| \leq \frac{1}{c(c_0 + c'_0)^2 \alpha^4},$$

the following inequalities hold. For any $l \in \{0, 1, \dots, N_+\}, r \in \{0, 1\}$,

$$(5.10) \quad \frac{h}{N} \left(|F_0^l| + \sum_{n=2}^{\infty} |T_0^{l,(n)}| \right) \leq \alpha^{-4},$$

$$(5.11) \quad c_0 \alpha^2 \left(\|F_2^l\|_{0,r} + \sum_{n=2}^{\infty} \|T_2^{l,(n)}\|_{0,r} \right) \leq 1,$$

$$(5.12) \quad M^{-2l} \sum_{m=2}^N c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m \left(\|F_m^l\|_{0,r} + \sum_{n=2}^{\infty} \|T_m^{l,(n)}\|_{0,r} \right) \leq 1.$$

PROOF. Let the symbol U_{max} denote $\sup_{\rho \in \mathcal{B}} |U_\rho|$ during the proof.

(5.11),(5.12): First let us prove the inequalities (5.11), (5.12) by induction with l . This part is close to the proof of [14, Proposition 4.1]. However, we present the full argument for self-containedness of the paper. Note that

$$F_m^{N_+}(\mathbf{X}) = -V_m^\delta(\mathbf{X}), \quad (\forall \mathbf{X} \in I^m, m \in \{2, 4\}),$$

where $V_m^\delta(\cdot) : I^m \rightarrow \mathbb{C} (m = 2, 4)$ are the anti-symmetric functions characterized in (2.31). From this we see that

$$\|F_m^{N_+}\|_{0,r} \leq U_{max}, \quad (\forall m \in \{2, 4\}, r \in \{0, 1\}).$$

Therefore, if $U_{max} \leq (2(c_0 + c'_0)^2 \alpha^4)^{-1}$,

$$c_0 \alpha^2 \|F_2^{N_+}\|_{0,r} \leq 1,$$

$$M^{-2N_+} \sum_{m \in \{2,4\}} c_0^{\frac{m}{2}} M^{\frac{N_+}{2}m} \alpha^m \|F_m^{N_+}\|_{0,r} \leq c_0 \alpha^2 U_{max} + c_0^2 \alpha^4 U_{max} \leq 1.$$

Thus, the inequalities (5.11), (5.12) for $l = N_+$ hold.

Let us fix $l \in \{0, 1, \dots, N_+ - 1\}$ and assume that (5.11), (5.12) hold for all $l' \in \{l+1, l+2, \dots, N_+\}$. Fix $r \in \{0, 1\}$. By combining (5.3), (5.4) with Lemma 3.8 (2) we have that for any $m \in \{2, 3, \dots, N\}$,

$$\begin{aligned} (5.13) \quad & \|T_m^{l,(n)}\|_{0,r} \\ & \leq 2^{-2m} c_0^{-\frac{m}{2}-n+1} \prod_{i=1}^n \binom{1}{q_i=0} \prod_{j=2}^n \binom{1}{r_j=0} 1_{\sum_{i=1}^n q_i + \sum_{j=2}^n r_j = r} \\ & \quad \cdot (c_0 M^{-(l+1)})^{n-1} \\ & \quad \cdot \prod_{k=1}^n \left(\sum_{m_k=2}^N 2^{3m_k} c_0^{\frac{m_k}{2}} \|J_{m_k}^{l+1}\|_{0,q_k} \right) 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m} \\ & = 2^{-2m} c_0^{-\frac{m}{2}} M^{-(l+1)(n-1)} \\ & \quad \cdot \prod_{i=1}^n \binom{1}{q_i=0} \prod_{j=2}^n \binom{1}{r_j=0} 1_{\sum_{i=1}^n q_i + \sum_{j=2}^n r_j = r} \\ & \quad \cdot \prod_{k=1}^n \left(\sum_{m_k=2}^N 2^{3m_k} c_0^{\frac{m_k}{2}} \|J_{m_k}^{l+1}\|_{0,q_k} \right) 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m}. \end{aligned}$$

By the assumption of induction,

$$\begin{aligned} (5.14) \quad & \sum_{m=2}^N 2^{3m} c_0^{\frac{m}{2}} \|J_m^{l+1}\|_{0,r} = 2^6 c_0 \|J_2^{l+1}\|_{0,r} + \sum_{m=4}^N 2^{3m} c_0^{\frac{m}{2}} \|J_m^{l+1}\|_{0,r} \\ & \leq c\alpha^{-2} + c\alpha^{-4} \leq c\alpha^{-2}. \end{aligned}$$

Substitution of (5.14) into (5.13) yields that

$$\|T_m^{l,(n)}\|_{0,r} \leq c^n c_0^{-\frac{m}{2}} M^{-(l+1)(n-1)} \alpha^{-2n}, \quad (\forall m \in \{2, 4\}).$$

Therefore, if $\alpha \geq c$,

$$(5.15) \quad \sum_{n=2}^{\infty} \|T_m^{l,(n)}\|_{0,r} \leq c c_0^{-\frac{m}{2}} M^{-l-1} \alpha^{-4}, \quad (\forall m \in \{2, 4\}).$$

It follows from (5.13) and (5.12) for $l + 1$ that if $M \geq c$,

$$\begin{aligned}
& M^{-2l} \sum_{m=2}^N c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m \|T_m^{l,(n)}\|_{0,r} \\
& \leq cM^{-2l} \cdot M^{-(l+1)(n-1)} \prod_{i=1}^n \binom{1}{q_i=0} \prod_{j=2}^n \binom{1}{r_j=0} 1_{\sum_{i=1}^n q_i + \sum_{j=2}^n r_j = r} \\
& \quad \cdot \prod_{k=1}^n \left(\sum_{m_k=2}^N 2^{3m_k} c_0^{\frac{m_k}{2}} \|J_{m_k}^{l+1}\|_{0,q_k} \right) \\
& \quad \cdot M^{\frac{l}{2}(\sum_{j=1}^n m_j - 2n + 2)} \alpha^{\sum_{j=1}^n m_j - 2n + 2} 2^{-2(\sum_{j=1}^n m_j - 2n + 2)} \\
& \leq cM\alpha^2 \prod_{i=1}^n \binom{1}{q_i=0} \prod_{j=2}^n \binom{1}{r_j=0} 1_{\sum_{i=1}^n q_i + \sum_{j=2}^n r_j = r} \\
& \quad \cdot \prod_{k=1}^n \left(2^4 M^{-2l-1} \alpha^{-2} \sum_{m_k=2}^N 2^{m_k} c_0^{\frac{m_k}{2}} M^{\frac{l}{2}m_k} \alpha^{m_k} \|J_{m_k}^{l+1}\|_{0,q_k} \right) \\
& \leq cM\alpha^2 \prod_{i=1}^n \binom{1}{q_i=0} \prod_{j=2}^n \binom{1}{r_j=0} 1_{\sum_{i=1}^n q_i + \sum_{j=2}^n r_j = r} (c\alpha^{-2})^n \\
& \leq cM\alpha^2 (c\alpha^{-2})^n,
\end{aligned}$$

where we especially used the inequality

$$\sum_{m=2}^N 2^m c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m \|J_m^{l+1}\|_{0,q} \leq 2^2 M^{2l+1}, \quad (\forall q \in \{0, 1\}).$$

Therefore, on the assumption $\alpha \geq c$,

$$(5.16) \quad M^{-2l} \sum_{m=2}^N c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m \sum_{n=2}^{\infty} \|T_m^{l,(n)}\|_{0,r} \leq cM\alpha^{-2}.$$

One implication of Lemma 3.1 is that for any $m \in \{6, 7, \dots, N\}$,

$$\|F_m^l\|_{0,r} \leq \sum_{n=m}^N 2^n c_0^{\frac{n-m}{2}} \|J_n^{l+1}\|_{0,r}.$$

Thus, by the assumption $M, \alpha \geq c$,

$$\begin{aligned}
 (5.17) \quad & M^{-2l} \sum_{m=6}^N c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m \|F_m^l\|_{0,r} \\
 & \leq M^{-2l} \sum_{n=6}^N \sum_{m=6}^n 2^n c_0^{\frac{n}{2}} M^{\frac{l}{2}m} \alpha^m \|J_n^{l+1}\|_{0,r} \\
 & \leq cM^{-2l} \sum_{n=6}^N 2^n c_0^{\frac{n}{2}} M^{\frac{l}{2}n} \alpha^n \|J_n^{l+1}\|_{0,r} \\
 & \leq cM^{-2l-3} \sum_{n=6}^N c_0^{\frac{n}{2}} M^{\frac{l+1}{2}n} \alpha^n \|J_n^{l+1}\|_{0,r} \\
 & \leq cM^{-1}.
 \end{aligned}$$

It remains to bound $\|F_m^l\|_{0,r}$ ($m \in \{2, 4\}$). Set

$$\hat{F}_4^l(\psi) := F_4^l(\psi) - F_4^{N+}(\psi).$$

Note that

$$\hat{F}_4^l(\psi) = \hat{F}_4^{l+1}(\psi) + T_4^{l+1}(\psi) + \mathcal{P}_4 \int \sum_{m=6}^N J_m^{l+1}(\psi + \psi^1) d\mu_{C_{o,l+1}}(\psi^1).$$

By using Lemma 3.1, (5.12), (5.15) for $l' \in \{l+1, l+2, \dots, N_+\}$ and the assumption $M, \alpha \geq c$ we deduce that

$$\begin{aligned}
 (5.18) \quad & \|\hat{F}_4^l\|_{0,r} \leq \|\hat{F}_4^{l+1}\|_{0,r} + \|T_4^{l+1}\|_{0,r} + \sum_{m=6}^N 2^m c_0^{\frac{m-4}{2}} \|J_m^{l+1}\|_{0,r} \\
 & \leq \|\hat{F}_4^{l+1}\|_{0,r} + \|T_4^{l+1}\|_{0,r} \\
 & \quad + cM^{-3(l+1)} \alpha^{-6} \sum_{m=6}^N c_0^{\frac{m-4}{2}} M^{\frac{l+1}{2}m} \alpha^m \|J_m^{l+1}\|_{0,r} \\
 & \leq \|\hat{F}_4^{l+1}\|_{0,r} + cc_0^{-2} M^{-l-1} \alpha^{-4} \leq cc_0^{-2} \sum_{j=l}^{N_+-1} M^{-j-1} \alpha^{-4} \\
 & \leq cc_0^{-2} M^{-l-1} \alpha^{-4},
 \end{aligned}$$

which implies that

$$(5.19) \quad c_0^2 \alpha^4 \|F_4^l\|_{0,r} \leq c_0^2 \alpha^4 U_{max} + cM^{-1}.$$

Next let us bound $\|F_2^l\|_{0,r}$. By definition,

$$(5.20) \quad \begin{aligned} F_2^l(\psi) &= F_2^{l+1}(\psi) + T_2^{l+1}(\psi) + \mathcal{P}_2 \int \hat{F}_4^{l+1}(\psi + \psi^1) d\mu_{C_{o,l+1}}(\psi^1) \\ &\quad + \mathcal{P}_2 \int F_4^{N+}(\psi + \psi^1) d\mu_{C_{o,l+1}}(\psi^1) \\ &\quad + \mathcal{P}_2 \int T_4^{l+1}(\psi + \psi^1) d\mu_{C_{o,l+1}}(\psi^1) \\ &\quad + \mathcal{P}_2 \int \sum_{m=6}^N J_m^{l+1}(\psi + \psi^1) d\mu_{C_{o,l+1}}(\psi^1). \end{aligned}$$

Application of Lemma 3.1, (5.1), (5.2), (5.3), (5.5), (5.12), (5.15), (5.18) for $l' \in \{l+1, l+2, \dots, N_+\}$ and the assumption $M \geq c$ gives that

$$\begin{aligned} \|F_2^l\|_{0,r} &\leq \|F_2^{l+1}\|_{0,r} + \|T_2^{l+1}\|_{0,r} + 2^4 c_0 \|\hat{F}_4^{l+1}\|_{0,r} \\ &\quad + U_{max} \max_{\rho \in \mathcal{B}} |C_{o,l+1}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\ &\quad + 2^4 c_0 \|T_4^{l+1}\|_{0,r} + \sum_{m=6}^N 2^m c_0^{\frac{m-2}{2}} \|J_m^{l+1}\|_{0,r} \\ &\leq \|F_2^{l+1}\|_{0,r} + U_{max} \max_{\rho \in \mathcal{B}} |C_{o,l+1}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| + cc_0^{-1} M^{-l-1} \alpha^{-4} \\ &\leq \|F_2^{N_+}\|_{0,r} + U_{max} \sum_{j=l}^{N_+-1} \max_{\rho \in \mathcal{B}} |C_{o,j+1}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\ &\quad + cc_0^{-1} \sum_{j=l}^{N_+-1} M^{-j-1} \alpha^{-4} \\ &\leq cc'_0 U_{max} + cc_0^{-1} M^{-l-1} \alpha^{-4}, \end{aligned}$$

or

$$(5.21) \quad c_0 \alpha^2 \|F_2^l\|_{0,r} \leq cc_0 c'_0 \alpha^2 U_{max} + cM^{-1} \alpha^{-2}.$$

The inequalities (5.15), (5.16), (5.17), (5.19), (5.21) ensure that

$$(5.22) \quad c_0 \alpha^2 \left(\|F_2^l\|_{0,r} + \sum_{n=2}^{\infty} \|T_2^{l,(n)}\|_{0,r} \right) \leq c c_0 c'_0 \alpha^2 U_{max} + c M^{-1} \alpha^{-2}.$$

$$(5.23) \quad \begin{aligned} & M^{-2l} \sum_{m=2}^N c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m \left(\|F_m^l\|_{0,r} + \sum_{n=2}^{\infty} \|T_m^{l,(n)}\|_{0,r} \right) \\ & \leq \sum_{m \in \{2,4\}} c_0^{\frac{m}{2}} \alpha^m \|F_m^l\|_{0,r} + M^{-2l} \sum_{m=6}^N c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m \|F_m^l\|_{0,r} \\ & \quad + M^{-2l} \sum_{m=2}^N c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m \sum_{n=2}^{\infty} \|T_m^{l,(n)}\|_{0,r} \\ & \leq c c_0 c'_0 \alpha^2 U_{max} + c_0^2 \alpha^4 U_{max} + c M^{-1} + c M \alpha^{-2}. \end{aligned}$$

On the assumption (5.9) the right-hand sides of (5.22) and (5.23) are less than 1. Thus, the induction concludes that the inequalities (5.11) and (5.12) hold for all $l \in \{0, 1, \dots, N_+\}$ and $r \in \{0, 1\}$.

(5.10): Let us prove the inequality (5.10), assuming that the inequalities (5.11), (5.12) are valid for all $l \in \{0, 1, \dots, N_+\}$. It follows from Lemma 3.8 (1), (5.3), (5.4) and (5.14) that

$$\frac{\hbar}{N} |T_0^{l,(n)}| \leq c_0^{-n+1} \cdot c_0^{n-1} M^{-(l+1)(n-1)} (c \alpha^{-2})^n = M^{l+1} (c M^{-l-1} \alpha^{-2})^n.$$

Thus, if $\alpha \geq c$,

$$(5.24) \quad \frac{\hbar}{N} \sum_{n=2}^{\infty} |T_0^{l,(n)}| \leq c M^{-l-1} \alpha^{-4}.$$

Define $\hat{F}_2^l(\psi) \in \bigwedge \mathcal{V}$ ($l \in \{0, 1, \dots, N_+\}$) by

$$\hat{F}_2^{N_+}(\psi) := 0,$$

$$\hat{F}_2^l(\psi) := F_2^l(\psi) - F_2^{N_+}(\psi) - \sum_{j=l+1}^{N_+} \mathcal{P}_2 \int F_4^{N_+}(\psi + \psi^1) d\mu_{C_{\alpha,j}}(\psi^1),$$

$$(\forall l \in \{0, 1, \dots, N_+ - 1\}).$$

Note that for any $l \in \{0, 1, \dots, N_+ - 1\}$,

$$(5.25) \quad \begin{aligned} \hat{F}_2^l(\psi) &= \hat{F}_2^{l+1}(\psi) + T_2^{l+1}(\psi) + \mathcal{P}_2 \int \hat{F}_4^{l+1}(\psi + \psi^1) d\mu_{C_{o,l+1}}(\psi^1) \\ &\quad + \mathcal{P}_2 \int T_4^{l+1}(\psi + \psi^1) d\mu_{C_{o,l+1}}(\psi^1) \\ &\quad + \mathcal{P}_2 \int \sum_{m=6}^N J_m^{l+1}(\psi + \psi^1) d\mu_{C_{o,l+1}}(\psi^1). \end{aligned}$$

By estimating in the same manner as in Lemma 3.1 and using (5.3) we can derive from (5.25) that for $r \in \{0, 1\}$,

$$\begin{aligned} \|\hat{F}_2^l\|_{0,r} &\leq \|\hat{F}_2^{l+1}\|_{0,r} + \|T_2^{l+1}\|_{0,r} + 2^4 c_0^{\frac{4-2}{2}} \|\hat{F}_4^{l+1}\|_{0,r} \\ &\quad + 2^4 c_0^{\frac{4-2}{2}} \|T_4^{l+1}\|_{0,r} + \sum_{m=6}^N 2^m c_0^{\frac{m-2}{2}} \|J_m^{l+1}\|_{0,r}. \end{aligned}$$

By (5.12), (5.15), (5.18) and the assumption $M \geq c$ we have that

$$(5.26) \quad \begin{aligned} \|\hat{F}_2^l\|_{0,r} &\leq \|\hat{F}_2^{l+1}\|_{0,r} + cc_0^{-1} M^{-l-1} \alpha^{-4} \leq cc_0^{-1} \sum_{j=l}^{N_+-1} M^{-j-1} \alpha^{-4} \\ &\leq cc_0^{-1} M^{-l-1} \alpha^{-4}, \quad (\forall r \in \{0, 1\}). \end{aligned}$$

Remark that for any $l \in \{0, 1, \dots, N_+ - 1\}$,

$$(5.27) \quad \begin{aligned} F_0^l &= F_0^{l+1} + T_0^{l+1} + \int \hat{F}_2^{l+1}(\psi) d\mu_{C_{o,l+1}}(\psi) + \int F_2^{N_+}(\psi) d\mu_{C_{o,l+1}}(\psi) \\ &\quad + 1_{l \leq N_+-2} \\ &\quad \cdot \int \left(\sum_{j=l+2}^{N_+} \mathcal{P}_2 \int F_4^{N_+}(\psi + \psi^1) d\mu_{C_{o,j}}(\psi^1) \right) d\mu_{C_{o,l+1}}(\psi) \\ &\quad + \int \hat{F}_4^{l+1}(\psi) d\mu_{C_{o,l+1}}(\psi) + \int F_4^{N_+}(\psi) d\mu_{C_{o,l+1}}(\psi) \\ &\quad + \sum_{m \in \{2,4\}} \int T_m^{l+1}(\psi) d\mu_{C_{o,l+1}}(\psi) + \sum_{m=6}^N \int J_m^{l+1}(\psi) d\mu_{C_{o,l+1}}(\psi) \\ &= F_0^{l+1} + T_0^{l+1} + \frac{\delta N}{4bh} \sum_{\rho \in \mathcal{B}} U_\rho C_{o,l+1}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0) \end{aligned}$$

$$\begin{aligned}
 & - 1_{l \leq N_+ - 2} \frac{N}{2bh} \sum_{\rho \in \mathcal{B}} U_\rho C_{o,l+1}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0) \sum_{j=l+2}^{N_+} C_{o,j}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0) \\
 & - \frac{N}{4bh} \sum_{\rho \in \mathcal{B}} U_\rho C_{o,l+1}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)^2 \\
 & + \sum_{m \in \{2,4\}} \int \hat{F}_m^{l+1}(\psi) d\mu_{C_{o,l+1}}(\psi) \\
 & + \sum_{m \in \{2,4\}} \int T_m^{l+1}(\psi) d\mu_{C_{o,l+1}}(\psi) + \sum_{m=6}^N \int J_m^{l+1}(\psi) d\mu_{C_{o,l+1}}(\psi).
 \end{aligned}$$

The equality (5.27) and the inequalities (5.3), (5.5), (5.12), (5.15), (5.18), (5.24), (5.26) imply that

$$\begin{aligned}
 (5.28) \quad & \frac{h}{N} |F_0^l| \\
 & \leq \frac{h}{N} |F_0^{l+1}| + \frac{h}{N} |T_0^{l+1}| + U_{max} \max_{\rho \in \mathcal{B}} |C_{o,l+1}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\
 & \quad + 1_{l \leq N_+ - 2} U_{max} \max_{\rho \in \mathcal{B}} |C_{o,l+1}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\
 & \quad \cdot \sum_{j=l+2}^{N_+} \max_{\eta \in \mathcal{B}} |C_{o,j}(\eta \mathbf{0} \uparrow 0, \eta \mathbf{0} \uparrow 0)| \\
 & \quad + U_{max} \max_{\rho \in \mathcal{B}} |C_{o,l+1}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)|^2 + \sum_{m \in \{2,4\}} c_0^{\frac{m}{2}} \|\hat{F}_m^{l+1}\|_{0,0} \\
 & \quad + \sum_{m \in \{2,4\}} c_0^{\frac{m}{2}} \|T_m^{l+1}\|_{0,0} + \sum_{m=6}^N c_0^{\frac{m}{2}} \|J_m^{l+1}\|_{0,0} \\
 & \leq \frac{h}{N} |F_0^{l+1}| + cM^{-l-1}\alpha^{-4} \\
 & \quad + c(c_0 + c'_0) U_{max} \max_{\rho \in \mathcal{B}} |C_{o,l+1}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\
 & \leq c \sum_{j=l}^{N_+ - 1} M^{-j-1}\alpha^{-4} \\
 & \quad + c(c_0 + c'_0) U_{max} \sum_{j=l}^{N_+ - 1} \max_{\rho \in \mathcal{B}} |C_{o,j+1}(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)|
 \end{aligned}$$

$$\leq cM^{-l-1}\alpha^{-4} + c(c_0 + c'_0)c'_0U_{max}.$$

By (5.24) and (5.28),

$$(5.29) \quad \frac{h}{N} \left(|F_0^l| + \sum_{n=2}^{\infty} |T_0^{l,(n)}| \right) \leq cM^{-1}\alpha^{-4} + c(c_0 + c'_0)c'_0U_{max}.$$

On the assumption (5.9) the right-hand side of (5.29) is bounded by α^{-4} from above. The proof is complete. \square

5.2. The generalized ultra-violet integration at different temperatures

Here we estimate the differences between Grassmann polynomials created by the multi-scale integration described in the previous subsection at 2 different temperatures. The analysis in this subsection is based on the inequalities developed in Section 4. We assume the condition (4.2) and that 2 families of covariances $\{C_{o,l}(\beta_a)\}_{l=1}^{N_+}$ ($a = 1, 2$) are given and they satisfy (5.1), (5.2), (5.3), (5.4), (5.5) as well as the following.

$$(5.30) \quad \begin{aligned} C_{o,l}(\beta_a)(\mathbf{X}) &= (-1)^{N\beta_a(\mathbf{X}+x)} C_{o,l}(\beta_a)(R_{\beta_a}(\mathbf{X} + x)), \\ (\forall \mathbf{X} \in I_0(\beta_a)^2, x \in (1/h)\mathbb{Z}, a \in \{1, 2\}, l \in \{1, 2, \dots, N_+\}), \end{aligned}$$

$$(5.31) \quad \begin{aligned} &|\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_{o,l}(\beta_1)(R_{\beta_1}(X_i, Y_j)))_{1 \leq i, j \leq n} \\ &\quad - \det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_{o,l}(\beta_2)(R_{\beta_2}(X_i, Y_j)))_{1 \leq i, j \leq n}| \leq \beta_1^{-\frac{1}{2}} c_0^n, \\ &(\forall r, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} \leq 1, \\ &\quad X_i, Y_i \in \hat{I}_0 \text{ (} i = 1, 2, \dots, n), l \in \{1, 2, \dots, N_+\}), \end{aligned}$$

$$(5.32) \quad |\widetilde{C_{o,l}(\beta_1)} - \widetilde{C_{o,l}(\beta_2)}|_0 \leq \beta_1^{-\frac{1}{2}} c_0 M^{-l}, \quad (\forall l \in \{1, 2, \dots, N_+\}),$$

where $\widetilde{C_{o,l}(\beta_a)} : I(\beta_a)^2 \rightarrow \mathbb{C}$ is the anti-symmetric extension of $C_{o,l}(\beta_a)$ ($a = 1, 2$) defined as in (3.2),

$$(5.33) \quad \sum_{l=1}^{N_+} \max_{\rho \in \mathcal{B}} |C_{o,l}(\beta_1)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0) - C_{o,l}(\beta_2)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \leq \beta_1^{-\frac{1}{2}} c'_0.$$

Here the parameters $M \in \mathbb{R}_{\geq 1}$, $w(0) \in \mathbb{R}_{>0}$, $r \in (0, 1]$ and the constants $c_0, c'_0 \in \mathbb{R}_{\geq 1}$ are the same as those in (5.1), (5.2), (5.3), (5.4), (5.5). Remind us that the parameters $w(0), r$ are also used in the measurement $|\cdot - \cdot|_0$.

With the covariances $\{C_{o,l}(\beta_a)\}_{l=1}^{N_+}$ let $F^l(\beta_a)(\psi), T^{l,(n)}(\beta_a)(\psi)$ ($n \in \mathbb{N}_{\geq 2}$), $T^l(\beta_a)(\psi), J^l(\beta_a)(\psi)$ ($\in \bigwedge \mathcal{V}(\beta_a)$) ($l = 0, 1, \dots, N_+$) be defined by (5.6), (5.7) for $a = 1, 2$ respectively.

One requirement of the analysis in Section 4 was that the kernels of Grassmann polynomials must satisfy the invariance (4.1). First let us confirm that this requirement is fulfilled in this situation.

LEMMA 5.3. *Assume that $J^l(\beta_a)(\psi)$ ($l \in \{0, 1, \dots, N_+\}, a \in \{1, 2\}$) are well-defined. Then,*

$$(5.34) \quad \begin{aligned} F_m^l(\beta_a)(\mathbf{X}) &= (-1)^{N_{\beta_a}(\mathbf{X}+x)} F_m^l(\beta_a)(R_{\beta_a}(\mathbf{X} + x)), \\ T_m^{l,(n)}(\beta_a)(\mathbf{X}) &= (-1)^{N_{\beta_a}(\mathbf{X}+x)} T_m^{l,(n)}(\beta_a)(R_{\beta_a}(\mathbf{X} + x)), \\ (\forall m \in \{1, 2, \dots, N(\beta_2)\}, \mathbf{X} \in I(\beta_a)^m, x \in (1/h)\mathbb{Z}, n \in \mathbb{N}_{\geq 2}, \\ & \quad l \in \{0, 1, \dots, N_+\}, a \in \{1, 2\}). \end{aligned}$$

PROOF. Fix $a \in \{1, 2\}$ and $x \in (1/h)\mathbb{Z}$. Let us define $S : I(\beta_a) \rightarrow I(\beta_a), Q : I(\beta_a) \rightarrow \mathbb{R}$ by

$$S(\mathbf{X}) := R_{\beta_a}(\mathbf{X} + x), \quad Q(\mathbf{X}) := \pi N_{\beta_a}(S^{-1}(\mathbf{X}) + x), \quad (\forall \mathbf{X} \in I(\beta_a)).$$

It follows from (5.30) and the definition of $F^{N_+}(\beta_a)(\psi)$ that

$$\begin{aligned} \widetilde{C}_{o,l}(\beta_a)(\mathbf{X}) &= e^{iQ_2(S_2(\mathbf{X}))} \widetilde{C}_{o,l}(\beta_a)(S_2(\mathbf{X})), \\ (\forall \mathbf{X} \in I(\beta_a)^2, l \in \{1, 2, \dots, N_+\}), \\ F^{N_+}(\beta_a)(\psi) &= F^{N_+}(\beta_a)(\mathcal{R}\psi), \end{aligned}$$

where we used the notations defined in Subsection 3.3. Thus, recursive application of Lemma 3.9 (1) with respect to l shows that

$$\begin{aligned} F^l(\beta_a)(\psi) &= F^l(\beta_a)(\mathcal{R}\psi), \quad T^{l,(n)}(\beta_a)(\psi) = T^{l,(n)}(\beta_a)(\mathcal{R}\psi), \\ (\forall l \in \{0, 1, \dots, N_+\}, n \in \mathbb{N}_{\geq 2}). \end{aligned}$$

By comparing the right-hand side of the equality

$$F_m^l(\beta_a)(\psi) = \left(\frac{1}{h}\right)^m \sum_{\mathbf{X} \in I(\beta_a)^m} F_m^l(\beta_a)(R_{\beta_a}(\mathbf{X} + x))\psi_{R_{\beta_a}(\mathbf{X} + x)}$$

with that of the equality

$$F_m^l(\beta_a)(\mathcal{R}\psi) = \left(\frac{1}{h}\right)^m \sum_{\mathbf{X} \in I(\beta_a)^m} (-1)^{N_{\beta_a}(\mathbf{X} + x)} F_m^l(\beta_a)(\mathbf{X})\psi_{R_{\beta_a}(\mathbf{X} + x)}$$

and by the uniqueness of anti-symmetric kernels we conclude that

$$\begin{aligned} F_m^l(\beta_a)(R_{\beta_a}(\mathbf{X} + x)) &= (-1)^{N_{\beta_a}(\mathbf{X} + x)} F_m^l(\beta_a)(\mathbf{X}), \\ (\forall m \in \{2, 3, \dots, N(\beta_2)\}, \mathbf{X} \in I(\beta_a)^m, x \in (1/h)\mathbb{Z}). \end{aligned}$$

The claimed equality concerning the kernels of $T^{l,(n)}(\beta_a)(\psi)$ can be derived in the same way. \square

The purpose of this subsection is to prove the following proposition.

PROPOSITION 5.4. *There exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that if the condition (5.9) holds with c , the following inequalities hold true. For any $l \in \{0, 1, \dots, N_+\}$, $r \in \{0, 1\}$,*

$$(5.35) \quad \left| \frac{h}{N(\beta_1)} F_0^l(\beta_1) - \frac{h}{N(\beta_2)} F_0^l(\beta_2) \right| + \sum_{n=2}^{\infty} \left| \frac{h}{N(\beta_1)} T_0^{l,(n)}(\beta_1) - \frac{h}{N(\beta_2)} T_0^{l,(n)}(\beta_2) \right| \leq \beta_1^{-\frac{1}{2}} \alpha^{-4},$$

$$(5.36) \quad c_0 \alpha^2 \left(|F_2^l(\beta_1) - F_2^l(\beta_2)|_0 + \sum_{n=2}^{\infty} |T_2^{l,(n)}(\beta_1) - T_2^{l,(n)}(\beta_2)|_0 \right) \leq \beta_1^{-\frac{1}{2}},$$

$$(5.37) \quad M^{-2l} \sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m \left(|F_m^l(\beta_1) - F_m^l(\beta_2)|_0 + \sum_{n=2}^{\infty} |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_0 \right) \leq \beta_1^{-\frac{1}{2}}.$$

PROOF. Set $U_{max} := \sup_{\rho \in \mathcal{B}} |U_\rho|$. We assume the condition (5.9) so that the inequalities (5.10), (5.11), (5.12) hold for β_1, β_2 .

(5.36),(5.37): First let us prove (5.36) and (5.37). The proof is made by induction with respect to l . We can see from (2.31) that $|F_m^{N_+}(\beta_1) - F_m^{N_+}(\beta_2)|_0 = 0$ ($\forall m \in \{2, 4\}$). Thus, the inequalities (5.36), (5.37) for $l = N_+$ hold true.

Let us fix $l \in \{0, 1, \dots, N_+ - 1\}$ and assume that (5.36), (5.37) hold for all $l' \in \{l+1, l+2, \dots, N_+\}$. By substituting (5.3), (5.4), (5.31), (5.32) into the inequality in Lemma 4.6 (2) we have

$$\begin{aligned}
 (5.38) \quad & |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_0 \\
 & \leq cn2^{-2m}c_0^{-\frac{m}{2}-n+1}(2c_0M^{-l-1})^{n-2} \\
 & \quad \cdot \prod_{j=2}^n \left(\sum_{m_j=2}^{N(\beta_2)} 2^{4m_j} c_0^{\frac{m_j}{2}} \sum_{a=1}^2 \|J_{m_j}^{l+1}(\beta_a)\|_{0,0} \right) \\
 & \quad \cdot \sum_{m_1=2}^{N(\beta_2)} 2^{4m_1} c_0^{\frac{m_1}{2}} \left(c_0 M^{-l-1} |J_{m_1}^{l+1}(\beta_1) - J_{m_1}^{l+1}(\beta_2)|_0 \right. \\
 & \quad \quad + \beta_1^{-1} c_0 M^{-l-1} \sum_{a=1}^2 \|J_{m_1}^{l+1}(\beta_a)\|_{0,1} \\
 & \quad \quad \left. + \beta_1^{-\frac{1}{2}} c_0 M^{-l-1} \sum_{a=1}^2 \|J_{m_1}^{l+1}(\beta_a)\|_{0,0} \right) \\
 & \quad \cdot \mathbf{1}_{\sum_{j=1}^n m_j - 2n + 2 \geq m} \\
 & \leq c^n 2^{-2m} c_0^{-\frac{m}{2}} M^{-(l+1)(n-1)} \prod_{j=2}^n \left(\sum_{m_j=2}^{N(\beta_2)} 2^{4m_j} c_0^{\frac{m_j}{2}} \sum_{a=1}^2 \|J_{m_j}^{l+1}(\beta_a)\|_{0,0} \right) \\
 & \quad \cdot \sum_{m_1=2}^{N(\beta_2)} 2^{4m_1} c_0^{\frac{m_1}{2}} \left(|J_{m_1}^{l+1}(\beta_1) - J_{m_1}^{l+1}(\beta_2)|_0 \right. \\
 & \quad \quad \left. + \beta_1^{-\frac{1}{2}} \sum_{r=0}^1 \sum_{a=1}^2 \|J_{m_1}^{l+1}(\beta_a)\|_{0,r} \right) \\
 & \quad \cdot \mathbf{1}_{\sum_{j=1}^n m_j - 2n + 2 \geq m}.
 \end{aligned}$$

The hypothesis of induction implies that

$$(5.39) \quad \sum_{m=2}^{N(\beta_2)} 2^{4m} c_0^{\frac{m}{2}} |J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_0 \leq c \beta_1^{-\frac{1}{2}} \alpha^{-2}.$$

Since the inequalities (5.11), (5.12) are available, we can also claim that

$$(5.40) \quad \sum_{m=2}^{N(\beta_2)} 2^{4m} c_0^{\frac{m}{2}} \sum_{a=1}^2 \|J_m^{l+1}(\beta_a)\|_{0,r} \leq c \alpha^{-2}, \quad (\forall r \in \{0, 1\}).$$

Using (5.39), (5.40), we obtain from (5.38) that for $m \in \{2, 4\}$,

$$|T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_0 \leq c^n \beta_1^{-\frac{1}{2}} c_0^{-\frac{m}{2}} M^{-(l+1)(n-1)} \alpha^{-2n}.$$

Moreover, by the assumption $\alpha \geq c$,

$$(5.41) \quad \sum_{n=2}^{\infty} |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_0 \leq c \beta_1^{-\frac{1}{2}} c_0^{-\frac{m}{2}} M^{-l-1} \alpha^{-4}, \quad (\forall m \in \{2, 4\}).$$

By using (5.12), (5.37) for $l+1$ we can derive from (5.38) that

$$\begin{aligned} & M^{-2l} \sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_0 \\ & \leq c^n M^{-2l} \cdot M^{-(l+1)(n-1)} \prod_{j=2}^n \left(\sum_{m_j=2}^{N(\beta_2)} 2^{4m_j} c_0^{\frac{m_j}{2}} \sum_{a=1}^2 \|J_{m_j}^{l+1}(\beta_a)\|_{0,0} \right) \\ & \quad \cdot \sum_{m_1=2}^{N(\beta_2)} 2^{4m_1} c_0^{\frac{m_1}{2}} \left(|J_{m_1}^{l+1}(\beta_1) - J_{m_1}^{l+1}(\beta_2)|_0 + \beta_1^{-\frac{1}{2}} \sum_{r=0}^1 \sum_{a=1}^2 \|J_{m_1}^{l+1}(\beta_a)\|_{0,r} \right) \\ & \quad \cdot M^{\frac{l}{2}(\sum_{j=1}^n m_j - 2n + 2)} \alpha^{\sum_{j=1}^n m_j - 2n + 2} 2^{-2(\sum_{j=1}^n m_j - 2n + 2)} \\ & \leq c^n M^{-2l} \prod_{j=2}^n \left(M^{-2l-1} \alpha^{-2} \sum_{m_j=2}^{N(\beta_2)} 2^{2m_j} c_0^{\frac{m_j}{2}} M^{\frac{l}{2}m_j} \alpha^{m_j} \sum_{a=1}^2 \|J_{m_j}^{l+1}(\beta_a)\|_{0,0} \right) \\ & \quad \cdot \sum_{m_1=2}^{N(\beta_2)} 2^{2m_1} c_0^{\frac{m_1}{2}} M^{\frac{l}{2}m_1} \alpha^{m_1} \end{aligned}$$

$$\begin{aligned} & \cdot \left(|J_{m_1}^{l+1}(\beta_1) - J_{m_1}^{l+1}(\beta_2)|_0 + \beta_1^{-\frac{1}{2}} \sum_{r=0}^1 \sum_{a=1}^2 \|J_{m_1}^{l+1}(\beta_a)\|_{0,r} \right) \\ & \leq c\beta_1^{-\frac{1}{2}} M(c\alpha^{-2})^{n-1}. \end{aligned}$$

Thus, on the assumption $\alpha \geq c$,

$$(5.42) \quad M^{-2l} \sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m \sum_{n=2}^{\infty} |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_0 \leq c\beta_1^{-\frac{1}{2}} M\alpha^{-2}.$$

On the other hand, Lemma 4.1 (2), (5.3) and (5.31) imply that for $m \in \{6, 7, \dots, N(\beta_2)\}$,

$$\begin{aligned} |F_m^l(\beta_1) - F_m^l(\beta_2)|_0 & \leq c \sum_{n=m}^{N(\beta_2)} 2^{2n} c_0^{\frac{n-m}{2}} \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|_0 \right. \\ & \quad \left. + \beta_1^{-\frac{1}{2}} \sum_{a=1}^2 \|J_n^{l+1}(\beta_a)\|_{0,0} + \beta_1^{-1} \sum_{a=1}^2 \|J_n^{l+1}(\beta_a)\|_{0,1} \right). \end{aligned}$$

Thus, by (5.12), (5.37) for $l+1$,

$$\begin{aligned} (5.43) \quad & M^{-2l} \sum_{m=6}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m |F_m^l(\beta_1) - F_m^l(\beta_2)|_0 \\ & \leq M^{-2l} \sum_{n=6}^{N(\beta_2)} \sum_{m=6}^n 2^{2n} c_0^{\frac{n}{2}} M^{\frac{l}{2}m} \alpha^m \\ & \quad \cdot \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|_0 + \beta_1^{-\frac{1}{2}} \sum_{r=0}^1 \sum_{a=1}^2 \|J_n^{l+1}(\beta_a)\|_{0,r} \right) \\ & \leq cM^{-2l-3} \sum_{n=6}^{N(\beta_2)} c_0^{\frac{n}{2}} M^{\frac{l+1}{2}n} \alpha^n \\ & \quad \cdot \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|_0 + \beta_1^{-\frac{1}{2}} \sum_{r=0}^1 \sum_{a=1}^2 \|J_n^{l+1}(\beta_a)\|_{0,r} \right) \\ & \leq c\beta_1^{-\frac{1}{2}} M^{-1}. \end{aligned}$$

Let us find an upper bound on $|F_m^l(\beta_1) - F_m^l(\beta_2)|_0$ ($m = 2, 4$). Lemma 4.1 (2), (5.3), (5.31) and the inequalities (5.12), (5.37), (5.41) for $l' \in \{l + 1, l + 2, \dots, N_+\}$ guarantee that

$$\begin{aligned}
(5.44) \quad & |F_4^l(\beta_1) - F_4^l(\beta_2)|_0 \\
& \leq |F_4^{l+1}(\beta_1) - F_4^{l+1}(\beta_2)|_0 + \sum_{n=2}^{\infty} |T_4^{l+1,(n)}(\beta_1) - T_4^{l+1,(n)}(\beta_2)|_0 \\
& \quad + c \sum_{n=6}^{N(\beta_2)} 2^{2n} c_0^{\frac{n-4}{2}} \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|_0 \right. \\
& \quad \quad \left. + \beta_1^{-\frac{1}{2}} \sum_{a=1}^2 \|J_n^{l+1}(\beta_a)\|_{0,0} + \beta_1^{-1} \sum_{a=1}^2 \|J_n^{l+1}(\beta_a)\|_{0,1} \right) \\
& \leq |F_4^{l+1}(\beta_1) - F_4^{l+1}(\beta_2)|_0 + c\beta_1^{-\frac{1}{2}} c_0^{-2} M^{-l-1} \alpha^{-4} \\
& \leq |F_4^{N_+}(\beta_1) - F_4^{N_+}(\beta_2)|_0 + c\beta_1^{-\frac{1}{2}} c_0^{-2} \sum_{j=l}^{N_+-1} M^{-j-1} \alpha^{-4} \\
& \leq c\beta_1^{-\frac{1}{2}} c_0^{-2} M^{-l-1} \alpha^{-4},
\end{aligned}$$

which implies that

$$(5.45) \quad c_0^2 \alpha^4 |F_4^l(\beta_1) - F_4^l(\beta_2)|_0 \leq c\beta_1^{-\frac{1}{2}} M^{-1}.$$

Remark that

$$\mathcal{P}_2 \int J_4^{N_+}(\beta_a)(\psi + \psi^1) d\mu_{C_{o,l+1}(\beta_a)}(\psi^1) = \left(\frac{1}{h}\right)^2 \sum_{\mathbf{x} \in I(\beta_a)^2} K_2^{l+1}(\beta_a)(\mathbf{X}) \psi_{\mathbf{x}}$$

with the anti-symmetric kernel $K_2^{l+1}(\beta_a)(\cdot) : I(\beta_a)^2 \rightarrow \mathbb{C}$ defined by

$$\begin{aligned}
& K_2^{l+1}(\beta_a)((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)) \\
& := -\frac{h}{2} U_\rho C_{o,l+1}(\beta_a)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0) \\
& \quad \cdot \mathbf{1}_{(\rho, \mathbf{x}, \sigma, x) = (\eta, \mathbf{y}, \tau, y)} (1_{(\theta, \xi) = (1, -1)} - 1_{(\theta, \xi) = (-1, 1)}).
\end{aligned}$$

We can see that

$$(5.46) \quad |K_2^{l+1}(\beta_1) - K_2^{l+1}(\beta_2)|_0$$

$$\leq U_{max} \sup_{\rho \in \mathcal{B}} |C_{o,l+1}(\beta_1)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0) - C_{o,l+1}(\beta_2)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)|.$$

It follows from (5.20) and Lemma 4.1 (2), (5.3), (5.31) that

$$\begin{aligned} & |F_2^l(\beta_1) - F_2^l(\beta_2)|_0 \\ & \leq |F_2^{l+1}(\beta_1) - F_2^{l+1}(\beta_2)|_0 + |T_2^{l+1}(\beta_1) - T_2^{l+1}(\beta_2)|_0 \\ & \quad + |K_2^{l+1}(\beta_1) - K_2^{l+1}(\beta_2)|_0 \\ & \quad + cc_0^{\frac{4-2}{2}} \left(|\hat{F}_4^{l+1}(\beta_1) - \hat{F}_4^{l+1}(\beta_2)|_0 + \beta_1^{-\frac{1}{2}} \sum_{a=1}^2 \|\hat{F}_4^{l+1}(\beta_a)\|_{0,0} \right. \\ & \quad \quad + \beta_1^{-1} \sum_{a=1}^2 \|\hat{F}_4^{l+1}(\beta_a)\|_{0,1} + |T_4^{l+1}(\beta_1) - T_4^{l+1}(\beta_2)|_0 \\ & \quad \quad \left. + \beta_1^{-\frac{1}{2}} \sum_{a=1}^2 \|T_4^{l+1}(\beta_a)\|_{0,0} + \beta_1^{-1} \sum_{a=1}^2 \|T_4^{l+1}(\beta_a)\|_{0,1} \right) \\ & \quad + c \sum_{n=6}^{N(\beta_2)} 2^{2n} c_0^{\frac{n-2}{2}} \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|_0 + \beta_1^{-\frac{1}{2}} \sum_{a=1}^2 \|J_n^{l+1}(\beta_a)\|_{0,0} \right. \\ & \quad \quad \left. + \beta_1^{-1} \sum_{a=1}^2 \|J_n^{l+1}(\beta_a)\|_{0,1} \right). \end{aligned}$$

Substitution of (5.12), (5.15), (5.18), (5.33), (5.37), (5.41), (5.44), (5.46) and the equality

$$(5.47) \quad |F_4^{l'}(\beta_1) - F_4^{l'}(\beta_2)|_0 = |\hat{F}_4^{l'}(\beta_1) - \hat{F}_4^{l'}(\beta_2)|_0$$

for $l' \in \{l+1, l+2, \dots, N_+\}$ yield that

$$\begin{aligned} & |F_2^l(\beta_1) - F_2^l(\beta_2)|_0 \\ & \leq |F_2^{l+1}(\beta_1) - F_2^{l+1}(\beta_2)|_0 + c\beta_1^{-\frac{1}{2}} c_0^{-1} M^{-l-1} \alpha^{-4} \\ & \quad + U_{max} \sup_{\rho \in \mathcal{B}} |C_{o,l+1}(\beta_1)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0) - C_{o,l+1}(\beta_2)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\ & \leq |F_2^{N_+}(\beta_1) - F_2^{N_+}(\beta_2)|_0 + c\beta_1^{-\frac{1}{2}} c_0^{-1} \sum_{j=l}^{N_+-1} M^{-j-1} \alpha^{-4} \\ & \quad + U_{max} \sum_{j=l}^{N_+-1} \sup_{\rho \in \mathcal{B}} |C_{o,j+1}(\beta_1)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0) - C_{o,j+1}(\beta_2)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \end{aligned}$$

$$\leq c\beta_1^{-\frac{1}{2}}c_0^{-1}M^{-l-1}\alpha^{-4} + U_{max}\beta_1^{-\frac{1}{2}}c'_0,$$

or

$$(5.48) \quad c_0\alpha^2|F_2^l(\beta_1) - F_2^l(\beta_2)|_0 \leq \beta_1^{-\frac{1}{2}}(c_0c'_0\alpha^2U_{max} + cM^{-1}\alpha^{-2}).$$

By (5.41), (5.42), (5.43), (5.45), (5.48) we have that

$$(5.49) \quad c_0\alpha^2 \left(|F_2^l(\beta_1) - F_2^l(\beta_2)|_0 + \sum_{n=2}^{\infty} |T_2^{l,(n)}(\beta_1) - T_2^{l,(n)}(\beta_2)|_0 \right) \\ \leq \beta_1^{-\frac{1}{2}}(c_0c'_0\alpha^2U_{max} + cM^{-1}\alpha^{-2}).$$

$$(5.50) \quad M^{-2l} \sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{1}{2}m} \alpha^m \left(|F_m^l(\beta_1) - F_m^l(\beta_2)|_0 \right. \\ \left. + \sum_{n=2}^{\infty} |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_0 \right) \\ \leq \sum_{m \in \{2,4\}} c_0^{\frac{m}{2}} \alpha^m |F_m^l(\beta_1) - F_m^l(\beta_2)|_0 \\ + M^{-2l} \sum_{m=6}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{1}{2}m} \alpha^m |F_m^l(\beta_1) - F_m^l(\beta_2)|_0 \\ + M^{-2l} \sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{1}{2}m} \alpha^m \sum_{n=2}^{\infty} |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_0 \\ \leq c\beta_1^{-\frac{1}{2}}(c_0c'_0\alpha^2U_{max} + M^{-1} + M\alpha^{-2}).$$

On the assumption (5.9) the right-hand sides of (5.49), (5.50) are less than $\beta_1^{-\frac{1}{2}}$. Thus, by induction the inequalities (5.36), (5.37) hold for all $l \in \{0, 1, \dots, N_+\}$.

(5.35): Let us prove the inequality (5.35), assuming that (5.36), (5.37) are true for all $l \in \{0, 1, \dots, N_+\}$. By substituting (5.3), (5.4), (5.14), (5.31), (5.32), (5.39) into the inequality in Lemma 4.6 (1) we obtain

$$\left| \frac{h}{N(\beta_1)} T_0^{l,(n)}(\beta_1) - \frac{h}{N(\beta_2)} T_0^{l,(n)}(\beta_2) \right|$$

$$\begin{aligned}
 &\leq cnc_0^{-n+1}(2c_0M^{-l-1})^{n-2} \left(\sum_{m=2}^{N(\beta_2)} 2^{3m} c_0^{\frac{m}{2}} \sum_{a=1}^2 \|J_m^{l+1}(\beta_a)\|_{0,0} \right)^{n-1} \\
 &\quad \cdot \sum_{m=2}^{N(\beta_2)} 2^{3m} c_0^{\frac{m}{2}} \left(\beta_1^{-1} c_0 M^{-l-1} \sum_{r=0}^1 \sum_{a=1}^2 \|J_m^{l+1}(\beta_a)\|_{0,r} \right. \\
 &\quad \left. + \beta_1^{-\frac{1}{2}} c_0 M^{-l-1} \sum_{a=1}^2 \|J_m^{l+1}(\beta_a)\|_{0,0} + c_0 M^{-l-1} |J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_0 \right) \\
 &\leq \beta_1^{-\frac{1}{2}} M^{l+1} (cM^{-l-1}\alpha^{-2})^n,
 \end{aligned}$$

which leads to

$$(5.51) \quad \sum_{n=2}^{\infty} \left| \frac{h}{N(\beta_1)} T_0^{l,(n)}(\beta_1) - \frac{h}{N(\beta_2)} T_0^{l,(n)}(\beta_2) \right| \leq c\beta_1^{-\frac{1}{2}} M^{-l-1} \alpha^{-4}.$$

It follows from (5.3), (5.25), (5.31) and Lemma 4.1 (2) that

$$\begin{aligned}
 &|\hat{F}_2^l(\beta_1) - \hat{F}_2^l(\beta_2)|_0 \\
 &\leq |\hat{F}_2^{l+1}(\beta_1) - \hat{F}_2^{l+1}(\beta_2)|_0 + |T_2^{l+1}(\beta_1) - T_2^{l+1}(\beta_2)|_0 \\
 &\quad + cc_0^{\frac{4-2}{2}} \left(|\hat{F}_4^{l+1}(\beta_1) - \hat{F}_4^{l+1}(\beta_2)|_0 + \beta_1^{-\frac{1}{2}} \sum_{a=1}^2 \|\hat{F}_4^{l+1}(\beta_a)\|_{0,0} \right. \\
 &\quad \left. + \beta_1^{-1} \sum_{a=1}^2 \|\hat{F}_4^{l+1}(\beta_a)\|_{0,1} + |T_4^{l+1}(\beta_1) - T_4^{l+1}(\beta_2)|_0 \right. \\
 &\quad \left. + \beta_1^{-\frac{1}{2}} \sum_{a=1}^2 \|T_4^{l+1}(\beta_a)\|_{0,0} + \beta_1^{-1} \sum_{a=1}^2 \|T_4^{l+1}(\beta_a)\|_{0,1} \right) \\
 &\quad + c \sum_{n=6}^{N(\beta_2)} 2^{2n} c_0^{\frac{n-2}{2}} \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|_0 + \beta_1^{-\frac{1}{2}} \sum_{a=1}^2 \|J_n^{l+1}(\beta_a)\|_{0,0} \right. \\
 &\quad \left. + \beta_1^{-1} \sum_{a=1}^2 \|J_n^{l+1}(\beta_a)\|_{0,1} \right).
 \end{aligned}$$

Using (5.12), (5.15), (5.18), (5.37), (5.41), (5.44), (5.47) for $l' \in \{l+1, l+2, \dots, N_+\}$, we can deduce that

$$(5.52) \quad |\hat{F}_2^l(\beta_1) - \hat{F}_2^l(\beta_2)|_0$$

$$\begin{aligned}
&\leq |\hat{F}_2^{l+1}(\beta_1) - \hat{F}_2^{l+1}(\beta_2)|_0 + c\beta_1^{-\frac{1}{2}}c_0^{-1}M^{-l-1}\alpha^{-4} \\
&\leq |\hat{F}_2^{N+}(\beta_1) - \hat{F}_2^{N+}(\beta_2)|_0 + c\beta_1^{-\frac{1}{2}}c_0^{-1}\sum_{j=l}^{N_+-1}M^{-j-1}\alpha^{-4} \\
&\leq c\beta_1^{-\frac{1}{2}}c_0^{-1}M^{-l-1}\alpha^{-4}.
\end{aligned}$$

By combining Lemma 4.1 (1) with (5.27) and inserting (5.3), (5.31) we obtain that

$$\begin{aligned}
&\left| \frac{h}{N(\beta_1)}F_0^l(\beta_1) - \frac{h}{N(\beta_2)}F_0^l(\beta_2) \right| \\
&\leq \left| \frac{h}{N(\beta_1)}F_0^{l+1}(\beta_1) - \frac{h}{N(\beta_2)}F_0^{l+1}(\beta_2) \right| \\
&\quad + \left| \frac{h}{N(\beta_1)}T_0^{l+1}(\beta_1) - \frac{h}{N(\beta_2)}T_0^{l+1}(\beta_2) \right| \\
&\quad + U_{max} \max_{\rho \in \mathcal{B}} |C_{o,l+1}(\beta_1)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0) - C_{o,l+1}(\beta_2)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\
&\quad + 1_{l \leq N_+-2} U_{max} \max_{\rho \in \mathcal{B}} |C_{o,l+1}(\beta_1)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0) - C_{o,l+1}(\beta_2)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\
&\quad \cdot \sum_{j=l+2}^{N_+} \max_{\eta \in \mathcal{B}} |C_{o,j}(\beta_1)(\eta \mathbf{0} \uparrow 0, \eta \mathbf{0} \uparrow 0)| \\
&\quad + 1_{l \leq N_+-2} U_{max} \max_{\rho \in \mathcal{B}} |C_{o,l+1}(\beta_2)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\
&\quad \cdot \sum_{j=l+2}^{N_+} \max_{\eta \in \mathcal{B}} |C_{o,j}(\beta_1)(\eta \mathbf{0} \uparrow 0, \eta \mathbf{0} \uparrow 0) - C_{o,j}(\beta_2)(\eta \mathbf{0} \uparrow 0, \eta \mathbf{0} \uparrow 0)| \\
&\quad + U_{max} \max_{\rho \in \mathcal{B}} |C_{o,l+1}(\beta_1)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)^2 - C_{o,l+1}(\beta_2)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)^2| \\
&\quad + c \sum_{m \in \{2,4\}} c_0^{\frac{m}{2}} \left(\left| \hat{F}_m^{l+1}(\beta_1) - \hat{F}_m^{l+1}(\beta_2) \right|_0 + \beta_1^{-\frac{1}{2}} \sum_{a=1}^2 \|\hat{F}_m^{l+1}(\beta_a)\|_{0,0} \right. \\
&\quad + \beta_1^{-1} \sum_{a=1}^2 \|\hat{F}_m^{l+1}(\beta_a)\|_{0,1} + |T_m^{l+1}(\beta_1) - T_m^{l+1}(\beta_2)|_0 \\
&\quad \left. + \beta_1^{-\frac{1}{2}} \sum_{a=1}^2 \|T_m^{l+1}(\beta_a)\|_{0,0} + \beta_1^{-1} \sum_{a=1}^2 \|T_m^{l+1}(\beta_a)\|_{0,1} \right)
\end{aligned}$$

$$\begin{aligned}
 & + c \sum_{m=6}^{N(\beta_2)} 2^m c_0^{\frac{m}{2}} \left(|J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_0 + \beta_1^{-\frac{1}{2}} \sum_{a=1}^2 \|J_m^{l+1}(\beta_a)\|_{0,0} \right. \\
 & \left. + \beta_1^{-1} \sum_{a=1}^2 \|J_m^{l+1}(\beta_a)\|_{0,1} \right).
 \end{aligned}$$

Moreover, substitution of (5.3), (5.5), (5.12), (5.15), (5.18), (5.26), (5.33), (5.37), (5.41), (5.44), (5.47), (5.51), (5.52) for $l' \in \{l+1, l+2, \dots, N_+\}$ gives that

$$\begin{aligned}
 (5.53) \quad & \left| \frac{h}{N(\beta_1)} F_0^l(\beta_1) - \frac{h}{N(\beta_2)} F_0^l(\beta_2) \right| \\
 & \leq \left| \frac{h}{N(\beta_1)} F_0^{l+1}(\beta_1) - \frac{h}{N(\beta_2)} F_0^{l+1}(\beta_2) \right| + c\beta_1^{-\frac{1}{2}} M^{-l-1} \alpha^{-4} \\
 & \quad + c(c_0 + c'_0) U_{max} \max_{\rho \in \mathcal{B}} |C_{o,l+1}(\beta_1)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0) \\
 & \quad - C_{o,l+1}(\beta_2)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\
 & \quad + c\beta_1^{-\frac{1}{2}} c'_0 U_{max} \max_{\rho \in \mathcal{B}} |C_{o,l+1}(\beta_2)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\
 & \leq c\beta_1^{-\frac{1}{2}} \sum_{j=l}^{N_+-1} M^{-j-1} \alpha^{-4} \\
 & \quad + c(c_0 + c'_0) U_{max} \\
 & \quad \cdot \sum_{j=l}^{N_+-1} \max_{\rho \in \mathcal{B}} |C_{o,j+1}(\beta_1)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0) - C_{o,j+1}(\beta_2)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\
 & \quad + c\beta_1^{-\frac{1}{2}} c'_0 U_{max} \sum_{j=l}^{N_+-1} \max_{\rho \in \mathcal{B}} |C_{o,j+1}(\beta_2)(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)| \\
 & \leq c\beta_1^{-\frac{1}{2}} (M^{-1} \alpha^{-4} + (c_0 + c'_0) c'_0 U_{max}).
 \end{aligned}$$

By coupling (5.51) with (5.53) and using the assumption (5.9) we conclude that

$$\begin{aligned}
 & \left| \frac{h}{N(\beta_1)} F_0^l(\beta_1) - \frac{h}{N(\beta_2)} F_0^l(\beta_2) \right| \\
 & \quad + \sum_{n=2}^{\infty} \left| \frac{h}{N(\beta_1)} T_0^{l,(n)}(\beta_1) - \frac{h}{N(\beta_2)} T_0^{l,(n)}(\beta_2) \right|
 \end{aligned}$$

$$\leq c\beta_1^{-\frac{1}{2}}(M^{-1}\alpha^{-4} + (c_0 + c'_0)c'_0U_{max}) \leq \beta_1^{-\frac{1}{2}}\alpha^{-4}. \quad \square$$

5.3. The generalized infrared integration

In this subsection we estimate Grassmann polynomials produced by a single-scale integration with a covariance which has different bound properties from those assumed in the previous subsection. Our aim here is to summarize a power-counting procedure of the infrared integration by giving a covariance with bound properties typical of a real covariance with infrared cut-off. In the model-dependent infrared integration regime in Section 7, we need to update the covariance by including the kernel of the quadratic part of a Grassmann polynomial created by the preceding integration. This means that in the IR integration, unlike in the UV integration, we cannot a priori give covariances for all the integration steps. For this reason here we construct estimates only for one integration step as a preliminary to the practical IR integration.

Let $l \in \mathbb{Z}_{<0}$. We assume that an exponent $r \in (0, 1]$ and weights $w(l)$, $w(l+1)$ satisfying $0 < w(l) \leq w(l+1)$ are given and a covariance $C_{o,l+1} : I_0^2 \rightarrow \mathbb{C}$ satisfies the following bound properties with constants $M, c_0 \in \mathbb{R}_{\geq 1}$, $a_1, a_2, a_3 \in \mathbb{R}_{\geq 0}$.

$$(5.54) \quad \begin{aligned} |\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}} C_{o,l+1}(X_i, Y_j))_{1 \leq i, j \leq n}| &\leq (c_0 M^{a_1(l+1)})^n, \\ (\forall r, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} &\leq 1, \\ X_i, Y_i \in I_0 \ (i = 1, 2, \dots, n)), \end{aligned}$$

$$(5.55) \quad \left\| \widetilde{C_{o,l+1}} \right\|_{l,r} \leq c_0 M^{-a_2(l+1) - ra_3(l+1)}, \quad (\forall r \in \{0, 1\}),$$

where $\widetilde{C_{o,l+1}} : I^2 \rightarrow \mathbb{C}$ is the anti-symmetric extension of $C_{o,l+1}$ defined as in (3.2). Recall that the parameters $(w(l), r)$, $(w(l+1), r)$ are used in the definition of $\|\cdot\|_{l,j}$, $\|\cdot\|_{l+1,j}$ respectively.

We assume that $J^{l+1}(\psi) (\in \bigwedge \mathcal{V})$ is given and it satisfies $J_m^{l+1}(\psi) = 0$ if $m \notin 2\mathbb{N}$ or $m \in \{0, 2\}$. Then, we define $F^l(\psi)$, $T^{l,(n)}(\psi)$ ($n \in \mathbb{N}_{\geq 2}$), $T^l(\psi) (\in \bigwedge \mathcal{V})$ by (5.7) with the input $J^{l+1}(\psi)$ and the covariance $C_{o,l+1}$ on the assumption that $\sum_{n=2}^{\infty} T^{l,(n)}(\psi)$ converges. We can prove the following lemma by the same argument as in the proof of Lemma 5.1.

LEMMA 5.5. *Assume that $J^l(\psi)$ is well-defined. Then, if $m \notin 2\mathbb{N} \cup \{0\}$,*

$$T_m^{l,(n)}(\psi) = F_m^l(\psi) = 0, \quad (\forall n \in \mathbb{N}_{\geq 2}).$$

This subsection is devoted to proving the following proposition.

PROPOSITION 5.6. *Assume that $a_4 \in \mathbb{R}_{\geq 0}$ and*

$$(5.56) \quad M^{-(a_1+a_2+a_4)(l+1)+ra_3(l+1)} \sum_{m=4}^N c_0^{\frac{m}{2}} M^{\frac{a_1(l+1)}{2}m} \alpha^m \|J_m^{l+1}\|_{l+1,r} \leq 1, \\ (\forall r \in \{0, 1\}).$$

Then, there exists a constant $c \in \mathbb{R}_{>1}$ independent of any parameter such that if the parameters $M, \alpha \in \mathbb{R}_{\geq 1}$ satisfy

$$(5.57) \quad M^{a_1-a_2-a_4} \geq c, \quad \alpha \geq cM^{a_1+a_2+a_4},$$

the following inequalities hold.

$$(5.58) \quad \frac{h}{N} \left(|F_0^l| + \sum_{n=2}^{\infty} |T_0^{l,(n)}| \right) \leq M^{(a_1+a_2+a_4)l} \alpha^{-3},$$

$$(5.59) \quad M^{-(a_1+a_2+a_4)l+ra_3l} \sum_{m=2}^N c_0^{\frac{m}{2}} M^{\frac{a_1l}{2}m} \alpha^m \\ \cdot \left(\|F_m^l\|_{l,r} + \sum_{n=2}^{\infty} \|T_m^{l,(n)}\|_{l,r} \right) \leq 1, \quad (\forall r \in \{0, 1\}).$$

PROOF. It follows from (5.56) and the inequalities $w(l) \leq w(l+1)$, $\alpha \geq c$, $M^{a_1} \geq c$ that

$$(5.60) \quad \sum_{m=4}^N 2^{3m} c_0^{\frac{m}{2}} M^{\frac{a_1(l+1)}{2}m} \|J_m^{l+1}\|_{l,r} \leq cM^{(a_1+a_2+a_4)(l+1)-ra_3(l+1)} \alpha^{-4},$$

$$(5.61) \quad \sum_{m=4}^N 2^{2m} c_0^{\frac{m}{2}} M^{\frac{a_1l}{2}m} \alpha^m \|J_m^{l+1}\|_{l,r} \leq cM^{(a_1+a_2+a_4)(l+1)-ra_3(l+1)-2a_1},$$

$(\forall r \in \{0, 1\})$.

We will use these inequalities not only in this proof but also in the proof of Proposition 5.9 in the next subsection.

(5.58): First let us prove the inequality (5.58). By Lemma 3.1, (5.54) and (5.60),

$$(5.62) \quad \frac{h}{N} |F_0^l| \leq \sum_{m=4}^N (c_0 M^{a_1(l+1)})^{\frac{m}{2}} \|J_m^{l+1}\|_{l,0} \leq cM^{(a_1+a_2+a_4)(l+1)} \alpha^{-4}.$$

On the other hand, by substituting (5.54), (5.55), (5.60) into the inequality in Lemma 3.8 (1) we have that

$$\begin{aligned} & \frac{h}{N} |T_0^{l,(n)}| \\ & \leq (c_0 M^{a_1(l+1)})^{-n+1} (c_0 M^{-a_2(l+1)})^{n-1} \left(\sum_{m=4}^N 2^{2m} (c_0 M^{a_1(l+1)})^{\frac{m}{2}} \|J_m^{l+1}\|_{l,0} \right)^n \\ & \leq M^{(a_1+a_2)(l+1)} (cM^{a_4(l+1)} \alpha^{-4})^n. \end{aligned}$$

Thus, by the assumption $\alpha \geq c$,

$$(5.63) \quad \frac{h}{N} \sum_{n=2}^{\infty} |T_0^{l,(n)}| \leq cM^{(a_1+a_2+2a_4)(l+1)} \alpha^{-8}.$$

By coupling (5.63) with (5.62) and using the assumption $\alpha \geq cM^{a_1+a_2+a_4}$ we obtain (5.58).

(5.59): Next let us show (5.59). It follows from Lemma 3.1 and (5.54) that

$$(5.64) \quad \begin{aligned} & \|F_m^l\|_{l,r} \\ & \leq 1_{m=2} \sum_{n=4}^N 2^n (c_0 M^{a_1(l+1)})^{\frac{n-2}{2}} \|J_n^{l+1}\|_{l,r} \\ & \quad + 1_{m \geq 4} \left(\|J_m^{l+1}\|_{l,r} + \sum_{n=m+2}^N 2^n (c_0 M^{a_1(l+1)})^{\frac{n-m}{2}} \|J_n^{l+1}\|_{l,r} \right). \end{aligned}$$

Moreover, by (5.60), (5.61) and the assumption $\alpha \geq cM^{a_1/2}$,

$$\begin{aligned}
 (5.65) \quad & M^{-(a_1+a_2+a_4)l+ra_3l} \sum_{m=2}^N c_0^{\frac{m}{2}} M^{\frac{a_1l}{2}m} \alpha^m \|F_m^l\|_{l,r} \\
 & \leq M^{-(a_1+a_2+a_4)l+ra_3l} c_0 M^{a_1l} \alpha^2 \sum_{n=4}^N 2^n (c_0 M^{a_1(l+1)})^{\frac{n-2}{2}} \|J_n^{l+1}\|_{l,r} \\
 & \quad + M^{-(a_1+a_2+a_4)l+ra_3l} \\
 & \quad \cdot \sum_{n=4}^N \sum_{m=4}^n c_0^{\frac{m}{2}} M^{\frac{a_1l}{2}m} \alpha^m 2^n (c_0 M^{a_1(l+1)})^{\frac{n-m}{2}} \|J_n^{l+1}\|_{l,r} \\
 & \leq M^{-a_1} \alpha^2 M^{-(a_1+a_2+a_4)l+ra_3l} \sum_{n=4}^N 2^n c_0^{\frac{n}{2}} M^{\frac{a_1(l+1)}{2}n} \|J_n^{l+1}\|_{l,r} \\
 & \quad + cM^{-(a_1+a_2+a_4)l+ra_3l} \sum_{n=4}^N 2^n c_0^{\frac{n}{2}} M^{\frac{a_1l}{2}n} \alpha^n \|J_n^{l+1}\|_{l,r} \\
 & \leq cM^{-a_1+a_2+a_4-ra_3}.
 \end{aligned}$$

On the other hand, insertion of (5.54), (5.55) into Lemma 3.8 (2) yields that

$$\begin{aligned}
 & \|T_m^{l,(n)}\|_{l,r} \\
 & \leq 2^{-2m} (c_0 M^{a_1(l+1)})^{-\frac{m}{2}-n+1} \prod_{i=1}^n \binom{1}{q_i=0} \prod_{j=2}^n \binom{1}{r_j=0} \mathbf{1}_{\sum_{i=1}^n q_i + \sum_{j=2}^n r_j = r} \\
 & \quad \cdot \prod_{k=2}^n (c_0 M^{-a_2(l+1)-a_3r_k(l+1)}) \prod_{p=1}^n \left(\sum_{m_p=4}^N 2^{3m_p} (c_0 M^{a_1(l+1)})^{\frac{m_p}{2}} \|J_{m_p}^{l+1}\|_{l,q_p} \right) \\
 & \quad \cdot \mathbf{1}_{\sum_{j=1}^n m_j - 2n + 2 \geq m} \\
 & = 2^{-2m} c_0^{-\frac{m}{2}} M^{-\frac{a_1(l+1)}{2}m + (a_1+a_2)(l+1) - ra_3(l+1)} \\
 & \quad \cdot \prod_{i=1}^n \binom{1}{q_i=0} \prod_{j=2}^n \binom{1}{r_j=0} \mathbf{1}_{\sum_{i=1}^n q_i + \sum_{j=2}^n r_j = r} \\
 & \quad \cdot \prod_{k=1}^n \left(M^{-(a_1+a_2)(l+1)+a_3q_k(l+1)} \sum_{m_k=4}^N 2^{3m_k} c_0^{\frac{m_k}{2}} M^{\frac{a_1(l+1)}{2}m_k} \|J_{m_k}^{l+1}\|_{l,q_k} \right)
 \end{aligned}$$

$$\cdot 1_{\sum_{j=1}^n m_j - 2n + 2 \geq m}.$$

By using the inequality $\alpha \geq cM^{a_1/2}$ and (5.61) we can derive from the inequality above that

$$\begin{aligned} & M^{-(a_1+a_2+a_4)l+ra_3l} \sum_{m=2}^N c_0^{\frac{m}{2}} M^{\frac{a_1l}{2}} m \alpha^m \|T_m^{l,(n)}\|_{l,r} \\ & \leq cM^{a_1+a_2-ra_3-a_4l} \prod_{i=1}^n \binom{1}{q_i=0} \prod_{j=2}^n \binom{1}{r_j=0} 1_{\sum_{i=1}^n q_i + \sum_{j=2}^n r_j = r} \\ & \quad \cdot \prod_{k=1}^n \left(M^{-(a_1+a_2)(l+1)+a_3q_k(l+1)} \sum_{m_k=4}^N 2^{3m_k} c_0^{\frac{m_k}{2}} M^{\frac{a_1(l+1)}{2}m_k} \|J_{m_k}^{l+1}\|_{l,q_k} \right) \\ & \quad \cdot 2^{-2(\sum_{j=1}^n m_j - 2n + 2)} M^{-\frac{a_1}{2}(\sum_{j=1}^n m_j - 2n + 2)} \alpha^{\sum_{j=1}^n m_j - 2n + 2} \\ & \leq cM^{a_2-ra_3-a_4l} \alpha^2 \prod_{i=1}^n \binom{1}{q_i=0} \prod_{j=2}^n \binom{1}{r_j=0} 1_{\sum_{i=1}^n q_i + \sum_{j=2}^n r_j = r} \\ & \quad \cdot \prod_{k=1}^n \left(cM^{a_1-(a_1+a_2)(l+1)+a_3q_k(l+1)} \alpha^{-2} \right. \\ & \quad \cdot \left. \sum_{m_k=4}^N 2^{m_k} c_0^{\frac{m_k}{2}} M^{\frac{a_1l}{2}m_k} \alpha^{m_k} \|J_{m_k}^{l+1}\|_{l,q_k} \right) \\ & \leq M^{a_2-ra_3-a_4l} \alpha^2 (cM^{-a_1+a_4(l+1)} \alpha^{-2})^n. \end{aligned}$$

Since $\alpha \geq c$,

$$(5.66) \quad \begin{aligned} & M^{-(a_1+a_2+a_4)l+ra_3l} \sum_{m=2}^N c_0^{\frac{m}{2}} M^{\frac{a_1l}{2}} m \alpha^m \sum_{n=2}^{\infty} \|T_m^{l,(n)}\|_{l,r} \\ & \leq cM^{-2a_1+a_2+2a_4-ra_3+a_4l} \alpha^{-2}. \end{aligned}$$

The inequalities (5.65), (5.66) imply that

$$\begin{aligned} & M^{-(a_1+a_2+a_4)l+ra_3l} \sum_{m=2}^N c_0^{\frac{m}{2}} M^{\frac{a_1l}{2}} m \alpha^m \left(\|F_m^l\|_{l,r} + \sum_{n=2}^{\infty} \|T_m^{l,(n)}\|_{l,r} \right) \\ & \leq c(M^{-a_1+a_2+a_4} + M^{-2a_1+a_2+a_4} \alpha^{-2}). \end{aligned}$$

By the assumption (5.57) the right-hand side of the inequality above is less than 1. \square

REMARK 5.7. Since Proposition 5.6 forms the basis of our IR integration process, it is important to know to which model the proposition does or does not apply. The covariance $C_{o,l+1}$ is a generalization of an effective covariance at the IR integration of scale $l + 1$, which is different from a computable free covariance with IR cut-off of scale $l + 1$. Therefore, by analyzing free covariances alone we cannot reach a rigorous statement on the applicability of the proposition. However, on the hypothesis that the IR singularity of an effective covariance is essentially same as that of a free covariance, let us try to extract at least some hints from calculations of free covariances. The condition (5.57) necessarily implies that $a_1 - a_2 > 0$. Let us investigate in which model the inequality $a_1 - a_2 > 0$ is unlikely to hold. Most studied many-electron models in constructive theories so far are single-band models having a non-empty free Fermi surface. So let us focus our attention on such models.

Assume that the free dispersion relation $\mathcal{E}(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}$ is Lipschitz continuous and satisfies the periodicity (2.2) and that $\{\mathbf{k} \in \mathbb{R}^d \mid \mathcal{E}(\mathbf{k}) = \mu\} \neq \emptyset$ with the chemical potential μ . For example, in the Hubbard model with nearest-neighbor hopping, without magnetic field, defined on a d -dimensional hyper-cubic lattice, the free dispersion relation is given by $\mathcal{E}(\mathbf{k}) = \sum_{j=1}^d \cos k_j$, apart from a multiplication of amplitude. With a non-negative smooth function $\chi(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ supported on the interval $[c_1, c_2]$, taking the value 1 on a subinterval of $[c_1, c_2]$, a free covariance $C_{o,l+1} : (\Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)_h)^2 \rightarrow \mathbb{C}$ with IR cut-off of scale $l + 1$ (< 0) typically takes the form that

$$\begin{aligned} & C_{o,l+1}(\mathbf{x}\sigma x, \mathbf{y}\tau y) \\ &= \frac{\delta_{\sigma,\tau}}{\beta L^d} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \frac{\chi(M^{-2(l+1)}(\omega^2 + (\mathcal{E}(\mathbf{k}) - \mu)^2))}{i\omega - \mathcal{E}(\mathbf{k}) + \mu}. \end{aligned}$$

On the assumption that h, L, β are sufficiently large we can choose $(\omega_0, \mathbf{k}_0) \in \mathcal{M}_h \times \Gamma^*$ so that $\chi(M^{-2(l+1)}(\omega_0^2 + (\mathcal{E}(\mathbf{k}_0) - \mu)^2)) = 1$ and thus

$$\left\| \widetilde{C_{o,l+1}} \right\|_{l,0} \geq \frac{1}{2h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} |C_{o,l+1}(\mathbf{0}\uparrow 0, \mathbf{x}\uparrow x)|$$

$$\begin{aligned}
&\geq \left| \frac{1}{2h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} e^{i\langle \mathbf{x}, \mathbf{k}_0 \rangle} e^{ix\omega_0} C_{o, l+1}(\mathbf{0} \uparrow 0, \mathbf{x} \uparrow x) \right| \\
&= \frac{\chi(M^{-2(l+1)}(\omega_0^2 + (\mathcal{E}(\mathbf{k}_0) - \mu)^2))}{2|i\omega_0 - \mathcal{E}(\mathbf{k}_0) + \mu|} \geq \text{const } M^{-l-1}.
\end{aligned}$$

This means that if $C_{o, l+1}$ satisfies (5.55) with small l , then $a_2 \geq 1$.

Estimation of the determinant of many-electron covariances is normally done by applying Gram's inequality. By following this standard approach we eventually have that

$$\begin{aligned}
&\text{(the left-hand side of (5.54) with } C_{o, l+1}\text{)} \\
&\leq \left(\frac{\text{const}}{\beta L^d} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} \frac{\chi(M^{-2(l+1)}(\omega^2 + (\mathcal{E}(\mathbf{k}) - \mu)^2))}{|i\omega - \mathcal{E}(\mathbf{k}) + \mu|} \right)^n.
\end{aligned}$$

So it comes down to estimating

$$(5.67) \quad \frac{1}{\beta L^d} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} \frac{\chi(M^{-2(l+1)}(\omega^2 + (\mathcal{E}(\mathbf{k}) - \mu)^2))}{|i\omega - \mathcal{E}(\mathbf{k}) + \mu|}.$$

If $\mathcal{E}(\mathbf{k}) = \mu$ ($\forall \mathbf{k} \in \mathbb{R}^d$),

$$\begin{aligned}
\text{(the term (5.67))} &\geq \text{const } M^{-l-1} \frac{1}{\beta} \sum_{\omega \in \mathcal{M}_h} \chi(M^{-2(l+1)}\omega^2) \geq \text{const} \\
&\geq \text{const } M^l.
\end{aligned}$$

Let us consider the case that $\mathcal{E}(\mathbf{k}) \neq \mu$ for some $\mathbf{k} \in \mathbb{R}^d$. Set

$$B := \left\{ \sum_{j=1}^d p_j \mathbf{v}_j \mid p_j \in [0, 2\pi] \ (j = 1, 2, \dots, d) \right\}.$$

It follows that $\text{esssup}_{\mathbf{k} \in B} |\nabla \mathcal{E}(\mathbf{k})| \neq 0$. In this case we further assume that there exists an interval (α_1, α_2) containing 0 such that

$$(5.68) \quad \inf_{\eta \in (\alpha_1, \alpha_2)} \mathcal{H}^{d-1}(\{\mathbf{k} \in B \mid \mathcal{E}(\mathbf{k}) - \mu = \eta\}) > 0,$$

where \mathcal{H}^{d-1} denotes the $d-1$ -dimensional Hausdorff measure on \mathbb{R}^d . For $l \in \mathbb{Z}_{<0}$ satisfying $[-c_2^{1/2}M^{l+1}, c_2^{1/2}M^{l+1}] \subset (\alpha_1, \alpha_2)$ we deduce by the coarea formula that

$$\begin{aligned} & \lim_{\substack{L \rightarrow \infty \\ L \in \mathbb{N}}} (\text{the term (5.67)}) \\ & \geq \text{const} \left(\text{esssup}_{\mathbf{k} \in B} |\nabla \mathcal{E}(\mathbf{k})| \right)^{-1} M^{-l-1} \\ & \quad \cdot \frac{1}{\beta} \sum_{\omega \in \mathcal{M}_h} \int_{-\infty}^{\infty} d\eta \chi(M^{-2(l+1)}(\omega^2 + \eta^2)) \mathcal{H}^{d-1}(\{\mathbf{k} \in B \mid \mathcal{E}(\mathbf{k}) - \mu = \eta\}) \\ & \geq \text{const} \left(\text{esssup}_{\mathbf{k} \in B} |\nabla \mathcal{E}(\mathbf{k})| \right)^{-1} \inf_{\eta \in (\alpha_1, \alpha_2)} \mathcal{H}^{d-1}(\{\mathbf{k} \in B \mid \mathcal{E}(\mathbf{k}) - \mu = \eta\}) \\ & \quad \cdot M^{l+1}. \end{aligned}$$

Therefore, if either $\{\mathbf{k} \in B \mid \mathcal{E}(\mathbf{k}) - \mu = 0\} = B$ or (5.68) holds, an estimation based on Gram's inequality can hardly yield the determinant bound (5.54) with $a_1 > 1$ for small l and large L . However, if $a_1 \leq 1$ and $a_2 \geq 1$, the inequality $a_1 - a_2 > 0$ cannot hold.

For example, if $\mathcal{E}(\mathbf{k}) = \sum_{j=1}^d \cos k_j$, $\mu \in (-d, d)$ and $B = [0, 2\pi]^d$, the condition (5.68) holds for some interval (α_1, α_2) containing 0. This suggests that an IR integration process based on the iteration of Proposition 5.6 does not instantly apply to the corresponding many-electron models. We should also remark that the role of $\mathcal{E}(\cdot)$ above is played by the smallest band spectrum whose zero set consists of a single point and the chemical potential μ is set to be zero when we analyze our 4-band model in Section 7. In this situation,

$$\{\mathbf{k} \in B \mid \mathcal{E}(\mathbf{k}) = 0\} \neq B, \quad \inf_{\eta \in (\alpha_1, \alpha_2)} \mathcal{H}^1(\{\mathbf{k} \in B \mid \mathcal{E}(\mathbf{k}) = \eta\}) = 0,$$

for any interval (α_1, α_2) containing 0. Thus, we cannot exclude the applicability of Proposition 5.6 by the above argument. In fact it will turn out that we can apply the proposition with the power $a_1 = 2$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1/2$.

REMARK 5.8. Let us study a possible reconstruction of Proposition 5.6 in the case that $a_1 > 0$, $a_1 - a_2 - a_4 = 0$. Assume that (5.56) holds,

$\alpha \geq cM^{a_1+a_2+a_4}$ and $M^{a_1} \geq c$ for a sufficiently large c . The only part which essentially needed the condition $M^{a_1-a_2-a_4} \geq c$ in the proof was the derivation of the upper bound on $\|F_4^l\|_{l,r}$. The bounds on the other terms can be obtained by using the conditions $\alpha \geq cM^{a_1+a_2+a_4}$ and $M^{a_1} \geq c$. Without using the condition $M^{a_1-a_2-a_4} \geq c$ we can derive from (5.56) and (5.64) that

$$M^{ra_3l}c_0^2\alpha^4\|F_4^l\|_{l,r} \leq M^{ra_3(l+1)}c_0^2\alpha^4\|J_4^{l+1}\|_{l+1,r} + c\alpha^{-2}.$$

Then, by combining with (5.66) we obtain

$$M^{ra_3l}c_0^2\alpha^4\left(\|F_4^l\|_{l,r} + \sum_{n=2}^{\infty}\|T_4^{l,(n)}\|_{l,r}\right) \leq M^{ra_3(l+1)}c_0^2\alpha^4\|J_4^{l+1}\|_{l+1,r} + c\alpha^{-2}.$$

If we assume that (5.54), (5.55), (5.56) hold for $l' \in \{l, l+1, \dots, -1\}$, we can repeatedly apply the above inequality to deduce that

$$M^{ra_3l}c_0^2\alpha^4\left(\|F_4^l\|_{l,r} + \sum_{n=2}^{\infty}\|T_4^{l,(n)}\|_{l,r}\right) \leq c_0^2\alpha^4\|J_4^0\|_{0,r} + c|l|\alpha^{-2}.$$

In practice the initial polynomial $J^0(\psi)$ is an output of the Matsubara UV integration. We see from (5.15) and (5.19) that the term $c_0^2\alpha^4\|J_4^0\|_{0,r}$ can be made less than $1/2$ by a minor assumption on M and $\sup_{\rho \in \mathcal{B}}|U_\rho|$ and thus

$$M^{ra_3l}c_0^2\alpha^4\left(\|F_4^l\|_{l,r} + \sum_{n=2}^{\infty}\|T_4^{l,(n)}\|_{l,r}\right) \leq \frac{1}{2} + c|l|\alpha^{-2}.$$

This inequality implies that if the term $|l|\alpha^{-2}$ is assumed to be small, the effective interaction $J_4^l(\psi)$ remains small and consequently the conclusions of Proposition 5.6 follow. As we will see in Section 7, the maximum value of $|l|$ in the IR integrations is proportional to $|\log \beta|$. Thus, we expect that in many-electron models where the quadratic kernels are qualitatively same as the free dispersion relation in a neighborhood of IR singularity and the marginal condition $a_1 - a_2 - a_4 = 0$ plus $a_1 > 0$ hold, an inductive IR integration procedure based on a variant of Proposition 5.6 can be justified under the additional assumption that $\alpha \geq c|\log \beta|^{1/2}$. The condition (5.9) suggests that the inequality $\alpha \geq c|\log \beta|^{1/2}$ eventually restricts the

allowed magnitude of the coupling to be less than some power of $|\log \beta|^{-1}$ after connecting the UV integration process to the IR integration process. Therefore, the resulting constructive theory in this case would be such that the domain of analyticity shrinks logarithmically with temperature.

5.4. The generalized infrared integration at different temperatures

Here we establish upper bounds on the differences between Grassmann polynomials produced by the single-scale integration introduced in the previous subsection at 2 different temperatures. To this end we need to assume that $l \in \mathbb{Z}_{<0}$, the condition (4.2) holds and the covariances $C_{o,l+1}(\beta_a) : I_0(\beta_a)^2 \rightarrow \mathbb{C}$ ($a = 1, 2$) satisfy (5.54), (5.55) and

$$(5.69) \quad \begin{aligned} C_{o,l+1}(\beta_a)(\mathbf{X}) &= (-1)^{N_{\beta_a}(\mathbf{X}+x)} C_{o,l+1}(\beta_a)(R_{\beta_a}(\mathbf{X}+x)), \\ (\forall \mathbf{X} \in I_0(\beta_a)^2, x \in (1/h)\mathbb{Z}, a \in \{1, 2\}), \end{aligned}$$

$$(5.70) \quad \begin{aligned} & \left| \det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_{o,l+1}(\beta_1)(R_{\beta_1}(X_i, Y_j)))_{1 \leq i, j \leq n} \right. \\ & \quad \left. - \det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_{o,l+1}(\beta_2)(R_{\beta_2}(X_i, Y_j)))_{1 \leq i, j \leq n} \right| \\ & \leq \beta_1^{-\frac{1}{2}} M^{-a_3(l+1)} (c_0 M^{a_1(l+1)})^n, \\ & (\forall r, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} \leq 1, \\ & \quad X_i, Y_i \in \hat{I}_0 \text{ (} i = 1, 2, \dots, n \text{)}), \end{aligned}$$

$$(5.71) \quad \left| \widetilde{C_{o,l+1}(\beta_1)} - \widetilde{C_{o,l+1}(\beta_2)} \right|_l \leq \beta_1^{-\frac{1}{2}} c_0 M^{-(a_2+a_3)(l+1)},$$

where $\widetilde{C_{o,l+1}(\beta_a)} : I(\beta_a)^2 \rightarrow \mathbb{C}$ is the anti-symmetric extension of $C_{o,l+1}(\beta_a)$ ($a = 1, 2$) defined as in (3.2). Let us note that the parameters $(w(l), r)$, $(w(l+1), r)$ are also used in the definition of $|\cdot - \cdot|_l$, $|\cdot - \cdot|_{l+1}$ respectively.

In addition, we assume that the input $J^{l+1}(\beta_a)(\psi) (\in \bigwedge \mathcal{V}(\beta_a))$ ($a = 1, 2$) satisfy $J_m^{l+1}(\beta_a)(\psi) = 0$ if $m \notin 2\mathbb{N}$ or $m \in \{0, 2\}$ and their kernels have the invariant property (4.1). Let $F^l(\beta_a)(\psi)$, $T^{l,(n)}(\beta_a)(\psi)$ ($n \in \mathbb{N}_{\geq 2}$), $T^l(\beta_a)(\psi)$, $J^l(\beta_a)(\psi) (\in \bigwedge \mathcal{V}(\beta_a))$ be defined by (5.7) with $J^{l+1}(\beta_a)(\psi)$, $C_{o,l+1}(\beta_a)$ for $a = 1, 2$ respectively. We prove the following.

PROPOSITION 5.9. *Let $a_4 \in \mathbb{R}_{\geq 0}$. Assume that $J^{l+1}(\beta_a)(\psi)$ ($a = 1, 2$) satisfy (5.56) and*

$$(5.72) \quad M^{-(a_1+a_2+a_4)(l+1)+a_3(l+1)} \sum_{m=4}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{a_1(l+1)}{2}m} \alpha^m \cdot |J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_{l+1} \leq \beta_1^{-\frac{1}{2}}.$$

Then, there exists a constant $c \in \mathbb{R}_{>1}$ independent of any parameter such that if the condition (5.57) holds with c , the following inequalities hold.

$$(5.73) \quad \left| \frac{h}{N(\beta_1)} F_0^l(\beta_1) - \frac{h}{N(\beta_2)} F_0^l(\beta_2) \right| + \sum_{n=2}^{\infty} \left| \frac{h}{N(\beta_1)} T_0^{l,(n)}(\beta_1) - \frac{h}{N(\beta_2)} T_0^{l,(n)}(\beta_2) \right| \leq \beta_1^{-\frac{1}{2}} M^{(a_1+a_2+a_4)l-a_3l} \alpha^{-3}.$$

$$(5.74) \quad M^{-(a_1+a_2+a_4)l+a_3l} \sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{a_1l}{2}m} \alpha^m \cdot \left(|F_m^l(\beta_1) - F_m^l(\beta_2)|_l + \sum_{n=2}^{\infty} |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_l \right) \leq \beta_1^{-\frac{1}{2}}.$$

PROOF. Note that the inequalities (5.72), $w(l) \leq w(l+1)$, $\alpha \geq c$, $M^{a_1} \geq c$ imply that

$$(5.75) \quad \sum_{m=4}^{N(\beta_2)} 2^{3m} c_0^{\frac{m}{2}} M^{\frac{a_1(l+1)}{2}m} |J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_l \leq c \beta_1^{-\frac{1}{2}} M^{(a_1+a_2+a_4)(l+1)-a_3(l+1)} \alpha^{-4},$$

$$(5.76) \quad \sum_{m=4}^{N(\beta_2)} 2^{2m} c_0^{\frac{m}{2}} M^{\frac{a_1l}{2}m} \alpha^m |J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_l \leq c \beta_1^{-\frac{1}{2}} M^{(a_1+a_2+a_4)(l+1)-a_3(l+1)-2a_1}.$$

(5.73): First we prove (5.73). By inserting (5.54), (5.60), (5.70), (5.75)

into the inequality in Lemma 4.1 (1) we observe that

$$\begin{aligned}
 (5.77) \quad & \left| \frac{\hbar}{N(\beta_1)} F_0^l(\beta_1) - \frac{\hbar}{N(\beta_2)} F_0^l(\beta_2) \right| \\
 & \leq c \sum_{m=4}^{N(\beta_2)} 2^{3m} c_0^{\frac{m}{2}} M^{\frac{a_1(l+1)}{2}m} \left(|J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_l \right. \\
 & \quad \left. + \beta_1^{-\frac{1}{2}} M^{-a_3(l+1)} \sum_{\delta=1}^2 \|J_m^{l+1}(\beta_\delta)\|_{l,0} + \beta_1^{-1} \sum_{\delta=1}^2 \|J_m^{l+1}(\beta_\delta)\|_{l,1} \right) \\
 & \leq c \beta_1^{-\frac{1}{2}} M^{(a_1+a_2+a_4)(l+1)-a_3(l+1)} \alpha^{-4}.
 \end{aligned}$$

By using (5.54), (5.55), (5.60), (5.70), (5.71), (5.75) we can deduce from Lemma 4.6 (1) that

$$\begin{aligned}
 & \left| \frac{\hbar}{N(\beta_1)} T_0^{l,(n)}(\beta_1) - \frac{\hbar}{N(\beta_2)} T_0^{l,(n)}(\beta_2) \right| \\
 & \leq cn(c_0 M^{a_1(l+1)})^{-n+1} (2c_0 M^{-a_2(l+1)})^{n-2} \\
 & \quad \cdot \left(\sum_{m=4}^{N(\beta_2)} 2^{3m} c_0^{\frac{m}{2}} M^{\frac{a_1(l+1)}{2}m} \sum_{\delta=1}^2 \|J_m^{l+1}(\beta_\delta)\|_{l,0} \right)^{n-1} \sum_{m=4}^{N(\beta_2)} 2^{3m} (c_0 M^{a_1(l+1)})^{\frac{m}{2}} \\
 & \quad \cdot \left(\beta_1^{-1} c_0 M^{-a_2(l+1)} \sum_{\delta=1}^2 \|J_m^{l+1}(\beta_\delta)\|_{l,1} \right. \\
 & \quad \quad + \beta_1^{-1} c_0 M^{-(a_2+a_3)(l+1)} \sum_{\delta=1}^2 \|J_m^{l+1}(\beta_\delta)\|_{l,0} \\
 & \quad \quad + c_0 M^{-a_2(l+1)} |J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_l \\
 & \quad \quad \left. + \beta_1^{-\frac{1}{2}} c_0 M^{-(a_2+a_3)(l+1)} \sum_{\delta=1}^2 \|J_m^{l+1}(\beta_\delta)\|_{l,0} \right) \\
 & \leq c^n (M^{-(a_1+a_2)(l+1)})^{n-1} \left(\sum_{m=4}^{N(\beta_2)} 2^{3m} c_0^{\frac{m}{2}} M^{\frac{a_1(l+1)}{2}m} \sum_{\delta=1}^2 \|J_m^{l+1}(\beta_\delta)\|_{l,0} \right)^{n-1} \\
 & \quad \cdot \sum_{m=4}^{N(\beta_2)} 2^{3m} c_0^{\frac{m}{2}} M^{\frac{a_1(l+1)}{2}m} \left(|J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_l \right.
 \end{aligned}$$

$$\begin{aligned}
& + \beta_1^{-\frac{1}{2}} \sum_{r=0}^1 M^{a_3(r-1)(l+1)} \sum_{\delta=1}^2 \|J_m^{l+1}(\beta_\delta)\|_{l,r} \Big) \\
& \leq c^n (M^{-(a_1+a_2)(l+1)})^{n-1} (M^{(a_1+a_2+a_4)(l+1)} \alpha^{-4})^{n-1} \\
& \quad \cdot \beta_1^{-\frac{1}{2}} M^{(a_1+a_2+a_4)(l+1)-a_3(l+1)} \alpha^{-4} \\
& \leq \beta_1^{-\frac{1}{2}} M^{(a_1+a_2-a_3)(l+1)} (cM^{a_4(l+1)} \alpha^{-4})^n.
\end{aligned}$$

Thus, by assuming that $\alpha \geq c$,

$$\begin{aligned}
(5.78) \quad & \sum_{n=2}^{\infty} \left| \frac{h}{N(\beta_1)} T_0^{l,(n)}(\beta_1) - \frac{h}{N(\beta_2)} T_0^{l,(n)}(\beta_2) \right| \\
& \leq c \beta_1^{-\frac{1}{2}} M^{(a_1+a_2+2a_4-a_3)(l+1)} \alpha^{-8}.
\end{aligned}$$

On the assumption $\alpha \geq cM^{a_1+a_2+a_4}$ the inequalities (5.77), (5.78) imply the inequality (5.73).

(5.74): Let us prove (5.74). By substituting (5.54), (5.70) into Lemma 4.1 (2) we obtain that

$$\begin{aligned}
& |F_m^l(\beta_1) - F_m^l(\beta_2)|_l \\
& \leq c_{1m=2} \sum_{n=4}^{N(\beta_2)} 2^{2n} (c_0 M^{a_1(l+1)})^{\frac{n-2}{2}} \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|_l \right. \\
& \quad \left. + \beta_1^{-\frac{1}{2}} \sum_{r=0}^1 M^{a_3(r-1)(l+1)} \sum_{\delta=1}^2 \|J_n^{l+1}(\beta_\delta)\|_{l,r} \right) \\
& + c_{1m \geq 4} \sum_{n=m}^{N(\beta_2)} 2^{2n} (c_0 M^{a_1(l+1)})^{\frac{n-m}{2}} \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|_l \right. \\
& \quad \left. + \beta_1^{-\frac{1}{2}} \sum_{r=0}^1 M^{a_3(r-1)(l+1)} \sum_{\delta=1}^2 \|J_n^{l+1}(\beta_\delta)\|_{l,r} \right).
\end{aligned}$$

Moreover, by (5.60), (5.61), (5.75), (5.76) and the condition $\alpha \geq cM^{a_1/2}$,

$$(5.79) \quad M^{-(a_1+a_2+a_4)l+a_3l} \sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{a_1l}{2}m} \alpha^m |F_m^l(\beta_1) - F_m^l(\beta_2)|_l$$

$$\begin{aligned}
 &\leq cM^{-(a_1+a_2+a_4)l+a_3l}c_0M^{a_1l}\alpha^2\sum_{n=4}^{N(\beta_2)}2^{2n}(c_0M^{a_1(l+1)})^{\frac{n-2}{2}} \\
 &\quad \cdot \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|l \right. \\
 &\quad \left. + \beta_1^{-\frac{1}{2}}\sum_{r=0}^1M^{a_3(r-1)(l+1)}\sum_{\delta=1}^2\|J_n^{l+1}(\beta_\delta)\|_{l,r} \right) \\
 &\quad + cM^{-(a_1+a_2+a_4)l+a_3l}\sum_{n=4}^{N(\beta_2)}\sum_{m=4}^n c_0^{\frac{m}{2}}M^{\frac{a_1l}{2}m}\alpha^m2^{2n}(c_0M^{a_1(l+1)})^{\frac{n-m}{2}} \\
 &\quad \cdot \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|l \right. \\
 &\quad \left. + \beta_1^{-\frac{1}{2}}\sum_{r=0}^1M^{a_3(r-1)(l+1)}\sum_{\delta=1}^2\|J_n^{l+1}(\beta_\delta)\|_{l,r} \right) \\
 &\leq cM^{-a_1}\alpha^2M^{-(a_1+a_2+a_4)l+a_3l}\sum_{n=4}^{N(\beta_2)}2^{2n}c_0^{\frac{n}{2}}M^{\frac{a_1(l+1)}{2}n} \\
 &\quad \cdot \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|l \right. \\
 &\quad \left. + \beta_1^{-\frac{1}{2}}\sum_{r=0}^1M^{a_3(r-1)(l+1)}\sum_{\delta=1}^2\|J_n^{l+1}(\beta_\delta)\|_{l,r} \right) \\
 &\quad + cM^{-(a_1+a_2+a_4)l+a_3l}\sum_{n=4}^{N(\beta_2)}2^{2n}c_0^{\frac{n}{2}}M^{\frac{a_1l}{2}n}\alpha^n \\
 &\quad \cdot \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|l \right. \\
 &\quad \left. + \beta_1^{-\frac{1}{2}}\sum_{r=0}^1M^{a_3(r-1)(l+1)}\sum_{\delta=1}^2\|J_n^{l+1}(\beta_\delta)\|_{l,r} \right) \\
 &\leq c\beta_1^{-\frac{1}{2}}M^{-a_1+a_2-a_3+a_4}.
 \end{aligned}$$

On the other hand, by substituting (5.54), (5.55), (5.70), (5.71) into the

inequality in Lemma 4.6 (2) we see that

$$\begin{aligned}
& |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_l \\
& \leq cn2^{-2m}(c_0M^{a_1(l+1)})^{-\frac{m}{2}-n+1}(2c_0M^{-a_2(l+1)})^{n-2} \\
& \quad \cdot \prod_{j=2}^n \left(\sum_{m_j=4}^{N(\beta_2)} 2^{4m_j} (c_0M^{a_1(l+1)})^{\frac{m_j}{2}} \sum_{\delta=1}^2 \|J_{m_j}^{l+1}(\beta_\delta)\|_{l,0} \right) \\
& \quad \cdot \sum_{m_1=4}^{N(\beta_2)} 2^{4m_1} (c_0M^{a_1(l+1)})^{\frac{m_1}{2}} \\
& \quad \cdot \left(\beta_1^{-1} c_0 M^{-a_2(l+1)} \sum_{\delta=1}^2 \|J_{m_1}^{l+1}(\beta_\delta)\|_{l,1} \right. \\
& \quad \quad + \beta_1^{-1} c_0 M^{-(a_2+a_3)(l+1)} \sum_{\delta=1}^2 \|J_{m_1}^{l+1}(\beta_\delta)\|_{l,0} \\
& \quad \quad + c_0 M^{-a_2(l+1)} |J_{m_1}^{l+1}(\beta_1) - J_{m_1}^{l+1}(\beta_2)|_l \\
& \quad \quad \left. + \beta_1^{-\frac{1}{2}} c_0 M^{-(a_2+a_3)(l+1)} \sum_{\delta=1}^2 \|J_{m_1}^{l+1}(\beta_\delta)\|_{l,0} \right) \mathbf{1}_{\sum_{j=1}^n m_j - 2n + 2 \geq m} \\
& \leq c^n 2^{-2m} c_0^{-\frac{m}{2}} M^{-\frac{a_1(l+1)}{2}m} (M^{-(a_1+a_2)(l+1)})^{n-1} \\
& \quad \cdot \prod_{j=2}^n \left(\sum_{m_j=4}^{N(\beta_2)} 2^{4m_j} c_0^{\frac{m_j}{2}} M^{\frac{a_1(l+1)}{2}m_j} \sum_{\delta=1}^2 \|J_{m_j}^{l+1}(\beta_\delta)\|_{l,0} \right) \\
& \quad \cdot \sum_{m_1=4}^{N(\beta_2)} 2^{4m_1} c_0^{\frac{m_1}{2}} M^{\frac{a_1(l+1)}{2}m_1} \\
& \quad \cdot \left(|J_{m_1}^{l+1}(\beta_1) - J_{m_1}^{l+1}(\beta_2)|_l + \beta_1^{-\frac{1}{2}} \sum_{r=0}^1 M^{a_3(r-1)(l+1)} \sum_{\delta=1}^2 \|J_{m_1}^{l+1}(\beta_\delta)\|_{l,r} \right) \\
& \quad \cdot \mathbf{1}_{\sum_{j=1}^n m_j - 2n + 2 \geq m}
\end{aligned}$$

Moreover, by (5.61), (5.76) and the condition $\alpha \geq cM^{a_1/2}$,

$$M^{-(a_1+a_2+a_4)l+a_3l} \sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{a_1}{2}l} \alpha^m |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_l$$

$$\begin{aligned}
 &\leq c^n M^{-(a_1+a_2+a_4)l+a_3l} (M^{-(a_1+a_2)(l+1)})^{n-1} \\
 &\quad \cdot \prod_{j=2}^n \left(\sum_{m_j=4}^{N(\beta_2)} 2^{4m_j} c_0^{\frac{m_j}{2}} M^{\frac{a_1(l+1)}{2}m_j} \sum_{\delta=1}^2 \|J_{m_j}^{l+1}(\beta_\delta)\|_{l,0} \right) \\
 &\quad \cdot \sum_{m_1=4}^{N(\beta_2)} 2^{4m_1} c_0^{\frac{m_1}{2}} M^{\frac{a_1(l+1)}{2}m_1} \\
 &\quad \cdot \left(|J_{m_1}^{l+1}(\beta_1) - J_{m_1}^{l+1}(\beta_2)|_l + \beta_1^{-\frac{1}{2}} \sum_{r=0}^1 M^{a_3(r-1)(l+1)} \sum_{\delta=1}^2 \|J_{m_1}^{l+1}(\beta_\delta)\|_{l,r} \right) \\
 &\quad \cdot 2^{-2(\sum_{j=1}^n m_j - 2n+2)} M^{-\frac{a_1}{2}(\sum_{j=1}^n m_j - 2n+2)} \alpha^{\sum_{j=1}^n m_j - 2n+2} \\
 &\leq c^n M^{-(a_1+a_2+a_4)l+a_3l} (M^{-a_1l-a_2(l+1)} \alpha^{-2})^{n-1} \\
 &\quad \cdot \prod_{j=2}^n \left(\sum_{m_j=4}^{N(\beta_2)} 2^{2m_j} c_0^{\frac{m_j}{2}} M^{\frac{a_1l}{2}m_j} \alpha^{m_j} \sum_{\delta=1}^2 \|J_{m_j}^{l+1}(\beta_\delta)\|_{l,0} \right) \\
 &\quad \cdot \sum_{m_1=4}^{N(\beta_2)} 2^{2m_1} c_0^{\frac{m_1}{2}} M^{\frac{a_1l}{2}m_1} \alpha^{m_1} \\
 &\quad \cdot \left(|J_{m_1}^{l+1}(\beta_1) - J_{m_1}^{l+1}(\beta_2)|_l + \beta_1^{-\frac{1}{2}} \sum_{r=0}^1 M^{a_3(r-1)(l+1)} \sum_{\delta=1}^2 \|J_{m_1}^{l+1}(\beta_\delta)\|_{l,r} \right) \\
 &\leq c^n M^{-(a_1+a_2+a_4)l+a_3l} (M^{-a_1l-a_2(l+1)} \alpha^{-2})^{n-1} (M^{(a_1+a_2+a_4)(l+1)-2a_1})^{n-1} \\
 &\quad \cdot \beta_1^{-\frac{1}{2}} M^{(a_1+a_2+a_4)(l+1)-a_3(l+1)-2a_1} \\
 &\leq \beta_1^{-\frac{1}{2}} M^{a_2-a_3-a_4l} \alpha^2 (cM^{-a_1+a_4(l+1)} \alpha^{-2})^n.
 \end{aligned}$$

Then, by the assumption $\alpha \geq c$ we have

$$\begin{aligned}
 (5.80) \quad &M^{-(a_1+a_2+a_4)l+a_3l} \sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{a_1l}{2}m} \alpha^m \sum_{n=2}^{\infty} |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_l \\
 &\leq c\beta_1^{-\frac{1}{2}} M^{-2a_1+a_2-a_3+2a_4+a_4l} \alpha^{-2} \leq c\beta_1^{-\frac{1}{2}} M^{-a_1+a_2+a_4}.
 \end{aligned}$$

By coupling (5.80) with (5.79) and using the condition $M^{a_1-a_2-a_4} \geq c$ we reach the inequality (5.74). \square

6. The Matsubara Ultra-Violet Integration

The results summarized in Subsection 5.1 and Subsection 5.2 have practical applications in the multi-scale integration over the Matsubara frequency, which we are going to present in this section. The purpose of the Matsubara UV integration in this paper is to find analytic continuations of the Grassmann polynomials $R^+(\psi)$, $R^-(\psi)$, which were defined in Lemma 2.8, into a (β, L, h) -independent domain of the multi-variables (U_1, U_2, \dots, U_b) around the origin. This will enable us to consider $(R^+(\psi) + R^-(\psi))/2$ as appropriate initial data for the forthcoming infrared integration. What we need to achieve our purpose is to show that the covariances used in the definition of $R^+(\psi)$, $R^-(\psi)$ can be decomposed into a sum of covariances satisfying the conditions required in Proposition 5.2 and Proposition 5.4. Then we can prove the existence of desired analytic continuations of $R^+(\psi)$, $R^-(\psi)$ by applying these propositions. The construction of this section is based on the assumption that the matrix-valued function $E : \mathbb{R}^d \rightarrow \text{Mat}(b, \mathbb{C})$ satisfies $E \in C^\infty(\mathbb{R}^d; \text{Mat}(b, \mathbb{C}))$, the properties (2.1), (2.2) and

$$(6.1) \quad \sup_{j \in \{1, 2, \dots, d\}} \sup_{(p_1, p_2, \dots, p_d) \in \mathbb{R}^d} \left\| \left(\frac{\partial}{\partial p_j} \right)^n E \left(\sum_{r=1}^d p_r \mathbf{v}_r \right) \right\|_{b \times b} \leq E_1 \cdot E_2^n n!,$$

$$(\forall n \in \mathbb{N} \cup \{0\})$$

with constants $E_1, E_2 \in \mathbb{R}_{\geq 0}$.

In order to shorten formulas, from this section we let the symbol $c(\alpha_1, \alpha_2, \dots, \alpha_n)$ denote a real positive constant depending only on parameters $\alpha_1, \alpha_2, \dots, \alpha_n$. For example, $c(\beta, L)$ denotes a positive constant depending only on β, L .

6.1. The covariance matrices with the Matsubara ultra-violet cut-off

First we have to specifically define a cut-off function on the Matsubara frequency. Motivated by [18, Appendix A], we construct the cut-off function from a suitable Gevrey-class function.

LEMMA 6.1. *There exists a function $\phi \in C^\infty(\mathbb{R})$ satisfying the following properties.*

$$\begin{aligned}\phi(x) &= 1, \quad (\forall x \in (-\infty, \pi^2/6]), \\ \phi(x) &= 0, \quad (\forall x \in [\pi^2/3, \infty)), \\ \frac{d}{dx}\phi(x) &\leq 0, \quad (\forall x \in \mathbb{R}),\end{aligned}$$

and

$$(6.2) \quad \left| \left(\frac{d}{dx} \right)^k \phi(x) \right| \leq 2^k (k!)^2, \quad (\forall x \in \mathbb{R}, k \in \mathbb{N} \cup \{0\}).$$

PROOF. Let us take the sequence $(a_j)_{j=0}^\infty$ in [11, Theorem 1.3.5] to be $((j+1)^{-2})_{j=0}^\infty$. Since $\sum_{j=0}^\infty a_j = \pi^2/6$, the theorem reads that there exists a function $u \in C_0^\infty(\mathbb{R})$ satisfying

$$\begin{aligned}u(x) &\geq 0, \quad (\forall x \in \mathbb{R}), \\ u(x) &= 0, \quad (\forall x \in \mathbb{R} \setminus [0, \pi^2/6]), \\ \left| \left(\frac{d}{dx} \right)^k u(x) \right| &\leq 2^k ((k+1)!)^2, \quad (\forall x \in \mathbb{R}, k \in \mathbb{N} \cup \{0\}), \\ \int_{-\infty}^\infty u(x) dx &= 1.\end{aligned}$$

Set

$$\phi(x) := \int_0^{-x + \frac{\pi^2}{3}} u(y) dy.$$

One can check that the function ϕ satisfies the claimed properties. \square

Take $M \in \mathbb{R}_{>1}$ and set

$$\begin{aligned}M_{UV} &:= \frac{2\sqrt{6}}{\pi}(E_1 + 1), \\ N_h &:= \max \left\{ \left\lceil \frac{\log(2h(\frac{\pi^2}{6})^{-1/2} M_{UV}^{-1})}{\log M} \right\rceil + 1, 1 \right\},\end{aligned}$$

where the symbol $[x]$ denotes the largest integer less than or equal to x for any $x \in \mathbb{R}$. We see that

$$h|1 - e^{i\frac{\omega}{h}}| \leq 2h \leq \left(\frac{\pi^2}{6}\right)^{\frac{1}{2}} M_{UV} M^{N_h}, \quad (\forall \omega \in \mathbb{R}).$$

Thus,

$$\phi(M_{UV}^{-2} M^{-2N_h} h^2 |1 - e^{i\frac{\omega}{h}}|^2) = 1, \quad (\forall \omega \in \mathbb{R}),$$

where $\phi(\cdot)$ is the function introduced in Lemma 6.1.

For any $\omega \in \mathbb{R}$ set

$$\begin{aligned} \chi_{h,0}(\omega) &:= \phi(M_{UV}^{-2} h^2 |1 - e^{i\frac{\omega}{h}}|^2), \\ \chi_{h,l}(\omega) &:= \phi(M_{UV}^{-2} M^{-2l} h^2 |1 - e^{i\frac{\omega}{h}}|^2) - \phi(M_{UV}^{-2} M^{-2(l-1)} h^2 |1 - e^{i\frac{\omega}{h}}|^2), \\ &\quad (l \in \{1, 2, \dots, N_h\}). \end{aligned}$$

Then, we have that

$$(6.3) \quad \chi_{h,0}(\omega) + \sum_{l=1}^{N_h} \chi_{h,l}(\omega) = 1, \quad (\forall \omega \in \mathbb{R}).$$

The values of $\chi_{h,0}(\cdot)$, $\chi_{h,l}(\cdot)$ are described as follows.

$$(6.4) \quad \chi_{h,0}(\omega) \begin{cases} = 1, & \text{if } h|1 - e^{i\frac{\omega}{h}}| \leq \frac{\pi}{\sqrt{6}} M_{UV}, \\ \in [0, 1], & \text{if } \frac{\pi}{\sqrt{6}} M_{UV} < h|1 - e^{i\frac{\omega}{h}}| < \frac{\pi}{\sqrt{3}} M_{UV}, \\ = 0, & \text{if } \frac{\pi}{\sqrt{3}} M_{UV} \leq h|1 - e^{i\frac{\omega}{h}}|, \end{cases}$$

$$\chi_{h,l}(\omega) \begin{cases} = 0, & \text{if } h|1 - e^{i\frac{\omega}{h}}| \leq \frac{\pi}{\sqrt{6}} M_{UV} M^{l-1}, \\ \in [0, 1], & \text{if } \frac{\pi}{\sqrt{6}} M_{UV} M^{l-1} < h|1 - e^{i\frac{\omega}{h}}| < \frac{\pi}{\sqrt{3}} M_{UV} M^l, \\ = 0, & \text{if } \frac{\pi}{\sqrt{3}} M_{UV} M^l \leq h|1 - e^{i\frac{\omega}{h}}|, \end{cases}$$

($\forall l \in \{1, 2, \dots, N_h\}, \omega \in \mathbb{R}$).

Using these cut-off functions, we define the covariances $C_l^+, C_l^- : I_0^2 \rightarrow \mathbb{C}$ ($l = 0, 1, \dots, N_h$) as follows. For any $(\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in I_0$,

$$(6.5) \quad \begin{aligned} &C_l^+(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) \\ &:= \frac{\delta_{\sigma, \tau}}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} e^{-i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \chi_{h,l}(\omega) h^{-1} \end{aligned}$$

$$\begin{aligned}
 & \cdot (I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1}(\rho, \eta), \\
 & C_l^-(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) \\
 & := \frac{\delta_{\sigma, \tau}}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} e^{-i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \chi_{h,l}(\omega) h^{-1} \\
 & \cdot (e^{i\frac{\omega}{h}I_b - \frac{1}{h}\overline{E(\mathbf{k})}} - I_b)^{-1}(\rho, \eta).
 \end{aligned}$$

We also define the covariances $C_{\leq 0}^+$, $C_{> 0}^+$, $C_{> 0}^-$ by (2.22), (2.23), (2.25) respectively by employing $\chi_{h,0}(\cdot)$ in place of $\chi(h|1 - e^{i\cdot/h}|)$. It follows from (6.3) that

$$C_{> 0}^+ = \sum_{l=1}^{N_h} C_l^+, \quad C_{> 0}^- = \sum_{l=1}^{N_h} C_l^-, \quad C = \sum_{l=0}^{N_h} C_l^+.$$

We show in the next lemma that the covariances C_l^+ , $C_l^- : I_0^2 \rightarrow \mathbb{C}$ ($l = 1, 2, \dots, N_h$) satisfy the bound properties required in Subsection 5.1. For this purpose let us introduce finite-difference operators. For any function $f : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$, set

$$\begin{aligned}
 \mathcal{D}_0 f(\omega, \mathbf{k}) &:= \frac{\beta}{2\pi} \left(f\left(\omega + \frac{2\pi}{\beta}, \mathbf{k}\right) - f(\omega, \mathbf{k}) \right), \\
 \mathcal{D}_j f(\omega, \mathbf{k}) &:= \frac{L}{2\pi} \left(f\left(\omega, \mathbf{k} + \frac{2\pi}{L} \mathbf{v}_j\right) - f(\omega, \mathbf{k}) \right), \\
 & (\forall j \in \{1, 2, \dots, d\}, (\omega, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^d).
 \end{aligned}$$

LEMMA 6.2. *Assume that $h \geq e^{2E_1}$. Then, there exist constants c_0 , $c'_0 \in \mathbb{R}_{\geq 1}$, which depend only on b , d , M_{UV} , M , E_2 , and a constant $c_w \in (0, 1]$ independent of any parameter such that the covariances C_l^+ , C_l^- ($l = 1, 2, \dots, N_h$) satisfy (5.3), (5.4) with c_0 , $N_+ = N_h$, the weight*

$$w(0) = c_w(d+1)^{-2} \min\{M_{UV}, (E_2+1)^{-1}\} M^{-2}$$

and the exponent $r = 1/2$, and (5.5) with c'_0 , $N_+ = N_h$.

PROOF. We prove the claims on C_l^+ . The boundedness of C_l^- can be proved in the same way. Since $(2/\pi)|\omega| \leq h|1 - e^{i\omega/h}| \leq |\omega|$ for any $\omega \in \mathbb{R}$ with $|\omega| \leq \pi h$, the conditions $\chi_{h,l}(\omega) \neq 0$ and $|\omega| \leq \pi h$ implies that

$$(6.6) \quad cM_{UV}M^{l-1} \leq |\omega| \leq cM_{UV}M^l, \quad (\forall l \in \{1, 2, \dots, N_h\}).$$

Therefore,

$$(6.7) \quad \frac{1}{\beta} \sum_{\omega \in \mathcal{M}_h} 1_{\chi_{h,l}(\omega) \neq 0} \leq cM_{UV}M^l, \\ \int_{-\pi h}^{\pi h} d\omega 1_{\chi_{h,l}(\omega) \neq 0} \leq cM_{UV}M^l, \quad (\forall l \in \{1, 2, \dots, N_h\}).$$

(Proof for (5.3)): Note that

$$(6.8) \quad h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1} \\ = h^{-1}(1 - e^{-i\frac{\omega}{h}})^{-1}(I_b - h^{-1}(e^{i\frac{\omega}{h}} - 1)^{-1}h(e^{\frac{1}{h}\overline{E(\mathbf{k})}} - I_b))^{-1}.$$

If $h \geq e^{E_1}$, $\|h(e^{\frac{1}{h}\overline{E(\mathbf{k})}} - I_b)\|_{b \times b} \leq E_1 + 1$. Thus, we can see from the definition of M_{UV} and (6.4) that

$$(6.9) \quad \|h^{-1}(e^{i\frac{\omega}{h}} - 1)^{-1}h(e^{\frac{1}{h}\overline{E(\mathbf{k})}} - I_b)\|_{b \times b} \leq \frac{\sqrt{6}}{\pi}M_{UV}^{-1}M^{-l+1}(E_1 + 1) \\ = \frac{1}{2}M^{-l+1},$$

($\forall \omega \in \mathbb{R}$ with $\chi_{h,l}(\omega) \neq 0$, $\mathbf{k} \in \mathbb{R}^d$).

It follows from (6.4), (6.8) and (6.9) that

$$(6.10) \quad \|h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1}\|_{b \times b} \leq cM_{UV}^{-1}M^{-l+1}, \\ (\forall \omega \in \mathbb{R} \text{ with } \chi_{h,l}(\omega) \neq 0, \mathbf{k} \in \mathbb{R}^d).$$

Recall the definition of the Hilbert space \mathcal{H} introduced in the proof of Lemma 2.4. For any $(\rho, \mathbf{x}, \sigma, x) \in I_0$ we define $f_{\rho\mathbf{x}\sigma x}^l, g_{\rho\mathbf{x}\sigma x}^l \in \mathcal{H}$ by

$$f_{\rho\mathbf{x}\sigma x}^l(\eta, \mathbf{k}, \tau, \omega) := \delta_{\rho,\eta}\delta_{\sigma,\tau}e^{i\langle \mathbf{x}, \mathbf{k} \rangle}e^{-ix\omega}\chi_{h,l}(\omega)^{\frac{1}{2}}, \\ g_{\rho\mathbf{x}\sigma x}^l(\eta, \mathbf{k}, \tau, \omega) := \delta_{\sigma,\tau}e^{i\langle \mathbf{x}, \mathbf{k} \rangle}e^{-ix\omega}\chi_{h,l}(\omega)^{\frac{1}{2}}h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1}(\eta, \rho).$$

Then, $C_l^+(X, Y) = \langle f_X^l, g_Y^l \rangle_{\mathcal{H}}$, ($\forall X, Y \in I_0$). Moreover, by (6.7) and (6.10),

$$(6.11) \quad \|f_{\rho\mathbf{x}\sigma x}^l\|_{\mathcal{H}} \leq c(M_{UV})M^{\frac{l}{2}}, \\ \|g_{\rho\mathbf{x}\sigma x}^l\|_{\mathcal{H}} \leq \left(\frac{1}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} \chi_{h,l}(\omega) \right)$$

$$\begin{aligned} & \cdot \left\| h^{-1} (I_b - e^{-i\frac{\omega}{\hbar} I_b + \frac{1}{\hbar} \overline{E(\mathbf{k})}})^{-1} \right\|_{b \times b}^2 \Big)^{\frac{1}{2}} \\ & \leq c(M_{UV}, M) M^{-\frac{l}{2}}, \quad (\forall (\rho, \mathbf{x}, \sigma, x) \in I_0). \end{aligned}$$

For any $r \in \mathbb{N}$ let $\mathbb{C}^r \otimes \mathcal{H}$ denote the tensor product of the Hilbert spaces $\mathbb{C}^r, \mathcal{H}$. Since

$$\langle \mathbf{p}, \mathbf{q} \rangle_{\mathbb{C}^r} C_l^+(X, Y) = \langle \mathbf{p} \otimes f_X^l, \mathbf{q} \otimes g_Y^l \rangle_{\mathbb{C}^r \otimes \mathcal{H}}, \quad (\forall \mathbf{p}, \mathbf{q} \in \mathbb{C}^r, X, Y \in I_0),$$

Gram's inequality in the Hilbert space $\mathbb{C}^r \otimes \mathcal{H}$ and (6.11) ensure that

$$\begin{aligned} |\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_l^+(X_i, Y_j))_{1 \leq i, j \leq n}| & \leq \prod_{i=1}^n \|\mathbf{p}_i \otimes f_{X_i}^l\|_{\mathbb{C}^r \otimes \mathcal{H}} \|\mathbf{q}_i \otimes g_{Y_i}^l\|_{\mathbb{C}^r \otimes \mathcal{H}} \\ & \leq \prod_{i=1}^n \|f_{X_i}^l\|_{\mathcal{H}} \|g_{Y_i}^l\|_{\mathcal{H}} \leq c(M_{UV}, M)^n, \\ & (\forall r, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} \leq 1, \\ & X_i, Y_i \in I_0 \ (i = 1, 2, \dots, n)). \end{aligned}$$

(Proof for (5.4)): By the assumption $h \geq e^{2E_1}$,

$$N_h = \left\lfloor \frac{\log(2h(\frac{\pi^2}{6})^{-1/2} M_{UV}^{-1})}{\log M} \right\rfloor + 1.$$

Thus,

$$(6.12) \quad M_{UV} M^{N_h - 1} \leq ch.$$

Using (6.12) and the inequality $E_1/h \leq 1$, we can check that

$$\begin{aligned} (6.13) \quad & \left\| \left(\frac{\partial}{\partial \omega} \right)^n h (I_b - e^{-i\frac{\omega}{\hbar} I_b + \frac{1}{\hbar} \overline{E(\mathbf{k})}}) \right\|_{b \times b} \\ & \leq ch^{1-n} \leq M_{UV} M^{l-1} (cM_{UV}^{-1} M^{1-l})^n n!, \quad (\forall n \in \mathbb{N}). \end{aligned}$$

We can apply Lemma C.3 (1) proved in Appendix C together with (6.1) and the assumption $h \geq e^{2E_1}$ to derive that for any $j \in \{1, 2, \dots, d\}$,

$$(6.14) \quad \left\| \left(\frac{\partial}{\partial p_j} \right)^n h (I_b - e^{-i\frac{\omega}{\hbar} I_b + \frac{1}{\hbar} \overline{E(\sum_{r=1}^d p_r \mathbf{v}_r)}}) \right\|_{b \times b}$$

$$\begin{aligned}
&\leq \sum_{m=1}^{\infty} \left(\frac{1}{h}\right)^{m-1} \frac{1}{m!} \left\| \left(\frac{\partial}{\partial p_j}\right)^n \overline{E \left(\sum_{r=1}^d p_r \mathbf{v}_r\right)^m} \right\|_{b \times b} \\
&\leq \sum_{m=1}^{\infty} \left(\frac{1}{h}\right)^{m-1} \frac{1}{m!} (2E_1)^m (2E_2)^n n! \\
&\leq (2E_1 + 1)(2E_2)^n n!, \quad (\forall n \in \mathbb{N}).
\end{aligned}$$

Taking into account (6.10), (6.13), we can substitute $s = cM_{UV}^{-1}M^{-l+1}$, $q = M_{UV}M^{l-1}$, $r = cM_{UV}^{-1}M^{-l+1}$, $t = 1$ into the inequality in Lemma C.3 (2) to obtain

$$\begin{aligned}
(6.15) \quad &\left\| \left(\frac{\partial}{\partial \omega}\right)^n h^{-1} (I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1} \right\|_{b \times b} \\
&\leq cM_{UV}^{-1}M^{-l+1} (cM_{UV}^{-1}M^{-l+1})^n n!, \\
&(\forall n \in \mathbb{N} \cup \{0\}, \omega \in \mathbb{R} \text{ with } \chi_{h,l}(\omega) \neq 0, \mathbf{k} \in \mathbb{R}^d).
\end{aligned}$$

Here we also used (6.10) to claim (6.15) for $n = 0$. By (6.10), (6.14) we can apply Lemma C.3 (2) with $s = cM_{UV}^{-1}M^{-l+1}$, $q = 2E_1 + 1$, $r = 2E_2$, $t = 1$ to deduce that

$$\begin{aligned}
(6.16) \quad &\left\| \left(\frac{\partial}{\partial p_j}\right)^n h^{-1} (I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\sum_{r=1}^d p_r \mathbf{v}_r)}})^{-1} \right\|_{b \times b} \\
&\leq cM_{UV}^{-2}M^{-2l+2} (E_1 + 1)(1 + c(E_1 + 1)M_{UV}^{-1}M^{-l+1})^{-1} \\
&\quad \cdot (cE_2(1 + c(E_1 + 1)M_{UV}^{-1}M^{-l+1}))^n n! \\
&\leq cM_{UV}^{-1}M^{-l+1} (cE_2)^n n!, \\
&(\forall n \in \mathbb{N}, j \in \{1, 2, \dots, d\}, \omega \in \mathbb{R} \text{ with } \chi_{h,l}(\omega) \neq 0, \\
&\quad (p_1, p_2, \dots, p_d) \in \mathbb{R}^d),
\end{aligned}$$

where we used the inequality $M_{UV} \geq E_1 + 1$ as well.

For any $\omega \in \mathbb{R}$ let $p_h(\omega)$ denote a number belonging to $[-\pi h, \pi h)$ and satisfying $\omega = p_h(\omega)$ in $\mathbb{R}/2\pi h\mathbb{Z}$. By using (6.6) we have for any $n \in \mathbb{N}$ and $\omega \in \mathbb{R}$ with $\chi_{h,l}(\omega) \neq 0$ that

$$\begin{aligned}
(6.17) \quad &\left| \left(\frac{d}{d\omega}\right)^n (M_{UV}^{-2}M^{-2(l-1)}h^2 |1 - e^{i\frac{\omega}{h}}|^2) \right| \\
&\leq cM_{UV}^{-2}M^{-2(l-1)}c^{n+1} |p_h(\omega)|^{2-n} n! \leq cM^2 (cM_{UV}^{-1}M^{-l+1})^n n!.
\end{aligned}$$

By (6.2), (6.17) we can substitute $q_1 = cM^2$, $r_1 = cM_{UV}^{-1}M^{-l+1}$, $q_2 = 1$, $r_2 = 2$, $t = 2$ into the result of Lemma C.1 to derive that

$$(6.18) \quad \left| \left(\frac{d}{d\omega} \right)^n \chi_{h,l}(\omega) \right| \leq cM^2(1+cM^2)^{-1}(cM_{UV}^{-1}M^{-l+1}(1+cM^2))^n(n!)^2 \leq c(cM_{UV}^{-1}M^{-l+3})^n(n!)^2, \quad (\forall n \in \mathbb{N} \cup \{0\}, \omega \in \mathbb{R}).$$

By the periodicity with the variable ω ,

$$(6.19) \quad \begin{aligned} & \left(\frac{\beta}{2\pi} \right)^n (e^{-i\frac{2\pi}{\beta}(x-y)} - 1)^n C_l^+(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) \\ &= \frac{\delta_{\sigma,\tau}}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} e^{-i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \\ & \quad \cdot \mathcal{D}_0^n(\chi_{h,l}(\omega) h^{-1} (I_b - e^{-i\frac{\omega}{h} I_b + \frac{1}{h} \overline{E(\mathbf{k})}})^{-1}) \\ &= \frac{\delta_{\sigma,\tau}}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} e^{-i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \prod_{r=1}^n \left(\frac{\beta}{2\pi} \int_0^{2\pi/\beta} d\omega_r \right) \left(\frac{\partial}{\partial \eta} \right)^n \\ & \quad \cdot (\chi_{h,l}(\eta) h^{-1} (I_b - e^{-i\frac{\eta}{h} I_b + \frac{1}{h} \overline{E(\mathbf{k})}})^{-1}) \Big|_{\eta=\omega+\sum_{r=1}^n \omega_r}. \end{aligned}$$

Note that by (6.15), (6.18),

$$(6.20) \quad \begin{aligned} & \left\| \left(\frac{\partial}{\partial \omega} \right)^n (\chi_{h,l}(\omega) h^{-1} (I_b - e^{-i\frac{\omega}{h} I_b + \frac{1}{h} \overline{E(\mathbf{k})}})^{-1}) \right\|_{b \times b} \\ & \leq \sum_{j=0}^n \binom{n}{j} \left| \left(\frac{d}{d\omega} \right)^j \chi_{h,l}(\omega) \right| \\ & \quad \cdot \left\| \left(\frac{\partial}{\partial \omega} \right)^{n-j} h^{-1} (I_b - e^{-i\frac{\omega}{h} I_b + \frac{1}{h} \overline{E(\mathbf{k})}})^{-1} \right\|_{b \times b} \\ & \leq \sum_{j=0}^n \binom{n}{j} (cM_{UV}^{-1}M^{-l+3})^j (j!)^2 \\ & \quad \cdot cM_{UV}^{-1}M^{-l+1} (cM_{UV}^{-1}M^{-l+1})^{n-j} (n-j)! \\ & \leq cM_{UV}^{-1}M^{-l+1} (cM_{UV}^{-1}M^{-l+3})^n (n!)^2, \\ & (\forall n \in \mathbb{N} \cup \{0\}, \omega \in \mathbb{R}, \mathbf{k} \in \mathbb{R}^d). \end{aligned}$$

By combining (6.7), (6.20) with (6.19) we obtain that

$$(6.21) \quad \left\| \left(\frac{\beta}{2\pi} \right)^n (e^{-i\frac{2\pi}{\beta}(x-y)} - 1)^n C_l^+(\cdot \mathbf{x}\sigma x, \cdot \mathbf{y}\tau y) \right\|_{b \times b} \\ \leq cM(cM_{UV}^{-1}M^{-l+3})^n (n!)^2, \quad (\forall n \in \mathbb{N} \cup \{0\}).$$

This implies that

$$(6.22) \quad \|C_l^+(\cdot \mathbf{x}\sigma x, \cdot \mathbf{y}\tau y)\|_{b \times b} \leq cM e^{-(c^{-1}M_{UV}M^{l-3}\frac{\beta}{2\pi}|e^{i(2\pi/\beta)(x-y)} - 1|)^{1/2}}, \\ (\forall (\mathbf{x}, \sigma, x), (\mathbf{y}, \tau, y) \in \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)_h).$$

By the periodic condition (2.2) we similarly have for any $j \in \{1, 2, \dots, d\}$ that

$$\left(\frac{L}{2\pi} \right)^n (e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y}, \mathbf{v}_j \rangle} - 1)^n C_l^+(\cdot \mathbf{x}\sigma x, \cdot \mathbf{y}\tau y) \\ = \frac{\delta_{\sigma, \tau}}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} e^{-i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \chi_{h,l}(\omega) \mathcal{D}_j^n (h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}}) - 1) \\ = \frac{\delta_{\sigma, \tau}}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} e^{-i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \chi_{h,l}(\omega) \prod_{r=1}^n \left(\frac{L}{2\pi} \int_0^{2\pi/L} dq_r \right) \left(\frac{\partial}{\partial p_j} \right)^n \\ \cdot (h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k}+p_j\mathbf{v}_j)}}) - 1) \Big|_{p_j = \sum_{r=1}^n q_r}.$$

Insertion of (6.7), (6.16) yields

$$\left\| \left(\frac{L}{2\pi} \right)^n (e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y}, \mathbf{v}_j \rangle} - 1)^n C_l^+(\cdot \mathbf{x}\sigma x, \cdot \mathbf{y}\tau y) \right\|_{b \times b} \\ \leq cM(cE_2)^n n!, \quad (\forall n \in \mathbb{N} \cup \{0\}),$$

where we also used (6.21) to claim this equality for $n = 0$. This leads to

$$(6.23) \quad \|C_l^+(\cdot \mathbf{x}\sigma x, \cdot \mathbf{y}\tau y)\|_{b \times b} \leq cM e^{-(c^{-1}(E_2+1)^{-1}\frac{L}{2\pi}|e^{i(2\pi/L)\langle \mathbf{x}-\mathbf{y}, \mathbf{v}_j \rangle} - 1|)^{1/2}}, \\ (\forall (\mathbf{x}, \sigma, x), (\mathbf{y}, \tau, y) \in \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)_h, j \in \{1, 2, \dots, d\}).$$

One can derive from (6.22), (6.23) that

$$\|C_l^+(\cdot \mathbf{x}\sigma x, \cdot \mathbf{y}\tau y)\|_{b \times b} \leq cM e^{-(c^{-1}(d+1)^{-2}M_{UV}M^{l-3}\frac{\beta}{2\pi}|e^{i(2\pi/\beta)(x-y)} - 1|)^{1/2}}$$

$$\cdot e^{-\sum_{j=1}^d (c^{-1}(d+1)^{-2}(E_2+1)^{-1} \frac{L}{2\pi} |e^{i(2\pi/L)\langle \mathbf{x}-\mathbf{y}, \mathbf{v}_j \rangle} - 1|)^{1/2}},$$

$$(\forall (\mathbf{x}, \sigma, x), (\mathbf{y}, \tau, y) \in \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)_h).$$

Here we may assume that $c \geq 1$ by taking a larger number if necessary. Set

$$c_w := \frac{1}{9} c^{-1},$$

$$w(0) := c_w (d+1)^{-2} \min\{M_{UV}, (E_2+1)^{-1}\} M^{-2},$$

with the constant $c \in \mathbb{R}_{\geq 1}$ appearing in the above inequality. Then, we have

$$(6.24) \quad |C_l^+(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y)|$$

$$\leq c(M) e^{-3(w(0)M^{l-1} \frac{\beta}{2\pi} |e^{i(2\pi/\beta)(x-y)} - 1|)^{1/2}}$$

$$\cdot e^{-3 \sum_{j=1}^d (w(0) \frac{L}{2\pi} |e^{i(2\pi/L)\langle \mathbf{x}-\mathbf{y}, \mathbf{v}_j \rangle} - 1|)^{1/2}},$$

$$(\forall (\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in I_0).$$

With this weight $w(0)$ and the exponent $r = 1/2$ we define the norm $\|\cdot\|_{0,0}$ and the semi-norm $\|\cdot\|_{0,1}$ by (3.3). We can check that

$$\|\widetilde{C}_l^+\|_{0,0} \leq c(M) \sup_{X \in I} \frac{1}{h} \sum_{Y \in I} e^{-(w(0)M^{l-1} d_0(X,Y))^{1/2}} \cdot e^{-\sum_{j=1}^d (w(0) d_j(X,Y))^{1/2}}$$

$$\leq c(M, b, d, w(0)) M^{-l},$$

$$\|\widetilde{C}_l^+\|_{0,1} \leq c(M) \sup_{i \in \{0,1, \dots, d\}} \sup_{X \in I} \frac{1}{h} \sum_{Y \in I} d_i(X, Y)$$

$$\cdot e^{-(w(0)M^{l-1} d_0(X,Y))^{1/2}} \cdot e^{-\sum_{j=1}^d (w(0) d_j(X,Y))^{1/2}}$$

$$\leq c(M, b, d, w(0)) M^{-l}.$$

These upper bounds can be derived even if we define c_w as $(1/4)c^{-1}$ so that the right-hand side of (6.24) contains the factor 2 in place of 3. However, we choose to define c_w as $(1/9)c^{-1}$ in order that (6.24) can be used in the next lemma, too.

(Proof for (5.5)): It follows from (6.8) that for any $\omega \in \mathbb{R}$ with $\chi_{h,l}(\omega) \neq 0$,

$$h^{-1} (I_b - e^{-i \frac{\omega}{h} I_b + \frac{1}{h} \overline{E(\mathbf{k})}})^{-1}$$

$$\begin{aligned}
&= h^{-1}(1 - e^{-i\frac{\omega}{h}})^{-1}I_b \\
&\quad + h^{-1}(1 - e^{-i\frac{\omega}{h}})^{-1} \sum_{n=1}^{\infty} (h^{-1}(e^{i\frac{\omega}{h}} - 1)^{-1}h(e^{\frac{1}{h}\overline{E(\mathbf{k})}} - I_b))^n.
\end{aligned}$$

By substituting this equality we obtain

$$\begin{aligned}
&C_l^+(\cdot\mathbf{0}\sigma\mathbf{0}, \cdot\mathbf{0}\sigma\mathbf{0}) \\
&= \frac{1}{2\beta h} \sum_{\omega \in \mathcal{M}_h} \chi_{h,l}(\omega)I_b + \frac{1}{\beta L^d} \sum_{\mathbf{k} \in \Gamma^*} \sum_{\omega \in \mathcal{M}_h} \chi_{h,l}(\omega)h^{-1}(1 - e^{-i\frac{\omega}{h}})^{-1} \\
&\quad \cdot \sum_{n=1}^{\infty} (h^{-1}(e^{i\frac{\omega}{h}} - 1)^{-1}h(e^{\frac{1}{h}\overline{E(\mathbf{k})}} - I_b))^n.
\end{aligned}$$

Moreover, by (6.4), (6.7), (6.9) and (6.12),

$$\begin{aligned}
\|C_l^+(\cdot\mathbf{0}\sigma\mathbf{0}, \cdot\mathbf{0}\sigma\mathbf{0})\|_{b \times b} &\leq c(M)M^{l-N_h} + c(M) \sum_{n=1}^{\infty} \left(\frac{1}{2}M^{-l+1}\right)^n \\
&\leq c(M)(M^{l-N_h} + M^{-l}),
\end{aligned}$$

which implies (5.5) with $N_+ = N_h$. \square

Next let us find upper bounds on the differences between the covariances defined at 2 different temperatures. These bounds were required in Subsection 5.2.

LEMMA 6.3. *Assume that (4.2) holds and $h \geq e^{2E_1}$. Let $w(0)$ be the weight introduced in Lemma 6.2. Then, there exist constants $c_0, c'_0 \in \mathbb{R}_{\geq 1}$, which depend only on b, d, M_{UV}, M, E_2 , such that the covariances $C_l^+(\beta_a), C_l^-(\beta_a)$ ($l = 1, 2, \dots, N_h, a = 1, 2$) satisfy (5.31), (5.32) with $c_0, N_+ = N_h$, the weight $w(0)$ and the exponent $r = 1/2$, and (5.33) with $c'_0, N_+ = N_h$.*

PROOF. We give the proof for C_l^+ . The claims for C_l^- can be proved in the same way. For any $(\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in \hat{I}_0, a \in \{1, 2\}$, set

$$\begin{aligned}
&C_{ont,l}(\beta_a)(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y) \\
&:= (-1)^{n_{\beta_a}(x) + n_{\beta_a}(y)} \frac{\delta_{\sigma,\tau}}{2\pi L^d} \sum_{\mathbf{k} \in \Gamma^*} \int_{-\pi h}^{\pi h} d\omega e^{-i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \chi_{h,l}(\omega)
\end{aligned}$$

$$\cdot h^{-1}(I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1}(\rho, \eta).$$

Since $n_{\beta_1}(x) = n_{\beta_2}(x)$ ($\forall x \in [-\beta_1/4, \beta_1/4)_h$), $C_{ont,l}(\beta_1) = C_{ont,l}(\beta_2)$. Note that for any $a \in \{1, 2\}$,

$$\begin{aligned} & C_{ont,l}(\beta_a)(\cdot \mathbf{x}\sigma x, \cdot \mathbf{y}\tau y) - C_l^+(\beta_a)(\cdot \mathbf{x}\sigma r_{\beta_a}(x), \cdot \mathbf{y}\tau r_{\beta_a}(y)) \\ &= (-1)^{n_{\beta_a}(x)+n_{\beta_a}(y)} \frac{\delta_{\sigma,\tau}}{2\pi L^d} \sum_{\mathbf{k} \in \Gamma^*} e^{-i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} \\ & \cdot \sum_{m=0}^{\frac{\beta_a h}{2}-1} \left(\int_{\frac{2\pi}{\beta_a}m + \frac{\pi}{\beta_a}}^{\frac{2\pi}{\beta_a}(m+1) + \frac{\pi}{\beta_a}} d\omega \int_{\frac{2\pi}{\beta_a}m + \frac{\pi}{\beta_a}}^{\omega} du \right. \\ & \quad \left. + \int_{-\frac{2\pi}{\beta_a}(m+1) - \frac{\pi}{\beta_a}}^{-\frac{2\pi}{\beta_a}m - \frac{\pi}{\beta_a}} d\omega \int_{-\frac{2\pi}{\beta_a}m - \frac{\pi}{\beta_a}}^{\omega} du \right) \\ & \cdot \frac{\partial}{\partial u} \left(e^{i(x-y)u} \chi_{h,l}(u) h^{-1} (I_b - e^{-i\frac{u}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1} \right) \\ & + (-1)^{n_{\beta_a}(x)+n_{\beta_a}(y)} \frac{\delta_{\sigma,\tau}}{2\pi L^d} \sum_{\mathbf{k} \in \Gamma^*} e^{-i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} \\ & \cdot \left(\int_{-\frac{\pi}{\beta_a}}^{\frac{\pi}{\beta_a}} d\omega - \int_{\pi h}^{\pi h + \frac{\pi}{\beta_a}} d\omega - \int_{-\pi h - \frac{\pi}{\beta_a}}^{-\pi h} d\omega \right) \\ & \cdot e^{i(x-y)\omega} \chi_{h,l}(\omega) h^{-1} (I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1}. \end{aligned}$$

On the assumption $h \geq e^{2E_1}$ we can apply (6.7), (6.20) to deduce that

$$\begin{aligned} (6.25) \quad & \|C_l^+(\beta_1)(\cdot \mathbf{x}\sigma r_{\beta_1}(x), \cdot \mathbf{y}\tau r_{\beta_1}(y)) - C_l^+(\beta_2)(\cdot \mathbf{x}\sigma r_{\beta_2}(x), \cdot \mathbf{y}\tau r_{\beta_2}(y))\|_{b \times b} \\ & \leq \sum_{a=1}^2 \|C_{ont,l}(\beta_a)(\cdot \mathbf{x}\sigma x, \cdot \mathbf{y}\tau y) - C_l^+(\beta_a)(\cdot \mathbf{x}\sigma r_{\beta_a}(x), \cdot \mathbf{y}\tau r_{\beta_a}(y))\|_{b \times b} \\ & \leq \sum_{a=1}^2 \frac{1}{\beta_a L^d} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_a}}^{\pi h + \frac{\pi}{\beta_a}} d\omega + \int_{-\pi h - \frac{\pi}{\beta_a}}^{-\frac{\pi}{\beta_a}} d\omega \right) \\ & \cdot \left(\| |x-y| \chi_{h,l}(\omega) \| h^{-1} (I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1} \|_{b \times b} \right. \\ & \quad \left. + \left\| \frac{\partial}{\partial \omega} (\chi_{h,l}(\omega) h^{-1} (I_b - e^{-i\frac{\omega}{h}I_b + \frac{1}{h}\overline{E(\mathbf{k})}})^{-1}) \right\|_{b \times b} \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{a=1}^2 \frac{1}{2\pi L^d} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{-\frac{\pi}{\beta_a}}^{\frac{\pi}{\beta_a}} d\omega + \int_{\pi h}^{\pi h + \frac{\pi}{\beta_a}} d\omega + \int_{-\pi h - \frac{\pi}{\beta_a}}^{-\pi h} d\omega \right) \\
& \quad \cdot \chi_{h,l}(\omega) \|h^{-1} (I_b - e^{-i\frac{\omega}{h} I_b + \frac{1}{h} \overline{E(\mathbf{k})}})^{-1}\|_{b \times b} \\
& \leq c(M_{UV}, M) \beta_1^{-1} (|x - y| + M^{-l}).
\end{aligned}$$

On the other hand, the inequalities (4.7), (6.24) imply that

$$\begin{aligned}
(6.26) \quad & |C_l^+(\beta_1)(\rho \mathbf{x} \sigma r_{\beta_1}(x), \eta \mathbf{y} \tau r_{\beta_1}(y)) - C_l^+(\beta_2)(\rho \mathbf{x} \sigma r_{\beta_2}(x), \eta \mathbf{y} \tau r_{\beta_2}(y))| \\
& \leq c(M) e^{-3(w(0)M^{l-1} \frac{1}{\pi} |x-y|)^{1/2} - 3 \sum_{j=1}^d (w(0) \frac{L}{2\pi} |e^{i(2\pi/L)\langle \mathbf{x}-\mathbf{y}, \mathbf{v}_j \rangle} - 1|)^{1/2}}.
\end{aligned}$$

By combining (6.26) with (6.25) we have

$$\begin{aligned}
(6.27) \quad & |C_l^+(\beta_1)(\rho \mathbf{x} \sigma r_{\beta_1}(x), \eta \mathbf{y} \tau r_{\beta_1}(y)) - C_l^+(\beta_2)(\rho \mathbf{x} \sigma r_{\beta_2}(x), \eta \mathbf{y} \tau r_{\beta_2}(y))| \\
& \leq c(M_{UV}, M) \beta_1^{-\frac{1}{2}} M^{-\frac{l}{2}} (M^l |x - y| + 1)^{\frac{1}{2}} \\
& \quad \cdot e^{-\frac{3}{2}(w(0)M^{l-1} \frac{1}{\pi} |x-y|)^{1/2} - \frac{3}{2} \sum_{j=1}^d (w(0) \frac{L}{2\pi} |e^{i(2\pi/L)\langle \mathbf{x}-\mathbf{y}, \mathbf{v}_j \rangle} - 1|)^{1/2}}, \\
& \quad (\forall (\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in \hat{I}_0).
\end{aligned}$$

The inequalities (5.32), (5.33) follow from (6.27).

To prove (5.31), take any $X_i, Y_i \in \hat{I}_0$ and $\mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r$ satisfying $\|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} \leq 1$ ($i = 1, 2, \dots, n$). Expanding the determinant along the 1st column and using (6.27), we observe that

$$\begin{aligned}
& |\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} (C_l^+(\beta_1)(R_{\beta_1}(X_i, Y_j)) - C_l^+(\beta_2)(R_{\beta_2}(X_i, Y_j))))_{1 \leq i, j \leq n}| \\
& \leq c(M_{UV}, M) \beta_1^{-\frac{1}{2}} \sum_{s=1}^n \\
& \quad \cdot |\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} (C_l^+(\beta_1)(R_{\beta_1}(X_i, Y_j)) - C_l^+(\beta_2)(R_{\beta_2}(X_i, Y_j))))_{\substack{1 \leq i, j \leq n \\ i \neq s, j \neq 1}}|.
\end{aligned}$$

Then, expanding the remaining determinant as in (2.29) and substituting the determinant bound (5.3) yield the result. \square

6.2. Application of the generalized ultra-violet integration

Since we have checked that the covariances C_l^δ ($l = 1, 2, \dots, N_h, \delta = +, -$) satisfy the desired bound properties, we can readily apply the propositions proved in Subsection 5.1 and Subsection 5.2 to complete the Matsubara UV integration. With $V^\delta(\psi) (\in \wedge \mathcal{V})$ ($\delta = +, -$) defined in (2.30),

set

$$\begin{aligned} F^{\delta, N_h}(\psi) &:= -V^\delta(\psi), \\ T^{\delta, N_h, (n)}(\psi) &:= 0, \quad (\forall n \in \mathbb{N}_{\geq 2}), \quad T^{\delta, N_h}(\psi) := 0, \\ J^{\delta, N_h}(\psi) &:= F^{\delta, N_h}(\psi) + T^{\delta, N_h}(\psi), \quad (\forall \delta \in \{+, -\}). \end{aligned}$$

Then, we inductively define $F^{\delta, l}(\psi)$, $T^{\delta, l, (n)}(\psi)$ ($n \in \mathbb{N}_{\geq 2}$), $T^{\delta, l}(\psi)$, $J^{\delta, l}(\psi) \in \bigwedge \mathcal{V}$ ($l \in \{0, 1, \dots, N_h - 1\}$) by (5.7) with the covariances $\{C_l^\delta\}_{l=1}^{N_h}$ for $\delta = +, -$ respectively.

PROPOSITION 6.4. *Let the weight $w(0)$ be the same as in Lemma 6.2, Lemma 6.3 and the exponent r be $1/2$. Assume that $h \geq e^{2E_1}$. Then, there exist constants $c_0, c'_0 \in \mathbb{R}_{\geq 1}$, which depend only on b, d, M_{UV}, M, E_2 , and a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that if the parameters $M, \alpha \in \mathbb{R}_{\geq 1}, U_\rho \in \mathbb{C}$ ($\rho \in \mathcal{B}$) satisfy*

$$(6.28) \quad M \geq c, \quad \alpha^2 \geq cM, \quad \sup_{\rho \in \mathcal{B}} |U_\rho| \leq \frac{1}{c(c_0 + c'_0)^2 \alpha^4},$$

the following statements hold true.

(1) For any $\delta \in \{+, -\}$, $r \in \{0, 1\}$ and $l \in \{0, 1, \dots, N_h\}$,

$$\begin{aligned} \frac{h}{N} \left(|F_0^{\delta, l}| + \sum_{n=2}^{\infty} |T_0^{\delta, l, (n)}| \right) &\leq \alpha^{-4}, \\ c_0 \alpha^2 \left(\|F_2^{\delta, l}\|_{0, r} + \sum_{n=2}^{\infty} \|T_2^{\delta, l, (n)}\|_{0, r} \right) &\leq 1, \\ M^{-2l} \sum_{m=2}^N c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m \left(\|F_m^{\delta, l}\|_{0, r} + \sum_{n=2}^{\infty} \|T_m^{\delta, l, (n)}\|_{0, r} \right) &\leq 1. \end{aligned}$$

(2) For any $\delta \in \{+, -\}$, $r \in \{0, 1\}$ and $l \in \{0, 1, \dots, N_h\}$, $J^{\delta, l}(\psi)$ is continuous with (U_1, U_2, \dots, U_b) in

$$\{(U_1, U_2, \dots, U_b) \in \mathbb{C}^b \mid |U_\rho| \leq (c(c_0 + c'_0)^2 \alpha^4)^{-1}, \quad (\forall \rho \in \mathcal{B})\}$$

and analytic with (U_1, U_2, \dots, U_b) in

$$\{(U_1, U_2, \dots, U_b) \in \mathbb{C}^b \mid |U_\rho| < (c(c_0 + c'_0)^2 \alpha^4)^{-1}, \quad (\forall \rho \in \mathcal{B})\}.$$

- (3) *There exists a (β, L) -dependent, h -independent constant $c' \in \mathbb{R}_{>0}$ such that if the inequality $\sup_{\rho \in \mathcal{B}} |U_\rho| \leq c'$ additionally holds,*

$$\operatorname{Re} \int e^{-V^\delta(\psi)} d\mu_{C_{>0}^\delta}(\psi) > 0$$

and

$$J^{\delta,0}(\psi) = \log \left(\int e^{-V^\delta(\psi+\psi^1)} d\mu_{C_{>0}^\delta}(\psi^1) \right)$$

for any $\delta \in \{+, -\}$.

- (4) *Assume that (4.2) holds. For any $\delta \in \{+, -\}$, $r \in \{0, 1\}$ and $l \in \{0, 1, \dots, N_h\}$,*

$$\begin{aligned} & \left| \frac{h}{N(\beta_1)} F_0^{\delta,l}(\beta_1) - \frac{h}{N(\beta_2)} F_0^{\delta,l}(\beta_2) \right| \\ & + \sum_{n=2}^{\infty} \left| \frac{h}{N(\beta_1)} T_0^{\delta,l,(n)}(\beta_1) - \frac{h}{N(\beta_2)} T_0^{\delta,l,(n)}(\beta_2) \right| \leq \beta_1^{-\frac{1}{2}} \alpha^{-4}, \\ c_0 \alpha^2 & \left(|F_2^{\delta,l}(\beta_1) - F_2^{\delta,l}(\beta_2)|_0 + \sum_{n=2}^{\infty} |T_2^{\delta,l,(n)}(\beta_1) - T_2^{\delta,l,(n)}(\beta_2)|_0 \right) \leq \beta_1^{-\frac{1}{2}}, \\ M^{-2l} & \sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}} M^{\frac{l}{2}m} \alpha^m \left(|F_m^{\delta,l}(\beta_1) - F_m^{\delta,l}(\beta_2)|_0 \right. \\ & \left. + \sum_{n=2}^{\infty} |T_m^{\delta,l,(n)}(\beta_1) - T_m^{\delta,l,(n)}(\beta_2)|_0 \right) \leq \beta_1^{-\frac{1}{2}}. \end{aligned}$$

REMARK 6.5. It will be shown in the proof below that the constant c_0 in Proposition 6.4 is equal to the maximum of c_0 appearing in Lemma 6.2 and Lemma 6.3. Since these lemmas hold for any larger constant and the generic constant c is independent of c_0 , we have the freedom to replace c_0 in Proposition 6.4 by any larger constant without changing the constant c . Such a replacement will be necessary when we connect the UV integration to the IR integration in Subsection 7.4.

REMARK 6.6. The definition of $J^{\delta,l}(\psi)$ ($l = 0, 1, \dots, N_h$) depends on the parameter M . This means that we have to fix M before introducing

these polynomials. However, the definition of these polynomials does not depend on the parameter α . For $J^{\delta,l}(\psi)$ ($l = 0, 1, \dots, N_h$) the results of Proposition 6.4 hold for any $\alpha \in \mathbb{R}_{\geq 1}$ satisfying (6.28). Bearing this fact in mind, we will use the results of (1), (2) for a large, (β, L, h) -dependent α to prove the claim (3) during the proof of the proposition below.

PROOF OF PROPOSITION 6.4. Let $c_0, c'_0 (\in \mathbb{R}_{\geq 1})$ be the maximum of c_0, c'_0 appearing in Lemma 6.2 and Lemma 6.3 respectively. Then, there exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that if (6.28) holds with c , the results of Proposition 5.2, Proposition 5.4 hold true. In the following we assume (6.28) with this c .

(1), (4): We can apply Proposition 5.2, Proposition 5.4 to justify (1), (4) respectively.

(2): Fix $\delta \in \{+, -\}$. Set

$$D := \{(U_1, U_2, \dots, U_b) \in \mathbb{C}^b \mid |U_\rho| < (c(c_0 + c'_0)^2 \alpha^4)^{-1}, (\forall \rho \in \mathcal{B})\}.$$

Apparently $J^{\delta, N_h}(\psi)$ is continuous in \overline{D} and analytic in D . Assume that $l \in \{0, 1, \dots, N_h - 1\}$ and $J^{\delta, l+1}(\psi)$ is continuous in \overline{D} and analytic in D . Then, so are $F^{\delta, l}(\psi), T^{\delta, l, (n)}(\psi)$ ($n \in \mathbb{N}_{\geq 2}$), since these consist of finite sums and products of $J^{\delta, l+1}(\psi)$. The claim (1) implies that $\sum_{n=2}^{\infty} T^{\delta, l, (n)}(\psi)$ converges uniformly with respect to (U_1, U_2, \dots, U_b) in \overline{D} . Therefore, $T^{\delta, l}(\psi)$ is continuous in \overline{D} and analytic in D , and thus so is $J^{\delta, l}(\psi)$. The induction with respect to l verifies the claim.

(3): Fix $\delta \in \{+, -\}$. First let us note that by definition there exists a (β, L, h) -dependent constant $\tilde{c} \in \mathbb{R}_{>0}$ such that if $\sup_{\rho \in \mathcal{B}} |U_\rho| \leq \tilde{c}$,

$$(6.29) \quad \operatorname{Re} \int e^{-V^\delta(\psi)} d\mu_{\sum_{j=l+1}^{N_h} C_j^\delta}(\psi) > 0, \quad (\forall l \in \{0, 1, \dots, N_h - 1\}).$$

It follows from (1) that

$$(6.30) \quad \begin{aligned} |J_0^{\delta, l}| &\leq \frac{N}{h} \alpha^{-4}, \\ \|J_m^{\delta, l}\|_{0,0} &\leq c_0^{-\frac{m}{2}} M^{-\frac{l}{2}m+2l} \alpha^{-m}, \\ &(\forall l \in \{0, 1, \dots, N_h\}, m \in \{2, 3, \dots, N\}). \end{aligned}$$

This implies that there exists a (β, L, h) -dependent constant $\tilde{c}' \in \mathbb{R}_{>0}$ such that if $\alpha \geq \tilde{c}'$,

$$\operatorname{Re} \int e^{zJ^{\delta, l+1}(\psi)} d\mu_{C_{l+1}^\delta}(\psi) > 0,$$

$$(\forall l \in \{0, 1, \dots, N_h - 1\}, z \in \mathbb{C} \text{ with } |z| \leq 2).$$

Thus, the Grassmann polynomials

$$\log \left(\int e^{zJ^{\delta, l+1}(\psi+\psi^1)} d\mu_{C_{l+1}^\delta}(\psi^1) \right) \quad (l = 0, 1, \dots, N_h - 1)$$

are analytic with z in $\{z \in \mathbb{C} \mid |z| < 2\}$ if

$$(6.31) \quad \sup_{\rho \in \mathcal{B}} |U_\rho| \leq \frac{1}{c(c_0 + c'_0)^2 \tilde{c}'^4}.$$

Therefore, if (6.31) holds,

$$\begin{aligned} (6.32) \quad & \log \left(\int e^{J^{\delta, l+1}(\psi+\psi^1)} d\mu_{C_{l+1}^\delta}(\psi^1) \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{d}{dz} \right)^n \log \left(\int e^{zJ^{\delta, l+1}(\psi+\psi^1)} d\mu_{C_{l+1}^\delta}(\psi^1) \right) \Big|_{z=0} \\ &= F^{\delta, l}(\psi) + \sum_{n=2}^{\infty} T^{\delta, l, (n)}(\psi) \\ &= J^{\delta, l}(\psi), \quad (\forall l \in \{0, 1, \dots, N_h - 1\}). \end{aligned}$$

Let us show that

$$(6.33) \quad J^{\delta, l}(\psi) = \log \left(\int e^{-V^\delta(\psi+\psi^1)} d\mu_{\sum_{j=l+1}^{N_h} C_j^\delta}(\psi^1) \right),$$

$$(\forall l \in \{0, 1, \dots, N_h - 1\})$$

on the assumption

$$(6.34) \quad \sup_{\rho \in \mathcal{B}} |U_\rho| \leq \min \left\{ \tilde{c}, \frac{1}{c(c_0 + c'_0)^2 \tilde{c}'^4} \right\}.$$

The equality (6.33) for $l = N_h - 1$ holds, since it is equivalent to (6.32) for $l = N_h - 1$. Assume that (6.33) holds for $l + 1$. By the condition (6.29) we can apply [14, Lemma C.2] to justify the equality

$$e^{J^{\delta, l+1}(\psi)} = \int e^{-V^\delta(\psi+\psi^1)} d\mu_{\sum_{j=l+2}^{N_h} C_j^\delta}(\psi^1).$$

Moreover, by (6.32) and [5, Proposition I.21],

$$\begin{aligned} J^{\delta,l}(\psi) &= \log \left(\int e^{J^{\delta,l+1}(\psi+\psi^1)} d\mu_{C_{l+1}^\delta}(\psi^1) \right) \\ &= \log \left(\int \int e^{-V^\delta(\psi+\psi^1+\psi^2)} d\mu_{\sum_{j=l+2}^{N_h} C_j^\delta}(\psi^2) d\mu_{C_{l+1}^\delta}(\psi^1) \right) \\ &= \log \left(\int e^{-V^\delta(\psi+\psi^1)} d\mu_{\sum_{j=l+1}^{N_h} C_j^\delta}(\psi^1) \right). \end{aligned}$$

Thus, the induction concludes that the equality (6.33) holds for all $l \in \{0, 1, \dots, N_h - 1\}$ on the assumption (6.34).

By Lemma 2.5 (1) there exists a (β, L) -dependent, h -independent constant $c' \in \mathbb{R}_{>0}$ such that if $\sup_{\rho \in \mathcal{B}} |U_\rho| \leq c'$,

$$\operatorname{Re} \int e^{-V^\delta(\psi)} d\mu_{C_{>0}^\delta}(\psi) > 0,$$

and thus the Grassmann polynomial

$$\log \left(\int e^{-V^\delta(\psi+\psi^1)} d\mu_{C_{>0}^\delta}(\psi^1) \right)$$

is analytic with (U_1, U_2, \dots, U_b) in $\{(U_1, U_2, \dots, U_b) \in \mathbb{C}^b \mid |U_\rho| < c', (\forall \rho \in \mathcal{B})\}$. On the other hand, by the claim (2) for $l = 0$ and taking the h -independent constant c' smaller if necessary we see that $J^{\delta,0}(\psi)$ is analytic in $\{(U_1, U_2, \dots, U_b) \in \mathbb{C}^b \mid |U_\rho| < c', (\forall \rho \in \mathcal{B})\}$. Therefore, the identity theorem ensures that the equality (6.33) for $l = 0$ holds for any $(U_1, U_2, \dots, U_b) \in \mathbb{C}^b$ satisfying $|U_\rho| < c' (\forall \rho \in \mathcal{B})$. \square

7. The Infrared Integration of the Model

Here we start the infrared analysis of the free energy density defined in Subsection 1.2. As we saw in Remark 1.5, the free energy density is independent of how to choose the argument $\theta_L(\cdot, \cdot) : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying (1.1) and (1.2). Therefore, let us focus on the model Hamiltonian with the argument θ_L simply defined by (1.4). The periodic properties of the hopping amplitude and the magnitude of the on-site coupling enable us to redefine the free energy density as that governed by a 4-band Hamiltonian, whose hopping amplitude and coupling constants are no longer dependent

on the position vector but on the band index. This 4-band many-electron system can be analyzed by means of the general estimations constructed so far. The essential ingredient of the IR integration process considered in this section is an extension of Giuliani-Mastropietro's RG method designed for the 2-band Hamiltonian ([9]). As in Giuliani-Mastropietro's RG we make use of the symmetries of Grassmann polynomials to show that covariances for the IR integration have good bound properties. Then, applying the framework developed in Subsection 5.3 and Subsection 5.4, we will move on to the proof of Theorem 1.1.

7.1. The four-band formulation

Let us set up a 4-band Hamiltonian whose free energy density is equal to that considered in Theorem 1.1. From now we assume that $d = 2$, $\mathbf{u}_1 = \mathbf{v}_1 = \mathbf{e}_1 (= (1, 0))$, $\mathbf{u}_2 = \mathbf{v}_2 = \mathbf{e}_2 (= (0, 1))$ so that

$$\Gamma = \left\{ \sum_{j=1}^2 m_j \mathbf{e}_j \mid m_j \in \{0, 1, \dots, L-1\} (j = 1, 2) \right\},$$

$$\Gamma^* = \left\{ \frac{2\pi}{L} \sum_{j=1}^2 m_j \mathbf{e}_j \mid m_j \in \{0, 1, \dots, L-1\} (j = 1, 2) \right\}.$$

Since we are going to define a 4-band model, we assume that $b = 4$ and $\mathcal{B} = \{1, 2, 3, 4\}$. The crystal lattice in a box is identified with $\mathcal{B} \times \Gamma$. In the case $L = 2$, the lattice $\mathcal{B} \times \Gamma$ can be pictured as in Figure 3. For any $\mathbf{x} \in \Gamma$ we assume that the site $(2, \mathbf{x})$ is right to the site $(1, \mathbf{x})$, the site $(3, \mathbf{x})$ is above $(1, \mathbf{x})$, and the site $(4, \mathbf{x})$ is right to the site $(3, \mathbf{x})$, as described in Figure 4.

With the parameters $t_{h,e}, t_{h,o}, t_{v,e}, t_{v,o} \in \mathbb{R}_{>0}$ we define $E(\cdot) : \mathbb{R}^2 \rightarrow \text{Mat}(4, \mathbb{C})$ by

$$(7.1) \quad E(\mathbf{k}) := \begin{pmatrix} 0 & t_{h,e}(1 + e^{-ik_1}) & t_{v,e}(1 + e^{-ik_2}) & 0 \\ t_{h,e}(1 + e^{ik_1}) & 0 & 0 & -t_{v,o}(1 + e^{-ik_2}) \\ t_{v,e}(1 + e^{ik_2}) & 0 & 0 & t_{h,o}(1 + e^{-ik_1}) \\ 0 & -t_{v,o}(1 + e^{ik_2}) & t_{h,o}(1 + e^{ik_1}) & 0 \end{pmatrix}.$$

We can see that $E(\cdot)$ satisfies (2.1) and (2.2). With this $E(\cdot)$ we define the free Hamiltonian H_0 by (2.3). For notational convenience, set $U_1 := U_{e,e}$,

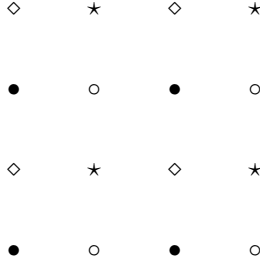


Fig. 3. The lattice $\mathcal{B} \times \Gamma$ for $L = 2$, where the symbol “●” denotes the sites of $\{1\} \times \Gamma$, the symbol “○” denotes the sites of $\{2\} \times \Gamma$, the symbol “◇” denotes the sites of $\{3\} \times \Gamma$, the symbol “★” denotes the sites of $\{4\} \times \Gamma$.

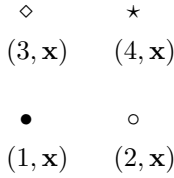


Fig. 4. The arrangement of 4 sites $(1, \mathbf{x}), (2, \mathbf{x}), (3, \mathbf{x}), (4, \mathbf{x})$.

$U_2 := U_{o,e}, U_3 := U_{e,o}, U_4 := U_{o,o}$ with the parameters $U_{e,e}, U_{o,e}, U_{e,o}, U_{o,o} \in \mathbb{R}$ introduced in Subsection 1.2. We define the interacting part V of the Hamiltonian by (2.4). Then, we define the Hamiltonian $H : F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})) \rightarrow F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$ by $H := H_0 + V$.

LEMMA 7.1. *The quantity*

$$-\frac{1}{\beta(2L)^2} \log(\text{Tr } e^{-\beta H})$$

derived from this Hamiltonian by the trace operation over $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$ is equal to the free energy density

$$-\frac{1}{\beta(2L)^2} \log(\text{Tr } e^{-\beta H})$$

considered in Theorem 1.1.

PROOF. For $\rho \in \mathcal{B}$ set

$$(7.2) \quad \mathbf{e}(\rho) := \begin{cases} \mathbf{0}, & \text{if } \rho = 1, \\ \mathbf{e}_1, & \text{if } \rho = 2, \\ \mathbf{e}_2, & \text{if } \rho = 3, \\ \mathbf{e}_1 + \mathbf{e}_2, & \text{if } \rho = 4. \end{cases}$$

Then, define the linear map $G : F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\})) \rightarrow F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$ by

$$\begin{aligned} G\Omega &:= \Omega_{2L}, \\ G(\psi_{\rho_1 \mathbf{x}_1 \sigma_1}^* \psi_{\rho_2 \mathbf{x}_2 \sigma_2}^* \cdots \psi_{\rho_n \mathbf{x}_n \sigma_n}^* \Omega) & \\ &:= \psi_{2\mathbf{x}_1 + \mathbf{e}(\rho_1)\sigma_1}^* \psi_{2\mathbf{x}_2 + \mathbf{e}(\rho_2)\sigma_2}^* \cdots \psi_{2\mathbf{x}_n + \mathbf{e}(\rho_n)\sigma_n}^* \Omega_{2L}, \\ &(\forall (\rho_j, \mathbf{x}_j, \sigma_j) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \ (j = 1, 2, \dots, n)), \end{aligned}$$

and by linearity. We can check that the map G is unitary. By definition,

$$(7.3) \quad \begin{aligned} H_0 &= \sum_{(\mathbf{x}, \sigma) \in \Gamma \times \{\uparrow, \downarrow\}} (t_{h,e} \psi_{1, \mathbf{x} \sigma}^* (\psi_{2, \mathbf{x} \sigma} + \psi_{2, \mathbf{x} - \mathbf{e}_1 \sigma}) \\ &\quad + t_{v,e} \psi_{1, \mathbf{x} \sigma}^* (\psi_{3, \mathbf{x} \sigma} + \psi_{3, \mathbf{x} - \mathbf{e}_2 \sigma}) \\ &\quad - t_{v,o} \psi_{2, \mathbf{x} \sigma}^* (\psi_{4, \mathbf{x} \sigma} + \psi_{4, \mathbf{x} - \mathbf{e}_2 \sigma}) \\ &\quad + t_{h,o} \psi_{3, \mathbf{x} \sigma}^* (\psi_{4, \mathbf{x} \sigma} + \psi_{4, \mathbf{x} - \mathbf{e}_1 \sigma})) + \text{h.c.}, \end{aligned}$$

where the notation ‘h.c’ means that the adjoint operator of the operator in front is placed. Note that

$$\begin{aligned} &GH_0G^* \\ &= \sum_{(\mathbf{x}, \sigma) \in \Gamma \times \{\uparrow, \downarrow\}} (t_{h,e} \psi_{2\mathbf{x}\sigma}^* (\psi_{2\mathbf{x} + \mathbf{e}_1 \sigma} + \psi_{2\mathbf{x} - \mathbf{e}_1 \sigma}) + t_{v,e} \psi_{2\mathbf{x}\sigma}^* (\psi_{2\mathbf{x} + \mathbf{e}_2 \sigma} + \psi_{2\mathbf{x} - \mathbf{e}_2 \sigma}) \\ &\quad - t_{v,o} \psi_{2\mathbf{x} + \mathbf{e}_1 \sigma}^* (\psi_{2\mathbf{x} + \mathbf{e}_1 + \mathbf{e}_2 \sigma} + \psi_{2\mathbf{x} + \mathbf{e}_1 - \mathbf{e}_2 \sigma}) \\ &\quad + t_{h,o} \psi_{2\mathbf{x} + \mathbf{e}_2 \sigma}^* (\psi_{2\mathbf{x} + \mathbf{e}_1 + \mathbf{e}_2 \sigma} + \psi_{2\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2 \sigma})) + \text{h.c} \\ &= H_0, \end{aligned}$$

where H_0 is the operator defined in (1.3) with the phase θ_L defined in (1.4). Similarly we can confirm that $GVG^* = V$, which was defined in (1.5).

Thus, we have that $\text{Tr} e^{-\beta H} = \text{Tr} e^{-\beta G H G^*} = \text{Tr} e^{-\beta \mathbf{H}}$ with the Hamiltonian \mathbf{H} containing the phase (1.4). Then, the claim follows from Remark 1.5. \square

Lemma 7.1 tells us that it suffices to prove the same statements as in Theorem 1.1 for

$$-\frac{1}{\beta L^2} \log(\text{Tr} e^{-\beta H})$$

with the Hamiltonian H defined above. Moreover, we will later confirm that the claims of Theorem 1.1 follow from the theorem proved under the assumption

$$(7.4) \quad \max\{t_{h,e}, t_{h,o}, t_{v,e}, t_{v,o}\} = 1.$$

Thus, from now we assume (7.4) unless otherwise stated.

7.2. The cut-off function for the infrared integration

Here we define a cut-off function whose support covers the zero set of the free dispersion relation. In order to choose such a cut-off function correctly, let us study properties of E first.

LEMMA 7.2. *The following inequalities hold.*

(1)

$$\left\| \left(\frac{\partial}{\partial k_j} \right)^n E(\mathbf{k}) \right\|_{4 \times 4} \leq 4, \quad (\forall \mathbf{k} \in \mathbb{R}^2, n \in \mathbb{N} \cup \{0\}, j \in \{1, 2\}).$$

(2)

$$\|(i\omega I_4 - E(\mathbf{k}))^{-1}\|_{4 \times 4} \leq \left(\omega^2 + f_{\mathbf{t}} \sum_{j=1}^2 (1 + \cos k_j) \right)^{-\frac{1}{2}}, \quad (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3),$$

where $f_{\mathbf{t}}$ is the quantity defined in (1.6).

PROOF. For any $\mathbf{k} \in \mathbb{R}^2$, $p, q \in \{1, -1\}$ set

$$(7.5) \quad \begin{aligned} X_{p,q}(\mathbf{k}) &:= p \left(A(\mathbf{k}) + q \sqrt{A(\mathbf{k})^2 - 4B(\mathbf{k})^2} \right)^{\frac{1}{2}}, \\ A(\mathbf{k}) &:= (t_{h,e}^2 + t_{h,o}^2)(1 + \cos k_1) + (t_{v,e}^2 + t_{v,o}^2)(1 + \cos k_2), \end{aligned}$$

$$B(\mathbf{k}) := t_{h,e}t_{h,o}(1 + \cos k_1) + t_{v,e}t_{v,o}(1 + \cos k_2).$$

A calculation shows that the eigen values of $E(\mathbf{k})$ are $X_{p,q}(\mathbf{k})$ ($p, q \in \{1, -1\}$). One can also check that the eigen values of $(\partial/\partial k_1)^n E(\mathbf{k})$ are $t_{h,e}$, $-t_{h,e}$, $t_{h,o}$, $-t_{h,o}$ and the eigen values of $(\partial/\partial k_2)^n E(\mathbf{k})$ are $t_{v,e}$, $-t_{v,e}$, $t_{v,o}$, $-t_{v,o}$ for any $n \in \mathbb{N}$.

(1): Using the assumption (7.4), we have that

$$\begin{aligned} \|E(\mathbf{k})\|_{4 \times 4} &\leq \max_{p,q \in \{1, -1\}} |X_{p,q}(\mathbf{k})| \leq \sqrt{2}|A(\mathbf{k})|^{\frac{1}{2}} \leq 4, \\ \left\| \left(\frac{\partial}{\partial k_j} \right)^n E(\mathbf{k}) \right\|_{4 \times 4} &\leq 1, \quad (\forall j \in \{1, 2\}, n \in \mathbb{N}). \end{aligned}$$

(2): Set

$$s := \max \left\{ \frac{t_{h,o}}{t_{h,e}} + \frac{t_{h,e}}{t_{h,o}}, \frac{t_{v,o}}{t_{v,e}} + \frac{t_{v,e}}{t_{v,o}} \right\}.$$

Since $A(\mathbf{k}) \leq sB(\mathbf{k})$, for any $p, q \in \{1, -1\}$,

$$\begin{aligned} |X_{p,q}(\mathbf{k})| &\geq \left| A(\mathbf{k}) - \sqrt{A(\mathbf{k})^2 - 4B(\mathbf{k})^2} \right|^{\frac{1}{2}} \\ &\geq \left| sB(\mathbf{k}) - \sqrt{s^2 B(\mathbf{k})^2 - 4B(\mathbf{k})^2} \right|^{\frac{1}{2}} = (s - \sqrt{s^2 - 4})^{\frac{1}{2}} B(\mathbf{k})^{\frac{1}{2}} \\ &\geq (s - \sqrt{s^2 - 4})^{\frac{1}{2}} (\min\{t_{h,e}t_{h,o}, t_{v,e}t_{v,o}\})^{\frac{1}{2}} \left(\sum_{j=1}^2 (1 + \cos k_j) \right)^{\frac{1}{2}}. \end{aligned}$$

Since

$$\begin{aligned} s - \sqrt{s^2 - 4} &\geq \min \left\{ \frac{t_{h,o}}{t_{h,e}}, \frac{t_{h,e}}{t_{h,o}}, \frac{t_{v,o}}{t_{v,e}}, \frac{t_{v,e}}{t_{v,o}} \right\}, \\ |X_{p,q}(\mathbf{k})| &\geq f_{\mathbf{t}}^{\frac{1}{2}} \left(\sum_{j=1}^2 (1 + \cos k_j) \right)^{\frac{1}{2}}, \quad (\forall p, q \in \{1, -1\}). \end{aligned}$$

Thus,

$$\begin{aligned} \|(i\omega I_4 - E(\mathbf{k}))^{-1}\|_{4 \times 4} &\leq \max_{p,q \in \{1, -1\}} |i\omega - X_{p,q}(\mathbf{k})|^{-1} \\ &\leq \left(\omega^2 + f_{\mathbf{t}} \sum_{j=1}^2 (1 + \cos k_j) \right)^{-\frac{1}{2}}. \quad \square \end{aligned}$$

Lemma 7.2 (1) implies that the inequality (6.1) holds with $E_1 = 4$, $E_2 = 1$. In this section we will apply the results of Section 6 for $E_1 = 4$, $E_2 = 1$. It follows that

$$M_{UV} = \frac{10\sqrt{6}}{\pi}$$

and the weight $w(0)$ originally set in Lemma 6.2 satisfies

$$(7.6) \quad w(0) = \frac{c_w}{18} M^{-2}.$$

In order to adjust the support of cut-off functions for the IR integration, from now we assume that

$$M > \sqrt{2}.$$

Let us set

$$M_{IR} := \frac{\sqrt{6}}{\pi} \left(\frac{\pi^2}{3} M_{UV}^2 + 4 \right)^{\frac{1}{2}}$$

and

$$N_\beta := \min \left\{ \left[\frac{\log \left(\frac{\pi}{\beta} \left(\frac{\pi}{\sqrt{3}} M_{IR} \right)^{-1} \right)}{\log M} \right], 0 \right\}.$$

Since $(\pi/\sqrt{3})M_{IR}M^{N_\beta} \leq \pi/\beta$,

$$(7.7) \quad \phi \left(M_{IR}^{-2} M^{-2N_\beta} \left(\omega^2 + f_{\mathbf{t}} \sum_{j=1}^2 (1 + \cos k_j) \right) \right) = 0, \\ (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3 \text{ with } |\omega| \geq \pi/\beta),$$

where ϕ is the smooth function introduced in Lemma 6.1. We define the functions $\chi_l : \mathbb{R}^3 \rightarrow \mathbb{R}$ ($l \in \{0, -1, \dots, N_\beta\}$) by

$$\chi_l(\omega, \mathbf{k}) \\ := \phi(M_{UV}^{-2}\omega^2) \left(\phi \left(M_{IR}^{-2} M^{-2(l+1)} \left(\omega^2 + f_{\mathbf{t}} \sum_{j=1}^2 (1 + \cos k_j) \right) \right) \right) \\ - \phi \left(M_{IR}^{-2} M^{-2l} \left(\omega^2 + f_{\mathbf{t}} \sum_{j=1}^2 (1 + \cos k_j) \right) \right), \quad ((\omega, \mathbf{k}) \in \mathbb{R}^3).$$

If $\phi(M_{UV}^{-2}\omega^2) \neq 0$,

$$\omega^2 + f_{\mathbf{t}} \sum_{j=1}^2 (1 + \cos k_j) \leq \frac{\pi^2}{3} M_{UV}^2 + 4f_{\mathbf{t}} \leq \frac{\pi^2}{3} M_{UV}^2 + 4,$$

and thus

$$(7.8) \quad \phi \left(M_{IR}^{-2} M^{-2} \left(\omega^2 + f_{\mathbf{t}} \sum_{j=1}^2 (1 + \cos k_j) \right) \right) = 1, \\ (\forall \omega \in \mathbb{R} \text{ with } \phi(M_{UV}^{-2}\omega^2) \neq 0, \mathbf{k} \in \mathbb{R}^2).$$

It follows from (7.7), (7.8) that

$$(7.9) \quad \sum_{l=0}^{N_\beta} \chi_l(\omega, \mathbf{k}) = \phi(M_{UV}^{-2}\omega^2), \quad (\forall \omega \in \mathcal{M}, \mathbf{k} \in \mathbb{R}^2).$$

The value of $\chi_l(\omega, \mathbf{k})$ is described as follows.

$$(7.10) \quad \chi_l(\omega, \mathbf{k}) \begin{cases} = 0, & \text{if } \left(\omega^2 + f_{\mathbf{t}} \sum_{j=1}^2 (1 + \cos k_j) \right)^{\frac{1}{2}} \\ & \leq \frac{\pi}{\sqrt{6}} M_{IR} M^l, \\ \in [0, 1], & \text{if } \frac{\pi}{\sqrt{6}} M_{IR} M^l < \left(\omega^2 + f_{\mathbf{t}} \sum_{j=1}^2 (1 + \cos k_j) \right)^{\frac{1}{2}} \\ & < \frac{\pi}{\sqrt{3}} M_{IR} M^{l+1}, \\ = 0, & \text{if } \left(\omega^2 + f_{\mathbf{t}} \sum_{j=1}^2 (1 + \cos k_j) \right)^{\frac{1}{2}} \\ & \geq \frac{\pi}{\sqrt{3}} M_{IR} M^{l+1}, \end{cases} \\ (\forall l \in \{0, -1, \dots, N_\beta\}, (\omega, \mathbf{k}) \in \mathbb{R}^3).$$

Since $M > \sqrt{2}$, $(\pi/\sqrt{3})M_{IR}M^{l-1} < (\pi/\sqrt{6})M_{IR}M^l$. This inequality implies that

$$(7.11) \quad \{(\omega, \mathbf{k}) \in \mathbb{R}^3 \mid \chi_l(\omega, \mathbf{k}) \neq 0\} \cap \{(\omega, \mathbf{k}) \in \mathbb{R}^3 \mid \chi_j(\omega, \mathbf{k}) \neq 0\} = \emptyset, \\ (\forall j, l \in \{0, -1, \dots, N_\beta\} \text{ with } |j - l| \geq 2).$$

We use $\chi_l : \mathbb{R}^3 \rightarrow \mathbb{R}$ ($l = 0, -1, \dots, N_\beta$) as the cut-off functions in the IR integration. For any $l \in \{0, -1, \dots, N_\beta\}$ set

$$\chi_{\leq l}(\omega, \mathbf{k}) := \sum_{j=l}^{N_\beta} \chi_j(\omega, \mathbf{k}),$$

$$\hat{\chi}_{\leq l}(\omega, \mathbf{k}) := \phi(M_{UV}^{-2}\omega^2)\phi\left(M_{IR}^{-2}M^{-2(l+1)}\left(\omega^2 + f_{\mathbf{t}}\sum_{j=1}^2(1 + \cos k_j)\right)\right),$$

$(\forall(\omega, \mathbf{k}) \in \mathbb{R}^3).$

Note that $\text{supp } \chi_l(\cdot) \subset \text{supp } \chi_{\leq l}(\cdot) \subset \text{supp } \hat{\chi}_{\leq l}(\cdot)$. Concerning the support of these cut-off functions, we will frequently use the following lemma.

LEMMA 7.3. *Let $l \in \{0, 1, \dots, N_\beta\}$. If $(\omega, \mathbf{k}) \in \mathbb{R} \times [0, 2\pi]^2$ satisfies $\hat{\chi}_{\leq l}(\omega, \mathbf{k}) \neq 0$, then,*

$$|\omega| \leq \frac{\pi}{\sqrt{3}}M_{IR}M^{l+1}, \quad |k_j - \pi| \leq \frac{\pi^2}{\sqrt{6}}f_{\mathbf{t}}^{-\frac{1}{2}}M_{IR}M^{l+1}, \quad (j = 1, 2).$$

PROOF. If $\hat{\chi}_{\leq l}(\omega, \mathbf{k}) \neq 0$,

$$\left(\omega^2 + f_{\mathbf{t}}\sum_{j=1}^2(1 + \cos k_j)\right)^{\frac{1}{2}} \leq \frac{\pi}{\sqrt{3}}M_{IR}M^{l+1}.$$

Then, by using the inequality $\sqrt{1 - \cos \theta} \geq (\sqrt{2}/\pi)|\theta|$ ($\forall \theta \in [-\pi, \pi]$) we can derive the claimed inequalities. \square

In order to indicate the dependency on β , we will sometimes write $\chi_{\leq l}(\beta)$ instead of $\chi_{\leq l}$. By (7.7) we see that

$$(7.12) \quad \begin{aligned} \chi_{\leq l}(\beta)(\omega, \mathbf{k}) &= \hat{\chi}_{\leq l}(\omega, \mathbf{k}), \\ (\forall(\omega, \mathbf{k}) \in \mathbb{R}^3 \text{ with } |\omega| \geq \pi/\beta, l \in \{0, -1, \dots, N_\beta\}), \\ \chi_{\leq l}(\beta_1)(\omega, \mathbf{k}) &= \chi_{\leq l}(\beta_2)(\omega, \mathbf{k}), \\ (\forall(\omega, \mathbf{k}) \in \mathbb{R}^3 \text{ with } |\omega| \geq \pi/\beta_1, l \in \{0, -1, \dots, N_{\beta_1}\}), \end{aligned}$$

if $\beta_1 \leq \beta_2$.

We define the weights $w(l)$ ($l \in \mathbb{Z}_{\leq 0}$) by

$$w(l) := w(0)M^l, \quad (\forall l \in \mathbb{Z}_{\leq 0}),$$

with the weight $w(0)$ characterized in (7.6). Moreover, we take the exponent r inside $\|\cdot\|_{l,0}$, $\|\cdot\|_{l,1}$, $|\cdot - \cdot|_l$ to be $1/2$ throughout this section. To organize

formulas systematically, for any differentiable function f with the variable (ω, k_1, k_2) let $(\partial/\partial k_0)f$ denote $(\partial/\partial\omega)f$.

LEMMA 7.4. *There exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that*

$$\left| \left(\frac{\partial}{\partial k_j} \right)^n \hat{\chi}_{\leq l}(\omega, \mathbf{k}) \right|, \left| \left(\frac{\partial}{\partial k_j} \right)^n \chi_l(\omega, \mathbf{k}) \right| \leq (c\omega(l)^{-1})^n (n!)^2, \\ (\forall(\omega, \mathbf{k}) \in \mathbb{R}^3, n \in \mathbb{N} \cup \{0\}, j \in \{0, 1, 2\}, l \in \{0, -1, \dots, N_\beta\}).$$

PROOF. Using the inequality $f_{\mathbf{t}} \leq 1$, we see that for any $n \in \mathbb{N}$,

$$\left| \left(\frac{d}{d\omega} \right)^n M_{UV}^{-2} \omega^2 \right| \leq cn!, \quad (\forall \omega \in \mathbb{R} \text{ with } \phi(M_{UV}^{-2} \omega^2) \neq 0),$$

$$\left| \left(\frac{\partial}{\partial \omega} \right)^n M_{IR}^{-2} M^{-2l} \left(\omega^2 + f_{\mathbf{t}} \sum_{i=1}^2 (1 + \cos k_i) \right) \right|$$

$$\leq \begin{cases} cM^{-l+1}, & \text{if } n = 1, \\ cM^{-2l}, & \text{if } n = 2, \\ 0, & \text{if } n \geq 3, \end{cases} \leq cM \cdot M^{-ln} n!,$$

$$\left| \left(\frac{\partial}{\partial k_j} \right)^n M_{IR}^{-2} M^{-2l} \left(\omega^2 + f_{\mathbf{t}} \sum_{i=1}^2 (1 + \cos k_i) \right) \right|$$

$$\leq \begin{cases} cM^{-l+1}, & \text{if } n = 1, \\ cM^{-2l}, & \text{if } n \geq 2, \end{cases} \leq cM \cdot M^{-ln} n!,$$

$(\forall(\omega, \mathbf{k}) \in \mathbb{R}^3$ satisfying

$$\phi \left(M_{IR}^{-2} M^{-2(l+1)} \left(\omega^2 + f_{\mathbf{t}} \sum_{i=1}^2 (1 + \cos k_i) \right) \right) \neq 0,$$

$\forall j \in \{1, 2\}$).

Thus, by (6.2), Lemma C.1 and the inequality $M \geq 1$,

$$\left| \left(\frac{d}{d\omega} \right)^n \phi(M_{UV}^{-2} \omega^2) \right| \leq c^n (n!)^2,$$

$$\left| \left(\frac{\partial}{\partial k_j} \right)^n \phi \left(M_{IR}^{-2} M^{-2l} \left(\omega^2 + f_{\mathbf{t}} \sum_{i=1}^2 (1 + \cos k_i) \right) \right) \right|,$$

$$\begin{aligned} & \left| \left(\frac{\partial}{\partial k_j} \right)^n \phi \left(M_{IR}^{-2} M^{-2(l+1)} \left(\omega^2 + f_{\mathbf{t}} \sum_{i=1}^2 (1 + \cos k_i) \right) \right) \right| \\ & \leq (cM^{-l+1})^n (n!)^2, \quad (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3, j \in \{0, 1, 2\}, n \in \mathbb{N} \cup \{0\}). \end{aligned}$$

By the condition $c_w \in (0, 1]$, $M^{-l+1} \leq w(l)^{-1}$. Using these inequalities, we can deduce that for any $(\omega, \mathbf{k}) \in \mathbb{R}^3$,

$$\begin{aligned} & \left| \left(\frac{\partial}{\partial k_j} \right)^n \hat{\chi}_{\leq l}(\omega, \mathbf{k}) \right| \\ & \leq \sum_{m=0}^n \binom{n}{m} \left| \left(\frac{\partial}{\partial k_j} \right)^m \phi(M_{UV}^{-2} \omega^2) \right| \\ & \quad \cdot \left| \left(\frac{\partial}{\partial k_j} \right)^{n-m} \phi \left(M_{IR}^{-2} M^{-2(l+1)} \left(\omega^2 + f_{\mathbf{t}} \sum_{j=1}^2 (1 + \cos k_j) \right) \right) \right| \\ & \leq \sum_{m=0}^n \binom{n}{m} c^m (m!)^2 (cM^{-l+1})^{n-m} ((n-m)!)^2 \leq (cw(l)^{-1})^n (n!)^2. \end{aligned}$$

The upper bound on $|(\partial/\partial k_j)^n \chi_l(\omega, \mathbf{k})|$ can be derived similarly. \square

By definition we have $(\pi/\sqrt{3})M_{IR}M^{N_\beta} \leq \pi/\beta$. The next lemma suggests that an opposite inequality is also available when we deal with covariances for the IR integration.

LEMMA 7.5. *If $\chi_{\leq 0}(\omega, \mathbf{k}) \neq 0$ for some $(\omega, \mathbf{k}) \in \mathbb{R}^3$ with $|\omega| \geq \pi/\beta$,*

$$\frac{1}{\beta} \leq M_{IR}M^{N_\beta+1}.$$

PROOF. By assumption $\phi(M_{UV}^{-2}(\pi/\beta)^2) \neq 0$, which implies that $1/\beta \leq M_{UV}/\sqrt{3} \leq M_{IR}/\sqrt{6}$. Thus, if $N_\beta = 0$, the claimed inequality holds. If $N_\beta < 0$,

$$N_\beta \geq \frac{\log \left(\frac{\pi}{\beta} \left(\frac{\pi}{\sqrt{3}} M_{IR} \right)^{-1} \right)}{\log M} - 1,$$

which implies that $1/\beta \leq (1/\sqrt{3})M_{IR}M^{N_\beta+1}$. \square

7.3. The covariance matrices in the infrared integration

Following the infrared integration scheme proposed in [18], [3], [9], we update the covariance by inserting the kernel of the quadratic Grassmann polynomial produced by the previous integration at every integration step. To simulate this procedure, we introduce a family of subsets of $\bigwedge \mathcal{V}$ consisting of polynomials satisfying certain bound properties and invariant properties. Then, we define a prototypical covariance by substituting the kernel of a polynomial belonging to one of these subsets and study its properties.

Let c_{IR} , $\alpha \in \mathbb{R}_{>0}$ and $D(\subset \mathbb{C}^4)$ be a domain satisfying that $\overline{\mathbf{U}} \in \overline{D}$ ($\forall \mathbf{U} \in \overline{D}$), where $\overline{\mathbf{U}}$ is the complex conjugate of \mathbf{U} and \overline{D} is the closure of D . For any $l \in \mathbb{Z}_{\leq 0}$ we define the subset $\mathcal{S}(l)$ of $\bigwedge \mathcal{V}$ as follows. A Grassmann polynomial $J(\psi)$ ($\in \bigwedge \mathcal{V}$) belongs to $\mathcal{S}(l)$ if and only if $J(\psi)$ is parameterized by $\mathbf{U} \in \overline{D}$ and satisfies the conditions (i), (ii), (iii), (iv).

(i) $J(\mathbf{U})(\psi)$ is continuous in \overline{D} and analytic in D with \mathbf{U} .

(ii)

$$(7.13) \quad \frac{h}{N} |J_0| \leq M^{\frac{7}{2}l} \alpha^{-3}, \quad (\forall \mathbf{U} \in \overline{D}),$$

$$(7.14) \quad M^{-\frac{7}{2}l+r} \sum_{m=2}^N c_{IR}^{\frac{m}{2}} M^{ml} \alpha^m \|J_m\|_{l,r} \leq 1, \quad (\forall \mathbf{U} \in \overline{D}, r \in \{0, 1\}).$$

(iii) With the notation introduced in Subsection 3.3,

$$J(\mathbf{U})(\psi) = J(\mathbf{U})(\mathcal{R}\psi), \quad (\forall \mathbf{U} \in \overline{D}),$$

for all $S : I \rightarrow I$ and $Q : I \rightarrow \mathbb{R}$ defined as follows.

$$(7.15) \quad \begin{aligned} S((\rho, \mathbf{x}, \sigma, x, \theta)) &:= (\rho, \mathbf{x}, \sigma, x, \theta), \\ Q((\rho, \mathbf{x}, \sigma, x, \theta)) &:= \frac{\pi}{2} \theta, \quad (\forall (\rho, \mathbf{x}, \sigma, x, \theta) \in I). \end{aligned}$$

$$(7.16) \quad \begin{aligned} S((\rho, \mathbf{x}, \sigma, x, \theta)) &:= (\rho, \mathbf{x}, \sigma, x, \theta), \\ Q((\rho, \mathbf{x}, \sigma, x, \theta)) &:= \pi 1_{\sigma=\uparrow}, \quad (\forall (\rho, \mathbf{x}, \sigma, x, \theta) \in I). \end{aligned}$$

$$(7.17) \quad S((\rho, \mathbf{x}, \sigma, x, \theta)) := (\rho, \mathbf{x}, -\sigma, x, \theta),$$

$$Q((\rho, \mathbf{x}, \sigma, x, \theta)) := 0, \quad (\forall (\rho, \mathbf{x}, \sigma, x, \theta) \in I).$$

$$(7.18) \quad \begin{aligned} S((\rho, \mathbf{x}, \sigma, x, \theta)) &:= (\rho, r_L(\mathbf{x} + \mathbf{z}), \sigma, r_\beta(x + s), \theta), \\ Q((\rho, \mathbf{x}, \sigma, x, \theta)) &:= \pi n_\beta(r_\beta(x - s) + s), \quad (\forall (\rho, \mathbf{x}, \sigma, x, \theta) \in I), \end{aligned}$$

where $\mathbf{z} \in \mathbb{Z}^2$ and $s \in (1/h)\mathbb{Z}$ are arbitrarily taken and fixed.

$$(7.19) \quad \begin{aligned} S((\rho, \mathbf{x}, \sigma, x, \theta)) &:= (\rho, r_L(-\mathbf{x} - \mathbf{e}(\rho)), \sigma, x, \theta), \\ Q((\rho, \mathbf{x}, \sigma, x, \theta)) &:= 0, \quad (\forall (\rho, \mathbf{x}, \sigma, x, \theta) \in I), \end{aligned}$$

where $\mathbf{e}(\rho)$ ($\rho \in \mathcal{B}$) are the vectors defined in (7.2).

(iv)

$$J(\mathbf{U})(\psi) = \overline{J(\overline{\mathbf{U}})}(\mathcal{R}\psi), \quad (\forall \mathbf{U} \in \overline{\mathcal{D}}),$$

for all $S : I \rightarrow I$ and $Q : I \rightarrow \mathbb{R}$ defined as follows.

$$(7.20) \quad \begin{aligned} S((\rho, \mathbf{x}, \sigma, x, \theta)) &:= (\rho, \mathbf{x}, \sigma, r_\beta(-x), -\theta), \\ Q((\rho, \mathbf{x}, \sigma, x, \theta)) &:= \pi(1_{\theta=1} + 1_{x \neq 0}), \quad (\forall (\rho, \mathbf{x}, \sigma, x, \theta) \in I). \end{aligned}$$

$$(7.21) \quad \begin{aligned} S((\rho, \mathbf{x}, \sigma, x, \theta)) &:= (\rho, \mathbf{x}, \sigma, x, -\theta), \\ Q((\rho, \mathbf{x}, \sigma, x, \theta)) &:= \pi 1_{\rho \in \{1,4\}}, \quad (\forall (\rho, \mathbf{x}, \sigma, x, \theta) \in I). \end{aligned}$$

We write $\mathcal{S}(l)(\beta)$ in place of $\mathcal{S}(l)$ when we want to indicate the dependency on β . On the assumption (4.2) we define the subset $\tilde{\mathcal{S}}(l)$ of $\mathcal{S}(l)(\beta_1) \times \mathcal{S}(l)(\beta_2)$ as follows. A pair of Grassmann polynomials $(J(\beta_1)(\psi), J(\beta_2)(\psi)) \in \mathcal{S}(l)(\beta_1) \times \mathcal{S}(l)(\beta_2)$ belongs to $\tilde{\mathcal{S}}(l)$ if and only if

$$(7.22) \quad \left| \frac{h}{N(\beta_1)} J_0(\beta_1) - \frac{h}{N(\beta_2)} J_0(\beta_2) \right| \leq \beta_1^{-\frac{1}{2}} M^{\frac{5}{2}l} \alpha^{-3}, \quad (\forall \mathbf{U} \in \overline{\mathcal{D}}).$$

$$(7.23) \quad M^{-\frac{5}{2}l} \sum_{m=2}^{N(\beta_2)} c_{IR}^{\frac{m}{2}} M^{ml} \alpha^m |J_m(\beta_1) - J_m(\beta_2)| \leq \beta_1^{-\frac{1}{2}}, \quad (\forall \mathbf{U} \in \overline{\mathcal{D}}).$$

Later in Subsection 7.4 we will see that the output of the infrared integration at scale $l + 1$ belongs to $\mathcal{S}(l)$ and a pair of the output at β_1 and β_2 belongs to $\tilde{\mathcal{S}}(l)$. The bound properties assumed in $\mathcal{S}(l)$ and $\tilde{\mathcal{S}}(l)$ correspond

to the resulting inequalities in Proposition 5.6 and Proposition 5.9 with $a_1 = 2$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1/2$. In fact we will apply these propositions with these exponents in the forthcoming IR analysis. The invariant properties listed in (iii), (iv) are especially needed in order that for $J(\psi) \in \mathcal{S}(l)$ the kernel of $J_2(\psi)$ has desirable symmetries for updating the covariance without changing the original infrared singularity.

As a preliminary, let us characterize the quadratic part of a polynomial belonging to this class in the momentum space. Let $l \in \mathbb{Z}_{\leq 0}$ and $J^l(\psi) \in \mathcal{S}(l)$. Using the kernel $J_2^l(\cdot) : I^2 \rightarrow \mathbb{C}$ of the quadratic part $\bar{J}_2^l(\psi)$, we define the map $W^l(\cdot, \cdot) : \mathcal{M} \times (2\pi/L)\mathbb{Z}^2 \rightarrow \text{Mat}(4, \mathbb{C})$ by

$$(7.24) \quad \begin{aligned} W^l(\omega, \mathbf{k})(\rho, \eta) &:= \frac{2}{h} \sum_{\mathbf{x} \in \Gamma} \sum_{x \in [0, \beta)_h} e^{-i\langle \mathbf{k}, \mathbf{x} \rangle} e^{-i\omega x} J_2^l((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1)), \\ &(\forall (\omega, \mathbf{k}) \in \mathcal{M} \times (2\pi/L)\mathbb{Z}^2, \rho, \eta \in \mathcal{B}). \end{aligned}$$

LEMMA 7.6. *The following statements hold true.*

(1)

$$\begin{aligned} J_2^l(\psi) &= \frac{1}{h^2} \sum_{\substack{(\rho, \mathbf{x}, \sigma, x), \\ (\eta, \mathbf{y}, \tau, y) \in I_0}} \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{k}, \mathbf{x} - \mathbf{y} \rangle} e^{i\omega(x-y)} W^l(\omega, \mathbf{k})(\rho, \eta) \psi_{\rho \mathbf{x} \sigma x} \bar{\psi}_{\eta \mathbf{y} \tau y}. \end{aligned}$$

(2) For any $(\omega, \mathbf{k}) \in \mathcal{M} \times (2\pi/L)\mathbb{Z}^2$, $\mathbf{U} \in \bar{D}$, $\rho, \eta \in \mathcal{B}$,

$$(7.25) \quad W^l(\mathbf{U})(\omega, \mathbf{k})(\rho, \eta) = \overline{W^l(\bar{\mathbf{U}})(-\omega, \mathbf{k})(\eta, \rho)},$$

$$(7.26) \quad W^l(\mathbf{U})(\omega, \mathbf{k})(\rho, \eta) = (-1)^{1_{\rho \in \{1,4\}} + 1_{\eta \in \{1,4\}}} \overline{W^l(\bar{\mathbf{U}})(\omega, \mathbf{k})(\eta, \rho)},$$

$$(7.27) \quad W^l(\mathbf{U})(\omega, \mathbf{k})(\rho, \eta) = e^{i\langle \mathbf{e}(\rho) - \mathbf{e}(\eta), \mathbf{k} \rangle} W^l(\mathbf{U})(\omega, -\mathbf{k})(\rho, \eta).$$

(3) *There exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that*

$$|W^l(\omega, \mathbf{k})(\rho, \eta)| \leq c \cdot c_{IR}^{-1} \left(|\omega| + \sum_{j=1}^2 |k_j - \pi| + \frac{1}{\beta} + \frac{1}{L} \right) M^{\frac{1}{2}l} \alpha^{-2},$$

$$(\forall(\omega, \mathbf{k}) \in \mathcal{M} \times (2\pi/L)\mathbb{Z}^2, \rho, \eta \in \mathcal{B}).$$

REMARK 7.7. The inequality in (3) suggests that $W^l(\omega, \mathbf{k})$ becomes negligibly small as (ω, \mathbf{k}) approaches $(0, \pi, \pi)$ and thus the point $(0, \pi, \pi)$ is essentially a zero-point of the perturbed matrix $i\omega I_4 - E(\mathbf{k}) - W^l(\omega, \mathbf{k})$. This is the crucial reason why the multi-scale IR integration around the point $(0, \pi, \pi)$ converges. We prove the inequality in (3) by making use of the invariant properties summarized in (2). Our argument based on the preserved symmetries is motivated by the preceding work [9] by Giuliani and Mastropietro and should be regarded as an extension of Giuliani-Mastropietro's RG method designed for the 2-band Hubbard model on the honeycomb lattice.

PROOF OF LEMMA 7.6. (1): By the invariance with S, Q defined in (7.15) we obtain that

$$(7.28) \quad \begin{aligned} & J_2^l((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)) \\ &= e^{i\frac{\pi}{2}(\theta+\xi)} J_2^l((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)), \\ & (\forall(\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi) \in I). \end{aligned}$$

By the invariance with S, Q defined in (7.16),

$$(7.29) \quad \begin{aligned} & J_2^l((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)) \\ &= (-1)^{1_{\sigma=\uparrow}+1_{\tau=\uparrow}} J_2^l((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)), \\ & (\forall(\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi) \in I). \end{aligned}$$

By the invariance with S, Q defined in (7.17),

$$(7.30) \quad \begin{aligned} & J_2^l((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)) = J_2^l((\rho, \mathbf{x}, -\sigma, x, \theta), (\eta, \mathbf{y}, -\tau, y, \xi)), \\ & (\forall(\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi) \in I). \end{aligned}$$

Note that for any $x \in [0, \beta)_h, s \in (1/h)\mathbb{Z}$,

$$n_\beta(r_\beta(r_\beta(x+s) - s) + s) = n_\beta(x+s).$$

Using this equality and the uniqueness of the anti-symmetric kernel, we can deduce from the invariance with S, Q defined in (7.18) that

$$(7.31) \quad J_2^l((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi))$$

$$\begin{aligned}
&= (-1)^{n_\beta(x+s)+n_\beta(y+s)} \\
&\quad \cdot J_2^l((\rho, r_L(\mathbf{x} + \mathbf{z}), \sigma, r_\beta(x + s), \theta), (\eta, r_L(\mathbf{y} + \mathbf{z}), \tau, r_\beta(y + s), \xi)), \\
&\quad (\forall(\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi) \in I, \mathbf{z} \in (2\pi/L)\mathbb{Z}^2, s \in (1/h)\mathbb{Z}).
\end{aligned}$$

Using the equalities (7.28), (7.29), (7.30), (7.31) in this order, we observe that

$$\begin{aligned}
(7.32) \quad &\frac{1}{h^2} \sum_{\substack{(\rho, \mathbf{x}, \sigma, x, \theta), \\ (\eta, \mathbf{y}, \tau, y, \xi) \in I}} J_2^l((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)) \psi_{\rho\mathbf{x}\sigma x} \psi_{\eta\mathbf{y}\tau y} \xi \\
&= \frac{2}{h^2} \sum_{\substack{(\rho, \mathbf{x}, \sigma, x), \\ (\eta, \mathbf{y}, \tau, y) \in I_0}} J_2^l((\rho, \mathbf{x}, \sigma, x, -1), (\eta, \mathbf{y}, \tau, y, 1)) \psi_{\rho\mathbf{x}\sigma x} \bar{\psi}_{\eta\mathbf{y}\tau y} \\
&= \frac{2}{h^2} \sum_{\substack{(\rho, \mathbf{x}, \sigma, x), \\ (\eta, \mathbf{y}, \tau, y) \in I_0}} \delta_{\sigma, \tau} J_2^l((\rho, \mathbf{x}, \sigma, x, -1), (\eta, \mathbf{y}, \tau, y, 1)) \psi_{\rho\mathbf{x}\sigma x} \bar{\psi}_{\eta\mathbf{y}\tau y} \\
&= \frac{2}{h^2} \sum_{\substack{(\rho, \mathbf{x}, \sigma, x), \\ (\eta, \mathbf{y}, \tau, y) \in I_0}} \delta_{\sigma, \tau} (-1)^{n_\beta(x-y)} \\
&\quad \cdot J_2^l((\rho, r_L(\mathbf{x} - \mathbf{y}), \uparrow, r_\beta(x - y), -1), (\eta, \mathbf{0}, \uparrow, 0, 1)) \psi_{\rho\mathbf{x}\sigma x} \bar{\psi}_{\eta\mathbf{y}\tau y}.
\end{aligned}$$

It follows from (7.24) that

$$\begin{aligned}
&\frac{1}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{k}, \mathbf{x} - \mathbf{y} \rangle} e^{i\omega(x-y)} W^l(\omega, \mathbf{k})(\rho, \eta) \\
&= (-1)^{n_\beta(x-y)} 2J_2^l((\rho, r_L(\mathbf{x} - \mathbf{y}), \uparrow, r_\beta(x - y), -1), (\eta, \mathbf{0}, \uparrow, 0, 1)).
\end{aligned}$$

By combining this equality with (7.32) we obtain the claimed equality.

(2): By the invariance with S, Q defined in (7.20),

$$\begin{aligned}
&\overline{J^l(\bar{\mathbf{U}})_2((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi))} \\
&= (-1)^{1_\theta = -1 + 1_\xi = -1 + 1_{x \neq 0} + 1_{y \neq 0}} \\
&\quad \cdot J^l(\mathbf{U})_2((\rho, \mathbf{x}, \sigma, r_\beta(-x), -\theta), (\eta, \mathbf{y}, \tau, r_\beta(-y), -\xi)), \\
&\quad (\forall(\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi) \in I).
\end{aligned}$$

By using this equality, (7.31) and the anti-symmetry of the kernel we have that

$$\overline{W^l(\bar{\mathbf{U}})(-\omega, \mathbf{k})(\eta, \rho)}$$

$$\begin{aligned}
 &= \frac{2}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} e^{i\langle \mathbf{k}, \mathbf{x} \rangle} e^{-i\omega x} \overline{J^l(\overline{\mathbf{U}})_2((\eta, \mathbf{x}, \uparrow, x, -1), (\rho, \mathbf{0}, \uparrow, 0, 1))} \\
 &= \frac{2}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} e^{i\langle \mathbf{k}, \mathbf{x} \rangle} e^{-i\omega x} (-1)^{1+1_{x \neq 0}} \\
 &\quad \cdot J^l(\mathbf{U})_2((\eta, \mathbf{x}, \uparrow, r_\beta(-x), 1), (\rho, \mathbf{0}, \uparrow, 0, -1)) \\
 &= \frac{2}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} e^{i\langle \mathbf{k}, \mathbf{x} \rangle} e^{-i\omega x} (-1) \\
 &\quad \cdot J^l(\mathbf{U})_2((\eta, \mathbf{0}, \uparrow, 0, 1), (\rho, r_L(-\mathbf{x}), \uparrow, x, -1)) \\
 &= W^l(\mathbf{U})(\omega, \mathbf{k})(\rho, \eta),
 \end{aligned}$$

which is (7.25).

By the invariance with S, Q defined in (7.21),

$$\begin{aligned}
 &\overline{J^l(\overline{\mathbf{U}})_2((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi))} \\
 &= (-1)^{1_{\rho \in \{1,4\}} + 1_{\eta \in \{1,4\}}} J^l(\mathbf{U})_2((\rho, \mathbf{x}, \sigma, x, -\theta), (\eta, \mathbf{y}, \tau, y, -\xi)), \\
 &(\forall (\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi) \in I).
 \end{aligned}$$

It follows from this equality, (7.31) and anti-symmetry that

$$\begin{aligned}
 &(-1)^{1_{\rho \in \{1,4\}} + 1_{\eta \in \{1,4\}} + 1} \overline{W^l(\overline{\mathbf{U}})(\omega, \mathbf{k})(\eta, \rho)} \\
 &= \frac{2}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} e^{i\langle \mathbf{k}, \mathbf{x} \rangle} e^{i\omega x} (-1)^{1_{\rho \in \{1,4\}} + 1_{\eta \in \{1,4\}} + 1} \\
 &\quad \cdot \overline{J^l(\overline{\mathbf{U}})_2((\eta, \mathbf{x}, \uparrow, x, -1), (\rho, \mathbf{0}, \uparrow, 0, 1))} \\
 &= \frac{2}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} e^{i\langle \mathbf{k}, \mathbf{x} \rangle} e^{i\omega x} (-1) J^l(\mathbf{U})_2((\eta, \mathbf{x}, \uparrow, x, 1), (\rho, \mathbf{0}, \uparrow, 0, -1)) \\
 &= \frac{2}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} e^{i\langle \mathbf{k}, \mathbf{x} \rangle} e^{i\omega x} (-1)^{n_\beta(-x)} \\
 &\quad \cdot J^l(\mathbf{U})_2((\rho, r_L(-\mathbf{x}), \uparrow, r_\beta(-x), -1), (\eta, \mathbf{0}, \uparrow, 0, 1)) \\
 &= W^l(\mathbf{U})(\omega, \mathbf{k})(\rho, \eta),
 \end{aligned}$$

which is (7.26).

By the invariance with S, Q defined in (7.19),

$$J_2^l((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi))$$

$$= J_2^l((\rho, r_L(-\mathbf{x} - \mathbf{e}(\rho)), \sigma, x, \theta), (\eta, r_L(-\mathbf{y} - \mathbf{e}(\eta)), \tau, y, \xi)), \\ (\forall(\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi) \in I).$$

By using this equality and (7.31) we can derive that

$$e^{i\langle \mathbf{e}(\rho) - \mathbf{e}(\eta), \mathbf{k} \rangle} W^l(\omega, -\mathbf{k})(\rho, \eta) \\ = \frac{2}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} e^{-i\langle \mathbf{k}, -\mathbf{x} - \mathbf{e}(\rho) + \mathbf{e}(\eta) \rangle} e^{-i\omega x} \\ \cdot J_2^l((\rho, r_L(-\mathbf{x} - \mathbf{e}(\rho)), \uparrow, x, -1), (\eta, r_L(-\mathbf{e}(\eta)), \uparrow, 0, 1)) \\ = W^l(\omega, \mathbf{k})(\rho, \eta),$$

which is (7.27).

(3): Take any $\rho, \eta \in \mathcal{B}$ satisfying $\rho, \eta \in \{1, 4\}$ or $\rho, \eta \in \{2, 3\}$. Since

$$(-1)^{1_{\rho \in \{1, 4\}} + 1_{\eta \in \{1, 4\}} + 1} = -1,$$

the equalities (7.25), (7.26) yield that

$$W^l(\omega, \mathbf{k})(\rho, \eta) + W^l(-\omega, \mathbf{k})(\rho, \eta) = 0, \quad (\forall(\omega, \mathbf{k}) \in \mathcal{M} \times (2\pi/L)\mathbb{Z}^2).$$

Especially,

$$W^l\left(\frac{\pi}{\beta}, \mathbf{k}\right)(\rho, \eta) + W^l\left(-\frac{\pi}{\beta}, \mathbf{k}\right)(\rho, \eta) = 0, \quad (\forall \mathbf{k} \in (2\pi/L)\mathbb{Z}^2).$$

Using this equality, we have that

$$(7.33) \quad |W^l(\omega, \mathbf{k})(\rho, \eta)| \leq \frac{1}{2} \left| W^l(\omega, \mathbf{k})(\rho, \eta) - W^l\left(\frac{\pi}{\beta}, \mathbf{k}\right)(\rho, \eta) \right| \\ + \frac{1}{2} \left| W^l(\omega, \mathbf{k})(\rho, \eta) - W^l\left(-\frac{\pi}{\beta}, \mathbf{k}\right)(\rho, \eta) \right| \\ \leq \left(|\omega| + \frac{c}{\beta} \right) \sup_{(\omega, \mathbf{k}) \in \mathcal{M} \times \frac{2\pi}{L}\mathbb{Z}^2} |\mathcal{D}_0 W^l(\omega, \mathbf{k})(\rho, \eta)|.$$

On the other hand, let us fix $\rho, \eta \in \mathcal{B}$ satisfying $\rho \in \{1, 4\}$ and $\eta \in \{2, 3\}$, or $\rho \in \{2, 3\}$ and $\eta \in \{1, 4\}$. First, consider the case that $L \in 2\mathbb{N}$. Since $(\pi, \pi) \in (2\pi/L)\mathbb{Z}^2$ in this case, the equality (7.27) ensures that $W^l(\omega, (\pi, \pi))(\rho, \eta) = 0$. We deduce from this equality that

$$(7.34) \quad |W^l(\omega, \mathbf{k})(\rho, \eta)| \leq |W^l(\omega, \mathbf{k})(\rho, \eta) - W^l(\omega, (\pi, k_2))(\rho, \eta)|$$

$$\begin{aligned}
 & + |W^l(\omega, (\pi, k_2))(\rho, \eta) - W^l(\omega, (\pi, \pi))(\rho, \eta)| \\
 & \leq \sum_{j=1}^2 |k_j - \pi| \sup_{(\omega, \mathbf{k}) \in \mathcal{M} \times \frac{2\pi}{L}\mathbb{Z}^2} |\mathcal{D}_j W^l(\omega, \mathbf{k})(\rho, \eta)|.
 \end{aligned}$$

Next, let us assume that $L \notin 2\mathbb{N}$. In this case, $\pi - \pi/L, \pi + \pi/L \in (2\pi/L)\mathbb{Z}$. Therefore, it follows from the equality (7.27) that

$$\begin{aligned}
 & \left| W^l\left(\omega, \left(\pi + \frac{\pi}{L}, \pi + \frac{\pi}{L}\right)\right)(\rho, \eta) + W^l\left(\omega, \left(\pi - \frac{\pi}{L}, \pi - \frac{\pi}{L}\right)\right)(\rho, \eta) \right| \\
 & \leq |e^{i\frac{\pi}{L}} - 1| \left| W^l\left(\omega, \left(\pi - \frac{\pi}{L}, \pi - \frac{\pi}{L}\right)\right)(\rho, \eta) \right|.
 \end{aligned}$$

Substituting this inequality, we see that

$$\begin{aligned}
 (7.35) \quad & |W^l(\omega, \mathbf{k})(\rho, \eta)| \\
 & \leq \frac{1}{2} \sum_{\delta \in \{1, -1\}} \left(\left| W^l(\omega, \mathbf{k})(\rho, \eta) - W^l\left(\omega, \left(\pi + \frac{\delta\pi}{L}, k_2\right)\right)(\rho, \eta) \right| \right. \\
 & \quad \left. + \left| W^l\left(\omega, \left(\pi + \frac{\delta\pi}{L}, k_2\right)\right)(\rho, \eta) \right. \right. \\
 & \quad \left. \left. - W^l\left(\omega, \left(\pi + \frac{\delta\pi}{L}, \pi + \frac{\delta\pi}{L}\right)\right)(\rho, \eta) \right| \right) \\
 & \quad + \frac{1}{2} \left| \sum_{\delta \in \{1, -1\}} W^l\left(\omega, \left(\pi + \frac{\delta\pi}{L}, \pi + \frac{\delta\pi}{L}\right)\right)(\rho, \eta) \right| \\
 & \leq \sum_{j=1}^2 \left(|k_j - \pi| + \frac{c}{L} \right) \sup_{(\omega, \mathbf{k}) \in \mathcal{M} \times \frac{2\pi}{L}\mathbb{Z}^2} |\mathcal{D}_j W^l(\omega, \mathbf{k})(\rho, \eta)| \\
 & \quad + \frac{c}{L} \sup_{(\omega, \mathbf{k}) \in \mathcal{M} \times \frac{2\pi}{L}\mathbb{Z}^2} |W^l(\omega, \mathbf{k})(\rho, \eta)|.
 \end{aligned}$$

The inequalities (7.33), (7.34), (7.35) lead to

$$\begin{aligned}
 (7.36) \quad & |W^l(\omega, \mathbf{k})(\rho, \eta)| \leq \left(|\omega| + \sum_{m=1}^2 |k_m - \pi| + \frac{c}{\beta} + \frac{c}{L} \right) \\
 & \quad \cdot \left(\sup_{j \in \{0, 1, 2\}} \sup_{(\omega, \mathbf{k}) \in \mathcal{M} \times \frac{2\pi}{L}\mathbb{Z}^2} |\mathcal{D}_j W^l(\omega, \mathbf{k})(\rho, \eta)| \right)
 \end{aligned}$$

$$+ \sup_{(\omega, \mathbf{k}) \in \mathcal{M} \times \frac{2\pi}{L} \mathbb{Z}^2} |W^l(\omega, \mathbf{k})(\rho, \eta)|, \\ (\forall (\omega, \mathbf{k}) \in \mathcal{M} \times (2\pi/L)\mathbb{Z}^2, \rho, \eta \in \mathcal{B}).$$

We can see from the definition of $W^l(\cdot)$ and (7.14) that

$$|W^l(\omega, \mathbf{k})(\rho, \eta)| \leq 2\|J_2^l\|_{l,0} \leq 2c_{IR}^{-1}M^{\frac{3}{2}l}\alpha^{-2}, \\ |\mathcal{D}_j W^l(\omega, \mathbf{k})(\rho, \eta)| \leq 2\|J_2^l\|_{l,1} \leq 2c_{IR}^{-1}M^{\frac{1}{2}l}\alpha^{-2}, \quad (\forall j \in \{0, 1, 2\}).$$

By combining these inequalities with (7.36) we obtain the inequality claimed in (3). \square

For later use we define an extension of the function $W^l(\cdot)$ with continuous variables. For this purpose we need a few more notations. For any $x \in [0, \beta]$ let

$$r'_\beta(x) := \begin{cases} x & \text{if } x \in [0, \beta/2), \\ x - \beta & \text{if } x \in [\beta/2, \beta). \end{cases}$$

For any $x \in [0, L]$ let

$$s_L(x) := \begin{cases} x & \text{if } x \in [0, L/2), \\ x - L & \text{if } x \in [L/2, L). \end{cases}$$

Then, for any $(x_1, x_2) \in [0, L]^2$ let $r'_L((x_1, x_2)) := (s_L(x_1), s_L(x_2))$. With these notations, set

$$(7.37) \quad \widehat{W}^l(\omega, \mathbf{k})(\rho, \eta) := \frac{2}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} e^{-i\langle \mathbf{k}, r'_L(\mathbf{x}) \rangle} e^{-i\omega r'_\beta(x)} (-1)^{n_\beta(r'_\beta(x))} \\ \cdot J_2^l((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1)), \\ (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3, \rho, \eta \in \mathcal{B}).$$

Note that

$$(7.38) \quad \widehat{W}^l(\omega, \mathbf{k}) = W^l(\omega, \mathbf{k}), \quad (\forall (\omega, \mathbf{k}) \in \mathcal{M} \times (2\pi/L)\mathbb{Z}^2),$$

Let us establish various inequalities involving this function in Lemma 7.8, Lemma 7.9, Lemma 7.10 and Lemma 7.11 below, step by step.

LEMMA 7.8. *Assume that*

$$(7.39) \quad \frac{1}{L} \leq \frac{1}{\beta} \leq M_{IR} M^{N_\beta+1}.$$

Then, there exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that the following inequalities hold for any $l \in \{0, -1, \dots, N_\beta\}$, $j \in \{0, -1, \dots, l\}$.

(1)

$$\begin{aligned} \|\widehat{W}^j(\omega, \mathbf{k})\|_{4 \times 4} &\leq c \cdot c_{IR}^{-1} f_{\mathbf{t}}^{-\frac{1}{2}} M^{\frac{1}{2}j+l+1} \alpha^{-2}, \\ (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3 \text{ satisfying } \chi_l(\omega, \mathbf{k}) \neq 0). \end{aligned}$$

(2)

$$\begin{aligned} \left\| \left(\frac{\partial}{\partial k_i} \right)^n \widehat{W}^j(\omega, \mathbf{k}) \right\|_{4 \times 4} &\leq c \cdot c_{IR}^{-1} M^{\frac{3}{2}j} \alpha^{-2} (c w(j)^{-1})^n (n!)^2, \\ (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3, i \in \{0, 1, 2\}, n \in \mathbb{N} \cup \{0\}). \end{aligned}$$

PROOF. (1): Take any $(\omega, \mathbf{k}) \in \mathbb{R}^3$ satisfying $\chi_l(\omega, \mathbf{k}) \neq 0$. By periodicity we may assume that $\mathbf{k} \in [0, 2\pi)^2$ without losing generality. Let $\hat{\omega} \in \mathcal{M}$, $\hat{\mathbf{k}} = (\hat{k}_1, \hat{k}_2) \in (2\pi/L)\mathbb{Z}^2$ be such that $\omega \in [\hat{\omega}, \hat{\omega} + 2\pi/\beta)$, $\mathbf{k} \in [\hat{k}_1, \hat{k}_1 + 2\pi/L) \times [\hat{k}_2, \hat{k}_2 + 2\pi/L)$. It follows from (7.14) that

$$\begin{aligned} (7.40) \quad &|\widehat{W}^j(\omega, \mathbf{k})(\rho, \eta) - W^j(\hat{\omega}, \hat{\mathbf{k}})(\rho, \eta)| \\ &\leq \frac{2}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} (|e^{i\omega r'_\beta(x)} - e^{i\hat{\omega} r'_\beta(x)}| + |e^{i\langle \mathbf{k}, r'_L(\mathbf{x}) \rangle} - e^{i\langle \hat{\mathbf{k}}, r'_L(\mathbf{x}) \rangle}|) \\ &\quad \cdot |J_2^j((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))| \\ &\leq \frac{c}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} \left(|\omega - \hat{\omega}| (1_{x < \frac{\beta}{2}} |x| + 1_{x \geq \frac{\beta}{2}} |x - \beta|) \right. \\ &\quad \left. + \sum_{m=1}^2 |k_m - \hat{k}_m| (1_{x_m < \frac{L}{2}} |x_m| + 1_{x_m \geq \frac{L}{2}} |x_m - L|) \right) \\ &\quad \cdot |J_2^j((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{c}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} \left(\frac{1}{\beta} d_0((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1)) \right. \\
&\quad \left. + \frac{1}{L} \sum_{m=1}^2 d_m((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1)) \right) \\
&\quad \cdot |J_2^j((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))| \\
&\leq c \left(\frac{1}{\beta} + \frac{1}{L} \right) \|J_2^j\|_{j,1} \leq c \left(\frac{1}{\beta} + \frac{1}{L} \right) c_{IR}^{-1} M^{\frac{1}{2}j} \alpha^{-2}.
\end{aligned}$$

By Lemma 7.3 and the inequality $f_{\mathbf{t}} \leq 1$,

$$|\hat{\omega}| + \sum_{m=1}^2 |\hat{k}_m - \pi| \leq c \left(\frac{1}{\beta} + \frac{1}{L} + f_{\mathbf{t}}^{-\frac{1}{2}} M^{l+1} \right).$$

We can combine this inequality with Lemma 7.6 (3), (7.39) and (7.40) to deduce that

$$\begin{aligned}
\|\widehat{W}^j(\omega, \mathbf{k})\|_{4 \times 4} &\leq \|\widehat{W}^j(\omega, \mathbf{k}) - W^j(\hat{\omega}, \hat{\mathbf{k}})\|_{4 \times 4} + \|W^j(\hat{\omega}, \hat{\mathbf{k}})\|_{4 \times 4} \\
&\leq c \cdot c_{IR}^{-1} \left(|\hat{\omega}| + \sum_{m=1}^2 |\hat{k}_m - \pi| + \frac{1}{\beta} + \frac{1}{L} \right) M^{\frac{1}{2}j} \alpha^{-2} \\
&\leq c \cdot c_{IR}^{-1} f_{\mathbf{t}}^{-\frac{1}{2}} M^{\frac{1}{2}j+l+1} \alpha^{-2}.
\end{aligned}$$

(2): By (7.14) and the inequality

$$|r'_\beta(x)| \leq \frac{\pi}{2} |d_0((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))|, \quad (\forall x \in [0, \beta)_h),$$

we have that

$$\begin{aligned}
\left\| \left(\frac{\partial}{\partial \omega} \right)^n \widehat{W}^j(\omega, \mathbf{k}) \right\|_{4 \times 4} &\leq 2 \left(\frac{\pi}{2} \right)^n \mathbf{w}(j)^{-n} (2n)! \|J_2^j\|_{j,0} \\
&\leq c^{n+1} \mathbf{w}(j)^{-n} (n!)^2 c_{IR}^{-1} M^{\frac{3}{2}j} \alpha^{-2}, \quad (\forall n \in \mathbb{N} \cup \{0\}).
\end{aligned}$$

Since

$$|s_L(x_i)| \leq \frac{\pi}{2} |d_i((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))|, \quad (\forall i \in \{1, 2\}, (x_1, x_2) \in \Gamma),$$

the upper bound on $\|(\partial/\partial k_i)^n \widehat{W}^j(\omega, \mathbf{k})\|_{4 \times 4}$ ($i \in \{1, 2\}$) can be obtained in the same way. \square

LEMMA 7.9. *Assume that (4.2) holds. Then, there exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that the following inequality holds true for any $j \in \{0, -1, \dots, N_{\beta_1}\}$, $(J^j(\beta_1)(\psi), J^j(\beta_2)(\psi)) \in \tilde{\mathcal{S}}(j)$.*

$$\begin{aligned} & \left\| \left(\frac{\partial}{\partial k_i} \right)^n (\widehat{W}^j(\beta_1)(\omega, \mathbf{k}) - \widehat{W}^j(\beta_2)(\omega, \mathbf{k})) \right\|_{4 \times 4} \\ & \leq c \beta_1^{-\frac{1}{2}} c_{IR}^{-1} M^{\frac{1}{2}j} \alpha^{-2} (c\omega(j)^{-1})^n (n!)^2, \\ & (\forall(\omega, \mathbf{k}) \in \mathbb{R}^3, i \in \{0, 1, 2\}, n \in \mathbb{N} \cup \{0\}). \end{aligned}$$

PROOF. Note that for any $a \in \{1, 2\}$,

$$\begin{aligned} & \widehat{W}^j(\beta_a)(\omega, \mathbf{k})(\rho, \eta) \\ & = \frac{2}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta_a)_h} e^{-i\langle \mathbf{k}, r'_L(\mathbf{x}) \rangle} (1_{x \in [0, \frac{\beta_1}{4})} e^{-i\omega x} + 1_{x \in [\frac{\beta_1}{4}, \frac{\beta_a}{2})} e^{-i\omega x} \\ & \quad - 1_{x \in [\frac{\beta_a}{2}, \beta_a - \frac{\beta_1}{4})} e^{-i\omega(x - \beta_a)} - 1_{x \in [\beta_a - \frac{\beta_1}{4}, \beta_a)} e^{-i\omega(x - \beta_a)}) \\ & \quad \cdot J_2^j(\beta_a)((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1)) \\ & = \frac{2}{h} \sum_{\mathbf{x} \in \Gamma} \sum_{x \in [-\frac{\beta_1}{4}, \frac{\beta_1}{4})_h} e^{-i\langle \mathbf{k}, r'_L(\mathbf{x}) \rangle} \\ & \quad \cdot (1_{x \in [0, \frac{\beta_1}{4})} e^{-i\omega x} J_2^j(\beta_a)((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1)) \\ & \quad - 1_{x \in [-\frac{\beta_1}{4}, 0)} e^{-i\omega x} J_2^j(\beta_a)((\rho, \mathbf{x}, \uparrow, x + \beta_a, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))) \\ & + \frac{2}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta_a)_h} e^{-i\langle \mathbf{k}, r'_L(\mathbf{x}) \rangle} (1_{x \in [\frac{\beta_1}{4}, \frac{\beta_a}{2})} e^{-i\omega x} - 1_{x \in [\frac{\beta_a}{2}, \beta_a - \frac{\beta_1}{4})} e^{-i\omega(x - \beta_a)}) \\ & \quad \cdot J_2^j(\beta_a)((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1)). \end{aligned}$$

Thus,

$$\left| \prod_{i=0}^2 \left(\frac{\partial}{\partial k_i} \right)^{n_i} (\widehat{W}^j(\beta_1)(\omega, \mathbf{k})(\rho, \eta) - \widehat{W}^j(\beta_2)(\omega, \mathbf{k})(\rho, \eta)) \right|$$

$$\begin{aligned}
&\leq \frac{2}{h} \sum_{\mathbf{x} \in \Gamma} \sum_{x \in [-\frac{\beta_1}{4}, \frac{\beta_1}{4}]_h} |x|^{n_0} |s_L(x_1)|^{n_1} |s_L(x_2)|^{n_2} \\
&\quad \cdot |J_2^j(\beta_1)(R_{\beta_1}((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))) \\
&\quad - J_2^j(\beta_2)(R_{\beta_2}((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1)))| \\
&\quad + \frac{2}{h} \sum_{a=1}^2 \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta_a]_h} (1_{x \in [\frac{\beta_1}{4}, \frac{\beta_a}{2}]}) |x|^{n_0} + 1_{x \in [\frac{\beta_a}{2}, \beta_a - \frac{\beta_1}{4}]} |x - \beta_a|^{n_0}) \\
&\quad \cdot |s_L(x_1)|^{n_1} |s_L(x_2)|^{n_2} |J_2^j(\beta_a)((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))| \\
&\leq \frac{2}{h} \sum_{\mathbf{x} \in \Gamma} \sum_{x \in [-\frac{\beta_1}{4}, \frac{\beta_1}{4}]_h} \left(\frac{\pi}{2}\right)^{n_1+n_2} \prod_{i=0}^2 |\hat{d}_i((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))|^{n_i} \\
&\quad \cdot |J_2^j(\beta_1)(R_{\beta_1}((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))) \\
&\quad - J_2^j(\beta_2)(R_{\beta_2}((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1)))| \\
&\quad + \frac{2}{h} \sum_{a=1}^2 \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta_a]_h} \left(\frac{\pi}{2}\right)^{n_1+n_2} \\
&\quad \cdot \left(1_{x \in [\frac{\beta_1}{4}, \frac{\beta_a}{2}]} \frac{|x|^{n_0}}{|d_0(\beta_a)((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))|^{n_0+1}} \right. \\
&\quad \quad \left. + 1_{x \in [\frac{\beta_a}{2}, \beta_a - \frac{\beta_1}{4}]} \frac{|x - \beta_a|^{n_0}}{|d_0(\beta_a)((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))|^{n_0+1}} \right) \\
&\quad \cdot |d_0(\beta_a)((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))| \\
&\quad \cdot \prod_{i=0}^2 |d_i((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))|^{n_i} \\
&\quad \cdot |J_2^j(\beta_a)((\rho, \mathbf{x}, \uparrow, x, -1), (\eta, \mathbf{0}, \uparrow, 0, 1))|.
\end{aligned}$$

Moreover, substitution of (7.14), (7.23) gives

$$\begin{aligned}
&\left\| \left(\frac{\partial}{\partial k_i} \right)^n (\widehat{W}^j(\beta_1)(\omega, \mathbf{k}) - \widehat{W}^j(\beta_2)(\omega, \mathbf{k})) \right\|_{4 \times 4} \\
&\leq c^{n+1} \mathbf{w}(j)^{-n} (2n)! |J_2^j(\beta_1) - J_2^j(\beta_2)|_j \\
&\quad + c^{n+1} \beta_1^{-1} \mathbf{w}(j)^{-n} (2n)! \sum_{a=1}^2 \|J_2^j(\beta_a)\|_{j,1}
\end{aligned}$$

$$\begin{aligned} &\leq c\beta_1^{-\frac{1}{2}}c_{IR}^{-1}M^{\frac{1}{2}j}\alpha^{-2}(cw(j)^{-1})^n(n!)^2, \\ &(\forall(\omega, \mathbf{k}) \in \mathbb{R}^3, i \in \{0, 1, 2\}, n \in \mathbb{N} \cup \{0\}). \quad \square \end{aligned}$$

LEMMA 7.10. *Assume that (7.39) holds. Then, there exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that the following inequalities hold true for any $l \in \{0, -1, \dots, N_\beta\}$.*

(1)

$$\begin{aligned} &\left\| \sum_{j=0}^{l'} \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \widehat{W}^j(\omega, \mathbf{k}) \right\|_{4 \times 4} \leq c \cdot c_{IR}^{-1} f_{\mathbf{t}}^{-\frac{1}{2}} M^{l+1} \alpha^{-2}, \\ &(\forall(\omega, \mathbf{k}) \in \mathbb{R}^3 \text{ satisfying } \chi_l(\omega, \mathbf{k}) \neq 0, l' \in \{0, -1, \dots, l\}). \end{aligned}$$

(2)

$$\begin{aligned} &\left\| (i\omega I_4 - E(\mathbf{k}))^{-1} \sum_{j=0}^{l'} \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \widehat{W}^j(\omega, \mathbf{k}) \right\|_{4 \times 4} \leq c \cdot c_{IR}^{-1} f_{\mathbf{t}}^{-\frac{1}{2}} M \alpha^{-2}, \\ &(\forall(\omega, \mathbf{k}) \in \mathbb{R}^3 \text{ satisfying } \chi_l(\omega, \mathbf{k}) \neq 0, l' \in \{0, -1, \dots, l\}). \end{aligned}$$

(3)

$$\begin{aligned} &\left\| \left(\frac{\partial}{\partial k_i} \right)^n \left(\sum_{j=0}^{l'} \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \widehat{W}^j(\omega, \mathbf{k}) \right) \right\|_{4 \times 4} \\ &\leq c \cdot c_{IR}^{-1} M^l \alpha^{-2} (cw(l)^{-1})^n (n!)^2, \\ &(\forall(\omega, \mathbf{k}) \in \mathbb{R}^3, i \in \{0, 1, 2\}, n \in \mathbb{N}, l' \in \{0, -1, \dots, l\}). \end{aligned}$$

PROOF. (1): It follows from Lemma 7.8 (1) and the assumption $M > \sqrt{2}$ that

$$\left\| \sum_{j=0}^{l'} \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \widehat{W}^j(\omega, \mathbf{k}) \right\|_{4 \times 4}$$

$$\leq c \cdot c_{IR}^{-1} f_{\mathbf{t}}^{-\frac{1}{2}} M^{l+1} \alpha^{-2} \sum_{j=0}^l M^{\frac{1}{2}j} \leq c \cdot c_{IR}^{-1} f_{\mathbf{t}}^{-\frac{1}{2}} M^{l+1} \alpha^{-2},$$

($\forall(\omega, \mathbf{k}) \in \mathbb{R}^3$ satisfying $\chi_l(\omega, \mathbf{k}) \neq 0, l' \in \{0, -1, \dots, l\}$).

(2): By Lemma 7.2 (2), (7.10) and the inequality in (1),

$$\begin{aligned} & \left\| (i\omega I_4 - E(\mathbf{k}))^{-1} \sum_{j=0}^{l'} \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \widehat{W}^j(\omega, \mathbf{k}) \right\|_{4 \times 4} \\ & \leq \|(i\omega I_4 - E(\mathbf{k}))^{-1}\|_{4 \times 4} \left\| \sum_{j=0}^{l'} \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \widehat{W}^j(\omega, \mathbf{k}) \right\|_{4 \times 4} \\ & \leq c \cdot c_{IR}^{-1} f_{\mathbf{t}}^{-\frac{1}{2}} M \alpha^{-2}, \\ & (\forall(\omega, \mathbf{k}) \in \mathbb{R}^3 \text{ satisfying } \chi_l(\omega, \mathbf{k}) \neq 0, l' \in \{0, -1, \dots, l\}). \end{aligned}$$

(3): One can see from Lemma 7.4, Lemma 7.8 (2), $w(j) = w(0)M^j$ and $M > \sqrt{2}$ that

$$\begin{aligned} & \left\| \left(\frac{\partial}{\partial k_i} \right)^n \left(\sum_{j=0}^{l'} \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \widehat{W}^j(\omega, \mathbf{k}) \right) \right\|_{4 \times 4} \\ & \leq \sum_{j=0}^{l'} \sum_{m=0}^n \binom{n}{m} \left\| \left(\frac{\partial}{\partial k_i} \right)^m \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \right\| \left\| \left(\frac{\partial}{\partial k_i} \right)^{n-m} \widehat{W}^j(\omega, \mathbf{k}) \right\|_{4 \times 4} \\ & \leq \sum_{j=0}^{l'} \sum_{m=0}^n \binom{n}{m} \\ & \quad \cdot (cw(j)^{-1})^m (m!)^2 c \cdot c_{IR}^{-1} M^{\frac{3}{2}j} \alpha^{-2} (cw(j)^{-1})^{n-m} ((n-m)!)^2 \\ & \leq c \cdot c_{IR}^{-1} \alpha^{-2} (cw(0)^{-1})^n \sum_{j=0}^l M^{\frac{3}{2}j-jn} \sum_{m=0}^n \binom{n}{m} (m!)^2 ((n-m)!)^2 \\ & \leq c \cdot c_{IR}^{-1} \alpha^{-2} (cw(0)^{-1})^n (n!)^2 M^{(1-n)l} \sum_{j=0}^l M^{\frac{1}{2}j} \\ & \leq c \cdot c_{IR}^{-1} M^l \alpha^{-2} (cw(l)^{-1})^n (n!)^2, \\ & (\forall(\omega, \mathbf{k}) \in \mathbb{R}^3, i \in \{0, 1, 2\}, n \in \mathbb{N}, l' \in \{0, -1, \dots, l\}). \quad \square \end{aligned}$$

LEMMA 7.11. *Assume that (4.2) holds. Then, there exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that the following inequality holds true for any $l \in \{0, -1, \dots, N_{\beta_1}\}$ and $(J^j(\beta_1)(\psi), J^j(\beta_2)(\psi)) \in \tilde{\mathcal{S}}(j)$ ($j = 0, -1, \dots, l$).*

$$\begin{aligned} & \left\| \left(\frac{\partial}{\partial k_i} \right)^n \left(\sum_{j=0}^l \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \widehat{W}^j(\beta_1)(\omega, \mathbf{k}) \right. \right. \\ & \quad \left. \left. - \sum_{j=0}^l \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \widehat{W}^j(\beta_2)(\omega, \mathbf{k}) \right) \right\|_{4 \times 4} \\ & \leq c \beta_1^{-\frac{1}{2}} c_{IR}^{-1} \alpha^{-2} (c w(l)^{-1})^n (n!)^2, \\ & (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3, i \in \{0, 1, 2\}, n \in \mathbb{N} \cup \{0\}). \end{aligned}$$

PROOF. By Lemma 7.4, Lemma 7.9 and the assumption $M > \sqrt{2}$,

$$\begin{aligned} & \left\| \left(\frac{\partial}{\partial k_i} \right)^n \left(\sum_{j=0}^l \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \widehat{W}^j(\beta_1)(\omega, \mathbf{k}) \right. \right. \\ & \quad \left. \left. - \sum_{j=0}^l \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \widehat{W}^j(\beta_2)(\omega, \mathbf{k}) \right) \right\|_{4 \times 4} \\ & \leq \sum_{j=0}^l \sum_{m=0}^n \binom{n}{m} \left| \left(\frac{\partial}{\partial k_i} \right)^m \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \right| \\ & \quad \cdot \left\| \left(\frac{\partial}{\partial k_i} \right)^{n-m} (\widehat{W}^j(\beta_1)(\omega, \mathbf{k}) - \widehat{W}^j(\beta_2)(\omega, \mathbf{k})) \right\|_{4 \times 4} \\ & \leq \sum_{j=0}^l \sum_{m=0}^n \binom{n}{m} (c w(j)^{-1})^m (m!)^2 \\ & \quad \cdot c \beta_1^{-\frac{1}{2}} c_{IR}^{-1} M^{\frac{1}{2}j} \alpha^{-2} (c w(j)^{-1})^{n-m} ((n-m)!)^2 \\ & \leq c \beta_1^{-\frac{1}{2}} c_{IR}^{-1} \alpha^{-2} (c w(l)^{-1})^n (n!)^2 \sum_{j=0}^l M^{\frac{1}{2}j} \\ & \leq c \beta_1^{-\frac{1}{2}} c_{IR}^{-1} \alpha^{-2} (c w(l)^{-1})^n (n!)^2, \end{aligned}$$

$(\forall(\omega, \mathbf{k}) \in \mathbb{R}^3, i \in \{0, 1, 2\}, n \in \mathbb{N} \cup \{0\}). \square$

Assuming that $l \in \{0, -1, \dots, N_\beta\}$ and $J^j(\psi) \in \mathcal{S}(j)$ ($j = 0, -1, \dots, l$), we can introduce a covariance which mimics a real covariance in the IR integration at the l th scale. Set

$$(7.41) \quad E_l(\omega, \mathbf{k}) := \sum_{j=0}^l \hat{\chi}_{\leq j}(\omega, \mathbf{k}) \widehat{W}^j(\omega, \mathbf{k}), \quad ((\omega, \mathbf{k}) \in \mathbb{R}^3).$$

By (7.12) and (7.38), if $(\omega, \mathbf{k}) \in \mathcal{M} \times \Gamma^*$,

$$(7.42) \quad E_l(\omega, \mathbf{k}) = \sum_{j=0}^l \chi_{\leq j}(\omega, \mathbf{k}) W^j(\omega, \mathbf{k}).$$

Define the covariance $C_l : I_0^2 \rightarrow \mathbb{C}$ by

$$(7.43) \quad C_l(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) := \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \\ \cdot \chi_l(\omega, \mathbf{k}) (i\omega I_4 - E(\mathbf{k}) - E_l(\omega, \mathbf{k}))^{-1}(\rho, \eta).$$

In the rest of this subsection we mainly study properties of C_l . To this end let us measure the support of the cut-off functions. Since $\text{supp } \chi_l(\cdot) \subset \text{supp } \chi_{\leq l}(\cdot) \subset \text{supp } \hat{\chi}_{\leq l}(\cdot)$, it is sufficient to measure $\text{supp } \hat{\chi}_{\leq l}(\cdot)$.

LEMMA 7.12. *Assume that (7.39) holds. Then, there exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that the following statements hold true for any $l \in \{0, -1, \dots, N_\beta\}$ and $(\omega', \mathbf{k}') \in \mathbb{R}^3$.*

$$(1) \quad \frac{1}{\beta L^2} \sum_{\omega \in \mathcal{M}} \sum_{\mathbf{k} \in \Gamma^*} 1_{\hat{\chi}_{\leq l}(\omega + \omega', \mathbf{k} + \mathbf{k}') \neq 0} \leq c f_{\mathbf{t}}^{-1} M_{IR}^3 M^{3l+3}.$$

$$(2) \quad \int_{-\infty}^{\infty} d\omega \frac{1}{L^2} \sum_{\mathbf{k} \in \Gamma^*} 1_{\hat{\chi}_{\leq l}(\omega, \mathbf{k} + \mathbf{k}') \neq 0} \leq c f_{\mathbf{t}}^{-1} M_{IR}^3 M^{3l+3}.$$

(3)

$$\frac{1}{L^2} \sum_{\mathbf{k} \in \Gamma^*} 1_{\hat{\chi}_{\leq l}(\omega', \mathbf{k} + \mathbf{k}') \neq 0} \leq c f_{\mathbf{t}}^{-1} M_{IR}^2 M^{2l+2}.$$

PROOF. All the claims are verified by Lemma 7.3, (7.39) and the inequality $f_{\mathbf{t}} \leq 1$. \square

LEMMA 7.13. *Assume that*

$$h \geq \frac{1}{\sqrt{3}} M_{UV}, \quad L \geq \beta.$$

Then, there exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that if

$$M \geq c, \quad \alpha^2 \geq cM,$$

the following statements hold true for any $l \in \{0, -1, \dots, N_\beta\}$ and $J^j(\psi) \in \mathcal{S}(j)$ ($j = 0, -1, \dots, l$).

- (1) *If $c_{IR} \geq f_{\mathbf{t}}^{-1}$ holds, $\mathbf{U} \mapsto C_l(\mathbf{U})(\mathbf{X})$ is continuous in \overline{D} and analytic in D ($\forall \mathbf{X} \in I_0^2$).*
- (2) *There exists a constant $c(M, c_w) \in \mathbb{R}_{\geq 1}$ depending only on M and c_w such that if $c_{IR} \geq c(M, c_w) f_{\mathbf{t}}^{-1}$ holds,*

$$\begin{aligned} |\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_l(\mathbf{U})(X_i, Y_j))_{1 \leq i, j \leq n}| &\leq (c_{IR} M^{2l})^n, \\ (\forall r, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} &\leq 1, \\ X_i, Y_i \in I_0 \text{ (} i = 1, 2, \dots, n), \mathbf{U} \in \overline{D}), & \\ \|\widetilde{C}_l(\mathbf{U})\|_{l-1, r} &\leq c_{IR} M^{-l-rl}, \quad (\forall r \in \{0, 1\}, \mathbf{U} \in \overline{D}). \end{aligned}$$

(3)

$$\widetilde{C}_l(\mathbf{U})(\mathbf{X}) = e^{iQ_2(S_2(\mathbf{X}))} \widetilde{C}_l(\mathbf{U})(S_2(\mathbf{X})), \quad (\forall \mathbf{X} \in I^2, \mathbf{U} \in \overline{D}),$$

for $S : I \rightarrow I$, $Q : I \rightarrow \mathbb{R}$ defined by (7.15), (7.16), (7.17), (7.18), (7.19) respectively.

(4)

$$\widetilde{C}_l(\mathbf{U})(\mathbf{X}) = e^{-iQ_2(S_2(\mathbf{X}))} \overline{\widetilde{C}_l(\overline{\mathbf{U}})(S_2(\mathbf{X}))}, \quad (\forall \mathbf{X} \in I^2, \mathbf{U} \in \overline{D}),$$

for $S : I \rightarrow I$, $Q : I \rightarrow \mathbb{R}$ defined by (7.20), (7.21) respectively.

In (2), (3), (4), $\widetilde{C}_l : I^2 \rightarrow \mathbb{C}$ is the anti-symmetric extension of C_l defined as in (3.2).

PROOF. Note that if $C_l(\mathbf{X}) = 0$ ($\forall \mathbf{X} \in I_0^2$), all the claims trivially hold true. If $C_l(\mathbf{X}) \neq 0$ for some $\mathbf{X} \in I_0^2$, there exists $(\omega, \mathbf{k}) \in \mathcal{M} \times \Gamma^*$ such that $\chi_{\leq 0}(\omega, \mathbf{k}) \neq 0$. Thus, by Lemma 7.5 we can always assume that $1/\beta \leq M_{IR} M^{N\beta+1}$ during the proof. Combined with the assumption $L \geq \beta$, this means that (7.39) holds and thus we can refer to the results of Lemma 7.10 and Lemma 7.12. In the following we prove (1), (3), (4) first and (2) in the end.

(1): Assume that $\alpha^2 \geq 2cM$ with the constant c appearing in the right-hand side of the inequality in Lemma 7.10 (2). Since $c_{IR}^{-1} \leq f_{\mathbf{t}}$ and $f_{\mathbf{t}} \leq 1$ by assumption,

$$\begin{aligned} & \| (i\omega I_4 - E(\mathbf{k}))^{-1} E_{l'}(\mathbf{U})(\omega, \mathbf{k}) \|_{4 \times 4} \leq \frac{1}{2} f_{\mathbf{t}} \cdot f_{\mathbf{t}}^{-\frac{1}{2}} \leq \frac{1}{2}, \\ & (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3 \text{ satisfying } \chi_{l'}(\omega, \mathbf{k}) \neq 0, l' \in \{0, -1, \dots, l\}, \mathbf{U} \in \overline{D}). \end{aligned}$$

Therefore, by Lemma 7.2 (2) and (7.10),

$$\begin{aligned} (7.44) \quad & \| (i\omega I_4 - E(\mathbf{k}) - E_{l'}(\mathbf{U})(\omega, \mathbf{k}))^{-1} \|_{4 \times 4} \\ & \leq \sum_{n=0}^{\infty} \| (i\omega I_4 - E(\mathbf{k}))^{-1} E_{l'}(\mathbf{U})(\omega, \mathbf{k}) \|_{4 \times 4}^n \| (i\omega I_4 - E(\mathbf{k}))^{-1} \|_{4 \times 4} \\ & \leq 2 \left(\frac{\pi}{\sqrt{6}} M_{IR} M^l \right)^{-1} \leq M^{-l}, \\ & (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3 \text{ satisfying } \chi_{l'}(\omega, \mathbf{k}) \neq 0, l' \in \{0, -1, \dots, l\}, \mathbf{U} \in \overline{D}). \end{aligned}$$

This implies the well-definedness of C_l . Since $\mathbf{U} \mapsto E_l(\mathbf{U})(\omega, \mathbf{k})$ is continuous in \overline{D} and analytic in D for any $(\omega, \mathbf{k}) \in \mathbb{R}^3$ by definition, so is the function $\mathbf{U} \mapsto C_l(\mathbf{U})(\mathbf{X})$ for any $\mathbf{X} \in I_0^2$.

(3): For any $S : I \rightarrow I$, $Q : I \rightarrow \mathbb{R}$ defined in (7.15), (7.16), (7.17), the claimed equality clearly holds.

For $S : I \rightarrow I$, $Q : I \rightarrow \mathbb{R}$ defined in (7.18) with $\mathbf{z} \in \mathbb{Z}^2$, $s \in (1/h)\mathbb{Z}$,

$$\begin{aligned} & e^{iQ_2(S_2((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)))} \widetilde{C}_l(S_2((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi))) \\ & = (-1)^{n_\beta(x+s) + n_\beta(y+s)} \\ & \quad \cdot \widetilde{C}_l((\rho, r_L(\mathbf{x} + \mathbf{z}), \sigma, r_\beta(x + s), \theta), (\eta, r_L(\mathbf{y} + \mathbf{z}), \tau, r_\beta(y + s), \xi)) \end{aligned}$$

$$= \widetilde{C}_l((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)), \quad (\forall (\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi) \in I).$$

To prove the invariance with $S : I \rightarrow I$, $Q : I \rightarrow \mathbb{R}$ defined in (7.19), let us define the map $Z : (2\pi/L)\mathbb{Z}^2 \rightarrow \text{Mat}(4, \mathbb{C})$ by $Z(\mathbf{k})(\rho, \eta) := e^{i\langle \mathbf{e}(\rho), \mathbf{k} \rangle} \delta_{\rho, \eta}$, ($\forall \mathbf{k} \in (2\pi/L)\mathbb{Z}^2, \rho, \eta \in \mathcal{B}$). By (7.1) and (7.27),

$$\begin{aligned} Z(\mathbf{k})^*(E(\mathbf{k}) + E_l(\omega, \mathbf{k}))Z(\mathbf{k}) &= E(-\mathbf{k}) + E_l(\omega, -\mathbf{k}), \\ (\forall (\omega, \mathbf{k}) \in \mathcal{M} \times (2\pi/L)\mathbb{Z}^2). \end{aligned}$$

Therefore,

$$\begin{aligned} & C_l((\rho, r_L(-\mathbf{x} - \mathbf{e}(\rho)), \sigma, x), (\eta, r_L(-\mathbf{y} - \mathbf{e}(\eta)), \tau, y)) \\ &= \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle -\mathbf{x} - \mathbf{e}(\rho) - (-\mathbf{y} - \mathbf{e}(\eta)), \mathbf{k} \rangle} e^{i(x-y)\omega} \\ & \quad \cdot \chi_l(\omega, \mathbf{k})(i\omega I_4 - E(\mathbf{k}) - E_l(\omega, \mathbf{k}))^{-1}(\rho, \eta) \\ &= \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle -\mathbf{x} + \mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \\ & \quad \cdot \chi_l(\omega, \mathbf{k})(Z(\mathbf{k})^*(i\omega I_4 - E(\mathbf{k}) - E_l(\omega, \mathbf{k}))Z(\mathbf{k}))^{-1}(\rho, \eta) \\ &= \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{x} - \mathbf{y}, -\mathbf{k} \rangle} e^{i(x-y)\omega} \\ & \quad \cdot \chi_l(\omega, -\mathbf{k})(i\omega I_4 - E(-\mathbf{k}) - E_l(\omega, -\mathbf{k}))^{-1}(\rho, \eta) \\ &= C_l((\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y)), \quad (\forall (\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in I_0). \end{aligned}$$

This implies the claimed invariance.

(4): It follows from definition and (7.25) that

$$(7.45) \quad \overline{E(\mathbf{k})(\rho, \eta)} = E(\mathbf{k})(\eta, \rho), \quad \overline{E_l(\overline{\mathbf{U}})(-\omega, \mathbf{k})(\rho, \eta)} = E_l(\mathbf{U})(\omega, \mathbf{k})(\eta, \rho), \\ (\forall (\omega, \mathbf{k}) \in \mathcal{M} \times (2\pi/L)\mathbb{Z}^2, \rho, \eta \in \mathcal{B}, \mathbf{U} \in \overline{D}).$$

Recall that for any $(X, \theta), (Y, \xi) \in I$,

$$\begin{aligned} & \widetilde{C}_l(\mathbf{U})((X, \theta), (Y, \xi)) \\ &= \frac{1}{2}(1_{(\theta, \xi)=(1, -1)}C_l(\mathbf{U})(X, Y) - 1_{(\theta, \xi)=(-1, 1)}C_l(\mathbf{U})(Y, X)). \end{aligned}$$

For $S : I \rightarrow I$, $Q : I \rightarrow \mathbb{R}$ defined in (7.20),

$$\begin{aligned}
(7.46) \quad & e^{-iQ_2(S_2((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)))} \overline{\widetilde{C}_l(\overline{\mathbf{U}})}(S_2((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi))) \\
&= e^{-i\pi(1_{\theta=-1}+1_{\xi=-1}+1_{x \neq 0}+1_{y \neq 0})} \\
&\quad \cdot \overline{\widetilde{C}_l(\overline{\mathbf{U}})}((\rho, \mathbf{x}, \sigma, r_\beta(-x), -\theta), (\eta, \mathbf{y}, \tau, r_\beta(-y), -\xi)) \\
&= \frac{1}{2}(-1)^{1_{x \neq 0}+1_{y \neq 0}} \\
&\quad \cdot \left(1_{(\theta, \xi)=(1, -1)} \overline{C_l(\overline{\mathbf{U}})}((\eta, \mathbf{y}, \tau, r_\beta(-y)), (\rho, \mathbf{x}, \sigma, r_\beta(-x))) \right. \\
&\quad \left. - 1_{(\xi, \theta)=(1, -1)} \overline{C_l(\overline{\mathbf{U}})}((\rho, \mathbf{x}, \sigma, r_\beta(-x)), (\eta, \mathbf{y}, \tau, r_\beta(-y))) \right).
\end{aligned}$$

Note that

$$\begin{aligned}
(7.47) \quad & (-1)^{1_{x \neq 0}+1_{y \neq 0}} \overline{C_l(\overline{\mathbf{U}})}((\rho, \mathbf{x}, \sigma, r_\beta(-x)), (\eta, \mathbf{y}, \tau, r_\beta(-y))) \\
&= (-1)^{1_{x \neq 0}+1_{y \neq 0}} \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{-i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} e^{-i(r_\beta(-x)-r_\beta(-y))\omega} \\
&\quad \cdot \chi_l(\omega, \mathbf{k}) \left(-i\omega I_4 - \overline{E(\mathbf{k})} - \overline{E_l(\overline{\mathbf{U}})}(\omega, \mathbf{k}) \right)^{-1} (\rho, \eta) \\
&= \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{y}-\mathbf{x}, \mathbf{k} \rangle} e^{i(y-x)\omega} \\
&\quad \cdot \chi_l(\omega, \mathbf{k}) \left(i\omega I_4 - \overline{E(\mathbf{k})} - \overline{E_l(\overline{\mathbf{U}})}(-\omega, \mathbf{k}) \right)^{-1} (\rho, \eta) \\
&= C_l(\mathbf{U})((\eta, \mathbf{y}, \tau, y), (\rho, \mathbf{x}, \sigma, x)),
\end{aligned}$$

where we used (7.45). By substituting (7.47) into (7.46) we obtain the claimed equality with $S : I \rightarrow I$, $Q : I \rightarrow \mathbb{R}$ defined in (7.20).

For $S : I \rightarrow I$, $Q : I \rightarrow \mathbb{R}$ defined in (7.21),

$$\begin{aligned}
(7.48) \quad & e^{-iQ_2(S_2((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)))} \overline{\widetilde{C}_l(\overline{\mathbf{U}})}(S_2((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi))) \\
&= \frac{1}{2}(-1)^{1_{\rho \in \{1, 4\}}+1_{\eta \in \{1, 4\}}+1} \\
&\quad \cdot \left(1_{(\theta, \xi)=(1, -1)} \overline{C_l(\overline{\mathbf{U}})}((\eta, \mathbf{y}, \tau, y), (\rho, \mathbf{x}, \sigma, x)) \right)
\end{aligned}$$

$$- 1_{(\xi, \theta) = (1, -1)} \overline{C_l(\overline{\mathbf{U}})((\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y))}.$$

Let us define $Y \in \text{Mat}(4, \mathbb{C})$ by

$$(7.49) \quad Y(\rho, \eta) := (-1)^{1_{\rho \in \{1,4\}}} \delta_{\rho, \eta}, \quad (\rho, \eta \in \mathcal{B}).$$

We can see from definition and (7.26) that

$$\begin{aligned} Y(E(\mathbf{k})^* + E_l(\overline{\mathbf{U}})(\omega, \mathbf{k})^*)Y &= -(E(\mathbf{k}) + E_l(\mathbf{U})(\omega, \mathbf{k})), \\ (\forall (\omega, \mathbf{k}) \in \mathcal{M} \times (2\pi/L)\mathbb{Z}^2, \mathbf{U} \in \overline{D}). \end{aligned}$$

Using this inequality, we observe that

$$\begin{aligned} (7.50) \quad & (-1)^{1_{\rho \in \{1,4\}} + 1_{\eta \in \{1,4\}}} \overline{C_l(\overline{\mathbf{U}})((\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y))} \\ &= (-1)^{1_{\rho \in \{1,4\}} + 1_{\eta \in \{1,4\}}} \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{-i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{-i(x-y)\omega} \\ & \quad \cdot \chi_l(\omega, \mathbf{k}) \left(-i\omega I_4 - \overline{E(\mathbf{k})} - \overline{E_l(\overline{\mathbf{U}})(\omega, \mathbf{k})} \right)^{-1} (\rho, \eta) \\ &= (-1)^{1_{\rho \in \{1,4\}} + 1_{\eta \in \{1,4\}}} \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{y} - \mathbf{x}, \mathbf{k} \rangle} e^{i(y-x)\omega} \\ & \quad \cdot \chi_l(\omega, \mathbf{k}) (-i\omega I_4 - E(\mathbf{k})^* - E_l(\overline{\mathbf{U}})(\omega, \mathbf{k})^*)^{-1} (\eta, \rho) \\ &= \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{y} - \mathbf{x}, \mathbf{k} \rangle} e^{i(y-x)\omega} \\ & \quad \cdot \chi_l(\omega, \mathbf{k}) (i\omega I_4 + Y(E(\mathbf{k})^* + E_l(\overline{\mathbf{U}})(\omega, \mathbf{k})^*)Y)^{-1} (\eta, \rho) \\ &= C_l(\mathbf{U})((\eta, \mathbf{y}, \tau, y), (\rho, \mathbf{x}, \sigma, x)). \end{aligned}$$

By combining (7.50) with (7.48) we obtain the claimed equality with $S : I \rightarrow I$, $Q : I \rightarrow \mathbb{R}$ defined in (7.21).

(2): Recall the definition of the Hilbert space \mathcal{H} given in the proof of Lemma 2.4. For any $(\rho, \mathbf{x}, \sigma, x) \in I_0$ let us define the vectors $u_{\rho \mathbf{x} \sigma x}^l, v_{\rho \mathbf{x} \sigma x}^l \in \mathcal{H}$ by

$$\begin{aligned} u_{\rho \mathbf{x} \sigma x}^l(\eta, \mathbf{k}, \tau, \omega) &:= \delta_{\rho, \eta} \delta_{\sigma, \tau} e^{-i\langle \mathbf{x}, \mathbf{k} \rangle} e^{-ix\omega} \chi_l(\omega, \mathbf{k})^{\frac{1}{2}}, \\ v_{\rho \mathbf{x} \sigma x}^l(\eta, \mathbf{k}, \tau, \omega) & \end{aligned}$$

$$:= \delta_{\sigma, \tau} e^{-i\langle \mathbf{x}, \mathbf{k} \rangle} e^{-ix\omega} \chi_l(\omega, \mathbf{k})^{\frac{1}{2}} (i\omega I_4 - E(\mathbf{k}) - E_l(\omega, \mathbf{k}))^{-1}(\eta, \rho),$$

$$(\forall (\eta, \mathbf{k}, \tau, \omega) \in \mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\} \times \mathcal{M}_h).$$

It follows that $C_l(X, Y) = \langle u_X^l, v_Y^l \rangle_{\mathcal{H}}$ ($\forall X, Y \in I_0$). Using Lemma 7.12 (1) and (7.44), we can derive that

$$\|u_X^l\|_{\mathcal{H}} \leq c(M) f_{\mathbf{t}}^{-\frac{1}{2}} M^{\frac{3}{2}l}, \quad \|v_X^l\|_{\mathcal{H}} \leq c(M) f_{\mathbf{t}}^{-\frac{1}{2}} M^{\frac{1}{2}l}, \quad (\forall X \in I_0).$$

Therefore, the standard argument based on Gram's inequality concludes that for any $r, n \in \mathbb{N}$, $\mathbf{p}_j, \mathbf{q}_j \in \mathbb{C}^r$ with $\|\mathbf{p}_j\|_{\mathbb{C}^r}, \|\mathbf{q}_j\|_{\mathbb{C}^r} \leq 1$, $X_j, Y_j \in I_0$ ($j = 1, 2, \dots, n$),

$$(7.51) \quad |\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_l(X_i, Y_j))_{1 \leq i, j \leq n}| \leq (c(M) f_{\mathbf{t}}^{-1} M^{2l})^n.$$

On the assumption $c_{IR} \geq c(M) f_{\mathbf{t}}^{-1}$ we obtain the claimed determinant bound.

Next let us prove the claimed upper bound on $\|\widetilde{C}_l\|_{l-1, r}$. By assumption, $\pi h \geq (\pi/\sqrt{3})M_{UV}$. Since $\chi_l(\omega, \mathbf{k}) = 0$ if $|\omega| \geq (\pi/\sqrt{3})M_{UV}$, we can replace \mathcal{M}_h by \mathcal{M} inside C_l . Then, by using the periodicity with \mathbf{k} we can justify the following transformation.

$$(7.52) \quad \left(\frac{\beta}{2\pi}\right)^{n_0} (e^{-i(x-y)\frac{2\pi}{\beta}} - 1)^{n_0} \prod_{j=1}^2 \left(\left(\frac{L}{2\pi}\right)^{n_j} (e^{-i(x_j-y_j)\frac{2\pi}{L}} - 1)^{n_j} \right)$$

$$\cdot C_l(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y)$$

$$= \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M} \times \Gamma^*} e^{i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega}$$

$$\cdot \prod_{j=0}^2 \mathcal{D}_j^{n_j} (\chi_l(\omega, \mathbf{k}) (i\omega I_4 - E(\mathbf{k}) - E_l(\omega, \mathbf{k}))^{-1})$$

$$= \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M} \times \Gamma^*} e^{i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega}$$

$$\cdot \prod_{j_0=1}^{n_0} \left(\frac{\beta}{2\pi} \int_0^{\frac{2\pi}{\beta}} d\omega_{j_0} \right) \prod_{j_1=1}^{n_1} \left(\frac{L}{2\pi} \int_0^{\frac{2\pi}{L}} dp_{1, j_1} \right)$$

$$\cdot \prod_{j_2=1}^{n_2} \left(\frac{L}{2\pi} \int_0^{\frac{2\pi}{L}} dp_{2, j_2} \right) \left(\frac{\partial}{\partial \omega'} \right)^{n_0} \prod_{q=1}^2 \left(\frac{\partial}{\partial k'_q} \right)^{n_q}$$

$$\begin{aligned} & \cdot (\chi_l(\omega', \mathbf{k}')(i\omega' I_4 - E(\mathbf{k}') - E_l(\omega', \mathbf{k}'))^{-1}) \\ & \cdot \Big|_{\omega'=\omega+\sum_{j_0=1}^{n_0} \omega_{j_0}, \mathbf{k}'=\mathbf{k}+\sum_{j_1=1}^{n_1} p_{1,j_1} \mathbf{e}_1+\sum_{j_2=1}^{n_2} p_{2,j_2} \mathbf{e}_2} \end{aligned}$$

Note that by Lemma 7.2 (1), the definition of $w(l)$ and the fact $c_w \leq 1$,

$$\begin{aligned} & \left\| \left(\frac{\partial}{\partial k_j} \right)^n (i\omega I_4 - E(\mathbf{k})) \right\|_{4 \times 4} \leq cM^{l-2} (w(l)^{-1})^n (n!)^2, \\ & (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3, j \in \{0, 1, 2\}, n \in \mathbb{N}). \end{aligned}$$

Thus, by Lemma 7.10 (3) and the inequality $c_{IR} \geq 1$,

$$\begin{aligned} (7.53) \quad & \left\| \left(\frac{\partial}{\partial k_j} \right)^n (i\omega I_4 - E(\mathbf{k}) - E_{l'}(\omega, \mathbf{k})) \right\|_{4 \times 4} \\ & \leq (cM^{-2} + c\alpha^{-2}) M^l (cw(l)^{-1})^n (n!)^2, \\ & (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3, j \in \{0, 1, 2\}, n \in \mathbb{N}, l' \in \{0, -1, \dots, l\}). \end{aligned}$$

By (7.44), (7.53) and the assumption $\alpha, M \geq c$ we can apply Lemma C.3 to deduce that

$$\begin{aligned} (7.54) \quad & \left\| \left(\frac{\partial}{\partial k_j} \right)^n (i\omega I_4 - E(\mathbf{k}) - E_{l'}(\omega, \mathbf{k}))^{-1} \right\|_{4 \times 4} \\ & \leq cM^{-l} (cw(l)^{-1})^n (n!)^2, \\ & (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3 \text{ satisfying } \chi_l(\omega, \mathbf{k}) \neq 0, j \in \{0, 1, 2\}, n \in \mathbb{N} \cup \{0\}, \\ & \quad l' \in \{0, -1, \dots, l\}). \end{aligned}$$

Lemma 7.4 and (7.54) yield

$$\begin{aligned} (7.55) \quad & \left\| \left(\frac{\partial}{\partial k_j} \right)^n (\chi_l(\omega, \mathbf{k})(i\omega I_4 - E(\mathbf{k}) - E_l(\omega, \mathbf{k}))^{-1}) \right\|_{4 \times 4} \\ & \leq cM^{-l} (cw(l)^{-1})^n (n!)^2, \quad (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3, j \in \{0, 1, 2\}, n \in \mathbb{N} \cup \{0\}). \end{aligned}$$

It follows from Lemma 7.12 (1), (7.52) and (7.55) that

$$|d_j(\mathbf{X})^n \widetilde{C}_l(\mathbf{X})| \leq c(M) f_{\mathbf{t}}^{-1} M^{2l} (cw(l)^{-1})^n (n!)^2,$$

which is also true for $n = 0$ by (7.51). Therefore, we reach

$$(7.56) \quad |\widetilde{C}_l(\mathbf{X})| \leq c(M) f_{\mathbf{t}}^{-1} M^{2l} e^{-\sum_{j=0}^2 (cw(l) d_j(\mathbf{X}))^{1/2}}, \quad (\forall \mathbf{X} \in I^2).$$

By the equality $w(l-1) = w(l)M^{-1}$ and the assumption $M \geq c$ we have

$$|e^{\sum_{j=0}^2 (w(l-1)d_j(\mathbf{X}))^{1/2}} \widetilde{C}_l(\mathbf{X})| \leq c(M) f_{\mathbf{t}}^{-1} M^{2l} e^{-\sum_{j=0}^2 (cw(l)d_j(\mathbf{X}))^{1/2}},$$

which implies that

$$\|\widetilde{C}_l\|_{l-1,r} \leq c(M) f_{\mathbf{t}}^{-1} M^{2l} w(l)^{-3-r} \leq c(M, c_w) f_{\mathbf{t}}^{-1} M^{-l-rl}.$$

Thus, if $c_{IR} \geq c(M, c_w) f_{\mathbf{t}}^{-1}$, we obtain the claimed upper bound. \square

LEMMA 7.14. *Assume that (4.2) holds and*

$$h \geq \frac{1}{\sqrt{3}} M_{UV}, \quad L \geq \beta_2.$$

Then, there exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that if

$$M \geq c, \quad \alpha^2 \geq cM,$$

the following statement holds true for any $l \in \{0, -1, \dots, N_{\beta_1}\}$ and $(J^j(\beta_1)(\psi), J^j(\beta_2)(\psi)) \in \widetilde{\mathcal{S}}(j)$ ($j = 0, -1, \dots, l$).

There exists a constant $c(M, c_w) \in \mathbb{R}_{\geq 1}$ depending only on M, c_w such that if $c_{IR} \geq c(M, c_w) f_{\mathbf{t}}^{-1}$ holds,

$$(7.57) \quad \begin{aligned} & |\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_l(\mathbf{U})(\beta_1)(R_{\beta_1}(X_i, Y_j)))_{1 \leq i, j \leq n} \\ & \quad - \det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_l(\mathbf{U})(\beta_2)(R_{\beta_2}(X_i, Y_j)))_{1 \leq i, j \leq n}| \\ & \leq \beta_1^{-\frac{1}{2}} M^{-l} (c_{IR} M^{2l})^n, \end{aligned}$$

$$(\forall r, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} \leq 1,$$

$$X_i, Y_i \in \hat{I}_0 \ (i = 1, 2, \dots, n), \mathbf{U} \in \overline{D}),$$

$$(7.58) \quad |\widetilde{C}_l(\mathbf{U})(\beta_1) - \widetilde{C}_l(\mathbf{U})(\beta_2)|_{l-1} \leq \beta_1^{-\frac{1}{2}} c_{IR} M^{-2l}, \quad (\forall \mathbf{U} \in \overline{D}).$$

PROOF. For any $(\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in \hat{I}_0, a \in \{1, 2\}$, set

$$\begin{aligned} & C_{ont,l}(\beta_a)(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) \\ & := (-1)^{n_{\beta_a}(x) + n_{\beta_a}(y)} \frac{\delta_{\sigma, \tau}}{2\pi L^2} \sum_{\mathbf{k} \in \Gamma^*} \int_{-\pi h}^{\pi h} d\omega e^{i(\mathbf{x}-\mathbf{y}, \mathbf{k})} e^{i(x-y)\omega} \end{aligned}$$

$$\cdot \chi_l(\omega, \mathbf{k})(i\omega I_4 - E(\mathbf{k}) - E_l(\beta_a)(\omega, \mathbf{k}))^{-1}(\rho, \eta),$$

Since $L \geq \beta_2 \geq \beta_1 \geq 1 > \sqrt{3}M_{IR}^{-1}$, it follows from the definition of N_β that $1/L \leq 1/\beta_a \leq M_{IR}M^{N_{\beta_a}+1}$ ($\forall a \in \{1, 2\}$). This means that the condition (7.39) holds for β_1 and β_2 . Thus, we can use the results of Lemma 7.12 for β_1 and β_2 . Note that

$$\begin{aligned} & C_{ont,l}(\beta_a)(\cdot \mathbf{x}\sigma x, \cdot \mathbf{y}\tau y) - C_l(\beta_a)(\cdot \mathbf{x}\sigma r_{\beta_a}(x), \cdot \mathbf{y}\tau r_{\beta_a}(y)) \\ &= (-1)^{n_{\beta_a}(x)+n_{\beta_a}(y)} \frac{\delta_{\sigma,\tau}}{2\pi L^2} \sum_{\mathbf{k} \in \Gamma^*} e^{i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} \\ &\cdot \sum_{m=0}^{\frac{\beta_a h}{2}-1} \left(\int_{\frac{2\pi}{\beta_a}m + \frac{\pi}{\beta_a}}^{\frac{2\pi}{\beta_a}(m+1) + \frac{\pi}{\beta_a}} d\omega \int_{\frac{2\pi}{\beta_a}m + \frac{\pi}{\beta_a}}^{\omega} du \right. \\ &\quad \left. + \int_{-\frac{2\pi}{\beta_a}m - \frac{\pi}{\beta_a}}^{-\frac{2\pi}{\beta_a}(m+1) - \frac{\pi}{\beta_a}} d\omega \int_{-\frac{2\pi}{\beta_a}m - \frac{\pi}{\beta_a}}^{\omega} du \right) \\ &\cdot \frac{\partial}{\partial u} (e^{i(x-y)u} \chi_l(u, \mathbf{k})(iuI_4 - E(\mathbf{k}) - E_l(\beta_a)(u, \mathbf{k}))^{-1}) \\ &+ (-1)^{n_{\beta_a}(x)+n_{\beta_a}(y)} \frac{\delta_{\sigma,\tau}}{2\pi L^2} \sum_{\mathbf{k} \in \Gamma^*} e^{i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} \\ &\cdot \left(\int_{-\frac{\pi}{\beta_a}}^{\frac{\pi}{\beta_a}} d\omega - \int_{\pi h}^{\pi h + \frac{\pi}{\beta_a}} d\omega - \int_{-\pi h - \frac{\pi}{\beta_a}}^{-\pi h} d\omega \right) \\ &\cdot e^{i(x-y)\omega} \chi_l(\omega, \mathbf{k})(i\omega I_4 - E(\mathbf{k}) - E_l(\beta_a)(\omega, \mathbf{k}))^{-1}. \end{aligned}$$

By Lemma 7.12 (2), (3) and (7.55) we see that

$$\begin{aligned} (7.59) \quad & \|C_{ont,l}(\beta_a)(\cdot \mathbf{x}\sigma x, \cdot \mathbf{y}\tau y) - C_l(\beta_a)(\cdot \mathbf{x}\sigma r_{\beta_a}(x), \cdot \mathbf{y}\tau r_{\beta_a}(y))\|_{4 \times 4} \\ &\leq \frac{1}{\beta_a L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_a}}^{\pi h + \frac{\pi}{\beta_a}} d\omega + \int_{-\pi h - \frac{\pi}{\beta_a}}^{-\frac{\pi}{\beta_a}} d\omega \right) \\ &\cdot \left(|x-y| \chi_l(\omega, \mathbf{k}) \|(i\omega I_4 - E(\mathbf{k}) - E_l(\beta_a)(\omega, \mathbf{k}))^{-1}\|_{4 \times 4} \right. \\ &\quad \left. + \left\| \frac{\partial}{\partial \omega} (\chi_l(\omega, \mathbf{k})(i\omega I_4 - E(\mathbf{k}) - E_l(\beta_a)(\omega, \mathbf{k}))^{-1}) \right\|_{4 \times 4} \right) \\ &+ \frac{1}{L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{-\frac{\pi}{\beta_a}}^{\frac{\pi}{\beta_a}} d\omega + \int_{\pi h}^{\pi h + \frac{\pi}{\beta_a}} d\omega + \int_{-\pi h - \frac{\pi}{\beta_a}}^{-\pi h} d\omega \right) \end{aligned}$$

$$\begin{aligned} & \cdot \chi_l(\omega, \mathbf{k}) \|(i\omega I_4 - E(\mathbf{k}) - E_l(\beta_a)(\omega, \mathbf{k}))^{-1}\|_{4 \times 4} \\ & \leq \frac{1}{\beta_1} c(M, c_w) f_{\mathbf{t}}^{-1} (|x - y| M^{2l} + M^l). \end{aligned}$$

Since $\pi h \geq (\pi/\sqrt{3})M_{UV}$, we can replace the integral over $[-\pi h, \pi h]$ inside $C_{ont,l}(\beta_a)$ by the integral over $(-\infty, \infty)$. Then, the integration by parts with ω and the periodicity with \mathbf{k} yield

$$\begin{aligned} & (x - y)^{n_0} \prod_{j=1}^2 \left(\left(\frac{L}{2\pi} \right)^{n_j} (e^{-i(x_j - y_j) \frac{2\pi}{L}} - 1)^{n_j} \right) C_{ont,l}(\beta_a)(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) \\ & = (-1)^{n_{\beta_a}(x) + n_{\beta_a}(y)} \frac{\delta_{\sigma, \tau}}{2\pi L^2} \sum_{\mathbf{k} \in \Gamma^*} \int_{-\infty}^{\infty} d\omega e^{i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \\ & \quad \cdot i^{n_0} \prod_{j_1=1}^{n_1} \left(\frac{L}{2\pi} \int_0^{\frac{2\pi}{L}} dp_{1,j_1} \right) \prod_{j_2=1}^{n_2} \left(\frac{L}{2\pi} \int_0^{\frac{2\pi}{L}} dp_{2,j_2} \right) \\ & \quad \cdot \left(\frac{\partial}{\partial \omega} \right)^{n_0} \prod_{q=1}^2 \left(\frac{\partial}{\partial k'_q} \right)^{n_q} (\chi_l(\omega, \mathbf{k}') (i\omega I_4 - E(\mathbf{k}') - E_l(\beta_a)(\omega, \mathbf{k}'))^{-1}) \\ & \quad \cdot \left| \begin{array}{c} \mathbf{k}' = \mathbf{k} + \sum_{j_1=1}^{n_1} p_{1,j_1} \mathbf{e}_1 + \sum_{j_2=1}^{n_2} p_{2,j_2} \mathbf{e}_2 \end{array} \right|. \end{aligned}$$

It follows from Lemma 7.12 (2), (7.55) and this equality that

$$\begin{aligned} & \left| \hat{d}_j(\mathbf{X})^n \widetilde{C_{ont,l}(\beta_a)}(\mathbf{X}) \right| \leq c(M) f_{\mathbf{t}}^{-1} M^{2l} (c_w(l)^{-1})^n (n!)^2, \\ & (\forall \mathbf{X} \in \hat{I}^2, j \in \{0, 1, 2\}, n \in \mathbb{N} \cup \{0\}), \end{aligned}$$

where

$$\begin{aligned} & \widetilde{C_{ont,l}(\beta_a)}(X\theta, Y\xi) \\ & := \frac{1}{2} (1_{(\theta, \xi) = (1, -1)} C_{ont,l}(\beta_a)(X, Y) - 1_{(\xi, \theta) = (1, -1)} C_{ont,l}(\beta_a)(Y, X)), \\ & (\forall X, Y \in \hat{I}_0, \theta, \xi \in \{1, -1\}). \end{aligned}$$

This leads to

$$\left| \widetilde{C_{ont,l}(\beta_a)}(\mathbf{X}) \right| \leq c(M) f_{\mathbf{t}}^{-1} M^{2l} e^{-\sum_{j=0}^2 (c_w(l) \hat{d}_j(\mathbf{X}))^{1/2}},$$

$$(\forall \mathbf{X} \in \hat{I}^2, a \in \{1, 2\}).$$

By combining this inequality with (7.56) and using the inequality

$$d_j(R_{\beta_a}(\mathbf{X})) \geq \frac{2}{\pi} \hat{d}_j(\mathbf{X}), \quad (\forall \mathbf{X} \in \hat{I}^2, j \in \{0, 1, 2\}),$$

we obtain

$$(7.60) \quad \begin{aligned} & \left| \widetilde{C_{ont,l}(\beta_a)}(\mathbf{X}) - \widetilde{C_l(\beta_a)}(R_{\beta_a}(\mathbf{X})) \right| \\ & \leq c(M) f_{\mathbf{t}}^{-1} M^{2l} e^{-\sum_{j=0}^2 (c_w(l) \hat{d}_j(\mathbf{X}))^{1/2}}, \\ & (\forall \mathbf{X} \in \hat{I}^2, a \in \{1, 2\}). \end{aligned}$$

It follows from (7.59), (7.60) that

$$(7.61) \quad \begin{aligned} & \left| \widetilde{C_{ont,l}(\beta_a)}(\mathbf{X}) - \widetilde{C_l(\beta_a)}(R_{\beta_a}(\mathbf{X})) \right| \\ & \leq \beta_1^{-\frac{1}{2}} c(M, c_w) f_{\mathbf{t}}^{-1} (M^{\frac{1}{2}l} w(l)^{\frac{1}{2}} \hat{d}_0(\mathbf{X})^{\frac{1}{2}} + M^{\frac{1}{2}l}) \\ & \quad \cdot M^l e^{-\sum_{j=0}^2 (c_w(l) \hat{d}_j(\mathbf{X}))^{1/2}} \\ & \leq \beta_1^{-\frac{1}{2}} c(M, c_w) f_{\mathbf{t}}^{-1} M^{\frac{3}{2}l} e^{-\sum_{j=0}^2 (c_w(l) \hat{d}_j(\mathbf{X}))^{1/2}}, \\ & (\forall \mathbf{X} \in \hat{I}^2, a \in \{1, 2\}). \end{aligned}$$

On the other hand, note that

$$(7.62) \quad \begin{aligned} & (x-y)^{n_0} \prod_{j=1}^2 \left(\frac{L}{2\pi} (e^{-i(x_j-y_j)\frac{2\pi}{L}} - 1) \right)^{n_j} \\ & \quad \cdot (C_{ont,l}(\beta_1)(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) - C_{ont,l}(\beta_2)(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y)) \\ & = (-1)^{n_{\beta_1}(x) + n_{\beta_1}(y)} \frac{\delta_{\sigma, \tau}}{2\pi L^2} \sum_{\mathbf{k} \in \Gamma^*} e^{i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} \\ & \quad \cdot \left(\int_{\frac{\pi}{\beta_1}}^{\infty} d\omega + \int_{-\infty}^{-\frac{\pi}{\beta_1}} d\omega \right) e^{i(x-y)\omega} i^{n_0} \\ & \quad \cdot \prod_{j_1=1}^{n_1} \left(\frac{L}{2\pi} \int_0^{\frac{2\pi}{L}} dp_{1,j_1} \right) \prod_{j_2=1}^{n_2} \left(\frac{L}{2\pi} \int_0^{\frac{2\pi}{L}} dp_{2,j_2} \right) \\ & \quad \cdot \left(\frac{\partial}{\partial \omega} \right)^{n_0} \prod_{q=1}^2 \left(\frac{\partial}{\partial k'_q} \right)^{n_q} \end{aligned}$$

$$\begin{aligned}
& \cdot \chi_l(\omega, \mathbf{k}') (i\omega I_4 - E(\mathbf{k}') - E_l(\beta_1)(\omega, \mathbf{k}'))^{-1} \\
& \cdot (E_l(\beta_1)(\omega, \mathbf{k}') - E_l(\beta_2)(\omega, \mathbf{k}')) (i\omega I_4 - E(\mathbf{k}') - E_l(\beta_2)(\omega, \mathbf{k}'))^{-1} \\
& \cdot \left| \begin{array}{c} \\ \\ \\ \mathbf{k}' = \mathbf{k} + \sum_{j_1=1}^{n_1} p_{1,j_1} \mathbf{e}_1 + \sum_{j_2=1}^{n_2} p_{2,j_2} \mathbf{e}_2 \\ \\ \end{array} \right. \\
& + (-1)^{n_{\beta_1}(x) + n_{\beta_1}(y)} \frac{\delta_{\sigma, \tau}}{2\pi L^2} \sum_{\mathbf{k} \in \Gamma^*} \int_{-\frac{\pi}{\beta_1}}^{\frac{\pi}{\beta_1}} d\omega e^{i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{i(x-y)\omega} \zeta^{n_0} \\
& \cdot \prod_{j_1=1}^{n_1} \left(\frac{L}{2\pi} \int_0^{\frac{2\pi}{L}} dp_{1,j_1} \right) \prod_{j_2=1}^{n_2} \left(\frac{L}{2\pi} \int_0^{\frac{2\pi}{L}} dp_{2,j_2} \right) \\
& \cdot \left(\frac{\partial}{\partial \omega} \right)^{n_0} \prod_{q=1}^2 \left(\frac{\partial}{\partial k'_q} \right)^{n_q} \\
& \cdot \chi_l(\omega, \mathbf{k}') ((i\omega I_4 - E(\mathbf{k}') - E_l(\beta_1)(\omega, \mathbf{k}'))^{-1} \\
& \quad - (i\omega I_4 - E(\mathbf{k}') - E_l(\beta_2)(\omega, \mathbf{k}'))^{-1}) \\
& \cdot \left| \begin{array}{c} \\ \\ \\ \mathbf{k}' = \mathbf{k} + \sum_{j_1=1}^{n_1} p_{1,j_1} \mathbf{e}_1 + \sum_{j_2=1}^{n_2} p_{2,j_2} \mathbf{e}_2 \\ \\ \end{array} \right. .
\end{aligned}$$

The inequalities (7.54), (7.55), $c_{IR} \geq 1$, $\alpha \geq c$ and Lemma 7.11 ensure that

$$\begin{aligned}
& \left\| \left(\frac{\partial}{\partial k_j} \right)^n (\chi_l(\omega, \mathbf{k}) (i\omega I_4 - E(\mathbf{k}) - E_l(\beta_1)(\omega, \mathbf{k}))^{-1} \right. \\
& \quad \cdot (E_l(\beta_1)(\omega, \mathbf{k}) - E_l(\beta_2)(\omega, \mathbf{k})) (i\omega I_4 - E(\mathbf{k}) - E_l(\beta_2)(\omega, \mathbf{k}))^{-1}) \left. \right\|_{4 \times 4} \\
& \leq \sum_{m_1=0}^n \binom{n}{m_1} \left\| \left(\frac{\partial}{\partial k_j} \right)^{m_1} \chi_l(\omega, \mathbf{k}) (i\omega I_4 - E(\mathbf{k}) - E_l(\beta_1)(\omega, \mathbf{k}))^{-1} \right\|_{4 \times 4} \\
& \quad \cdot \sum_{m_2=0}^{n-m_1} \binom{n-m_1}{m_2} \left\| \left(\frac{\partial}{\partial k_j} \right)^{m_2} (E_l(\beta_1)(\omega, \mathbf{k}) - E_l(\beta_2)(\omega, \mathbf{k})) \right\|_{4 \times 4} \\
& \quad \cdot \left\| \left(\frac{\partial}{\partial k_j} \right)^{n-m_1-m_2} (i\omega I_4 - E(\mathbf{k}) - E_l(\beta_2)(\omega, \mathbf{k}))^{-1} \right\|_{4 \times 4} \\
& \leq c\beta_1^{-\frac{1}{2}} c_{IR}^{-1} \alpha^{-2} M^{-2l} (c\omega(l)^{-1})^n \sum_{m_1=0}^n \binom{n}{m_1} (m_1!)^2
\end{aligned}$$

$$\begin{aligned} & \cdot \sum_{m_2=0}^{n-m_1} \binom{n-m_1}{m_2} (m_2!)^2 ((n-m_1-m_2)!)^2 \\ & \leq c\beta_1^{-\frac{1}{2}} M^{-2l} (cw(l)^{-1})^n (n!)^2, \quad (\forall j \in \{0, 1, 2\}, n \in \mathbb{N} \cup \{0\}). \end{aligned}$$

Using this inequality, Lemma 7.12 (2),(3) and (7.55), we can derive from (7.62) that

$$\begin{aligned} & \left| \hat{d}_j(\mathbf{X})^n \left(\widetilde{C_{ont,l}(\beta_1)}(\mathbf{X}) - \widetilde{C_{ont,l}(\beta_2)}(\mathbf{X}) \right) \right| \\ & \leq \beta_1^{-\frac{1}{2}} c(M) f_{\mathbf{t}}^{-1} M^l (cw(l)^{-1})^n (n!)^2, \quad (\forall \mathbf{X} \in \hat{I}^2, j \in \{0, 1, 2\}, n \in \mathbb{N} \cup \{0\}), \end{aligned}$$

which leads to

$$\begin{aligned} (7.63) \quad & \left| \widetilde{C_{ont,l}(\beta_1)}(\mathbf{X}) - \widetilde{C_{ont,l}(\beta_2)}(\mathbf{X}) \right| \\ & \leq \beta_1^{-\frac{1}{2}} c(M) f_{\mathbf{t}}^{-1} M^l e^{-\sum_{j=0}^2 (cw(l)\hat{d}_j(\mathbf{X}))^{1/2}}, \quad (\forall \mathbf{X} \in \hat{I}^2). \end{aligned}$$

By combining (7.63) with (7.61) we have

$$\begin{aligned} (7.64) \quad & \left| \widetilde{C_l}(\beta_1)(R_{\beta_1}(\mathbf{X})) - \widetilde{C_l}(\beta_2)(R_{\beta_2}(\mathbf{X})) \right| \\ & \leq \beta_1^{-\frac{1}{2}} c(M, c_w) f_{\mathbf{t}}^{-1} M^l e^{-\sum_{j=0}^2 (cw(l)\hat{d}_j(\mathbf{X}))^{1/2}}, \quad (\forall \mathbf{X} \in \hat{I}^2). \end{aligned}$$

By the equality $w(l-1) = w(l)M^{-1}$ and the assumption $M \geq c$ we can derive from (7.64) that

$$\left| \widetilde{C_l}(\beta_1) - \widetilde{C_l}(\beta_2) \right|_{l-1} \leq \beta_1^{-\frac{1}{2}} c(M, c_w) f_{\mathbf{t}}^{-1} M^{-2l}.$$

On the assumption $c_{IR} \geq c(M, c_w) f_{\mathbf{t}}^{-1}$, the above inequality gives (7.58).

To prove the determinant bound, let us take any $r, n \in \mathbb{N}$, $\mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r$ satisfying $\|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} \leq 1$ and $X_i, Y_i \in \hat{I}_0$ ($i = 1, 2, \dots, n$). By expanding along the 1st column and using (7.64),

$$\begin{aligned} & \left| \det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} (C_l(\beta_1)(R_{\beta_1}(X_i, Y_j)) - C_l(\beta_2)(R_{\beta_2}(X_i, Y_j))))_{1 \leq i, j \leq n} \right| \\ & \leq \beta_1^{-\frac{1}{2}} c(M, c_w) f_{\mathbf{t}}^{-1} M^l \sum_{s=1}^n \\ & \quad \cdot \left| \det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} (C_l(\beta_1)(R_{\beta_1}(X_i, Y_j)) - C_l(\beta_2)(R_{\beta_2}(X_i, Y_j))))_{\substack{1 \leq i, j \leq n \\ i \neq s, j \neq 1}} \right|. \end{aligned}$$

By expanding the remaining determinants by means of the Cauchy-Binet formula as in (2.29) and using (7.51) we obtain that

$$\begin{aligned} & |\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} (C_l(\beta_1)(R_{\beta_1}(X_i, Y_j)) - C_l(\beta_2)(R_{\beta_2}(X_i, Y_j)))_{1 \leq i, j \leq n})| \\ & \leq \beta_1^{-\frac{1}{2}} M^{-l} (c(M, c_w) f_{\mathbf{t}}^{-1} M^{2l})^n. \end{aligned}$$

On the assumption $c_{IR} \geq c(M, c_w) f_{\mathbf{t}}^{-1}$, this inequality yields (7.57). \square

Though the next lemma is not used in our proof of Theorem 1.1, it enlightens us about why the infrared integration is necessary to achieve our goal. The discussion in Remark 1.8 was based on this lemma. Here we temporarily lift the imposition of (7.4).

LEMMA 7.15. *Set $t_{max} := \max\{t_{h,e}, t_{h,o}, t_{v,e}, t_{v,o}\}$. Assume that $t_{max}\beta \geq 1$. Then, there exist constants $c_1, c_2, c_3 > 0$ independent of any physical parameter such that the following inequality holds for any $L \in \mathbb{N}$ satisfying $L \geq c_1 t_{max}\beta$ and $\sigma \in \{\uparrow, \downarrow\}$.*

$$c_2\beta \leq \int_0^\beta dx \sum_{\mathbf{x} \in \Gamma} \|C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0)\|_{4 \times 4} \leq c_3 t_{max}^{\frac{1}{2}} f_{\mathbf{t}}^{-1} \beta,$$

where $C : (\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta])^2 \rightarrow \mathbb{C}$ is the free covariance (2.5) with the free Hamiltonian H_0 given by (7.3).

PROOF. First we prove the claim under the assumption (7.4). Let $C_{\leq 0}^+ : I_0^2 \rightarrow \mathbb{C}$ be defined by (2.22) with $\phi(M_{UV}^{-2} h^2 |1 - e^{i\frac{\omega}{h}}|^2)$ in place of $\chi(h|1 - e^{i\frac{\omega}{h}}|)$ and $E(\cdot)$ defined by (7.1). Since now the inequality (6.1) holds with $E_1 = 4$, $E_2 = 1$, the results of Lemma 6.2 for $E_1 = 4$, $E_2 = 1$ are available. Then, by recalling Lemma 2.1 and (6.3) we have that

$$(7.65) \quad \begin{aligned} \sup_{Y \in I_0} \frac{1}{h} \sum_{X \in I_0} |C(X, Y) - C_{\leq 0}^+(X, Y)| & \leq c(M, c_w) \sum_{l=1}^{N_h} M^{-l} \\ & \leq c(M, c_w), \end{aligned}$$

if $h \geq c$. Define the covariances $C'_l : I_0^2 \rightarrow \mathbb{C}$ ($l = 0, -1, \dots, N_\beta$) by

$$C'_l(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y)$$

$$:= \frac{\delta_{\sigma,\tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i(\mathbf{x}-\mathbf{y}, \mathbf{k})} e^{i(x-y)\omega} \chi_l(\omega, \mathbf{k}) (i\omega I_4 - E(\mathbf{k}))^{-1}(\rho, \eta).$$

The covariance C'_l is equal to C_l with $J^j(\psi) = 0 \in \mathcal{S}(j)$ ($j = 0, -1, \dots, l$). Thus, Lemma 7.13 (2) ensures that

$$\sup_{Y \in I_0} \frac{1}{h} \sum_{X \in I_0} |C'_l(X, Y)| \leq c(M, c_w) f_{\mathbf{t}}^{-1} M^{-l}$$

on the assumption that $h \geq c$, $L \geq \beta$ and $M \geq c$. The assumption $\beta \geq 1$ implies that $M^{-N_\beta} \leq c(M)\beta$. Therefore, if $h \geq c$, $1 \leq \beta \leq L$,

$$(7.66) \quad \sup_{Y \in I_0} \frac{1}{h} \sum_{X \in I_0} \left| \sum_{l=0}^{N_\beta} C'_l(X, Y) \right| \leq c(M, c_w) f_{\mathbf{t}}^{-1} \beta.$$

By (7.9) we can deduce that for any $(\mathbf{x}, \sigma, x) \in \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)_h$,

$$(7.67) \quad \begin{aligned} & \left\| C_{\leq 0}^+(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0) - \sum_{l=0}^{N_\beta} C'_l(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0) \right\|_{4 \times 4} \\ & \leq \frac{1}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} \\ & \quad \cdot (|\phi(M_{UV}^{-2} h^2 |1 - e^{i\frac{\omega}{h}}|^2) - \phi(M_{UV}^{-2} \omega^2)| \\ & \quad \cdot \|h^{-1}(I_4 - e^{-i\frac{\omega}{h}} I_4 + \frac{1}{h} E(\mathbf{k}))^{-1}\|_{4 \times 4} \\ & \quad + \phi(M_{UV}^{-2} \omega^2) \|h^{-1}(I_4 - e^{-i\frac{\omega}{h}} I_4 + \frac{1}{h} E(\mathbf{k}))^{-1}\|_{4 \times 4} \\ & \quad \cdot \|(i\omega I_4 - E(\mathbf{k}))^{-1}\|_{4 \times 4} \\ & \quad \cdot \|h(I_4 - e^{-i\frac{\omega}{h}} I_4 + \frac{1}{h} E(\mathbf{k})) - (i\omega I_4 - E(\mathbf{k}))\|_{4 \times 4}) \\ & \leq \frac{1}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} \mathbf{1}_{\phi(M_{UV}^{-2} h^2 |1 - e^{i\frac{\omega}{h}}|^2) \neq 0 \vee \phi(M_{UV}^{-2} \omega^2) \neq 0} \\ & \quad \cdot (c\beta h^{-2} + c\beta^2 h^{-1}) \\ & \leq c\beta h^{-2} + c\beta^2 h^{-1}. \end{aligned}$$

Combination of (7.65), (7.66), (7.67) yields that

$$\frac{1}{h} \sum_{(\mathbf{x}, x) \in \Gamma \times [0, \beta)_h} \|C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0)\|_{4 \times 4}$$

$$\leq c(M, c_w) f_{\mathbf{t}}^{-1} \beta + c(M, c_w) + cL^2 \beta^2 h^{-2} + cL^2 \beta^3 h^{-1}.$$

Then, sending $h \rightarrow \infty$ and using the inequality $f_{\mathbf{t}}^{-1} \beta \geq 1$ we reach the inequality

$$(7.68) \quad \int_0^\beta dx \sum_{\mathbf{x} \in \Gamma} \|C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0)\|_{4 \times 4} \leq c(M, c_w) f_{\mathbf{t}}^{-1} \beta$$

on the assumption $1 \leq \beta \leq L$.

Let us prove the lower bound. By (D.1) in Appendix D,

$$C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0) = \frac{1}{L^2} \sum_{\mathbf{k} \in \Gamma^*} e^{i\langle \mathbf{x}, \mathbf{k} \rangle} e^{xE(\mathbf{k})} (I_4 + e^{\beta E(\mathbf{k})})^{-1}$$

for any $(\mathbf{x}, x) \in \Gamma \times [0, \beta)$. Consider the case that $L \in 2\mathbb{N}$. Since $(\pi, \pi) \in \Gamma^*$ and $E(\pi, \pi) = 0$, we have that for any $x \in [0, \beta)$

$$\sum_{\mathbf{x} \in \Gamma} e^{-i\langle \mathbf{x}, (\pi, \pi) \rangle} C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0) = \frac{1}{2} I_4.$$

Therefore,

$$(7.69) \quad \begin{aligned} & \int_0^\beta dx \sum_{\mathbf{x} \in \Gamma} \|C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0)\|_{4 \times 4} \\ & \geq \int_0^\beta dx \left\| \sum_{\mathbf{x} \in \Gamma} e^{-i\langle \mathbf{x}, (\pi, \pi) \rangle} C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0) \right\|_{4 \times 4} \\ & = \frac{\beta}{2}. \end{aligned}$$

On the other hand, if $L \in 2\mathbb{N} + 1$, $(\pi - \frac{\pi}{L}, \pi - \frac{\pi}{L}) \in \Gamma^*$. For any $x \in [0, \beta)$,

$$\begin{aligned} & \sum_{\mathbf{x} \in \Gamma} e^{-i\langle \mathbf{x}, (\pi - \frac{\pi}{L}, \pi - \frac{\pi}{L}) \rangle} C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0) \\ & = e^{xE(\pi - \frac{\pi}{L}, \pi - \frac{\pi}{L})} (I_4 + e^{\beta E(\pi - \frac{\pi}{L}, \pi - \frac{\pi}{L})})^{-1}. \end{aligned}$$

Moreover, by using Lemma 7.2 (1) we can prove that

$$\left\| \sum_{\mathbf{x} \in \Gamma} e^{-i\langle \mathbf{x}, (\pi - \frac{\pi}{L}, \pi - \frac{\pi}{L}) \rangle} C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0) \right\|_{4 \times 4}$$

$$\begin{aligned}
 &\geq \frac{1}{2} - \left\| e^{xE(\pi-\frac{\pi}{L},\pi-\frac{\pi}{L})} (I_4 + e^{\beta E(\pi-\frac{\pi}{L},\pi-\frac{\pi}{L})})^{-1} - \frac{1}{2} I_4 \right\|_{4 \times 4} \\
 &\geq \frac{1}{2} - \int_{-\frac{\pi}{L}}^0 dr \left\| \frac{d}{dr} e^{xE(\pi+r,\pi+r)} (I_4 + e^{\beta E(\pi+r,\pi+r)})^{-1} \right\|_{4 \times 4} \\
 &\geq \frac{1}{2} - c\beta L^{-1}.
 \end{aligned}$$

This implies that

$$(7.70) \quad \int_0^\beta dx \sum_{\mathbf{x} \in \Gamma} \|C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0)\|_{4 \times 4} \geq \frac{\beta}{2} - c\beta^2 L^{-1}.$$

By assuming that $1 \leq \beta$ and $c\beta \leq L$ we obtain from (7.68), (7.69), (7.70) that

$$c\beta \leq \int_0^\beta dx \sum_{\mathbf{x} \in \Gamma} \|C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0)\|_{4 \times 4} \leq c(M, c_w) f_{\mathbf{t}}^{-1} \beta.$$

Thus, the claim has been proved under the assumption (7.4).

Let us admit that the claim is true under the assumption (7.4) and derive the result in the general case. Let $\hat{C}(\beta) : (\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta])^2 \rightarrow \mathbb{C}$ be the covariance defined by (2.5) with the free Hamiltonian $\frac{1}{t_{max}} H_0$. It follows that if $\beta \geq 1$ and $L \geq c_1 \beta$,

$$c_2 \beta \leq \int_0^\beta dx \sum_{\mathbf{x} \in \Gamma} \|\hat{C}(\beta)(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0)\|_{4 \times 4} \leq c_3 f_{\mathbf{t}/t_{max}}^{-1} \beta.$$

We can see from the definition that

$$\hat{C}(\beta)(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0) = C(\beta/t_{max})(\cdot \mathbf{x} \sigma(x/t_{max}), \cdot \mathbf{0} \sigma 0)$$

for any $(\mathbf{x}, \sigma, x) \in \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta]$. Therefore, if $t_{max}\beta \geq 1$ and $L \geq c_1 t_{max}\beta$,

$$c_2 t_{max} \beta \leq \int_0^{t_{max}\beta} dx \sum_{\mathbf{x} \in \Gamma} \|C(\beta)(\cdot \mathbf{x} \sigma(x/t_{max}), \cdot \mathbf{0} \sigma 0)\|_{4 \times 4} \leq c_3 f_{\mathbf{t}/t_{max}}^{-1} t_{max} \beta,$$

or

$$c_2 \beta \leq \int_0^\beta dx \sum_{\mathbf{x} \in \Gamma} \|C(\beta)(\cdot \mathbf{x} \sigma x, \cdot \mathbf{0} \sigma 0)\|_{4 \times 4} \leq c_3 f_{\mathbf{t}/t_{max}}^{-1} \beta = c_3 t_{max}^{\frac{1}{2}} f_{\mathbf{t}}^{-1} \beta. \quad \square$$

7.4. Application of the generalized infrared integration

Since we have studied the properties of the prototypical covariance for the infrared integration in the previous subsection, we can apply the general infrared analysis summarized in Subsection 5.3 and Subsection 5.4 to prove Theorem 1.1. We follow several steps until we reach the proof of the main theorem. Since some technicalities arise in how to choose the constant c_{IR} and the domain D of \mathbb{C}^4 , let us write $\mathcal{S}(c_{IR}, D)(l)$, $\tilde{\mathcal{S}}(c_{IR}, D)(l)$ in place of $\mathcal{S}(l)$, $\tilde{\mathcal{S}}(l)$ respectively. We also write $\mathcal{S}(\beta)(c_{IR}, D)(l)$ instead of $\mathcal{S}(c_{IR}, D)(l)$ when we want to indicate the dependency on β as well. Throughout this subsection we assume that

$$(7.71) \quad L \geq \beta, \quad h \geq e^8.$$

These are the conditions required in Proposition 6.4, Lemma 7.13 and Lemma 7.14, since now $E_1 = 4$, $M_{UV} = 10\sqrt{6}/\pi$.

We use the output of the UV integration as the input of the infrared integration. Thus, we need to confirm the next statement as the first step.

LEMMA 7.16. *Let c ($\in \mathbb{R}_{>0}$) be the generic positive constant and c_0, c'_0 ($\in \mathbb{R}_{\geq 1}$) be the M -dependent constants appearing in Proposition 6.4. Assume that*

$$M \geq \max\{M_{UV}, c\}, \quad \alpha^2 \geq cM$$

and set

$$D_{UV} := \{(U_1, U_2, U_3, U_4) \in \mathbb{C}^4 \mid |U_\rho| < (c(c_0 + c'_0)^2 \alpha^4)^{-1}, (\forall \rho \in \mathcal{B})\}.$$

Let $J^{+,0}(\psi), J^{-,0}(\psi) \in \bigwedge \mathcal{V}$ be the Grassmann polynomials defined in the beginning of Subsection 6.2. Set

$$J^0(\psi) := \frac{1}{2}(J^{+,0}(\psi) + J^{-,0}(\psi)).$$

Then,

$$J^0(\psi) \in \mathcal{S}(c_0, D_{UV})(0).$$

Moreover, on the assumption (4.2),

$$(J^0(\beta_1)(\psi), J^0(\beta_2)(\psi)) \in \tilde{\mathcal{S}}(c_0, D_{UV})(0).$$

REMARK 7.17. One necessary condition for a Grassmann polynomial to belong to $\mathcal{S}(c_0, D_{UV})(0)$ is the invariance with $S : I \rightarrow I, Q : I \rightarrow \mathbb{R}$ defined in (7.21). It is shown below that the polynomial $J^0(\psi)$ satisfies this invariance, while $J^{+,0}(\psi)$ or $J^{-,0}(\psi)$ cannot be proved to have this invariance by itself. For this reason it is more convenient to deal with $J^0(\psi)$ than $J^{+,0}(\psi), J^{-,0}(\psi)$ as the input to the infrared integration. The adoption of $J^0(\psi)$ in place of $J^{+,0}(\psi), J^{-,0}(\psi)$ is justified by Lemma 2.10.

PROOF OF LEMMA 7.16. By Proposition 6.4 (1), (2) we see that $J^0(\psi)$ satisfies the conditions (i), (ii) of $\mathcal{S}(c_0, D_{UV})(0)$. Moreover, on the assumption (4.2), Proposition 6.4 (4) implies that $(J^0(\beta_1)(\psi), J^0(\beta_2)(\psi)) \in \tilde{\mathcal{S}}(c_0, D_{UV})(0)$.

It remains to check that $J^0(\psi)$ satisfies the invariant properties (iii), (iv). Let $J^{\delta,l}(\psi)$ ($l \in \{0, 1, \dots, N_h\}, \delta \in \{+, -\}$) be the Grassmann polynomials defined in the beginning of Subsection 6.2. Since $J^{\delta, N_h}(\psi) = -V^\delta(\psi)$, by recalling (2.30) we can check that $J^{\delta, N_h}(\psi)$ satisfies the invariant properties (iii), (iv) except the invariance with $S : I \rightarrow I, Q : I \rightarrow \mathbb{R}$ defined in (7.21). In the same way as in the proof of Lemma 7.13 (3), (4) we can show that for any $l \in \{0, 1, \dots, N_h\}, \delta \in \{+, -\}$,

$$\widetilde{C}_l^\delta(\mathbf{U})(\mathbf{X}) = e^{iQ_2(S_2(\mathbf{X}))} \widetilde{C}_l^\delta(\mathbf{U})(S_2(\mathbf{X})), \quad (\forall \mathbf{X} \in I^2, \mathbf{U} \in \overline{D_{UV}}),$$

with $S : I \rightarrow I, Q : I \rightarrow \mathbb{R}$ defined in (7.15), (7.16), (7.17), (7.18), (7.19) respectively, and

$$\widetilde{C}_l^\delta(\mathbf{U})(\mathbf{X}) = e^{-iQ_2(S_2(\mathbf{X}))} \overline{\widetilde{C}_l^\delta(\overline{\mathbf{U}})(S_2(\mathbf{X}))}, \quad (\forall \mathbf{X} \in I^2, \mathbf{U} \in \overline{D_{UV}}),$$

with $S : I \rightarrow I, Q : I \rightarrow \mathbb{R}$ defined in (7.20). Therefore, we can inductively apply Lemma 3.9 to conclude that $J^{\delta,0}(\psi)$ satisfies the invariant properties (iii), (iv) except the invariance with $S : I \rightarrow I, Q : I \rightarrow \mathbb{R}$ defined in (7.21), and so does $J^0(\psi)$ by definition.

In the following let $S : I \rightarrow I, Q : I \rightarrow \mathbb{R}$ be those defined in (7.21). We see that

$$(7.72) \quad V^\delta(\mathcal{R}\psi) = V^{-\delta}(\psi), \quad (\forall \delta \in \{+, -\}).$$

Moreover, for any $(\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi) \in I,$

$$(7.73) \quad e^{-iQ_2(S_2((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)))} \overline{C_{>0}^\delta(S_2((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi)))}$$

$$= \frac{1}{2} (-1)^{1_{\rho \in \{1,4\}} + 1_{\eta \in \{1,4\}}} + 1 \left(1_{(\theta, \xi) = (1, -1)} \overline{C_{>0}^\delta((\eta, \mathbf{y}, \tau, y), (\rho, \mathbf{x}, \sigma, x))} \right. \\ \left. - 1_{(\xi, \theta) = (1, -1)} \overline{C_{>0}^\delta((\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y))} \right).$$

For the matrix $Y \in \text{Mat}(4, \mathbb{C})$ defined in (7.49), $YE(\mathbf{k})Y = -E(\mathbf{k})$. Using this equality and $E(\mathbf{k})^* = E(\mathbf{k})$ we can derive that

$$\begin{aligned} & (-1)^{1_{\rho \in \{1,4\}} + 1_{\eta \in \{1,4\}}} + 1 \overline{C_{>0}^+((\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y))} \\ &= \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{-i(x-y)\omega} \sum_{l=1}^{N_h} \chi_{h,l}(\omega) \\ & \quad \cdot h^{-1} (-I_4 + e^{i\frac{\omega}{h} I_4 + \frac{1}{h} Y E(\mathbf{k}) Y})^{-1}(\rho, \eta) \\ &= \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{-i(x-y)\omega} \sum_{l=1}^{N_h} \chi_{h,l}(\omega) \\ & \quad \cdot h^{-1} (e^{i\frac{\omega}{h} I_4 - \frac{1}{h} E(\mathbf{k})^*} - I_4)^{-1}(\rho, \eta) \\ &= \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{-i\langle \mathbf{y} - \mathbf{x}, \mathbf{k} \rangle} e^{i(y-x)\omega} \sum_{l=1}^{N_h} \chi_{h,l}(\omega) \\ & \quad \cdot h^{-1} (e^{i\frac{\omega}{h} I_4 - \frac{1}{h} \overline{E(\mathbf{k})}} - I_4)^{-1}(\eta, \rho) \\ &= C_{>0}^-((\eta, \mathbf{y}, \tau, y), (\rho, \mathbf{x}, \sigma, x)), \end{aligned}$$

which also implies that

$$\begin{aligned} & (-1)^{1_{\rho \in \{1,4\}} + 1_{\eta \in \{1,4\}}} + 1 \overline{C_{>0}^-((\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y))} \\ &= C_{>0}^+((\eta, \mathbf{y}, \tau, y), (\rho, \mathbf{x}, \sigma, x)). \end{aligned}$$

Substituting these equalities into (7.73) yields

$$(7.74) \quad e^{-iQ_2(S_2(\mathbf{X}))} \overline{C_{>0}^\delta(S_2(\mathbf{X}))} = \widetilde{C_{>0}^{-\delta}(\mathbf{X})}, \quad (\forall \mathbf{X} \in I^2, \delta \in \{+, -\}).$$

We see from the definition of Grassmann Gaussian integral and (7.74) that for any $\mathbf{X} \in I^n$,

$$\int \psi_{\mathbf{X}} d\mu_{C_{>0}^\delta}(\psi) = e^{-\sum_{\mathbf{Y} \in I^2} \widetilde{C_{>0}^\delta(\mathbf{Y})} \frac{\partial}{\partial \psi_{\mathbf{Y}}} \psi_{\mathbf{X}}} \Big|_{\psi=0}$$

$$\begin{aligned}
 &= e^{-\sum_{\mathbf{Y} \in I^2} e^{iQ_2(\mathbf{Y})} \overline{C_{>0}^\delta}(S_2^{-1}(\mathbf{Y})) \frac{\partial}{\partial \psi_{\mathbf{Y}}} e^{-iQ_n(S_n(\mathbf{X}))} \psi_{S_n(\mathbf{X})} \Big|_{\psi=0} \\
 &= e^{-\sum_{\mathbf{Y} \in I^2} \overline{C_{>0}^\delta}(\mathbf{Y}) \frac{\partial}{\partial \psi_{\mathbf{Y}}} e^{-iQ_n(S_n(\mathbf{X}))} \psi_{S_n(\mathbf{X})} \Big|_{\psi=0} \\
 &= \int e^{-iQ_n(S_n(\mathbf{X}))} \psi_{S_n(\mathbf{X})} d\mu_{\overline{C_{>0}^\delta}}(\psi),
 \end{aligned}$$

or

$$\begin{aligned}
 (7.75) \quad \int \psi_{\mathbf{X}} d\mu_{\overline{C_{>0}^\delta}}(\psi) &= \int e^{iQ_n(S_n(\mathbf{X}))} \psi_{S_n(\mathbf{X})} d\mu_{\overline{C_{>0}^\delta}}(\psi) \\
 &= \int (\mathcal{R}\psi)_{\mathbf{X}} d\mu_{\overline{C_{>0}^\delta}}(\psi).
 \end{aligned}$$

Proposition 6.4 (3) states that if $\sup_{\rho \in B} |U_\rho|$ is sufficiently small,

$$J^{\delta,0}(\psi) = \log \left(\int e^{-V^\delta(\psi+\psi^1)} d\mu_{\overline{C_{>0}^\delta}}(\psi^1) \right).$$

By (7.75),

$$\begin{aligned}
 \overline{J^{\delta,0}(\overline{\mathbf{U}})}(\psi) &= \log \left(\int e^{-V^\delta(\psi+\psi^1)} d\mu_{\overline{C_{>0}^\delta}}(\psi^1) \right) \\
 &= \log \left(\int e^{-V^\delta(\psi+\mathcal{R}\psi^1)} d\mu_{\overline{C_{>0}^\delta}}(\psi^1) \right).
 \end{aligned}$$

Then, by (7.72) we obtain

$$\overline{J^{\delta,0}(\overline{\mathbf{U}})}(\mathcal{R}\psi) = J^{-\delta,0}(\mathbf{U})(\psi).$$

Since $J^{-\delta,0}(\mathbf{U})$, $\overline{J^{\delta,0}(\overline{\mathbf{U}})}(\mathcal{R}\psi)$ are continuous in $\overline{D_{UV}}$ and analytic in D_{UV} with \mathbf{U} , the identity theorem and the continuity guarantee this equality for all $\mathbf{U} \in \overline{D_{UV}}$. Therefore, we have

$$\overline{J^0(\overline{\mathbf{U}})}(\mathcal{R}\psi) = J^0(\mathbf{U})(\psi), \quad (\forall \mathbf{U} \in \overline{D_{UV}}). \quad \square$$

The aim of our IR integration is to find an analytic continuation of

$$-\frac{1}{\beta L^2} \log \left(\int e^{J^0(\psi)} d\mu_{C_{\leq 0}^\infty}(\psi) \right)$$

into a (β, L, h) -independent domain of \mathbf{U} around the origin. Here the covariance $C_{\leq 0}^\infty : I_0^2 \rightarrow \mathbb{C}$ is defined by (2.24) with $\phi(M_{UV}^{-2}\omega^2)$ in place of $\chi(|\omega|)$. The next lemma explains how this aim is achieved.

LEMMA 7.18. *There exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that if $M \geq c$, we can choose $J^l(\psi) \in \bigwedge \mathcal{V}$ ($l \in \{-1, -2, \dots, N_\beta - 1\}$) so that the following statements hold true.*

- (1) *There exist constants $c(M, c_w), c'(M, c_w) \in \mathbb{R}_{\geq 1}$ depending only on M, c_w such that*

$$\begin{aligned} J^l(\psi) &\in \mathcal{S}(c_{IR}, D_{IR})(l), \\ (\forall l \in \{0, -1, \dots, N_\beta - 1\}, \alpha \in \mathbb{R}_{>0} \text{ with } \alpha^2 \geq cM^7), \end{aligned}$$

where $J^0(\psi)$ is the polynomial set in Lemma 7.16,

$$\begin{aligned} c_{IR} &:= c(M, c_w) f_{\mathbf{t}}^{-1}, \\ D_{IR} &:= \left\{ (U_1, U_2, U_3, U_4) \in \mathbb{C}^4 \mid |U_\rho| < \frac{f_{\mathbf{t}}^2}{c'(M, c_w)\alpha^4}, (\forall \rho \in \mathcal{B}) \right\}. \end{aligned}$$

- (2)

$$\begin{aligned} \operatorname{Re}\{\det(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k}))^{-1} \chi_{\leq l}(\omega, \mathbf{k}) W^l(\omega, \mathbf{k}))\} &> 0, \\ (\forall (\omega, \mathbf{k}) \in \mathcal{M} \times \Gamma^*, l \in \{0, -1, \dots, N_\beta\}, \alpha \in \mathbb{R}_{>0} \text{ with } \alpha^2 \geq cM^7, \\ \mathbf{U} \in \overline{D_{IR}}), \end{aligned}$$

where $E_1(\omega, \mathbf{k}) := 0$, $W^l(\omega, \mathbf{k})$ and $E_{l+1}(\omega, \mathbf{k})$ ($l \in \{0, -1, \dots, N_\beta\}$) are derived from $J^l(\psi)$ by (7.24) and (7.41) respectively.

- (3) *There exists a constant $c(\beta, L, M) \in \mathbb{R}_{>0}$ depending only on β, L, M such that if $\alpha \geq c(\beta, L, M)$ additionally holds,*

$$\int e^{J^0(\psi)} d\mu_{C_{\leq 0}^\infty}(\psi) \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$$

and

$$(7.76) \quad -\frac{1}{\beta L^2} \log \left(\int e^{J^0(\psi)} d\mu_{C_{\leq 0}^\infty}(\psi) \right)$$

$$\begin{aligned}
 &= -\frac{1}{\beta L^2} \sum_{l=0}^{N_\beta-1} J_0^l \\
 &\quad - \sum_{l=0}^{N_\beta} \frac{2}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M} \times \Gamma^*} \\
 &\quad \cdot \log \left(\det \left(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k}))^{-1} \right. \right. \\
 &\quad \left. \left. \cdot \chi_{\leq l}(\omega, \mathbf{k}) W^l(\omega, \mathbf{k}) \right) \right), \\
 &(\forall \mathbf{U} \in \overline{D_{IR}}).
 \end{aligned}$$

(4) Assume that (4.2) holds and $L \geq \beta_2$. Then,

$$\begin{aligned}
 &(J^l(\beta_1)(\psi), J^l(\beta_2)(\psi)) \in \tilde{\mathcal{S}}(c_{IR}, D_{IR})(l), \\
 &(\forall l \in \{0, -1, \dots, N_{\beta_1}\}, \alpha \in \mathbb{R}_{>0} \text{ with } \alpha^2 \geq cM^7).
 \end{aligned}$$

PROOF. In the following we will apply Proposition 5.6, Proposition 5.9 for $a_1 = 2$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1/2$, Lemma 7.13, Lemma 7.14 and Lemma 7.16. Note that there exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that for any $M, \alpha \in \mathbb{R}_{\geq 1}$ satisfying

$$(7.77) \quad M^{\frac{1}{2}} \geq c, \quad \alpha \geq cM^{\frac{7}{2}},$$

the claims of these propositions and lemmas hold. We assume the condition (7.77) during the proof. We can replace c in (7.77) by a larger generic constant without altering the statements of this lemma. Such a replacement will be necessary in the proof of the claim (2).

In order to organize the argument, we introduce the sets $\mathcal{R}(l)$ ($l \in \mathbb{Z}_{\leq 0}$) of covariances as follows. For a constant $c_1 \in \mathbb{R}_{\geq 1}$ and a domain $D_o(\subset \mathbb{C}^4)$ satisfying that $\overline{\mathbf{U}} \in \overline{D_o}$ ($\forall \mathbf{U} \in \overline{D_o}$), where $\overline{D_o}$ is the closure of D_o , a function $C_o : I_0^2 \rightarrow \mathbb{C}$ belongs to $\mathcal{R}(c_1, D_o)(l)$ if and only if C_o is parameterized by $\mathbf{U} \in \overline{D_o}$ and satisfies the following conditions.

(i) $\mathbf{U} \mapsto C_o(\mathbf{U})(\mathbf{X})$ is continuous in $\overline{D_o}$ and analytic in D_o ($\forall \mathbf{X} \in I_0^2$).

(ii)

$$\begin{aligned}
 &|\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_o(\mathbf{U})(X_i, Y_j))_{1 \leq i, j \leq n}| \leq (c_1 M^{2l})^n, \\
 &(\forall r, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} \leq 1, \\
 &X_i, Y_i \in I_0 \text{ (} i = 1, 2, \dots, n), \mathbf{U} \in \overline{D_o}).
 \end{aligned}$$

(iii)

$$\|\widetilde{C}_o(\mathbf{U})\|_{l-1,r} \leq c_1 M^{-l-rl}, \quad (\forall r \in \{0, 1\}, \mathbf{U} \in \overline{D_o}).$$

(iv) Under the notation introduced in Subsection 3.3,

$$\widetilde{C}_o(\mathbf{U})(\mathbf{X}) = e^{iQ_2(S_2(\mathbf{X}))} \widetilde{C}_o(\mathbf{U})(S_2(\mathbf{X})), \quad (\forall \mathbf{X} \in I^2, \mathbf{U} \in \overline{D_o}),$$

for $S : I \rightarrow I$, $Q : I \rightarrow \mathbb{R}$ defined by (7.15), (7.16), (7.17), (7.18), (7.19) respectively.

(v)

$$\widetilde{C}_o(\mathbf{U})(\mathbf{X}) = e^{-iQ_2(S_2(\mathbf{X}))} \overline{\widetilde{C}_o(\overline{\mathbf{U}})(S_2(\mathbf{X}))}, \quad (\forall \mathbf{X} \in I^2, \mathbf{U} \in \overline{D_o}),$$

for $S : I \rightarrow I$, $Q : I \rightarrow \mathbb{R}$ defined by (7.20), (7.21) respectively.

Here $\widetilde{C}_o : I^2 \rightarrow \mathbb{C}$ is the anti-symmetric extension of C_o defined as in (3.2). We will also write $\mathcal{R}(\beta)(c_1, D_o)(l)$ in place of $\mathcal{R}(c_1, D_o)(l)$ when we want to indicate its β -dependency.

Let us inductively construct $J^l(\psi) \in \mathcal{S}(l)$ ($l = 0, -1, \dots, N_\beta - 1$), $C_j \in \mathcal{R}(l)$ ($l = 0, -1, \dots, N_\beta$). Let $c(M, c_w)$ be the maximum of the constants with the same notation appearing in Lemma 7.13 and Lemma 7.14. Recall that $c(M, c_w)$ depends only on M , c_w and that the constant $c_0 (\in \mathbb{R}_{\geq 1})$ appearing in Lemma 7.16 stems from Proposition 6.4 and depends only on M , since now b, d, M_{UV}, E_2 are fixed constants. As remarked in Remark 6.5, we can replace c_0 by $\max\{c(M, c_w), c_0\} f_{\mathbf{t}}^{-1}$ in Proposition 6.4. Accordingly Lemma 7.16 ensures that

$$J^0(\psi) \in \mathcal{S}(\max\{c(M, c_w), c_0\} f_{\mathbf{t}}^{-1}, D)(0),$$

and on the assumption (4.2),

$$(J^0(\beta_1)(\psi), J^0(\beta_2)(\psi)) \in \widetilde{\mathcal{S}}(\max\{c(M, c_w), c_0\} f_{\mathbf{t}}^{-1}, D)(0),$$

where

$$D := \left\{ (U_1, U_2, U_3, U_4) \in \mathbb{C}^4 \mid \right.$$

$$|U_\rho| < \frac{1}{c(\max\{c(M, c_w), c_0\} f_{\mathbf{t}}^{-1} + c'_0)^2 \alpha^4}, \quad (\forall \rho \in \mathcal{B}) \Big\}$$

with the constant c appearing in (7.77). Now, set

$$\begin{aligned} c_{IR} &:= \max\{c(M, c_w), c_0\} f_{\mathbf{t}}^{-1}, \\ D_{IR} &:= \left\{ (U_1, U_2, U_3, U_4) \in \mathbb{C}^4 \mid \right. \\ &\quad \left. |U_\rho| < \frac{f_{\mathbf{t}}^2}{c(\max\{c(M, c_w), c_0\} + c'_0)^2 \alpha^4}, \quad (\forall \rho \in \mathcal{B}) \right\}. \end{aligned}$$

Since $D_{IR} \subset D$, we have $J^0(\psi) \in \mathcal{S}(c_{IR}, D_{IR})(0)$, $(J^0(\beta_1)(\psi), J^0(\beta_2)(\psi)) \in \tilde{\mathcal{S}}(c_{IR}, D_{IR})(0)$.

Assume that $l \in \{0, -1, \dots, N_\beta\}$ and

$$J^j(\psi) \in \mathcal{S}(c_{IR}, D_{IR})(j), \quad (\forall j \in \{0, -1, \dots, l\}).$$

Using $J^j(\psi)$ ($j = 0, -1, \dots, l$), we define the covariance $C_l : I_0^2 \rightarrow \mathbb{C}$ by (7.43). It follows from Lemma 7.13 and the inequality $c_{IR} \geq c(M, c_w) f_{\mathbf{t}}^{-1}$ that $C_l \in \mathcal{R}(c_{IR}, D_{IR})(l)$. Then, we define the Grassmann polynomials $F^{l-1}(\psi)$, $T^{l-1, (n)}(\psi)$ ($n \in \mathbb{N}_{\geq 2}$), $T^{l-1}(\psi)$, $J^{l-1}(\psi) \in \bigwedge \mathcal{V}$ by (5.7) with the covariance C_l and the input $J^l(\psi) - J_0^l - J_2^l(\psi)$. We can apply Proposition 5.6 for $a_1 = 2$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1/2$ and Lemma 3.9 to prove that $J^{l-1}(\psi) \in \mathcal{S}(c_{IR}, D_{IR})(l-1)$. The continuity and the analyticity of $J^{l-1}(\psi)$ with \mathbf{U} can be proved by the same argument as in the proof of Proposition 6.4 (2). Thus, we have inductively constructed

$$\begin{aligned} J^l(\psi) &\in \mathcal{S}(c_{IR}, D_{IR})(l), \quad (l = 0, -1, \dots, N_\beta - 1), \\ C_l &\in \mathcal{R}(c_{IR}, D_{IR})(l), \quad (l = 0, -1, \dots, N_\beta). \end{aligned}$$

Thus, the claim (1) holds with $c'(M, c_w) := c(\max\{c(M, c_w), c_0\} + c'_0)^2$. Before proving the claims (2), (3), let us show that the claim (4) holds true.

(4): Define the subset $\tilde{\mathcal{R}}(c_{IR}, D_{IR})(l)$ of $\mathcal{R}(\beta_1)(c_{IR}, D_{IR})(l) \times \mathcal{R}(\beta_2)(c_{IR}, D_{IR})(l)$ ($l = 0, -1, \dots, N_{\beta_1}$) as follows. $(C_o(\beta_1), C_o(\beta_2)) \in \mathcal{R}(\beta_1)(c_{IR}, D_{IR})(l) \times \mathcal{R}(\beta_2)(c_{IR}, D_{IR})(l)$ belongs to $\tilde{\mathcal{R}}(c_{IR}, D_{IR})(l)$ if and only if

(i)

$$\begin{aligned}
& |\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_o(\mathbf{U})(\beta_1)(R_{\beta_1}(X_i, Y_j)))_{1 \leq i, j \leq n} \\
& \quad - \det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^r} C_o(\mathbf{U})(\beta_2)(R_{\beta_2}(X_i, Y_j)))_{1 \leq i, j \leq n}| \\
& \leq \beta_1^{-\frac{1}{2}} M^{-l} (c_{IR} M^{2l})^n, \\
& (\forall r, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^r \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^r}, \|\mathbf{q}_i\|_{\mathbb{C}^r} \leq 1, \\
& \quad X_i, Y_i \in \hat{I}_0 \ (i = 1, 2, \dots, n), \mathbf{U} \in \overline{D_{IR}}).
\end{aligned}$$

(ii)

$$|\widetilde{C}_o(\mathbf{U})(\beta_1) - \widetilde{C}_o(\mathbf{U})(\beta_2)|_l \leq \beta_1^{-\frac{1}{2}} c_{IR} M^{-2l}, \ (\forall \mathbf{U} \in \overline{D_{IR}}),$$

where $\widetilde{C}_o(\beta_j) : I(\beta_j)^2 \rightarrow \mathbb{C}$ is the anti-symmetric extension of $C_o(\beta_j)$ defined as in (3.2) for $j = 1, 2$.

We have already seen that $(J^0(\beta_1)(\psi), J^0(\beta_2)(\psi)) \in \tilde{\mathcal{S}}(c_{IR}, D_{IR})(0)$. By Lemma 7.14 and the inequality $c_{IR} \geq c(M, c_w) f_{\mathbf{t}}^{-1}$, $(C_0(\beta_1), C_0(\beta_2)) \in \tilde{\mathcal{R}}(c_{IR}, D_{IR})(0)$.

Assume that $l \in \{0, -1, \dots, N_{\beta_1}\}$ and

$$\begin{aligned}
& (J^j(\beta_1)(\psi), J^j(\beta_2)(\psi)) \in \tilde{\mathcal{S}}(c_{IR}, D_{IR})(j), \\
& (C_j(\beta_1), C_j(\beta_2)) \in \tilde{\mathcal{R}}(c_{IR}, D_{IR})(j), \ (\forall j \in \{0, -1, \dots, l\}).
\end{aligned}$$

Then, we can apply Proposition 5.9 for $a_1 = 2, a_2 = 1, a_3 = 1, a_4 = 1/2$ to conclude that

$$(J^{l-1}(\beta_1)(\psi), J^{l-1}(\beta_2)(\psi)) \in \tilde{\mathcal{S}}(c_{IR}, D_{IR})(l-1).$$

Moreover, if $l \geq N_{\beta_1} + 1$, we can again apply Lemma 7.14 to prove that

$$(C_{l-1}(\beta_1), C_{l-1}(\beta_2)) \in \tilde{\mathcal{R}}(c_{IR}, D_{IR})(l-1).$$

Therefore, the claim (4) has been proved by induction with l .

(2): Take any $(\omega, \mathbf{k}) \in \mathbb{R}^3$ satisfying $|\omega| \geq \pi/\beta$ and $\chi_{\leq l}(\omega, \mathbf{k}) \neq 0$. By the assumption $L \geq \beta$ and Lemma 7.5, the condition (7.39) holds. By Lemma 7.8 (1), (7.44) and the inequalities $c_{IR}^{-1} \leq f_{\mathbf{t}}, f_{\mathbf{t}} \leq 1$,

$$\|(i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k}))^{-1} \chi_{\leq l}(\omega, \mathbf{k}) \widehat{W}^l(\omega, \mathbf{k})\|_{4 \times 4}$$

$$\begin{aligned}
 &\leq \sum_{j=l}^{N_\beta} \chi_j(\omega, \mathbf{k}) \|(i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k}))^{-1}\|_{4 \times 4} \|\widehat{W}^l(\omega, \mathbf{k})\|_{4 \times 4} \\
 &\leq \sum_{j=l}^{N_\beta} \chi_j(\omega, \mathbf{k}) M^{-j} c \cdot c_{IR}^{-1} f_{\mathbf{t}}^{-\frac{1}{2}} M^{\frac{1}{2}l+j+1} \alpha^{-2} \\
 &\leq c f_{\mathbf{t}} \cdot f_{\mathbf{t}}^{-\frac{1}{2}} M^{\frac{1}{2}l+1} \alpha^{-2} \leq c M \alpha^{-2}.
 \end{aligned}$$

Since the left-hand side of the above inequality vanishes if $\chi_{\leq l}(\omega, \mathbf{k}) = 0$, we eventually have

$$\begin{aligned}
 (7.78) \quad &\|(i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k}))^{-1} \chi_{\leq l}(\omega, \mathbf{k}) \widehat{W}^l(\omega, \mathbf{k})\|_{4 \times 4} \\
 &\leq c M \alpha^{-2}, \quad (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3 \text{ with } |\omega| \geq \pi/\beta).
 \end{aligned}$$

By the condition (7.77),

$$\begin{aligned}
 (7.79) \quad &\left| \det(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k}))^{-1} \chi_{\leq l}(\omega, \mathbf{k}) \widehat{W}^l(\omega, \mathbf{k})) - 1 \right| \\
 &\leq c M \alpha^{-2}, \quad (\forall (\omega, \mathbf{k}) \in \mathbb{R}^3 \text{ with } |\omega| \geq \pi/\beta).
 \end{aligned}$$

The claim follows from this inequality and the replacement of the constant c in (7.77) by a larger constant if necessary.

(3): By (7.13) and (7.14),

$$\begin{aligned}
 |J_0^l| &\leq \frac{N}{h} \alpha^{-3}, \\
 \left(\frac{1}{h}\right)^m \sum_{\mathbf{X} \in I^m} |J_m^l(\mathbf{X})| &\leq \frac{N}{h} c_{IR}^{-\frac{m}{2}} M^{-lm} \alpha^{-m}, \quad (\forall m \in \mathbb{N}).
 \end{aligned}$$

Using these inequalities, we have that

$$\begin{aligned}
 |e^{J_0^l} - 1| &\leq e^{\frac{N}{h} \alpha^{-3}} - 1. \\
 \left| \int e^{z \sum_{m=4}^N J_m^l(\psi)} d\mu_{C_l}(\psi) - 1 \right| \\
 &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{j=1}^n \left(2 \sum_{m_j=4}^N \left(\frac{1}{h}\right)^{m_j} \sum_{\mathbf{X} \in I^{m_j}} |J_{m_j}^l(\mathbf{X})| \right) (c_{IR} M^{2l})^{\frac{1}{2} \sum_{j=1}^n m_j} \\
 &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \left(2 \frac{N}{h} \sum_{m=4}^N \alpha^{-m} \right)^n \leq e^{2 \frac{N}{h} \frac{\alpha^{-4}}{1-\alpha^{-1}}} - 1, \quad (\forall z \in \mathbb{C} \text{ with } |z| < 2).
 \end{aligned}$$

The above inequalities imply that

$$(7.80) \quad \begin{aligned} \operatorname{Re} e^{J_0^l} &> 0, \quad (\forall l \in \{0, -1, \dots, N_\beta - 1\}), \\ \operatorname{Re} \int e^{z \sum_{m=4}^N J_m^l(\psi)} d\mu_{C_l}(\psi) &> 0, \\ (\forall l \in \{0, -1, \dots, N_\beta\}, z \in \mathbb{C} \text{ with } |z| < 2), \end{aligned}$$

if α is larger than or equal to a constant $c(\beta, L) \in \mathbb{R}_{>0}$ depending only on β and L . We can especially see from (7.80) that

$$\log \left(\int e^{z \sum_{m=4}^N J_m^l(\psi + \psi^1)} d\mu_{C_l}(\psi^1) \right)$$

is analytic with z in $\{z \in \mathbb{C} \mid |z| < 2\}$, and thus

$$(7.81) \quad \begin{aligned} &\log \left(\int e^{\sum_{m=4}^N J_m^l(\psi + \psi^1)} d\mu_{C_l}(\psi^1) \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{d}{dz} \right)^n \log \left(\int e^{z \sum_{m=4}^N J_m^l(\psi + \psi^1)} d\mu_{C_l}(\psi^1) \right) \Big|_{z=0} \\ &= J^{l-1}(\psi), \quad (\forall l \in \{0, -1, \dots, N_\beta\}), \end{aligned}$$

if $\alpha \geq c(\beta, L)$.

Define the covariances $C_{\leq l}, D_{\leq l} : I_0^2 \rightarrow \mathbb{C}$ ($l = 0, -1, \dots, N_\beta$) by

$$\begin{aligned} C_{\leq l}(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) &:= \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{k}, \mathbf{x} - \mathbf{y} \rangle} e^{i\omega(x-y)} \\ &\quad \cdot \chi_{\leq l}(\omega, \mathbf{k}) (i\omega I_4 - E(\mathbf{k}) - E_l(\omega, \mathbf{k}))^{-1}, \\ D_{\leq l}(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) &:= \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{k}, \mathbf{x} - \mathbf{y} \rangle} e^{i\omega(x-y)} \\ &\quad \cdot \chi_{\leq l}(\omega, \mathbf{k}) (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k}))^{-1}, \end{aligned}$$

where $E_1(\omega, \mathbf{k}) := 0$. By (7.44), $C_{\leq l}, D_{\leq l}$ are well-defined. Note that by (7.9), $D_{\leq 0}$ is equal to $C_{\leq 0}^\infty$. Moreover,

$$(7.82) \quad C_l(\mathbf{X}) + D_{\leq l-1}(\mathbf{X}) = C_{\leq l}(\mathbf{X}), \quad (\forall \mathbf{X} \in I_0^2, l \in \{0, -1, \dots, N_\beta + 1\}).$$

Introduce the functions $J_{0,2}^l : I_0^2 \rightarrow \mathbb{C}$ ($l \in \{0, -1, \dots, N_\beta\}$) by

$$\begin{aligned} & J_{0,2}^l(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) \\ & := \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} \mathbf{1}_{\chi_{\leq l}(\omega, \mathbf{k})=0} e^{i\langle \mathbf{k}, \mathbf{x}-\mathbf{y} \rangle} e^{i\omega(x-y)} W^l(\omega, \mathbf{k})(\rho, \eta). \end{aligned}$$

Then, let us define $\tilde{J}_2^l(\psi) \in \bigwedge \mathcal{V}$ by

$$\tilde{J}_2^l(\psi) := \left(\frac{1}{h}\right)^2 \sum_{X, Y \in I_0} J_{0,2}^l(X, Y) \psi_X \bar{\psi}_Y.$$

Since

$$\begin{aligned} \sum_{X \in I_0} J_{0,2}^l(X, Y) D_{\leq l}(Z, X) &= \sum_{X \in I_0} J_{0,2}^l(X, Y) D_{\leq l}(X, Z) = 0, \\ (\forall Y, Z \in I_0, l \in \{0, -1, \dots, N_\beta\}), \end{aligned}$$

we can derive from the definition of the Grassmann Gaussian integral that

$$(7.83) \quad \int e^{J^l(\psi)} d\mu_{D_{\leq l}}(\psi) = \int e^{J^l(\psi) - \tilde{J}_2^l(\psi)} d\mu_{D_{\leq l}}(\psi),$$

$$(\forall l \in \{0, -1, \dots, N_\beta\}).$$

To approximate $C_{\leq l}$, $D_{\leq l}$ by invertible matrices, let us take $\varepsilon \in \mathbb{R}_{>0}$. Then, define the covariances $C_{\leq l}^\varepsilon, D_{\leq l}^\varepsilon : I_0^2 \rightarrow \mathbb{C}$ ($l \in \{0, -1, \dots, N_\beta\}$) by

$$\begin{aligned} & C_{\leq l}^\varepsilon(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) \\ & := C_{\leq l}(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) + \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{k}, \mathbf{x}-\mathbf{y} \rangle} e^{i\omega(x-y)} \mathbf{1}_{\chi_{\leq l}(\omega, \mathbf{k})=0} \varepsilon I_4, \\ & D_{\leq l}^\varepsilon(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) \\ & := D_{\leq l}(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) + \frac{\delta_{\sigma, \tau}}{\beta L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{k}, \mathbf{x}-\mathbf{y} \rangle} e^{i\omega(x-y)} \mathbf{1}_{\chi_{\leq l}(\omega, \mathbf{k})=0} \varepsilon I_4. \end{aligned}$$

Let $C_{\leq l}^{\varepsilon, -1}, D_{\leq l}^{\varepsilon, -1} : I_0^2 \rightarrow \mathbb{C}$ be the inverse matrix of $C_{\leq l}^\varepsilon, D_{\leq l}^\varepsilon$ respectively. We can see that

$$C_{\leq l}^{\varepsilon, -1}(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y)$$

$$\begin{aligned}
&= \frac{\delta_{\sigma,\tau}}{\beta L^2 h^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{k}, \mathbf{x} - \mathbf{y} \rangle} e^{i\omega(x-y)} \\
&\quad \cdot \left(1_{\chi_{\leq l}(\omega, \mathbf{k}) \neq 0} \chi_{\leq l}(\omega, \mathbf{k})^{-1} (i\omega I_4 - E(\mathbf{k}) - E_l(\omega, \mathbf{k})) + 1_{\chi_{\leq l}(\omega, \mathbf{k}) = 0} \varepsilon^{-1} I_4 \right), \\
&D_{\leq l}^{\varepsilon, -1}(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) \\
&= \frac{\delta_{\sigma,\tau}}{\beta L^2 h^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} e^{i\langle \mathbf{k}, \mathbf{x} - \mathbf{y} \rangle} e^{i\omega(x-y)} \\
&\quad \cdot \left(1_{\chi_{\leq l}(\omega, \mathbf{k}) \neq 0} \chi_{\leq l}(\omega, \mathbf{k})^{-1} (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k})) + 1_{\chi_{\leq l}(\omega, \mathbf{k}) = 0} \varepsilon^{-1} I_4 \right).
\end{aligned}$$

By Lemma 7.6 (1) and (7.42) we have

$$\begin{aligned}
(7.84) \quad &J_2^l(\psi) - \tilde{J}_2^l(\psi) - \sum_{X, Y \in I_0} D_{\leq l}^{\varepsilon, -1}(X, Y) \psi_X \bar{\psi}_Y \\
&= - \sum_{X, Y \in I_0} C_{\leq l}^{\varepsilon, -1}(X, Y) \psi_X \bar{\psi}_Y, \\
&(\forall l \in \{0, -1, \dots, N_\beta\}).
\end{aligned}$$

Here let us recall some basics of Grassmann integration. We can number each element of I_0 so that $I_0 = \{X_j\}_{j=1}^{8L^2\beta h}$. We define the linear map $\int \cdot d\psi d\bar{\psi} : \Lambda(\mathcal{V} \oplus \mathcal{V}^1) \rightarrow \Lambda \mathcal{V}^1$ as follows. For any $f(\psi^1) \in \Lambda \mathcal{V}^1$,

$$\begin{aligned}
&\int f(\psi^1) \bar{\psi}_{X_1} \bar{\psi}_{X_2} \cdots \bar{\psi}_{X_{8L^2\beta h}} \psi_{X_1} \psi_{X_2} \cdots \psi_{X_{8L^2\beta h}} d\psi d\bar{\psi} := f(\psi^1), \\
&\int f(\psi^1) \psi_{\mathbf{X}} d\psi d\bar{\psi} := 0, \quad (\forall \mathbf{X} \in I^m, m \neq 16L^2\beta h).
\end{aligned}$$

Then, for any $g \in \Lambda(\mathcal{V} \oplus \mathcal{V}^1)$, $\int g d\psi d\bar{\psi} (\in \Lambda \mathcal{V}^1)$ is uniquely determined by linearity and anti-symmetry. Let $C_o : I_0^2 \rightarrow \mathbb{C}$ be an invertible covariance matrix. By taking the Grassmann variables a_1, a_2, \dots, a_D to be $\bar{\psi}_{X_1}, \bar{\psi}_{X_2}, \dots, \bar{\psi}_{X_{8L^2\beta h}}, \psi_{X_1}, \psi_{X_2}, \dots, \psi_{X_{8L^2\beta h}}$ and the skew symmetric matrix S to be

$$\begin{pmatrix} 0 & C_o \\ -C_o^t & 0 \end{pmatrix}$$

in [5, Lemma I.10] we obtain the equality that

$$\int e^{\sum_{X \in I_0} (\bar{\psi}_X \bar{\psi}_X + \psi_X^1 \psi_X)} e^{-\sum_{X, Y \in I_0} C_o^{-1}(X, Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi}$$

$$\begin{aligned}
 & \cdot / \int e^{-\sum_{X,Y \in I_0} C_o^{-1}(X,Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi} \\
 & = e^{-\sum_{X,Y \in I_0} C_o(X,Y) \bar{\psi}_X^1 \psi_Y^1}.
 \end{aligned}$$

By applying the left derivative

$$\frac{\partial}{\partial \bar{\psi}_{X_{i_1}}^1} \cdots \frac{\partial}{\partial \bar{\psi}_{X_{i_l}}^1} \frac{\partial}{\partial \psi_{X_{j_m}}^1} \cdots \frac{\partial}{\partial \psi_{X_{j_1}}^1}$$

to both sides of the above equality and comparing the constant terms we obtain that

$$\begin{aligned}
 & \int \bar{\psi}_{X_{i_1}} \cdots \bar{\psi}_{X_{i_l}} \psi_{X_{j_m}} \cdots \psi_{X_{j_1}} e^{-\sum_{X,Y \in I_0} C_o^{-1}(X,Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi} \\
 & \cdot / \int e^{-\sum_{X,Y \in I_0} C_o^{-1}(X,Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi} \\
 & = \begin{cases} \det(C_o(X_{i_p}, X_{j_q}))_{1 \leq p, q \leq l} & \text{if } l = m, \\ 0 & \text{if } l \neq m \end{cases} \\
 & = \int \bar{\psi}_{X_{i_1}} \cdots \bar{\psi}_{X_{i_l}} \psi_{X_{j_m}} \cdots \psi_{X_{j_1}} d\mu_{C_o}(\psi).
 \end{aligned}$$

By linearity,

$$\begin{aligned}
 (7.85) \quad & \int f(\psi) d\mu_{C_o}(\psi) = \int f(\psi) e^{-\sum_{X,Y \in I_0} C_o^{-1}(X,Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi} \\
 & \cdot / \int e^{-\sum_{X,Y \in I_0} C_o^{-1}(X,Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi}, \\
 & (\forall f(\psi) \in \bigwedge \mathcal{V}).
 \end{aligned}$$

Note that

$$\begin{aligned}
 (7.86) \quad & \int e^{-\sum_{X,Y \in I_0} C_o^{-1}(X,Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi} = (-1)^{8L^2\beta h(8L^2\beta h-1)/2} \det C_o^{-1} \\
 & = \det C_o^{-1}.
 \end{aligned}$$

By combining (7.83), (7.84) with the general equalities (7.85), (7.86) we observe that

$$(7.87) \quad \int e^{J^0(\psi)} d\mu_{C_{\leq 0}^\infty}(\psi) = \int e^{J^0(\psi)} d\mu_{D_{\leq 0}}(\psi) = \int e^{J^0(\psi) - \bar{J}_2^0(\psi)} d\mu_{D_{\leq 0}}(\psi)$$

$$\begin{aligned}
&= \lim_{\varepsilon \searrow 0} \int e^{J^0(\psi) - \tilde{J}_2^0(\psi)} e^{-\sum_{X,Y \in I_0} D_{\leq 0}^{\varepsilon, -1}(X,Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi} \\
&\quad \cdot \int e^{-\sum_{X,Y \in I_0} D_{\leq 0}^{\varepsilon, -1}(X,Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi} \\
&= e^{J_0^0} \lim_{\varepsilon \searrow 0} \int e^{\sum_{m=4}^N J_m^0(\psi)} e^{-\sum_{X,Y \in I_0} C_{\leq 0}^{\varepsilon, -1}(X,Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi} \\
&\quad \cdot \int e^{-\sum_{X,Y \in I_0} D_{\leq 0}^{\varepsilon, -1}(X,Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi} \\
&= e^{J_0^0} \lim_{\varepsilon \searrow 0} \int e^{-\sum_{X,Y \in I_0} C_{\leq 0}^{\varepsilon, -1}(X,Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi} \\
&\quad \cdot \int e^{-\sum_{X,Y \in I_0} D_{\leq 0}^{\varepsilon, -1}(X,Y) \psi_X \bar{\psi}_Y} d\psi d\bar{\psi} \\
&\quad \cdot \int e^{\sum_{m=4}^N J_m^0(\psi)} d\mu_{C_{\leq 0}^{\varepsilon}}(\psi) \\
&= e^{J_0^0} \lim_{\varepsilon \searrow 0} \\
&\quad \cdot \prod_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} \left(\det \left(1_{\chi_{\leq 0}(\omega, \mathbf{k}) \neq 0} \chi_{\leq 0}(\omega, \mathbf{k})^{-1} \right. \right. \\
&\quad \quad \cdot (i\omega I_4 - E(\mathbf{k}) - E_0(\omega, \mathbf{k})) \\
&\quad \quad \quad \left. \left. + 1_{\chi_{\leq 0}(\omega, \mathbf{k}) = 0} \varepsilon^{-1} I_4 \right)^2 \right) \\
&\quad \cdot \int e^{\sum_{m=4}^N J_m^0(\psi)} d\mu_{C_{\leq 0}^{\varepsilon}}(\psi) \\
&\quad \cdot \prod_{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^*} \left(\det \left(1_{\chi_{\leq 0}(\omega, \mathbf{k}) \neq 0} \chi_{\leq 0}(\omega, \mathbf{k})^{-1} (i\omega I_4 - E(\mathbf{k})) \right. \right. \\
&\quad \quad \quad \left. \left. + 1_{\chi_{\leq 0}(\omega, \mathbf{k}) = 0} \varepsilon^{-1} I_4 \right)^2 \right) \\
&= e^{J_0^0} \prod_{\substack{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^* \\ \text{with } \chi_{\leq 0}(\omega, \mathbf{k}) \neq 0}} \left(\det \left(I_4 - (i\omega I_4 - E(\mathbf{k}))^{-1} \right. \right. \\
&\quad \quad \quad \left. \left. \cdot \chi_{\leq 0}(\omega, \mathbf{k}) W^0(\omega, \mathbf{k}) \right)^2 \right) \\
&\quad \cdot \int e^{\sum_{m=4}^N J_m^0(\psi)} d\mu_{C_{\leq 0}}(\psi) \\
&= e^{J_0^0} \prod_{\substack{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^* \\ \text{with } \chi_{\leq 0}(\omega, \mathbf{k}) \neq 0}} \left(\det \left(I_4 - (i\omega I_4 - E(\mathbf{k}))^{-1} \right. \right.
\end{aligned}$$

$$\begin{aligned} & \cdot \chi_{\leq 0}(\omega, \mathbf{k}) W^0(\omega, \mathbf{k}))^2 \\ & \cdot \int e^{J^{-1}(\psi)} d\mu_{D_{\leq -1}}(\psi). \end{aligned}$$

In the last line of (7.87) we did the following transformation, which can be justified by (7.80), (7.81), (7.82), [5, Proposition I.21] and [14, Lemma C.2].

$$\begin{aligned} \int e^{\sum_{m=4}^N J_m^0(\psi)} d\mu_{C_{\leq 0}}(\psi) &= \int \int e^{\sum_{m=4}^N J_m^0(\psi+\psi^1)} d\mu_{C_0}(\psi^1) d\mu_{D_{\leq -1}}(\psi) \\ &= \int e^{\log(\int e^{\sum_{m=4}^N J_m^0(\psi+\psi^1)} d\mu_{C_0}(\psi^1))} d\mu_{D_{\leq -1}}(\psi) \\ &= \int e^{J^{-1}(\psi)} d\mu_{D_{\leq -1}}(\psi). \end{aligned}$$

By repeating this procedure and using the fact that $C_{\leq N_\beta} = C_{N_\beta}$ we obtain the following.

$$\begin{aligned} (7.88) \quad & \int e^{J^0(\psi)} d\mu_{C_{\leq 0}^\infty}(\psi) \\ &= e^{J_0^0+J_0^{-1}+\dots+J_0^{N_\beta+1}} \prod_{l=0}^{N_\beta+1} \left(\prod_{\substack{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^* \\ \text{with } \chi_{\leq l}(\omega, \mathbf{k}) \neq 0}} \right. \\ & \quad \cdot \det(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k})))^{-1} \\ & \quad \left. \cdot \chi_{\leq l}(\omega, \mathbf{k}) W^l(\omega, \mathbf{k})^2 \right) \\ & \quad \cdot \int e^{J^{N_\beta}(\psi)} d\mu_{D_{\leq N_\beta}}(\psi) \\ &= e^{J_0^0+J_0^{-1}+\dots+J_0^{N_\beta}} \prod_{l=0}^{N_\beta} \left(\prod_{\substack{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^* \\ \text{with } \chi_{\leq l}(\omega, \mathbf{k}) \neq 0}} \right. \\ & \quad \cdot \det(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k})))^{-1} \\ & \quad \left. \cdot \chi_{\leq l}(\omega, \mathbf{k}) W^l(\omega, \mathbf{k})^2 \right) \\ & \quad \cdot \int e^{\sum_{m=4}^N J_m^{N_\beta}(\psi)} d\mu_{C_{\leq N_\beta}}(\psi) \end{aligned}$$

$$\begin{aligned}
&= e^{J_0^0 + J_0^{-1} + \dots + J_0^{N_\beta - 1}} \prod_{l=0}^{N_\beta} \left(\prod_{\substack{(\omega, \mathbf{k}) \in \mathcal{M}_h \times \Gamma^* \\ \text{with } \chi_{\leq l}(\omega, \mathbf{k}) \neq 0}} \right. \\
&\quad \cdot \det \left(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k}))^{-1} \right. \\
&\quad \left. \left. \cdot \chi_{\leq l}(\omega, \mathbf{k}) W^l(\omega, \mathbf{k}) \right)^2 \right).
\end{aligned}$$

It follows from (7.13) that

$$(7.89) \quad \left| \sum_{l=0}^{N_\beta - 1} J_0^l \right| \leq \frac{N}{h} c\alpha^{-3}, \quad \left| e^{\sum_{l=0}^{N_\beta - 1} J_0^l} - 1 \right| \leq e^{\frac{N}{h} c\alpha^{-3}} - 1.$$

Set

$$\begin{aligned}
K &:= \{(l, \omega, \mathbf{k}, \sigma) \in \{0, -1, \dots, N_\beta\} \times \mathcal{M}_h \times \Gamma^* \times \{\uparrow, \downarrow\} \mid \\
&\quad \chi_{\leq l}(\omega, \mathbf{k}) \neq 0\}, \\
K' &:= \{(l, \omega, \mathbf{k}, \sigma) \in \{0, -1, \dots, N_\beta\} \times \mathcal{M} \times \Gamma^* \times \{\uparrow, \downarrow\} \mid \\
&\quad \phi(M_{UV}^{-2} \omega^2) \neq 0\}.
\end{aligned}$$

Using (7.78) and (7.79), we see that for any $Q \subset K$ with $Q \neq \emptyset$,

$$\begin{aligned}
(7.90) \quad &\left| \prod_{(l, \omega, \mathbf{k}, \sigma) \in Q} \det \left(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k}))^{-1} \right. \right. \\
&\quad \left. \left. \cdot \chi_{\leq l}(\omega, \mathbf{k}) W^l(\omega, \mathbf{k}) \right) - 1 \right| \\
&= \left| \prod_{(l, \omega, \mathbf{k}, \sigma) \in Q} \det \left(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k}))^{-1} \right. \right. \\
&\quad \left. \left. \cdot \chi_{\leq l}(\omega, \mathbf{k}) W^l(\omega, \mathbf{k}) \right) \right. \\
&\quad \left. - \prod_{(l, \omega, \mathbf{k}, \sigma) \in Q} 1 \right| \\
&\leq cM\alpha^{-2} \#Q (1 + cM\alpha^{-2})^{\#Q - 1} \leq cM\alpha^{-2} \#K' (1 + cM\alpha^{-2})^{\#K' - 1}.
\end{aligned}$$

By (7.89) and (7.90),

$$\begin{aligned}
 (7.91) \quad & \operatorname{Im} \sum_{l=0}^{N_\beta-1} J_0^l \in (-\pi, \pi), \\
 & \operatorname{Re} e^{\sum_{l=0}^{N_\beta-1} J_0^l} > 0, \\
 & \operatorname{Re} \prod_{(l, \omega, \mathbf{k}, \sigma) \in Q} \det(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\omega, \mathbf{k}))^{-1} \\
 & \qquad \qquad \qquad \cdot \chi_{\leq l}(\omega, \mathbf{k}) W^l(\omega, \mathbf{k})) \\
 & > 0, \quad (\forall Q \subset K \text{ with } Q \neq \emptyset),
 \end{aligned}$$

if α is larger than a constant $c(\beta, L, M) \in \mathbb{R}_{>0}$ depending only on β, L, M . Since $\log e^z = z$ ($\forall z \in \mathbb{C}$ with $\operatorname{Im} z \in (-\pi, \pi)$), $z_1 z_2 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, $\log(z_1 z_2) = \log z_1 + \log z_2$ ($\forall z_1, z_2 \in \mathbb{C}$ with $\operatorname{Re} z_1 > 0, \operatorname{Re} z_2 > 0$), the condition (7.91) enables us to deduce the claim (3) from (7.88). \square

On the assumption that $M \geq c, \alpha^2 \geq cM^7$ with the constant $c \in \mathbb{R}_{>0}$ appearing in Lemma 7.18, we define the function $J_{\text{end}}(\cdot) : \overline{D_{IR}} \rightarrow \mathbb{C}$ by the right-hand side of (7.76). In the next two lemmas we study properties of J_{end} .

LEMMA 7.19. *There exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that if*

$$M \geq c, \alpha^2 \geq cM^7,$$

the following statements hold true.

- (1) $\mathbf{U} \mapsto J_{\text{end}}(\mathbf{U})$ is continuous in $\overline{D_{IR}}$ and analytic in D_{IR} .
- (2) There exists a constant $c(M) \in \mathbb{R}_{>0}$ depending only on M such that

$$|J_{\text{end}}(\mathbf{U})| \leq c(M) f_{\mathbf{t}}^{-1} \alpha^{-2}, \quad (\forall \mathbf{U} \in \overline{D_{IR}}).$$

- (3) Additionally assume that (4.2) holds and $L \geq \beta_2$. Then, there exists a constant $c''(M, c_w) \in \mathbb{R}_{>0}$ depending only on M, c_w such that

$$|J_{\text{end}}(\beta_1)(\mathbf{U}) - J_{\text{end}}(\beta_2)(\mathbf{U})| \leq c''(M, c_w) \beta_1^{-\frac{1}{2}} f_{\mathbf{t}}^{-1} \alpha^{-2}, \quad (\forall \mathbf{U} \in \overline{D_{IR}}).$$

PROOF. (1): Since $J^l(\psi) \in \mathcal{S}(c_{IR}, D_{IR})(l)$ ($\forall l \in \{0, -1, \dots, N_\beta - 1\}$),

$$\mathbf{U} \mapsto -\frac{1}{\beta L^2} \sum_{l=0}^{N_\beta-1} J_0^l(\mathbf{U})$$

is continuous in $\overline{D_{IR}}$ and analytic in D_{IR} . By Lemma 7.18 (2) and the definitions (7.24), (7.41), the second term of J_{end} is also seen to be continuous in $\overline{D_{IR}}$ and analytic in D_{IR} .

(2): To estimate the second term of J_{end} , we can apply Lemma 7.12 (1), since the condition (7.39) is ensured by Lemma 7.5. Using (7.13), (7.79), Lemma 7.12 (1) and the inequality $f_{\mathbf{t}} \leq 1$, we have that

$$\begin{aligned} |J_{end}| &\leq c \sum_{l=0}^{N_\beta-1} M^{\frac{7}{2}l} \alpha^{-3} + c \sum_{l=0}^{N_\beta} f_{\mathbf{t}}^{-1} M^{3l+3} \sum_{n=1}^{\infty} \frac{1}{n} (cM\alpha^{-2})^n \\ &\leq c\alpha^{-3} + cM^4 f_{\mathbf{t}}^{-1} \alpha^{-2} \leq c(M) f_{\mathbf{t}}^{-1} \alpha^{-2}. \end{aligned}$$

(3): Note that by the condition on L, β_1, β_2 , the inequality (7.39) holds for β_1 and β_2 . This means that we can use Lemma 7.8, Lemma 7.10 and Lemma 7.12 (2),(3), in which (7.39) is assumed, in the following.

We can decompose $J_{end}(\beta_2)$ as follows.

$$J_{end}(\beta_2) = J_{end}^1(\beta_2) + J_{end}^2(\beta_2),$$

where

$$\begin{aligned} J_{end}^1(\beta_2) &:= -\frac{1}{\beta_2 L^2} \sum_{l=0}^{N_{\beta_1}-1} J_0^l(\beta_2) - \sum_{l=0}^{N_{\beta_1}} \frac{2}{\beta_2 L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}(\beta_2) \times \Gamma^*} \\ &\quad \cdot \log \left(\det \left(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(\omega, \mathbf{k}))^{-1} \right. \right. \\ &\quad \left. \left. \cdot \chi_{\leq l}(\beta_2)(\omega, \mathbf{k}) W^l(\beta_2)(\omega, \mathbf{k}) \right) \right), \\ J_{end}^2(\beta_2) &:= -\frac{1}{\beta_2 L^2} \sum_{l=N_{\beta_1}-2}^{N_{\beta_2}-1} J_0^l(\beta_2) - \sum_{l=N_{\beta_1}-1}^{N_{\beta_2}} \frac{2}{\beta_2 L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}(\beta_2) \times \Gamma^*} \\ &\quad \cdot \log \left(\det \left(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(\omega, \mathbf{k}))^{-1} \right. \right. \\ &\quad \left. \left. \cdot \chi_{\leq l}(\beta_2)(\omega, \mathbf{k}) W^l(\beta_2)(\omega, \mathbf{k}) \right) \right). \end{aligned}$$

By using (7.13), (7.79) and Lemma 7.12 (1) and recalling the definition of N_{β_1} we can deduce that

$$\begin{aligned}
 (7.92) \quad |J_{end}^2(\beta_2)| &\leq c \sum_{l=N_{\beta_1}-2}^{N_{\beta_2}-1} M^{\frac{7}{2}l} \alpha^{-3} \\
 &\quad + c \sum_{l=N_{\beta_1}-1}^{N_{\beta_2}} f_{\mathbf{t}}^{-1} M^{3l+3} \sum_{n=1}^{\infty} \frac{1}{n} (cM\alpha^{-2})^n \\
 &\leq c(M)\beta_1^{-3} f_{\mathbf{t}}^{-1} \alpha^{-2}.
 \end{aligned}$$

Next let us find an upper bound on $|J_{end}(\beta_1) - J_{end}^1(\beta_2)|$. Note that

$$\begin{aligned}
 (7.93) \quad &|J_{end}(\beta_1) - J_{end}^1(\beta_2)| \\
 &\leq \sum_{l=0}^{N_{\beta_1}-1} \left| \frac{1}{\beta_1 L^2} J_0^l(\beta_1) - \frac{1}{\beta_2 L^2} J_0^l(\beta_2) \right| + 2 \sum_{l=0}^{N_{\beta_1}} \sum_{a=1}^2 \\
 &\quad \cdot \left| \frac{1}{\beta_a L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}(\beta_a) \times \Gamma^*} \right. \\
 &\quad \quad \cdot \log \left(\det \left(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\beta_a)(\omega, \mathbf{k}))^{-1} \right. \right. \\
 &\quad \quad \quad \left. \left. \cdot \chi_{\leq l}(\beta_a)(\omega, \mathbf{k}) \widehat{W}^l(\beta_a)(\omega, \mathbf{k}) \right) \right) \\
 &\quad - \frac{1}{2\pi L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_a}}^{\pi h} du + \int_{-\pi h}^{-\frac{\pi}{\beta_a}} du \right) \\
 &\quad \quad \cdot \log \left(\det \left(I_4 - (iu I_4 - E(\mathbf{k}) - E_{l+1}(\beta_a)(u, \mathbf{k}))^{-1} \right. \right. \\
 &\quad \quad \quad \left. \left. \cdot \chi_{\leq l}(\beta_a)(u, \mathbf{k}) \widehat{W}^l(\beta_a)(u, \mathbf{k}) \right) \right) \left. \right| \\
 &+ \sum_{l=0}^{N_{\beta_1}} \frac{1}{\pi L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_1}}^{\pi h} du + \int_{-\pi h}^{-\frac{\pi}{\beta_1}} du \right) \\
 &\quad \cdot \left| \log \left(\det \left(I_4 - (iu I_4 - E(\mathbf{k}) - E_{l+1}(\beta_1)(u, \mathbf{k}))^{-1} \right. \right. \right. \\
 &\quad \quad \left. \left. \cdot \chi_{\leq l}(\beta_1)(u, \mathbf{k}) \widehat{W}^l(\beta_1)(u, \mathbf{k}) \right) \right) \\
 &\quad - \log \left(\det \left(I_4 - (iu I_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k}))^{-1} \right. \right. \\
 &\quad \quad \left. \left. \cdot \chi_{\leq l}(\beta_2)(u, \mathbf{k}) \widehat{W}^l(\beta_2)(u, \mathbf{k}) \right) \right) \left. \right|
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{l=0}^{N_{\beta_1}} \frac{1}{\pi L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_2}}^{\frac{\pi}{\beta_1}} du + \int_{-\frac{\pi}{\beta_1}}^{-\frac{\pi}{\beta_2}} du \right) \\
& \cdot \left| \log \left(\det \left(I_4 - (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k})) \right)^{-1} \right. \right. \\
& \quad \left. \left. \cdot \chi_{\leq l}(\beta_2)(u, \mathbf{k}) \widehat{W}^l(\beta_2)(u, \mathbf{k}) \right) \right|.
\end{aligned}$$

The assumption $h \geq e^8$ implies that $\pi h \geq (\pi/\sqrt{3})M_{UV}$, or $\phi(M_{UV}^{-2}u^2) = 0$ ($\forall u \in \mathbb{R}$ with $|u| \geq \pi h$). This explains why the integrals with u over a domain outside $[-\pi h, \pi h]$ vanish in the following calculations. By using (7.79), (7.78), Lemma 7.12 (2), (7.54), Lemma 7.4, Lemma 7.8 (2) and the inequality $c_{IR} \geq f_t^{-1}$ in this order,

$$\begin{aligned}
(7.94) \quad & \left| \frac{1}{\beta_a L^2} \sum_{(\omega, \mathbf{k}) \in \mathcal{M}(\beta_a) \times \Gamma^*} \right. \\
& \cdot \log \left(\det \left(I_4 - (i\omega I_4 - E(\mathbf{k}) - E_{l+1}(\beta_a)(\omega, \mathbf{k})) \right)^{-1} \right. \\
& \quad \left. \cdot \chi_{\leq l}(\beta_a)(\omega, \mathbf{k}) W^l(\beta_a)(\omega, \mathbf{k}) \right) \\
& - \frac{1}{2\pi L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_a}}^{\pi h} du + \int_{-\pi h}^{-\frac{\pi}{\beta_a}} du \right) \\
& \quad \cdot \log \left(\det \left(I_4 - (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_a)(u, \mathbf{k})) \right)^{-1} \right. \\
& \quad \left. \left. \cdot \chi_{\leq l}(\beta_a)(u, \mathbf{k}) \widehat{W}^l(\beta_a)(u, \mathbf{k}) \right) \right| \\
& \leq \frac{1}{2\pi L^2} \sum_{\mathbf{k} \in \Gamma^*} \sum_{m=0}^{\frac{\beta_a h}{2} - 1} \\
& \quad \cdot \left(\int_{\frac{2\pi}{\beta_a} m + \frac{\pi}{\beta_a}}^{\frac{2\pi}{\beta_a} (m+1) + \frac{\pi}{\beta_a}} d\omega \int_{\frac{2\pi}{\beta_a} m + \frac{\pi}{\beta_a}}^{\omega} du \right. \\
& \quad \left. + \int_{-\frac{2\pi}{\beta_a} (m+1) - \frac{\pi}{\beta_a}}^{-\frac{2\pi}{\beta_a} m - \frac{\pi}{\beta_a}} d\omega \int_{-\frac{2\pi}{\beta_a} m - \frac{\pi}{\beta_a}}^{\omega} du \right) \\
& \quad \cdot \left| \frac{\partial}{\partial u} \log \left(\det \left(I_4 - (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_a)(u, \mathbf{k})) \right)^{-1} \right) \right|
\end{aligned}$$

$$\begin{aligned}
 & \left| \cdot \chi_{\leq l}(\beta_a)(u, \mathbf{k}) \widehat{W}^l(\beta_a)(u, \mathbf{k}) \right| \\
 & + \frac{1}{2\pi L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\pi h}^{\pi h + \frac{\pi}{\beta_a}} du + \int_{-\pi h - \frac{\pi}{\beta_a}}^{-\pi h} du \right) \\
 & \cdot \left| \log \left(\det \left(I_4 - (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_a)(u, \mathbf{k}))^{-1} \right. \right. \right. \\
 & \left. \left. \left. \cdot \chi_{\leq l}(\beta_a)(u, \mathbf{k}) \widehat{W}^l(\beta_a)(u, \mathbf{k}) \right) \right| \\
 & \leq \frac{c}{\beta_a L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_a}}^{\pi h} du + \int_{-\pi h}^{-\frac{\pi}{\beta_a}} du \right) \\
 & \cdot \left| \frac{\partial}{\partial u} \det \left(I_4 - (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_a)(u, \mathbf{k}))^{-1} \right. \right. \\
 & \left. \left. \cdot \chi_{\leq l}(\beta_a)(u, \mathbf{k}) \widehat{W}^l(\beta_a)(u, \mathbf{k}) \right) \right| \\
 & \leq \frac{c}{\beta_1 L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_a}}^{\pi h} du + \int_{-\pi h}^{-\frac{\pi}{\beta_a}} du \right) \sum_{j=l}^{N_{\beta_a}} \\
 & \cdot \left\| \frac{\partial}{\partial u} \left((iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_a)(u, \mathbf{k}))^{-1} \right. \right. \\
 & \left. \left. \cdot \chi_j(u, \mathbf{k}) \widehat{W}^l(\beta_a)(u, \mathbf{k}) \right) \right\|_{4 \times 4} \\
 & \leq \frac{c}{\beta_1 L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_a}}^{\pi h} du + \int_{-\pi h}^{-\frac{\pi}{\beta_a}} du \right) \sum_{j=l}^{N_{\beta_a}} \\
 & \cdot \sum_{n=0}^1 \left\| \left(\frac{\partial}{\partial u} \right)^n \left((iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_a)(u, \mathbf{k}))^{-1} \right) \right\|_{4 \times 4} \\
 & \cdot \sum_{m=0}^{1-n} \left\| \left(\frac{\partial}{\partial u} \right)^m \chi_j(u, \mathbf{k}) \right\| \left\| \left(\frac{\partial}{\partial u} \right)^{1-n-m} \widehat{W}^l(\beta_a)(u, \mathbf{k}) \right\|_{4 \times 4} \\
 & \leq c(M) \beta_1^{-1} \sum_{j=l}^{N_{\beta_a}} f_{\mathbf{t}}^{-1} M^{3j} \sum_{n=0}^1 M^{-j} (cw(j)^{-1})^n
 \end{aligned}$$

$$\begin{aligned}
& \cdot \sum_{m=0}^{1-n} (c_w(j)^{-1})^m c_{IR}^{-1} M^{\frac{3}{2}l} \alpha^{-2} (c_w(l)^{-1})^{1-n-m} \\
& \leq c''(M, c_w) \beta_1^{-1} M^{\frac{3}{2}l} \alpha^{-2} \sum_{j=l}^{N_{\beta_a}} M^j \\
& \leq c''(M, c_w) \beta_1^{-1} M^{\frac{5}{2}l} \alpha^{-2}.
\end{aligned}$$

Also, by recalling (7.12) we see that

$$\begin{aligned}
(7.95) \quad & \frac{1}{L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_1}}^{\pi h} du + \int_{-\pi h}^{-\frac{\pi}{\beta_1}} du \right) \\
& \cdot \left| \log \left(\det \left(I_4 - (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_1)(u, \mathbf{k})) \right)^{-1} \right. \right. \\
& \quad \cdot \chi_{\leq l}(\beta_1)(u, \mathbf{k}) \widehat{W}^l(\beta_1)(u, \mathbf{k}) \left. \right) \\
& \quad - \log \left(\det \left(I_4 - (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k})) \right)^{-1} \right. \\
& \quad \cdot \chi_{\leq l}(\beta_2)(u, \mathbf{k}) \widehat{W}^l(\beta_2)(u, \mathbf{k}) \left. \right) \left. \right| \\
& \leq \frac{1}{L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_1}}^{\pi h} du + \int_{-\pi h}^{-\frac{\pi}{\beta_1}} du \right) 1_{\chi_{\leq l}(\beta_1)(u, \mathbf{k}) \neq 0} \\
& \cdot \left| \log \left(\det \left(I_4 + (I_4 - (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k})) \right)^{-1} \right. \right. \\
& \quad \cdot \chi_{\leq l}(\beta_2)(u, \mathbf{k}) \widehat{W}^l(\beta_2)(u, \mathbf{k}) \left. \right)^{-1} \\
& \quad \cdot \left((iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k}))^{-1} \chi_{\leq l}(\beta_2)(u, \mathbf{k}) \widehat{W}^l(\beta_2)(u, \mathbf{k}) \right. \\
& \quad \left. - (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_1)(u, \mathbf{k}))^{-1} \right. \\
& \quad \left. \cdot \chi_{\leq l}(\beta_1)(u, \mathbf{k}) \widehat{W}^l(\beta_1)(u, \mathbf{k}) \right) \left. \right) \left. \right| \\
& = \frac{1}{L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_1}}^{\pi h} du + \int_{-\pi h}^{-\frac{\pi}{\beta_1}} du \right) 1_{\chi_{\leq l}(\beta_1)(u, \mathbf{k}) \neq 0} \\
& \cdot \left| \log \left(\det \left(I_4 + (I_4 - (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k})) \right)^{-1} \right. \right. \\
& \quad \cdot \chi_{\leq l}(\beta_2)(u, \mathbf{k}) \widehat{W}^l(\beta_2)(u, \mathbf{k}) \left. \right)^{-1} \\
& \quad \cdot \left((iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k}))^{-1} \chi_{\leq l}(\beta_1)(u, \mathbf{k}) \right. \\
& \quad \cdot (\widehat{W}^l(\beta_2)(u, \mathbf{k}) - \widehat{W}^l(\beta_1)(u, \mathbf{k})) \\
& \quad \left. + (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k}))^{-1} \right.
\end{aligned}$$

$$\begin{aligned} & \cdot (E_{l+1}(\beta_2)(u, \mathbf{k}) - E_{l+1}(\beta_1)(u, \mathbf{k})) \\ & \cdot (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_1)(u, \mathbf{k}))^{-1} \\ & \cdot \chi_{\leq l}(\beta_1)(u, \mathbf{k}) \widehat{W}^l(\beta_1)(u, \mathbf{k})) \Big|, \end{aligned}$$

where we used the equalities of the form

$$\begin{aligned} & \log(\det(I_4 - A)) - \log(\det(I_4 - B)) \\ & = \log(\det(I_4 + (I_4 - B)^{-1}(B - A))), \\ & (C - A)^{-1} - (C - B)^{-1} = (C - A)^{-1}(A - B)(C - B)^{-1}, \end{aligned}$$

for $A, B, C \in \text{Mat}(4, \mathbb{C})$. Moreover, by (7.44), the inequality $c_{IR} \geq f_{\mathbf{t}}^{-1}$, Lemma 7.8 (1), Lemma 7.9, Lemma 7.10 (1) and Lemma 7.11 we have for any $j \in \{l, l-1, \dots, N_{\beta_1}\}$, $(u, \mathbf{k}) \in \mathbb{R}^3$ with $\chi_j(u, \mathbf{k}) \neq 0$ that

$$\begin{aligned} (7.96) \quad & \left\| (I_4 - (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_1)(u, \mathbf{k}))^{-1} \right. \\ & \cdot \chi_{\leq l}(\beta_2)(u, \mathbf{k}) \widehat{W}^l(\beta_2)(u, \mathbf{k}))^{-1} \\ & \cdot ((iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k}))^{-1} \chi_{\leq l}(\beta_1)(u, \mathbf{k}) \\ & \cdot (\widehat{W}^l(\beta_2)(u, \mathbf{k}) - \widehat{W}^l(\beta_1)(u, \mathbf{k})) \\ & + (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k}))^{-1} \\ & \cdot (E_{l+1}(\beta_2)(u, \mathbf{k}) - E_{l+1}(\beta_1)(u, \mathbf{k})) \\ & \cdot (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_1)(u, \mathbf{k}))^{-1} \\ & \cdot \chi_{\leq l}(\beta_1)(u, \mathbf{k}) \widehat{W}^l(\beta_1)(u, \mathbf{k})) \Big\|_{4 \times 4} \\ & \leq c \|(iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k}))^{-1} \chi_{\leq l}(\beta_1)(u, \mathbf{k}) \\ & \cdot (\widehat{W}^l(\beta_2)(u, \mathbf{k}) - \widehat{W}^l(\beta_1)(u, \mathbf{k}))\|_{4 \times 4} \\ & + c \|(iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k}))^{-1} \\ & \cdot (E_{l+1}(\beta_2)(u, \mathbf{k}) - E_{l+1}(\beta_1)(u, \mathbf{k}))\|_{4 \times 4} \\ & \leq cM^{-j} \|\widehat{W}^l(\beta_2)(u, \mathbf{k}) - \widehat{W}^l(\beta_1)(u, \mathbf{k})\|_{4 \times 4} \\ & + cM^{-j} \|E_{l+1}(\beta_2)(u, \mathbf{k}) - E_{l+1}(\beta_1)(u, \mathbf{k})\|_{4 \times 4} \\ & \leq \min\{cM\alpha^{-2}, c\beta_1^{-\frac{1}{2}}M^{-j}\alpha^{-2}\} < 1. \end{aligned}$$

By taking into account Lemma 7.12 (2) and (7.96) we observe that

$$(7.97) \quad (\text{the right-hand side of (7.95)})$$

$$\begin{aligned}
 &\leq \sum_{j=l}^{N_{\beta_1}} \frac{1}{L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_1}}^{\pi h} du + \int_{-\pi h}^{-\frac{\pi}{\beta_1}} du \right) 1_{\chi_j(u, \mathbf{k}) \neq 0} \\
 &\quad \cdot \sum_{n=1}^{\infty} \frac{1}{n} (\min\{cM\alpha^{-2}, c\beta_1^{-\frac{1}{2}} M^{-j} \alpha^{-2}\})^n \\
 &\leq c(M) \sum_{j=l}^{N_{\beta_1}} f_{\mathbf{t}}^{-1} M^{3j} \min\{cM\alpha^{-2}, c\beta_1^{-\frac{1}{2}} M^{-j} \alpha^{-2}\} \\
 &\leq c(M) \beta_1^{-\frac{1}{2}} f_{\mathbf{t}}^{-1} M^{2l} \alpha^{-2}.
 \end{aligned}$$

Similarly, by using Lemma 7.12 (3) and (7.79) we can derive that

$$\begin{aligned}
 (7.98) \quad &\frac{1}{L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_2}}^{\frac{\pi}{\beta_1}} du + \int_{-\frac{\pi}{\beta_1}}^{-\frac{\pi}{\beta_2}} du \right) \\
 &\quad \cdot \left| \log \left(\det \left(I_4 - (iuI_4 - E(\mathbf{k}) - E_{l+1}(\beta_2)(u, \mathbf{k}))^{-1} \right. \right. \right. \\
 &\quad \quad \left. \left. \left. \cdot \chi_{\leq l}(\beta_2)(u, \mathbf{k}) \widehat{W}^l(\beta_2)(u, \mathbf{k}) \right) \right) \right| \\
 &\leq \frac{1}{L^2} \sum_{\mathbf{k} \in \Gamma^*} \left(\int_{\frac{\pi}{\beta_2}}^{\frac{\pi}{\beta_1}} du + \int_{-\frac{\pi}{\beta_1}}^{-\frac{\pi}{\beta_2}} du \right) 1_{\chi_{\leq l}(\beta_2)(u, \mathbf{k}) \neq 0} \sum_{n=1}^{\infty} \frac{1}{n} (cM\alpha^{-2})^n \\
 &\leq c(M) \beta_1^{-1} f_{\mathbf{t}}^{-1} M^{2l} \alpha^{-2}.
 \end{aligned}$$

Combining (7.22), (7.94), (7.97), (7.98) with (7.93) yields

$$\begin{aligned}
 (7.99) \quad &|J_{end}(\beta_1) - J_{end}^1(\beta_2)| \leq c \sum_{l=0}^{N_{\beta_1}-1} \beta_1^{-\frac{1}{2}} M^{\frac{5}{2}l} \alpha^{-3} \\
 &\quad + \sum_{l=0}^{N_{\beta_1}} c''(M, c_w) \beta_1^{-\frac{1}{2}} f_{\mathbf{t}}^{-1} M^{2l} \alpha^{-2} \\
 &\leq c''(M, c_w) \beta^{-\frac{1}{2}} f_{\mathbf{t}}^{-1} \alpha^{-2}.
 \end{aligned}$$

Finally, by coupling (7.99) with (7.92) we reach the claimed inequality. \square

In order to indicate the dependency on the parameters β, L, h , let us write $J_{end}(\beta, L, h)(\mathbf{U})$ in place of $J_{end}(\mathbf{U})$ in the following. For any compact

set K of \mathbb{C}^4 let $C(K; \mathbb{C})$ denote the Banach space of all complex-valued continuous functions on K , equipped with the uniform norm. To prove the next lemma, we need Lemma D.1 proved in Appendix D.

LEMMA 7.20. *For any non-empty compact set K of \mathbb{C}^4 satisfying $K \subset D_{IR}$ the following statements hold true.*

- (1) *For any $\beta \in \mathbb{R}_{>0}$, $L \in \mathbb{N}$ with $L \geq \beta$, $J_{end}(\beta, L, h)(\cdot)$ converges in $C(K; \mathbb{C})$ as $h \rightarrow \infty$ ($h \in 2\mathbb{N}/\beta$).*
- (2) *Set $J(\beta, L) := \lim_{h \rightarrow \infty, h \in 2\mathbb{N}/\beta} J_{end}(\beta, L, h)$. For any $\beta \in \mathbb{R}_{>0}$, $J(\beta, L)(\cdot)$ converges in $C(K; \mathbb{C})$ as $L \rightarrow \infty$ ($L \in \mathbb{N}$).*
- (3) *Set $J(\beta) := \lim_{L \rightarrow \infty, L \in \mathbb{N}} J(\beta, L)$. $J(\beta)(\cdot)$ converges in $C(K; \mathbb{C})$ as $\beta \rightarrow \infty$ ($\beta \in \mathbb{N}$).*

PROOF. (1),(2): Take any $U_0 \in (0, f_{\mathbf{t}}^2(c'(M, c_w)\alpha^4)^{-1})$ and small $\varepsilon \in \mathbb{R}_{>0}$, where $c'(M, c_w)$ is the constant appearing in the definition of D_{IR} in Lemma 7.18 (1). Set

$$D_\varepsilon := \{(U_1, U_2, U_3, U_4) \in \mathbb{C}^4 \mid |U_\rho| < U_0 - \varepsilon, (\forall \rho \in \mathcal{B})\}.$$

Note that $z\mathbf{U} \in D_{IR}$ for any $\mathbf{U} \in \overline{D_\varepsilon}$ and $z \in \mathbb{C}$ with $|z| \leq U_0/(U_0 - \varepsilon)$. By Lemma 7.19 (1) and Cauchy's integral formula we can justify the following equalities.

$$(7.100) \quad \begin{aligned} J_{end}(\beta, L, h)(\mathbf{U}) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial}{\partial z} \right)^n J_{end}(\beta, L, h)(z\mathbf{U}) \Big|_{z=0} \\ &= \sum_{n=0}^{\infty} a_n(\beta, L, h)(\mathbf{U}), \quad (\forall \mathbf{U} \in \overline{D_\varepsilon}), \end{aligned}$$

where

$$a_n(\beta, L, h)(\mathbf{U}) := \frac{1}{2\pi i} \oint_{|z|=\frac{U_0}{U_0-\varepsilon}} dz \frac{J_{end}(\beta, L, h)(z\mathbf{U})}{z^{n+1}}.$$

By Lemma 7.19 (2),

$$(7.101) \quad |a_n(\beta, L, h)(\mathbf{U})| \leq c(M) f_{\mathbf{t}}^{-1} \alpha^{-2} \left(\frac{U_0 - \varepsilon}{U_0} \right)^n,$$

$$(\forall \mathbf{U} \in \overline{D_\varepsilon}, n \in \mathbb{N} \cup \{0\}).$$

By Lemma 7.18 (3) there exists a constant $c(\beta, L, M) \in \mathbb{R}_{>0}$ depending only on β, L, M such that

$$-\frac{1}{\beta L^2} \log \left(\int e^{J^0(\mathbf{U})(\psi)} d\mu_{C_{\leq 0}^\infty}(\psi) \right) = J_{\text{end}}(\beta, L, h)(\mathbf{U}),$$

$$(\forall \mathbf{U} \in \mathbb{C}^4 \text{ with } |U_\rho| \leq f_{\mathbf{t}}^2(c'(M, c_w)c(\beta, L, M)^4)^{-1}, (\forall \rho \in \mathcal{B})).$$

By this equality and Lemma 2.10 (1) we can choose constants $c_1, h_0 \in \mathbb{R}_{>0}$, which may depend on β, L but are independent of h , so that for any $\mathbf{U} \in \overline{D_\varepsilon}$, $h \in 2\mathbb{N}/\beta$ with $h \geq h_0$,

$$(7.102) \quad a_n(\beta, L, h)(\mathbf{U})$$

$$= \frac{1}{2\pi i} \oint_{|z|=c_1} dz \frac{1}{z^{n+1}} \left(-\frac{1}{\beta L^2} \log \left(\int e^{J^0(z\mathbf{U})(\psi)} d\mu_{C_{\leq 0}^\infty}(\psi) \right) \right.$$

$$\left. + \frac{1}{\beta L^2} \log \left(\int e^{-V(z\mathbf{U})(\psi)} d\mu_C(\psi) \right) \right)$$

$$- \frac{1}{\beta L^2} \frac{1}{n!} \left(\frac{\partial}{\partial z} \right)^n \log \left(\int e^{-V(z\mathbf{U})(\psi)} d\mu_C(\psi) \right) \Big|_{z=0}.$$

By Lemma D.1 proved in Appendix D the last term in the right-hand side of (7.102) uniformly converges with respect to $\mathbf{U} \in \overline{D_\varepsilon}$ as we send h to infinity first, then L to infinity next. Moreover, Lemma 2.10 (2) and Proposition 6.4 (3) imply that

$$\lim_{\substack{h \rightarrow \infty \\ h \in 2\mathbb{N}/\beta}} \sup_{\mathbf{U} \in \overline{D_\varepsilon}} \left| \frac{1}{2\pi i} \oint_{|z|=c_1} dz \frac{1}{z^{n+1}} \left(\frac{1}{\beta L^2} \log \left(\int e^{J^0(z\mathbf{U})(\psi)} d\mu_{C_{\leq 0}^\infty}(\psi) \right) \right. \right.$$

$$\left. \left. - \frac{1}{\beta L^2} \log \left(\int e^{-V(z\mathbf{U})(\psi)} d\mu_C(\psi) \right) \right) \right|$$

$$= 0.$$

Therefore, there exist $\{a_n(\beta, L)\}_{n=0}^\infty, \{a_n(\beta)\}_{n=0}^\infty \subset C(\overline{D_\varepsilon}; \mathbb{C})$ such that

$$(7.103) \quad \lim_{\substack{h \rightarrow \infty \\ h \in 2\mathbb{N}/\beta}} \sup_{\mathbf{U} \in \overline{D_\varepsilon}} |a_n(\beta, L, h)(\mathbf{U}) - a_n(\beta, L)(\mathbf{U})| = 0,$$

$$(7.104) \quad \lim_{\substack{L \rightarrow \infty \\ L \in \mathbb{N}}} \sup_{\mathbf{U} \in \overline{D_\varepsilon}} |a_n(\beta, L)(\mathbf{U}) - a_n(\beta)(\mathbf{U})| = 0, \quad (\forall n \in \mathbb{N} \cup \{0\}).$$

Since the right-hand side of (7.101) is summable with n over $\mathbb{N} \cup \{0\}$, we can apply the dominated convergence theorem for $l^1(\mathbb{N} \cup \{0\})$ together with (7.103), (7.104) to deduce from (7.100) that

$$\begin{aligned} \lim_{\substack{h \rightarrow \infty \\ h \in 2\mathbb{N}/\beta}} \sup_{\mathbf{U} \in \overline{D_\varepsilon}} \left| J_{end}(\beta, L, h)(\mathbf{U}) - \sum_{n=0}^{\infty} a_n(\beta, L)(\mathbf{U}) \right| &= 0, \\ \lim_{\substack{L \rightarrow \infty \\ L \in \mathbb{N}}} \sup_{\mathbf{U} \in \overline{D_\varepsilon}} \left| \sum_{n=0}^{\infty} a_n(\beta, L)(\mathbf{U}) - \sum_{n=0}^{\infty} a_n(\beta)(\mathbf{U}) \right| &= 0. \end{aligned}$$

For any compact set K of \mathbb{C}^4 with $K \subset D_{IR}$ we can choose $U_0 \in (0, f_{\mathbf{t}}^2(c'(M, c_w)\alpha^4)^{-1})$ and $\varepsilon \in \mathbb{R}_{>0}$ so that $K \subset \overline{D_\varepsilon}$. Thus, the claims (1), (2) have been proved.

(3): By sending h and L to infinity in Lemma 7.19 (3) we obtain

$$\sup_{\mathbf{U} \in K} |J(\beta_1)(\mathbf{U}) - J(\beta_2)(\mathbf{U})| \leq c''(M, c_w)\beta_1^{-\frac{1}{2}} f_{\mathbf{t}}^{-1}\alpha^{-2},$$

for any $\beta_1, \beta_2 \in \mathbb{N}$ with $\beta_2 \geq \beta_1$, which implies that $(J(\beta))_{\beta \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $C(K; \mathbb{C})$. Therefore, the claim (3) holds true. \square

Here we can give the proof of Theorem 1.1, admitting lemmas proved in Appendix E.

PROOF OF THEOREM 1.1. First of all let us assume that Theorem 1.1 is true if $\max\{t_{h,e}, t_{h,o}, t_{v,e}, t_{v,o}\} = 1$. Set $t_{max} := \max\{t_{h,e}, t_{h,o}, t_{v,e}, t_{v,o}\}$. By the theorem for the Hamiltonian $\mathbf{H}_0/t_{max} + \mathbf{V}$ there exist a generic constant $c \in \mathbb{R}_{>0}$ and continuous functions $F_{\beta,L}(\cdot), F_\beta(\cdot), F(\cdot) : \overline{D_{\mathbf{t}/t_{max}}(c)}^4 \rightarrow \mathbb{C}$ such that $F_{\beta,L}(\cdot)$ is analytic in $D_{\mathbf{t}/t_{max}}(c)^4$ and

$$\begin{aligned} F_{\beta,L}(\mathbf{U}) &= -\frac{1}{\beta(2L)^2} \log(\text{Tr } e^{-\beta(\mathbf{H}_0/t_{max} + \mathbf{V})}), \\ (\forall \mathbf{U} \in \overline{D_{\mathbf{t}/t_{max}}(c)}^4 \cap \mathbb{R}^4, \beta \in \mathbb{R}_{>0}, L \in \mathbb{N} \text{ with } L \geq \beta), \\ \lim_{\substack{L \rightarrow \infty \\ L \in \mathbb{N}}} \sup_{\mathbf{z} \in \overline{D_{\mathbf{t}/t_{max}}(c)}^4} |F_{\beta,L}(\mathbf{z}) - F_\beta(\mathbf{z})| &= 0, \quad (\forall \beta \in \mathbb{R}_{>0}), \end{aligned}$$

$$\lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{\mathbf{z} \in \overline{D_{\mathbf{t}/t_{max}}(c)}^4} |F_{\beta}(\mathbf{z}) - F(\mathbf{z})| = 0.$$

By replacing β by $t_{max}\beta$ and taking into account the equality $D_{\mathbf{t}/t_{max}}(c) = (1/t_{max})\overline{D_{\mathbf{t}}(c)}$ we have that

$$\begin{aligned} t_{max}F_{t_{max}\beta, L} \left(\frac{1}{t_{max}}\mathbf{U} \right) &= -\frac{1}{\beta(2L)^2} \log(\text{Tr } e^{-\beta\mathbf{H}}), \\ (\forall \mathbf{U} \in \overline{D_{\mathbf{t}}(c)}^4 \cap \mathbb{R}^4, \beta \in \mathbb{R}_{>0}, L \in \mathbb{N} \text{ with } L \geq t_{max}\beta), \\ \lim_{\substack{L \rightarrow \infty \\ L \in \mathbb{N}}} \sup_{\mathbf{z} \in \overline{D_{\mathbf{t}}(c)}^4} \left| t_{max}F_{t_{max}\beta, L} \left(\frac{1}{t_{max}}\mathbf{z} \right) - t_{max}F_{t_{max}\beta} \left(\frac{1}{t_{max}}\mathbf{z} \right) \right| &= 0, \\ (\forall \beta \in \mathbb{R}_{>0}), \\ \lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{\mathbf{z} \in \overline{D_{\mathbf{t}}(c)}^4} \left| t_{max}F_{t_{max}\beta} \left(\frac{1}{t_{max}}\mathbf{z} \right) - t_{max}F \left(\frac{1}{t_{max}}\mathbf{z} \right) \right| &= 0. \end{aligned}$$

Since the functions $F_{t_{max}\beta, L}(\cdot/t_{max})$, $F_{t_{max}\beta}(\cdot/t_{max})$, $F(\cdot/t_{max})$ are continuous in $\overline{D_{\mathbf{t}}(c)}^4$ and $F_{t_{max}\beta, L}(\cdot/t_{max})$ is analytic in $D_{\mathbf{t}}(c)^4$, the claims for the Hamiltonian \mathbf{H} hold true. Therefore, it suffices to prove the theorem on the assumption that $t_{max} = 1$.

In the following we assume that $\beta \in \mathbb{R}_{>0}$, $L \in \mathbb{N}$ satisfies $L \geq \beta$ and the parameters $M, \alpha \in \mathbb{R}_{>0}$ satisfy the conditions required in Lemma 7.18 and Lemma 7.19. By Lemma 7.20 there exist functions $J(\beta, L)(\cdot)$, $J(\beta)(\cdot)$, $J(\cdot) : D_{IR} \rightarrow \mathbb{C}$ such that

$$\begin{aligned} J(\beta, L)(\mathbf{U}) &= \lim_{\substack{h \rightarrow \infty \\ h \in 2\mathbb{N}/\beta}} J_{end}(\beta, L, h)(\mathbf{U}), \\ J(\beta)(\mathbf{U}) &= \lim_{L \in \mathbb{N}} \lim_{\substack{L \rightarrow \infty \\ h \in 2\mathbb{N}/\beta}} J_{end}(\beta, L, h)(\mathbf{U}), \\ J(\mathbf{U}) &= \lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{N}}} \lim_{L \in \mathbb{N}} \lim_{\substack{L \rightarrow \infty \\ h \in 2\mathbb{N}/\beta}} J_{end}(\beta, L, h)(\mathbf{U}), \quad (\forall \mathbf{U} \in D_{IR}). \end{aligned}$$

(1): By Lemma 2.2, Lemma 2.10, Proposition 6.4 (3) and Lemma 7.18 (3), there exists a constant $c_1 \in \mathbb{R}_{>0}$ which may depend on β, L but not on h such that for any $\mathbf{U} \in \mathbb{R}^4$ with $|U_{\rho}| \leq c_1$ ($\forall \rho \in \mathcal{B}$),

$$(7.105) \quad J(\beta, L)(\mathbf{U})$$

$$\begin{aligned}
 &= \lim_{\substack{h \rightarrow \infty \\ h \in 2\mathbb{N}/\beta}} \left(-\frac{1}{\beta L^2} \log \left(\int e^{J^0(\psi)} d\mu_{C_{\leq 0}^\infty}(\psi) \right) \right. \\
 &\quad \left. + \frac{1}{\beta L^2} \log \left(\int e^{-V(\psi)} d\mu_C(\psi) \right) \right) \\
 &\quad + \lim_{\substack{h \rightarrow \infty \\ h \in 2\mathbb{N}/\beta}} \left(-\frac{1}{\beta L^2} \log \left(\int e^{-V(\psi)} d\mu_C(\psi) \right) \right) \\
 &= -\frac{1}{\beta L^2} \log \left(\frac{\text{Tr } e^{-\beta H}}{\text{Tr } e^{-\beta H_0}} \right).
 \end{aligned}$$

By Lemma 7.19 (1) and Lemma 7.20 (1), $\mathbf{U} \mapsto J(\beta, L)(\mathbf{U})$ is analytic in D_{IR} . On the other hand, by Lemma E.3 proved in Appendix E there exists a domain $O \subset \mathbb{C}^4$ such that $D_{IR} \cap \mathbb{R}^4 \subset O$ and $\mathbf{U} \mapsto \log(\text{Tr } e^{-\beta H})$ is analytic in O . Therefore, by the identity theorem we obtain that

$$(7.106) \quad J(\beta, L)(\mathbf{U}) = -\frac{1}{\beta L^2} \log \left(\frac{\text{Tr } e^{-\beta H}}{\text{Tr } e^{-\beta H_0}} \right), \quad (\forall \mathbf{U} \in D_{IR} \cap \mathbb{R}^4),$$

or by Lemma 7.1 and Lemma E.1 proved in Appendix E,

$$\begin{aligned}
 (7.107) \quad & -\frac{1}{\beta(2L)^2} \log(\text{Tr } e^{-\beta H}) \\
 &= \frac{1}{4} J(\beta, L)(\mathbf{U}) - \frac{1}{2\beta L^2} \sum_{\mathbf{k} \in \Gamma^*} \sum_{p, q \in \{1, -1\}} \log(1 + e^{-\beta X_{p, q}(\mathbf{k})}), \\
 & (\forall \mathbf{U} \in D_{IR} \cap \mathbb{R}^4),
 \end{aligned}$$

where $X_{p, q}(\mathbf{k})$ ($p, q \in \{1, -1\}$) are the eigen values of $E(\mathbf{k})$ given in (7.5). The equality (7.107) implies that the claim (1) holds for $D_{\mathbf{t}}(c')$ with any $c' \in (0, (c'(M, c_w)\alpha^4)^{-1})$, where $c'(M, c_w) \in \mathbb{R}_{>0}$ is the M, c_w -dependent constant appearing in the definition of D_{IR} in Lemma 7.18 (1). In the following we fix c' to be $(2c'(M, c_w)\alpha^4)^{-1}$ so that $\overline{D_{\mathbf{t}}(c')}^4 \subset D_{IR}$.

(2): Set

$$\begin{aligned}
 F_{\beta, L}(\mathbf{U}) &:= \frac{1}{4} J(\beta, L)(\mathbf{U}) - \frac{1}{2\beta L^2} \sum_{\mathbf{k} \in \Gamma^*} \sum_{p, q \in \{1, -1\}} \log(1 + e^{-\beta X_{p, q}(\mathbf{k})}), \\
 F_{\beta}(\mathbf{U}) &:= \frac{1}{4} J(\beta)(\mathbf{U}) - \frac{1}{2\beta(2\pi)^2} \sum_{p, q \in \{1, -1\}} \int_{[-\pi, \pi]^2} d\mathbf{k} \log(1 + e^{-\beta X_{p, q}(\mathbf{k})}),
 \end{aligned}$$

$(\forall \mathbf{U} \in D_{IR}).$

We can see from Lemma 7.20 (2) and the continuity of the function $\mathbf{k} \mapsto X_{p,q}(\mathbf{k})$ that

$$\lim_{\substack{L \rightarrow \infty \\ L \in \mathbb{N}}} \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')^4}} |F_{\beta,L}(\mathbf{U}) - F_{\beta}(\mathbf{U})| = 0, \quad (\forall \beta \in \mathbb{R}_{>0}).$$

Thus, the claim (2) is true.

(3): By Lemma E.2 proved in Appendix E, Lemma 7.19 (2), (7.106) and continuity,

$$\begin{aligned} (7.108) \quad & \left| \frac{1}{\beta L^2} \log \left(\frac{\text{Tr } e^{-\beta H}}{\text{Tr } e^{-\beta H_0}} \right) - \frac{1}{[\beta] L^2} \log \left(\frac{\text{Tr } e^{-[\beta] H}}{\text{Tr } e^{-[\beta] H_0}} \right) \right| \\ & \leq \int_{[\beta]}^{\beta} d\gamma \left| \frac{1}{\gamma^2 L^2} \left(\frac{\text{Tr } e^{-\gamma H}}{\text{Tr } e^{-\gamma H_0}} \right) \right| \\ & \quad + c \left(1 + \sup_{\mathbf{U} \in \overline{D_{IR}}} \|\mathbf{U}\|_{\mathbb{C}^4} \right) \log \left(\frac{\beta}{[\beta]} \right) \\ & \leq c \left(c(M) f_{\mathbf{t}}^{-1} \alpha^{-2} + 1 + \sup_{\mathbf{U} \in \overline{D_{IR}}} \|\mathbf{U}\|_{\mathbb{C}^4} \right) \log \left(\frac{\beta}{[\beta]} \right), \\ & (\forall \mathbf{U} \in \overline{D_{IR}} \cap \mathbb{R}^4), \end{aligned}$$

where $c(M) \in \mathbb{R}_{>0}$ is the M -dependent constant appearing in Lemma 7.19 (2). Set

$$\begin{aligned} F(\mathbf{U}) &:= \frac{1}{4} J(\mathbf{U}) + \frac{1}{2(2\pi)^2} \sum_{p,q \in \{1,-1\}} \int_{[-\pi,\pi]^2} d\mathbf{k} \mathbf{1}_{X_{p,q}(\mathbf{k}) < 0} X_{p,q}(\mathbf{k}), \\ & (\forall \mathbf{U} \in D_{IR}). \end{aligned}$$

Lemma 7.20 implies that the function $\mathbf{U} \mapsto F(\mathbf{U})$ is analytic in D_{IR} . We can derive from (7.106) and (7.108) that for any $\beta \in \mathbb{R}_{\geq 1}$,

$$\begin{aligned} & \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')^4} \cap \mathbb{R}^4} |F_{\beta}(\mathbf{U}) - F(\mathbf{U})| \\ & \leq c \left(c(M) f_{\mathbf{t}}^{-1} \alpha^{-2} + 1 + \sup_{\mathbf{U} \in \overline{D_{IR}}} \|\mathbf{U}\|_{\mathbb{C}^4} \right) \log \left(\frac{\beta}{[\beta]} \right) \end{aligned}$$

$$\begin{aligned}
 & + \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')^4}} |J([\beta])(\mathbf{U}) - J(\mathbf{U})| \\
 & + \left| -\frac{1}{2\beta(2\pi)^2} \sum_{p,q \in \{1,-1\}} \int_{[-\pi,\pi]^2} d\mathbf{k} \log(1 + e^{-\beta X_{p,q}(\mathbf{k})}) \right. \\
 & \quad \left. - \frac{1}{2(2\pi)^2} \sum_{p,q \in \{1,-1\}} \int_{[-\pi,\pi]^2} d\mathbf{k} 1_{X_{p,q}(\mathbf{k}) < 0} X_{p,q}(\mathbf{k}) \right|.
 \end{aligned}$$

Thus, by Lemma 7.20 (3),

$$(7.109) \quad \lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')^4} \cap \mathbb{R}^4} |F_\beta(\mathbf{U}) - F(\mathbf{U})| = 0.$$

To complete the proof, we need to show that

$$(7.110) \quad \lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')^4}} |F_\beta(\mathbf{U}) - F(\mathbf{U})| = 0.$$

The convergence property (7.110) can be proved by a basic argument. However, we present the proof for completeness. Note that by Lemma 7.19 (2),

$$(7.111) \quad \sup_{\mathbf{U} \in D_{IR}} |F_\beta(\mathbf{U})| \leq \tilde{c}, \quad (\forall \beta \in \mathbb{R}_{\geq 1}), \quad \sup_{\mathbf{U} \in D_{IR}} |F(\mathbf{U})| \leq \tilde{c},$$

with some constant $\tilde{c} (\in \mathbb{R}_{>0})$ independent of β . For any $j \in \{1, 2, 3, 4\}$, $n \in \mathbb{N} \cup \{0\}$, $\mathbf{U} \in \overline{D_{\mathbf{t}}(c')^4}$, set

$$a_{\beta,j,n}(\mathbf{U}) := \frac{1}{n!} \left(\frac{\partial}{\partial U_j} \right)^n F_\beta(\mathbf{U}), \quad a_{j,n}(\mathbf{U}) := \frac{1}{n!} \left(\frac{\partial}{\partial U_j} \right)^n F(\mathbf{U}).$$

Since $F_\beta(\cdot)$, $F(\cdot)$ are analytic in D_{IR} ,

$$\begin{aligned}
 (7.112) \quad F_\beta(\mathbf{U}) &= \sum_{n=0}^{\infty} a_{\beta,j,n}(U_1, \dots, U_{j-1}, 0, U_{j+1}, \dots, U_4) U_j^n, \\
 F(\mathbf{U}) &= \sum_{n=0}^{\infty} a_{j,n}(U_1, \dots, U_{j-1}, 0, U_{j+1}, \dots, U_4) U_j^n, \\
 & (\forall j \in \{1, 2, 3, 4\}, \mathbf{U} \in \overline{D_{\mathbf{t}}(c')^4}, \beta \in \mathbb{R}_{\geq 1}).
 \end{aligned}$$

Moreover, by (7.111) Cauchy's integral formula gives that

$$(7.113) \quad \begin{aligned} |a_{\beta,j,n}(U_1, \dots, U_{j-1}, 0, U_{j+1}, \dots, U_4)| &\leq \tilde{c} \left(\frac{3}{2} c' f_{\mathbf{t}}^2 \right)^{-n}, \\ |a_{j,n}(U_1, \dots, U_{j-1}, 0, U_{j+1}, \dots, U_4)| &\leq \tilde{c} \left(\frac{3}{2} c' f_{\mathbf{t}}^2 \right)^{-n}, \\ (\forall j \in \{1, 2, 3, 4\}, n \in \mathbb{N} \cup \{0\}, \\ (U_1, \dots, U_{j-1}, U_{j+1}, \dots, U_4) &\in \overline{D_{\mathbf{t}}(c')}^3, \beta \in \mathbb{R}_{\geq 1}). \end{aligned}$$

Let us prove that

$$(7.114) \quad \lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')}^3 \cap \mathbb{R}^3} |a_{\beta,1,n}(0, \mathbf{U}) - a_{1,n}(0, \mathbf{U})| = 0, \quad (\forall n \in \mathbb{N} \cup \{0\})$$

by induction with n . By (7.109),

$$\begin{aligned} &\lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')}^3 \cap \mathbb{R}^3} |a_{\beta,1,0}(0, \mathbf{U}) - a_{1,0}(0, \mathbf{U})| \\ &= \lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')}^3 \cap \mathbb{R}^3} |F_{\beta}(0, \mathbf{U}) - F(0, \mathbf{U})| = 0. \end{aligned}$$

Next, let us assume that there exists $m \in \mathbb{N} \cup \{0\}$ such that (7.114) holds for all $n \in \mathbb{N} \cup \{0\}$ with $n \leq m$. Suppose that there exist $\delta \in \mathbb{R}_{>0}$ and a sequence $(\beta_l)_{l=1}^{\infty}$ such that $\beta_l \rightarrow \infty$ as $l \rightarrow \infty$ and

$$\sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')}^3 \cap \mathbb{R}^3} |a_{\beta_l,1,m+1}(0, \mathbf{U}) - a_{1,m+1}(0, \mathbf{U})| \geq \delta, \quad (\forall l \in \mathbb{N}).$$

Then, we can see from (7.112), (7.113) that

$$\begin{aligned} \delta |U|^{m+1} &\leq \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')}^3 \cap \mathbb{R}^3} |\overline{a_{\beta_l,1,m+1}}(0, \mathbf{U}) - a_{1,m+1}(0, \mathbf{U})| |U|^{m+1} \\ &\leq \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')}^3 \cap \mathbb{R}^3} |F_{\beta_l}(|U|, \mathbf{U}) - F(|U|, \mathbf{U})| \\ &\quad + \sum_{n=0}^m \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')}^3 \cap \mathbb{R}^3} |a_{\beta_l,1,n}(0, \mathbf{U}) - a_{1,n}(0, \mathbf{U})| |U|^n \\ &\quad + 2\tilde{c} \sum_{n=m+2}^{\infty} \left(\frac{3}{2} c' f_{\mathbf{t}}^2 \right)^{-n} |U|^n, \quad (\forall \mathbf{U} \in \overline{D_{\mathbf{t}}(c')}). \end{aligned}$$

By (7.109) and the induction hypothesis the first term and the second term in the right-hand side of the above inequality converge to 0 as $l \rightarrow \infty$. Then, by dividing both sides by $|U|^{m+1}$ we obtain that

$$\begin{aligned} \delta &\leq 2\tilde{c} \sum_{n=m+2}^{\infty} \left(\frac{3}{2}c'f_{\mathbf{t}}^2\right)^{-n} |U|^{n-m-1} \\ &\leq 2\tilde{c}|U| \left(\frac{3}{2}c'f_{\mathbf{t}}^2\right)^{-m-2} \sum_{n=m+2}^{\infty} \left(\frac{3}{2}\right)^{-n+m+2}, \quad (\forall U \in \overline{D_{\mathbf{t}}(c')}\setminus\{0\}). \end{aligned}$$

Sending U to 0 yields $\delta \leq 0$, which is a contradiction. Thus,

$$\lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')^3} \cap \mathbb{R}^3} |a_{\beta,1,m+1}(0, \mathbf{U}) - a_{1,m+1}(0, \mathbf{U})| = 0.$$

By induction, the convergence property (7.114) holds true.

It follows from (7.112), (7.113), (7.114) and the dominated convergence theorem that

$$(7.115) \quad \lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{\substack{U \in \overline{D_{\mathbf{t}}(c')} \\ \mathbf{U} \in \overline{D_{\mathbf{t}}(c')^3} \cap \mathbb{R}^3}} |F_{\beta}(U, \mathbf{U}) - F(U, \mathbf{U})| = 0.$$

By using (7.115) in place of (7.109) in an inductive argument parallel to that above we can prove that

$$(7.116) \quad \lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{\substack{U \in \overline{D_{\mathbf{t}}(c')} \\ \mathbf{U} \in \overline{D_{\mathbf{t}}(c')^2} \cap \mathbb{R}^2}} |a_{\beta,2,n}(U, 0, \mathbf{U}) - a_{2,n}(U, 0, \mathbf{U})| = 0, \\ (\forall n \in \mathbb{N} \cup \{0\}).$$

Then, combination of (7.112), (7.113), (7.116) and the dominated convergence theorem concludes that

$$\lim_{\substack{\beta \rightarrow \infty \\ \beta \in \mathbb{R}_{>0}}} \sup_{\substack{\mathbf{U} \in \overline{D_{\mathbf{t}}(c')^2} \\ \mathbf{U}' \in \overline{D_{\mathbf{t}}(c')^2} \cap \mathbb{R}^2}} |F_{\beta}(\mathbf{U}, \mathbf{U}') - F(\mathbf{U}, \mathbf{U}')| = 0.$$

By repeating this argument twice more we reach (7.110). The proof of the theorem is complete. \square

Appendix A. The Flux Phase Problem

A sufficient condition to be a minimizer of the flux phase problem for the half-filled Hubbard model on a square lattice was essentially given by Lieb in [15]. In order to support readers' verification of Corollary 1.2, here we state Lieb's theorem with some supplementary arguments concerning the repeated reflection, which are not explicit in the short letter [15]. Since the proof below is based on the proved lemmas [15, Lemma] and [16, Lemma 2.1], it will present no more than a review to readers who are already familiar with this subject. Apart from the condition on the flux per plaquette, we need a condition on the flux through the large circles around the periodic lattice in order to define a Hamiltonian minimizing the free energy under the periodic boundary condition. The sufficiency of these conditions was discussed by Macris and Nachtergaele in [17]. The statement of the theorem below is also implied by [17, Theorem 1.4, Remarks (a)], which provides a generalized version of Lieb's theorem with an alternative proof. The proof below merely uses the original key lemmas [15, Lemma] and [16, Lemma 2.1].

First let us consider the problem in a general setting. Assume that the hopping amplitude $t(\cdot, \cdot) : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}_{\geq 0}$ and the magnitude of on-site interaction $U(\cdot) : \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfy the following.

$$\begin{aligned} t(\mathbf{x}, \mathbf{y}) &= t(\mathbf{y}, \mathbf{x}) = t(\mathbf{x} + 2mL\mathbf{e}_1 + 2nL\mathbf{e}_2, \mathbf{y}), \\ t(\mathbf{x}, \mathbf{y}) &= 0 \text{ if } \mathbf{x} - \mathbf{y} \neq \mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2 \text{ in } (\mathbb{Z}/2L\mathbb{Z})^2, \\ U(\mathbf{x}) &= U(\mathbf{x} + 2mL\mathbf{e}_1 + 2nL\mathbf{e}_2), \quad (\forall \mathbf{x}, \mathbf{y} \in \mathbb{Z}^2, m, n \in \mathbb{Z}). \end{aligned}$$

We minimize the free energy with respect to the argument of the hopping matrix elements. The argument is represented by a function $\phi(\cdot, \cdot) : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying that

$$\begin{aligned} \text{(A.1)} \quad \phi(\mathbf{x}, \mathbf{y}) &= -\phi(\mathbf{y}, \mathbf{x}) \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \\ \phi(\mathbf{x} + 2mL\mathbf{e}_1 + 2nL\mathbf{e}_2, \mathbf{y}) &= \phi(\mathbf{x}, \mathbf{y}) \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{Z}^2, m, n \in \mathbb{Z}). \end{aligned}$$

With ϕ satisfying (A.1), define the Hamiltonian $H(\phi)$ on $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$ by

$$H(\phi) := \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)} t(\mathbf{x}, \mathbf{y}) e^{i\phi(\mathbf{x}, \mathbf{y})} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma}$$

$$+ \sum_{\mathbf{x} \in \Gamma(2L)} U(\mathbf{x}) \left(\psi_{\mathbf{x}\uparrow}^* \psi_{\mathbf{x}\downarrow}^* \psi_{\mathbf{x}\downarrow} \psi_{\mathbf{x}\uparrow} - \frac{1}{2} \sum_{\sigma \in \{\uparrow, \downarrow\}} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x}\sigma} \right).$$

The flux phase problem is to find a phase ϕ satisfying (A.1) such that

$$-\frac{1}{\beta} \log(\text{Tr } e^{-\beta H(\phi)}) = \min_{\substack{\eta: \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R} \\ \text{satisfying (A.1)}}} \left(-\frac{1}{\beta} \log(\text{Tr } e^{-\beta H(\eta)}) \right).$$

For a phase ϕ , the flux per plaquette $f_p(\phi)(\cdot) : \mathbb{Z}^2 \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} f_p(\phi)(\mathbf{x}) := & \phi(\mathbf{x} + \mathbf{e}_1, \mathbf{x}) + \phi(\mathbf{x} + \mathbf{e}_1 + \mathbf{e}_2, \mathbf{x} + \mathbf{e}_1) \\ & + \phi(\mathbf{x} + \mathbf{e}_2, \mathbf{x} + \mathbf{e}_1 + \mathbf{e}_2) + \phi(\mathbf{x}, \mathbf{x} + \mathbf{e}_2), \quad (\forall \mathbf{x} \in \mathbb{Z}^2). \end{aligned}$$

The flux through horizontal circle $f_h(\phi)(\cdot) : \mathbb{Z} \rightarrow \mathbb{R}$ and the flux through vertical circle $f_v(\phi)(\cdot) : \mathbb{Z} \rightarrow \mathbb{R}$ are defined by

$$\begin{aligned} f_h(\phi)(x) &:= \sum_{j=0}^{2L-1} \phi((j+1, x), (j, x)), \\ f_v(\phi)(x) &:= \sum_{j=0}^{2L-1} \phi((x, j+1), (x, j)), \quad (\forall x \in \mathbb{Z}). \end{aligned}$$

Before stating the theorem, let us confirm the fact that the free energy depends on a phase ϕ only through the flux $f_p(\phi)(\cdot)$, $f_h(\phi)(\cdot)$, $f_v(\phi)(\cdot)$. Assume that phases $\phi_1(\cdot, \cdot)$, $\phi_2(\cdot, \cdot) : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfy (A.1) and

$$\begin{aligned} f_p(\phi_1)(\mathbf{x}) &= f_p(\phi_2)(\mathbf{x}) \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall \mathbf{x} \in \mathbb{Z}^2), \\ f_h(\phi_1)(x) &= f_h(\phi_2)(x) \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \\ f_v(\phi_1)(x) &= f_v(\phi_2)(x) \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall x \in \mathbb{Z}). \end{aligned}$$

Our aim is to prove that $\text{Tr } e^{-\beta H(\phi_1)} = \text{Tr } e^{-\beta H(\phi_2)}$. To reach the conclusion, let us follow a few lemmas. In the following let $\|\cdot\|_{\mathbb{R}^2}$ denote the euclidean norm of \mathbb{R}^2 and $(\phi_1 - \phi_2)(\mathbf{x}, \mathbf{y})$ denote $\phi_1(\mathbf{x}, \mathbf{y}) - \phi_2(\mathbf{x}, \mathbf{y})$.

LEMMA A.1. *Assume that $n \geq 2$, $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in \Gamma(2L)$, $\|\mathbf{x}_j - \mathbf{x}_{j+1}\|_{\mathbb{R}^2} = 1$ ($j = 1, 2, \dots, n-2$), $\|\mathbf{x}_{n-1} - \mathbf{x}_n\|_{\mathbb{R}^2} = 2L-1$ and $\mathbf{x}_{n-1} - \mathbf{x}_n = \pm \mathbf{e}_1, \pm \mathbf{e}_2$ in $(\mathbb{Z}/2L\mathbb{Z})^2$. Then, there exist $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_m \in$*

$\Gamma(2L)$ such that $\mathbf{y}_1 = \mathbf{x}_1$, $\mathbf{y}_m = \mathbf{x}_n$, $\|\mathbf{y}_j - \mathbf{y}_{j+1}\|_{\mathbb{R}^2} = 1$ ($j = 1, 2, \dots, m-1$) and

$$\sum_{j=1}^{m-1} (\phi_1 - \phi_2)(\mathbf{y}_{j+1}, \mathbf{y}_j) = \sum_{j=1}^{n-1} (\phi_1 - \phi_2)(\mathbf{x}_{j+1}, \mathbf{x}_j) \text{ in } \mathbb{R}/2\pi\mathbb{Z}.$$

PROOF. We can choose $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_m \in \Gamma(2L)$ satisfying that $\mathbf{y}_1 = \mathbf{x}_1$, $\mathbf{y}_m = \mathbf{x}_n$, $\|\mathbf{y}_j - \mathbf{y}_{j+1}\|_{\mathbb{R}^2} = 1$ ($j = 1, 2, \dots, m-1$). Then, by using the equality $f_p(\phi_1 - \phi_2)(\mathbf{x}) = 0$ ($\mathbf{x} \in \mathbb{Z}^2$) repeatedly and either $f_h(\phi_1 - \phi_2)(x) = 0$ ($x \in \mathbb{Z}$) or $f_v(\phi_1 - \phi_2)(x) = 0$ ($x \in \mathbb{Z}$) only once we can deduce that

$$\begin{aligned} & (\phi_1 - \phi_2)(\mathbf{y}_{m-1}, \mathbf{y}_m) + (\phi_1 - \phi_2)(\mathbf{y}_{m-2}, \mathbf{y}_{m-1}) + \dots + (\phi_1 - \phi_2)(\mathbf{y}_1, \mathbf{y}_2) \\ & + \sum_{j=1}^{n-1} (\phi_1 - \phi_2)(\mathbf{x}_{j+1}, \mathbf{x}_j) = 0 \text{ in } \mathbb{R}/2\pi\mathbb{Z}. \quad \square \end{aligned}$$

LEMMA A.2. Assume that $n \geq 2$, $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in \Gamma(2L)$ and $\mathbf{x}_j - \mathbf{x}_{j+1} = \pm \mathbf{e}_1, \pm \mathbf{e}_2$ in $(\mathbb{Z}/2L\mathbb{Z})^2$ ($j = 1, 2, \dots, n$) where $\mathbf{x}_{n+1} := \mathbf{x}_1$. Then,

$$\sum_{j=1}^n (\phi_1 - \phi_2)(\mathbf{x}_{j+1}, \mathbf{x}_j) = 0 \text{ in } \mathbb{R}/2\pi\mathbb{Z}.$$

PROOF. If $\|\mathbf{x}_j - \mathbf{x}_{j+1}\|_{\mathbb{R}^2} = 1$ ($j = 1, 2, \dots, n$), we can prove the claimed equality only by using that $f_p(\phi_1 - \phi_2)(\mathbf{x}) = 0$ ($\mathbf{x} \in \mathbb{Z}^2$). Let us consider the case that there are $j_1, j_2, \dots, j_l \in \{1, 2, \dots, n\}$ such that $j_1 < j_2 < \dots < j_l$, $\|\mathbf{x}_{j_i} - \mathbf{x}_{j_i+1}\|_{\mathbb{R}^2} = 2L - 1$ ($i = 1, 2, \dots, l$) and $\|\mathbf{x}_j - \mathbf{x}_{j+1}\|_{\mathbb{R}^2} = 1$ ($\forall j \in \{1, 2, \dots, n\} \setminus \{j_1, j_2, \dots, j_l\}$). By Lemma A.1 there exist $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{m_1} \in \Gamma(2L)$ such that $\mathbf{y}_1 = \mathbf{x}_1$, $\mathbf{y}_{m_1} = \mathbf{x}_{j_1+1}$, $\|\mathbf{y}_j - \mathbf{y}_{j+1}\|_{\mathbb{R}^2} = 1$ ($j = 1, 2, \dots, m_1 - 1$) and

$$\sum_{j=1}^{m_1-1} (\phi_1 - \phi_2)(\mathbf{y}_{j+1}, \mathbf{y}_j) = \sum_{j=1}^{j_1} (\phi_1 - \phi_2)(\mathbf{x}_{j+1}, \mathbf{x}_j) \text{ in } \mathbb{R}/2\pi\mathbb{Z}.$$

Then, we can apply Lemma A.1 to the sequence $\mathbf{y}_1, \dots, \mathbf{y}_{m_1}, \mathbf{x}_{j_1+2}, \dots, \mathbf{x}_{j_2}, \mathbf{x}_{j_2+1}$. By repeating this procedure we conclude that there are $\mathbf{z}_1, \mathbf{z}_2,$

$\dots, \mathbf{z}_m \in \Gamma(2L)$ such that $\|\mathbf{z}_j - \mathbf{z}_{j+1}\|_{\mathbb{R}^2} = 1$ ($j = 1, 2, \dots, m$), $\mathbf{z}_{m+1} = \mathbf{z}_1$ and

$$\sum_{j=1}^m (\phi_1 - \phi_2)(\mathbf{z}_{j+1}, \mathbf{z}_j) = \sum_{j=1}^n (\phi_1 - \phi_2)(\mathbf{x}_{j+1}, \mathbf{x}_j) \text{ in } \mathbb{R}/2\pi\mathbb{Z}.$$

We have already seen that the left-hand side of the above equality is 0 (mod 2π) in the beginning. \square

The idea of the next lemma is essentially the same as [16, Lemma 2.1].

LEMMA A.3 ([16, Lemma 2.1]). *There exists a function $\theta(\cdot) : \Gamma(2L) \rightarrow \mathbb{R}$ such that for any $\mathbf{x}, \mathbf{y} \in \Gamma(2L)$ with $\mathbf{x} - \mathbf{y} = \pm \mathbf{e}_1, \pm \mathbf{e}_2$ in $(\mathbb{Z}/2L\mathbb{Z})^2$,*

$$\phi_1(\mathbf{x}, \mathbf{y}) = \phi_2(\mathbf{x}, \mathbf{y}) + \theta(\mathbf{x}) - \theta(\mathbf{y}) \text{ in } \mathbb{R}/2\pi\mathbb{Z}.$$

PROOF. For any $(x, y) \in \Gamma(2L)$ set

$$\begin{aligned} \theta((x, y)) := & 1_{x \geq 1} \sum_{j=0}^{x-1} (\phi_1 - \phi_2)((j+1, 0), (j, 0)) \\ & + 1_{y \geq 1} \sum_{j=0}^{y-1} (\phi_1 - \phi_2)((x, j+1), (x, j)). \end{aligned}$$

Then, it follows from Lemma A.2 that

$$\theta(\mathbf{x}) + (\phi_1 - \phi_2)(\mathbf{y}, \mathbf{x}) - \theta(\mathbf{y}) = 0 \text{ in } \mathbb{R}/2\pi\mathbb{Z}. \quad \square$$

LEMMA A.4.

$$\text{Tr } e^{-\beta H(\phi_1)} = \text{Tr } e^{-\beta H(\phi_2)}.$$

PROOF. Let $\theta(\cdot) : \Gamma(2L) \rightarrow \mathbb{R}$ be the function introduced in Lemma A.3. We can extend the domain of $\theta(\cdot)$ to \mathbb{Z}^2 so that

$$\theta(\mathbf{x} + 2mL\mathbf{e}_1 + 2nL\mathbf{e}_2) = \theta(\mathbf{x}), \quad (\forall \mathbf{x} \in \mathbb{Z}^2, m, n \in \mathbb{Z}),$$

$$e^{i\phi_1(\mathbf{x}, \mathbf{y})} = e^{i(\phi_2(\mathbf{x}, \mathbf{y}) + \theta(\mathbf{x}) - \theta(\mathbf{y}))},$$

$$(\forall \mathbf{x}, \mathbf{y} \in \mathbb{Z}^2 \text{ with } \mathbf{x} - \mathbf{y} = \pm \mathbf{e}_1, \pm \mathbf{e}_2 \text{ in } (\mathbb{Z}/2L\mathbb{Z})^2).$$

Let us define the transform B on $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$ by

$$B\Omega_{2L} := \Omega_{2L},$$

$$B(\psi_{\mathbf{x}_1\sigma_1}^* \psi_{\mathbf{x}_2\sigma_2}^* \cdots \psi_{\mathbf{x}_n\sigma_n}^* \Omega_{2L}) = e^{i\sum_{j=1}^n \theta(\mathbf{x}_j)} \psi_{\mathbf{x}_1\sigma_1}^* \psi_{\mathbf{x}_2\sigma_2}^* \cdots \psi_{\mathbf{x}_n\sigma_n}^* \Omega_{2L},$$

$$(\forall (\mathbf{x}_j, \sigma_j) \in \Gamma(2L) \times \{\uparrow, \downarrow\} \ (j = 1, 2, \dots, n))$$

and by linearity. The transform B is unitary and satisfies the equality $BH(\phi_2)B^* = H(\phi_1)$, which yields that $\text{Tr } e^{-\beta H(\phi_1)} = \text{Tr } e^{-\beta H(\phi_2)}$. \square

The following theorem was essentially proved in [15].

THEOREM A.5 ([15]). *Assume that $U \in \mathbb{R}$, $t_h, t_v \in \mathbb{R}_{>0}$ and*

$$t(\mathbf{x}, \mathbf{y}) = \begin{cases} t_h & \text{if } \mathbf{x} - \mathbf{y} = \mathbf{e}_1, -\mathbf{e}_1 \text{ in } (\mathbb{Z}/2L\mathbb{Z})^2, \\ t_v & \text{if } \mathbf{x} - \mathbf{y} = \mathbf{e}_2, -\mathbf{e}_2 \text{ in } (\mathbb{Z}/2L\mathbb{Z})^2, \\ 0 & \text{otherwise,} \end{cases}$$

$$U(\mathbf{x}) = U, \ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{Z}^2).$$

Moreover, assume that $\phi : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfies (A.1) and

$$(A.2) \quad \begin{aligned} f_p(\phi)(\mathbf{x}) &= \pi \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \ (\forall \mathbf{x} \in \mathbb{Z}^2), \\ f_h(\phi)(x) &= f_v(\phi)(x) = \pi(L - 1) \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \ (\forall x \in \mathbb{Z}). \end{aligned}$$

Then,

$$-\frac{1}{\beta} \log(\text{Tr } e^{-\beta H(\phi)}) = \min_{\substack{\eta: \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R} \\ \text{satisfying (A.1)}}} \left(-\frac{1}{\beta} \log(\text{Tr } e^{-\beta H(\eta)}) \right).$$

PROOF. Lemma A.4 implies that if there exists a minimizer $\phi : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying (A.1) and (A.2), then any $\eta : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying (A.1) and (A.2) is also a minimizer. Thus, it is sufficient to prove the existence of a minimizer satisfying (A.2).

Since this is a minimization problem of a continuous function defined on $[0, 2\pi]^{2(2L)^2}$, a minimizer exists. Assume that $\phi(\cdot, \cdot) : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfies (A.1) and gives the minimum. Set

$$\Gamma(2L)_a := \{(x_1, x_2) \in \Gamma(2L) \mid 1 \leq x_2 \leq L\},$$

$$\Gamma(2L)_b := \Gamma(2L) \setminus \Gamma(2L)_a.$$

Define $\theta(\cdot) : \mathbb{Z}^2 \rightarrow \mathbb{R}$ by

$$\theta((x, y)) := \begin{cases} -\phi((x, y), (x, y) + \mathbf{e}_2) & \text{if } y = 0 \text{ or } L \text{ in } \mathbb{Z}/2L\mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\forall (x, y) \in \mathbb{Z}^2).$$

Then, define the transform τ on $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$ by

$$\begin{aligned} & \tau(\psi_{\mathbf{x}_1\sigma_1}^* \psi_{\mathbf{x}_2\sigma_2}^* \cdots \psi_{\mathbf{x}_n\sigma_n}^* \Omega_{2L}) \\ & := e^{i(\theta(\mathbf{x}_1) + \theta(\mathbf{x}_2) + \cdots + \theta(\mathbf{x}_n))} \psi_{\mathbf{x}_1\sigma_1}^* \psi_{\mathbf{x}_2\sigma_2}^* \cdots \psi_{\mathbf{x}_n\sigma_n}^* \Omega_{2L}, \\ & (\forall (\mathbf{x}_j, \sigma_j) \in \Gamma(2L) \times \{\uparrow, \downarrow\} \ (j = 1, 2, \dots, n)), \end{aligned}$$

and by linearity. We can see that τ is unitary, $\tau H(\phi) \tau^* = H(\phi')$ with $\phi' : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying (A.1) and that if $y = 0$ or L in $\mathbb{Z}/2L\mathbb{Z}$,

$$(A.3) \quad \phi'((x, y), (x, y) + \mathbf{e}_2) = 0, \quad (\forall x \in \mathbb{Z}).$$

For any $\mathbf{x} \in \mathbb{Z}^2$ let $\text{Ref}(\mathbf{x})$ denote the point of \mathbb{Z}^2 obtained from \mathbf{x} by reflection with respect to the horizontal line $\{(x, 1/2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ in \mathbb{R}^2 . For conciseness let $V(\mathbf{x})$ denote the operator

$$U \left(\psi_{\mathbf{x}\uparrow}^* \psi_{\mathbf{x}\downarrow}^* \psi_{\mathbf{x}\downarrow} \psi_{\mathbf{x}\uparrow} - \frac{1}{2} \sum_{\sigma \in \{\uparrow, \downarrow\}} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{x}\sigma} \right).$$

We decompose $H(\phi')$ as follows.

$$H(\phi') = H_a + H_b + H_{int},$$

where

$$\begin{aligned} H_a & := \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)_a} t(\mathbf{x}, \mathbf{y}) e^{i\phi'(\mathbf{x}, \mathbf{y})} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma} + \sum_{\mathbf{x} \in \Gamma(2L)_a} V(\mathbf{x}), \\ H_b & := \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)_b} t(\mathbf{x}, \mathbf{y}) e^{i\phi'(\mathbf{x}, \mathbf{y})} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma} + \sum_{\mathbf{x} \in \Gamma(2L)_b} V(\mathbf{x}), \\ H_{int} & := H(\phi') - H_a - H_b. \end{aligned}$$

Moreover, set

$$\begin{aligned}\Xi(H_a) &:= \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)_b} t(\mathbf{x}, \mathbf{y}) e^{i(\phi'(\text{Ref}(\mathbf{x}), \text{Ref}(\mathbf{y})) + \pi)} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma} \\ &\quad + \sum_{\mathbf{x} \in \Gamma(2L)_b} V(\mathbf{x}), \\ \Xi(H_b) &:= \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)_a} t(\mathbf{x}, \mathbf{y}) e^{i(\phi'(\text{Ref}(\mathbf{x}), \text{Ref}(\mathbf{y})) + \pi)} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma} \\ &\quad + \sum_{\mathbf{x} \in \Gamma(2L)_a} V(\mathbf{x}).\end{aligned}$$

Since the property (A.3) holds, we can apply [15, Lemma] concerning the reflection with respect to the horizontal line $\{(x, 1/2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. Since $t(\cdot, \cdot)$ is invariant under this reflection, the transformation of H_a, H_b in [15, Lemma] yields $\Xi(H_a), \Xi(H_b)$ respectively. The result is that

$$(\text{Tr} e^{-\beta H(\phi')})^2 \leq \text{Tr} e^{-\beta(H_a + \Xi(H_a) + H_{int})} \text{Tr} e^{-\beta(H_b + \Xi(H_b) + H_{int})}.$$

Since $\text{Tr} e^{-\beta H(\phi)}$ is maximum and $\text{Tr} e^{-\beta H(\phi)} = \text{Tr} e^{-\beta H(\phi')}$, we can derive from the above inequality that

$$\text{Tr} e^{-\beta H(\phi)} = \text{Tr} e^{-\beta(H_b + \Xi(H_b) + H_{int})}.$$

There exists a phase $\eta : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying (A.1) such that for any $\mathbf{x}, \mathbf{y} \in \Gamma(2L)$,

$$(A.4) \quad \eta(\mathbf{x}, \mathbf{y}) = \begin{cases} \phi'(\text{Ref}(\mathbf{x}), \text{Ref}(\mathbf{y})) + \pi & \text{if } \mathbf{x}, \mathbf{y} \in \Gamma(2L)_a, \\ \phi'(\mathbf{x}, \mathbf{y}) & \text{otherwise,} \end{cases}$$

and $H_b + \Xi(H_b) + H_{int} = H(\eta)$. By (A.3) and (A.4) we observe that for any $x \in \mathbb{Z}$

$$\begin{aligned}f_v(\eta)(x) &= \phi'((x, 1), (x, 0)) + \sum_{j=1}^{L-1} (\phi'(\text{Ref}((x, j+1)), \text{Ref}((x, j))) + \pi) \\ &\quad + \sum_{j=L}^{2L-1} \phi'((x, j+1), (x, j))\end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=1}^{L-1} (\phi'((x, 2L - j), (x, 2L + 1 - j)) + \pi) + \sum_{j=L+1}^{2L-1} \phi'((x, j + 1), (x, j)) \\
 &= \pi(L - 1) \text{ in } \mathbb{R}/2\pi\mathbb{Z}.
 \end{aligned}$$

In the following we repeat the reflection with vertical lines until we obtain a phase minimizing the free energy and satisfy the conditions (A.1), (A.2). For $s \in \{0, 1, \dots, L - 1\}$ set

$$\begin{aligned}
 \Gamma(2L)_r^s &:= \{(x_1, x_2) \in \Gamma(2L) \mid s + 1 \leq x_1 \leq s + L\}, \\
 \Gamma(2L)_l^s &:= \Gamma(2L) \setminus \Gamma(2L)_r^s.
 \end{aligned}$$

Define $\theta_s(\cdot) : \mathbb{Z}^2 \rightarrow \mathbb{R}$ by

$$\begin{aligned}
 \theta_s((x, y)) &:= \begin{cases} -\eta((x, y), (x, y) + \mathbf{e}_1) & \text{if } x = s \text{ in } \mathbb{Z}/2L\mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases} \\
 (\forall (x, y) \in \mathbb{Z}^2).
 \end{aligned}$$

Then, we define the transform $\tau_s(\eta)$ on $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$ by

$$\begin{aligned}
 &\tau_s(\eta)(\psi_{\mathbf{x}_1\sigma_1}^* \psi_{\mathbf{x}_2\sigma_2}^* \cdots \psi_{\mathbf{x}_n\sigma_n}^* \Omega_{2L}) \\
 &:= e^{i(\theta_s(\mathbf{x}_1) + \theta_s(\mathbf{x}_2) + \cdots + \theta_s(\mathbf{x}_n))} \psi_{\mathbf{x}_1\sigma_1}^* \psi_{\mathbf{x}_2\sigma_2}^* \cdots \psi_{\mathbf{x}_n\sigma_n}^* \Omega_{2L}, \\
 &(\forall (\mathbf{x}_j, \sigma_j) \in \Gamma(2L) \times \{\uparrow, \downarrow\} \ (j = 1, 2, \dots, n)),
 \end{aligned}$$

and by linearity. Remark that $\tau_s(\eta)$ is unitary, $\tau_s(\eta)H(\eta)\tau_s(\eta)^* = H(\eta')$ with $\eta' : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying (A.1),

$$(A.5) \quad f_v(\eta')(x) = \pi(L - 1) \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall x \in \mathbb{Z})$$

and that if $x = s$ in $\mathbb{Z}/2L\mathbb{Z}$,

$$(A.6) \quad \eta'((x, y), (x, y) + \mathbf{e}_1) = 0, \quad (\forall y \in \mathbb{Z}).$$

For any $\mathbf{x} \in \mathbb{Z}^2$ let $\text{Ref}_s(\mathbf{x})$ be the point of \mathbb{Z}^2 obtained from \mathbf{x} by reflection with respect to the line $\{(s + 1/2, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$ in \mathbb{R}^2 .

When $s = 0$, let us decompose $H(\eta')$ as follows.

$$H(\eta') = H_r^0 + H_l^0 + H_{int}^0,$$

where

$$\begin{aligned} H_r^0 &:= \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)_r^0} t(\mathbf{x}, \mathbf{y}) e^{i\eta'(\mathbf{x}, \mathbf{y})} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma} + \sum_{\mathbf{x} \in \Gamma(2L)_r^0} V(\mathbf{x}), \\ H_l^0 &:= \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)_l^0} t(\mathbf{x}, \mathbf{y}) e^{i\eta'(\mathbf{x}, \mathbf{y})} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma} + \sum_{\mathbf{x} \in \Gamma(2L)_l^0} V(\mathbf{x}), \\ H_{int}^0 &:= H(\eta') - H_r^0 - H_l^0. \end{aligned}$$

Moreover, set

$$\begin{aligned} \Xi(H_r^0) &:= \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)_r^0} t(\mathbf{x}, \mathbf{y}) e^{i(\eta'(\text{Ref}_0(\mathbf{x}), \text{Ref}_0(\mathbf{y})) + \pi)} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma} \\ &\quad + \sum_{\mathbf{x} \in \Gamma(2L)_r^0} V(\mathbf{x}), \\ \Xi(H_l^0) &:= \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)_l^0} t(\mathbf{x}, \mathbf{y}) e^{i(\eta'(\text{Ref}_0(\mathbf{x}), \text{Ref}_0(\mathbf{y})) + \pi)} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma} \\ &\quad + \sum_{\mathbf{x} \in \Gamma(2L)_l^0} V(\mathbf{x}). \end{aligned}$$

The property (A.6) for $s = 0$ enables us to apply [15, Lemma] concerning the reflection with respect to the line $\{(1/2, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$. By the invariant property of $t(\cdot, \cdot)$ the transformation of H_l^0 , H_r^0 in [15, Lemma] gives $\Xi(H_l^0)$, $\Xi(H_r^0)$ respectively. As the result,

$$(\text{Tr} e^{-\beta H(\eta')})^2 \leq \text{Tr} e^{-\beta(H_l^0 + \Xi(H_l^0) + H_{int}^0)} \text{Tr} e^{-\beta(H_r^0 + \Xi(H_r^0) + H_{int}^0)}.$$

Since $\text{Tr} e^{-\beta H(\phi)}$ is maximum and $\text{Tr} e^{-\beta H(\phi)} = \text{Tr} e^{-\beta H(\eta')}$, the above inequality implies that

$$\text{Tr} e^{-\beta H(\phi)} = \text{Tr} e^{-\beta(H_l^0 + \Xi(H_l^0) + H_{int}^0)}.$$

There exists a phase $\phi_0 : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying (A.1), (A.5) such that $H_l^0 + \Xi(H_l^0) + H_{int}^0 = H(\phi_0)$. Moreover, by (A.6) for $s = 0$, if $x = 0$ in $\mathbb{Z}/2L\mathbb{Z}$,

$$\begin{aligned} \text{(A.7)} \quad f_p(\phi_0)((x, y)) &= \eta'((x, y) + \mathbf{e}_1, (x, y)) \end{aligned}$$

$$\begin{aligned}
 & + \eta'(\text{Ref}_0((x, y) + \mathbf{e}_1 + \mathbf{e}_2), \text{Ref}_0((x, y) + \mathbf{e}_1)) + \pi \\
 & + \eta'((x, y) + \mathbf{e}_2, (x, y) + \mathbf{e}_1 + \mathbf{e}_2) + \eta'((x, y), (x, y) + \mathbf{e}_2) \\
 & = \eta'((0, y) + \mathbf{e}_2, (0, y)) + \pi + \eta'((0, y), (0, y) + \mathbf{e}_2) \\
 & = \pi \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall y \in \mathbb{Z}).
 \end{aligned}$$

If $L = 1$, the equalities (A.7), (A.1) imply (A.2). Thus, ϕ_0 is the desired minimizer.

Let us assume that $L \geq 2$, $s \in \{0, 1, \dots, L-2\}$ and $\text{Tr} e^{-\beta H(\phi)} = \text{Tr} e^{-\beta H(\phi_s)}$ with $\phi_s : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying (A.1), (A.5) and that if $x = j$ in $\mathbb{Z}/2L\mathbb{Z}$ for some $j \in \{0, 1, \dots, s\}$,

$$(A.8) \quad f_p(\phi_s)((x, y)) = \pi \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall y \in \mathbb{Z}).$$

We can write that

$$\tau_{s+1+L}(\phi_s)\tau_{s+1}(\phi_s)H(\phi_s)\tau_{s+1}(\phi_s)^*\tau_{s+1+L}(\phi_s)^* = H(\phi'_s)$$

with a phase $\phi'_s : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying (A.1), (A.5), (A.8) and that if $x = s+1$ or $s+1+L$ in $\mathbb{Z}/2L\mathbb{Z}$,

$$(A.9) \quad \phi'_s((x, y), (x, y) + \mathbf{e}_1) = 0, \quad (\forall y \in \mathbb{Z}).$$

We can decompose $H(\phi'_s)$ as follows.

$$H(\phi'_s) = H_r^{s+1} + H_l^{s+1} + H_{int}^{s+1},$$

where

$$\begin{aligned}
 H_r^{s+1} & := \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)_r^{s+1}} t(\mathbf{x}, \mathbf{y}) e^{i\phi'_s(\mathbf{x}, \mathbf{y})} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma} + \sum_{\mathbf{x} \in \Gamma(2L)_r^{s+1}} V(\mathbf{x}), \\
 H_l^{s+1} & := \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)_l^{s+1}} t(\mathbf{x}, \mathbf{y}) e^{i\phi'_s(\mathbf{x}, \mathbf{y})} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma} + \sum_{\mathbf{x} \in \Gamma(2L)_l^{s+1}} V(\mathbf{x}), \\
 H_{int}^{s+1} & := H(\phi'_s) - H_r^{s+1} - H_l^{s+1}.
 \end{aligned}$$

Define the operators $\Xi(H_r^{s+1})$, $\Xi(H_l^{s+1})$ by

$$\Xi(H_r^{s+1}) := \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)_l^{s+1}} t(\mathbf{x}, \mathbf{y}) e^{i(\phi'_s(\text{Ref}_{s+1}(\mathbf{x}), \text{Ref}_{s+1}(\mathbf{y})) + \pi)} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma}$$

$$\begin{aligned}
 & + \sum_{\mathbf{x} \in \Gamma(2L)_l^{s+1}} V(\mathbf{x}), \\
 \Xi(H_l^{s+1}) := & \sum_{\sigma \in \{\uparrow, \downarrow\}} \sum_{\mathbf{x}, \mathbf{y} \in \Gamma(2L)_r^{s+1}} t(\mathbf{x}, \mathbf{y}) e^{i(\phi'_s(\text{Ref}_{s+1}(\mathbf{x}), \text{Ref}_{s+1}(\mathbf{y})) + \pi)} \psi_{\mathbf{x}\sigma}^* \psi_{\mathbf{y}\sigma} \\
 & + \sum_{\mathbf{x} \in \Gamma(2L)_r^{s+1}} V(\mathbf{x}).
 \end{aligned}$$

Again the property (A.9) enables us to apply [15, Lemma] concerning the reflection with respect to the line $\{(s + 3/2, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$. Since $t(\cdot, \cdot)$ is invariant under the reflection, we have that

$$(\text{Tr } e^{-\beta H(\phi'_s)})^2 \leq \text{Tr } e^{-\beta(H_l^{s+1} + \Xi(H_l^{s+1}) + H_{int}^{s+1})} \text{Tr } e^{-\beta(H_r^{s+1} + \Xi(H_r^{s+1}) + H_{int}^{s+1})}.$$

This inequality implies that

$$\text{Tr } e^{-\beta H(\phi)} = \text{Tr } e^{-\beta(H_l^{s+1} + \Xi(H_l^{s+1}) + H_{int}^{s+1})},$$

because $\text{Tr } e^{-\beta H(\phi)}$ is maximum and $\text{Tr } e^{-\beta H(\phi)} = \text{Tr } e^{-\beta H(\phi'_s)}$. There exists a phase $\phi_{s+1} : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying (A.1) such that for any $\mathbf{x}, \mathbf{y} \in \Gamma(2L)$,

$$\begin{aligned}
 \text{(A.10)} \quad & \phi_{s+1}(\mathbf{x}, \mathbf{y}) \\
 & = \begin{cases} \phi'_s(\text{Ref}_{s+1}(\mathbf{x}), \text{Ref}_{s+1}(\mathbf{y})) + \pi & \text{if } \mathbf{x}, \mathbf{y} \in \Gamma(2L)_r^{s+1}, \\ \phi'_s(\mathbf{x}, \mathbf{y}) & \text{otherwise,} \end{cases}
 \end{aligned}$$

and $H(\phi_{s+1}) = H_l^{s+1} + \Xi(H_l^{s+1}) + H_{int}^{s+1}$. Using (A.5), (A.8), (A.9) for ϕ'_s and considering (A.10), we can check that ϕ_{s+1} satisfies (A.5) and if $x = j$ in $\mathbb{Z}/2L\mathbb{Z}$ for some $j \in \{0, 1, \dots, s\}$,

$$f_p(\phi_{s+1})((x, y)) = f_p(\phi'_s)((x, y)) = \pi \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall y \in \mathbb{Z}),$$

if $x = s + 1$ in $\mathbb{Z}/2L\mathbb{Z}$,

$$\begin{aligned}
 & f_p(\phi_{s+1})((x, y)) \\
 & = \phi'_s((x, y) + \mathbf{e}_1, (x, y)) \\
 & \quad + \phi'_s(\text{Ref}_{s+1}((x, y) + \mathbf{e}_1 + \mathbf{e}_2), \text{Ref}_{s+1}((x, y) + \mathbf{e}_1)) + \pi \\
 & \quad + \phi'_s((x, y) + \mathbf{e}_2, (x, y) + \mathbf{e}_1 + \mathbf{e}_2) + \phi'_s((x, y), (x, y) + \mathbf{e}_2) \\
 & = \phi'_s((s + 1, y) + \mathbf{e}_2, (s + 1, y)) + \pi + \phi'_s((s + 1, y), (s + 1, y) + \mathbf{e}_2)
 \end{aligned}$$

$$= \pi \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall y \in \mathbb{Z}),$$

if $x = j$ in $\mathbb{Z}/2L\mathbb{Z}$ for some $j \in \{s+2, s+3, \dots, 2s+2\}$,

$$\begin{aligned} f_p(\phi_{s+1})((x, y)) &= \phi'_s(\text{Ref}_{s+1}((x, y) + \mathbf{e}_1), \text{Ref}_{s+1}((x, y))) \\ &\quad + \phi'_s(\text{Ref}_{s+1}((x, y) + \mathbf{e}_1 + \mathbf{e}_2), \text{Ref}_{s+1}((x, y) + \mathbf{e}_1)) \\ &\quad + \phi'_s(\text{Ref}_{s+1}((x, y) + \mathbf{e}_2), \text{Ref}_{s+1}((x, y) + \mathbf{e}_1 + \mathbf{e}_2)) \\ &\quad + \phi'_s(\text{Ref}_{s+1}((x, y)), \text{Ref}_{s+1}((x, y) + \mathbf{e}_2)) \\ &= -f_p(\phi'_s)(\text{Ref}_{s+1}((x, y)) - \mathbf{e}_1) \\ &= \pi \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall y \in \mathbb{Z}). \end{aligned}$$

In summary, if $x = j$ in $\mathbb{Z}/2L\mathbb{Z}$ for some $j \in \{0, 1, \dots, 2s+2\}$,

$$f_p(\phi_{s+1})((x, y)) = \pi \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall y \in \mathbb{Z}).$$

By induction with s we have that $\text{Tr} e^{-\beta H(\phi)} = \text{Tr} e^{-\beta H(\phi_{L-1})}$ with a phase $\phi_{L-1} : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{R}$ satisfying (A.1), (A.5) and that if $x = j$ in $\mathbb{Z}/2L\mathbb{Z}$ for some $j \in \{0, 1, \dots, 2L-2\}$,

$$f_p(\phi_{L-1})((x, y)) = \pi \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall y \in \mathbb{Z}).$$

Moreover, by (A.9) and (A.10) for $s = L-2$, if $x = 2L-1$ in $\mathbb{Z}/2L\mathbb{Z}$,

$$\begin{aligned} f_p(\phi_{L-1})((x, y)) &= \phi'_{L-2}((x, y) + \mathbf{e}_1, (x, y)) \\ &\quad + \phi'_{L-2}((x, y) + \mathbf{e}_1 + \mathbf{e}_2, (x, y) + \mathbf{e}_1) \\ &\quad + \phi'_{L-2}((x, y) + \mathbf{e}_2, (x, y) + \mathbf{e}_1 + \mathbf{e}_2) \\ &\quad + \phi'_{L-2}(\text{Ref}_{L-1}((x, y)), \text{Ref}_{L-1}((x, y) + \mathbf{e}_2)) + \pi \\ &= \phi'_{L-2}((0, y) + \mathbf{e}_2, (0, y)) + \phi'_{L-2}((0, y), (0, y) + \mathbf{e}_2) + \pi \\ &= \pi \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall y \in \mathbb{Z}). \end{aligned}$$

Thus, the phase ϕ_{L-1} has the flux $\pi \pmod{2\pi}$ per plaquette. Furthermore, by (A.9) and (A.10) for $s = L-2$,

$$\begin{aligned} f_h(\phi_{L-1})(x) &= \sum_{j=0}^{L-1} \phi'_{L-2}((j+1, x), (j, x)) \\ &\quad + \sum_{j=L}^{2L-2} (\phi'_{L-2}(\text{Ref}_{L-1}((j+1, x)), \text{Ref}_{L-1}((j, x))) + \pi) \end{aligned}$$

$$\begin{aligned}
& + \phi'_{L-2}((0, x), (2L-1, x)) \\
& = \sum_{j=0}^{L-2} \phi'_{L-2}((j+1, x), (j, x)) \\
& \quad + \sum_{j=L}^{2L-2} (\phi'_{L-2}((2L-2-j, x), (2L-1-j, x)) + \pi) \\
& = \pi(L-1) \text{ in } \mathbb{R}/2\pi\mathbb{Z}, \quad (\forall x \in \mathbb{Z}).
\end{aligned}$$

Therefore, the phase ϕ_{L-1} is a minimizer satisfying (A.2). \square

Appendix B. L^1 -Estimates of Kernels of Grassmann Polynomials

Here we prove several lemmas used in Subsection 2.5. These lemmas concern estimations of Grassmann polynomials with respect to the L^1 -norm $\|\cdot\|_{L^1}$ on their anti-symmetric kernels. Though we know the unique existence of anti-symmetric kernels, it is not always trivial to characterize the kernels explicitly. First of all we confirm that we can estimate anti-symmetric kernels without characterizing them.

LEMMA B.1. *Assume that $W_m(\psi) \in \mathcal{P}_m(\wedge \mathcal{V})$ is written as*

$$W_m(\psi) = \left(\frac{1}{h}\right)^m \sum_{\mathbf{X} \in I^m} \hat{W}_m(\mathbf{X}) \psi_{\mathbf{X}},$$

where the function $\hat{W}_m : I^m \rightarrow \mathbb{C}$ is not necessarily anti-symmetric. Then,

$$\|W_m\|_{L^1} \leq \|\hat{W}_m\|_{L^1}.$$

PROOF. By the uniqueness of anti-symmetric kernels we have that

$$W_m(\mathbf{X}) = \frac{1}{m!} \sum_{\sigma \in \mathbb{S}_m} \text{sgn}(\sigma) \hat{W}_m(\mathbf{X}_\sigma), \quad (\forall \mathbf{X} \in I^m).$$

Thus,

$$\|W_m\|_{L^1} \leq \frac{1}{m!} \sum_{\sigma \in \mathbb{S}_m} \|\hat{W}_m\|_{L^1} = \|\hat{W}_m\|_{L^1}. \quad \square$$

We summarize necessary bounds on polynomials produced by Grassmann Gaussian integrals in the next lemma.

LEMMA B.2. *Assume that a covariance $A : I_0^2 \rightarrow \mathbb{C}$ and a covariance $A^{(\varepsilon)} : I_0^2 \rightarrow \mathbb{C}$ parameterized by $\varepsilon \in [0, 1)$ satisfy that*

$$\begin{aligned} |\det(A(X_i, Y_j))_{1 \leq i, j \leq n}| &\leq c_A^n, \\ |\det(A^{(\varepsilon)}(X_i, Y_j))_{1 \leq i, j \leq n}| &\leq \varepsilon \cdot c_A^n, \\ (\forall n \in \mathbb{N}, X_j, Y_j \in I_0 \ (j = 1, 2, \dots, n), \varepsilon \in [0, 1)), \end{aligned}$$

with a constant $c_A \in \mathbb{R}_{\geq 0}$. With $W(\psi), W^{(1)}(\psi), W^{(2)}(\psi) \in \bigwedge \mathcal{V}$ set

$$\begin{aligned} S(\psi) &:= \int e^{W(\psi+\psi^1)} d\mu_A(\psi^1), \\ S^{(j)}(\psi) &:= \int e^{W^{(j)}(\psi+\psi^1)} d\mu_A(\psi^1), \ (j = 1, 2), \\ S^{(\varepsilon)}(\psi) &:= \int e^{W(\psi+\psi^1)} d\mu_{A+A^{(\varepsilon)}}(\psi^1), \ (\varepsilon \in [0, 1)). \end{aligned}$$

Then, the following inequalities hold true.

(1)

$$|S_0 - e^{W_0}| \leq e^{|W_0|} \left(e^{\sum_{m=1}^N c_A^{\frac{m}{2}} \|W_m\|_{L^1}} - 1 \right).$$

(2) For any $\alpha \in \mathbb{R}_{\geq 0}$,

$$\sum_{m=0}^N \alpha^m c_A^{\frac{m}{2}} \|S_m\|_{L^1} \leq e^{\sum_{m=0}^N (\alpha+1)^m c_A^{\frac{m}{2}} \|W_m\|_{L^1}}.$$

(3) For any $\alpha \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} &\sum_{m=0}^N \alpha^m c_A^{\frac{m}{2}} \|S_m^{(1)} - S_m^{(2)}\|_{L^1} \\ &\leq \left(|e^{W_0^{(1)}} - e^{W_0^{(2)}}| + e^{|W_0^{(2)}|} \sum_{m=1}^N (\alpha+1)^m c_A^{\frac{m}{2}} \|W_m^{(1)} - W_m^{(2)}\|_{L^1} \right) \\ &\quad \cdot e^{\sup_{j \in \{1,2\}} \sum_{m=1}^N (\alpha+1)^m c_A^{\frac{m}{2}} \|W_m^{(j)}\|_{L^1}}. \end{aligned}$$

(4) For any $\alpha \in \mathbb{R}_{\geq 0}$,

$$\sum_{m=0}^N \alpha^m c_A^{\frac{m}{2}} \|S_m - S_m^{(\varepsilon)}\|_{L^1} \leq \varepsilon e^{|W_0|} \left(e^{\sum_{m=1}^N (\alpha+2)^m c_A^{\frac{m}{2}} \|W_m\|_{L^1}} - 1 \right).$$

PROOF. By anti-symmetry we have that

$$\begin{aligned} \text{(B.1)} \quad S_m(\psi) = & e^{W_0} \left(1_{m=0} + \sum_{n=1}^N \frac{1}{n!} \right. \\ & \cdot \prod_{l=1}^n \left(\sum_{m_l=1}^N \left(\frac{1}{h} \right)^{m_l} \sum_{k_l=0}^{m_l} \binom{m_l}{k_l} \right) \\ & \cdot \sum_{\mathbf{X}_l \in I^{m_l-k_l}} \sum_{\mathbf{Y}_l \in I^{k_l}} W_{m_l}(\mathbf{X}_l, \mathbf{Y}_l) \Big) \\ & \cdot \varepsilon_{\pm} 1_{\sum_{l=1}^n k_l = m} \\ & \cdot \int \psi_{\mathbf{X}_1}^1 \psi_{\mathbf{X}_2}^1 \cdots \psi_{\mathbf{X}_n}^1 d\mu_A(\psi^1) \psi_{\mathbf{Y}_1} \psi_{\mathbf{Y}_2} \cdots \psi_{\mathbf{Y}_n} \Big), \end{aligned}$$

where the factor $\varepsilon_{\pm} \in \{1, -1\}$ depends only on $(m_l)_{l=1}^n, (k_l)_{l=1}^n$.

(1): We can derive from (B.1) that

$$\begin{aligned} \text{(B.2)} \quad |S_0 - e^{W_0}| & \leq e^{|W_0|} \sum_{n=1}^N \frac{1}{n!} \left(\sum_{m=1}^N c_A^{\frac{m}{2}} \|W_m\|_{L^1} \right)^n \\ & = e^{|W_0|} \left(e^{\sum_{m=1}^N c_A^{\frac{m}{2}} \|W_m\|_{L^1}} - 1 \right). \end{aligned}$$

(2): It follows from Lemma B.1 and (B.1) that

$$\begin{aligned} & \alpha^m c_A^{\frac{m}{2}} \|S_m\|_{L^1} \\ & \leq e^{|W_0|} \\ & \cdot \left(1_{m=0} + \sum_{n=1}^N \frac{1}{n!} \prod_{l=1}^n \left(\sum_{m_l=1}^N c_A^{\frac{m_l}{2}} \|W_{m_l}\|_{L^1} \sum_{k_l=0}^{m_l} \binom{m_l}{k_l} \alpha^{k_l} \right) 1_{\sum_{l=1}^n k_l = m} \right). \end{aligned}$$

Thus,

$$\begin{aligned}
 & \sum_{m=0}^N \alpha^m c_A^{\frac{m}{2}} \|S_m\|_{L^1} \\
 & \leq e^{|W_0|} \left(1 + \sum_{n=1}^N \frac{1}{n!} \prod_{l=1}^n \left(\sum_{m_l=1}^N c_A^{\frac{m_l}{2}} \|W_{m_l}\|_{L^1} \sum_{k_l=0}^{m_l} \binom{m_l}{k_l} \alpha^{k_l} \right) \right) \\
 & \leq e^{\sum_{m=0}^N (\alpha+1)^m c_A^{\frac{m}{2}} \|W_m\|_{L^1}}.
 \end{aligned}$$

(3): From (B.1) we deduce that

$$\begin{aligned}
 & \alpha^m c_A^{\frac{m}{2}} \|S_m^{(1)} - S_m^{(2)}\|_{L^1} \\
 & \leq |e^{W_0^{(1)}} - e^{W_0^{(2)}}| \\
 & \quad \cdot \left(1_{m=0} + \sum_{n=1}^N \frac{1}{n!} \prod_{l=1}^n \left(\sum_{m_l=1}^N c_A^{\frac{m_l}{2}} \|W_{m_l}^{(1)}\|_{L^1} \sum_{k_l=0}^{m_l} \binom{m_l}{k_l} \alpha^{k_l} \right) 1_{\sum_{l=1}^n k_l=m} \right) \\
 & \quad + e^{|W_0^{(2)}|} \sum_{n=1}^N \frac{1}{n!} \prod_{l=1}^n \\
 & \quad \cdot \left(\sum_{m_l=1}^N c_A^{\frac{m_l}{2}} \sum_{k_l=0}^{m_l} \binom{m_l}{k_l} \alpha^{k_l} \left(\frac{1}{h}\right)^{m_l} \sum_{\mathbf{X}_l \in I^{m_l}} \right) 1_{\sum_{l=1}^n k_l=m} \\
 & \quad \cdot \left| \prod_{l=1}^n W_{m_l}^{(1)}(\mathbf{X}_l) - \prod_{l=1}^n W_{m_l}^{(2)}(\mathbf{X}_l) \right|.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \sum_{m=0}^N \alpha^m c_A^{\frac{m}{2}} \|S_m^{(1)} - S_m^{(2)}\|_{L^1} \\
 & \leq |e^{W_0^{(1)}} - e^{W_0^{(2)}}| e^{\sum_{m=1}^N (\alpha+1)^m c_A^{\frac{m}{2}} \|W_m^{(1)}\|_{L^1}} \\
 & \quad + e^{|W_0^{(2)}|} \sum_{m=1}^N (\alpha+1)^m c_A^{\frac{m}{2}} \|W_m^{(1)} - W_m^{(2)}\|_{L^1} \\
 & \quad \cdot \sum_{n=1}^N \frac{1}{(n-1)!} \left(\sup_{j \in \{1,2\}} \sum_{m=1}^N (\alpha+1)^m c_A^{\frac{m}{2}} \|W_m^{(j)}\|_{L^1} \right)^{n-1}
 \end{aligned}$$

$$\leq \left(|e^{W_0^{(1)}} - e^{W_0^{(2)}}| + e^{|W_0^{(2)}}| \sum_{m=1}^N (\alpha + 1)^m c_A^{\frac{m}{2}} \|W_m^{(1)} - W_m^{(2)}\|_{L^1} \right) \cdot e^{\sup_{j \in \{1,2\}} \sum_{m=1}^N (\alpha+1)^m c_A^{\frac{m}{2}} \|W_m^{(j)}\|_{L^1}}.$$

(4): By applying the Cauchy-Binet formula in the same way as in (2.29) and substituting the determinant bounds on $A, A^{(\varepsilon)}$ we observe that for any $n \in \mathbb{N}, X_j, Y_j \in I_0 (j = 1, 2, \dots, n),$

$$\begin{aligned} & \left| \det((A + A^{(\varepsilon)})(X_i, Y_j))_{1 \leq i, j \leq n} - \det(A(X_i, Y_j))_{1 \leq i, j \leq n} \right| \\ & \leq \sum_{\substack{\phi: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, 2n\} \\ \text{with } \phi(1) < \phi(2) < \dots < \phi(n)}} \mathbf{1}_{\phi(n) > n} \\ & \quad \cdot \left| \det((A_{(n)}, I_n)(i, \phi(j)))_{1 \leq i, j \leq n} \right| \left| \det \left(\begin{pmatrix} I_n \\ A_{(n)}^{(\varepsilon)} \end{pmatrix} (\phi(i), j) \right)_{1 \leq i, j \leq n} \right| \\ & \leq \sum_{m=0}^{n-1} \sum_{\substack{\phi: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, 2n\} \\ \text{with } \phi(1) < \phi(2) < \dots < \phi(n)}} \mathbf{1}_{\phi(m) \leq n < \phi(m+1)} c_A^m \varepsilon c_A^{n-m} \leq \varepsilon (2^2 c_A)^n, \end{aligned}$$

where

$$\begin{aligned} A_{(n)}(i, j) & := A(X_i, Y_j), \quad A_{(n)}^{(\varepsilon)}(i, j) := A^{(\varepsilon)}(X_i, Y_j), \quad (\forall i, j \in \{1, 2, \dots, n\}), \\ \phi(0) & := n. \end{aligned}$$

By using this inequality and Lemma B.1 we can derive from (B.1) that for any $m \in \{0, 1, \dots, N\},$

$$\begin{aligned} & \alpha^m c_A^{\frac{m}{2}} \|S_m - S_m^{(\varepsilon)}\|_{L^1} \\ & \leq \alpha^m c_A^{\frac{m}{2}} e^{|W_0|} \sum_{n=1}^N \frac{1}{n!} \prod_{l=1}^n \\ & \quad \cdot \left(\sum_{m_l=1}^N \left(\frac{1}{h}\right)^{m_l} \sum_{k_l=0}^{m_l} \binom{m_l}{k_l} \sum_{\mathbf{X}_l \in I^{m_l - k_l}} \sum_{\mathbf{Y}_l \in I^{k_l}} |W_{m_l}(\mathbf{X}_l, \mathbf{Y}_l)| \right) \mathbf{1}_{\sum_{l=1}^n k_l = m} \\ & \quad \cdot \left| \int \psi_{\mathbf{X}_1}^1 \psi_{\mathbf{X}_2}^1 \cdots \psi_{\mathbf{X}_n}^1 d\mu_A(\psi^1) - \int \psi_{\mathbf{X}_1}^1 \psi_{\mathbf{X}_2}^1 \cdots \psi_{\mathbf{X}_n}^1 d\mu_{A+A^{(\varepsilon)}}(\psi^1) \right| \end{aligned}$$

$$\leq \varepsilon e^{|W_0|} \sum_{n=1}^N \frac{1}{n!} \prod_{l=1}^n \cdot \left(\sum_{m_l=1}^N (2^2 c_A)^{\frac{m_l}{2}} \|W_{m_l}\|_{L^1} \sum_{k_l=0}^{m_l} \binom{m_l}{k_l} 2^{-k_l} \alpha^{k_l} \right) 1_{\sum_{l=1}^n k_l = m},$$

which implies that

$$\begin{aligned} \sum_{m=0}^N \alpha^m c_A^{\frac{m}{2}} \|S_m - S_m^{(\varepsilon)}\|_{L^1} &\leq \varepsilon e^{|W_0|} \sum_{n=1}^N \frac{1}{n!} \left(\sum_{m=1}^N (\alpha + 2)^m c_A^{\frac{m}{2}} \|W_m\|_{L^1} \right)^n \\ &\leq \varepsilon e^{|W_0|} \left(e^{\sum_{m=1}^N (\alpha+2)^m c_A^{\frac{m}{2}} \|W_m\|_{L^1}} - 1 \right). \quad \square \end{aligned}$$

We also need upper bounds on logarithm of Grassmann polynomials.

LEMMA B.3. *With $W(\psi), W^{(1)}(\psi), W^{(2)}(\psi) \in \bigwedge \mathcal{V}$ satisfying $|W_0 - 1|, |W_0^{(j)} - 1| < 1$ ($j = 1, 2$) set $Q(\psi) := \log W(\psi)$, $Q^{(j)}(\psi) := \log W^{(j)}(\psi)$ ($j = 1, 2$). Then, the following inequalities hold.*

(1)

$$|Q_0| \leq -\log(1 - |W_0 - 1|).$$

(2) For any $\alpha \in \mathbb{R}_{\geq 0}$ satisfying

$$(B.3) \quad |W_0|^{-1} \sum_{m=1}^N \alpha^m \|W_m\|_{L^1} < 1,$$

$$\sum_{m=1}^N \alpha^m \|Q_m\|_{L^1} \leq -\log \left(1 - |W_0|^{-1} \sum_{m=1}^N \alpha^m \|W_m\|_{L^1} \right).$$

(3)

$$|Q_0^{(1)} - Q_0^{(2)}| \leq |\log(W_0^{(1)}) - \log(W_0^{(2)})|.$$

(4) For any $\alpha \in \mathbb{R}_{\geq 0}$ satisfying

$$(B.4) \quad \sup_{j \in \{1,2\}} \sum_{m=1}^N \alpha^m \|W_m^{(j)}\|_{L^1} < \inf_{j \in \{1,2\}} |W_0^{(j)}|,$$

$$\begin{aligned} & \sum_{m=1}^N \alpha^m \|Q_m^{(1)} - Q_m^{(2)}\|_{L^1} \\ & \leq \left(1 - \left(\inf_{j \in \{1,2\}} |W_0^{(j)}| \right)^{-1} \sup_{j \in \{1,2\}} \sum_{m=1}^N \alpha^m \|W_m^{(j)}\|_{L^1} \right)^{-1} \\ & \quad \cdot |W_0^{(1)} W_0^{(2)}|^{-1} \sum_{m=0}^N \alpha^m \|W_m^{(1)}\|_{L^1} \sum_{n=0}^N \alpha^n \|W_n^{(1)} - W_n^{(2)}\|_{L^1}. \end{aligned}$$

PROOF. (1),(3): Since $Q_0 = \log W_0$, $Q_0^{(j)} = \log W_0^{(j)}$ ($j = 1, 2$), the claimed inequalities are true.

(2): Note that for any $m \in \{1, 2, \dots, N\}$,

$$(B.5) \quad Q_m(\psi) = \sum_{n=1}^m \frac{(-1)^{n-1}}{n} W_0^{-n} \prod_{l=1}^n \left(\sum_{m_l=1}^N \left(\frac{1}{h} \right)^{m_l} \sum_{\mathbf{X}_l \in I^{m_l}} W_{m_l}(\mathbf{X}_l) \right) \\ \cdot \psi_{\mathbf{X}_1} \psi_{\mathbf{X}_2} \cdots \psi_{\mathbf{X}_n} 1_{\sum_{l=1}^n m_l = m}.$$

Thus, it follows from Lemma B.1 and the assumption (B.3) that

$$\begin{aligned} \sum_{m=1}^N \alpha^m \|Q_m\|_{L^1} & \leq \sum_{m=1}^N \sum_{n=1}^m \frac{1}{n} |W_0|^{-n} \prod_{l=1}^n \left(\sum_{m_l=1}^N \alpha^{m_l} \|W_{m_l}\|_{L^1} \right) 1_{\sum_{l=1}^n m_l = m} \\ & \leq \sum_{n=1}^N \frac{1}{n} |W_0|^{-n} \left(\sum_{m=1}^N \alpha^m \|W_m\|_{L^1} \right)^n \\ & \leq -\log \left(1 - |W_0|^{-1} \sum_{m=1}^N \alpha^m \|W_m\|_{L^1} \right). \end{aligned}$$

(4): By (B.5) and Lemma B.1 we have that

$$\begin{aligned} & \|Q_m^{(1)} - Q_m^{(2)}\|_{L^1} \\ & \leq \sum_{n=1}^m \frac{1}{n} |W_0^{(1)-n} - W_0^{(2)-n}| \prod_{l=1}^n \left(\sum_{m_l=1}^N \|W_{m_l}^{(1)}\|_{L^1} \right) 1_{\sum_{l=1}^n m_l = m} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{n=1}^m \frac{1}{n} |W_0^{(2)}|^{-n} \prod_{l=1}^n \left(\sum_{m_l=1}^N \left(\frac{1}{h} \right)^{m_l} \sum_{\mathbf{X}_l \in I^{m_l}} \right) \\
 & \quad \cdot \left| \prod_{j=1}^n W_{m_j}^{(1)}(\mathbf{X}_j) - \prod_{j=1}^n W_{m_j}^{(2)}(\mathbf{X}_j) \right| 1_{\Sigma_{l=1}^n m_l=m}.
 \end{aligned}$$

Therefore, on the assumption (B.4),

$$\begin{aligned}
 & \sum_{m=1}^N \alpha^m \|Q_m^{(1)} - Q_m^{(2)}\|_{L^1} \\
 & \leq \sum_{n=1}^N \sum_{m=n}^N \frac{1}{n} |W_0^{(1)-n} - W_0^{(2)-n}| \prod_{l=1}^n \left(\sum_{m_l=1}^N \alpha^{m_l} \|W_{m_l}^{(1)}\|_{L^1} \right) 1_{\Sigma_{l=1}^n m_l=m} \\
 & \quad + \sum_{n=1}^N \sum_{m=n}^N \frac{1}{n} |W_0^{(2)}|^{-n} \prod_{l=1}^n \left(\sum_{m_l=1}^N \alpha^{m_l} \left(\frac{1}{h} \right)^{m_l} \sum_{\mathbf{X}_l \in I^{m_l}} \right) \\
 & \quad \cdot \left| \prod_{j=1}^n W_{m_j}^{(1)}(\mathbf{X}_j) - \prod_{j=1}^n W_{m_j}^{(2)}(\mathbf{X}_j) \right| 1_{\Sigma_{l=1}^n m_l=m} \\
 & \leq \sum_{n=1}^N \frac{1}{n} |W_0^{(1)-n} - W_0^{(2)-n}| \left(\sum_{m=1}^N \alpha^m \|W_m^{(1)}\|_{L^1} \right)^n \\
 & \quad + \sum_{n=1}^N \frac{1}{n} |W_0^{(2)}|^{-n} \prod_{l=1}^n \left(\sum_{m_l=1}^N \alpha^{m_l} \left(\frac{1}{h} \right)^{m_l} \sum_{\mathbf{X}_l \in I^{m_l}} \right) \\
 & \quad \cdot \left| \prod_{j=1}^n W_{m_j}^{(1)}(\mathbf{X}_j) - \prod_{j=1}^n W_{m_j}^{(2)}(\mathbf{X}_j) \right| \\
 & \leq |W_0^{(1)-1} - W_0^{(2)-1}| \sum_{n=1}^N \left(\inf_{j \in \{1,2\}} |W_0^{(j)}| \right)^{-n+1} \left(\sum_{m=1}^N \alpha^m \|W_m^{(1)}\|_{L^1} \right)^n \\
 & \quad + \sum_{m=1}^N \alpha^m \|W_m^{(1)} - W_m^{(2)}\|_{L^1} \sum_{n=1}^N |W_0^{(2)}|^{-n} \left(\sup_{j \in \{1,2\}} \sum_{k=1}^N \alpha^k \|W_k^{(j)}\|_{L^1} \right)^{n-1} \\
 & \leq \left(|W_0^{(1)-1} - W_0^{(2)-1}| \sum_{m=1}^N \alpha^m \|W_m^{(1)}\|_{L^1} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{m=1}^N \alpha^m \|W_m^{(1)} - W_m^{(2)}\|_{L^1} |W_0^{(2)}|^{-1} \\
 & \cdot \left(1 - \left(\inf_{j \in \{1,2\}} |W_0^{(j)}| \right)^{-1} \sup_{j \in \{1,2\}} \sum_{m=1}^N \alpha^m \|W_m^{(j)}\|_{L^1} \right)^{-1},
 \end{aligned}$$

which leads to the claimed inequality. \square

Appendix C. Estimation of Gevrey-Class Functions

Here we establish some estimates on functions and matrix-valued functions whose local regularity is that of Gevrey-class. We use these estimates to derive decay bounds on covariance matrices containing a Gevrey-class cut-off function. We intend not to expand our analysis more than what we need for our purposes. More general calculus of Gevrey-class functions are found in, e.g., [8], [21].

LEMMA C.1. *Assume that $O_1, O_2 (\subset \mathbb{R})$ are open intervals, $f_j \in C^\infty(O_j; \mathbb{R})$ ($j = 1, 2$) and $f_1(O_1) \subset O_2$. Moreover, assume that $x_0 \in O_1$ and*

$$\begin{aligned}
 & \left| \left(\frac{d}{dx} \right)^n f_1(x) \Big|_{x=x_0} \right| \leq q_1 r_1^n n!, \\
 & \left| \left(\frac{d}{dx} \right)^n f_2(x) \Big|_{x=f_1(x_0)} \right| \leq q_2 r_2^n (n!)^t, \quad (\forall n \in \mathbb{N}),
 \end{aligned}$$

with constants $q_j, r_j \in \mathbb{R}_{\geq 0}$ ($j = 1, 2$), $t \in \mathbb{R}_{\geq 1}$. Then,

$$\left| \left(\frac{d}{dx} \right)^n f_2(f_1(x)) \Big|_{x=x_0} \right| \leq \frac{q_1 q_2 r_2}{1 + q_1 r_2} (r_1 (1 + q_1 r_2))^n (n!)^t, \quad (\forall n \in \mathbb{N}).$$

REMARK C.2. A systematic estimation of the composition of Gevrey-class functions was presented in [8, Section I]. Here we provide another basic estimation motivated by [21, Proposition 1.4.6].

PROOF OF LEMMA C.1. Fix $n \in \mathbb{N}$. By Taylor's theorem, for any $x \in O_1$,

$$f_1(x) = \sum_{l=0}^n \frac{1}{l!} \left(\frac{d}{dy} \right)^l f_1(y) \Big|_{y=x_0} (x - x_0)^l$$

$$+ \frac{1}{n!} \int_{x_0}^x (x - y)^n \left(\frac{d}{dy}\right)^{n+1} f_1(y) dy.$$

Thus, for any $m \in \mathbb{N}$,

$$\begin{aligned} & \left(\frac{d}{dx}\right)^n (f_1(x) - f_1(x_0))^m \Big|_{x=x_0} \\ &= \left(\frac{d}{dx}\right)^n \left(\sum_{l=1}^n \frac{1}{l!} \left(\frac{d}{dy}\right)^l f_1(y) \Big|_{y=x_0} (x - x_0)^l \right)^m \Big|_{x=x_0} \\ &= n! \prod_{j=1}^m \left(\sum_{l_j=1}^n \frac{1}{l_j!} \left(\frac{d}{dy}\right)^{l_j} f_1(y) \Big|_{y=x_0} \right) 1_{\sum_{j=1}^m l_j=n}. \end{aligned}$$

Moreover, by Taylor's theorem, for any $x \in O_1$,

$$\begin{aligned} & f_2(f_1(x)) \\ &= \sum_{m=0}^n \frac{1}{m!} \left(\frac{d}{dy}\right)^m f_2(y) \Big|_{y=f_1(x_0)} (f_1(x) - f_1(x_0))^m \\ & \quad + \frac{1}{n!} \int_{f_1(x_0)}^{f_1(x)} (f_1(x) - y)^n \left(\frac{d}{dy}\right)^{n+1} f_2(y) dy. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left(\frac{d}{dx}\right)^n f_2(f_1(x)) \Big|_{x=x_0} \\ &= \sum_{m=1}^n \frac{1}{m!} \left(\frac{d}{dy}\right)^m f_2(y) \Big|_{y=f_1(x_0)} \left(\frac{d}{dx}\right)^n (f_1(x) - f_1(x_0))^m \Big|_{x=x_0} \\ &= \sum_{m=1}^n \frac{1}{m!} \left(\frac{d}{dy}\right)^m f_2(y) \Big|_{y=f_1(x_0)} \\ & \quad \cdot n! \prod_{j=1}^m \left(\sum_{l_j=1}^n \frac{1}{l_j!} \left(\frac{d}{dx}\right)^{l_j} f_1(y) \Big|_{y=x_0} \right) 1_{\sum_{j=1}^m l_j=n}. \end{aligned}$$

By substituting the assumed upper bounds we have

$$\left| \left(\frac{d}{dx}\right)^n f_2(f_1(x)) \Big|_{x=x_0} \right| \leq \sum_{m=1}^n \frac{1}{m!} q_2 r_2^m (m!)^t n! \prod_{j=1}^m \left(\sum_{l_j=1}^n q_1 r_1^{l_j} \right) 1_{\sum_{j=1}^m l_j=n}$$

$$\begin{aligned}
 &\leq q_2 r_1^n (n!)^t \sum_{m=1}^n (q_1 r_2)^m \prod_{j=1}^m \left(\sum_{l_j=1}^n \right) 1_{\sum_{j=1}^m l_j = n} \\
 &= q_2 r_1^n (n!)^t \frac{1}{n!} \left(\frac{d}{dz} \right)^n \Big|_{z=0} \sum_{m=1}^n (q_1 r_2)^m \left(\frac{z}{1-z} \right)^m \\
 &= q_2 r_1^n (n!)^t \frac{1}{n!} \left(\frac{d}{dz} \right)^n \Big|_{z=0} \frac{\frac{q_1 r_2 z}{1-z}}{1 - \frac{q_1 r_2 z}{1-z}} \\
 &= q_1 q_2 r_2 (1 + q_1 r_2)^{-1} (r_1 (1 + q_1 r_2))^n (n!)^t. \quad \square
 \end{aligned}$$

In the rest of this section we find upper bounds on matrix-valued functions.

LEMMA C.3. *Assume that $O(\subset \mathbb{R})$ is an open interval, $x_0 \in O$, $A \in C^\infty(O; \text{Mat}(b, \mathbb{C}))$ and*

$$\left\| \left(\frac{d}{dx} \right)^n A(x) \Big|_{x=x_0} \right\|_{b \times b} \leq q r^n (n!)^t, \quad (\forall n \in \mathbb{N}),$$

with constants $q, r \in \mathbb{R}_{\geq 0}$, $t \in \mathbb{R}_{\geq 1}$. Then, the following statements hold true.

(1) If $t = 1$,

$$\left\| \left(\frac{d}{dx} \right)^n A(x)^m \Big|_{x=x_0} \right\|_{b \times b} \leq (2q)^m (2r)^n n!, \quad (\forall m, n \in \mathbb{N}).$$

(2) If $A(x)$ is invertible for any $x \in O$ and $\|A(x_0)^{-1}\|_{b \times b} \leq s$ with a constant $s \in \mathbb{R}_{\geq 0}$,

$$\left\| \left(\frac{d}{dx} \right)^n A(x)^{-1} \Big|_{x=x_0} \right\|_{b \times b} \leq \frac{s^2 q}{(1 + (sq)^{\frac{1}{t}})^t} (r(1 + (sq)^{\frac{1}{t}})^t)^n (n!)^t,$$

($\forall n \in \mathbb{N}$).

PROOF. (1): For any $x \in O$,

$$A(x) = \sum_{l=0}^n \frac{1}{l!} \left(\frac{d}{dy} \right)^l A(y) \Big|_{y=y_0} (x - x_0)^l$$

$$+ \frac{1}{n!} \int_{x_0}^x (x - y)^n \left(\frac{d}{dy}\right)^{n+1} A(y) dy.$$

Thus, for any $m \in \mathbb{N}$,

$$\begin{aligned} \left(\frac{d}{dx}\right)^n A(x)^m \Big|_{x=x_0} &= \left(\frac{d}{dx}\right)^n \left(\sum_{l=0}^n \frac{1}{l!} \left(\frac{d}{dy}\right)^l A(y) \Big|_{y=x_0} (x - x_0)^l \right)^m \Big|_{x=x_0} \\ &= n! \prod_{\substack{j=1 \\ \text{order}}}^m \left(\sum_{l_j=0}^n \frac{1}{l_j!} \left(\frac{d}{dy}\right)^{l_j} A(y) \Big|_{y=x_0} \right) 1_{\sum_{j=1}^m l_j = n}. \end{aligned}$$

Then, by using the assumed upper bounds and Cauchy’s integral formula we observe that

$$\begin{aligned} \left\| \left(\frac{d}{dx}\right)^n A(x)^m \Big|_{x=x_0} \right\|_{b \times b} &\leq n! \prod_{j=1}^m \left(\sum_{l_j=0}^n q r^{l_j} \right) 1_{\sum_{j=1}^m l_j = n} \\ &= q^{m_r n} \left(\frac{d}{dz}\right)^n \Big|_{z=0} \left(\frac{1}{1-z}\right)^m = q^{m_r n} \frac{n!}{2\pi i} \oint_{|z|=\frac{1}{2}} dz \frac{1}{z^{n+1}(1-z)^m} \\ &\leq (2q)^m (2r)^n n!. \end{aligned}$$

(2): First let us prove the equality that for any $n \in \mathbb{N}$, $x \in O$,

$$\begin{aligned} \text{(C.1)} \quad &\left(\frac{d}{dx}\right)^n A(x)^{-1} \\ &= \sum_{l=1}^n \prod_{j=1}^l \left(\sum_{m_j=1}^n \right) 1_{\sum_{j=1}^l m_j = n} c^{(n)}(l, m_1, m_2, \dots, m_l) \\ &\quad \cdot (-1)^l \prod_{\substack{k=1 \\ \text{order}}}^l \left(A(x)^{-1} \left(\frac{d}{dx}\right)^{m_k} A(x) \right) A(x)^{-1}, \end{aligned}$$

where the coefficients $c^{(n)}(l, m_1, m_2, \dots, m_l) \in \mathbb{N}$ ($\forall l \in \{1, 2, \dots, n\}$, $m_j \in \{1, 2, \dots, n\}$ ($j = 1, 2, \dots, l$) with $\sum_{j=1}^l m_j = n$) are inductively defined as follows.

$$\begin{aligned} c^{(1)}(1, 1) &:= 1, \\ c^{(n)}(l, m_1, m_2, \dots, m_l) \end{aligned}$$

$$:= \sum_{i=1}^l (1_{l \geq 2} 1_{m_i=1} c^{(n-1)}(l-1, m_1, \dots, m_{i-1}, \widehat{m_i}, m_{i+1}, \dots, m_l) + 1_{l \leq n-1} 1_{m_i \neq 1} c^{(n-1)}(l, m_1, \dots, m_{i-1}, m_i-1, m_{i+1}, \dots, m_l)).$$

Here $(m_1, \dots, m_{i-1}, \widehat{m_i}, m_{i+1}, \dots, m_l)$ denotes $(m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_l)$.

Since

$$\frac{d}{dx} A(x)^{-1} = -A(x)^{-1} \frac{d}{dx} A(x) \cdot A(x)^{-1},$$

the equality (C.1) holds for $n = 1$. Assume that (C.1) holds for $n - 1$. Then,

$$\begin{aligned} & \left(\frac{d}{dx}\right)^n A(x)^{-1} \\ &= \sum_{l=1}^{n-1} \prod_{j=1}^l \left(\sum_{m_j=1}^{n-1}\right) 1_{\sum_{j=1}^l m_j = n-1} c^{(n-1)}(l, m_1, m_2, \dots, m_l) (-1)^l \\ & \cdot \left(\sum_{i=1}^l \prod_{\substack{k=1 \\ \text{order}}}^{i-1} \left(A(x)^{-1} \left(\frac{d}{dx}\right)^{m_k} A(x)\right)\right) \\ & \cdot \left(-A(x)^{-1} \frac{d}{dx} A(x) \cdot A(x)^{-1} \left(\frac{d}{dx}\right)^{m_i} A(x)\right. \\ & \quad \left.+ A(x)^{-1} \left(\frac{d}{dx}\right)^{m_i+1} A(x)\right) \\ & \cdot \prod_{\substack{p=i+1 \\ \text{order}}}^l \left(A(x)^{-1} \left(\frac{d}{dx}\right)^{m_p} A(x)\right) A(x)^{-1} \\ & \quad - \prod_{\substack{k=1 \\ \text{order}}}^l \left(A(x)^{-1} \left(\frac{d}{dx}\right)^{m_k} A(x)\right) A(x)^{-1} \frac{d}{dx} A(x) \cdot A(x)^{-1} \\ &= \sum_{l=1}^{n-1} \prod_{j=1}^{l+1} \left(\sum_{m_j=1}^n\right) 1_{\sum_{j=1}^{l+1} m_j = n} (-1)^{l+1} \\ & \cdot \sum_{i=1}^{l+1} 1_{m_i=1} c^{(n-1)}(l, m_1, \dots, m_{i-1}, \widehat{m_i}, m_{i+1}, \dots, m_{l+1}) \end{aligned}$$

$$\begin{aligned}
 & \cdot \prod_{\substack{k=1 \\ \text{order}}}^{l+1} \left(A(x)^{-1} \left(\frac{d}{dx} \right)^{m_k} A(x) \right) A(x)^{-1} \\
 & + \sum_{l=1}^{n-1} \prod_{j=1}^l \left(\sum_{m_j=1}^n \right) 1_{\sum_{j=1}^l m_j=n} (-1)^l \\
 & \cdot \sum_{i=1}^l 1_{m_i \neq 1} c^{(n-1)}(l, m_1, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots, m_l) \\
 & \cdot \prod_{\substack{k=1 \\ \text{order}}}^l \left(A(x)^{-1} \left(\frac{d}{dx} \right)^{m_k} A(x) \right) A(x)^{-1} \\
 = & \sum_{l=1}^n \prod_{j=1}^l \left(\sum_{m_j=1}^n \right) 1_{\sum_{j=1}^l m_j=n} (-1)^l \sum_{i=1}^l \\
 & \cdot (1_{l \geq 2} 1_{m_i=1} c^{(n-1)}(l-1, m_1, \dots, m_{i-1}, \widehat{m_i}, m_{i+1}, \dots, m_l) \\
 & \quad + 1_{l \leq n-1} 1_{m_i \neq 1} c^{(n-1)}(l, m_1, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots, m_l)) \\
 & \cdot \prod_{\substack{k=1 \\ \text{order}}}^l \left(A(x)^{-1} \left(\frac{d}{dx} \right)^{m_k} A(x) \right) A(x)^{-1},
 \end{aligned}$$

which is equal to the right-hand side of (C.1) for n . Thus, by induction the equality (C.1) holds for all $n \in \mathbb{N}$.

It follows from (C.1) and the assumed upper bounds that

$$\begin{aligned}
 \text{(C.2)} \quad & \left\| \left(\frac{d}{dx} \right)^n A(x)^{-1} \Big|_{x=x_0} \right\|_{b \times b} \\
 & \leq \sum_{l=1}^n \prod_{j=1}^l \left(\sum_{m_j=1}^n \right) 1_{\sum_{j=1}^l m_j=n} c^{(n)}(l, m_1, m_2, \dots, m_l) s^{l+1} \\
 & \cdot \prod_{k=1}^l (qr^{m_k} (m_k!)^t) \\
 & = sr^n \sum_{l=1}^n (sq)^l \prod_{j=1}^l \left(\sum_{m_j=1}^n (m_j!)^t \right) 1_{\sum_{j=1}^l m_j=n} c^{(n)}(l, m_1, m_2, \dots, m_l)
 \end{aligned}$$

$$\leq sr^n \left(\sum_{l=1}^n (sq)^{\frac{l}{t}} \prod_{j=1}^l \left(\sum_{m_j=1}^n m_j! \right) \cdot 1_{\sum_{j=1}^l m_j=n} c^{(n)}(l, m_1, m_2, \dots, m_l) \right)^t.$$

For any $X \in \mathbb{R}_{\geq 0}$ let us compute the sum

$$(C.3) \quad \sum_{l=1}^n X^l \prod_{j=1}^l \left(\sum_{m_j=1}^n m_j! \right) 1_{\sum_{j=1}^l m_j=n} c^{(n)}(l, m_1, m_2, \dots, m_l).$$

Set

$$f(x) := \frac{X}{1+X} \sum_{n=0}^{\infty} x^n.$$

We see that

$$\begin{aligned} & \text{(the sum (C.3))} \\ &= \frac{1}{X+1} \sum_{l=1}^n \prod_{j=1}^l \left(\sum_{m_j=1}^n m_j! \right) 1_{\sum_{j=1}^l m_j=n} c^{(n)}(l, m_1, m_2, \dots, m_l) (-1)^l \\ & \cdot \prod_{k=1}^l \left(\frac{1}{1-f(0)} \left(\frac{d}{dx} \right)^{m_k} (1-f(x)) \Big|_{x=0} \right) \frac{1}{1-f(0)}. \end{aligned}$$

Then, by applying the formula (C.1) with $A(x) = 1 - f(x)$ we obtain

$$\text{(the sum (C.3))} = \frac{1}{1+X} \left(\frac{d}{dx} \right)^n \frac{1}{1-f(x)} \Big|_{x=0} = X(1+X)^{n-1} n!.$$

Substitution of this equality with $X = (sq)^{\frac{1}{t}}$ into (C.2) gives

$$\begin{aligned} & \left\| \left(\frac{d}{dx} \right)^n A(x)^{-1} \Big|_{x=x_0} \right\|_{b \times b} \\ & \leq sr^n ((sq)^{\frac{1}{t}} (1 + (sq)^{\frac{1}{t}})^{n-1} n!)^t = s^2 q (1 + (sq)^{\frac{1}{t}})^{-t} (r(1 + (sq)^{\frac{1}{t}})^t)^n (n!)^t. \quad \square \end{aligned}$$

Appendix D. The Time-Continuum, Infinite-Volume Limit of the Truncated Grassmann Integral Formulation

In this section we prove that for any $n \in \mathbb{N}$,

$$-\frac{1}{\beta L^d} \left(\frac{d}{dz}\right)^n \log \left(\int e^{-zV(\psi)} d\mu_C(\psi) \right) \Big|_{z=0}$$

converges uniformly with respect to the coupling constants as $h \rightarrow \infty$, $L \rightarrow \infty$. This convergence property itself does not imply the convergence of the full formulation

$$-\frac{1}{\beta L^d} \log \left(\int e^{-V(\psi)} d\mu_C(\psi) \right).$$

It only guarantees the convergence of any finite truncation of the Taylor series of the function

$$z \mapsto -\frac{1}{\beta L^d} \log \left(\int e^{-zV(\psi)} d\mu_C(\psi) \right)$$

around $z = 0$ as $h, L \rightarrow \infty$. However, once we know the analyticity of the Grassmann integral formulation with the coupling constants on an (h, L) -independent domain containing the origin, we can use the result of this section to prove the uniform convergence of the full formulation as $h, L \rightarrow \infty$.

We need this type of convergence result only in Subsection 7.4, where the model Hamiltonian is specifically analyzed. However, we set up the problem in a general setting without specifying the kinetic term of the Hamiltonian. We assume that

$$E \in C^{d+1}(\mathbb{R}^d; \text{Mat}(b, \mathbb{C}))$$

and (2.1), (2.2) are valid. For any $n \in \mathbb{N}$, $\mathbf{U} \in \mathbb{C}^b$ set

$$a_n(\beta, L, h)(\mathbf{U}) := -\frac{1}{\beta L^d} \frac{1}{n!} \left(\frac{d}{dz}\right)^n \log \left(\int e^{-zV(\psi)} d\mu_C(\psi) \right) \Big|_{z=0}$$

with the covariance C defined by (2.5) and $V(\psi) (\in \wedge \mathcal{V})$ defined by (2.12).

LEMMA D.1. *For any non-empty compact set K of \mathbb{C}^b and $n \in \mathbb{N}$ the following statements hold true.*

- (1) $a_n(\beta, L, h)(\cdot)$ converges in $C(K; \mathbb{C})$ as $h \rightarrow \infty$ ($h \in 2\mathbb{N}/\beta$).
- (2) Set $a_n(\beta, L) := \lim_{h \rightarrow \infty, h \in 2\mathbb{N}/\beta} a_n(\beta, L, h)$. $a_n(\beta, L)(\cdot)$ converges in $C(K; \mathbb{C})$ as $L \rightarrow \infty$ ($L \in \mathbb{N}$).

PROOF. The claims can be proved by following the same idea as in [13, Appendix B], [14, Appendix D]. However, we present the proof in a self-contained style. Here we use the notations introduced in the proof of Lemma 2.1. Let us define the matrix-valued function $A : \mathbb{R}^d \rightarrow \text{Mat}(b, \mathbb{C})$ by $A(\mathbf{k}) := (\alpha_\rho(\mathbf{k})\delta_{\rho,\eta})_{1 \leq \rho, \eta \leq b}$. By using (2.7) we can deduce from (2.10) that

$$\begin{aligned}
\text{(D.1)} \quad & C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) \\
&= \frac{\delta_{\sigma, \tau}}{L^d} \sum_{\mathbf{k} \in \Gamma^*} e^{-i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} \overline{U(\mathbf{k})} e^{(x-y)A(\mathbf{k})} \\
&\quad \cdot (1_{x \geq y} (I_b + e^{\beta A(\mathbf{k})})^{-1} - 1_{x < y} (I_b + e^{-\beta A(\mathbf{k})})^{-1}) U(\mathbf{k})^t \\
&= \frac{\delta_{\sigma, \tau}}{L^d} \sum_{\mathbf{k} \in \Gamma^*} e^{-i\langle \mathbf{x} - \mathbf{y}, \mathbf{k} \rangle} e^{(x-y)\overline{E(\mathbf{k})}} \\
&\quad \cdot (1_{x \geq y} (I_b + e^{\beta \overline{E(\mathbf{k})}})^{-1} - 1_{x < y} (I_b + e^{-\beta \overline{E(\mathbf{k})}})^{-1}), \\
& (\forall (\mathbf{x}, \sigma, x), (\mathbf{y}, \tau, y) \in \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta]).
\end{aligned}$$

Set

$$\begin{aligned}
& C_\infty(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) \\
&:= \frac{\delta_{\sigma, \tau}}{(2\pi)^d} \int_{[0, 2\pi]^d} d\mathbf{p} e^{-i\langle \mathbf{x} - \mathbf{y}, \sum_{j=1}^d \mathbf{p}_j \mathbf{v}_j \rangle} e^{(x-y)\overline{E(\sum_{j=1}^d \mathbf{p}_j \mathbf{v}_j)}} \\
&\quad \cdot (1_{x \geq y} (I_b + e^{\beta \overline{E(\sum_{j=1}^d \mathbf{p}_j \mathbf{v}_j)}})^{-1} - 1_{x < y} (I_b + e^{-\beta \overline{E(\sum_{j=1}^d \mathbf{p}_j \mathbf{v}_j)}})^{-1}), \\
& (\forall (\mathbf{x}, \sigma, x), (\mathbf{y}, \tau, y) \in \Gamma_\infty \times \{\uparrow, \downarrow\} \times [0, \beta]).
\end{aligned}$$

It follows from the continuity of the function $\mathbf{k} \mapsto E(\mathbf{k})$ that

$$\begin{aligned}
\text{(D.2)} \quad & \lim_{\substack{L \rightarrow \infty \\ L \in \mathbb{N}}} C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) = C_\infty(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y), \\
& (\forall (\mathbf{x}, \sigma, x), (\mathbf{y}, \tau, y) \in \Gamma_\infty \times \{\uparrow, \downarrow\} \times [0, \beta]).
\end{aligned}$$

Using the periodicity with the variable \mathbf{k} , we observe that for any $\mathbf{x}, \mathbf{y} \in \Gamma_\infty$ and $n \in \{1, 2, \dots, d+1\}$,

$$\begin{aligned} & \left(\frac{L}{2\pi} \left(e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y}, \mathbf{v}_j \rangle} - 1 \right) \right)^n C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y) \\ &= \frac{\delta_{\sigma, \tau}}{L^d} \sum_{\mathbf{k} \in \Gamma^*} e^{-i\langle \mathbf{x}-\mathbf{y}, \mathbf{k} \rangle} \prod_{m=1}^n \left(\frac{L}{2\pi} \int_0^{\frac{2\pi}{L}} dq_m \right) \\ & \cdot \left(\frac{\partial}{\partial p_j} \right)^n \left(e^{(x-y)\overline{E(\mathbf{k}+p_j \mathbf{v}_j)}} (1_{x \geq y} (I_b + e^{\beta \overline{E(\mathbf{k}+p_j \mathbf{v}_j)})})^{-1} \right. \\ & \quad \left. - 1_{x < y} (I_b + e^{-\beta \overline{E(\mathbf{k}+p_j \mathbf{v}_j)})})^{-1} \right) \Big|_{p_j = \sum_{m=1}^n q_m}. \end{aligned}$$

Set

$$E_{max} := \sup_{\substack{l \in \{1, 2, \dots, d\} \\ m \in \{0, 1, \dots, d+1\}}} \sup_{\mathbf{p} \in \mathbb{R}^d} \left\| \left(\frac{\partial}{\partial p_l} \right)^m E \left(\sum_{j=1}^d p_j \mathbf{v}_j \right) \right\|_{b \times b}.$$

From the above equality we can derive that

$$\begin{aligned} \text{(D.3)} \quad & \|C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y)\|_{b \times b} \leq \frac{c(\beta, d, E_{max})}{1 + \sum_{j=1}^d \left| \frac{L}{2\pi} \left(e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y}, \mathbf{v}_j \rangle} - 1 \right) \right|^{d+1}}, \\ & (\forall (\mathbf{x}, \sigma, x), (\mathbf{y}, \tau, y) \in \Gamma_\infty \times \{\uparrow, \downarrow\} \times [0, \beta)), \end{aligned}$$

where the constant $c(\beta, d, E_{max}) (\in \mathbb{R}_{>0})$ depends only on β, d, E_{max} . Especially we have

$$\begin{aligned} \text{(D.4)} \quad & \|C(\cdot \mathbf{x} \sigma x, \cdot \mathbf{y} \tau y)\|_{b \times b} \leq \frac{c(\beta, d, E_{max})}{1 + \left(\frac{2}{\pi}\right)^{d+1} \sum_{j=1}^d |\langle \mathbf{x} - \mathbf{y}, \mathbf{v}_j \rangle|^{d+1}}, \\ & (\forall (\mathbf{x}, \sigma, x), (\mathbf{y}, \tau, y) \in \Gamma_\infty \times \{\uparrow, \downarrow\} \times [0, \beta)) \\ & \text{with } |\langle \mathbf{x} - \mathbf{y}, \mathbf{v}_j \rangle| \leq L/2 \ (\forall j \in \{1, 2, \dots, d\}). \end{aligned}$$

Note that

$$\begin{aligned} a_1(\beta, L, h)(\mathbf{U}) &= \frac{1}{\beta L^d} \int V(\psi) d\mu_C(\psi) \\ &= \sum_{\rho \in \mathcal{B}} U_\rho (C(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)^2 - C(\rho \mathbf{0} \uparrow 0, \rho \mathbf{0} \uparrow 0)). \end{aligned}$$

By (D.2) the claims (1), (2) for $n = 1$ hold true.

To prove the claims for $n \in \mathbb{N}_{\geq 2}$, we need to introduce a few more notations. For any $T \in \mathbb{T}_n$, $j \in \{1, 2, \dots, n\}$ set

$$G_j^1(T) := \{v \in \{1, 2, \dots, n\} \mid v \text{ is on the shortest path connecting } 1 \text{ to } j \text{ in } T\},$$

$$\tilde{G}_j^1(T) := G_j^1(T) \setminus \{1\}.$$

Note that $1, j \in G_j^1(T)$. In the following we use the notations introduced in Subsection 3.2 plus (2.32). For any $T \in \mathbb{T}_n$, $((\sigma_l, \theta_l))_{l \in T} \in \prod_{l \in T} (\{\uparrow, \downarrow\} \times \{1, -1\})$, $(\rho_j, \mathbf{x}_j, x_j) \in \mathcal{B} \times \Gamma_\infty \times [0, \beta)$ ($j = 1, 2, \dots, n$), set

$$F_{T, ((\sigma_l, \theta_l))_{l \in T}}(\rho_1 \mathbf{x}_1 x_1, \rho_2 \mathbf{x}_2 x_2, \dots, \rho_n \mathbf{x}_n x_n)$$

$$:= \prod_{\substack{j \in \{1, 2, \dots, n\} \\ \text{with } L_j^1(T) \neq \emptyset}} \prod_{\{j, s\} \in L_j^1(T)} \cdot (-2\tilde{C}(\rho_j \mathbf{x}_j \sigma_{\{j, s\}} x_j \theta_{\{j, s\}}, \rho_s \mathbf{x}_s (-\sigma_{\{j, s\}}) x_s (-\theta_{\{j, s\}}))),$$

$$F'_{T, ((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n))$$

$$:= \prod_{\substack{j \in \{1, 2, \dots, n\} \\ \text{with } L_j^1(T) \neq \emptyset}} \prod_{\{j, s\} \in L_j^1(T)} (-2\tilde{C}(\rho_j \mathbf{0} \sigma_{\{j, s\}} x_j \theta_{\{j, s\}}, \rho_s \mathbf{x}_s (-\sigma_{\{j, s\}}) x_s (-\theta_{\{j, s\}}))),$$

where

$$\tilde{C}(\rho \mathbf{x} \sigma x \theta, \eta \mathbf{y} \tau y \xi)$$

$$:= \frac{1}{2} (1_{(\theta, \xi) = (1, -1)} C(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y) - 1_{(\theta, \xi) = (-1, 1)} C(\eta \mathbf{y} \tau y, \rho \mathbf{x} \sigma x)),$$

$$(\forall (\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi) \in \mathcal{B} \times \Gamma_\infty \times \{\uparrow, \downarrow\} \times [0, \beta) \times \{1, -1\}).$$

Moreover, for any $(\rho_j, \mathbf{x}_j, x_j) \in \mathcal{B} \times \Gamma \times [0, \beta)_h$ ($j = 1, 2, \dots, n$) set

$$H_{T, ((\sigma_l, \theta_l))_{l \in T}}(\rho_1 \mathbf{x}_1 x_1, \rho_2 \mathbf{x}_2 x_2, \dots, \rho_n \mathbf{x}_n x_n)$$

$$:= \prod_{\substack{j \in \{1, 2, \dots, n\} \\ \text{with } L_j^1(T) \neq \emptyset}} \prod_{\{j, s\} \in L_j^1(T)} \frac{\partial}{\partial \psi^j_{\rho_j \mathbf{x}_j \sigma_{\{j, s\}} x_j \theta_{\{j, s\}}}} \frac{\partial}{\partial \psi^s_{\rho_s \mathbf{x}_s (-\sigma_{\{j, s\}}) x_s (-\theta_{\{j, s\}})}}.$$

Recall the definition (2.32) of the polynomial $V_{\rho \mathbf{x}}^+(\psi) \in \bigwedge \mathcal{V}$. By the invariant property

$$(D.5) \quad C((\rho, \mathbf{x} + \mathbf{z}, \sigma, x), (\eta, \mathbf{y} + \mathbf{z}, \tau, y)) = C((\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y)),$$

$$(\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \Gamma_\infty)$$

we see that

$$\begin{aligned}
 (D.6) \quad & ope(T, C) H_{T, ((\sigma_l, \theta_l))_{l \in T}} (\rho_1 \mathbf{x}_1 x_1, \rho_2 \mathbf{x}_2 x_2, \dots, \rho_n \mathbf{x}_n x_n) \\
 & \cdot \prod_{j=1}^n V_{\rho_j \mathbf{x}_j x_j}^+(\psi^j) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\
 & = ope(T, C) \\
 & \quad \cdot H_{T, ((\sigma_l, \theta_l))_{l \in T}} (\rho_1 r_L(\mathbf{x}_1 + \mathbf{y}) x_1, \rho_2 r_L(\mathbf{x}_2 + \mathbf{y}) x_2, \dots, \\
 & \quad \quad \quad \rho_n r_L(\mathbf{x}_n + \mathbf{y}) x_n) \\
 & \quad \cdot \prod_{j=1}^n V_{\rho_j r_L(\mathbf{x}_j + \mathbf{y}) x_j}^+(\psi^j) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}, \quad (\forall \mathbf{y} \in \Gamma_\infty).
 \end{aligned}$$

Using (3.9), (D.5), (D.6), we have that

$$\begin{aligned}
 (D.7) \quad & a_n(\beta, L, h)(\mathbf{U}) = \frac{(-1)^{n+1}}{n! \beta L^d} \sum_{T \in \mathbb{T}_n} Ope(T, C) \prod_{j=1}^n V(\psi^j) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\
 & = \frac{(-1)^{n+1}}{n! \beta L^d} \sum_{T \in \mathbb{T}_n} \prod_{l \in T} \left(\sum_{\substack{(\sigma_l, \theta_l) \\ \in \{\uparrow, \downarrow\} \times \{1, -1\}}} \right) \prod_{i=1}^n \left(\frac{1}{h} \sum_{\substack{(\rho_i, \mathbf{x}_i, x_i) \\ \in \mathcal{B} \times \Gamma \times [0, \beta]_h}} U_{\rho_i} \right) \\
 & \quad \cdot ope(T, C) \\
 & \quad \cdot F_{T, ((\sigma_l, \theta_l))_{l \in T}} (\rho_1 \mathbf{x}_1 x_1, \rho_2 \mathbf{x}_2 x_2, \dots, \rho_n \mathbf{x}_n x_n) \\
 & \quad \cdot H_{T, ((\sigma_l, \theta_l))_{l \in T}} (\rho_1 \mathbf{x}_1 x_1, \rho_2 \mathbf{x}_2 x_2, \dots, \rho_n \mathbf{x}_n x_n) \\
 & \quad \cdot \prod_{j=1}^n V_{\rho_j \mathbf{x}_j x_j}^+(\psi^j) \Bigg|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\
 & = \frac{(-1)^{n+1}}{n! \beta L^d} \sum_{T \in \mathbb{T}_n} \prod_{l \in T} \left(\sum_{\substack{(\sigma_l, \theta_l) \\ \in \{\uparrow, \downarrow\} \times \{1, -1\}}} \right) \prod_{i=1}^n \left(\frac{1}{h} \sum_{\substack{(\rho_i, \mathbf{x}_i, x_i) \\ \in \mathcal{B} \times \Gamma \times [0, \beta]_h}} U_{\rho_i} \right) \\
 & \quad \cdot ope(T, C) \\
 & \quad \cdot F_{T, ((\sigma_l, \theta_l))_{l \in T}} \left(\rho_1 \mathbf{x}_1 x_1, \rho_2 r_L \left(\sum_{v \in G_2^1(T)} \mathbf{x}_v \right) x_2, \dots, \right.
 \end{aligned}$$

$$\begin{aligned}
 & \rho_n r_L \left(\sum_{v \in G_n^1(T)} \mathbf{x}_v \right) x_n \\
 \cdot & H_{T,((\sigma_l, \theta_l))_{l \in T}} \left(\rho_1 \mathbf{x}_1 x_1, \rho_2 r_L \left(\sum_{v \in G_2^1(T)} \mathbf{x}_v \right) x_2, \dots, \right. \\
 & \left. \rho_n r_L \left(\sum_{v \in G_n^1(T)} \mathbf{x}_v \right) x_n \right) \\
 \cdot & \prod_{j=1}^n V_{\rho_j r_L(\sum_{v \in G_j^1(T)} \mathbf{x}_v) x_j}^+(\psi^j) \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\
 = & \frac{(-1)^{n+1}}{n! \beta L^d} \sum_{T \in \mathbb{T}_n} \prod_{l \in T} \left(\sum_{\substack{(\sigma_l, \theta_l) \\ \in \{\uparrow, \downarrow\} \times \{1, -1\}}} \right) \prod_{i=1}^n \left(\frac{1}{h} \sum_{\substack{(\rho_i, \mathbf{x}_i, x_i) \\ \in \mathcal{B} \times \Gamma \times [0, \beta)_h}} U_{\rho_i} \right) \\
 \cdot & ope(T, C) \\
 \cdot & F'_{T,((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n)) \\
 \cdot & H_{T,((\sigma_l, \theta_l))_{l \in T}} \left(\rho_1 \mathbf{x}_1 x_1, \rho_2 r_L \left(\sum_{v \in G_2^1(T)} \mathbf{x}_v \right) x_2, \dots, \right. \\
 & \left. \rho_n r_L \left(\sum_{v \in G_n^1(T)} \mathbf{x}_v \right) x_n \right) \\
 \cdot & \prod_{j=1}^n V_{\rho_j r_L(\sum_{v \in G_j^1(T)} \mathbf{x}_v) x_j}^+(\psi^j) \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}} \\
 = & \frac{(-1)^{n+1}}{n! \beta} \sum_{T \in \mathbb{T}_n} \prod_{l \in T} \left(\sum_{\substack{(\sigma_l, \theta_l) \\ \in \{\uparrow, \downarrow\} \times \{1, -1\}}} \right) \left(\frac{1}{h} \sum_{\substack{(\rho_1, x_1) \\ \in \mathcal{B} \times [0, \beta)_h}} U_{\rho_1} \right) \\
 \cdot & \prod_{i=2}^n \left(\frac{1}{h} \sum_{\substack{(\rho_i, \mathbf{x}_i, x_i) \\ \in \mathcal{B} \times \Gamma \times [0, \beta)_h}} U_{\rho_i} \right) ope(T, C) \\
 \cdot & F'_{T,((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n))
 \end{aligned}$$

$$\begin{aligned} & \cdot H_{T,((\sigma_l, \theta_l))_{l \in T}} \left(\rho_1 \mathbf{0} x_1, \rho_2 r_L \left(\sum_{v \in \tilde{G}_2^1(T)} \mathbf{x}_v \right) x_2, \dots, \right. \\ & \qquad \qquad \qquad \left. \rho_n r_L \left(\sum_{v \in \tilde{G}_n^1(T)} \mathbf{x}_v \right) x_n \right) \\ & \cdot V_{\rho_1 \mathbf{0} x_1}^+(\psi^1) \prod_{j=2}^n V_{\rho_j r_L(\sum_{v \in \tilde{G}_j^1(T)} \mathbf{x}_v) x_j}^+(\psi^j) \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}. \end{aligned}$$

For any $(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n-1}) \in \Gamma_\infty^{n-1}$ set

$$\begin{aligned} & \chi_L(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n-1}) \\ & := 1_{L \in 2\mathbb{N}^1_{(\mathbf{x}_i, \mathbf{v}_j) \in \{-\frac{L}{2}, -\frac{L}{2}+1, \dots, \frac{L}{2}-1\}, (\forall i \in \{1, 2, \dots, n-1\}, j \in \{1, 2, \dots, d\})}} \\ & \quad + 1_{L \notin 2\mathbb{N}^1_{(\mathbf{x}_i, \mathbf{v}_j) \in \{-\frac{L-1}{2}, -\frac{L-1}{2}+1, \dots, \frac{L-1}{2}\}, (\forall i \in \{1, 2, \dots, n-1\}, j \in \{1, 2, \dots, d\})}}. \end{aligned}$$

For any $(\rho_1, \rho_2, \dots, \rho_n) \in \mathcal{B}^n$, $(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n) \in \Gamma_\infty^{n-1}$, $(x_1, x_2, \dots, x_n) \in [0, \beta]_h^n$, set

$$\begin{aligned} & H'_{T,((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n)) \\ & := \text{ope}(T, C) \\ & \cdot H_{T,((\sigma_l, \theta_l))_{l \in T}} \left(\rho_1 \mathbf{0} x_1, \rho_2 r_L \left(\sum_{v \in \tilde{G}_2^1(T)} \mathbf{x}_v \right) x_2, \dots, \rho_n r_L \left(\sum_{v \in \tilde{G}_n^1(T)} \mathbf{x}_v \right) x_n \right) \\ & \cdot V_{\rho_1 \mathbf{0} x_1}^+(\psi^1) \prod_{j=2}^n V_{\rho_j r_L(\sum_{v \in \tilde{G}_j^1(T)} \mathbf{x}_v) x_j}^+(\psi^j) \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1, 2, \dots, n\})}}. \end{aligned}$$

Moreover, for any $x \in [0, \beta)$ let \hat{x} denote an element of $[0, \beta)_h$ satisfying $x \in [\hat{x}, \hat{x} + 1/h)$. With these notations we obtain from (D.7) that

$$\begin{aligned} & a_n(\beta, L, h)(\mathbf{U}) \\ & = \frac{(-1)^{n+1}}{n! \beta} \sum_{T \in \mathbb{T}_n} \prod_{l \in T} \left(\sum_{\substack{(\sigma_l, \theta_l) \\ \in \{\uparrow, \downarrow\} \times \{1, -1\}}} \right) \sum_{\rho_1 \in \mathcal{B}} U_{\rho_1} \int_0^\beta dx_1 \\ & \cdot \prod_{i=2}^n \left(\sum_{\substack{(\rho_i, \mathbf{x}_i) \\ \in \mathcal{B} \times \Gamma_\infty}} U_{\rho_i} \int_0^\beta dx_i \right) \chi_L(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n) \end{aligned}$$

$$\begin{aligned} &\cdot F'_{T,((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)) \\ &\cdot H'_{T,((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)). \end{aligned}$$

Using (D.3), (3.8) and the properties of the matrix $M_{at}(T, \xi, \mathbf{s})$ inside $ope(T, C)$, we can derive that

$$\begin{aligned} (D.8) \quad &|H'_{T,((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n))| \\ &\leq \prod_{i=1}^n \left(\sum_{m_i \in \{2,4\}} \right) \Big| ope(T, C) \\ &\quad \cdot H_{T,((\sigma_l, \theta_l))_{l \in T}} \left(\rho_1 \mathbf{0} \hat{x}_1, \rho_2 r_L \left(\sum_{v \in \tilde{G}_2^1(T)} \mathbf{x}_v \right) \hat{x}_2, \dots, \right. \\ &\quad \left. \rho_n r_L \left(\sum_{v \in \tilde{G}_n^1(T)} \mathbf{x}_v \right) \hat{x}_n \right) \\ &\quad \cdot \mathcal{P}_{m_1} V_{\rho_1 \mathbf{0} \hat{x}_1}^+(\psi^1) \prod_{j=2}^n \mathcal{P}_{m_j} V_{\rho_j r_L(\sum_{v \in \tilde{G}_j^1(T)} \mathbf{x}_v) \hat{x}_j}^+(\psi^j) \Big|_{\substack{\psi^j=0 \\ (\forall j \in \{1,2,\dots,n\})}} \Big| \\ &\leq \prod_{i=1}^n \left(\sum_{m_i \in \{2,4\}} 1_{n_i(T) \leq m_i} n_i(T)! \binom{m_i}{n_i(T)} \right) \\ &\quad \cdot \sup_{\substack{\mathbf{p}_j, \mathbf{q}_j \in \mathbb{C}^n \text{ with } \|\mathbf{p}_j\|_{\mathbb{C}^n}, \|\mathbf{q}_j\|_{\mathbb{C}^n} \leq 1 \\ (j=1,2,\dots, \frac{1}{2} \sum_{k=1}^n m_k - n + 1)}} \sup_{\substack{X_j, Y_j \in I_0 \\ (j=1,2,\dots, \frac{1}{2} \sum_{k=1}^n m_k - n + 1)}} \\ &\quad \cdot |\det(\langle \mathbf{p}_i, \mathbf{q}_j \rangle_{\mathbb{C}^n} C(X_i, Y_j))_{1 \leq i, j \leq \frac{1}{2} \sum_{k=1}^n m_k - n + 1}| \\ &\leq \prod_{i=1}^n \left(\sum_{m_i \in \{2,4\}} 1_{n_i(T) \leq m_i} n_i(T)! \binom{m_i}{n_i(T)} \right) \\ &\quad \cdot \left(\frac{1}{2} \sum_{k=1}^n m_k - n + 1 \right)! c(\beta, d, E_{max})^{\frac{1}{2} \sum_{k=1}^n m_k - n + 1}. \end{aligned}$$

With these preparations we can prove the claims (1), (2) for $n \in \mathbb{N}_{\geq 2}$. Let us take any non-empty compact set K of \mathbb{C}^b .

(1): Since it consists of finite sums and products of the covariance $C : (\mathcal{B} \times \Gamma_\infty \times \{\uparrow, \downarrow\} \times [0, \beta])^2 \rightarrow \mathbb{C}$, the domain of the function

$$H'_{T,((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \dots, \rho_n), \cdot, \cdot)$$

can be naturally extended to $(\Gamma_\infty)^{n-1} \times [0, \beta]^n$. Note that $(x, y) \mapsto C(\rho \mathbf{x} \sigma x, \eta \mathbf{y} \tau y)$ is continuous a.e. in $[0, \beta]^2$, and thus

$$(x_1, x_2, \dots, x_n) \mapsto H'_{T, ((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \dots, \rho_n), (\mathbf{x}_2, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n))$$

is continuous a.e. in $[0, \beta]^n$. These imply that

$$\begin{aligned} & \lim_{\substack{h \rightarrow \infty \\ h \in 2\mathbb{N}/\beta}} F'_{T, ((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \dots, \rho_n), (\mathbf{x}_2, \dots, \mathbf{x}_n), (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)) \\ &= F'_{T, ((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \dots, \rho_n), (\mathbf{x}_2, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n)), \\ & \lim_{\substack{h \rightarrow \infty \\ h \in 2\mathbb{N}/\beta}} H'_{T, ((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \dots, \rho_n), (\mathbf{x}_2, \dots, \mathbf{x}_n), (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)) \\ &= H'_{T, ((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \dots, \rho_n), (\mathbf{x}_2, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n)) \end{aligned}$$

for a.e. $(x_1, x_2, \dots, x_n) \in [0, \beta]^n$.

By these convergence properties and the uniform bounds (D.3), (D.8) we can apply the dominated convergence theorem for $L^1([0, \beta]^n)$ to conclude that

$$\lim_{\substack{h \rightarrow \infty \\ h \in 2\mathbb{N}/\beta}} a_n(\beta, L, h) = a_n(\beta, L) \text{ in } C(K; \mathbb{C}),$$

where

$$\begin{aligned} & a_n(\beta, L)(\mathbf{U}) \\ &:= \frac{(-1)^{n+1}}{n! \beta} \sum_{T \in \mathbb{T}_n} \prod_{l \in T} \left(\sum_{\substack{(\sigma_l, \theta_l) \\ \in \{\uparrow, \downarrow\} \times \{1, -1\}}} \right) \sum_{\rho_1 \in \mathcal{B}} U_{\rho_1} \int_0^\beta dx_1 \\ & \cdot \prod_{i=2}^n \left(\sum_{\substack{(\rho_i, \mathbf{x}_i) \\ \in \mathcal{B} \times \Gamma_\infty}} U_{\rho_i} \int_0^\beta dx_i \right) \chi_L(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n) \\ & \cdot F'_{T, ((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n)) \\ & \cdot H'_{T, ((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n)). \end{aligned}$$

(2): Substitution of (D.4) and (D.8) yields that

$$(D.9) \quad \left| \prod_{i=1}^n U_{\rho_i} \chi_L(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n) \right|$$

$$\begin{aligned}
 & \cdot F'_{T,((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n)) \\
 & \cdot H'_{T,((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n)) \Big| \\
 & \leq \prod_{i=1}^n |U_{\rho_i}| \prod_{j=2}^n \left(\frac{c(\beta, d, E_{max})}{1 + (\frac{2}{\pi})^{d+1} \sum_{k=1}^d |\langle \mathbf{x}_j, \mathbf{v}_k \rangle|^{d+1}} \right) \\
 & \cdot \prod_{k=1}^n \left(\sum_{m_k \in \{2,4\}} 1_{n_k(T) \leq m_k} n_k(T)! \binom{m_k}{n_k(T)} \right) \\
 & \cdot \left(\frac{1}{2} \sum_{k=1}^n m_k - n + 1 \right)! c(\beta, d, E_{max})^{\frac{1}{2} \sum_{k=1}^n m_k - n + 1}, \\
 & (\forall (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n) \in \Gamma_\infty^{n-1}, (x_1, x_2, \dots, x_n) \in [0, \beta]^n).
 \end{aligned}$$

The right-hand side of the inequality above is integrable over $\Gamma_\infty^{n-1} \times [0, \beta]^n$. Note that

$$\lim_{\substack{L \rightarrow \infty \\ L \in \mathbb{N}}} \chi_L(\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n) = 1, \quad (\forall (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n) \in \Gamma_\infty^{n-1}).$$

Moreover, we can see from (D.2) that

$$\begin{aligned}
 & \lim_{\substack{L \rightarrow \infty \\ L \in \mathbb{N}}} F'_{T,((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n)) \\
 & \cdot H'_{T,((\sigma_l, \theta_l))_{l \in T}}((\rho_1, \rho_2, \dots, \rho_n), (\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n))
 \end{aligned}$$

exists for any $((\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n), (x_1, x_2, \dots, x_n)) \in \Gamma_\infty^{n-1} \times [0, \beta]^n$. By these convergence properties and the inequality (D.9) we can apply the dominated convergence theorem for $L^1(\Gamma_\infty^{n-1} \times [0, \beta]^n)$ to conclude that $a_n(\beta, L)$ converges in $C(K; \mathbb{C})$ as $L \rightarrow \infty$ ($L \in \mathbb{N}$). \square

Appendix E. Direct Treatment of the Free Energy Density

In this part of Appendix we prove some lemmas concerning the free energy density in direct ways without going through the Grassmann integral formulation. Though the results of this section are needed only in the model-dependent analysis in Subsection 7.4, here we consider the problem in a general configuration. Let H_0, V, H be the operators on

$F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$ defined in Subsection 2.1 with $E \in C(\mathbb{R}^d; \text{Mat}(b, \mathbb{C}))$ satisfying (2.1), (2.2). In the following id denotes the identity map on $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$.

LEMMA E.1. *For any $\mathbf{k} \in \Gamma^*$ let $\alpha_\rho(\mathbf{k})$ ($\rho \in \mathcal{B}$) be the eigen values of $E(\mathbf{k})$. Then,*

$$-\frac{1}{\beta L^d} \log(\text{Tr } e^{-\beta H_0}) = -\frac{2}{\beta L^d} \sum_{\rho \in \mathcal{B}} \sum_{\mathbf{k} \in \Gamma^*} \log(1 + e^{-\beta \alpha_\rho(\mathbf{k})}).$$

PROOF. With the unitary matrix $U(\mathbf{k}) \in \text{Mat}(b, \mathbb{C})$ satisfying (2.7), set

$$\psi_{\rho \mathbf{k} \sigma} := \frac{1}{L^{\frac{d}{2}}} \sum_{\mathbf{x} \in \Gamma} e^{-i\langle \mathbf{k}, \mathbf{x} \rangle} \sum_{\eta \in \mathcal{B}} U(\mathbf{k})^*(\rho, \eta) \psi_{\eta \mathbf{x} \sigma}, \quad ((\rho, \mathbf{k}, \sigma) \in \mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\}).$$

We can number each element of $\mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\}$ so that $\mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\} = \{K_j\}_{j=1}^{2bL^d}$. The anti-commutation relation holds as follows.

$$\begin{aligned} \psi_{K_i} \psi_{K_j} + \psi_{K_j} \psi_{K_i} &= 0, \\ \psi_{K_i}^* \psi_{K_j} + \psi_{K_j} \psi_{K_i}^* &= \delta_{i,j} id, \quad (\forall i, j \in \{1, 2, \dots, 2bL^d\}). \end{aligned}$$

By (2.7),

$$H_0 = \sum_{(\rho, \mathbf{k}, \sigma) \in \mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\}} \alpha_\rho(\mathbf{k}) \psi_{\rho \mathbf{k} \sigma}^* \psi_{\rho \mathbf{k} \sigma},$$

and thus,

$$H_0 \psi_{\rho_1 \mathbf{k}_1 \sigma_1}^* \psi_{\rho_2 \mathbf{k}_2 \sigma_2}^* \cdots \psi_{\rho_n \mathbf{k}_n \sigma_n}^* \Omega = \sum_{j=1}^n \alpha_{\rho_j}(\mathbf{k}_j) \psi_{\rho_1 \mathbf{k}_1 \sigma_1}^* \psi_{\rho_2 \mathbf{k}_2 \sigma_2}^* \cdots \psi_{\rho_n \mathbf{k}_n \sigma_n}^* \Omega,$$

$$(\forall (\rho_j, \mathbf{k}_j, \sigma_j) \in \mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\} \quad (j = 1, 2, \dots, n)).$$

This implies that

$$\text{Tr } e^{-\beta H_0} = 1 + \sum_{\substack{S \subset \mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\} \\ \text{with } S \neq \emptyset}} e^{-\beta \sum_{(\rho, \mathbf{k}, \sigma) \in S} \alpha_\rho(\mathbf{k})} = \prod_{\rho \in \mathcal{B}} \prod_{\mathbf{k} \in \Gamma^*} (1 + e^{-\beta \alpha_\rho(\mathbf{k})})^2. \quad \square$$

We use the following lemma to approximate the normalized free energy density at β by that at $[\beta](\in \mathbb{N})$.

LEMMA E.2. *For any $\beta \in \mathbb{R}_{\geq 1}$,*

$$\begin{aligned} & \left| \frac{1}{\beta L^d} \log \left(\frac{\text{Tr } e^{-\beta H}}{\text{Tr } e^{-\beta H_0}} \right) - \frac{1}{[\beta] L^d} \log \left(\frac{\text{Tr } e^{-[\beta] H}}{\text{Tr } e^{-[\beta] H_0}} \right) \right| \\ & \leq \int_{[\beta]}^{\beta} d\gamma \frac{1}{\gamma^2 L^d} \left| \log \left(\frac{\text{Tr } e^{-\gamma H}}{\text{Tr } e^{-\gamma H_0}} \right) \right| \\ & \quad + 2b \left(2 \sup_{\mathbf{k} \in \mathbb{R}^d} \|E(\mathbf{k})\|_{b \times b} + \sup_{\rho \in \mathcal{B}} |U_\rho| \right) \log \left(\frac{\beta}{[\beta]} \right). \end{aligned}$$

PROOF. Let $\sigma(H)$ denote the set of all eigen values of H . We can take an orthonormal basis B of $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$ consisting of eigen vectors of H . Then,

$$(E.1) \quad |\text{Tr}(H e^{-\beta H})| \leq \max_{\alpha \in \sigma(H)} |\alpha| \sum_{v \in B} \langle v, e^{-\beta H} v \rangle_{F_f} = \|H\|_{\mathfrak{B}(F_f)} \text{Tr } e^{-\beta H}.$$

Since the eigen values of H_0 are

$$\left\{ \sum_{(\rho, \mathbf{k}, \sigma) \in S} \alpha_\rho(\mathbf{k}) \mid S \subset \mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\}, S \neq \emptyset \right\} \cup \{0\},$$

$$(E.2) \quad \|H_0\|_{\mathfrak{B}(F_f)} \leq \sum_{(\rho, \mathbf{k}, \sigma) \in \mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\}} |\alpha_\rho(\mathbf{k})| \leq 2bL^d \sup_{\mathbf{k} \in \mathbb{R}^d} \|E(\mathbf{k})\|_{b \times b}.$$

Since

$$\|\psi_{\rho \mathbf{x} \sigma}^* \psi_{\rho \mathbf{x} \sigma}\|_{\mathfrak{B}(F_f)} = 1, \quad (\forall (\rho, \mathbf{x}, \sigma) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}),$$

$$(E.3) \quad \|V\|_{\mathfrak{B}(F_f)} \leq \sum_{(\rho, \mathbf{x}) \in \mathcal{B} \times \Gamma} |U_\rho| \|\psi_{\rho \mathbf{x} \uparrow}^* \psi_{\rho \mathbf{x} \downarrow}^* \psi_{\rho \mathbf{x} \downarrow} \psi_{\rho \mathbf{x} \uparrow}\|_{\mathfrak{B}(F_f)} \\ + \frac{1}{2} \sum_{(\rho, \mathbf{x}, \sigma) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}} |U_\rho| \|\psi_{\rho \mathbf{x} \sigma}^* \psi_{\rho \mathbf{x} \sigma}\|_{\mathfrak{B}(F_f)}$$

$$\leq 2bL^d \sup_{\rho \in \mathcal{B}} |U_\rho|.$$

By combining (E.2), (E.3) with (E.1) we obtain

$$(E.4) \quad \left| \frac{\text{Tr}(e^{-\beta H} H)}{\text{Tr} e^{-\beta H}} \right| \leq 2bL^d \left(\sup_{\mathbf{k} \in \mathbb{R}^d} \|E(\mathbf{k})\|_{b \times b} + \sup_{\rho \in \mathcal{B}} |U_\rho| \right).$$

Note that

$$(E.5) \quad \begin{aligned} & \frac{1}{\beta L^d} \log \left(\frac{\text{Tr} e^{-\beta H}}{\text{Tr} e^{-\beta H_0}} \right) - \frac{1}{[\beta] L^d} \log \left(\frac{\text{Tr} e^{-[\beta] H}}{\text{Tr} e^{-[\beta] H_0}} \right) \\ &= \int_{[\beta]}^{\beta} d\gamma \frac{d}{d\gamma} \left(\frac{1}{\gamma L^d} \log \left(\frac{\text{Tr} e^{-\gamma H}}{\text{Tr} e^{-\gamma H_0}} \right) \right) \\ &= - \int_{[\beta]}^{\beta} d\gamma \frac{1}{\gamma^2 L^d} \log \left(\frac{\text{Tr} e^{-\gamma H}}{\text{Tr} e^{-\gamma H_0}} \right) - \int_{[\beta]}^{\beta} d\gamma \frac{1}{\gamma L^d} \frac{\text{Tr}(e^{-\gamma H} H)}{\text{Tr} e^{-\gamma H}} \\ & \quad + \int_{[\beta]}^{\beta} d\gamma \frac{1}{\gamma L^d} \frac{\text{Tr}(e^{-\gamma H_0} H_0)}{\text{Tr} e^{-\gamma H_0}}. \end{aligned}$$

Using (E.4), we can derive the claimed inequality from (E.5). \square

We use the next lemma to relate the output of the infrared integration to the free energy density by means of the identity theorem.

LEMMA E.3. *For any $r \in \mathbb{R}_{>0}$ there exists a domain $O(\subset \mathbb{C})$ such that $(-r, r) \subset O$ and the function $\mathbf{U} \mapsto \log(\text{Tr} e^{-\beta H})$ is analytic in $O^b(= O \times O \times \cdots \times O)$.*

PROOF. Take any $\delta_\rho \in [-1, 1]$ ($\rho \in \mathcal{B}$). Define the operator V_0 on $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$ by

$$V_0 := \sum_{(\rho, \mathbf{x}) \in \mathcal{B} \times \Gamma} \delta_\rho \psi_{\rho \mathbf{x} \uparrow}^* \psi_{\rho \mathbf{x} \downarrow}^* \psi_{\rho \mathbf{x} \downarrow} \psi_{\rho \mathbf{x} \uparrow} - \frac{1}{2} \sum_{(\rho, \mathbf{x}, \sigma) \in \mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}} \delta_\rho \psi_{\rho \mathbf{x} \sigma}^* \psi_{\rho \mathbf{x} \sigma}.$$

Take any $r \in \mathbb{R}_{>0}$ and assume that $\mathbf{U} \in (-r, r)^b$. Then, for any $\delta \in [0, 1]$,

$$\left| \text{Tr} e^{-\beta(H+i\delta V_0)} - \text{Tr} e^{-\beta H} \right| \leq \int_0^\delta d\varepsilon \left| \frac{d}{d\varepsilon} \text{Tr} e^{-\beta(H+i\varepsilon V_0)} \right|$$

$$\leq \delta\beta 2^{2bL^d} \|V_0\|_{\mathfrak{B}(F_f)} e^{\beta(\|H\|_{\mathfrak{B}(F_f)} + \|V_0\|_{\mathfrak{B}(F_f)})},$$

where we used the equality $(d/d\varepsilon) \operatorname{Tr} e^{-\beta(H+i\varepsilon V_0)} = -i\beta \operatorname{Tr}(e^{-\beta(H+i\varepsilon V_0)} V_0)$. This equality can be justified by, e.g., [12, Lemma 2.3]. Therefore,

$$\begin{aligned} & \operatorname{Re} \operatorname{Tr} e^{-\beta(H+i\delta V_0)} \\ & \geq \operatorname{Tr} e^{-\beta H} - \delta\beta 2^{2bL^d} \|V_0\|_{\mathfrak{B}(F_f)} e^{\beta(\|H\|_{\mathfrak{B}(F_f)} + \|V_0\|_{\mathfrak{B}(F_f)})} \\ & \geq 1 - \delta\beta 2^{2bL^d} \sup_{\substack{\mathbf{U} \in [-r,r]^b, \\ \delta_\rho \in [-1,1] (\rho \in \mathcal{B})}} \left\{ \|V_0\|_{\mathfrak{B}(F_f)} e^{\beta(\|H\|_{\mathfrak{B}(F_f)} + \|V_0\|_{\mathfrak{B}(F_f)})} \right\}. \end{aligned}$$

We can conclude from the above inequality that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$\operatorname{Re} \operatorname{Tr} e^{-\beta(H+iV_0)} > 0, \quad (\forall \mathbf{U} \in (-r, r)^b, \delta_\rho \in (-\varepsilon, \varepsilon) (\rho \in \mathcal{B})).$$

This implies that the function $\mathbf{U} \mapsto \log(\operatorname{Tr} e^{-\beta H})$ is analytic in the domain

$$\{x + iy \mid x \in (-r, r), y \in (-\varepsilon, \varepsilon)\}^b \subset \mathbb{C}^b). \quad \square$$

Acknowledgments. The author wishes to thank Fumihiko Nakano for interesting conversations. This work was supported by JSPS KAKENHI Grant Number 26870110.

Notation

Parameters and constants

Notation	Description	Reference
L	size of the spatial lattice	Subsection 1.2
$t_{h,e}, t_{h,o},$ $t_{v,e}, t_{v,o}$	magnitude of the hopping matrix elements	Subsection 1.2
$U_{e,e}, U_{o,e},$ $U_{e,o}, U_{o,o}$	coupling constants	Subsection 1.2
β	inverse temperature	Subsection 1.2
$f_{\mathbf{t}}$	parameter depending only on $t_{h,e}, t_{h,o},$ $t_{v,e}, t_{v,o}$	Subsection 1.2

d	spatial dimension	Subsection 2.1
b	number of sites in a primitive unit cell	Subsection 2.1
h	element of $(2/\beta)\mathbb{N}$, step size of the discretization of $[0, \beta]$	Subsection 2.2
N	$4b\beta hL^d$, cardinality of I	Subsection 2.2
$w(l)$	scale-dependent weight	beginning of Section 3
r	number belonging to $(0, 1]$, exponent inside $\ \cdot\ _{l,0}$, $\ \cdot\ _{l,1}$, $ \cdot - \cdot _l$	beginning of Section 3
c	real positive constant independent of any parameter	beginning of Section 5
M	parameter to control the upper bounds of covariances	Subsection 5.1
$c(\alpha_1, \dots, \alpha_n)$	real positive constant depending only on parameters $\alpha_1, \dots, \alpha_n$	beginning of Section 6
M_{UV}	parameter to control the size of support of UV cut-off functions	Subsection 6.1
N_h	largest scale in the UV integration	Subsection 6.1
c_w	constant ($\in (0, 1]$) inside $w(0)$ independent of any parameter	Subsection 6.1
M_{IR}	parameter to control the size of support of IR cut-off functions	Subsection 7.2
N_β	smallest scale in the IR integration	Subsection 7.2

Sets and spaces

Notation	Description	Reference
$\Gamma(2L)$	$\{0, 1, \dots, 2L - 1\}^2$	Subsection 1.2
$F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$	Fermionic Fock space	Subsection 1.2
$D_t(c)$	subset of \mathbb{C} which depends on $t_{h,e}$, $t_{h,o}$, $t_{v,e}$, $t_{v,o}$	Subsection 1.2
Γ	spatial lattice for a generalized system	Subsection 2.1
Γ^*	momentum lattice for a generalized system	Subsection 2.1

\mathcal{B}	$\{1, 2, \dots, b\}$	Subsection 2.1
$F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$	Fermionic Fock space	Subsection 2.1
$\text{Mat}(n, \mathbb{C})$	set of all $n \times n$ matrices	Subsection 2.1
$[0, \beta)_h$	$\{0, 1/h, \dots, \beta - 1/h\}$	Subsection 2.2
I_0	$\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times [0, \beta)_h$	Subsection 2.2
I	$I_0 \times \{1, -1\}$	Subsection 2.2
\mathcal{V}	complex vector space spanned by the basis $\{\psi_X\}_{X \in I}$	Subsection 2.2
\mathcal{V}_p	complex vector space spanned by the basis $\{\psi_X^p\}_{X \in I}$	Subsection 2.2
$\wedge \mathcal{V}$	Grassmann algebra generated by $\{\psi_X\}_{X \in I}$	Subsection 2.2
\mathbb{S}_n	set of all permutations over $\{1, 2, \dots, n\}$	Subsection 2.2
\mathcal{M}	$(\pi/\beta)(2\mathbb{Z} + 1)$	Subsection 2.3
\mathcal{M}_h	$\{\omega \in \mathcal{M} \mid \omega < \pi h\}$	Subsection 2.3
\mathcal{H}	Hilbert space $L^2(\mathcal{B} \times \Gamma^* \times \{\uparrow, \downarrow\} \times \mathcal{M}_h)$	Subsection 2.5
\mathbb{T}_n	set of all trees over $\{1, 2, \dots, n\}$	Subsection 3.2
$L_q^p(T)$	subgraph of tree T	Subsection 3.2
$I_{0, \infty}$	$\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times (1/h)\mathbb{Z}$	beginning of Section 4
I_∞	$I_{0, \infty} \times \{1, -1\}$	beginning of Section 4
Γ_∞	$\{\sum_{j=1}^d m_j \mathbf{u}_j \mid m_j \in \mathbb{Z} (j = 1, 2, \dots, d)\}$	beginning of Section 4
$[-\beta_1/4, \beta_1/4)_h$	$\{-\beta_1/4, -\beta_1/4 + 1/h, \dots, \beta_1/4 - 1/h\}$	beginning of Section 4
$[\beta_1/4, \beta_a - \beta_1/4)_h$	$\{\beta_1/4, \beta_1/4 + 1/h, \dots, \beta_a - \beta_1/4 - 1/h\}$	beginning of Section 4
\hat{I}_0	$\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times [-\beta_1/4, \beta_1/4)_h$	beginning of Section 4
\hat{I}	$\hat{I}_0 \times \{1, -1\}$	beginning of Section 4
I_0^0	$\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\} \times \{0\}$	beginning of Section 4

I^0	$I_0^0 \times \{1, -1\}$	beginning of Section 4
$\mathcal{S}(l)$	subset of $\bigwedge \mathcal{V}$	Subsection 7.3
$\tilde{\mathcal{S}}(l)$	subset of $\mathcal{S}(l)(\beta_1) \times \mathcal{S}(l)(\beta_2)$	Subsection 7.3

Functions and maps

Notation	Description	Reference
\mathbf{H}	1-band Hamiltonian on $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$	Subsection 1.2
\mathbf{H}_0	kinetic part of \mathbf{H}	Subsection 1.2
\mathbf{V}	interacting part of \mathbf{H}	Subsection 1.2
H	b -band Hamiltonian on $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$	Subsection 2.1
H_0	kinetic part of H	Subsection 2.1
V	interacting part of H	Subsection 2.1
$E(\cdot)$	the generalized hopping matrix in the momentum space	Subsection 2.1
\mathcal{P}_n	projection from $\bigwedge \mathcal{V}$ to $\bigwedge^n \mathcal{V}$	Subsection 2.2
$\partial/\partial\psi_X$	Grassmann left derivative	Subsection 2.2
$C(\cdot)$	full covariance	Subsection 2.3
I_n	$n \times n$ unit matrix	Subsection 2.3
$C_{\leq 0}^\infty(\cdot)$	h -independent covariance matrix with Matsubara UV cut-off	Subsection 2.5
$d_j(\cdot)$	function to measure the difference between 2 elements of I	beginning of Section 3
$ope(T, C_o)$	operator made of Grassmann left-derivatives	Subsection 3.2
$Ope(T, C_o)$	$ope(T, C_o) \prod_{\{p,q\} \in T} (\Delta_{p,q}(C_o) + \Delta_{q,p}(C_o))$	Subsection 3.2
$r_\beta(\cdot)$	map from $(1/h)\mathbb{Z}$ to $[0, \beta)_h$ satisfying $x = n_\beta(x)\beta + r_\beta(x)$, $(\forall x \in (1/h)\mathbb{Z})$	beginning of Section 4
$n_\beta(\cdot)$	map from $(1/h)\mathbb{Z}$ to \mathbb{Z} satisfying $x = n_\beta(x)\beta + r_\beta(x)$, $(\forall x \in (1/h)\mathbb{Z})$	beginning of Section 4
$R_\beta(\cdot)$	map from $I_{0,\infty}^n$ to I_0^n , or from I_∞^n to I^n	beginning of Section 4

$N_\beta(\cdot)$	map from $I_{0,\infty}^n$ to \mathbb{Z} , or from I_∞^n to \mathbb{Z}	beginning of Section 4
$r_L(\cdot)$	map from Γ_∞ to Γ	beginning of Section 4
$\hat{d}_j(\cdot)$	function to measure the difference between 2 elements of \hat{I}	beginning of Section 4
$\phi(\cdot)$	Gevrey-class function used to construct cut-off functions	Subsection 6.1
$\chi_{h,l}(\cdot)$	UV cut-off function	Subsection 6.1
$C_l^+(\cdot)$, $C_l^-(\cdot)$	covariance matrices for the UV integration	Subsection 6.1
\mathcal{D}_j	finite difference operator	Subsection 6.1
$\chi_l(\cdot)$	IR cut-off function	Subsection 7.2
$\chi_{\leq l}(\cdot)$	$\sum_{j=l}^{N_\beta} \chi_j(\cdot)$	Subsection 7.2
$\hat{\chi}_{\leq l}(\cdot)$	variant of $\chi_{\leq l}(\cdot)$	Subsection 7.2
$C_l(\cdot)$	covariance matrix for the IR integration	Subsection 7.3

Inner products, norms and semi-norms

Notation	Description	Reference
$\langle \cdot, \cdot \rangle$	standard inner product of \mathbb{R}^d	Subsection 2.1
$\ \cdot \ _{b \times b}$	operator norm for $b \times b$ -matrices	Subsection 2.1
$\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$	standard inner product of \mathbb{C}^n	Subsection 2.1
$\ \cdot \ _{\mathbb{C}^n}$	norm of \mathbb{C}^n induced by $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$	Subsection 2.1
$\langle \cdot, \cdot \rangle_{F_f}$	inner product of $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$	Subsection 2.4
$\ \cdot \ _{F_f}$	norm of $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$ induced by $\langle \cdot, \cdot \rangle_{F_f}$	Subsection 2.4
$\ \cdot \ _{\mathfrak{B}(F_f)}$	operator norm for linear transforms on $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$	Subsection 2.4
$\langle \cdot, \cdot \rangle_{\mathcal{H}}$	inner product of the Hilbert space \mathcal{H}	Subsection 2.5
$\ \cdot \ _{\mathcal{H}}$	norm of \mathcal{H} induced by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$	Subsection 2.5
$\ \cdot \ _{L^1}$	L^1 -norm for functions on I^n	Subsection 2.5
$\ \cdot \ _{l,0}$	scale-dependent norm for anti-symmetric functions	beginning of Section 3

$\ \cdot\ _{l,1}$	scale-dependent semi-norm for anti-symmetric functions	beginning of Section 3
$ \cdot - \cdot _l$	scale-dependent measurement of the difference between two anti-symmetric functions defined at β_1 and β_2	beginning of Section 4

Other notations

Notation	Description	Reference
$\mathbf{e}_1, \mathbf{e}_2$	$\mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1)$	Subsection 1.2
Ω_{2L}	vacuum of $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$	Subsection 1.2
Ω	vacuum of $F_f(L^2(\mathcal{B} \times \Gamma \times \{\uparrow, \downarrow\}))$	Subsection 2.3
$\mathbf{e}(\rho)$	$\mathbf{e}(1) = (0, 0), \mathbf{e}(2) = (1, 0),$	Subsection 7.1
$(\rho = 1, 2, 3, 4)$	$\mathbf{e}(3) = (0, 1), \mathbf{e}(4) = (1, 1)$	

References

[1] Afchain, S., Magnen, J. and V. Rivasseau, Renormalization of the 2-point function of the Hubbard model at half-filling, *Ann. Henri Poincaré.* **6** (2005), 399–448.

[2] Afchain, S., Magnen, J. and V. Rivasseau, The Hubbard model at half-filling, part III: the lower bound on the self-energy, *Ann. Henri Poincaré.* **6** (2005), 449–483.

[3] Benfatto, G., Giuliani, A. and V. Mastropietro, Fermi liquid behavior in the 2D Hubbard model at low temperatures, *Ann. Henri Poincaré.* **7** (2006), 809–898.

[4] Feldman, J., Knörrer, H. and E. Trubowitz, A representation for Fermionic correlation functions, *Commun. Math. Phys.* **195** (1998), 465–493.

[5] Feldman, J., Knörrer, H. and E. Trubowitz, Fermionic functional integrals and the renormalization group, CRM monograph series No. 16, American Mathematical Society, Providence, R.I., 2002.

[6] Feldman, J., Knörrer, H. and E. Trubowitz, A two dimensional Fermi liquid, *Commun. Math. Phys.* **247** (2004), 1–319.

[7] Feldman, J., Knörrer, H. and E. Trubowitz, Single scale analysis of many fermion systems, *Rev. Math. Phys.* **15** (2003), 949–1169.

- [8] Gevrey, M., Sur la nature analytique des solutions des équations aux dérivées partielles. Premier mémoire, *Ann. Sci. Ec. Norm. Sup.* **35** (1918), 129–190.
- [9] Giuliani, A. and V. Mastropietro, The two-dimensional Hubbard model on the honeycomb lattice, *Commun. Math. Phys.* **293** (2010), 301–346.
- [10] Giuliani, A., Mastropietro, V. and M. Porta, Universality of conductivity in interacting graphene, *Commun. Math. Phys.* **311** (2012), 317–355.
- [11] Hörmander, L., The analysis of linear partial differential operators I: Distribution theory and Fourier analysis, *Classics in mathematics*, Springer-Verlag, Berlin, Heidelberg, New York, 2003.
- [12] Kashima, Y., A rigorous treatment of the perturbation theory for many-electron systems, *Rev. Math. Phys.* **21** (2009), 981–1044.
- [13] Kashima, Y., Exponential decay of correlation functions in many-electron systems, *J. Math. Phys.* **51** (2010), 063521.
- [14] Kashima, Y., Exponential decay of equal-time four-point correlation functions in the Hubbard model on the copper-oxide lattice, *Ann. Henri Poincaré.* **15** (2014), 1453–1522.
- [15] Lieb, E. H., Flux phase of the half-filled band, *Phys. Rev. Lett.* **73** (1994), 2158.
- [16] Lieb, E. H. and M. Loss, Fluxes, Laplacians, and Kasteleyn’s theorem, *Duke. Math. J.* **71** (1993), 337–363.
- [17] Macris, N. and B. Nachtergaele, On the flux phase conjecture at half-filling: an improved proof, *J. Stat. Phys.* **85** (1996), 745–761.
- [18] Pedra, W., Zur mathematischen Theorie der Fermiflüssigkeiten bei positiven Temperaturen, PhD thesis, the University of Leipzig, Leipzig, 2005.
- [19] Pedra, W. and M. Salmhofer, Determinant bounds and the Matsubara UV problem of many-fermion systems, *Commun. Math. Phys.* **282** (2008), 797–818.
- [20] Rivasseau, V., The two dimensional Hubbard model at half-filling. I. Convergent contributions, *J. Stat. Phys.* **106** (2002), 693–722.
- [21] Rodino, L., *Linear partial differential operators in Gevrey spaces*, World Scientific, Singapore, New Jersey, London, 1993.
- [22] Salmhofer, M. and C. Wierczkowski, Positivity and convergence in fermionic quantum field theory, *J. Stat. Phys.* **99** (2000), 557–586.

(Received May 8, 2014)

(Revised March 24, 2015)

Graduate School of Mathematical Sciences
University of Tokyo
Komaba, Tokyo 153-8914, Japan
E-mail: kashima@ms.u-tokyo.ac.jp