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Global Existence and Asymptotic Behavior of Solutions to Some Nonlinear Systems of Schrödinger Equations

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Dedicated to Professor Nakao Hayashi on his sixtieth birthday

Abstract. We study the global existence and the large time behavior of solutions to the coupled system of the Schrödinger equations with cubic nonlinearities in one space dimension. We construct modified wave operators to the system for small final data.

1. Introduction

We study the global existence and the large time behavior of solutions to the coupled system of the Schrödinger equations with cubic nonlinearities in one space dimension. In the present paper, we construct a modified wave operator to the system for small final data. We consider the Cauchy problem at infinite initial time of the following coupled system of the Schrödinger equations with cubic nonlinearities in one space dimension (for given final data):

$$\begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\partial_x^2 u_1 = F_1(u_1, u_2), \\ i\partial_t u_2 + \dfrac{1}{2m_2}\partial_x^2 u_2 = F_2(u_1, u_2), \end{cases} \tag{1.1}$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}$, u_1 and u_2 are complex valued unknown functions of (t, x), $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$ and m_1 and m_2 are positive constants. The nonlinearities are given by

$$F_j(u_1, u_2) = g_j(u_1, u_2)u_j + N_j(u_1, u_2),$$

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where

$$g_j(u_1,u_2) = \mu_{j,1}|u_1|^2 + \mu_{j,2}|u_2|^2 + \delta_{m_1,m_2}\mu_{j,3}(u_1\bar{u}_2 + \bar{u}_1 u_2), \tag{1.2}$$

$$N_j(u_1,u_2) = \sum_{\alpha;|\alpha|=3,(1.4),(1.5)} \lambda_{j,\alpha} u_1^{\alpha_1}\bar{u}_1^{\alpha_2}u_2^{\alpha_3}\bar{u}_2^{\alpha_4}, \tag{1.3}$$

$\delta_{p,q} = 1$ if $p = q$, and $\delta_{p,q} = 0$ if $p \neq q$, $\lambda_{j,\alpha} \in \mathbb{C}$ and $\mu_{j,k} \in \mathbb{R}$, and the summation in (1.3) is taken by any multi-indeces $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}_+^4$ satisfying

$$(\alpha_1 - \alpha_2)m_1 + (\alpha_3 - \alpha_4)m_2 \neq m_j, \tag{1.4}$$

$$(\alpha_1 - \alpha_2)m_1 + (\alpha_3 - \alpha_4)m_2 \neq 0. \tag{1.5}$$

(For a condition $(*)$, $\sum_{\alpha;\ (*)}$ denotes a summation taken over multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}_+^4$ satisfying the condition $(*)$.) In addition, when $m_1 = m_2$, we assume "$\mu_{1,3} = \mu_{2,3} = 0$" or "$\mu_{1,k} = \mu_{2,k}$ for $k = 1, 2, 3$" (see Remark 1.3 below). The nonlinearities $g_j(u_1, u_2)u_j$ and $N_j(u_1, u_2)$ are cubic. We construct modified wave operators for the system (1.1) for small final data in Theorem 1.1.

A lot of works have been devoted to the global existence and the asymptotic behavior of solutions to the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = f(u), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{1.6}$$

where Δ is the Laplace operator with respect to the space variable x.

Following Section 1 of [22], we recall several known results on the asymptotic behavior of solutions to the nonlinear Schrödinger equation (1.6) with a nonlinearity $f(u) = \lambda|u|^{p-1}u$, where $p > 1$ and $\lambda \in \mathbb{C} \setminus \{0\}$. This nonlinearity is gauge invariant, that is, $f(\cdot)$ satisfies $f(e^{i\theta}z) = e^{i\theta}f(z)$ for $\theta \in \mathbb{R}$ and $z \in \mathbb{C}$. It is well-known that if $p > 1+2/n$, then the nonlinearity $\lambda|u|^{p-1}u$ is a short-range interaction, that is, contribution of the nonlinearity is negligible for large time. (For results on the short-range scattering for the equation (1.6), see, e.g., Ginibre [2] and Nakanishi-Ozawa [19].) On the other hand, if $p \leq 1+2/n$, then the nonlinearity $\lambda|u|^{p-1}u$ is a long-range interaction, that is, contribution of the nonlinear term is not negligible for large time. More precisely, in Barab [1], it was shown that there does not exist an asymptotically free solution for the equation (1.6) if $1 \leq p \leq 1+2/n$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

Therefore we see that for the equation (1.6), the case $p = 1 + 2/n$ is the borderline between the short-range case and the long-range one. Recall that the solution to the Cauchy problem of the free Schrödinger equation

$$\begin{cases} i\partial_t v + \dfrac{1}{2}\Delta v = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ v(0,x) = \phi(x), & x \in \mathbb{R}^n \end{cases}$$

is given by $U(t)\phi$, where $U(t) = e^{it\Delta/2}$, and it decays as $\|U(t)\phi\|_{L^q} \le Ct^{-n(1/2-1/q)}\|\phi\|_{L^{q'}}$, where $q \ge 2$ and $1/q + 1/q' = 1$. We consider the equation (1.6) with the critical exponent $p = 1 + 2/n$ and $\lambda \in \mathbb{R}\setminus\{0\}$. In this case, the modified wave operators to the equation (1.6) were constructed by Ozawa [20] for $n = 1$ and Ginibre-Ozawa [3] for $n = 2$ or 3 for small final data u_+ by a suitable phase shift, more precisely, the solution u behaves like the modified free profile $U(t)e^{-iS(t,-i\nabla)}u_+$, where $S(t,x) = \lambda|\hat{u}_+(x)|^{2/n}\log t$. Ginibre and Velo [5] proved the existence of modified wave operators to the equation (1.6) in the case $n = 1$ without any size restriction of the final state u_+ and extended the above results. For $p = 1 + 2/n$ and $n \le 3$, Hayashi and Naumkin [9] showed that the small global solution for the initial value problem of the equation (1.6) with $\lambda \in \mathbb{R}\setminus\{0\}$ satisfies the time decay estimate $\|u(t)\|_{L^\infty_x} = O(t^{-n/2})$, and that the solution has a modified free profile with the above phase shift. Furthermore, in [10], they improved the above result. In the case of $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda < 0$, $p \le 1 + 2/n$ and $n \le 3$, the large time behavior of solutions to the initial value problem of the equation (1.6) was studied in [16], [17] and [24]. In this case, the equation (1.6) has a long-range nonlinear dissipation, that is, the global solution to the nonlinear equation (1.6) decays faster than the free solution as $t \to +\infty$.

Following Section 1 of [23], we introduce several results on the large time behavior of solutions for the equation (1.6) with a critical non-gauge invariant nonlinearity. To investigate the large time behavior of solutions for the equation (1.6), non-gauge-invariant nonlinearities are more complicated than gauge invariant ones. We consider the nonlinearities

$$f(u) = \lambda_1 u^3 + \lambda_2 u\bar{u}^2 + \lambda_3\bar{u}^3, \quad \text{when } n = 1, \tag{1.7}$$

$$f(u) = \lambda_1 u^2 + \lambda_2\bar{u}^2, \quad \text{when } n = 2, \tag{1.8}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$. As we mentioned above, it is expected that the cubic nonlinear Schrödinger equation in one space dimension and the quadratic nonlinear Schrödinger equation in two space dimensions are on the borderline between the short-range and the long-range cases. In [18], [25] and [11], the existence of wave operators for the equation (1.6) with these nonlinearities for small final data u_+ was showed under the suitable assumptions including $\hat{u}_+(0) = 0$. In [11] and [25], the critical gauge invariant nonlinearities are included and the existence of modified wave operators was proved. These results mean that the critical non-gauge invariant nonlinearities (1.7)–(1.8) are short range interactions under the assumptions such as $\hat{u}_+(0) = 0$. Since each term of the non-gauge invariant nonlinearities (1.7)–(1.8) has a different oscillation and they do not resonate with the linear part, it could be shown that their contribution vanishes in large time under the condition $\hat{u}_+(0) = 0$ in the above results. On the other hand, in the case of $f(u) = \lambda|u|^2$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $n = 2$, non-existence of asymptotically free solutions in L^2 to the equation (1.6) was shown in [23] and [26]. In the same case, Ikeda and Wakasugi [13] proved the existence of finite time blow-up solutions in L^2 to the initial value problem of the equation (1.6) with arbitrarily small initial data.

Recently, some systems of Schrödinger equations have been investigated. Hayashi, Li and Naumkin [6, 7, 8] studied the scattering theory for the following system of nonlinear Schrödinger equations:

$$\begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\Delta u_1 = \gamma \bar{u}_1 u_2, \\ i\partial_t u_2 + \dfrac{1}{2m_2}\Delta u_2 = u_1^2 \end{cases} \tag{1.9}$$

in $\mathbb{R}^2$. Here, γ is a given complex number with $|\gamma| = 1$. (For related results on (1.9), see, e.g., [12, 14, 21].)

In this paper, we study the global existence and the large time behavior of solutions to the system (1.1) of the Schrödinger equation with cubic nonlinearity in one space dimension by constructing modified wave operators for small final data. The nonlinearity $F_j(u_1, u_2)$ of the system (1.1) has a gauge invariant term $g_j(u_1, u_2)u_j$ and a non-gauge-invariant term $N_j(u_1, u_2)$. To control interaction of u_1 and u_2, we restrict cubic nonlinear terms depending on the constants m_1 and m_2 (see Remark 1.1). The proof is mainly based on the methods of [11].

Before stating our main result, we introduce several notations.

Notation. We denote the Schwartz space on $\mathbb{R}^n$ by $\mathcal{S}$. Let $\mathcal{S}'$ be the set of tempered distributions on $\mathbb{R}^n$. For $w \in \mathcal{S}'$, we denote the Fourier transform of w by $\hat{w}$. For $w \in L^1(\mathbb{R}^n)$, $\hat{w}$ is represented as

$$\hat{w}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} w(x) e^{-ix\cdot\xi}\, dx.$$

For $s, m \in \mathbb{R}$, we introduce the weighted Sobolev spaces $H^{s,m}$ corresponding to the Lebesgue space L^2 as follows:

$$H^{s,m} \equiv \{\psi \in \mathcal{S}' \colon \|\psi\|_{H^{s,m}} \equiv \|(1+|x|^2)^{m/2}(1-\Delta)^{s/2}\psi\|_{L^2} < \infty\}.$$

We also denote $H^{s,0}$ by H^s. For $1 \le p \le \infty$ and a positive integer k, we define the Sobolev space W_p^k corresponding to the Lebesgue space L^p by

$$W_p^k \equiv \left\{ \psi \in L^p \colon \|\psi\|_{W_p^k} \equiv \sum_{|\alpha| \le k} \|\partial^\alpha \psi\|_{L^p} < \infty \right\}.$$

Note that for a positive integer k, $H^k = W_2^k$ and the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{W_2^k}$ are equivalent.

For $s \in \mathbb{R}$, we define the homogeneous Sobolev spaces $\dot{H}^s$ by

$$\dot{H}^s \equiv \{w \in \mathcal{S}' \colon (-\Delta)^{s/2} w \in L^2\}$$

with the semi-norm

$$\|w\|_{\dot{H}^s} \equiv \|(-\Delta)^{s/2} w\|_{L^2}.$$

We introduce the following operators:

$$U_m(t) \equiv e^{it\Delta/2m}, \quad \mathcal{L}_m = i\partial_t + \frac{1}{2m}\partial_x^2.$$

C denotes a constant and so forth. They may differ from line to line, when it does not cause any confusion.

Remark 1.1. Recall that the solution for the free Schrödinger equation $\mathcal{L}_k u = 0$ oscillates as $e^{ik|x|^2/2t}$. Roughly speaking, nonlinear term $u_1^{\alpha_1}\bar{u}_1^{\alpha_2}u_2^{\alpha_3}\bar{u}_2^{\alpha_4}$, which appears in the definitions of the nonlinearities $F_1(u_1, u_2)$ and $F_2(u_1, u_2)$, oscillate as $\exp(i((\alpha_1-\alpha_2)m_1+(\alpha_3-\alpha_4)m_2)|x|^2/$

$2t$). If $(\alpha_1-\alpha_2)m_1+(\alpha_3-\alpha_4)m_2=m_1$, then this nonlinear term resonates with the free solution of $\mathcal{L}_{m_1}u=0$. If $(\alpha_1-\alpha_2)m_1+(\alpha_3-\alpha_4)m_2=0$, then an oscillation of this nonlinearity vanishes. (Our method is not applicable to nonlinearities which do not oscillate.) The assumptions (1.4) and (1.5) mean that the the nonlinearity $u_1^{\alpha_1}\bar{u}_1^{\alpha_2}u_2^{\alpha_3}\bar{u}_2^{\alpha_4}$ does not resonate with the linear part, and its oscillation does not vanish.

The non-resonance condition $(\alpha_1-\alpha_2)m_1+(\alpha_3-\alpha_4)m_2\neq m_1$ for $F_1(u_1,u_2)$, which is the inequality (1.4) for $j=1$, does not hold if and only if

- $\alpha=(2,1,0,0)$ or $\alpha=(1,0,1,1)$ for arbitrary $m_1,m_2>0$, that is, $u_1^2\bar{u}_1$ or $u_1u_2\bar{u}_2$, respectively,
- $\alpha=(2,1,0,0)$, $\alpha=(1,0,1,1)$, $\alpha=(1,1,1,0)$, $\alpha=(0,0,2,1)$, $\alpha=(0,1,2,0)$ or $\alpha=(2,0,0,1)$ when $m_1=m_2$, that is, $u_1^2\bar{u}_1$, $u_1u_2\bar{u}_2$, $u_1\bar{u}_1u_2$, $u_2^2\bar{u}_2$, $\bar{u}_1u_2^2$ or $u_1^2\bar{u}_2$, respectively,
- $\alpha=(2,1,0,0)$, $\alpha=(1,0,1,1)$ or $\alpha=(0,2,1,0)$ when $m_2=3m_1$, that is, $u_1^2\bar{u}_1$, $u_1u_2\bar{u}_2$ or $\bar{u}_1^2u_2$, respectively,
- $\alpha=(0,0,3,0)$ when $m_2=m_1/3$, that is, u_2^3.

The non-resonance condition $(\alpha_1-\alpha_2)m_1+(\alpha_3-\alpha_4)m_2\neq m_2$ for $F_2(u_1,u_2)$, which is the inequality (1.4) for $j=2$, does not hold if and only if

- $\alpha=(0,0,2,1)$ or $\alpha=(1,1,1,0)$ for arbitrary $m_1,m_2>0$, that is, $u_2^2\bar{u}_2$ or $u_1\bar{u}_1u_2$, respectively,
- $\alpha=(0,0,2,1)$, $\alpha=(1,1,1,0)$, $\alpha=(1,0,1,1)$, $\alpha=(2,1,0,0)$, $\alpha=(2,0,0,1)$ or $\alpha=(0,1,2,0)$ when $m_1=m_2$, that is, $u_2^2\bar{u}_2$, $u_1\bar{u}_1u_2$, $u_1u_2\bar{u}_2$, $u_1^2\bar{u}_1$, $u_1^2\bar{u}_2$ or $\bar{u}_1u_2^2$, respectively,
- $\alpha=(0,0,2,1)$, $\alpha=(1,1,1,0)$ or $\alpha=(1,0,0,2)$ when $m_2=m_1/3$, that is, $u_2^2\bar{u}_2$, $u_1\bar{u}_1u_2$ or $u_1\bar{u}_2^2$, respectively,
- $\alpha=(3,0,0,0)$ when $m_2=3m_1$, that is, u_1^3.

The condition $(\alpha_1-\alpha_2)m_1+(\alpha_3-\alpha_4)m_2=0$, which implies vanishing oscillation, holds if and only if "$\alpha=(2,0,0,1)$ or $\alpha=(0,2,1,0)$, that is, $u_1^2\bar{u}_2$ or $\bar{u}_1^2u_2$ for $m_2=2m_1$", or "$\alpha=(1,0,0,2)$ or $\alpha=(0,1,2,0)$, that is, $u_1\bar{u}_2^2$ or $\bar{u}_1u_2^2$ for $m_2=m_1/2$".

The main result is the following:

THEOREM 1.1. *Let $u_{1+}, u_{2+} \in H^{0,2} \cap \dot{H}^{-b}$, where $1/2 < b < 3/2$, and $\|u_{1+}\|_{H^{0,2}\cap\dot{H}^{-b}} + \|u_{2+}\|_{H^{0,2}\cap\dot{H}^{-b}}$ be sufficiently small. In addition, when $m_1 = m_2$, assume that "$\mu_{1,3} = \mu_{2,3} = 0$" or "$\mu_{1,k} = \mu_{2,k}$ for $k = 1, 2, 3$". Then the system (1.1) has a unique solution (u_1, u_2) satisfying*

$$\begin{aligned} &(u_1, u_2) \in C([0,\infty); L^2) \oplus C([0,\infty); L^2), \\ &\sup_{t\geq 1} t^{b/2}(\|(u_1(t), u_2(t)) - (u_{1a}(t), u_{2a}(t))\|_{L^2} \\ &\qquad + \|(u_1, u_2) - (u_{1a}, u_{2a})\|_{L^4([t,\infty);L^\infty)}) < \infty, \end{aligned}$$

where

$$\begin{aligned} u_{ja}(t,x) &= (U_{m_j}(t) e^{-im_j|x|^2/2t} e^{-iS_j(t,-i\nabla)} u_{j+})(x) \\ &= \left(\frac{m_j}{it}\right)^{1/2} \hat{u}_{j+}\left(\frac{m_j x}{t}\right) e^{im_j|x|^2/2t - iS_j(t, m_j x/t)}, \end{aligned} \tag{1.10}$$

$$S_j(t,x) = g_j\left(\hat{u}_{1+}\left(\frac{m_1}{m_j}x\right), \hat{u}_{2+}\left(\frac{m_2}{m_j}x\right)\right)\log t \tag{1.11}$$

for $j = 1, 2$. Furthermore the modified wave operator

$$W_+ : (u_{1+}, u_{2+}) \mapsto (u_1(0), u_2(0))$$

is well-defined.

A similar result holds for the negative time.

REMARK 1.2. It is well-known that the asymptotics $u_{ja}(t,\cdot) = U_{m_j}(t)e^{-im_j|x|^2/2t}e^{-iS_j(t,-i\nabla)}u_{j+}$ in Theorem 1.1 behaves as $U_{m_j}(t)e^{-iS_j(t,-i\nabla)}u_{j+}$ as $t \to \infty$ for $j = 1, 2$.

REMARK 1.3. When $m_1 = m_2$, we assumed that "$\mu_{1,3} = \mu_{2,3} = 0$" or "$\mu_{1,k} = \mu_{2,k}$ for $k = 1, 2, 3$". This assumption means the following. Assume that $m_1 = m_2$ and "$\mu_{1,3} \neq 0$ or $\mu_{2,3} \neq 0$". Then the third term of (1.2) is not always zero, and we have to treat the functions $u_{1a}\bar{u}_{2a}$ and $\bar{u}_{1a}u_{2a}$ to prove Lemma 3.1 below. The function $u_{1a}\bar{u}_{2a}$ involves the factor $e^{-iS_1(t,m_1x/t)}\overline{e^{-iS_2(t,m_2x/t)}}$ and it is not easy to treat this factor. Under the condition $\mu_{1,k} = \mu_{2,k}$ for $k = 1, 2, 3$, we see that $g_1(u_1, u_2) = g_2(u_1, u_2)$, $S_1(t,x) = g_1(\hat{u}_{1+}(x), \hat{u}_{2+}(x))\log t = S_2(t,x)$, and

$e^{-iS_1(t,m_1x/t)}\overline{e^{-iS_2(t,m_2x/t)}} = e^{-iS_1(t,m_1x/t)}e^{iS_1(t,m_1x/t)} = 1$. Hence, oscillation in $u_{1a}\bar{u}_{2a}$ caused by $e^{-iS_1(t,m_1x/t)}\overline{e^{-iS_2(t,m_2x/t)}}$ vanishes.

REMARK 1.4. In Theorem 1.1, we assumed the condition $u_{1+}, u_{2+} \in \dot{H}^{-b}$ with $1/2 < b < 3/2$ in order to treat nonlinearities with non-resonance conditions (1.4) or (1.5). Therefore if the nonlinearities F_1 and F_2 do not involve these terms, then this assumption is not needed.

REMARK 1.5. If $\phi \in H^{0,2}$ and $\hat{\phi}(0) = 0$, then $\phi \in \dot{H}^{-\alpha}$ with $0 \leq \alpha < 1 + n/2$. (See Remark 1.5 in [11].)

This paper is organized as follows. In Section 2, we solve the abstract final value problem around an asymptotic function which decays like $t^{-1/2}$ in L^∞ and approximates the system (1.1) suitably in large time. In Section 3, we show our asymptotics (u_{1a}, u_{2a}) satisfies the assumptions of the Cauchy problem at infinite initial time in Section 2, and we prove Theorem 1.1.

2. The Cauchy Problem at Infinite Initial Time

In this section, we construct a global solution (u_1, u_2) for the system (1.1) such that (u_1, u_2) approachs a given modified free dynamics (u_{1a}, u_{2a}), which decays as the free solution and approximates the system (1.1) suitably, as $t \to \infty$.

First we introduce the Strichartz estimate for the free Schrödinger equation obtained by Yajima [27]. We define the linear operator

$$(G_k h)(t) = \int_t^\infty U_k(t-s)h(s)\,ds, \tag{2.1}$$

where h is a function of (t, x) and $k > 0$.

LEMMA 2.1. *Let n denote the space dimension, and let (q, r) and $(\tilde{q}, \tilde{r})$ be pairs of positive numbers satisfying $2/q = n(1/2 - 1/r)$, $2 < q \leq \infty$, $2/\tilde{q} = n(1/2 - 1/\tilde{r})$ and $2 < \tilde{q} \leq \infty$. Then G_k is a bounded operator from $L_t^{\tilde{q}'}((T_0, \infty); L_x^{\tilde{r}'}(\mathbb{R}^n))$ into $L_t^q((T_0, \infty); L_x^r(\mathbb{R}^n))$ with norm uniformly bounded with respect to T_0, where $(\tilde{q}', \tilde{r}')$ is a pair of positive numbers satisfying $1/\tilde{q} + 1/\tilde{q}' = 1$ and $1/\tilde{r} + 1/\tilde{r}' = 1$. Furthermore, if $h \in L_t^{\tilde{q}'}((T_0, \infty); L_x^{\tilde{r}'}(\mathbb{R}^n))$, then $G_k h \in C_t([T_0, \infty); L_x^2(\mathbb{R}^n))$.*

Let (u_{1a}, u_{2a}) be a given asymptotic profile, that is, an approximate solution for large time, which will be determined explicitly in Section 3. We introduce the following functions:

$$R_j(u_{1a}, u_{2a}) = \mathcal{L}_{m_j} u_{ja} - F_j(u_{1a}, u_{2a})$$

for $j = 1, 2$.

For $T > 0$ and $\rho > 0$, we introduce the following function spaces

$$X_T = \{(w_1, w_2) \in C([T, \infty); L^2) \oplus C([T, \infty); L^2);\ \|(w_1, w_2)\|_{X_T} < \infty\},$$
$$B_T(\rho) = \{(w_1, w_2) \in X_T;\ \|(w_1, w_2)\|_{X_T} \leq \rho\},$$

where

$$\|(w_1, w_2)\|_{X_T} = \sup_{t \geq T} (1 + t)^d (\|(w_1(t), w_2(t))\|_{L^2} + \|(w_1, w_2)\|_{L^4((t,\infty); L^\infty_x)}).$$

X_T is a Banach space with the norm $\|\cdot\|_{X_T}$, and $B_T(\rho)$ is a complete metric space with the X_T-metric.

Proposition 2.1. *Assume that there exists a constant $\eta' > 0$ such that for $t \geq 0$,*

$$\|u_{1a}(t)\|_{L^\infty_x} + \|u_{2a}(t)\|_{L^\infty_x} \leq \eta'(1 + t)^{-1/2},$$
$$\sum_{j=1}^{2} (\left\|(G_{m_j} R_j(u_{1a}, u_{2a}))(t)\right\|_{L^2_x} + \left\|G_{m_j} R_j(u_{1a}, u_{2a})\right\|_{L^4_\tau((t,\infty); L^\infty_x)})$$
$$\leq \eta'(1 + t)^{-d},$$

where $1/4 < d < 1$, and that η' is sufficiently small. Then the system (1.1) *has a unique solution (u_1, u_2) satisfying*

$$(u_1, u_2) \in C([0, \infty); L^2) \oplus C([0, \infty); L^2),$$
$$\sup_{t \geq 1} t^d (\|(u_1(t), u_2(t)) - (u_{1a}(t), u_{2a}(t))\|_{L^2}$$
$$+ \|(u_1, u_2) - (u_{1a}, u_{2a})\|_{L^4([t,\infty); L^\infty)}) < \infty.$$

PROOF. We may assume that $0 < \eta' \le 1$. Let $(w_1, w_2) = (u_1 - u_{1a}, u_2 - u_{2a})$. The system (1.1) is equivalent to

$$\begin{cases} \mathcal{L}_{m_1} w_1 = F_1(w_1 + u_{1a}, w_2 + u_{2a}) - F_1(u_{1a}, u_{2a}) - R_1, \\ \mathcal{L}_{m_2} w_2 = F_2(w_1 + u_{1a}, w_2 + u_{2a}) - F_2(u_{1a}, u_{2a}) - R_2. \end{cases} \tag{2.2}$$

The associate integral equations to the system (2.2) is

$$\begin{cases} w_1(t) = & i\{G_{m_1}(F_1(w_1 + u_{1a}, w_2 + u_{2a}) - F_1(u_{1a}, u_{2a}) \\ & - R_1(u_{1a}, u_{2a}))\}(t), \\ w_2(t) = & i\{G_{m_2}(F_2(w_1 + u_{1a}, w_2 + u_{2a}) - F_2(u_{1a}, u_{2a}) \\ & - R_2(u_{1a}, u_{2a}))\}(t). \end{cases} \tag{2.3}$$

In order to prove this proposition, it is sufficient to show the existence and uniqueness of solutions for the system (2.3) in X_0 for sufficiently small η'.

We first prove the existence of a solution (w_1, w_2) to the system (2.3) in $B_0(\rho)$ for sufficiently small η' and ρ. We define the operator

$$K(w_1, w_2) = (K_1(w_1, w_2), K_2(w_1, w_2)),$$

where

$$\begin{aligned} K_j(w_1, w_2) = & i\{G_{m_j}(F_j(w_1 + u_{1a}, w_2 + u_{2a}) - F_j(u_{1a}, u_{2a}) \\ & - R_j(u_{1a}, u_{2a}))\}(t). \end{aligned}$$

We show that the operator K is a contraction map on $B_0(\rho)$ if η' and ρ are sufficiently small. Let $\rho > 0$ be determined later and let $(w_1, w_2) \in B_0(\rho)$. We evaluate $K_1(w_1, w_2)$. It is easy to see that

$$\begin{aligned} & |F_1(w_1 + u_{1a}, w_2 + u_{2a}) - F_1(u_{1a}, u_{2a})| \\ \le & C\{(|w_1|^2 + |w_2|^2)(|w_1| + |w_2|) + (|u_{1a}|^2 + |u_{2a}|^2)(|w_1| + |w_2|)\}. \end{aligned} \tag{2.4}$$

From the assumptions, the estimate (2.4), Lemma 2.1 and the facts $1/4 < d < 1$ and $(w_1, w_2) \in B_0(\rho)$, it follows that

$$\begin{aligned} & \|K_1(w_1, w_2)(t)\|_{L^2} + \|K_1(w_1, w_2)\|_{L^4((t,\infty);L_x^\infty)} \\ \le & C(\|(|w_1|^2 + |w_2|^2)(|w_1| + |w_2|)\|_{L^{4/3}((t,\infty);L_x^1)} \\ & + \|(|u_{1a}|^2 + |u_{2a}|^2)(|w_1| + |w_2|)\|_{L^1((t,\infty);L_x^2)} + \eta'(1+t)^{-d}) \end{aligned}$$

$$
\begin{aligned}
\le & C\{(\|w_1\|^2_{L^4((t,\infty);L^2_x)} + \|w_2\|^2_{L^4((t,\infty);L^2_x)}) \\
& \times (\|w_1\|_{L^4((t,\infty);L^\infty_x)} + \|w_2\|_{L^4((t,\infty);L^\infty_x)}) \\
& + \|(\|u_{1a}\|^2_{L^\infty_x} + \|u_{2a}\|^2_{L^\infty_x})(\|w_1\|_{L^2_x} + \|w_2\|_{L^2_x})\|_{L^1(t,\infty)} + \eta'(1+t)^{-d}\} \\
\le & C(1+t)^{-d}(\rho^3(1+t)^{-2d+1/2} + \rho\eta'^2 + \eta') \\
\le & C(1+t)^{-d}(\rho^3 + \rho\eta'^2 + \eta').
\end{aligned}
$$

This implies

$$
\begin{aligned}
& \sup_{t\ge 0}(1+t)^d(\|K_1(w_1,w_2)(t)\|_{L^2} + \|K_1(w_1,w_2)\|_{L^4((t,\infty);L^\infty_x)}) \\
& \quad \le C(\rho^3 + \rho\eta'^2 + \eta').
\end{aligned}
$$

In the same way, we have the same estimate for $K_2(w_1,w_2)$. Therefore we have

$$
\|K(w_1,w_2)\|_{X_0} \le C(\rho^3 + \rho\eta'^2 + \eta'). \tag{2.5}
$$

Similarly, for $(w_1,w_2),(z_1,z_2) \in B_0(\rho)$, we have

$$
\|K(w_1,w_2) - K(z_1,z_2)\|_{X_0} \le C(\rho^2 + \eta'^2)\|(w_1,w_2) - (z_1,z_2)\|_{X_0}. \tag{2.6}
$$

Note that there exist sufficiently small η' and ρ such that

$$
\begin{aligned}
C(\rho^3 + \rho\eta'^2 + \eta') &\le \rho, \\
C(\rho^2 + \eta'^2) &\le \frac{1}{2}.
\end{aligned}
$$

Hence, from the estimates (2.5) and (2.6), we see that the operator K is a contraction map on $B_0(\rho)$ if η' and ρ are sufficiently small. Therefore the system (2.3) has a unique solution (w_1,w_2) in $B_0(\rho)$ for sufficiently small η' and ρ.

Uniqueness of solutions for the system (2.3) in X_0 can be proved exactly in the same way as Proposition 2.1 in [25]. □

3. Remainder Estimates and Proof of Theorem 1.1

In this section, we show that the asymptotic profile (u_{1a},u_{2a}) in Theorem 1.1 satisfies the assumptions in Proposition 2.1, and we prove Theorem 1.1 as a consequence of Proposition 2.1. In order to check the assumptions in Proposition 2.1, it is sufficient to show that for $t \ge 1$, the estimates

$$\|u_{1a}(t)\|_{L^2_x} + \|u_{2a}(t)\|_{L^2_x} \leq \eta', \tag{3.1}$$

$$\|u_{1a}(t)\|_{L^\infty_x} + \|u_{2a}(t)\|_{L^\infty_x} \leq \eta' t^{-1/2}, \tag{3.2}$$

$$\begin{aligned}&\sum_{j=1}^{2}(\big\|(G_{m_j}R_j(u_{1a},u_{2a}))(t)\big\|_{L^2_x} + \big\|G_{m_j}R_j(u_{1a},u_{2a})\big\|_{L^4_\tau((t,\infty);L^\infty_x)})\\ &\leq \eta' t^{-d}\end{aligned} \tag{3.3}$$

hold, where $1/4 < d < 1$, and the operators G_{m_j} is defined by (2.1). In fact, in order to avoid the singularity at $t = 0$, multiplying a cut off function $\theta \in C^\infty_t$ such that $\theta(t) = 0$ if $t \leq 1/2$ and $\theta(t) = 1$ if $t \geq 3/4$ to u_{1a} and u_{2a}, we easily see from the estimates (3.1)–(3.3) that the resulting functions satisfy the assumptions in Proposition 2.1. Throughout this section, we assume all the assumptions in Theorem 1.1. We put

$$\eta = \|u_{1+}\|_{H^{0,2}\cap\dot{H}^{-b}} + \|u_{2+}\|_{H^{0,2}\cap\dot{H}^{-b}}. \tag{3.4}$$

We may assume $\eta \leq 1$ because η is sufficiently small.

We define the modified free dynamics (u_{1a}, u_{2a}) with a phase shift by (1.10). In order to handle long range effects caused by the nonlinear terms $g_1(u_1,u_2)u_1$ and $g_2(u_1,u_2)u_2$ resonating with the linear part, we have introduced the phase functions S_1 and S_2 defined by (1.11) as in Ozawa [20] and Ginibre-Ozawa [3]. Noting the assumption "$\mu_{j,3} = 0$ for $j = 1,2$" or "$\mu_{1,k} = \mu_{2,k}$ for $k = 1,2,3$" when $m_1 = m_2$, we can apply their method to these nonlinearities.

The following lemma holds:

LEMMA 3.1. *Assume that $u_{1+}, u_{2+} \in H^{0,2}$. Then there exists a constant $C > 0$ such that for $t \geq 1$,*

$$\|u_{ja}(t)\|_{L^2} = \|\hat{u}_{j+}\|_{L^2} = \|u_{j+}\|_{L^2},$$

$$\|u_{ja}(t)\|_{L^\infty} \leq Ct^{-1/2}\|\hat{u}_{j+}\|_{L^\infty},$$

$$\begin{aligned}&\|\mathcal{L}_{m_j}u_{ja}(t) - g_j(u_{1a},u_{2a})u_{ja}\|_{L^2}\\ &\leq C\frac{(\log t)^2}{t^2}\|u_{j+}\|_{H^{0,2}}(1 + \|u_{1+}\|^2_{H^{0,2}} + \|u_{2+}\|^2_{H^{0,2}})\end{aligned}$$

for $j = 1,2$.

Let $a \in \mathbb{R} \setminus \{0\}$, f and V be complex and real valued functions of $x \in \mathbb{R}$, respectively, and

$$\widetilde{N}_a(t,x) = \frac{1}{t}\left(\frac{a}{it}\right)^{1/2} f\left(\frac{ax}{t}\right) e^{ia|x|^2/2t - iV(ax/t)\log t}. \tag{3.5}$$

In order to treat the nonlinear terms $N_1(u_1,u_2)$ and $N_2(u_1,u_2)$ which do not resonate with the linear part, we estimate the integral $\int_t^\infty U_k(t-s)\widetilde{N}_a(s)\,ds$. We introduce the following lemmas. We can show Lemma 3.2 in the same way as Lemma 2.4 in [11].

It is well-known that

$$U_k(t) = M_k(t)D_k(t)\mathcal{F}M_k(t), \quad k > 0, \tag{3.6}$$

where M_k and D_k are the following operators:

$$(M_k f)(t,x) = e^{ik|x|^2/2t} f(x), \quad (D_k g)(t,x) = \left(\frac{k}{it}\right)^{1/2} g\left(t, \frac{kx}{t}\right).$$

By the definition of $\widetilde{N}_a$ and the relation (3.6), we have

$$\begin{aligned} \widetilde{N}_a(t,x) =& \frac{1}{t}\left(\frac{a}{it}\right)^{1/2} f\left(\frac{ax}{t}\right) e^{ia|x|^2/2t - iV(ax/t)\log t} \\ =& \frac{1}{t}(U_a(t)M_a(t)^{-1}e^{-iV(-i\partial)\log t}\check{f})(x), \end{aligned} \tag{3.7}$$

where $\check{f}$ is the inverse Fourier transform of f.

Lemma 3.2. *Let $a \in \mathbb{R}$, $1/2 < \delta < 2$, and f and V be complex and real valued functions of $x \in \mathbb{R}$, respectively. Let δ_1 and δ_2 be constants such that $\delta_1 = 1$ if $\delta < 1$, $\delta < \delta_1 < 2$ if $\delta \geq 1$, and $\delta_2 = 3/2$ if $\delta < 3/2$, $\delta < \delta_2 < 2$ if $\delta \geq 3/2$. Let $\widetilde{N}_a$ be defined by (3.5). Then there exists a constant $C > 0$ such that for $t \geq 1$,*

$$\begin{aligned} \left\| \int_t^\infty \widetilde{N}_a(s)\,ds \right\|_{L^2} \leq& Ct^{-\delta/2}(\||\cdot|^{-\delta} f\|_{L^2} + \||\cdot|^{-\delta} fV\|_{L^2}) \\ &+ Ct^{-\delta_1/2}(\||\cdot|^{1-\delta_1}\partial f\|_{L^2} \\ &\qquad + \||\cdot|^{1-\delta_1} f\partial V\|_{L^2}(1+\log t)), \end{aligned}$$

$$\left\| \int_s^\infty \widetilde{N}_a(s)\,ds \right\|_{L_s^4((t,\infty);L_x^\infty)}$$
$$\le Ct^{-\delta/2}(\||\cdot|^{-\delta} f\|_{L^2} + \||\cdot|^{-\delta}\partial f\|_{L^2} + \||\cdot|^{-\delta} fV\|_{L^2} + \||\cdot|^{-\delta}\partial(fV)\|_{L^2})$$
$$+ Ct^{-\delta_2/2}(\||\cdot|^{3/2-\delta_2}\partial f\|_{L^\infty} + \||\cdot|^{3/2-\delta_2} f\partial V\|_{L^\infty}(1+\log t)).$$

PROOF. By the integration by parts, we see that

$$\begin{aligned}
&\left| \int_t^\infty \widetilde{N}_a(s)\,ds \right| \\
&= \left| \int_t^\infty \frac{|a|^{1/2}}{s^{3/2}} f\left(\frac{ax}{s}\right) e^{-iV(ax/s)\log s} \frac{1}{1-(ia|x|^2/2s)} \partial_s(se^{ia|x|^2/2s})\,ds \right| \\
&\le \left| \frac{|a|^{1/2}}{t^{1/2}} f\left(\frac{ax}{t}\right) \frac{1}{1-(ia|x|^2/2t)} \right| \\
&\quad + \left| \int_t^\infty \partial_s\left(\frac{|a|^{1/2}}{s^{3/2}} f\left(\frac{ax}{s}\right) e^{-iV(ax/s)\log s} \right.\right. \\
&\qquad \left.\left. \times \frac{1}{1-(ia|x|^2/2s)} \right) se^{ia|x|^2/2s}\,ds \right| \\
&\le C\{B_1(t)+B_2(t)\},
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
B_1(t) &= \frac{1}{t^{1/2}} \left| f\left(\frac{ax}{t}\right) \right| \frac{1}{1+(|a||x|^2/t)}, \\
B_2(t) &= \int_t^\infty \left\{ \frac{1}{s^{3/2}} \frac{1}{1+(|a||x|^2/s)} \left(\left| f\left(\frac{ax}{s}\right) \right| \right.\right. \\
&\quad + \frac{|a||x|}{s} \left| (\partial f)\left(\frac{ax}{s}\right) \right| + \frac{|a||x|}{s} \left| f\left(\frac{ax}{s}\right)(\partial V)\left(\frac{ax}{s}\right) \right| \log s \\
&\quad + \left| f\left(\frac{ax}{s}\right) V\left(\frac{ax}{s}\right) \right| \\
&\quad \left.\left. + \frac{|a||x|^2}{s} \left| f\left(\frac{ax}{s}\right) \right| \frac{1}{1+(|a||x|^2/s)} \right) \right\} ds.
\end{aligned} \tag{3.9}$$

We estimate the L^2-norm. Since $1/2 < \delta < 2$, we obtain

$$\begin{aligned}
\|B_1(t)\|_{L_x^2} &= Ct^{-1/2-\delta/2} \left\| \left(\frac{|a||x|}{t} \right)^{-\delta} f\left(\frac{ax}{t}\right) \frac{(\sqrt{|a|/t}|x|)^\delta}{1+(|a||x|^2/t)} \right\|_{L^2} \\
&\le Ct^{-\delta/2} \||\cdot|^{-\delta} f\|_{L^2}.
\end{aligned} \tag{3.10}$$

Similarly, we can estimate $\|B_2(t)\|_{L^2}$, and hence we have the first inequality of this lemma.

We next consider the $L^4_s((t,\infty);L^\infty_x)$-norm. By the Gagliardo-Nirenberg inequality and $\delta > 1/2$,

$$\begin{aligned}
&\|B_1(t)\|_{L^\infty_x} \\
\leq & t^{-1/2}\left\| f\left(\sqrt{\frac{a}{t}}x\right)\frac{1}{1+x^2}\right\|_{L^\infty} \\
\leq & Ct^{-1/2}\left\| f\left(\sqrt{\frac{a}{t}}x\right)\frac{1}{1+x^2}\right\|_{L^2}^{1/2}\left\|\partial\left(f\left(\sqrt{\frac{a}{t}}x\right)\frac{1}{1+x^2}\right)\right\|_{L^2}^{1/2} \\
\leq & Ct^{-1/2}\left\| f\left(\sqrt{\frac{a}{t}}x\right)\frac{1}{1+x^2}\right\|_{L^2}^{1/2} \\
&\times\left\{\left\| f\left(\sqrt{\frac{a}{t}}x\right)\frac{x}{1+x^4}\right\|_{L^2}+t^{-1/2}\left\|(\partial f)\left(\sqrt{\frac{a}{t}}x\right)\frac{1}{1+x^2}\right\|_{L^2}\right\}^{1/2} \\
\leq & Ct^{-1/2}(t^{1/4-\delta/2}\||\cdot|^{-\delta}f\|_{L^2})^{1/2} \\
&\times(t^{1/4-\delta/2}\||\cdot|^{-\delta}f\|_{L^2}+t^{-1/4-\delta/2}\||\cdot|^{-\delta}\partial f\|_{L^2})^{1/2} \\
\leq & Ct^{-\delta/2-1/4}\||\cdot|^{-\delta}f\|_{L^2}^{1/2}(\||\cdot|^{-\delta}f\|_{L^2}+\||\cdot|^{-\delta}\partial f\|_{L^2})^{1/2}.
\end{aligned}$$

We have obtained the fourth inequality in the same way as in the estimate (3.10). Taking the L^4-norm with respect to time variable, we have

$$\|B_1(t)\|_{L^4_s((t,\infty);L^\infty_x)} \leq Ct^{-\delta/2}\||\cdot|^{-\delta}f\|_{L^2}^{1/2}(\||\cdot|^{-\delta}f\|_{L^2}+\||\cdot|^{-\delta}\partial f\|_{L^2})^{1/2}.$$

In the same way as above, we can estimate the first, fourth and fifth terms of the integrand in the definition (3.9) of B_2. We describe the estimate for the second term. When $1/2 < \delta < 3/2$, we can easily see

$$\begin{aligned}
&\left\|\int_t^\infty \frac{1}{s^{3/2}}\frac{1}{1+(|a||x|^2/s)}\frac{|a||x|}{s}\left|(\partial f)\left(\frac{ax}{s}\right)\right|\,ds\right\|_{L^\infty_x} \\
=&\left\|\int_t^\infty \frac{|a|^{1/2}}{s^2}\frac{\sqrt{|a|/s}|x|}{1+(|a||x|^2/s)}\left|(\partial f)\left(\frac{ax}{s}\right)\right|\,ds\right\|_{L^\infty_x} \\
\leq & Ct^{-1}\|\partial f\|_{L^\infty}.
\end{aligned}$$

Taking the L^4-norm with respect to time variable, we have

$$
\begin{aligned}
(3.11)\qquad &\left\| \int_s^\infty \frac{1}{\tau^{3/2}} \frac{1}{1+(|a||x|^2/\tau)} \frac{|a||x|}{\tau} \left|(\partial f)\left(\frac{ax}{\tau}\right)\right| \, d\tau \right\|_{L_s^4((t,\infty);L_x^\infty)} \\
&\le C t^{-3/4} \|\partial f\|_{L^\infty}.
\end{aligned}
$$

On the other hand, when $\delta \ge 3/2$, in the same way as in the estimate (3.10), we have

$$
\begin{aligned}
&\left\| \int_t^\infty \frac{1}{s^{3/2}} \frac{1}{1+(|a||x|^2/s)} \frac{|a||x|}{s} \left|(\partial f)\left(\frac{ax}{s}\right)\right| \, ds \right\|_{L_x^\infty} \\
=&\left\| \int_t^\infty \frac{1}{s^{3/2}} \frac{1}{1+(|x|^2)} \sqrt{\frac{|a|}{s}}|x| \left|(\partial f)\left(\sqrt{\frac{|a|}{s}}x\right)\right| \, ds \right\|_{L_x^\infty} \\
=&\left\| \int_t^\infty \frac{|a|^{-1/4+\delta_2/2}}{s^{5/4+\delta_2/2}} \frac{|x|^{\delta_2-1/2}}{1+(|x|^2)} \left(\sqrt{\frac{|a|}{s}}|x|\right)^{3/2-\delta_2} \left|(\partial f)\left(\sqrt{\frac{|a|}{s}}x\right)\right| \, ds \right\|_{L_x^\infty} \\
\le& C t^{-\delta_2/2-1/4} \||\cdot|^{3/2-\delta_2}\partial f\|_{L^\infty}.
\end{aligned}
$$

where $\delta < \delta_2 < 2$. Taking the L^4-norm with respect to time variable, we have

$$
\begin{aligned}
(3.12)\qquad &\left\| \int_s^\infty \frac{1}{\tau^{3/2}} \frac{1}{1+(|a||x|^2/\tau)} \frac{|a||x|}{\tau} \left|(\partial f)\left(\frac{ax}{\tau}\right)\right| \, d\tau \right\|_{L_s^4((t,\infty);L_x^\infty)} \\
&\le C t^{-\delta_2/2} \||\cdot|^{3/2-\delta_2}\partial f\|_{L^\infty}.
\end{aligned}
$$

From the estimate (3.11) and (3.12), the estimate for this term is completed. The third term on the right hand side of (3.9) can be evaluated in the same way. Summing up the above estimates, we have the second inequality of this lemma. □

We estimate $G_k\widetilde{N}_a$.

LEMMA 3.3. *Let $k>0$, $1/2<\delta<2$, $a\in\mathbb{R}$ such that $a\neq 0$ and $a\neq k$, and f and V be complex and real valued functions of $x\in\mathbb{R}$, respectively. Let δ_1 and δ_2 be constants such that $\delta_1=1$ if $\delta<1$, $\delta<\delta_1<2$ if $\delta\ge 1$, and $\delta_2=3/2$ if $\delta<3/2$, $\delta<\delta_2<2$ if $\delta\ge 3/2$. Let $\widetilde{N}_a$ be defined by* (3.5).

Then there exists a constant $C > 0$ such that for $t \geq 1$,

$$
\begin{aligned}
&\|(G_k\widetilde{N}_a)(t)\|_{L^2_x} + \|G_k\widetilde{N}_a\|_{L^4((t,\infty);L^\infty_x)} \\
\leq & Ct^{-\delta/2}\Bigg\{\sum_{(\tilde{f},\widetilde{V})=(f,V),(fV,V)} (\||\cdot|^{-\delta}\tilde{f}\|_{L^2} + \||\cdot|^{-\delta}\partial\tilde{f}\|_{L^2} \\
& \quad + \||\cdot|^{-\delta}\tilde{f}\widetilde{V}\|_{L^2} + \||\cdot|^{1-\delta_1}\partial\tilde{f}\|_{L^2} + \||\cdot|^{1-\delta_1}\tilde{f}\partial\widetilde{V}\|_{L^2}) \\
& \quad + \||\cdot|^{3/2-\delta_2}\partial f\|_{L^\infty} + \||\cdot|^{3/2-\delta_2} f\partial V\|_{L^\infty} \\
& \quad + \|\partial^2(fe^{-iV\log t})\|_{L^2}\Bigg\},
\end{aligned}
\tag{3.13}
$$

where the operator G_k is defined by (2.1)*.*

Proof. Note the equality

$$
\begin{aligned}
U_k(-s)\widetilde{N}_a(s) =& \partial_s\left(U_k(-s)\int_T^s \widetilde{N}_a(\tau)\,d\tau\right) \\
&+ \frac{i}{2k}U_k(-s)\left(\int_T^s \partial^2\widetilde{N}_a(\tau)\,d\tau\right),
\end{aligned}
\tag{3.14}
$$

where $t < s < T$. Integrating the equality (3.14) over the interval (t, T), applying $U(t)$ to the resulting equality and letting $T \to \infty$, we have

$$
\begin{aligned}
(G_k\widetilde{N}_a)(t) =& \int_t^\infty \widetilde{N}_a(\tau)\,d\tau \\
& - \frac{i}{2k}\int_t^\infty U_k(t-s)\left(\int_s^\infty \partial^2\widetilde{N}_a(\tau)\,d\tau\right) ds.
\end{aligned}
\tag{3.15}
$$

We calculate $\partial^2\widetilde{N}_a$:

$$
\partial^2\widetilde{N}_a(t) = -2ai\partial_t\widetilde{N}_a(t) + 2a\mathcal{L}_a\widetilde{N}_a(t). \tag{3.16}
$$

Noting (3.7), we see that

$$\begin{aligned}\mathcal{L}_a\widetilde{N}_a(t) =& -\frac{i}{t^2}U_a(t)M_a(t)^{-1}e^{-iV(-i\partial)\log t}\check{f}\\ &+\frac{i}{t}U_a(t)\partial_t(M_a(t)^{-1}e^{-iV(-i\partial)\log t}\check{f})\\ =& -\frac{i}{t^2}U_a(t)M_a(t)^{-1}e^{-iV(-i\partial)\log t}\check{f}\\ &+\frac{i}{t}U_a(t)M_a(t)^{-1}\left(\frac{ia|x|^2}{2t^2}-\frac{i}{t}V(-i\partial)\right)e^{-iV(-i\partial)\log t}\check{f}\\ =&P(t)+r(t),\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}P(t) &= -\frac{i}{t}\widetilde{N}_a(t)+\frac{1}{t}\widetilde{N}_a(t)V\left(\frac{ax}{t}\right),\\ r(t) &= -\frac{a}{2t^3}U_a(t)(M_a(t)^{-1}|x|^2e^{-iV(-i\partial)\log t}\check{f}).\end{aligned}$$

Note that r decays faster than P. From the equalities (3.15), (3.16) and (3.17), we obtain

$$\begin{aligned}(G_k\widetilde{N}_a)(t) =& \int_t^\infty \widetilde{N}_a(\tau)\,d\tau+\frac{a}{k}\int_t^\infty U_k(t-s)\widetilde{N}_a(s)\,ds\\ &-\frac{ia}{k}\int_t^\infty U_k(t-s)\left(\int_s^\infty P(\tau)\,d\tau\right)ds\\ &-\frac{ia}{k}\int_t^\infty U_k(t-s)\left(\int_s^\infty r(\tau)\,d\tau\right)ds.\end{aligned} \tag{3.18}$$

Since the integral in the second term on the right hand side of (3.18) is $G_k\widetilde{N}_a$ and $k \neq a$, we obtain

$$(G_k\widetilde{N}_a)(t) = I_1(t)+I_2(t)+I_3(t), \tag{3.19}$$

where

$$\begin{aligned}I_1(t) &= \frac{k}{k-a}\int_t^\infty \widetilde{N}_a(\tau)\,d\tau,\\ I_2(t) &= -\frac{ia}{k-a}\int_t^\infty U_k(t-s)\left(\int_s^\infty P(\tau)\,d\tau\right)ds,\end{aligned}$$

$$I_3(t) = -\frac{ia}{k-a}\int_t^\infty U_k(t-s)\left(\int_s^\infty r(\tau)\,d\tau\right)ds.$$

From Lemmas 2.1 and 3.2, it follows that

$$\begin{aligned}
&\sum_{j=1}^{2}(\|I_j(t)\|_{L^2_x} + \|I_j\|_{L^4((t,\infty);L^\infty_x)}) \\
&\le \left\|\int_t^\infty \widetilde{N}_a(s)\,ds\right\|_{L^2_x} + \left\|\int_s^\infty \widetilde{N}_a(\tau)\,d\tau\right\|_{L^4((t,\infty);L^\infty_x)} \\
&\quad + \int_t^\infty \left\|\int_\infty^s P(\tau)\,d\tau\right\|_{L^2_x} ds \\
&\le Ct^{-\delta/2}\Bigg\{ \sum_{(\tilde f,\widetilde V)=(f,V),(fV,V)} (\||\cdot|^{-\delta}\tilde f\|_{L^2} + \||\cdot|^{-\delta}\partial\tilde f\|_{L^2} \\
&\quad + \||\cdot|^{-\delta}\tilde f\widetilde V\|_{L^2} + \||\cdot|^{1-\delta_1}\partial\tilde f\|_{L^2} + \||\cdot|^{1-\delta_1}\tilde f\partial\widetilde V\|_{L^2}) \\
&\quad + \||\cdot|^{3/2-\delta_2}\partial f\|_{L^\infty} + \||\cdot|^{3/2-\delta_2}f\partial V\|_{L^\infty}\Bigg\}.
\end{aligned} \tag{3.20}$$

Using Lemma 2.1 and noting the estimate

$$\|r(t)\|_{L^2} \le C\frac{(\log t)^2}{t^3}\|\partial^2(fe^{-iV\log t})\|_{L^2},$$

we have

$$\begin{aligned}
\|I_3(t)\|_{L^2_x} + \|I_3\|_{L^4((t,\infty);L^\infty_x)} &\le \int_t^\infty\int_s^\infty \|r(\tau)\|_{L^2}\,d\tau\,ds \\
&\le C\frac{(\log t)^2}{t}\|\partial^2(fe^{-iV\log t})\|_{L^2}.
\end{aligned} \tag{3.21}$$

From the equality (3.19), the estimates (3.20), (3.21) and the fact $1/2 < \delta < 2$, we obtain the inequality (3.13). □

REMARK 3.1. Let $1/2 < \delta < 3/2$. By Hölder's inequality, the Gagloardo-Nirenberg inequality $\|\psi\|_{L^\infty(\mathbb{R})} \le C\|\psi\|_{L^2(\mathbb{R})}^{1/2}\|\partial\psi\|_{L^2(\mathbb{R})}^{1/2}$ and the facts $-\delta < 1-\delta_1$ and $\delta_2 = 3/2$ when $1/2 \le \delta < 3/2$, the norms on the right hand side of (3.13) is dominated by

$$\begin{aligned}
C(&\|f\|_{H^2} + \||\cdot|^{-\delta}f\|_{L^2} + \||\cdot|^{-\delta}\partial f\|_{L^2} \\
&+ \|V\|_{H^2} + \||\cdot|^{-\delta}V\|_{L^2} + \||\cdot|^{-\delta}\partial V\|_{L^2})
\end{aligned}$$

if these norms are less than or equal to 1.

Proof of Theorem 1.1. Assume all the assumptions in Theorem 1.1. Let u_{1a} and u_{2a} be the functions defined by (1.10). According to Proposition 2.1, as mentioned before, it is sufficient to show the estimates (3.1)–(3.3). The estimates (3.1) and (3.2) immediately follow from the definitions of u_{1a} and u_{2a}. We prove the estimate (3.3). We rewrite $R_j(u_{1a}, u_{2a})$:

$$R_j(u_{1a}, u_{2a}) = \mathcal{L}_{m_j} u_{ja} - g_j(u_{1a}, u_{2a}) u_{ja} - N_j(u_{1a}, u_{2a})$$

for $j = 1, 2$. By Lemmas 2.1, 3.1 and 3.3, we have

$$\begin{aligned}
&\|G_{m_j} R_j(u_{1a}, u_{2a})(t)\|_{L^2_x} + \|G_{m_j} R_j(u_{1a}, u_{2a})\|_{L^4_\tau((t,\infty);L^\infty_x)} \\
&\le C \int_t^\infty \|\mathcal{L}_{m_j} u_{ja}(s) - g_j(u_{1a}(s), u_{2a}(s)) u_{ja}(s)\|_{L^2} \, ds \\
&\quad + \|G_{m_j} N_j(u_{1a}, u_{2a})(t)\|_{L^2_x} + \|G_{m_j} N_j(u_{1a}, u_{2a})\|_{L^4_\tau((t,\infty);L^\infty_x)} \\
&\le C\eta t^{-b/2},
\end{aligned}$$

where $1/2 < b < 3/2$ appearing in the assumption of Theorem 1.1, and η is defined by (3.4). Here we have applied Lemma 3.3 to each term of $N_j(u_{1a}, u_{2a})$. Taking $\eta' = C\eta$ and $d = b/2$, we see that the assumptions in Proposition 2.1 are satisfied. This completes the proof of Theorem 1.1. □

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