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Notes on Generalizations of Local Ogus-Vologodsky Correspondence

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Abstract. Given a smooth scheme over $\mathbb{Z}/p^n\mathbb{Z}$ with a lift of relative Frobenius to $\mathbb{Z}/p^{n+1}\mathbb{Z}$, we construct a functor from the category of Higgs modules to that of modules with integrable connection as the composite of the level raising inverse image functors from the category of modules with integrable p^m-connection to that of modules with integrable p^{m-1}-connection for $1 \leq m \leq n$. In the case $m = 1$, we prove that the level raising inverse image functor is an equivalence when restricted to quasi-nilpotent objects, which generalizes a local result of Ogus-Vologodsky. We also prove that the above level raising inverse image functor for a smooth p-adic formal scheme induces an equivalence of $\mathbb{Q}$-linearized categories for general m when restricted to nilpotent objects (in strong sense), under a strong condition on Frobenius lift. We also prove a similar result for the category of modules with integrable p^m-Witt-connection.

Contents

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Introduction

For a proper smooth algebraic variety X over $\mathbb{C}$, the equivalence of the category of modules endowed with integrable connection on X and the category of Higgs modules on X (with semistability and vanishing Chern number condition) is established by Simpson [10]. In search for the analogue of it in chatacteristic $p > 0$, Ogus and Vologodsky proved in [9] similar equivalences for a smooth scheme X_1 over a scheme S_1 of characteristic $p > 0$. One of their results [9, 2.11] is described as follows: Denote the structure morphism $X_1 \longrightarrow S_1$ by f_1, let $F_{S_1} : S_1 \longrightarrow S_1$ be the absolute Frobenius morphism, put $X_1^{(1)} := S_1 \times_{F_{S_1},S_1} X_1$, denote the projection $X_1^{(1)} \longrightarrow S_1$ by $f_1^{(1)}$ and let $F_{X_1/S_1} : X_1 \longrightarrow X_1^{(1)}$ be the relative Frobenius morphism. Assume that we are given smooth lifts $f_2 : X_2 \longrightarrow S_2, f_2^{(1)} : X_2^{(1)} \longrightarrow S_2$ of $f_1, f_1^{(1)}$ to morphisms of flat $\mathbb{Z}/p^2\mathbb{Z}$-schemes and a lift $F_2 : X_2 \longrightarrow X_2^{(1)}$ of the morphism F_{X_1/S_1} which is a morphism over S_2. Then there exists an equivalence between the category of quasi-nilpotent Higgs modules on $X_1^{(1)}$ and that of modules with quasi-nilpotent integrable connection on X_1. (There is also a version [9, 2.8] which does not assume the existence of f_2 and F_2. In this case, the categories they treat are a little more restricted in some sense.) The key of their proof is the Azumaya algebra property of the sheaf $\mathcal{D}^{(0)}_{X_1/S_1}$ of differential operators of level 0 on X_1 over S_1. Also, in the above situation, one can give an explicit description of this functor as the inverse image by 'divided Frobenius'. (See [9, 2.11.2], [4, 5.9, 6.5] or Remark 1.12 in this paper.)

The purpose of this paper is to construct a functor from Higgs modules to modules with integrable connection for smooth schemes over some flat $\mathbb{Z}/p^n\mathbb{Z}$-schemes and study the properties of this functor and related functors. Let us fix $n \in \mathbb{N}$ and let S_{n+1} be a scheme flat over $\mathbb{Z}/p^{n+1}\mathbb{Z}$. Let us put $S_j := S_{n+1} \otimes \mathbb{Z}/p^j\mathbb{Z}\,(j \in \mathbb{N}, \leq n+1)$, let f_1, F_{S_1} be as above and for $0 \leq m \leq n$, let us put $X_1^{(m)} := S_1 \times_{F_{S_1}^m,S_1} X_1$, denote the projection $X_1^{(m)} \longrightarrow S_1$ by $f_1^{(m)}$ and for $1 \leq m \leq n$, let $F^{(m)}_{X_1/S_1} : X_1^{(m-1)} \longrightarrow X_1^{(m)}$ be the relative Frobenius morphism for $f_1^{(m)}$. Moreover, assume that we are given a smooth lift $f_{n+1} : X_{n+1} \longrightarrow S_{n+1}$ of f_1, smooth lifts $f_{n+1}^{(m)} : X_{n+1}^{(m)} \longrightarrow S_{n+1}$ of $f_1^{(m)}$ $(0 \leq m \leq n)$ with $f_1^{(0)} = f_1$ and lifts $F_{n+1}^{(m)} : X_{n+1}^{(m-1)} \longrightarrow X_{n+1}^{(m)}$ of the morphisms

$F^{(m)}_{X_1/S_1}$ $(1 \leq m \leq n)$ which are morphisms over S_{n+1}. Furthermore, let $f_n : X_n \longrightarrow S_n$, $f^{(m)}_n : X^{(m)}_n \longrightarrow S_n$ $(0 \leq m \leq n)$, $F^{(m)}_n : X^{(m-1)}_n \longrightarrow X^{(m)}_n$ be $f_{n+1} \otimes \mathbb{Z}/p^n\mathbb{Z}$, $f^{(m)}_{n+1} \otimes \mathbb{Z}/p^n\mathbb{Z}$, $F^{(m)}_{n+1} \otimes \mathbb{Z}/p^n\mathbb{Z}$, respectively. Under this assumption, first we construct the functor $\Psi^{\mathrm{qn}} : \mathrm{HIG}(X^{(n)}_n)^{\mathrm{qn}} \longrightarrow \mathrm{MIC}(X_n)^{\mathrm{qn}}$ from the category $\mathrm{HIG}(X^{(n)}_n)^{\mathrm{qn}}$ of quasi-nilpotent Higgs modules on $X^{(n)}_n$ to the category $\mathrm{MIC}(X_n)^{\mathrm{qn}}$ of modules with quasi-nilpotent integrable connection on X_n.

The construction of the functor Ψ^{qn} is done in the following way: For $0 \leq m \leq n$, let $\mathrm{MIC}^{(m)}(X^{(m)}_n)^{\mathrm{qn}}$ be the category of modules with quasi-nilpotent integrable p^m-connection (also called 'quasi-nilpotent integrable connection of level $-m$') on $X^{(m)}_n$, that is, the category of pairs $(\mathcal{E}, \nabla)$ consisting of an $\mathcal{O}_{X^{(m)}_n}$-module $\mathcal{E}$ endowed with an additive map $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{X^{(m)}_n/S_n}$ with the property $\nabla(fe) = f\nabla(e) + p^m e \otimes df$ $(e \in \mathcal{E}, f \in \mathcal{O}_{X^{(m)}_n})$, the integrability and the quasi-nilpotence condition. (The notion of p^m-connection is the case $\lambda = p^m$ of the notion of λ-connection of Simpson [11].) Note that we have $\mathrm{HIG}(X^{(n)}_n)^{\mathrm{qn}} = \mathrm{MIC}^{(n)}(X^{(n)}_n)^{\mathrm{qn}}$ since $p^n = 0$ on $X^{(n)}_n$. We construct the functor Ψ^{qn} as the composite of functors $F^{(m),*,\mathrm{qn}}_{n+1} : \mathrm{MIC}^{(m)}(X^{(m)}_n)^{\mathrm{qn}} \longrightarrow \mathrm{MIC}^{(m-1)}(X^{(m-1)}_n)^{\mathrm{qn}}$ for $1 \leq m \leq n$, which are defined as the inverse image by divided Frobenius (which we call the level raising inverse image) associated to $F^{(m)}_{n+1}$, as in the case of [9, 2.11.2], [4, 5.9].

The first naive question might be whether the functor Ψ^{qn} is an equivalence or not. (Note that, since the sheaf of differential operators of level 0 on X_n over S_n does not seem to have Azumaya algebra property, it would be hard to generalize the method of Ogus-Vologodsky in this case.) Unfortunately, the functors $F^{(m),*,\mathrm{qn}}$ are not equivalences (not full, not essentially surjective) for $m \geq 2$ and so the functor Ψ^{qn} is not an equivalence either. So an interesting question would be to construct nice functors from the functors $F^{(m),*,\mathrm{qn}}_{n+1}$. As a first answer to this question, we prove that the functor $F^{(1),*,\mathrm{qn}}_{n+1}$ is an equivalence, under the assumption that there exists a closed immersion $S_{n+1} \hookrightarrow S$ of S_{n+1} into a p-adic formal scheme S flat over $\mathbb{Z}_p$. This generalizes [9, 2.11] under the existence of S. To prove this, we may work locally and so we may assume the existence of a smooth lift $f : X \longrightarrow S$ of f_{n+1}, a smooth lift $f^{(1)} : X^{(1)} \longrightarrow S$ of $f^{(1)}_1$ and a lift $F^{(1)} : X \longrightarrow X^{(1)}$ of the morphism $F^{(1)}_{X_1/S_1}$. In this situation, we introduce

the notion of the sheaf of p-adic differential operators $\mathcal{D}^{(-1)}_{X^{(1)}/S}$ of level -1 of $X^{(1)}$ over S, which is a level -1 version of the sheaf of p-adic differential operators defined and studied by Berthelot [1], [2]. Then we prove that the category $\mathrm{MIC}^{(1)}(X_n^{(1)})^{\mathrm{qn}}$ is equivalent to the category of p^n-torsion quasi-nilpotent left $\mathcal{D}^{(-1)}_{X^{(1)}/S}$-modules and re-define the functor $F^{(1),*,\mathrm{qn}}$ as the level raising inverse image functor from the category of p^n-torsion quasi-nilpotent left $\mathcal{D}^{(-1)}_{X^{(1)}/S}$-modules to the category of p^n-torsion quasi-nilpotent left $\mathcal{D}^{(0)}_{X/S}$-modules. This is a negative level version of the level raising inverse image functor defined in [2, 2.2]. (We also give a definition of the sheaf of p-adic differential operators of level $-m$ and give the interpretation of (quasi-nilpotent) modules with integrable p^m-connection and (level raising) inverse image functors of them in terms of $\mathcal{D}$-modules for general $m \in \mathbb{N}$.) Then, one can prove the equivalence of the functor $F^{(1),*,\mathrm{qn}}$ by following the proof of Frobenius descent by Berthelot [2, 2.3].

To explain our next result, let us fix $m \in \mathbb{N}$ and assume that we are given smooth lifts $f : X \longrightarrow S$ of f_1, $f^{(1)} : X^{(1)} \longrightarrow S$ of $f_1^{(1)}$ and a 'nice' lift $F : X \longrightarrow X^{(1)}$ of F_{X_1/S_1} which is a morphism over S. Then, under certain assumption on nilpotence condition, we prove that the i-th de Rham cohomology of an object in the category $\mathrm{MIC}^{(m)}(X^{(1)})$ of modules with integrable p^m-connection on $X^{(1)}$ is isomorphic to the i-th de Rham cohomology of its level raising inverse image by F (which is an object in the category $\mathrm{MIC}^{(m-1)}(X)$ of modules with integrable p^{m-1}-connection on X) when $i = 0$, and isomorphic modulo torsion for general i. This implies that the level raising inverse image by F induces a fully faithful functor between a full subcategory of $\mathrm{MIC}^{(m)}(X^{(1)})$ and that of $\mathrm{MIC}^{(m-1)}(X)$ satisfying certain nilpotence condition and that it induces an equivalence of $\mathbb{Q}$-linearized categories (again under nilpotence condition), which gives a second answer to the question we raised in the previous paragraph.

Also, we prove a Witt version of the result in the previous paragraph: We introduce the category of modules with integrable p^m-Witt connection $\mathrm{MIWC}^{(m)}(X_1)$ on X_1 for a smooth morphism $X_1 \longrightarrow S_1$ of chatacteristic $p > 0$ with S_1 perfect and the level raising inverse image functor $F_* : \mathrm{MIWC}^{(m)}(X_1^{(1)}) \longrightarrow \mathrm{MIWC}^{(m-1)}(X_1)$ for $X_1 \longrightarrow S_1$ as above and $X_1^{(1)} := S_1 \times_{F_{S_1},S_1} X_1$. We prove that the functor F_* induces a fully faithful functor between a full subcategory of $\mathrm{MIWC}^{(m)}(X_1^{(1)})$ and that of $\mathrm{MIWC}^{(m-1)}(X_1)$

satisfying certain nilpotence condition and that it induces an equivalence of $\mathbb{Q}$-linearized categories (again under nilpotence condition). This result has an advantage that we need no assumption on the liftability of objects and Frobenius morphisms.

The content of each section is as follows: In Section 1, we introduce the notion of modules with integrable p^m-connection, (level raising) inverse image functors and define the functor Ψ (Ψ^{qn}) from the category of (quasi-nilpotent) Higgs modules to the category of modules with (quasi-nilpotent) integrable connection as the iteration of level raising inverse image functors. We also give an example which shows that the functor Ψ, Ψ^{qn} are not equivalences.

In Section 2, we introduce the sheaf of p-adic differential operators of negative level and prove basic properties of it. In particular, we prove the equivalence between the category of (quasi-nilpotent) left $\mathcal{D}$-modules of level $-m$ and the category of (quasi-nilpotent) modules with integrable p^m-connection ($m \in \mathbb{N}$). In the case of schemes over $\mathbb{Z}/p^n\mathbb{Z}$, we also introduce certain crystalized categories to describe the categories of (quasi-nilpotent) modules with integrable p^m-connection, and prove certain crystalline property of them. In Section 3, we prove that the level raising inverse image functor from the category of modules with integrable p-connection to that of modules with integrable connection defined in Section 1 is an equivalence of categories when restricted to quasi-nilpotent objects. In Section 4, we compare the de Rham cohomology of certain modules with integrable p^m-connection over smooth p-adic formal schemes and that of the pull-back of them by the level raising inverse image functor, and deduce the full-faithfulness (resp. the equivalence) of the functor from the category (resp. the $\mathbb{Q}$-linearized category) of modules with integrable p^m-connection to that of modules with integrable p^{m-1}-connection satisfying nilpotent conditions. In Section 4, we introduce the notion of modules with integrable p^m-Witt connection and the level raising inverse image functor from the category of modules with integrable p^m-Witt connection to that of modules with integrable p^{m-1}-Witt connection. We compare the de Rham cohomology of certain modules with integrable p^m-Witt-connection and the de Rham cohomology of the pull-back of them by the level raising inverse image functor, and deduce the full-faithfulness (resp. the equivalence) of the functor from the category (resp. the $\mathbb{Q}$-linearized category) of modules with inte-

grable p^m-Witt-connection to that of modules with integrable p^{m-1}-Witt-connection satisfying nilpotent conditions.

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Convention

Throughout this paper, p is a fixed prime number. Fiber products and tensor products in this paper are p-adically completed ones, unless otherwise stated. A local section of a sheaf $\mathcal{E}$ on a (formal) scheme means a section of $\mathcal{E}$ on an affine open (formal) subscheme.

1. Modules with Integrable p^m-Connection

In this section, we define the notion of modules with integrable p^m-connection. Also, we define the inverse image functor of the categories of modules with integrable p^m-connection, and the 'level raising inverse image functor' from the category of modules with integrable p^m-connection to that of modules with integrable p^{m-1}-connection for a lift of Frobenius morphism. As a composite of the level raising inverse image functors, we define the functor from the category of Higgs modules to the category of modules with integrable connection, which is a generalization of the inverse of local Cartier transform of Ogus-Vologodsky [9, 2.11].

First we give a definition of p^m-connection, which is a special case (the case $\lambda = p^m$) of the notion of λ-connection of Simpson [11].

DEFINITION 1.1. Let $X \longrightarrow S$ be a smooth morphism of schemes over $\mathbb{Z}/p^n\mathbb{Z}$ or p-adic formal schemes and let $m \in \mathbb{N}$. A p^m-connection on an $\mathcal{O}_X$-module $\mathcal{E}$ is an additive map $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ satisfying $\nabla(fe) = f\nabla(e) + p^m e \otimes df$ for any local sections $e \in \mathcal{E}, f \in \mathcal{O}_X$. We refer to a p^m-connection also as a connection of level $-m$.

When we are given an $\mathcal{O}_X$-module with p^m-connection $(\mathcal{E}, \nabla)$, we can define the additive map $\nabla_k : \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^k_{X/S} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{k+1}_{X/S}$ which is characterized by $\nabla_k(e \otimes \omega) = \nabla(e) \wedge \omega + p^m e \otimes d\omega$.

Definition 1.2. With the notation above, we call $(\mathcal{E}, \nabla)$ integrable if we have $\nabla_1 \circ \nabla = 0$. We denote the category of $\mathcal{O}_X$-modules with integrable p^m-connection by $\mathrm{MIC}^{(m)}(X)$.

When $m = 0$, the notion of modules with integrable p^m-connection is nothing but that of modules with integrable connection. In this case, we denote the category $\mathrm{MIC}^{(m)}(X)$ also by $\mathrm{MIC}(X)$. Also, when X, S are schemes over $\mathbb{Z}/p^n\mathbb{Z}$ with $n \leq m$, the notion of modules with integrable p^m-connection is nothing but that of Higgs modules. In this case, we denote the category $\mathrm{MIC}^{(m)}(X)$ also by $\mathrm{HIG}(X)$.

Remark 1.3. For a smooth morphism $f : X \longrightarrow S$ of p-adic formal schemes and $n \in \mathbb{N}$, we denote the full subcategory of $\mathrm{MIC}^{(m)}(X)$ consisting of p^n-torsion objects by $\mathrm{MIC}^{(m)}(X)_n$. If we denote the morphism $f \otimes \mathbb{Z}/p^n\mathbb{Z}$ by $X_n \longrightarrow S_n$, the direct image by the canonical closed immersion $X_n \hookrightarrow X$ induces the equivalence of categories $\mathrm{MIC}^{(m)}(X_n) \xrightarrow{=} \mathrm{MIC}^{(m)}(X)_n$.

Let us assume given a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ S & \longrightarrow & T \end{array} \tag{1.1}$$

of schemes over $\mathbb{Z}/p^n\mathbb{Z}$ or p-adic formal schemes with smooth vertical arrows and and an object $(\mathcal{E}, \nabla)$ in $\mathrm{MIC}^{(m)}(Y)$ (where $\mathrm{MIC}^{(m)}(Y)$ is defined for the morphism $Y \longrightarrow T$). Then we can endow a structure of an integrable p^m-connection $g^*\nabla$ on $g^*\mathcal{E}$ by $g^*\nabla(fg^*(e)) = fg^*(\nabla(e)) + p^m g^*(e) \otimes df$ ($e \in \mathcal{E}, f \in \mathcal{O}_X$). So we have the inverse image functor

$$g^* : \mathrm{MIC}^{(m)}(Y) \longrightarrow \mathrm{MIC}^{(m)}(X); \quad (\mathcal{E}, \nabla) \mapsto g^*(\mathcal{E}, \nabla) := (g^*\mathcal{E}, g^*\nabla).$$

Remark 1.4. Let us assume given the commutative diagram (1.1) of p-adic formal schemes with smooth vertical arrows and let us denote the morphism $g \otimes \mathbb{Z}/p^n\mathbb{Z}$ by $g_n : X_n \longrightarrow Y_n$. Then the inverse image functor g^* above

induces the functor $g^* : \mathrm{MIC}^{(m)}(Y)_n \longrightarrow \mathrm{MIC}^{(m)}(X)_n$, and it coincides with the inverse image functor $g_n^* : \mathrm{MIC}^{(m)}(Y_n) \longrightarrow \mathrm{MIC}^{(m)}(X_n)$ associated to g_n via the equivalences $\mathrm{MIC}^{(m)}(X_n) \stackrel{=}{\longrightarrow} \mathrm{MIC}^{(m)}(X)_n, \mathrm{MIC}^{(m)}(Y_n) \stackrel{=}{\longrightarrow} \mathrm{MIC}^{(m)}(Y)_n$ of Remark 1.3.

Next we introduce the notion of quasi-nilpotence. Let $X \longrightarrow S$ be a smooth morphism of schemes over $\mathbb{Z}/p^n\mathbb{Z}$ which admits a local coordinate $t_1, ..., t_d$. Then, for $(\mathcal{E}, \nabla) \in \mathrm{MIC}^{(m)}(X)$, we can write ∇ as $\nabla(e) = \sum_{i=1}^d \theta_i(e) dt_i$ for some additive maps $\theta_i : \mathcal{E} \longrightarrow \mathcal{E}\, (1 \leq i \leq d)$. Then we have

$$0 = (\nabla_1 \circ \nabla)(e) = \sum_{i<j} (\theta_i\theta_j - \theta_j\theta_i)(e) dt_i \wedge dt_j.$$

So we have $\theta_i\theta_j = \theta_j\theta_i$. Therefore, for $a = (a_1, ..., a_d) \in \mathbb{N}^d$, the map $\theta^a := \prod_{i=1}^d \theta_i^{a_i}$ is well-defined.

DEFINITION 1.5. With the above situation, we call $(\mathcal{E}, \nabla)$ quasi-nilpotent with respect to $(t_1, ..., t_d)$ if, for any local section $e \in \mathcal{E}$, there exists some $N \in \mathbb{N}$ such that $\theta^a(e) = 0$ for any $a \in \mathbb{N}^d$ with $|a| \geq N$.

LEMMA 1.6. *The above definition of quasi-nilpotence does not depend on the local coordinate* $(t_1, ..., t_d)$.

PROOF. When $m = 0$, this is classical [3]. Here we prove the lemma in the case $m > 0$. (The proof is easier in this case.) First, let us note that, for $f \in \mathcal{O}_X$, we have the equality

$$\sum_i \theta_i(fe) dt_i = \nabla(fe) = f\nabla(e) + p^m e df = \sum_i (f\theta_i(e) + p^m \frac{\partial f}{\partial t_i}) dt_i.$$

So we have the equality

$$\theta_i f = f\theta_i + p^m \frac{\partial f}{\partial t_i}. \tag{1.2}$$

Now let us take another local coordinate $t'_1, ..., t'_d$, and write ∇ as $\nabla(e) = \sum_{i=1}^d \theta'_i(e) dt'_i$. Then we have

$$\sum_i \theta_i(e) dt_i = \sum_{i,j} \theta_i(e) \frac{\partial t_i}{\partial t'_j} dt'_j = \sum_j (\sum_i \frac{\partial t_i}{\partial t'_j} \theta_i(e)) dt'_j.$$

Hence we have $\theta'_j = \sum_i \dfrac{\partial t_i}{\partial t'_j}\theta_i$.

Let us prove that, for any local section $e \in \mathcal{E}$ and for any $a \in \mathbb{N}^d$, there exist some $f_{a,b} \in \mathcal{O}_X$ ($b \in \mathbb{N}^d, |b| \leq |a|$) with

$$\theta'^a(e) = \sum_{|b|\leq|a|} p^{m(|a|-|b|)} f_{a,b}\theta^b(e), \tag{1.3}$$

by induction on a: Indeed, this is trivially true when $a = 0$. If this is true for a, we have

$$\begin{aligned}\theta'_j\theta'^a(e) &= (\sum_i \frac{\partial t_i}{\partial t'_j}\theta_i)(\sum_{|b|\leq|a|} p^{m(|a|-|b|)} f_{a,b}\theta^b)(e)\\ &= \sum_{i,b}(p^{m(|a|-|b|)}\frac{\partial t_i}{\partial t'_j} f_{a,b}\theta^{b+e_i}(e)\\ &\quad + p^{m(|a|-|b|+1)}\frac{\partial f_{a,b}}{\partial t_i}\theta^b(e)) \quad \text{(by (1.2))}\end{aligned}$$

and from this equation, we see that the claim is true for $a + e_j$.

Now let us assume that $(\mathcal{E}, \nabla)$ is quasi-nilpotent with respect to $(t_1, ..., t_d)$, and take a local section $e \in \mathcal{E}$. Then there exists some $N \in \mathbb{N}$ such that $\theta^b(e) = 0$ for any $b \in \mathbb{N}^d, |b| \geq N$. Then, for any $a \in \mathbb{N}^d, |a| \geq N + n$, we have either $|b| \geq N$ or $|a| - |b| \geq n$ for any $b \in \mathbb{N}^d$. Hence we have either $p^{m(|a|-|b|)} = 0$ or $\theta^b(e) = 0$ on the right hand side of (1.3) and so we have $\theta'^a(e) = 0$. So we have shown that $(\mathcal{E}, \nabla)$ is quasi-nilpotent with respect to $(t'_1, ..., t'_d)$ and so we are done. □

Remark 1.7. By (1.2), we have

$$\theta^a(fe) = \sum_{0\leq b\leq a} p^{m|b|}\frac{\partial^b f}{\partial t^b}\theta^{a-b}(e)$$

for $e \in \mathcal{E}, f \in \mathcal{O}_X$, and we have $p^{m|b|}\dfrac{\partial^b f}{\partial t^b} \in p^{m|b|}b!\mathcal{O}_X$. Hence, if we have $\theta^a(e) = 0$ for any $a \in \mathbb{N}^d, |a| \geq N$, we have $\theta^a(fe) = 0$ for any $a \in \mathbb{N}^d, |a| \geq N + p^n d$. Therefore, to check the quasi-nilpotence of $(\mathcal{E}, \nabla)$ (with respect to some local coordinate $t_1, ..., t_d$), it suffices to check that, for some local generator $e_1, ..., e_r$ of $\mathcal{E}$, there exists some $N \in \mathbb{N}$ such that $\theta^a(e_i) = 0$ for

any $a \in \mathbb{N}^d, |a| \geq N$ and $1 \leq i \leq r$. Also, we can take $N \in \mathbb{N}$ such that $\theta^a(e) = 0$ for any $a \in \mathbb{N}^d, |a| \geq N$ and any local section $e \in \mathcal{E}$.

When a given morphism does not admit a local coordinate globally, we define the notion of quasi-nilpotence as follows:

DEFINITION 1.8.

(1) Let $X \longrightarrow S$ be a smooth morphism of schemes over $\mathbb{Z}/p^n\mathbb{Z}$. Then an object $(\mathcal{E}, \nabla)$ in $\mathrm{MIC}^{(m)}(X)$ is called quasi-nilpotent if, locally on X, there exists a local coordinate $t_1, ..., t_d$ of X over S such that $(\mathcal{E}, \nabla)$ is quasi-nilpotent with respect to $(t_1, ..., t_d)$. (Note that, by Lemma 1.6, this definition is independent of the choice of $t_1, ..., t_d$.)

(2) Let $X \longrightarrow S$ be a smooth morphism of p-adic formal schemes. Then an object $(\mathcal{E}, \nabla)$ in $\mathrm{MIC}^{(m)}(X)$ is called quasi-nilpotent if it is contained in $\mathrm{MIC}^{(m)}(X)_n$ for some n and the object in $\mathrm{MIC}^{(m)}(X_n)$ (where $X_n := X \otimes \mathbb{Z}/p^n\mathbb{Z}$) corresponding to $(\mathcal{E}, \nabla)$ via the equivalence in Remark 1.3 is quasi-nilpotent.

We denote the full subcategory of $\mathrm{MIC}^{(m)}(X)$ consisting of quasi-nilpotent objects by $\mathrm{MIC}^{(m)}(X)^{\mathrm{qn}}$, and in the case of (2), we denote the category $\mathrm{MIC}^{(m)}(X)_n \cap \mathrm{MIC}^{(m)}(X)^{\mathrm{qn}}$ by $\mathrm{MIC}^{(m)}(X)^{\mathrm{qn}}_n$.

Next we prove the functoriality of quasi-nilpotence.

PROPOSITION 1.9. *Let us assume given a commutative diagram* (1.1) *of smooth morphism of p-adic formal schemes or schemes over $\mathbb{Z}/p^n\mathbb{Z}$ with smooth vertical arrows. Then the inverse image functor $g^* : \mathrm{MIC}^{(m)}(Y) \longrightarrow \mathrm{MIC}^{(m)}(X)$ induces the functor $g^{*,\mathrm{qn}} : \mathrm{MIC}^{(m)}(Y)^{\mathrm{qn}} \longrightarrow \mathrm{MIC}^{(m)}(X)^{\mathrm{qn}}$, that is, g^* sends quasi-nilpotent objects to quasi-nilpotent objects.*

PROOF. In view of Remark 1.4, it suffices to prove the case of schemes over $\mathbb{Z}/p^n\mathbb{Z}$. When $m = 0$, the proposition is classical ([3], [2]). So we may assume $m > 0$. Since the quasi-nilpotence is a local property, we may assume that there exists a local coordinate $t_1, ..., t_d$ (resp. $t'_1, ..., t'_{d'}$) of X over S (resp. Y over T). Let us take an object $(\mathcal{E}, \nabla)$ in $\mathrm{MIC}^{(m)}(Y)^{\mathrm{qn}}$ and write

the map $\nabla, g^*\nabla$ as $\nabla(e) = \sum_j \theta'_j(e)dt'_j$, $g^*\nabla(fg^*(e)) = \sum_i \theta_i(fg^*(e))dt_i$. Let us write $g^*(dt'_j) = \sum_i a_{ij}dt_i$. Then we have

$$\begin{aligned} g^*\nabla(fg^*(e)) &= fg^*(\nabla(e)) + p^m g^*(e) \otimes df \\ &= \sum_{i,j}(a_{ij}fg^*(\theta'_j(e)) + p^m g^*(e)\frac{\partial f}{\partial t_i})dt_i \end{aligned}$$

and so we have $\theta_i(fg^*(e)) = \sum_j a_{ij}fg^*(\theta'_j(e)) + p^m \dfrac{\partial f}{\partial t_i}g^*(e)$. Let us prove that, for any local sections $e \in \mathcal{E}, f \in \mathcal{O}_X$ and $a \in \mathbb{N}^d$, there exist some $f_{a,b} \in \mathcal{O}_X$ $(b \in \mathbb{N}^d, |b| \leq |a|)$ (which depends on e, f) with

$$\theta^a(fg^*(e)) = \sum_{|b|\leq|a|} p^{m(|a|-|b|)} f_{a,b} g^*(\theta'^b(e)), \tag{1.4}$$

by induction on a: Indeed, this is trivially true when $a = 0$. If this is true for a, we have

$$\begin{aligned} \theta_i\theta^a(fg^*(e)) &= \theta_i(\sum_{|b|\leq|a|} p^{m(|a|-|b|)} f_{a,b} g^*(\theta'^b(e))) \\ &= \sum_{j,b}(a_{ij}p^{m(|a|-|b|)} f_{a,b} g^*(\theta'^{b+e_j}(e)) \\ &\quad + p^{m(|a|-|b|+1)}\frac{\partial f_{a,b}}{\partial t_i} g^*(\theta'^b(e))) \end{aligned}$$

and from this equation, we see that the claim is true for $a + e_i$. From (1.4), we can prove the quasi-nilpotence of $(g^*\mathcal{E}, g^*\nabla)$ as in the proof of Lemma 1.6. So we are done. □

Before we define the level raising inverse image functor, we give the following definition to fix the situation.

Definition 1.10. In this definition, 'a scheme flat over $\mathbb{Z}/p^\infty\mathbb{Z}$' means a p-adic formal scheme flat over $\mathbb{Z}_p$.

For $a, b, c \in \mathbb{N} \cup \{\infty\}$ with $a \geq b \geq c$, we mean by $\mathbf{Hyp}(a, b, c)$ the following hypothesis: We are given a scheme S_a flat over $\mathbb{Z}/p^a\mathbb{Z}$, and for $j \in \mathbb{N}, j \leq a$, S_j denotes the scheme $S_a \otimes \mathbb{Z}/p^j\mathbb{Z}$. We are also given a smooth morphism of finite type $f_1 : X_1 \longrightarrow S_1$, and let $F_{X_1} : X_1 \longrightarrow X_1, F_{S_1} :$

$S_1 \longrightarrow S_1$ be the Frobenius endomorphism (p-th power endomorphism). Let us put $X_1^{(1)} := S_1 \times_{F_{S_1},S_1} X_1$ and denote the projection $X_1^{(1)} \longrightarrow S_1$ by $f_1^{(1)}$. Then the morphism F_{X_1} induces the relative Frobenius morphism $F_{X_1/S_1} : X_1 \longrightarrow X_1^{(1)}$. We assume that we are given a smooth lift $f_b : X_b \longrightarrow S_b$ of f_1, a smooth lift $f_b^{(1)} : X_b^{(1)} \longrightarrow S_b$ of $f_1^{(1)}$, and for $j \in \mathbb{N}, j \leq b$, denote the morphism $f_b \otimes \mathbb{Z}/p^j\mathbb{Z}$, $f_b^{(1)} \otimes \mathbb{Z}/p^j\mathbb{Z}$ by $f_j : X_j \longrightarrow S_j$, $f_j^{(1)} : X_j^{(1)} \longrightarrow S_j$, respectively. Also, we assume that we are given a lift $F_c : X_c \longrightarrow X_c^{(1)}$ of the morphism F_{X_1/S_1} which is a morphism over S_c. For $j \in \mathbb{N}, j \leq c$, let $F_j : X_j \longrightarrow X_j^{(1)}$ be $F_c \otimes \mathbb{Z}/p^j\mathbb{Z}$. Finally, when $a = \infty$ (resp. $b = \infty$, $c = \infty$), we denote S_a (resp. $f_b : X_b \longrightarrow S_b$ and $f_b^{(1)} : X_b^{(1)} \longrightarrow S_b$, $F_c : X_c \longrightarrow X_c^{(1)}$) simply by S (resp. $f : X \longrightarrow S$ and $f^{(1)} : X^{(1)} \longrightarrow S$, $F : X \longrightarrow X^{(1)}$).

Roughly speaking, $\mathbf{Hyp}(a, b, c)$ means that S_1 is lifted to a scheme S_a flat over $\mathbb{Z}/p^a\mathbb{Z}$, $f_1 : X_1 \longrightarrow S_1$ and $f_1^{(1)} : X_1^{(1)} \longrightarrow S_1$ are lifted to morphisms over $S_b = S_a \otimes \mathbb{Z}/p^b\mathbb{Z}$ and the relative Frobenius $F_{X_1/S_1} : X_1 \longrightarrow X_1^{(1)}$ is lifted to a morphism over $S_c = S_a \otimes \mathbb{Z}/p^c\mathbb{Z}$.

Now we define the level raising inverse image functor for a lift of Frobenius. Let $n \in \mathbb{N}$ and assume that we are in the situation of $\mathbf{Hyp}(n+1, n+1, n+1)$. When we work locally, we can take a local coordinate $t_1, ..., t_d$ of X_{n+1} over S_{n+1} and a local coordinate $t'_1, ..., t'_d$ of $X_{n+1}^{(1)}$ over S_{n+1} such that $F_{n+1}^*(t'_i) = t_i^p + pa_i$ for some $a_i \in \mathcal{O}_{X_{n+1}}$. Hence we have $F_{n+1}^*(dt'_i) = p(t_i^{p-1}dt_i + da_i)$, that is, the image of the homomorphism $F_{n+1}^* : \Omega^1_{X_{n+1}^{(1)}/S_{n+1}} \longrightarrow \Omega^1_{X_{n+1}/S_{n+1}}$ is contained in $p\Omega^1_{X_{n+1}/S_{n+1}}$. So there exists a unique morphism $\overline{F}_{n+1}^* : \Omega^1_{X_n^{(1)}/S_n} \longrightarrow \Omega^1_{X_n/S_n}$ which makes the following diagram commutative

$$\begin{array}{ccc} \Omega^1_{X_{n+1}^{(1)}/S_{n+1}} & \xrightarrow{F_{n+1}^*} & \Omega^1_{X_{n+1}/S_{n+1}} \\ {\scriptstyle \text{proj.}}\downarrow & & \uparrow{\scriptstyle \underline{p}} \\ \Omega^1_{X_n^{(1)}/S_n} & \xrightarrow{\overline{F}_{n+1}^*} & \Omega^1_{X_n/S_n}, \end{array}$$

where proj. denotes the natural projection and $\underline{p}$ denotes the map naturally induced by the multiplication by p on $\Omega^1_{X_{n+1}/S_{n+1}}$. Using this, we define the

level raising inverse image functor

$$F^*_{n+1} : \mathrm{MIC}^{(m)}(X^{(1)}_n) \longrightarrow \mathrm{MIC}^{(m-1)}(X_n) \tag{1.5}$$

as follows: an object $(\mathcal{E}, \nabla)$ in $\mathrm{MIC}^{(m)}(X^{(1)}_n)$ is sent by F^*_{n+1} to the object $(F^*_n\mathcal{E}, F^*_n\nabla)$, where $F^*_n\nabla$ is the additive map characterized by $(F^*_n\nabla)(fF^*_n(e)) = f\overline{F}^*_{n+1}(\nabla(e)) + p^{m-1}F^*_n(e) \otimes df$ for $e \in \mathcal{E}, f \in \mathcal{O}_{X_n}$. (Here, by abuse of notation, we denoted the map

$$\mathcal{E} \otimes \Omega^1_{X^{(1)}_n/S_n} \longrightarrow F^*_n\mathcal{E} \otimes \Omega^1_{X_n/S_n};\ \ e \otimes \omega \mapsto F^*_n(e) \otimes \overline{F}^*_{n+1}(\omega)$$

also by $\overline{F}^*_{n+1}$.)

We need to check that the above definition of functor (1.5) is well-defined. First note that the map $F^*_n\nabla$ is well-defined, because we have for $e \in \mathcal{E}, f \in \mathcal{O}_{X_n}, g \in \mathcal{O}_{X^{(1)}_n}$ the equality

$$\begin{aligned}
&(F^*_n\nabla)(fF^*_n(ge)) = f\overline{F}^*_{n+1}(\nabla(ge)) + p^{m-1}F^*_n(ge) \otimes df \\
&= f\overline{F}^*_{n+1}(g\nabla(e) + p^m e \otimes dg) + p^{m-1}F^*_n(ge) \otimes df \\
&= fF^*_n(g)\overline{F}^*_{n+1}(\nabla(e)) + p^{m-1}(fF^*_n(e) \otimes p\overline{F}^*_{n+1}(dg) + F^*_n(ge) \otimes df) \\
&= fF^*_n(g)\overline{F}^*_{n+1}(\nabla(e)) + p^{m-1}F^*_n(e) \otimes d(fF^*_n(g)) \\
&= (F^*_n\nabla)((fF^*_n(g))F^*_n(e)).
\end{aligned}$$

Next we check that $(F^*_n\mathcal{E}, F^*_n\nabla)$ is a p^{m-1}-connection. This follows from the fact that, for $e \in F^*_n\mathcal{E}$ with $e = \sum_i f_iF^*_n(e_i)\,(f_i \in \mathcal{O}_{X_n}, e_i \in \mathcal{E})$ and $f \in \mathcal{O}_{X_n}$, we have the equality

$$\begin{aligned}
&(F^*_n\nabla)(fe) = (F^*_n\nabla)(\sum_i ff_i\nabla(e_i)) \\
&= \sum_i (ff_i\overline{F}^*_{n+1}(\nabla(e_i)) + p^{m-1}F^*_n(e_i) \otimes d(ff_i)) \\
&= f\sum_i f_i\overline{F}^*_{n+1}(\nabla(e_i)) + f\sum_i p^{m-1}F^*_n(e_i) \otimes df_i + \sum_i p^{m-1}f_iF^*_n(e_i) \otimes df \\
&= fF^*_n(e) + p^{m-1}e \otimes df.
\end{aligned}$$

Finally, we need to check the integrability of $(F^*_n\mathcal{E}, F^*_n\nabla)$. This is done as follows: Let us take the local coordinate $t_1, ..., t_d$ of X_{n+1} over S_{n+1} and the

local coordinate $t'_1, ..., t'_d$ of $X^{(1)}_{n+1}$ over S_{n+1} as above (so that $F^*_{n+1}(t'_i) = t_i^p + pa_i$), take a local section $e \in \mathcal{E}$ and write

$$\nabla(e) = \sum_i e_i \otimes dt'_i, \quad \nabla(e_i) = \sum_j e_{ij} \otimes dt'_j.$$

The integrability of $(\mathcal{E}, \nabla)$ implies that $e_{ij} = e_{ji}$. Let $(F^*_n\nabla)_1 : F^*_n\mathcal{E} \otimes \Omega^1_{X_n/S_n} \longrightarrow F^*_n\mathcal{E} \otimes \Omega^2_{X_n/S_n}$ be the morphism defined by $(F^*_n\nabla)_1(g \otimes \omega) = (F^*_n\nabla)(g) \wedge \omega + p^{m-1}g \otimes d\omega$. Then our task is to show $((F^*_n\nabla)_1 \circ (F^*_n\nabla))(fF^*_n(e)) = 0$ for $f \in \mathcal{O}_{X_n}$. This is actually calculated as follows:

$$\begin{aligned}
&((F^*_n\nabla)_1 \circ (F^*_n\nabla))(fF^*_n(e)) \\
&= (F^*_n\nabla)_1(f\overline{F}^*_{n+1}(\nabla(e)) + p^{m-1}F^*_n(e)df) \\
&= (F^*_n\nabla)_1(\sum_i fF^*_n(e_i) \otimes \overline{F}^*_{n+1}(dt'_i) + p^{m-1}F^*_n(e)df) \\
&= \sum_i f\overline{F}^*_{n+1}(\nabla(e_i)) \wedge \overline{F}^*_{n+1}(dt'_i) + \sum_i p^{m-1}F^*_n(e_i)df \wedge \overline{F}^*_{n+1}(dt'_i) \\
&\quad + \sum_i p^{m-1}fF^*_n(e_i) \otimes d\overline{F}^*_{n+1}(dt'_i) + p^{m-1}\overline{F}^*_{n+1}(\nabla(e)) \wedge df \\
&= f\sum_i \overline{F}^*_{n+1}(\nabla(e_i)) \wedge \overline{F}^*_{n+1}(dt'_i) + p^{m-1}f\sum_i F^*_n(e_i) \otimes d\overline{F}^*_{n+1}(dt'_i) \\
&= f\sum_{i,j} F^*_n(e_{ij}) \otimes (\overline{F}^*_{n+1}(dt'_j) \wedge \overline{F}^*_{n+1}(dt'_i)) \\
&\quad + p^{m-1}f\sum_i F^*_n(e_i) \otimes d(t_i^{p-1}dt_i + \sum_j \frac{\partial a_i}{\partial t_j}dt_j) \\
&= p^{m-1}f\sum_i F^*_n(e_i) \otimes \left(\sum_{j,k} \frac{\partial^2 a_i}{\partial t_k \partial t_j} dt_k \wedge dt_j\right) = 0.
\end{aligned}$$

So we have checked the integrability of $(F^*_n\mathcal{E}, F^*_n\nabla)$ and so the functor (1.5) is well-defined.

Also, in the situation of $\mathbf{Hyp}(\infty, \infty, \infty)$, we have the homomorphism

$\overline{F}^* : \Omega^1_{X^{(1)}/S} \longrightarrow \Omega^1_{X/S}$ which makes the diagram

$$\begin{array}{ccc} \Omega^1_{X^{(1)}/S} & \xrightarrow{F^*} & \Omega^1_{X/S} \\ \downarrow & & \uparrow{\scriptstyle p} \\ \Omega^1_{X^{(1)}/S} & \xrightarrow{\overline{F}^*} & \Omega^1_{X/S} \end{array}$$

commutative, and using this, we can define the level raising inverse image functor

$$F^* : \mathrm{MIC}^{(m)}(X^{(1)}) \longrightarrow \mathrm{MIC}^{(m-1)}(X) \tag{1.6}$$

in the same way.

Remark 1.11. Assume we are in the situation of $\mathbf{Hyp}(\infty, \infty, \infty)$ and put $X^{(1)}_n := X^{(1)} \otimes \mathbb{Z}/p^n\mathbb{Z}, X_n := X \otimes \mathbb{Z}/p^n\mathbb{Z}$. Then the functor (1.6) induces the functor $F^* : \mathrm{MIC}^{(m)}(X^{(1)})_n \longrightarrow \mathrm{MIC}^{(m-1)}(X)_n$, and this coincides with the functor (1.5) via the equivalences $\mathrm{MIC}^{(m)}(X^{(1)}_n) \xrightarrow{=} \mathrm{MIC}^{(m)}(X^{(1)})_n, \mathrm{MIC}^{(m-1)}(X_n) \xrightarrow{=} \mathrm{MIC}^{(m-1)}(X)_n$ of Remark 1.3.

Remark 1.12. Assume that we are in the situation of $\mathbf{Hyp}(2, 2, 2)$. Then the level raising inverse image functor for $m = 1$ is written as $F_2^* : \mathrm{HIG}(X^{(1)}_1) \longrightarrow \mathrm{MIC}(X_1)$. Let us see how it is calculated locally. Let us take a local coordinate $t_1, ..., t_d$ of X_2 and a local coordinate $t'_1, ..., t'_d$ of $X^{(1)}_2$ with $F_2^*(t'_i) = t_i^p + pa_i$. Then $\overline{F}_2^* : \Omega^1_{X^{(1)}_1/S_1} \longrightarrow \Omega^1_{X_1/S_1}$ is written as $\overline{F}_2^*(dt'_i) = t_i^{p-1}dt_i + da_i$ and the functor F_2^* is defined by using it. So we obtain the following expression of the functor F_2^*: A Higgs module $(\mathcal{E}, \theta)$ on $X^{(1)}_1$ of the form $\theta(e) = \sum_{i=1}^d \theta_i(e) \otimes dt'_i$ is sent to the integrable connection $(F^*_{X_1/S_1}\mathcal{E}, \nabla)$ such that, if we write $\nabla = \sum_{i=1}^d \partial_i dt_i$, we have

$$\partial_i(1 \otimes e) = t_i^{p-1} \otimes \theta_i(e) + \sum_{j=1}^d \frac{\partial a_j}{\partial t_i} \otimes \theta_j(e).$$

Let $\iota : \mathrm{HIG}(X^{(1)}_1) \longrightarrow \mathrm{HIG}(X^{(1)}_1)$ be the functor $(\mathcal{E}, \theta) \mapsto (\mathcal{E}, -\theta)$. Then, by the above expression, we see that the functor $F_2^* \circ \iota$ coincides with a special case of the functor defined in [4, 5.8] (the case $m = 0$ in the notation of [4])

for quasi-nilpotent objects. (The underlying sheaf $F_1^*\mathcal{E}$ is globally the same as the image of the functor in [4, 5.8], and the connections coincide because they coincide locally.) Hence, by [4, 6.5], it coincides with the functor in [9, 2.11] for quasi-nilpotent objects.

We have the functoriality of quasi-nilpotence with respect to level raising inverse image functors, as follows:

PROPOSITION 1.13. *Assume that we are in the situation of* $\mathbf{Hyp}(n+1, n+1, n+1)$ $(n \in \mathbb{N})$ *(resp.* $\mathbf{Hyp}(\infty, \infty, \infty)$*). Then the level raising inverse image functor* $F_{n+1}^* : \mathrm{MIC}^{(m)}(X_n^{(1)}) \longrightarrow \mathrm{MIC}^{(m-1)}(X_n)$ *(resp.* $F^* : \mathrm{MIC}^{(m)}(X^{(1)}) \longrightarrow \mathrm{MIC}^{(m-1)}(X)$*) induces the functor* $F_{n+1}^{*,\mathrm{qn}} : \mathrm{MIC}^{(m)}(X_n^{(1)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}^{(m-1)}(X_n)^{\mathrm{qn}}$ *(resp.* $F^{*,\mathrm{qn}} : \mathrm{MIC}^{(m)}(X^{(1)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}^{(m-1)}(X)^{\mathrm{qn}}$*), that is,* F_{n+1}^* *(resp.* F^**) sends quasi-nilpotent objects to quasi-nilpotent objects.*

PROOF. In view of Remark 1.11, it suffices to prove the proposition for F_{n+1}^*. In the case $m = n = 1$, the functor $F_2^* \circ \iota$ (ι is as in Remark 1.12) coincides with the functor in [9, 2.11]. Hence it sends quasi-nilpotent objects to quasi-nilpotent objects. Since ι induces an auto-equivalence of $\mathrm{MIC}^{(1)}(X_1^{(1)})^{\mathrm{qn}}$, we see that F_2^* sends quasi-nilpotent objects to quasi-nilpotent objects.

Next, let us prove the proposition in the case $m = 1$ and n general, by induction on n. Let us take an object $(\mathcal{E}, \nabla)$ in $\mathrm{MIC}^{(1)}(X_n^{(1)})$. Then we have the exact sequence

$$0 \longrightarrow (p\mathcal{E}, \nabla|_{p\mathcal{E}}) \longrightarrow (\mathcal{E}, \nabla) \longrightarrow (\mathcal{E}/p\mathcal{E}, \overline{\nabla}) \longrightarrow 0,$$

where $\overline{\nabla}$ is the p-connection on $\mathcal{E}/p\mathcal{E}$ induced by ∇. Since $F_n : X_n \longrightarrow X_n^{(1)}$ is finite flat, the above exact sequence induces the following exact sequence:

$$0 \longrightarrow F_{n+1}^*(p\mathcal{E}, \nabla|_{p\mathcal{E}}) \longrightarrow F_{n+1}^*(\mathcal{E}, \nabla) \longrightarrow F_{n+1}^*(\mathcal{E}/p\mathcal{E}, \overline{\nabla}) \longrightarrow 0.$$

Then, since $F_{n+1}^*(p\mathcal{E}, \nabla|_{p\mathcal{E}}) = F_n^*(p\mathcal{E}, \nabla|_{p\mathcal{E}})$ and $F_{n+1}^*(\mathcal{E}/p\mathcal{E}, \overline{\nabla}) = F_2^*(\mathcal{E}/p\mathcal{E}, \overline{\nabla})$, they are quasi-nilpotent by induction hypothesis. Then, if we work locally, take a local coordinate $t_1, ..., t_d$ of X_{n+1} and write the connection on $F_{n+1}^*(\mathcal{E}, \nabla)$ as $e \mapsto \sum_i \theta(e) \otimes dt_i$ $(e \in F_{n+1}^*\mathcal{E})$, there exists some $N \in \mathbb{N}$ such that $\theta^a(e)$ is zero in $F_{n+1}^*(\mathcal{E}/p\mathcal{E})$ for any $a \in \mathbb{N}^d$ with $|a| \geq N$

and any local section $e \in F_{n+1}^*\mathcal{E}$, by Remark 1.7. Then, since $\theta^a(e)$ is contained in $F_{n+1}^*(p\mathcal{E})$, there exists some $M \in \mathbb{N}$ such that $\theta^{a+b}(e) = 0$ for any $b \in \mathbb{N}^d$ with $|b| \geq M$ and any local section $e \in F_{n+1}^*\mathcal{E}$, again by Remark 1.7. Hence $F_{n+1}^*(\mathcal{E}, \nabla)$ is also quasi-nilpotent, as desired.

Finally we prove the proposition in the case $m \geq 2$. Let us take a local coordinate $t_1, ..., t_d$ of X_{n+1}, a local coordinate $t'_1, ..., t'_d$ of $X_{n+1}^{(1)}$ with $F_{n+1}^*(t'_i) = t_i^p + pa_i$. Take an object $(\mathcal{E}, \nabla)$ in $\mathrm{MIC}^{(m)}(X_n^{(1)})$ and write $\nabla(e) := \sum_i \theta'(e)dt'_i$, $F_n^*\nabla(fF_n^*(e)) = \sum_i \theta(fF_n^*(e))dt_i$. Then we can prove that, for any local sections $e \in \mathcal{E}, f \in \mathcal{O}_{X_n}$ and $a \in \mathbb{N}^d$, there exist some $f_{a,b} \in \mathcal{O}_{X_n}$ $(b \in \mathbb{N}^d, |b| \leq |a|)$ (which depends on e, f) with

$$\theta^a(f\overline{F}_{n+1}^*(e)) = \sum_{|b| \leq |a|} p^{(m-1)(|a|-|b|)} f_{a,b}\overline{F}_{n+1}^*(\theta'^b(e)), \tag{1.7}$$

in the same way as the proof of Proposition 1.9. From this we see the quasi-nilpotence of $F_{n+1}^*(\mathcal{E}, \nabla) = (F_n^*\mathcal{E}, F_n^*\nabla)$ again in the same way as the proof of Proposition 1.9. □

Remark 1.14. In the above proof, we used the results in [9]. Later, we give another proof of Proposition 1.13 which does not use any results in [9] under a slightly stronger hypothesis $\mathbf{Hyp}(\infty, n+1, n+1)$ or $\mathbf{Hyp}(\infty, \infty, \infty)$.

Now we define a functor from the category of (quasi-nilpotent) Higgs modules to the category of modules with (quasi-nilpotent) integrable connection as a composite of level raising inverse image functors. Let us consider the following hypothesis.

Hypothesis 1.15. Let us fix $n \in \mathbb{N}$ and let S_{n+1} be a scheme flat over $\mathbb{Z}/p^{n+1}\mathbb{Z}$. For $j \in \mathbb{N}, j \leq n+1$, let us put $S_j := S_{n+1} \otimes \mathbb{Z}/p^j\mathbb{Z}$. Let $f_1 : X_1 \longrightarrow S_1$ be a smooth morphism and let $F_{S_1} : S_1 \longrightarrow S_1$ be the Frobenius endomorphism. For $0 \leq m \leq n$, let us put $X_1^{(m)} := S_1 \times_{F_{S_1}^m, S_1} X_1$, denote the projection $X_1^{(m)} \longrightarrow S_1$ by $f_1^{(m)}$ and for $1 \leq m \leq n$, let $F_{X_1/S_1}^{(m)} : X_1^{(m-1)} \longrightarrow X_1^{(m)}$ be the relative Frobenius morphism for $f_1^{(m-1)}$.

Assume that we are given a smooth lift $f_{n+1} : X_{n+1} \longrightarrow S_{n+1}$ of f_1, smooth lifts $f_{n+1}^{(m)} : X_{n+1}^{(m)} \longrightarrow S_{n+1}$ of $f_1^{(m)}$ $(0 \leq m \leq n)$ with $f_{n+1}^{(0)} = f_{n+1}$ and lifts $F_{n+1}^{(m)} : X_{n+1}^{(m-1)} \longrightarrow X_{n+1}^{(m)}$ of the morphism $F_{X_1/S_1}^{(m)}$ $(1 \leq$

$m \leq n$) which are morphisms over S_{n+1}. Finally, let $f_n : X_n \longrightarrow S_n$, $f_n^{(m)} : X_n^{(m)} \longrightarrow S_n$, $F_n^{(m)} : X_n^{(m-1)} \longrightarrow X_n^{(m)}$ be $f_{n+1} \otimes \mathbb{Z}/p^n\mathbb{Z}, f_{n+1}^{(m)} \otimes \mathbb{Z}/p^n\mathbb{Z}, F_{n+1}^{(m)} \otimes \mathbb{Z}/p^n\mathbb{Z}$, respectively.

Then we define the functor as follows:

DEFINITION 1.16. Assume that we are in the situation of Hypothesis 1.15. Then we define the functors

$$\Psi : \mathrm{HIG}(X_n^{(n)}) \longrightarrow \mathrm{MIC}(X_n), \quad \Psi^{\mathrm{qn}} : \mathrm{HIG}(X_n^{(n)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}(X_n)^{\mathrm{qn}}$$

as the composite $F_{n+1}^{(1),*} \circ F_{n+1}^{(2),*} \circ \cdots \circ F_{n+1}^{(n),*}, F_{n+1}^{(1),*,\mathrm{qn}} \circ F_{n+1}^{(2),*,\mathrm{qn}} \circ \cdots \circ F_{n+1}^{(n),*,\mathrm{qn}}$ of level raising inverse image functors

$$F_{n+1}^{(m),*} : \mathrm{MIC}^{(m)}(X_n^{(m)}) \longrightarrow \mathrm{MIC}^{(k)}(X_n^{(m-1)}) \quad (1 \leq m \leq n),$$
$$F_{n+1}^{(m),*,\mathrm{qn}} : \mathrm{MIC}^{(m)}(X_n^{(m)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}^{(m-1)}(X_n^{(m-1)})^{\mathrm{qn}} \quad (1 \leq m \leq n),$$

respectively.

Since the morphisms $F_n^{(m)}$ are finite flat, we see that the functors Ψ, Ψ^{qn} are exact and faithful. However, we see in the following example that the functors Ψ, Ψ^{qn} are not so good as one might expect.

Example 1.17. In this example, let us put $S_{n+1} = \mathrm{Spec}\, \mathbb{Z}/p^{n+1}\mathbb{Z}$ and let $S_j := S_{n+1} \otimes \mathbb{Z}/p^j\mathbb{Z} = \mathrm{Spec}\, \mathbb{Z}/p^j\mathbb{Z}, X_j := \mathrm{Spec}\, (\mathbb{Z}/p^j\mathbb{Z})[t^{\pm 1}]$ for $j \in \mathbb{N}, j \leq n+1$. Also, put $X_j^{(m)} := \mathrm{Spec}\, (\mathbb{Z}/p^j\mathbb{Z})[t^{\pm 1}]$ for all $m \in \mathbb{N}, j \in \mathbb{N}, j \leq n+1$ and let $F_j^{(m)} : X_j^{(m-1)} \longrightarrow X_j^{(m)}$ be the morphism defined by $t \mapsto t^p$. Then $\overline{F}_{n+1}^{(m),*} : \Omega^1_{X_n^{(m)}/S_n} \longrightarrow \Omega^1_{X_n^{(m-1)}/S_n}$ sends $f(t)t^{-1}dt$ to $f(t^p)t^{-p}t^{p-1}dt = f(t^p)t^{-1}dt$ and the level raising inverse image functor $F_{n+1}^{(m),*} : \mathrm{MIC}^{(m)}(X_n^{(m)}) \longrightarrow \mathrm{MIC}^{(m-1)}(X_n^{(m-1)})$ is defined as 'the pull-back by $\overline{F}_{n+1}^{(m),*}$'.

For $m \in \mathbb{N}$ and $f(t) \in \Gamma(X_n^{(m)}, \mathcal{O}_{X_n^{(m)}})$, we define the p^m-connection $(\mathcal{O}_{X_n^{(m)}}, \nabla_{f(t)})$ by $\nabla_{f(t)} = p^m d + f(t)t^{-1}dt$. It is locally free of rank 1. Since any locally free sheaf of rank 1 on $X_n^{(m)}$ is free, any p^m-connection on $X_n^{(m)}$ which is locally free of rank 1 has the form $(\mathcal{O}_{X_n^{(m)}}, \nabla_{f(t)})$ for some $f(t)$. For

a p^m-connection $(\mathcal{O}_{X_n^{(m)}}, \nabla_{f(t)})$, the p^{m-1}-connection $F_{n+1}^{(m),*}(\mathcal{O}_{X_n^{(m)}}, \nabla_{f(t)})$ is equal to $(\mathcal{O}_{X_n^{(m-1)}}, \nabla_{f(t^p)})$ thanks to the description of $F_{n+1}^{(m),*}$ given in the previous paragraph.

Let us make some more calculation on the p^m-connection $(\mathcal{O}_{X_n^{(m)}}, \nabla_{f(t)})$. It is easy to see that we have an isomorphism $(\mathcal{O}_{X_n^{(m)}}, \nabla_{f(t)}) \cong (\mathcal{O}_{X_n^{(m)}}, \nabla_0)$ if and only if $(\mathcal{O}_{X_n^{(m)}}, \nabla_{f(t)})$ is generated as $\mathcal{O}_{X_n^{(m)}}$-module by a horizontal element. Since we have

$$\nabla_{f(t)}(g(t)) = p^m \frac{dg}{dt}dt + gft^{-1}dt = g(p^m t g^{-1}\frac{dg}{dt} + f)t^{-1}dt,$$

we see that $(\mathcal{O}_{X^{(m)}}, \nabla_{f(t)})$ is isomorphic to $(\mathcal{O}_{X_n^{(m)}}, \nabla_0)$ if and only if there exists an element $g \in \Gamma(X_n^{(m)}, \mathcal{O}^{\times}_{X_n^{(m)}})$ with $f = -p^m t g^{-1}\frac{dg}{dt}$. If g is an element in $\Gamma(X^{(m)}, \mathcal{O}^{\times}_{X_n^{(m)}})$, it has the form $g = c(t^N + ph_1)$ for some $c \in (\mathbb{Z}/p^n\mathbb{Z})^{\times}$, $N \in \mathbb{Z}$ and $h_1 \in \Gamma(X^{(m)}, \mathcal{O}_{X^{(m)}})$, and in this case, g^{-1} has the form $c^{-1}(t^{-N} + ph_2)$ for some element h_2 in $\Gamma(X_n^{(m)}, \mathcal{O}_{X_n^{(m)}})$. Then we have

$$\begin{aligned} -p^m t g^{-1}\frac{dg}{dt} &= -p^m t(t^{-N} + ph_2)(Nt^{N-1} + p\frac{dh_1}{dt}) \\ &= -p^m N + p^{m+1}h(t) \end{aligned}$$

for some $h \in \Gamma(X_n^{(m)}, \mathcal{O}_{X_n^{(m)}})$. Therefore, we have shown that if $(\mathcal{O}_{X_n^{(m)}}, \nabla_{f(t)})$ is isomorphic to $(\mathcal{O}_{X_n^{(m)}}, \nabla_0)$, f has the form $-p^m N + p^{m+1}h(t)$.

Now, to investigate the functor $F_{n+1}^{(1),*} : \mathrm{MIC}^{(1)}(X_n^{(1)}) \longrightarrow \mathrm{MIC}(X_n)$, first let us consider the p-connection $(\mathcal{O}_{X_n^{(1)}}, \nabla_1)$. Then, since there does not exist $N \in \mathbb{N}, h \in \Gamma(X_n^{(1)}, \mathcal{O}_{X_n^{(1)}})$ with $1 = -pN + p^2 h$, it is not isomorphic to $(\mathcal{O}_{X_n^{(1)}}, \nabla_0)$. On the other hand, we see that the connection $F_{n+1}^{(1),*}(\mathcal{O}_{X_n^{(1)}}, \nabla_1) = (\mathcal{O}_{X_n}, \nabla_1)$ is isomorphic to $F_{n+1}^{(1),*}(\mathcal{O}_{X_n^{(1)}}, \nabla_0) = (\mathcal{O}_{X_n}, \nabla_0)$ because we have $1 = -tg^{-1}\frac{dg}{dt}$ when $g = t^{-1}$. So we can conclude that the functor $F_{n+1}^{(1),*}$ is not full. Secondly, let us consider the connection $(\mathcal{O}_{X_n}, \nabla_t)$. If it is contained in the essential image of $F_{n+1}^{(1),*}$, we should have $(\mathcal{O}_{X_n}, \nabla_t) \cong F_{n+1}^{(1),*}(\mathcal{O}_{X_n^{(1)}}, \nabla_{f(t)}) = (\mathcal{O}_{X_n}, \nabla_{f(t^p)})$ for some $f(t)$. Then we have $f(t^p) - t = -N + ph$ for some $N \in \mathbb{Z}$ and $h \in \Gamma(X_n, \mathcal{O}_{X_n})$, but it is impossible. Hence we see that the functor $F_{n+1}^{(1),*}$ is not essentially surjective.

Next, let us investigate the functors $F_{n+1}^{(m),*} : \mathrm{MIC}^{(m)}(X_n^{(m)}) \longrightarrow \mathrm{MIC}^{(m-1)}(X_n^{(m-1)})$, $F_{n+1}^{(m),*,\mathrm{qn}} : \mathrm{MIC}^{(m)}(X_n^{(m)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}^{(m-1)}(X_n^{(m-1)})^{\mathrm{qn}}$ for $m \geq 2$. First let us consider the p^m-connection $(\mathcal{O}_{X_n^{(m)}}, \nabla_{p^{m-1}})$. We see as in the previous paragraph that it is not isomorphic to $(\mathcal{O}_{X_n^{(m)}}, \nabla_0)$, and that $F_{n+1}^{(m),*}(\mathcal{O}_{X_n^{(m)}}, \nabla_{p^{m-1}}) = (\mathcal{O}_{X_n^{(m-1)}}, \nabla_{p^{m-1}})$ is isomorphic to $F_{n+1}^{(m),*}(\mathcal{O}_{X_n^{(m)}}, \nabla_0) = (\mathcal{O}_{X_n^{(m-1)}}, \nabla_0)$. If we put $\nabla_{p^{m-1}}(e) = \partial(e)dt$, we can see easily by induction that $\partial^l(1) = (\prod_{i=0}^{l-1}(p^{m-1} - ip^m))/t^l$. By this and Remark 1.7, we see that $(\mathcal{O}_{X_n^{(m)}}, \nabla_{p^{m-1}})$ is quasi-nilpotent. So the functors $F_{n+1}^{(m),*}$, $F_{n+1}^{(m),*,\mathrm{qn}}$ are not full. Secondly, let us consider the connection $(\mathcal{O}_{X_n^{(m-1)}}, \nabla_{pt})$. If it is contained in the essential image of $F_{n+1}^{(m),*}$, we should have $(\mathcal{O}_{X_n^{(m-1)}}, \nabla_{pt}) \cong F_{n+1}^{(m),*}(\mathcal{O}_{X_n^{(m)}}, \nabla_{f(t)}) = (\mathcal{O}_{X_n^{(m-1)}}, \nabla_{f(t^p)})$ for some $f(t)$. Then we have $f(t^p) - pt = -p^{m-1}N + p^m h$ for some $N \in \mathbb{N}$ and $h \in \Gamma(X_n^{(m-1)}, \mathcal{O}_{X_n^{(m-1)}})$, but it is impossible. Also, if we put $\nabla_{pt}(e) = \partial(e)dt$, we can see easily by induction that $\partial^l(1) = p^l$. By this and Remark 1.7, we see that $(\mathcal{O}_{X_n^{(m-1)}}, \nabla_{pt})$ is quasi-nilpotent. Hence the functors $F_{n+1}^{(m),*}$, $F_{n+1}^{(m),*,\mathrm{qn}}$ are not essentially surjective.

In conclusion, Ψ is not full, not essentially surjective for any $m \geq 1$, and Ψ^{qn} is not full, not essentially surjective for any $m \geq 2$.

In view of the above example, we would like to ask the following question.

QUESTION 1.18. Is it possible to construct some nice functor (a fully faithful functor or an equivalence) from the functors $F_{n+1}^{(m),*}, F_{n+1}^{(m),*,\mathrm{qn}}$, possibly under some more assumption?

Several answers to this question will be given in Sections 3, 4 and 5.

2. p-Adic Differential Operators of Negative Level

In this section, first we introduce the sheaf of p-adic differential operators of level $-m$ ($m \in \mathbb{N}$), which is a 'negative level version' of the sheaf of p-adic differential operators of level m defined by Berthelot, for a smooth morphism of p-adic formal schemes flat over $\mathbb{Z}_p$. We prove the equivalence of the notion of left $\mathcal{D}$-modules in this sense and that of modules with

integrable p^m-connection. We also define the inverse image functors and the level raising inverse image functors for left $\mathcal{D}$-modules, which are compatible with the corresponding notion for modules with integrable p^m-connection over p-adic formal schemes.

The definition of the sheaf of p-adic differential operators of level $-m$ ($m \in \mathbb{N}$) is possible only for smooth morphisms of p-adic formal schemes, because we use the formal blow-up with respect to an ideal containing p^m in the definition. In the case of smooth morphisms $X_n \longrightarrow S_n$ of schemes flat over $\mathbb{Z}/p^n\mathbb{Z}$, we give a similar description by considering all the local lifts of X_n to smooth p-adic formal scheme and consider the 'crystalized' category of $\mathcal{D}$-modules.

We also consider a variant of the 'crystalized' category of $\mathcal{D}$-modules, which is also related to the category of modules with integrable p^m-connection. As a consequence, we prove certain crystalline property for the category of modules with integrable p^m-connection: When $f_n : X_n \longrightarrow S_n$ is a smooth morphism of flat $\mathbb{Z}/p^n\mathbb{Z}$-schemes and if we denote the morphism $f_n \otimes \mathbb{Z}/p\mathbb{Z}$ by $X_1 \longrightarrow S_1$, we know that the category $\mathrm{MIC}(X_n)^{\mathrm{qn}}$, which is equivalent to the category of crystals on the crystalline site $(X_1/S_n)_{\mathrm{crys}}$, depends only on the diagram $X_1 \longrightarrow S_1 \hookrightarrow S_n$. We prove here similar results for the categories of modules with integrable p^m-connection, although the result in the case $m > 0$ is weaker than that in the case $m = 0$.

2.1. The case of p-adic formal schemes

Let S be a p-adic formal scheme flat over $\mathrm{Spf}\,\mathbb{Z}_p$ and let X be a p-adic formal scheme smooth over S. For a positive integer r, we denote the r-fold fiber product of X over S by X^r. For positive integers m, r, let $\widetilde{T}_{X,(-m)}(r)$ be the formal blow-up of X^{r+1} along the ideal $p^m\mathcal{O}_{X^{r+1}} + \mathrm{Ker}\Delta(r)^*$, where $\Delta(r)^* : \mathcal{O}_{X^{r+1}} \longrightarrow \mathcal{O}_X$ denotes the homomorphism induced by the diagonal map $\Delta(r) : X \hookrightarrow X^{r+1}$. Let $T_{X,(-m)}(r)$ be the open formal subscheme of $\widetilde{T}_{X,(-m)}(r)$ defined by

$$T_{X,(-m)}(r) := \{x \in \widetilde{T}_{X,(-m)}(r) \mid p^m\mathcal{O}_{\widetilde{T}_{X,(-m)}(r),x} = ((p^m\mathcal{O}_{X^{r+1}} + \mathrm{Ker}\Delta(r)^*)\mathcal{O}_{\widetilde{T}_{X,(-m)}(r)})_x\}.$$

Then, since we have $(p^m\mathcal{O}_{X^{r+1}} + \mathrm{Ker}\Delta(r)^*)|_X = p^m\mathcal{O}_X$, the diagonal map $\Delta(r)$ factors through a morphism $\widetilde{\Delta}(r) : X \hookrightarrow T_{X,(-m)}(r)$ by the universality of formal blow-up. Let us put $I_{X,(-m)}(r) := \mathrm{Ker}\widetilde{\Delta}(r)^*$. Let

$(\mathcal{P}_{X,(-m)}(r), \overline{I}_{X,(-m)}(r))$ be the PD-envelope of $\mathcal{O}_{T_{X,(-m)}(r)}$ with respect to the ideal $I_{X,(-m)}(r)$, and let us put $P_{X,(-m)}(r) := \underline{\mathrm{Spf}}_{\mathcal{O}_X} \mathcal{P}_{X,(-m)}(r)$. Also, for $k \in \mathbb{N}$, let $\mathcal{P}^k_{X,(-m)}(r), P^k_{X,(-m)}(r)$ be $\mathcal{P}_{X,(-m)}(r), P_{X,(-m)}(r)$ modulo $\overline{I}_{X,(-m)}(r)^{[k+1]}$. In the case $r = 1$, we drop the symbol (r) from the notation.

Note that $\mathcal{P}_{X,(-m)}$ admits two $\mathcal{O}_X$-algebra ($\mathcal{O}_X$-module) structure induced by the 0-th and 1-st projection $X^2 \longrightarrow X$, which we call the left $\mathcal{O}_X$-algebra ($\mathcal{O}_X$-module) structure and the right $\mathcal{O}_X$-algebra ($\mathcal{O}_X$-module) structure, respectively. Note also that, for $m' \leq m$, we have the canonical morphism $\mathcal{P}_{X,(-m')} \longrightarrow \mathcal{P}_{X,(-m)}$.

Locally, $\mathcal{P}_{X,(-m)}(r)$ is described in the following way. Assume that X admits a local parameter $t_1, ..., t_d$ over S. Then, if we denote the q-th projection $X^{r+1} \longrightarrow X$ by π_i $(0 \leq q \leq r)$ and if we put $\tau_{i,q} := \pi^*_{q+1} t_i - \pi^*_q t_i$, $\mathrm{Ker}\Delta(r)^*$ is generated by $\tau_{i,q}$'s $(1 \leq i \leq d, 0 \leq q \leq r-1)$ and we have $T_{X,(-m)}(r) = \underline{\mathrm{Spf}}_{\mathcal{O}_X} \mathcal{O}_X \{\tau_{i,q}/p^m\}_{i,q}$, where $\{-\}$ means the p-adically completed polynomial algebra. So we have $\mathcal{P}_{X,(-m)}(r) = \mathcal{O}_X \langle \tau_{i,q}/p^m \rangle_{i,q}$, where $\langle - \rangle$ means the p-adically completed PD-polynomial algebra.

We see easily that the identity map $X^{r+r'+1} \longrightarrow X^{r+1} \times_X X^{r'+1}$ naturally induces the isomorphism $\mathcal{P}_{X,(-m)}(r) \otimes_{\mathcal{O}_X} \mathcal{P}_{X,(-m)}(r') \longrightarrow \mathcal{P}_{X,(-m)}(r + r')$ and in the local situation, the element $\tau_{i,q}/p^m \otimes 1$ (resp. $1 \otimes \tau_{i,q}/p^m$) on the left hand side corresponds to the element $\tau_{i,q}/p^m$ (resp. $\tau_{i,q+r}/p^m$) on the right hand side. Then, the projection $X^3 \longrightarrow X^2$ to the $(0,2)$-th factor induces the homomorphism

$$\delta : \mathcal{P}_{X,(-m)} \longrightarrow \mathcal{P}_{X,(-m)}(2) \cong \mathcal{P}_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{P}_{X,(-m)}$$

with $\delta(\tau_i/p^m) = \tau_i/p^m \otimes 1 + 1 \otimes \tau_i/p^m$ (here we denoted $\tau_{i,0}$ simply by τ_i) and so it induces the homomorphism

$$\delta^{k,k'} : \mathcal{P}^{k+k'}_{X,(-m)} \longrightarrow \mathcal{P}^k_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{k'}_{X,(-m)}.$$

Using these, we define the sheaf of p-adic differential operators of negative level as follows:

DEFINITION 2.1. Let X, S be as above. Then we define the sheaf $\mathcal{D}^{(-m)}_{X/S,k}$ of p-adic differential operators of level $-m$ and order $\leq k$ by

$\mathcal{D}^{(-m)}_{X/S,k} := \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}^k_{X,(-m)}, \mathcal{O}_X)$ and the sheaf $\mathcal{D}^{(-m)}_{X/S}$ of p-adic differential operators of level $-m$ by $\mathcal{D}^{(-m)}_{X/S} := \bigcup_{k=0}^{\infty} \mathcal{D}^{(-m)}_{X/S,k}$. We define the product

$$\mathcal{D}^{(-m)}_{X/S,k} \times \mathcal{D}^{(-m)}_{X/S,k'} \longrightarrow \mathcal{D}^{(-m)}_{X/S,k+k'}$$

by sending (P, P') to the homomorphism

$$\mathcal{P}^{k+k'}_{X,(-m)} \xrightarrow{\delta^{k,k'}} \mathcal{P}^k_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{k'}_{X,(-m)} \xrightarrow{\mathrm{id}\otimes P'} \mathcal{P}^k_{X,(-m)} \xrightarrow{P} O_X.$$

By definition, $\mathcal{D}^{(-m)}_{X/S}$ also admits two $\mathcal{O}_X$-module structures, which are defined as the multiplication by the elements in $\mathcal{D}^{(-m)}_{X/S,0} = \mathcal{O}_X$ from left and from right. We call these the left and the right $\mathcal{O}_X$-module strucrure of $\mathcal{D}^{(-m)}_{X/S}$. Note that $P \in \mathcal{D}^{(-m)}_{X/S,k}$ acts on $\mathcal{O}_X$ as the composite

$$\mathcal{O}_X \longrightarrow \mathcal{P}^k_{X,(-m)} \xrightarrow{P} \mathcal{O}_X$$

(where the first map is defined by $f \mapsto 1 \otimes f$), and this defines the action of $\mathcal{D}^{(-m)}_{X/S}$ on $\mathcal{O}_X$. For $m' \leq m$, the canonical map $\mathcal{P}_{X,(-m')} \longrightarrow \mathcal{P}_{X,(-m)}$ induces the homomorphism of rings $\rho_{-m',-m} : \mathcal{D}^{(-m)}_{X/S} \longrightarrow \mathcal{D}^{(-m')}_{X/S}$.

Assume that X admits a local parameter $t_1, ..., t_d$ over S and put $\tau_i := 1 \otimes t_i - t_i \otimes 1 \in \mathcal{O}_{X^2}$. Then, as we saw before, we have $\mathcal{P}_{X,(-m)} = \mathcal{O}_X \langle \tau_i / p^m \rangle_i$ and so $\mathcal{P}^k_{X,(-m)}$ admits a basis $\{(\tau/p^m)^{[l]}\}_{|l| \leq k}$ as $\mathcal{O}_X$-module. (Here and after, we use multi-index notation.) We denote the dual basis of it in $\mathcal{D}^{(-m)}_{X/S,k}$ by $\{\partial^{\langle l \rangle}\}_{|l| \leq k}$. When $l = (0, ..., 1, ..., 0)$ (1 is placed in the i-th entry), $\partial^{\langle l \rangle}$ is denoted also by ∂_i. When we would like to clarify the level, we denote the element $\partial^{\langle l \rangle}$ by $\partial^{\langle l \rangle_{-m}}$. Since the canonical map $\mathcal{P}_{X,(-m')} \longrightarrow \mathcal{P}_{X,(-m)}$ sends $(\tau/p^{m'})^{[l]}$ to $p^{(m-m')|l|}(\tau/p^m)^{[l]}$, we have $\rho_{-m',-m}(\partial^{\langle l \rangle_{-m}}) = p^{(m-m')|l|}\partial^{\langle l \rangle_{-m'}}$.

We prove some formulas which are the analogues of the ones in [1, 2.2.4]:

Proposition 2.2. *With the above notation, we have the following*:

(1) *For* $f \in \mathcal{O}_X$, $1 \otimes f = \sum_{|l| \leq k} \partial^{\langle l \rangle}(f)(\tau/p^m)^{[l]}$ *in* $\mathcal{P}^k_{X,(-m)}$.

(2) $\partial^{\langle l\rangle}(t^i) = l!\binom{i}{l}p^{m|l|}t^{i-l}$.

(3) $\partial^{\langle l\rangle}\partial^{\langle l'\rangle} = \partial^{\langle l+l'\rangle}$.

(4) *For* $f \in \mathcal{O}_X$, $\partial^{\langle k\rangle}f = \sum_{k'+k''=k}\binom{k}{k'}\partial^{\langle k'\rangle}(f)\partial^{\langle k''\rangle}$.

PROOF. (1) is immediate from definition. By looking at the coefficient of $(\tau/p^m)^{[l]}$ of $1\otimes t^i = (t+p^m(\tau/p^m))^i$, we obtain (2). From the equality

$$\begin{aligned}
&(\partial^{\langle l\rangle}\partial^{\langle l'\rangle})((\tau/p^m)^{[i]})\\
&=(\partial^{\langle l\rangle}(\mathrm{id}\otimes\partial^{\langle l'\rangle})\delta^{|l|,|l'|})((\tau/p^m)^{[i]})\\
&=(\partial^{\langle l\rangle}(\mathrm{id}\otimes\partial^{\langle l'\rangle}))(\sum_{a+b=i}(\tau/p^m)^{[a]}\otimes(\tau/p^m)^{[b]}) = \begin{cases} 1, & \text{if } i=l+l',\\ 0, & \text{otherwise,}\end{cases}
\end{aligned}$$

we see the assertion (3). From the equality

$$\begin{aligned}
(\partial^{\langle k\rangle}f)((\tau/p^m)^{[i]}) &= \partial^{\langle k\rangle}((1\otimes f)(\tau/p^m)^{[i]})\\
&= \partial^{\langle k\rangle}(\sum_l \partial^{\langle l\rangle}(f)(\tau/p^m)^{[l]}(\tau/p^m)^{[i]})\\
&= \partial^{\langle k\rangle}(\sum_l \binom{l+i}{l}\partial^{\langle l\rangle}(f)(\tau/p^m)^{[l+i]}),
\end{aligned}$$

we see the assertion (4). □

REMARK 2.3. Let $\mathcal{D}_{X/S}$ be the formal scheme version of the sheaf of usual differential operators and let us take a local basis $\{\partial^{[l]}\}_{l\in\mathbb{N}^d}$ of $\mathcal{D}_{X/S}$, which can be defined in the same way as $\{\partial^{\langle l\rangle}\}$ above. Then $\mathcal{O}_X$ admits the natural action of $\mathcal{D}_{X/S}$ and we see, for $l\in\mathbb{N}^d$, $m\in\mathbb{N}$ and $f\in\mathcal{O}_X$, the equalities

$$\partial^{\langle l\rangle_{-m}}(f) = p^{m|l|}\partial^{\langle l\rangle_0}(f) = l!p^{m|l|}\partial^{[l]}(f).$$

In particular, we have $\partial^{\langle l\rangle_{-m}}(f)\to 0$ as $|l|\to\infty$.

Next we define the notion of $(-m)$-PD-stratification and compare it with the notion of left $\mathcal{D}^{(-m)}_{X/S}$-module.

DEFINITION 2.4. A $(-m)$-PD-stratification on an $\mathcal{O}_X$-module $\mathcal{E}$ is a compatible family of $\mathcal{P}^k_{X,(-m)}$-linear isomorphisms $\{\epsilon_k : \mathcal{P}^k_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{=} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^k_{X,(-m)}\}_k$ with $\epsilon_0 = \mathrm{id}$ such that the following diagram is commutative for any $k, k' \in \mathbb{N}$:

$$\begin{array}{ccc} \mathcal{P}^k_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{k'}_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E} & \xrightarrow[\mathrm{id}\otimes\epsilon_{k'}]{} & \mathcal{P}^k_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^{k'}_{X,(-m)} \\ & \searrow^{\delta^{k,k',*}(\epsilon_{k+k'})} & \downarrow{\scriptstyle \epsilon_k\otimes\mathrm{id}} \\ & & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^k_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{k'}_{X,(-m)}. \end{array}$$

The conditions put on $\{\epsilon_k\}_k$ in the above definition is called the cocycle condition. It is easy to see that the cocycle condition is equivalent to the condition $q^{k*}_{02}(\epsilon_k) = q^{k*}_{01}(\epsilon_k) \circ q^{k*}_{12}(\epsilon_k)$ for $k \in \mathbb{N}$, where q^k_{ij} denotes the homomorphism $\mathcal{P}^k_{X,(-m)} \longrightarrow \mathcal{P}^k_{X,(-m)}(2)$ induced by the (i,j)-th projection $X^3 \longrightarrow X^2$.

We have the following equivalence, which is an analogue of [1, 2.3.2]:

PROPOSITION 2.5. *For an $\mathcal{O}_X$-module $\mathcal{E}$, the following three data are equivalent.*

(a) *A left $\mathcal{D}^{(-m)}_{X/S}$-module structure on $\mathcal{E}$ which extends the given $\mathcal{O}_X$-module structure.*

(b) *A compatible family of $\mathcal{O}_X$-linear homomorphisms $\{\theta_k : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^k_{X,(-m)}\}_k$ (where we regard $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^k_{X,(-m)}$ as $\mathcal{O}_X$-module by using the right $\mathcal{O}_X$-module structure of $\mathcal{P}^k_{X,(-m)}$) with $\theta_0 = \mathrm{id}$ such that the following diagram is commutative for any $k, k' \in \mathbb{N}$:*

$$\begin{array}{ccc} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^{k+k'}_{X,(-m)} & \xrightarrow{\mathrm{id}\otimes\delta^{k,k'}} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^k_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{P}^{k'}_{X,(-m)} \\ {\scriptstyle \theta_{k+k'}}\uparrow & & \uparrow{\scriptstyle \theta_k\otimes\mathrm{id}} \\ \mathcal{E} & \xrightarrow{\theta_{k'}} & \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^{k'}_{X,(-m)}. \end{array} \tag{2.1}$$

(c) *A $(-m)$-PD-stratification $\{\epsilon_k\}_k$ on $\mathcal{E}$.*

PROOF. Since the proof is identical with the classical case, we only give a brief sketch. The data in (a) is equivalent to a compatible family of homomorphisms $\mu_k : \mathcal{D}^{(-m)}_{X/S,k} \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{E}$ satisfying the condition coming from the product structure of $\mathcal{D}^{(-m)}_{X/S}$, and μ_k's induce the homomorphisms

$$\theta_k : \mathcal{E} \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{D}^{(-m)}_{X/S,k}, \mathcal{E}) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^k_{X,(-m)} \quad (k \in \mathbb{N})$$

which satisfy the conditions in (b). So the data in (a) gives the data (b), and we see easily that they are in fact equivalent. When we are given the data in (b), we obtain the $\mathcal{P}^k_{X,(-m)}$-linear homomorphism $\epsilon_k : \mathcal{P}^k_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^k_{X,(-m)}$ by taking $\mathcal{P}^k_{X,(-m)}$-linearization of θ_k. Since $\epsilon_k(1 \otimes x) = \theta_k(x)$ is written locally as $\sum_{|l| \leq k} \partial^{\langle l \rangle}(x) \otimes (\tau/p^m)^{[l]}$, we see that ϵ_k is actually an isomorphism because the inverse of it is given locally by $x \otimes 1 \mapsto \sum_{|l| \leq k} (-1)^{|l|} (\tau/p^m)^{[l]} \otimes \partial^{\langle l \rangle}(x)$. The cocycle condition for $\{\epsilon_k\}_k$ follows from the commutative diagram (2.1) for $\{\theta_k\}_k$ and so the data in (b) gives the data in (c). Again we see easily that they are in fact equivalent. □

Next we relate the notion of left $\mathcal{D}^{(-m)}_{X/S}$-modules and that of modules with p^m-connection. Let $X \longrightarrow S$ be as above. Recall that a p^m-connection on an $\mathcal{O}_X$-module $\mathcal{E}$ is an additive map $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ satisfying $\nabla(fe) = f\nabla(e) + p^m e \otimes df$ $(e \in \mathcal{E}, f \in \mathcal{O}_X)$. To give another description of p^m-connection, let us put $J^1_{X/S} := \mathrm{Ker}(\mathcal{P}^1_{X,(-m)} \longrightarrow \mathcal{O}_X)$. Then we have a natural map $\alpha : \Omega^1_{X/S} \longrightarrow J^1_{X/S}$ induced by the map $\mathcal{P}^1_{X,(0)} \longrightarrow \mathcal{P}^1_{X,(-m)}$, and locally α is given by $dt_i = \tau_i \mapsto p^m(\tau_i/p^m)$. So α is injective and the image is equal to $p^m J^1_{X/S}$. Hence we have the unique isomorphism $\beta : \Omega^1_{X/S} \xrightarrow{=} J^1_{X/S}$ satisfying $p^m\beta = \alpha$. Via the identification by β, a p^m-connection on $\mathcal{E}$ is equivalent to an additive map $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} J^1_{X/S}$ satisfying $\nabla(fe) = f\nabla(e) + e \otimes df$ for any $e \in \mathcal{E}, f \in \mathcal{O}_X$. (Attention: the element $df \in J^1_{X/S}$ here is the element $1 \otimes f - f \otimes 1 \in \mathcal{P}^1_{X,(-m)}$, not the element $\beta(1 \otimes f - f \otimes 1)$.)

The following proposition is the analogue of [3, 2.9].

PROPOSITION 2.6. *Let $X \longrightarrow S$ be as above. For an $\mathcal{O}_X$-module $\mathcal{E}$, the following data are equivalent*:

(a) *A p^m-connection $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} J^1_{X/S}$ on $\mathcal{E}$.*

(b) *A* $\mathcal{P}^1_{X,(-m)}$*-linear isomorphism* $\epsilon_1 : \mathcal{P}^1_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E} \overset{=}{\longrightarrow} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X,(-m)}$ *which is equal to identity modulo* $J^1_{X/S}$.

Proof. Since the proof is again identical with [3, 2.9], we only give a brief sketch. First assume that we are given the isomorphism ϵ_1 as in (b). Then, if we define $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} J^1_{X/S}$ by $\nabla(e) := \epsilon(1 \otimes e) - e \otimes 1$, it gives a p^m-connection. Conversely, if we are given a p^m-connection $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} J^1_{X/S}$, let us define the $\mathcal{P}^1_{X,(-m)}$-linear homomorphism $\epsilon_1 : \mathcal{P}^1_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X,(-m)}$ by $\epsilon(1 \otimes e) = \nabla(e) + e \otimes 1$. Then it is easy to see that ϵ_1 is equal to identity modulo $J^1_{X/S}$. To show that ϵ_1 is an isomorphism, let us consider the isomomorphism $t : \mathcal{P}^1_{X,(-m)} \longrightarrow \mathcal{P}^1_{X,(-m)}$ induced by the morphism $X^2 \longrightarrow X^2; (x, y) \mapsto (y, x)$ and let $s : \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X,(-m)} \longrightarrow \mathcal{P}^1_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E}$ be the isomorphism $x \otimes \xi \mapsto t(\xi) \otimes x$. Then we see that $(s \circ \epsilon_1)^2 : \mathcal{P}^1_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{P}^1_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E}$ is a $\mathcal{P}^1_{X,(-m)}$-linear endomorphism which is equal to the identity modulo $J^1_{X/S}$. Hence it is an isomorphism and we see from this that ϵ_1 is also an isomorphism. □

As for the integrability, we have the following proposition.

Proposition 2.7. *Let* $\mathcal{E}$ *be an* $\mathcal{O}_X$*-module and let* $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \overset{\mathrm{id}\otimes\beta,=}{\longrightarrow} \mathcal{E} \otimes_{\mathcal{O}_X} J^1_{X/S}$ *be a* p^m*-connection. Let* $\epsilon_1 : \mathcal{P}^1_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X,(-m)}$ *be the* $\mathcal{P}^1_{X,(-m)}$*-linear isomorphism corresponding to* ∇ *by the equivalence in Proposition 2.6 and let* $\mu_1 : \mathcal{D}^{(-m)}_{X/S,1} \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{E}$ *be the homomorphism induced by the composite*

$$\mathcal{E} \hookrightarrow \mathcal{P}^1_{X,(-1)} \otimes_{\mathcal{O}_X} \mathcal{E} \overset{\epsilon_1}{\longrightarrow} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X,(-1)} \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{D}^{(-m)}_{X/S,1}, \mathcal{E}).$$

Then the following conditions are equivalent.

(1) $(\mathcal{E}, \nabla)$ *is integrable.*

(2) μ_1 *is (uniquely) extendable to a* $\mathcal{D}^{(-m)}_{X/S}$*-module structure on* $\mathcal{E}$ *which extends the given* $\mathcal{O}_X$*-module structure.*

Proof. We may work locally. So we can write $\nabla(e) = \sum_i \theta_i(e) dt_i \overset{\mathrm{id}\otimes\beta}{\mapsto} \sum_i \theta_i(e)(\tau_i/p^m)$, using local coordinate. Then we have $\mu_1(\partial_i \otimes e) = \theta_i(e)$.

First assume the condition (2). Then, since $[\partial_i, \partial_j](e) = 0$ for any $e \in \mathcal{E}$, we have $[\theta_i, \theta_j](e) = 0$ for any e and so $(\mathcal{E}, \nabla)$ is integrable. So the condition (1) is satisfied.

On the other hand, let us assume the condition (1). Then, we define the action of $\partial^{\langle k \rangle} \in \mathcal{D}^{(-m)}_{X/S}$ on $e \in \mathcal{E}$ by $\partial^{\langle k \rangle}(e) := \prod_{i=1}^{d} \theta_i^{k_i}(e)$, where $k = (k_1, ..., k_d)$. To see that this action acturally defines a $\mathcal{D}^{(-m)}_{X/S}$-module structure on $\mathcal{E}$, we have to check the following equalities for local sections $e \in \mathcal{E}$ and $f \in \mathcal{O}_X$ (see (3), (4) in Proposition 2.2):

$$\partial^{\langle k \rangle} \partial^{\langle k' \rangle}(e) = \partial^{\langle k+k' \rangle}(e), \tag{2.2}$$

$$\partial^{\langle k \rangle}(fe) = \sum_{k'+k''=k} \binom{k}{k'} \partial^{\langle k' \rangle}(f) \partial^{\langle k'' \rangle}(e). \tag{2.3}$$

By the definition of the action of $\partial^{\langle k \rangle}$'s on e given above, the equality (2.2) is reduced to the equality $\theta_i \theta_j(e) = \theta_j \theta_i(e)$, that is, the integrability of $(\mathcal{E}, \nabla)$. In view of the equality (2.2), the proof of the equality (2.3) is reduced to the case $|k| = 1$, and in this case, it is rewritten as

$$\theta_i(fe) = f\theta_i(e) + \partial_i(f)e \quad (1 \leq i \leq d).$$

This is equlvalent to the equality $\nabla(fe) = f\nabla(e) + e \otimes df$ in $\mathcal{E} \otimes_{\mathcal{O}_X} J^1_{X/S}$, which is true by the definition of p^m-connection. So we have the well-defined $\mathcal{D}^{(-m)}_{X/S}$-module structure on $\mathcal{E}$ and hence the condition (2) is satisfied. So we are done. □

COROLLARY 2.8. *For an $\mathcal{O}_X$-module $\mathcal{E}$, the following three data are equivalent.*

(a) *An integrable p^m-connection on $\mathcal{E}$.*

(b) *A $\mathcal{D}^{(m)}_{X/S}$-module structure on $\mathcal{E}$ which extends the given $\mathcal{O}_X$-module structure.*

(c) *A $(-m)$-PD-stratification $\{\epsilon_k\}_k$ on $\mathcal{E}$.*

In particular, we have the equivalence

$$\mathrm{MIC}^{(m)}(X) \xrightarrow{=} (\text{left } \mathcal{D}^{(-m)}_{X/S}\text{-modules})$$

and it induces the equivalence

$$\mathrm{MIC}^{(m)}(X)_n \xrightarrow{=} (\text{left } \mathcal{D}^{(-m)}_{X/S} \otimes \mathbb{Z}/p^n\mathbb{Z}\text{-modules}).$$

Proof. It suffices to prove the equivalence of (a) and (b). When we are given an integrable p^m-connection on $\mathcal{E}$, we have the desired $\mathcal{D}^{(m)}_{X/S}$-module structure on $\mathcal{E}$ thanks to Proposition 2.7. Conversely, when we are given a $\mathcal{D}^{(m)}_{X/S}$-module structure on $\mathcal{E}$ which extends the given $\mathcal{O}_X$-module structure, we have the induced homomorphism $\mu_1 : \mathcal{D}^{(-m)}_{X/S,1} \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{E}$. It gives the $\mathcal{O}_X$-linear homomorphism

$$\mathcal{E} \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{D}^{(-m)}_{X/S,1}, \mathcal{E}) \cong \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X,(-m)}$$

(where the $\mathcal{O}_X$-module structure on the target is induced by the right $\mathcal{O}_X$-module structure on $\mathcal{P}^1_{X,(-m)}$), and by taking the $\mathcal{P}^1_{X,(-m)}$-linearization of it, we obtain the homomorphism $\epsilon_1 : \mathcal{P}^1_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X,(-m)}$ which is equal to the identity modulo $J^1_{X/S}$. It is automatically an isomorphism by the last argument in the proof of Proposition 2.6, and it gives a p^m-connection ∇ by Proposition 2.6. Then, ∇ gives rise to the homomorphism μ_1 by the recipe given in the statement of Proposition 2.7. Since μ_1 is extendable to the $\mathcal{D}^{(-m)}_{X/S}$-module structure by assumption, we see by Proposition 2.7 that ∇ is integrable. So we obtain the integrable p^m-connection ∇ and so we are done. □

Next we give a $\mathcal{D}$-module theoretic interpretation of the quasi-nilpotence for objects in $\mathrm{MIC}^{(m)}(X)_n$. The following proposition is the analogue of [1, 2.3.7].

Proposition 2.9. *Let $f : X \longrightarrow S$ be a smooth morphism of p-adic formal schemes flat over $\mathbb{Z}_p$. Let $m \in \mathbb{N}$ and let $\mathcal{E} := (\mathcal{E}, \nabla)$ be an object in $\mathrm{MIC}^{(m)}(X)_n$, regarded as a left $\mathcal{D}^{(-m)}_{X/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-module. Then the following conditions are equivalent.*

(a) *$(\mathcal{E}, \nabla)$ is quasi-nilpotent as an object in $\mathrm{MIC}^{(m)}(X)_n$.*

(b) *Locally on X, f admits a local coordinate such that the following condition is satisfied: For any local section $e \in \mathcal{E}$, there exists some $N \in \mathbb{N}$*

such that $\partial^{\langle k \rangle}(e) = 0$ *for any* k *with* $|k| \geq N$, *where* $\partial^{\langle k \rangle}$ *is the element in* $\mathcal{D}^{(-m)}_{X/S}$ *defined by using the fixed local coordinate.*

(c) *The condition given in* (b) *is satisfied for any local coordinate.*

(d) *There exists* (*uniquely*) *a* $\mathcal{P}_{X,(-m)}$*-linear isomorphism* $\epsilon : \mathcal{P}_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E} \overset{=}{\longrightarrow} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X,(-m)}$ *satisfying the cocycle condition on* $\mathcal{P}_{X,(-m)}(2)$ *which induces the* $(-m)$*-PD stratification* $\{\epsilon_k\}_k$ *on* $\mathcal{E}$ *associated to the* $\mathcal{D}^{(-m)}_{X/S}$*-module structure on* $\mathcal{E}$ *via Proposition 2.5.*

(*We call the isomorphism* ϵ *in* (d) *the* $(-m)$*-HPD-stratification associated to* $\mathcal{E}$.)

PROOF. The proof is similar to that of [1, 2.3.7]. First, let us work locally on X, take a local coordinate $t_1, ..., t_d$ of f and write ∇ as $\nabla(e) = \sum_i \theta_i(e) dt_i$. Then, in the notation in (b), we have $\theta^k = \partial^{\langle k \rangle}$ for any $k \in \mathbb{N}^d$. Hence we have the equivalence of (a) and (b). When the condition (b) is satisfied, we can define the morphism $\theta : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X,(-m)}$ by $\theta(e) = \sum_k \partial^{\langle k \rangle}(e) \otimes (\tau/p^m)^{[k]}$ and by $\mathcal{P}_{X,(-m)}$-linearizing it, we obtain the homomorphism $\epsilon : \mathcal{P}_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X,(-m)}$ which induces the stratification $\{\epsilon_k\}_k$. The cocycle condition for ϵ follows from that for $\{\epsilon_k\}_k$ and the uniqueness is clear. Also, if we define $\theta' : \mathcal{E} \longrightarrow \mathcal{P}_{X,(-m)} \otimes_{\mathcal{O}_X} \mathcal{E}$ by $\theta'(e) := \sum_k (-1)^{|k|} (\tau/p^m)^{[k]} \otimes \partial^{\langle k \rangle}(e)$, we see that the $\mathcal{P}_{X,(-m)}$-linearization of it gives the inverse of ϵ. So ϵ is an isomorphism and thus defines a $(-m)$-HPD-stratification. Conversely, if we are given a $(-m)$-HPD-stratification ϵ associated to $\mathcal{E}$, the coefficient of $(\tau/p^m)^{[k]}$ of the elememt $\epsilon(1 \otimes e) \in \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_{X,(-m)} = \bigoplus_k \mathcal{E}(\tau/p^m)^{[k]}$ is equal to $\partial^{\langle k \rangle}(e)$, by Proposition 2.5. Hence the condition (b) is satisfied. Finally, since the condition (d) is independent of the choice of the local coordinate, we have the equivalence of the conditions (c) and (d). □

DEFINITION 2.10. Let $f : X \longrightarrow S$ be as above. Then, a left $\mathcal{D}^{(-m)}_{X/S}$-module $\mathcal{E}$ is said to be quasi-nilpotent if it is p^n-torsion for some n and that it satisfies the condition (d) of Proposition 2.9. By Proposition 2.9, we have the equivalence

$$\mathrm{MIC}^{(m)}(X)^{\mathrm{qn}} \overset{=}{\longrightarrow} (\text{quasi-nilpotent left } \mathcal{D}^{(-m)}_{X/S}\text{-modules}),$$

which is induced by

$$\mathrm{MIC}^{(m)}(X)_n^{\mathrm{qn}} \xrightarrow{=} (\text{quasi-nilpotent left } \mathcal{D}_{X/S}^{(-m)} \otimes \mathbb{Z}/p^n\mathbb{Z}\text{-modules}) \quad (n \in \mathbb{N}).$$

Next we give a definition of the inverse image functor for left $\mathcal{D}_{-/-}^{(-m)}$-modules. Let

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array} \tag{2.4}$$

be a commutative diagram of p-adic formal schemes flat over $\operatorname{Spf}\mathbb{Z}_p$ such that the vertical arrows are smooth. Then, for any $m, k \in \mathbb{N}$, it induces the commutative diagram

$$\begin{array}{ccccc} P^k_{X',(-m)} & \xrightarrow{\subset} & P_{X',(-m)} & \xrightarrow{p'_i} & X' \\ {\scriptstyle g^k}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ P^k_{X,(-m)} & \xrightarrow{\subset} & P_{X,(-m)} & \xrightarrow{p_i} & X \end{array} \tag{2.5}$$

for $i = 0, 1$, where p'_i, p_i denotes the morphism induced by the i-th projection $X'^2 \longrightarrow X', X^2 \longrightarrow X$, respectively. So, if $\mathcal{E}$ is an $\mathcal{O}_X$-module endowed with a $(-m)$-PD-stratification $\{\epsilon_k\}_k$, $f^*\mathcal{E}$ is naturally endowed with the $(-m)$-PD-stratification $\{g^{k*}\epsilon_k\}_k$. Hence, in view of Proposition 2.5, we have the functor

$$\begin{aligned} f^* :&(\text{left } \mathcal{D}_{X/S}^{(-m)}\text{-modules}) \longrightarrow (\text{left } \mathcal{D}_{X'/S'}^{(-m)}\text{-modules}); \\ &(\mathcal{E}, \{\epsilon_k\}_k) \quad \mapsto \quad (f^*\mathcal{E}, \{g^{k*}\epsilon_k\}_k), \end{aligned} \tag{2.6}$$

and this induces also the functor

$$\begin{aligned} f^* : &(\text{left } \mathcal{D}_{X/S}^{(-m)} \otimes \mathbb{Z}/p^n\mathbb{Z}\text{-modules}) \\ &\longrightarrow (\text{left } \mathcal{D}_{X'/S'}^{(-m)} \otimes \mathbb{Z}/p^n\mathbb{Z}\text{-modules}). \end{aligned} \tag{2.7}$$

As for the quasi-nilpotence, we have the following:

PROPOSITION 2.11. *With the above notation, assume that $\mathcal{E}$ is quasi-nilpotent. Then $f^*\mathcal{E}$ is also quasi-nilpotent.*

PROOF. When $\mathcal{E}$ is quasi-nilpotent, the $(-m)$-PD-stratification $\{\epsilon_k\}_k$ associated to $\mathcal{E}$ is induced from a $(-m)$-HPD-stratification ϵ. Then the $(-m)$-PD-stratification $\{g^{k^*}\epsilon_k\}_k$ associated to $f^*\mathcal{E}$ is induced from the $(-m)$-HPD-stratification $g^*\epsilon$ by the commutativity of the diagram (2.5). So $f^*\mathcal{E}$ is also quasi-nilpotent. □

The inverse image functor here is equivalent to the inverse image functor in the previous section in the following sense.

PROPOSITION 2.12. *With the above notation, the inverse image functor* (2.6) *is equal to the inverse image functor $f^* : \mathrm{MIC}^{(m)}(X) \longrightarrow \mathrm{MIC}^{(m)}(X')$ defined in the previous section via the equivalence in Corollary 2.8.* (*Hence the inverse image functor* (2.7) *is equal to the inverse image functor $f^* : \mathrm{MIC}^{(m)}(X)_n \longrightarrow \mathrm{MIC}^{(m)}(X')_n$ defined in the previous section.*)

PROOF. Assume given an object $(\mathcal{E}, \nabla) \in \mathrm{MIC}^{(m)}(X)$ and let $(f^*\mathcal{E}, f^*\nabla) \in \mathrm{MIC}^{(m)}(X')$ be the inverse image of it defined in the previous section. On the other hand, let $(\mathcal{E}, \{\epsilon_k\})$ be the $(-m)$-PD-stratification associated to $(\mathcal{E}, \nabla)$, let $(f^*\mathcal{E}, \{g^{k^*}\epsilon_k\})$ be the inverse image of it defined above and let $(f^*\mathcal{E}, \widetilde{\nabla})$ be the object in $\mathrm{MIC}^{(m)}(X')$ associated to $(f^*\mathcal{E}, \{g^{k^*}\epsilon_k\})$ via the equivalence in Corollary 2.8. Since the underlying $\mathcal{O}_{X'}$-module $f^*\mathcal{E}$ of $(f^*\mathcal{E}, f^*\nabla)$ and that of $(f^*\mathcal{E}, \widetilde{\nabla})$ are the same, it suffices to prove the coincidence of p^m-connections $f^*\nabla$ and $\widetilde{\nabla}$. To see this, we may work locally.

Take a local section $e \in \mathcal{E}$ and let us put $\nabla(e) = \sum_i e_i da_i$. Then we have $f^*\nabla(e) = \sum f^*e_i \otimes df^*(a_i)$.

On the other hand, if we denote the composite $\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_{X/S} \xrightarrow{\mathrm{id}\otimes\beta} \mathcal{E} \otimes J^1_{X/S}$ by ∇', we have $\nabla'(e) = \sum_i e_i(da_i/p^m)$. Hence $\epsilon^1 : \mathcal{P}^1_{X,(-m)} \otimes \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{P}^1_{X,(-m)}$ is written as

$$\epsilon(1 \otimes e) = e \otimes 1 + \nabla'(e) = e \otimes 1 + \sum_i e_i \otimes da_i/p^m.$$

Since $g^{1*} : \mathcal{P}^1_{X,(-m)} \longrightarrow \mathcal{P}^1_{X',(-m)}$ sends da_i/p^m to $df^*(a_i)/p^m$, we have

$$\begin{aligned} g^{1*}\epsilon(1 \otimes f^*e) &= f^*e \otimes 1 + \sum_i f^*e_i \otimes g^{1*}(da_i/p^m) \\ &= f^*e \otimes 1 + \sum_i f^*e_i \otimes df^*(a_i)/p^m, \end{aligned}$$

and so we have $\widetilde{\nabla}(f^*e) = \sum_i f^*e_i \otimes df^*(a_i)$. Therefore we have $f^*\nabla = \widetilde{\nabla}$, as desired. □

Remark 2.13. Proposition 2.11 together with Proposition 2.12 gives another proof of Proposition 1.9 in the case of p-adic formal schemes flat over $\mathbb{Z}_p$ (at least in the case $m \geq 1$).

Next we define the level raising inverse image functor for $\mathcal{D}^{(-)}_{-/-}$-modules. The following proposition is an analogue of [2, 2.2.2].

Proposition 2.14. *In the situation of* $\mathbf{Hyp}(\infty, \infty, \infty)$*, F induces naturally a PD-morphism $\Phi : P_{X,(-m+1)} \longrightarrow P_{X^{(1)},(-m)}$ (with respect to the PD-ideal on the defining ideal of $X \hookrightarrow P_{X,(-m+1)}, X^{(1)} \hookrightarrow P_{X^{(1)},(-m)}$, respectively).*

Proof. We may work locally. So we may assume that there exist a local parameter $t_1, ..., t_d$ of X and a local parameter $t'_1, ..., t'_d$ of $X^{(1)}$ such that $F^*(t'_i) = t_i^p + pa_i$ for some $a_i \in \mathcal{O}_X$ $(1 \leq i \leq d)$. Let us put $\tau_i := 1 \otimes t_i - t_i \otimes 1 \in \mathcal{O}_{X^2}$, $\tau'_i := 1 \otimes t'_i - t'_i \otimes 1 \in \mathcal{O}_{(X^{(1)})^2}$ and let us denote the morphism $F \times F : X^2 \longrightarrow (X^{(1)})^2$ simply by F^2. Then we have

$$\begin{aligned} F^{2*}(\tau'_i) &= 1 \otimes t_i^p - t_i^p \otimes 1 + p(1 \otimes a_i - a_i \otimes 1) \\ &= (\tau_i + t_i \otimes 1)^p - t_i^p \otimes 1 + p(1 \otimes a_i - a_i \otimes 1) \\ &= \tau_i^p + \sum_{k=1}^{p-1} \binom{p}{k} t_i^{p-k}\tau_i^k + p(1 \otimes a_i - a_i \otimes 1). \end{aligned} \tag{2.8}$$

Hence there exists an element $\sigma_i \in I := \mathrm{Ker}(\mathcal{O}_{X^2} \to \mathcal{O}_X)$ such that $F^{2*}(\tau'_i) = \tau_i^p + p\sigma_i$. So, when $m \geq 2$, the image of $F^{2*}(\tau'_i)$ in $\mathcal{O}_{T_{X,(-m+1)}}$ belongs to $I^p + pI \subseteq (p^{m-1}\mathcal{O}_{T_{X,(-m+1)}})^p + p(p^{m-1}\mathcal{O}_{T_{X,(-m+1)}}) = p^m\mathcal{O}_{T_{X,(-m+1)}}$. So, by the universality of formal blow-up, F^2 induces the morphism

$T_{X,(-m+1)} \longrightarrow T_{X^{(1)},(-m)}$ and by the universality of the PD-envelope, it induces the PD-morphism $\Phi : P_{X,(-m+1)} \longrightarrow P_{X',(-m)}$, as desired. When $m = 1$, the image of $F^{2*}(\tau_i')$ in $\mathcal{O}_{P_{X,(0)}}$ is equal to $p!\tau_i^{[p]} + p\sigma_i$ and so it belongs to $p\mathcal{O}_{P_{X,(0)}}$. Hence F^2 induces the morphism $P_{X,(0)} \longrightarrow T_{X^{(1)},(-1)}$ and then it induces the PD-morphism $\Phi : P_{X,(0)} \longrightarrow P_{X^{(1)},(-1)}$, as desired. □

REMARK 2.15. In the same way as the above proof, we can prove also that, for $r \in \mathbb{N}$, the morphism $F^{r+1} : X^{r+1} \longrightarrow (X^{(1)})^{r+1}$ naturally induces the PD-morphism $\Phi : P_{X,(-m+1)}(r) \longrightarrow P_{X^{(1)},(-m)}(r)$.

Let the situation be as in $\mathbf{Hyp}(\infty, \infty, \infty)$ and let $m \in \mathbb{N}$. Then, by Proposition 2.14, we have the commutative diagrams

$$\begin{array}{ccccc} P^k_{X,(-m+1)} & \xrightarrow{\subset} & P_{X,(-m+1)} & \xrightarrow{p_i} & X \\ {\scriptstyle \Phi^k}\downarrow & & {\scriptstyle \Phi}\downarrow & & {\scriptstyle F}\downarrow \\ P^k_{X^{(1)},(-m)} & \xrightarrow{\subset} & P_{X^{(1)},(-m)} & \xrightarrow{p_i^{(1)}} & X^{(1)} \end{array} \tag{2.9}$$

for $i = 0, 1$, where $p_i, p_i^{(1)}$ denotes the morphism induced by the i-th projection $X^2 \longrightarrow X, (X^{(1)})^2 \longrightarrow X^{(1)}$ respectively and Φ^k is the morphism naturally induced by Φ. So, if $\mathcal{E}$ is an $\mathcal{O}_{X^{(1)}}$-module endowed with a $(-m)$-PD-stratification $\{\epsilon_k\}_k$, $F^*\mathcal{E}$ is endowed with a $(-m+1)$-PD-stratification $\{\Phi^{k*}\epsilon_k\}$. Hence, in view of Proposition 2.5, we have the functor

$$\begin{aligned} F^* : &(\text{left } \mathcal{D}^{(-m)}_{X^{(1)}/S}\text{-modules}) \longrightarrow (\text{left } \mathcal{D}^{(-m+1)}_{X/S}\text{-modules}); \\ &(\mathcal{E}, \{\epsilon_k\}_k) \quad \mapsto \quad (f^*\mathcal{E}, \{\Phi^{k*}\epsilon_k\}_k), \end{aligned} \tag{2.10}$$

and this induces also the functor

$$\begin{aligned} F^* : &(\text{left } \mathcal{D}^{(-m)}_{X^{(1)}/S} \otimes \mathbb{Z}/p^n\mathbb{Z}\text{-modules}) \\ &\longrightarrow (\text{left } \mathcal{D}^{(-m+1)}_{X/S} \otimes \mathbb{Z}/p^n\mathbb{Z}\text{-modules}). \end{aligned} \tag{2.11}$$

By the existence of the diagram (2.9), we can prove the following in the same way as Proposition 2.11 (so we omit the proof):

PROPOSITION 2.16. *With the above notation, assume that $\mathcal{E}$ is quasi-nilpotent. Then $F^*\mathcal{E}$ is also quasi-nilpotent.*

The level raising inverse image functor here is equivalent to that in the previous section in the following sense.

PROPOSITION 2.17. *With the above notation, the inverse image functor* (2.10) *is equal to the level raising inverse image functor* $F^* : \mathrm{MIC}^{(m)}(X^{(1)}) \longrightarrow \mathrm{MIC}^{(m-1)}(X)$ *defined in the previous section via the equivalence in Corollary 2.8.* (*Hence the inverse image functor* (2.11) *is equal to the inverse image functor* $F^* : \mathrm{MIC}^{(m)}(X^{(1)})_n \longrightarrow \mathrm{MIC}^{(m-1)}(X)_n$ *defined in the previous section.*)

PROOF. Assume given an object $(\mathcal{E}, \nabla) \in \mathrm{MIC}^{(m)}(X^{(1)})$ and let $(F^*\mathcal{E}, F^*\nabla)$ be the level raising inverse image of it defined in the previous section. On the other hand, let $(\mathcal{E}, \{\epsilon_k\})$ be the $(-m)$-PD-stratification associated to $(\mathcal{E}, \nabla)$, let $(F^*\mathcal{E}, \{\Phi^{k*}\epsilon_k\})$ be the level raising inverse image of it defined above and let $(F^*\mathcal{E}, \widetilde{\nabla})$ be the object in $\mathrm{MIC}^{(m-1)}(X)$ associated to $(F^*\mathcal{E}, \{\Phi^{k*}\epsilon_k\})$ via the equivalence in Corollary 2.8. Since the underlying $\mathcal{O}_X$-module $F^*\mathcal{E}$ of $(F^*\mathcal{E}, F^*\nabla)$ and that of $(F^*\mathcal{E}, \widetilde{\nabla})$ are the same, it suffices to prove the coincidence of p^m-connections $F^*\nabla$ and $\widetilde{\nabla}$. To see this, we may work locally. So we can take t_i, t_i', a_i as in the proof of Proposition 2.14. Take a local section $e \in \mathcal{E}$ and let us put $\nabla(e) = \sum_i e_i dt_i'$. Then, by definition, $F^*\nabla(e) = \sum F^*e_i(t_i^{p-1}dt_i + da_i)$.

On the other hand, if we denote the composite $\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_{X/S} \xrightarrow{\mathrm{id}\otimes\beta} \mathcal{E} \otimes J^1_{X/S}$ by ∇', $\nabla'(e) = \sum_i e_i(dt_i'/p^m)$. Hence $\epsilon^1 : \mathcal{P}^1_{X,(-m)} \otimes \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{P}^1_{X,(-m)}$ is written as

$$\epsilon(1 \otimes e) = e \otimes 1 + \nabla'(e) = e \otimes 1 + \sum_i e_i \otimes dt_i'/p^m.$$

Since $\Phi^{1*} : \mathcal{P}^1_{X,(-m)} \longrightarrow \mathcal{P}^1_{X',(-m)}$ sends dt_i'/p^m to $t_i^{p-1}(dt_i/p^{m-1}) + da_i/p^{m-1}$ by the calculation (2.8), we have

$$\begin{aligned}\Phi^{1*}\epsilon(1 \otimes F^*e) &= F^*e \otimes 1 + \sum_i F^*e_i \otimes \Phi^{1*}(dt_i'/p^m) \\ &= F^*e \otimes 1 + \sum_i F^*e_i \otimes (t_i^{p-1}(dt_i/p^{m-1}) + da_i/p^{m-1}),\end{aligned}$$

and so we have $\widetilde{\nabla}(F^*e) = \sum_i f^*e_i \otimes (t_i^{p-1}dt_i + da_i)$. Therefore we have $f^*\nabla = \widetilde{\nabla}$, as desired. □

REMARK 2.18. Proposition 2.16 together with Proposition 2.17 gives another proof of Proposition 1.13 under $\mathbf{Hyp}(\infty, \infty, \infty)$, as promised in Remark 1.14.

2.2. The case of schemes over $\mathbb{Z}/p^n\mathbb{Z}$

In the previous subsection, we defined the sheaf of p-adic differential operators of level $-m$ for smooth morphisms of p-adic formal schemes flat over $\mathbb{Z}_p$. The construction there does not work well for smooth morphisms of schemes flat over $\mathbb{Z}/p^n\mathbb{Z}$ because we needed the formal blow-up with respect to certain ideal containing p^m in the construction. In this subsection, we explain how to interpret the notion of the category of modules with integrable p^m-connection for smooth morphisms of schemes $X_n \longrightarrow S_n$ flat over $\mathbb{Z}/p^n\mathbb{Z}$ and the (level raising) inverse image functors between them in terms of $\mathcal{D}$-modules, under the assumption that S_n is liftable to a p-adic formal scheme S flat over $\mathbb{Z}_p$. (Note that X_n is not necessarily liftable to a smooth p-adic formal scheme over S globally). The key point is to consider all the local lifts of X_n to a smooth p-adic formal scheme over S and consider the 'crystalized' category.

DEFINITION 2.19. Let S be a p-adic formal scheme flat over $\mathbb{Z}_p$, let $S_n := S \otimes \mathbb{Z}/p^n\mathbb{Z}$ and let $f : X_n \longrightarrow S_n$ be a smooth morphism. Then we define the category $C(X_n/S)$ as follows: An object is a triple (U_n, U, i_U) consisting of an open subscheme U_n of X_n, a smooth formal scheme U over S and a closed immersion $i_U : U_n \hookrightarrow U$ which makes the following diagram Cartesian:

$$\begin{array}{ccccc} U_n & \xrightarrow{\subset} & X_n & \xrightarrow{f} & S_n \\ \downarrow{\scriptstyle i_U} & & & & \downarrow{\scriptstyle \cap} \\ U & & \longrightarrow & & S. \end{array}$$

A morphism $\varphi : (U_n, U, i_U) \longrightarrow (V_n, V, i_V)$ in $C(X_n/S)$ is defined to be a pair of a morphism $\varphi_n : U_n \longrightarrow V_n$ over X_n and a morphism $\varphi : U \longrightarrow V$ over S such that the square

$$\begin{array}{ccc} U_n & \xrightarrow{i_U} & U \\ {\scriptstyle \varphi_n}\downarrow & & \downarrow{\scriptstyle \varphi} \\ V_n & \xrightarrow{i_V} & V \end{array}$$

is Cartesian.

LEMMA 2.20. *Let S be a p-adic formal scheme flat over $\mathbb{Z}_p$ and let $f, f' : U \longrightarrow V$ be morphisms of smooth p-adic formal schemes over S which coincide modulo p^n. Then, for a $\mathcal{D}^{(-m)}_{V/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-module $\mathcal{E}$, there exists a canonical isomorphism $\tau_{f,f'} : f'^*\mathcal{E} \xrightarrow{=} f^*\mathcal{E}$ of $\mathcal{D}^{(-m)}_{U/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-modules.*

PROOF. Let $\{\epsilon_k\}_k$ be the $(-m)$-PD-stratification associated to $\mathcal{E}$ and let $f^*_k, f'^*_k : \mathcal{P}^k_{V,(-m)} \longrightarrow \mathcal{P}^k_{U,(-m)}$ be the morphism induced by f, f', respectively.

First let us prove that f^*_k is equal to f'^*_k modulo p^n. Since $\mathcal{P}^k_{V,(-m)}$ is topologically generated by $\mathcal{O}_V$ and the elements of the form $(1 \otimes a - a \otimes 1)/p^m$ $(a \in \mathcal{O}_V)$, it suffices to check that the images of these elements by f^*_k coincides with those by f'^*_k modulo p^n. For the elements in $\mathcal{O}_V$, this is clear since f and f' are equal modulo p^n. Let us consider the images of the element $(1 \otimes a - a \otimes 1)/p^m$. If we put $f^*(a) - f'^*(a) =: p^n b$, we have

$$\begin{aligned} &f^*_k((1 \otimes a - a \otimes 1)/p^m) - f'^*_k((1 \otimes a - a \otimes 1)/p^m) \\ =&((1 \otimes f^*(a) - f^*(a) \otimes 1)/p^m) - (1 \otimes f'^*(a) - f'^*(a) \otimes 1)/p^m) \\ =&p^n(1 \otimes b - b \otimes 1)/p^m. \end{aligned}$$

Hence f^*_k is equal to f'^*_k modulo p^n, as desired.

Let us put $\overline{f} = f \bmod p^n = f' \bmod p^n, \overline{f}^*_k = f^*_k \bmod p^n = f'^*_k \bmod p^n$. Then we have the canonical isomorphism

$$\tau_{f,f'} : f'^*\mathcal{E} \xrightarrow{=} \overline{f}^*\mathcal{E} \xrightarrow{=} f^*\mathcal{E},$$

and since we have $f'^*_k\epsilon_k = \overline{f}^*_k\epsilon_k = f^*_k\epsilon_k$, $\tau_{f,f'}$ gives an isomorphism as $\mathcal{D}^{(-m)}_{U/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-modules. □

Using these, we give the following definition.

DEFINITION 2.21. Let us take $n, n', m \in \mathbb{N}$ with $n \leq n'$, let S be a p-adic formal scheme flat over $\mathbb{Z}_p$, let $S_{n'} := S \otimes \mathbb{Z}/p^{n'}\mathbb{Z}$ and let $f : X_{n'} \longrightarrow S_{n'}$ be a smooth morphism. Then we define the category $D^{(-m)}(X_{n'}/S)_n$ as the category of pairs

$$(2.12)\quad ((\mathcal{E}_U)_{U:=(U_{n'},U,i_U)\in C(X_{n'}/S)}, (\alpha_\varphi)_{\varphi:(U_{n'},U,i_U)\to(V_{n'},V,i_V)\in \mathrm{Mor}\, C(X_{n'}/S)}),$$

where $\mathcal{E}_U$ is a $\mathcal{D}^{(-m)}_{U/S}\otimes\mathbb{Z}/p^n\mathbb{Z}$-module and α_φ is an isomorphism $\varphi^*\mathcal{E}_V \overset{=}{\longrightarrow} \mathcal{E}_U$ as $\mathcal{D}^{(-m)}_{U/S}\otimes\mathbb{Z}/p^n\mathbb{Z}$-modules satisfying the following conditions:

(1) $\alpha_{\mathrm{id}} = \mathrm{id}$.

(2) $\alpha_{\varphi\circ\psi} = \alpha_\varphi \circ \varphi^*(\alpha_\psi)$ for morphisms $\varphi : (U_{n'}, U, i_U) \longrightarrow (V_{n'}, V, i_V), \psi : (V_{n'}, V, i_V) \longrightarrow (W_{n'}, W, i_W)$ in $C(X_{n'}/S)$.

(3) For two morphisms $\varphi, \psi : (U_{n'}, U, i_U) \longrightarrow (V_{n'}, V, i_V)$ in $C(X_{n'}/S)$, the isomorphism $\alpha_\psi^{-1}\circ\alpha_\varphi : \varphi^*\mathcal{E}_V \overset{=}{\longrightarrow} \mathcal{E}_U \overset{=}{\longrightarrow} \psi^*\mathcal{E}$ coincides with $\tau_{\psi,\varphi}$ defined in Lemma 2.20.

We denote the object (2.12) simply by $((\mathcal{E}_U)_U, (\alpha_\varphi)_\varphi)$ or $(\mathcal{E}_U)_U$.

We call an object $(\mathcal{E}_U)_U \in D^{(-m)}(X_{n'}/S)_n$ quasi-nilpotent if each $\mathcal{E}_U$ is a quasi-nilpotent $\mathcal{D}^{(-m)}_{U/S}\otimes\mathbb{Z}/p^n\mathbb{Z}$-module, and denote the full subcategory of $D^{(-m)}(X_{n'}/S)_n$ consisting of quasi-nilpotent objects by $D^{(-m)}(X_{n'}/S)^{\mathrm{qn}}_n$.

Then we have the following, which is a $\mathcal{D}$-module theoretic interpretation of the category $\mathrm{MIC}^{(m)}(X_n)$:

PROPOSITION 2.22. *Let us take $n, n', m \in \mathbb{N}$ with $n \le n'$ and let S be a p-adic formal scheme flat over $\mathbb{Z}_p$. Let $S_{n'} := S\otimes\mathbb{Z}/p^{n'}\mathbb{Z}$, let $f : X_{n'} \longrightarrow S_{n'}$ be a smooth morphism and let $X_n \longrightarrow S_n$ be $f\otimes\mathbb{Z}/p^n\mathbb{Z}$. Then there exists the canonical equivalence of categories*

$$\mathrm{MIC}^{(m)}(X_n) \overset{=}{\longrightarrow} D^{(-m)}(X_{n'}/S)_n, \quad \mathrm{MIC}^{(m)}(X_n)^{\mathrm{qn}} \overset{=}{\longrightarrow} D^{(-m)}(X_{n'}/S)^{\mathrm{qn}}_n.$$

PROOF. Assume we are given an object $(\mathcal{E}, \nabla)$ in $\mathrm{MIC}^{(m)}(X_n)$. Let us take an object $(U_{n'}, U, i_U)$ of $C(X_{n'}/S)$, let us put $U_n := U_{n'}\otimes\mathbb{Z}/p^n\mathbb{Z}$ and denote the composite $U_n \hookrightarrow U_{n'} \overset{i_U}{\hookrightarrow} U$ by $\widetilde{i_U}$. Then, via $\widetilde{i_U}$, we can regard $\mathcal{E}_U := (\mathcal{E}, \nabla)|_{U_n}$ as an object in $\mathrm{MIC}^{(m)}(U)_n$ by Remark 1.3, and by Corollary 2.8, we can regard it as a $\mathcal{D}^{(-m)}_{U/S}\otimes\mathbb{Z}/p^n\mathbb{Z}$-module. Also, for a morphism $\varphi : (U_{n'}, U, i_U) \longrightarrow (V_{n'}, V, i_V)$ in $C(X_{n'}/S)$, the induced morphism $U_n \longrightarrow V_n := V\otimes\mathbb{Z}/p^n\mathbb{Z}$ gives the isomorphism $\alpha_\varphi : \varphi^*\mathcal{E}_V \overset{=}{\longrightarrow} \mathcal{E}_U$ as objects in $\mathrm{MIC}^{(m)}(U_n)$ and it induces the isomorphim as $\mathcal{D}^{(-m)}_{U/S}\otimes\mathbb{Z}/p^n\mathbb{Z}$-modules

by Remarks 1.3, 1.4, Corollary 2.8 and Proposition 2.12. We can check easily that the pair $((\mathcal{E}_U)_U, (\alpha_\varphi)_\varphi)$ defines an object in $D^{(-m)}(X_{n'}/S)_n$. (The condition (3) in Definition 2.21 comes from the fact that the isomorphism α_φ defined above depends only on φ modulo p^n.)

Conversely, if we are given an object $((\mathcal{E}_U)_U, (\alpha_\varphi)_\varphi)$ in $D^{(-m)}(X_{n'}/S)_n$, each $\mathcal{E}_U$ naturally defines an object in $\mathrm{MIC}^{(m)}(U_{n'})_n$, and the transition maps α_φ's depend only on φ modulo $p^{n'}$ up to canonical isomorphism. Hence $\mathcal{E}_U$'s glue to give an object in $\mathrm{MIC}^{(m)}(X_{n'})_n = \mathrm{MIC}^{(m)}(X_n)$. We can check that these functors give the inverse of each other, and so obtain the first equivalence.

To obtain the second equivalence, it suffices to see the consistence of the definitions of quasi-nilpotence. This follows from Proposition 2.9. □

Next we define the inverse image functor for the categories $D^{(-m)}(-/-)$. Let $n, n', m \in \mathbb{N}$ with $n \leq n'$ and assume given the following commutative diagram

$$\begin{array}{ccccc} X_{n'} & \longrightarrow & S_{n'} & \xrightarrow{\subseteq} & S \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ Y_{n'} & \longrightarrow & T_{n'} & \xrightarrow{\subseteq} & T, \end{array}$$

where S, T are p-adic formal schemes flat over $\mathbb{Z}_p$, $S_{n'} = S \otimes \mathbb{Z}/p^{n'}\mathbb{Z}, T_{n'} = T \otimes \mathbb{Z}/p^{n'}\mathbb{Z}$, the right top arrow and the right bottom arrow are canonical closed immersion, the left top arrow and the left bottom arrow are smooth. Under this situation, we define the inverse image functor

$$f^* : D^{(-m)}(Y_{n'}/T)_n \longrightarrow D^{(-m)}(X_{n'}/S)_n$$

as follows: Let us take an object $\mathcal{E} := (\mathcal{E}_V)_V$ in $D^{(-m)}(Y_{n'}/T)_n$ and $(U_{n'}, U, i_U) \in C(X_{n'}/S)$. Then, locally on U, there exists an object $(V_{n'}, V, i_V) \in C(Y_{n'}/T)$, a morphism $\varphi_{n'} : U_{n'} \longrightarrow V_{n'}$ over f, a morphism $\varphi : U \longrightarrow V$ over $S \longrightarrow T$ with $\varphi \circ i_U = i_V \circ \varphi_{n'}$. Then, we define the $\mathcal{D}^{(-m)}_{U/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-module $(f^*\mathcal{E})_U$ by $(f^*\mathcal{E})_U := \varphi^*\mathcal{E}_V$. When there exists another object $(V'_{n'}, V', i_{V'}) \in C(Y_{n'}/T)$ with morphisms $\varphi'_{n'} : U_{n'} \longrightarrow V'_{n'}$, $\varphi' : U \longrightarrow V'$ as above, there exists an isomorphism $\iota : V \xrightarrow{=} V'$ locally on V'. Then, since $\iota \circ \varphi$ and φ' are equal modulo $p^{n'}$, we have the isomorphism

$$\varphi^*\mathcal{E}_V \xrightarrow{=} (\iota \circ \varphi)^*\mathcal{E}_{V'} \xrightarrow{\tau_{\varphi', \iota\circ\varphi}} \varphi'^*\mathcal{E}_{V'},$$

and this is indendent of the choice of ι because, when we are given another isomorphism $\iota' : V \xrightarrow{=} V'$, we have the commutative diagram

$$\begin{array}{ccccc} \varphi^*\mathcal{E}_V & \xrightarrow{=} & (\iota\circ\varphi)^*\mathcal{E}_{V'} & \xrightarrow{\tau_{\varphi',\iota\circ\varphi}} & \varphi'^*\mathcal{E}_{V'} \\ \| & & \downarrow{\scriptstyle \tau_{\iota'\circ\varphi,\iota\circ\varphi}} & & \| \\ \varphi^*\mathcal{E}_V & \xrightarrow{=} & (\iota'\circ\varphi)^*\mathcal{E}_{V'} & \xrightarrow{\tau_{\varphi',\iota'\circ\varphi}} & \varphi'^*\mathcal{E}_{V'} \end{array}$$

(the commutativity of the left square is the pull-back of the property (3) in Definition 2.21 by φ^* and that of the right square comes from the definition of $\tau_{-/-}$). Therefore, we can glue the local definition $(f^*\mathcal{E})_U := \varphi^*\mathcal{E}_V$ and define the $\mathcal{D}^{(-m)}_{U/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-module $(f^*\mathcal{E})_U$ globally. We can also check that the $(f^*\mathcal{E})_U$'s for $(U_{n'}, U, i_U) \in C(X_{n'}/S)$ form an object $f^*\mathcal{E} := ((f^*\mathcal{E})_U)_U$ in $D^{(-m)}(X_{n'}/S)_n$ in the same way. By the correspondence $\mathcal{E} \mapsto f^*\mathcal{E}$, the inverse image functor

$$f^* : D^{(-m)}(Y_{n'}/T)_n \longrightarrow D^{(-m)}(X_{n'}/S)_n$$

is defined. Because this functor is defined locally as the inverse image functor of $\mathcal{D}^{(-m)}_{-/-} \otimes \mathbb{Z}/p^n\mathbb{Z}$-modules, it induces the functor

$$f^{*,\mathrm{qn}} : D^{(-m)}(Y_{n'}/T)^{\mathrm{qn}}_n \longrightarrow D^{(-m)}(X_{n'}/S)^{\mathrm{qn}}_n.$$

Also, we see by the construction that the inverse image functors $f^*, f^{*,\mathrm{qn}}$ here are equal to the inverse image functors of modules with integrable p^m-connection

$$\begin{aligned} f^*_n &: \mathrm{MIC}^{(m)}(Y_n) \xrightarrow{=} \mathrm{MIC}^{(m)}(X_n), \\ f^{*,\mathrm{qn}}_n &: \mathrm{MIC}^{(m)}(Y_n)^{\mathrm{qn}} \xrightarrow{=} \mathrm{MIC}^{(m)}(X_n)^{\mathrm{qn}} \end{aligned}$$

(where $X_n := X_{n'} \otimes \mathbb{Z}/p^n\mathbb{Z}, Y_n := Y_{n'} \otimes \mathbb{Z}/p^n\mathbb{Z}, f_n := f \otimes \mathbb{Z}/p^n\mathbb{Z}$) defined in Section 1 via the equivalences in Proposition 2.22. (See also Proposition 2.12.)

Next we define the level raising inverse image functor. First we prove the following lemma, which is an analogue of Lemma 2.20.

LEMMA 2.23. *Assume we are in the situation of* $\mathbf{Hyp}(\infty, n+1, n+1)$. *Let* $f, f' : U \longrightarrow V$ *be morphisms of smooth p-adic formal schemes over* S

which coincide modulo p^{n+1} *and put* $\overline{f} := f \otimes \mathbb{Z}/p^{n+1}\mathbb{Z} = f' \otimes \mathbb{Z}/p^{n+1}\mathbb{Z} : U_{n+1} \longrightarrow V_{n+1}$. *Assume moreover that this morphism fits into the following commutative diagram*

$$\begin{array}{ccc} U_{n+1} & \xrightarrow{\overline{f}} & V_{n+1} \\ \cap\downarrow & & \cap\downarrow \\ X_{n+1} & \xrightarrow{F_{n+1}} & X^{(1)}_{n+1}, \end{array}$$

where the vertical arrows are open immersions. (So f, f' *are local lifts of* F_{n+1} *and so we can define the level raising inverse image functor for* f, f'.*) Then, for a* $\mathcal{D}^{(-m)}_{V/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$*-module* $\mathcal{E}$*, the canonical isomorphism* $\tau_{f,f'} : f'^*\mathcal{E} \xrightarrow{=} f^*\mathcal{E}$ *in Lemma 2.20 (which is a priori an isomorphism as* $\mathcal{D}^{(-m)}_{U/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$*-modules) is an isomorphism of* $\mathcal{D}^{(-m+1)}_{U/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$*-modules.*

Proof. Since F_{n+1} is a homeomorphism, we may shrink V so that $f, f' : U \longrightarrow V$ are homeomorphisms. Then we can replace U_{n+1}, V_{n+1} by X_{n+1}, $X^{(1)}_{n+1}$ and then by putting $X := U, X^{(1)} := V$, we may assume that the situation is as in $\mathbf{Hyp}(\infty, \infty, \infty)$ (but we have two lifts f, f' instead of F there).

Let $\{\epsilon_k\}_k$ be the $(-m)$-PD-stratification associated to $\mathcal{E}$ and let $\Phi^*_k, \Phi'^*_k : \mathcal{P}^k_{X^{(1)},(-m)} \longrightarrow \mathcal{P}^k_{X,(-m)}$ be the morphism induced by f, f' respectively, by Proposition 2.14. (See also the diagram (2.9).)

Let us prove that Φ^*_k is equal to Φ'^*_k modulo p^n. To prove this, we may work locally. So we assume that there exist $t_i, t'_i, \tau_i, \tau'_i$ as in the proof of Proposition 2.14. Note that f and f' coincide modulo p^{n+1}. So, by the calculation similar to (2.8), we see that $f^{2*}(\tau'_i) - f'^{2*}(\tau'_i)$ can be written as an element of the form $p^{n+1}(1 \otimes b_i - b_i \otimes 1)$. Then, by definition of Φ^*_k, Φ'^*_k, we have $\Phi^*_k(\tau_i/p^m) - \Phi'^*_k(\tau_i/p^m) = p^n\{(1 \otimes b_i - b_i \otimes 1)/p^{m-1}\}$. Since $\mathcal{P}^k_{X^{(1)},(-m)}$ is topologically generated by $\mathcal{O}_{X^{(1)}}$ and τ'_i/p^m's, we see the coincidence of Φ^*_k and Φ'^*_k modulo p^n, as desired.

Let us put $\overline{\Phi}^*_k = \Phi^*_k \bmod p^n = \Phi'^*_k \bmod p^n$. Then, since we have $\Phi'^*_k\epsilon_k = \overline{\Phi}^*_k\epsilon_k = \Phi^*_k\epsilon_k$, the isomorphism $\tau_{f,f'}$ in Lemma 2.20 gives an isomorphism as $\mathcal{D}^{(-m)}_{X/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-modules. So we are done. □

Under $\mathbf{Hyp}(\infty, n', n')$ with $n' \in \mathbb{N}, n' \geq n+1$, we define the level raising

inverse image functor

$$F_{n'}^* : D^{(-m)}(X_{n'}^{(1)}/T)_n \longrightarrow D^{(-m+1)}(X_{n'}/S)_n$$

as follows: Let us take an object $\mathcal{E} := (\mathcal{E}_V)_V$ in $D^{(-m)}(X_{n'}^{(1)}/T)_n$ and $(U_{n'}, U, i_U) \in C(X_{n'}/S)$. Then, locally on U, there exists an object $(V_{n'}, V, i_V) \in C(X_{n'}^{(1)}/T)$, a morphism $\varphi_{n'} : U_{n'} \longrightarrow V_{n'}$ over $F_{n'}$, a morphism $\varphi : U \longrightarrow V$ over S with $\varphi \circ i_U = i_V \circ \varphi_{n'}$. Then, we define the $\mathcal{D}_{U/S}^{(-m+1)} \otimes \mathbb{Z}/p^n\mathbb{Z}$-module $(F_{n'}^*\mathcal{E})_U$ by $(F_{n'}^*\mathcal{E})_U := \varphi^*\mathcal{E}_V$, where the right hand side denotes the level raising inverse image by φ. When there exists another object $(V'_{n'}, V', i_{V'}) \in C(X_{n'}^{(1)}/T)$ with morphisms $\varphi'_{n'} : U_{n'} \longrightarrow V'_{n'}$, $\varphi' : U \longrightarrow V'$ as above, there exists an isomorphism $\iota : V \xrightarrow{=} V'$ locally on V'. Then, since $\iota \circ \varphi$ and φ' are equal modulo $p^{n'}$, we have the isomorphism

$$\varphi^*\mathcal{E}_V \xrightarrow{=} (\iota \circ \varphi)^*\mathcal{E}_{V'} \xrightarrow{\tau_{\varphi', \iota\circ\varphi}} \varphi'^*\mathcal{E}_{V'},$$

and this is indendent of the choice of ι as before. Therefore, we can glue the local definition $(F_{n'}^*\mathcal{E})_U := \varphi^*\mathcal{E}_V$ and define the $\mathcal{D}_{U/S}^{(-m)} \otimes \mathbb{Z}/p^n\mathbb{Z}$-module $(F_{n'}^*\mathcal{E})_U$ globally. We can also check that the $(F_{n'}^*\mathcal{E})_U$'s for $(U_{n'}, U, i_U) \in C(X_{n'}/S)$ form an object $F_{n'}^*\mathcal{E} := ((F_{n'}^*\mathcal{E})_U)_U$ in $D^{(-m)}(X_{n'}/S)_n$ in the same way. By the correspondence $\mathcal{E} \mapsto F_{n'}^*\mathcal{E}$, the level raising inverse image functor

$$F_{n'}^* : D^{(-m)}(X_{n'}^{(1)}/S)_n \longrightarrow D^{(-m+1)}(X_{n'}/S)_n$$

is defined. Because this functor is defined locally as the level raising inverse image functor of $\mathcal{D}_{-/-}^{(-)} \otimes \mathbb{Z}/p^n\mathbb{Z}$-modules, it induces the functor

$$F_{n'}^{*,\mathrm{qn}} : D^{(-m)}(X_{n'}^{(1)}/T)_n^{\mathrm{qn}} \longrightarrow D^{(-m+1)}(X_{n'}/S)_n^{\mathrm{qn}}.$$

Also, we see by the construction that the inverse image functors $F_{n'}^*, F_{n'}^{*,\mathrm{qn}}$ here are equal to the level raising inverse image functors of modules with integrable p^m-connection

$$F_{n+1}^* : \mathrm{MIC}^{(m)}(X_n^{(1)}) \longrightarrow \mathrm{MIC}^{(m-1)}(X_n),$$
$$F_{n+1}^{*,\mathrm{qn}} : \mathrm{MIC}^{(m)}(X_n^{(1)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}^{(m-1)}(X_n)^{\mathrm{qn}}$$

defined in Section 1 via the equivalences in Proposition 2.22. (See also Proposition 2.17.) In particular, we have given another proof of Proposition 1.13 under the assumption $\mathbf{Hyp}(\infty, n+1, n+1)$, as promised in Remark 1.14.

2.3. Crystalline property of integrable p^m-connections

In this subsection, we prove a crystalline property for the categories $\mathrm{MIC}^{(m)}(X_n), \mathrm{MIC}^{(m)}(X_n)^{\mathrm{qn}}$ for a smooth morphism $X_n \longrightarrow S_n$ satisfying certain liftability condition. We prove a similar result also for the categories $\mathrm{MIC}^{(m)}(X), \mathrm{MIC}^{(m)}(X)^{\mathrm{qn}}$ for a smooth morphism $X \longrightarrow S$ of p-adic formal schemes flat over $\mathbb{Z}_p$. The key construction in the former case is a variant $\overline{D}^{(-m)}(X_{m+e}/S)_n$ (where $e = 1$ or 2) of the category of the form $D^{(-)}(-/-)_-$ defined in the previous subsection and the (level raising) inverse image functor for it.

The starting point is the following, which is an analogue of [2, 2.1.5]:

PROPOSITION 2.24. *Let*

$$\begin{array}{ccc} X & & Y \\ \downarrow & & \downarrow \\ S & \longrightarrow & T \end{array}$$

be a diagram of p-adic formal schemes flat over $\operatorname{Spf}\mathbb{Z}_p$ *such that the vertical arrows are smooth. Let $m, e \in \mathbb{N}$, let $\mathcal{E}$ be a left $\mathcal{D}^{(-m)}_{Y/T}$-module and assume one of the following*:

(a) *$p \geq 3, e = 1$ or $p = 2, e = 2$.*

(b) *$\mathcal{E}$ is quasi-nilpotent and $e = 1$.*

Let us put $X_{m+e} := X \otimes \mathbb{Z}/p^{m+e}\mathbb{Z}$ and let $f_{m+e} : X_{m+e} \longrightarrow Y$ be a morphism over T. Assume that $f, f' : X \longrightarrow Y$ are morphisms over T which lift f_{m+e}. Then there exists a canonical $\mathcal{D}^{(-m)}_{X/S}$-linear isomorphism $\overline{\tau}_{f,f'} : f'^*\mathcal{E} \xrightarrow{=} f^*\mathcal{E}$.

Moreover, when $\mathcal{E}$ is p^n-torsion and f and f' are equal modulo p^{n+m} for some $n \geq 1$, the isomorphism $\overline{\tau}_{f,f'}$ is equal to the isomorphism $\tau_{f,f'}$ defined in Lemma 2.20.

PROOF. By assumption, we have the commutative diagram

$$\begin{array}{ccc} X_{m+e} & \xrightarrow{f_{m+e}} & Y \\ \cap\downarrow & & \downarrow\mathrm{diag.} \\ X & \xrightarrow{(f,f')} & Y \times_T Y, \end{array} \tag{2.13}$$

and by the universality of formal blow-up and PD-envelope, the lower horizontal arrow factors as

$$X \longrightarrow T_{Y,(-m-e)} \longrightarrow T_{Y,(-m)} \longrightarrow Y \times_T Y. \tag{2.14}$$

Let us denote the morphism $X \longrightarrow T_{Y,(-m)}$ in the diagram (2.14) (the composite of the first two arrows) by g' and let us denote the composite $X_e := X \otimes \mathbb{Z}/p^e\mathbb{Z} \hookrightarrow X_{m+e} \overset{f_{m+e}}{\longrightarrow} Y$ by f_e. Then the diagram

$$\begin{array}{ccc} X_e & \xrightarrow{f_e} & Y \\ {\scriptstyle\cap}\downarrow & & \downarrow \\ X & \xrightarrow{g'} & T_{Y,(-m)} \end{array} \tag{2.15}$$

(where the right vertical arrow is the morphism induced by the diagonal map) is commutative: Indeed, using the fact that $\mathcal{O}_{T_{Y,(-m)}}$ is locally topologically generated by sections a with $p^m a \in \mathcal{O}_{Y\times_T Y}$, the commutativity of the diagram (2.13) induces that of (2.15). Then, by the universality of PD-envelope, the diagram (2.15) induces the commutative diagram

$$\begin{array}{ccc} X_e & \xrightarrow{f_e} & Y \\ {\scriptstyle\cap}\downarrow & & \downarrow{\scriptstyle\cap} \\ X & \xrightarrow{g} & P_{Y,(-m)}. \end{array} \tag{2.16}$$

In the case (a), the defining ideal $p^e\mathcal{O}_X$ of the closed immersion $X_e \hookrightarrow X$ is topologically PD-nilpotent. So the morphism $X \longrightarrow P_{Y,(-m)}$ in (2.16) factors as $X \overset{g_r}{\longrightarrow} P^r_{Y,(-m)} \hookrightarrow P_{Y,(-m)}$ for some $r \in \mathbb{N}$. Then we define $\overline{\tau}_{f,f'} := g_r^*(\eta_r)$, where η_r is the isomorphism of $(-m)$-PD-stratification associated to $\mathcal{E}$ on $P^r_{Y,(-m)}$. In the case (b), we define $\overline{\tau}_{f,f'} := g^*(\eta)$, where η is the isomorphism of $(-m)$-HPD-stratification associated to $\mathcal{E}$ on $P_{Y,(-m)}$. Then we see that $\overline{\tau}_{f,f'}$ is an $\mathcal{O}_X$-linear isomorphism.

So it suffices to prove that $\overline{\tau}_{f,f'}$ is $\mathcal{D}^{(-m)}_{X/S}$-linear. Let us put $\mathcal{F} := f^*\mathcal{E}, \mathcal{F}' := f'^*\mathcal{E}$, let us take $l \in \mathbb{N}$, let $p_i : P^l_{X,(-m)} \longrightarrow X\,(i = 0, 1)$ be the morphism induced by the i-th projection $X^2 \longrightarrow X$ and let $\epsilon_l : p_1^*\mathcal{F} \overset{=}{\longrightarrow} p_0^*\mathcal{F}, \epsilon'_l : p_1^*\mathcal{F}' \overset{=}{\longrightarrow} p_0^*\mathcal{F}'$ be the isomorphism of $(-m)$-PD-stratification for

$\mathcal{F}, \mathcal{F}'$ on $P^l_{X,(-m)}$. Then it suffices to prove the commutativity of the following diagram of sheaves on $P^l_{X,(-m)}$:

$$\begin{array}{ccc} p_1^*\mathcal{F}' & \xrightarrow{p_1^*(\overline{\tau}_{f',f})} & p_1^*\mathcal{F} \\ {\scriptstyle \epsilon'_l}\downarrow & & {\scriptstyle \epsilon_l}\downarrow \\ p_0^*\mathcal{F}' & \xrightarrow{p_0^*(\overline{\tau}_{f',f})} & p_0^*\mathcal{F}. \end{array} \tag{2.17}$$

Let us consider the following commutative diagram

$$\begin{array}{ccc} X_{m+e} & \xrightarrow{f_{m+e}} & Y \\ \downarrow & & \downarrow \\ X^2 & \xrightarrow{(f\times f, f'\times f')} & Y^4, \end{array} \tag{2.18}$$

where the vertical arrows are the diagonal embeddings. Then, by the universality of formal blow-up, the composite $P_{X,(-m)} \longrightarrow X^2 \xrightarrow{(f\times f, f'\times f')} Y^4$ factors as $P_{X,(-m)} \xrightarrow{h'} T_{Y,(-m)}(3) \longrightarrow Y^4$, and we see (in the same way as the proof of the commutativity of (2.15)) that the commutativity of the diagram (2.18) induces that of the following diagram:

$$\begin{array}{ccc} X_e & \xrightarrow{f_e} & Y \\ \downarrow & & \downarrow \\ P_{X,(-m)} & \xrightarrow{h'} & T_{Y,(-m)}(3). \end{array} \tag{2.19}$$

Then, noting that the defining ideal of the closed immersion $X_e \hookrightarrow X \hookrightarrow P_{X,(-m)}$ admits a PD-structure canonically, we see that the diagram (2.19) gives rise to the morphism $h : P^l_{X,(-m)} \longrightarrow P_{Y,(-m)}(3)$. Also, the defining ideal of the closed immersion $X_e \hookrightarrow P^l_{X,(-m)}$ still admits a PD-structure canonically.

Assume that we are in the case (a). Then the PD-structure on the defining ideal of $X_e \hookrightarrow P^l_{X,(-m)}$ is topologically PD-nilpotent. So the morphism h factors as $P^l_{X,(-m)} \xrightarrow{h_s} P^s_{Y,(-m)}(3) \longrightarrow P_{Y,(-m)}(3)$ for some $s \in \mathbb{N}, s \geq l, r$. Let $q_{ij} : P^s_{Y,(-m)}(3) \longrightarrow P^s_{Y,(-m)}$ $(0 \leq i < j \leq 3)$ be the

morphism induced by the (i,j)-th projection $X^4 \longrightarrow X^2$. Then we have $p_0^*(\overline{\tau}_{f',f}) = p_0^* g_s^*(\eta_s) = h_s^* q_{02}^*(\eta_s)$ and $p_1^*(\overline{\tau}_{f',f}) = h_s^* q_{13}^*(\eta_s)$. Also, when we denote the morphism $P^l_{X,(-m)} \longrightarrow P^s_{Y,(-m)}$ induced by $f \times f : X^2 \longrightarrow Y^2$ by φ, we have $\epsilon_l = \varphi^*(\eta_s) = h^* q_{01}^*(\eta_s)$ and similarly we have $\epsilon'_l = h^* q_{23}(\eta_s)$. So The commutativity of the diagram (2.17) follows from the cocycle condition for η_s. So we are done in the case (a). In the case (b), we can prove the commutativity of the diagram (2.17) in the same way, by replacing η_s by η and $P^s_{Y,(-m)}$ by $P_{Y,(-m)}$.

Finally, let us assume that $\mathcal{E}$ is p^n-torsion and that f and f' are equal modulo p^{n+m} for some $n \geq 1$. Let us consider locally and take a local coordinate $t_1, ..., t_d$ of Y over T. Then, by definition, the isomorphism $\tau_{f,f'} : f'^*\mathcal{E} \stackrel{=}{\longrightarrow} f^*\mathcal{E}$ is written as

$$f'^*(e) \mapsto \sum_{k \in \mathbb{N}^d} ((f'^*(t) - f^*(t))/p^m)^{[k]} f^*(\partial^{\langle k \rangle} e),$$

and the k-th term on the right hand side is contained in $(k!)^{-1} p^{|k|n} f^*\mathcal{E} \subseteq p^n f^*\mathcal{E} = 0$ when $k \neq 0$. Hence we have $\overline{\tau}_{f,f'}(f'^*(e)) = f^*(e)$ and this implies the equality $\overline{\tau}_{f,f'} = \tau_{f,f'}$. $\square$

REMARK 2.25. When $\mathcal{E}$ is p^n-torsion and f, f' are equal only modulo p^n, the isomorphism $\overline{\tau}_{f,f'}$ above is not necessarily equal to the isomorphism $\tau_{f,f'}$ in Lemma 2.20 unless $m = 0$.

By the argument in [2, 2.1.6], we have the following immediate corollary of Proposition 2.24 (we omit the proof):

COROLLARY 2.26.

(1) *Let us put $e = 1$ if $p \geq 3$ and $e = 2$ if $p = 2$. let $f : X \longrightarrow S$ be a smooth morphism of formal schemes flat over $\mathbb{Z}_p$ and let $X_{m+e} \longrightarrow S_{m+e}$ be $f \otimes \mathbb{Z}/p^{m+e}\mathbb{Z}$. Then the category*

$$\mathrm{MIC}^{(m)}(X) = (\text{left } \mathcal{D}^{(-m)}_{X/S}\text{-modules})$$

depends only on the diagram $X_{m+e} \longrightarrow S_{m+e} \hookrightarrow S$ and functorial with respect to this diagram.

(2) *let $f : X \longrightarrow S$ be a smooth morphism of formal schemes flat over $\mathbb{Z}_p$ and let $X_{m+1} \longrightarrow S_{m+1}$ be $f \otimes \mathbb{Z}/p^{m+1}\mathbb{Z}$. Then the category*

$$\mathrm{MIC}^{(m)}(X)^{\mathrm{qn}} = (\text{quasi-nilpotent left } \mathcal{D}^{(-m)}_{X/S}\text{-modules})$$

depends only on the diagram $X_{m+1} \longrightarrow S_{m+1} \hookrightarrow S$ and functorial with respect to this diagram.

Next we consider the case of p^n-torsion objects. First, using Proposition 2.24, we give the following definition:

Definition 2.27.

(1) Let us put $e = 1$ if $p \geq 3$ and $e = 2$ if $p = 2$. Let us take $n, n', m \in \mathbb{N}$ with $m + e \leq n'$, let S be a p-adic formal scheme flat over $\mathbb{Z}_p$, let $S_{n'} := S \otimes \mathbb{Z}/p^{n'}\mathbb{Z}$ and let $f : X_{n'} \longrightarrow S_{n'}$ be a smooth morphism. Then we define the category $\overline{D}^{(-m)}(X_{n'}/S)_n$ as the category of pairs

$$\begin{aligned}&((\mathcal{E}_U)_{U:=(U_{n'},U,i_U)\in C(X_{n'}/S)},\\ &\quad(\alpha_\varphi)_{\varphi:(U_{n'},U,i_U)\to(V_{n'},V,i_V)\in \mathrm{Mor}\, C(X_{n'}/S)}),\end{aligned} \tag{2.20}$$

where $\mathcal{E}_U$ is a $\mathcal{D}^{(-m)}_{U/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-module and α_φ is an isomorphism $\varphi^*\mathcal{E}_V \xrightarrow{=} \mathcal{E}_U$ as $\mathcal{D}^{(-m)}_{U/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-modules satisfying the following conditions:

(a) $\alpha_{\mathrm{id}} = \mathrm{id}$.

(b) $\alpha_{\varphi\circ\psi} = \alpha_\varphi \circ \varphi^*(\alpha_\psi)$ for morphisms $\varphi : (U_{n'}, U, i_U) \longrightarrow (V_{n'}, V, i_V), \psi : (V_{n'}, V, i_V) \longrightarrow (W_{n'}, W, i_W)$ in $C(X_{n'}/S)$.

(c) For two morphisms $\varphi, \psi : (U_{n'}, U, i_U) \longrightarrow (V_{n'}, V, i_V)$ in $C(X_{n'}/S)$, the isomorphism $\alpha_\psi^{-1} \circ \alpha_\varphi : \varphi^*\mathcal{E}_V \xrightarrow{=} \mathcal{E}_U \xrightarrow{=} \psi^*\mathcal{E}$ coincides with $\overline{\tau}_{\psi,\varphi}$ defined in Proposition 2.24.

We denote the object (2.20) simply by $((\mathcal{E}_U)_U, (\alpha_\varphi)_\varphi)$ or $(\mathcal{E}_U)_U$.

(2) Let us put $e = 1$ and let us take $n, n', m \in \mathbb{N}$, $S, f : X_{n'} \longrightarrow S_{n'}$ as in (1). Then we define the category $\overline{D}^{(-m)}(X_{n'}/S)^{\mathrm{qn}}_n$ as the category of pairs (2.20) where $\mathcal{E}_U$ is a quasi-nilpotent $\mathcal{D}^{(-m)}_{U/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-module and α_φ is an isomorphism $\varphi^*\mathcal{E}_V \xrightarrow{=} \mathcal{E}_U$ as $\mathcal{D}^{(-m)}_{U/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-modules satisfying the conditions $(a), (b), (c)$ in (1).

When $n' \geq \max(m+n, m+e)$ (where e is as in Definition 2.27), we have the equalities $\overline{D}^{(m)}(X_{n'}/S)_n = D^{(m)}(X_{n'}/S)_n$, $\overline{D}^{(m)}(X_{n'}/S)_n^{\mathrm{qn}} = D^{(m)}(X_{n'}/S)_n^{\mathrm{qn}}$, because the isomorphisms $\overline{\tau}_{\psi,\varphi}$ used in Definition 2.27 are equal to the isomorphisms $\tau_{\psi,\varphi}$ used in Definition 2.21. Now let us note that, for $n', n'' \in \mathbb{N}$ with $m+e \leq n' \leq n''$, the functor

$$\begin{aligned} r' : C(X_{n''}/S) &\longrightarrow C(X_{n'}/S); \\ (U_{n''}, U, i_U) &\mapsto (U_{n'} := U_{n''} \otimes \mathbb{Z}/p^{n'}\mathbb{Z}, U, U_{n'} \hookrightarrow U_{n''} \overset{i_U}{\hookrightarrow} U) \end{aligned}$$

induces the functors $r : \overline{D}^{(m)}(X_{n'}/S)_n \longrightarrow \overline{D}^{(m)}(X_{n''}/S)_n$, $r : \overline{D}^{(m)}(X_{n'}/S)_n^{\mathrm{qn}} \longrightarrow \overline{D}^{(m)}(X_{n''}/S)_n^{\mathrm{qn}}$. Hence we obtain the functor

$$R : \overline{D}^{(m)}(X_{m+e}/S)_n \xrightarrow{r} \overline{D}^{(m)}(X_{n'}/S)_n \xrightarrow{=} D^{(m)}(X_{n'}/S)_n, \tag{2.21}$$

$$R : \overline{D}^{(m)}(X_{m+e}/S)_n^{\mathrm{qn}} \xrightarrow{r} \overline{D}^{(m)}(X_{n'}/S)_n^{\mathrm{qn}} \xrightarrow{=} D^{(m)}(X_{n'}/S)_n^{\mathrm{qn}} \tag{2.22}$$

for $n' \geq \max(m+n, m+e)$. Then we have the following:

PROPOSITION 2.28.

(1) *Let us put $e = 1$ if $p \geq 3$ and $e = 2$ if $p = 2$. Let us take $n, n', m \in \mathbb{N}$ with $n' \geq \max(m+n, m+e)$ and let S be a p-adic formal scheme flat over $\mathbb{Z}_p$. Let $S_{n'} := S \otimes \mathbb{Z}/p^{n'}\mathbb{Z}$, let $f : X_{n'} \longrightarrow S_{n'}$ be a smooth morphism and let $X_{m+e} \longrightarrow S_{m+e}$ be $f \otimes \mathbb{Z}/p^{m+e}\mathbb{Z}$. Then the functor* (2.21) *is an equivalence of categories.*

(2) *Let us put $e = 1$ and let the other notations be as in* (1)*. Then the functor* (2.22) *is an equivalence of categories.*

PROOF. Since the proof is the same, we only prove (1). To do so, it suffices to construct the inverse of the functor $r : \overline{D}^{(m)}(X_{m+e}/S)_n \longrightarrow \overline{D}^{(m)}(X_{n'}/S)_n$. So let us take $\mathcal{E} := (\mathcal{E}_V)_V \in \overline{D}^{(m)}(X_{n'}/S)_n$ and take an object (U_{m+e}, U, i_U) in $C(X_{m+e}/S)$. Then, locally on U, there exists an object $(V_{n'}, V, i_V)$ in $C(X_{n'}/S)$ and a morphism $\varphi : U \longrightarrow V$ over S inducing $\overline{\varphi} : U_{m+e} \longrightarrow V_{n'}$ which is a morphism over the canonical closed immersion $X_{m+e} \hookrightarrow X_{n'}$: Indeed, if we put $U_{n'} := U \otimes \mathbb{Z}/p^{n'}\mathbb{Z}$, it is a smooth lift of $U_{m+e} \hookrightarrow X_{m+e} \longrightarrow S_{m+e}$ over $S_{n'}$. Hence we have the isomorphism

$U_{n'} \overset{\sim}{=} V_{n'}$ between $U_{n'}$ and some open subscheme $V_{n'}$ of $X_{n'}$ locally on U. Then, locally on U, $V_{n'}$ admits a smooth lift $i_V : V_{n'} \hookrightarrow V$ over S and the isomorphism $U_{n'} \overset{\sim}{=} V_{n'}$ is liftable to an isomorphism $\varphi : U \longrightarrow V$ over S, as desired.

Taking $(V_{n'}, V, i_V)$ and $\varphi : U \longrightarrow V$ as in the previous paragraph, we define a $\mathcal{D}^{(-m)}_{U/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-module $r^{-1}(\mathcal{E})_U$ by $r^{-1}(\mathcal{E})_U := \varphi^*\mathcal{E}_V$. When there exists another object $(V'_{n'}, V', i_{V'})$ and another isomorphism $\varphi' : U \longrightarrow V'$, there exists an isomorphism $\iota : V \overset{=}{\longrightarrow} V'$ locally on V'. Then we have the isomorphism

$$\varphi^*\mathcal{E}_V \overset{=}{\longrightarrow} (\iota \circ \varphi)^*\mathcal{E}_{V'} \overset{\overline{\tau}_{\varphi', \iota\circ\varphi}}{\longrightarrow} \varphi'^*\mathcal{E}_{V'},$$

and this is indendent of the choice of ι because, when we are given another isomorphism $\iota' : V \overset{=}{\longrightarrow} V'$, we have the commutative diagram

$$\begin{array}{ccccc}
\varphi^*\mathcal{E}_V & \overset{=}{\longrightarrow} & (\iota \circ \varphi)^*\mathcal{E}_{V'} & \overset{\overline{\tau}_{\varphi', \iota\circ\varphi}}{\longrightarrow} & \varphi'^*\mathcal{E}_{V'} \\
\| & & \overset{\overline{\tau}_{\iota'\circ\varphi, \iota\circ\varphi}}{\downarrow} & & \| \\
\varphi^*\mathcal{E}_V & \overset{=}{\longrightarrow} & (\iota' \circ \varphi)^*\mathcal{E}_{V'} & \overset{\overline{\tau}_{\varphi', \iota'\circ\varphi}}{\longrightarrow} & \varphi'^*\mathcal{E}_{V'}.
\end{array}$$

Therefore, we can glue the local definition $r^{-1}(\mathcal{E})_U := \varphi^*\mathcal{E}_V$ and define the $\mathcal{D}^{(-m)}_{U/S} \otimes \mathbb{Z}/p^n\mathbb{Z}$-module $r^{-1}(\mathcal{E})_U$ globally. We can also check that the $r^{-1}(\mathcal{E})_U$'s for $(U_{m+e}, U, i_U) \in C(X_{m+e}/S)$ form an object $r^{-1}\mathcal{E} := ((r^{-1}\mathcal{E})_U)_U$ in $\overline{D}^{(-m)}(X_{m+e}/S)_n$ in the same way, and so we have defined the functor $r^{-1} : \overline{D}^{(m)}(X_{n'}/S)_n \longrightarrow \overline{D}^{(m)}(X_{m+e}/S)_n$, which is easily seen to be the inverse of the functor r. So we are done. □

Let us put $e = 1$ if $p \geq 3$ and $e = 2$ if $p = 2$. Let us take $n, m \in \mathbb{N}$ and assume given the following commutative diagram

$$\begin{array}{ccccc}
X_{m+e} & \longrightarrow & S_{m+e} & \overset{\subseteq}{\longrightarrow} & S \\
{\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\
Y_{m+e} & \longrightarrow & T_{m+e} & \overset{\subseteq}{\longrightarrow} & T,
\end{array} \tag{2.23}$$

where S, T are p-adic formal schemes flat over $\mathbb{Z}_p$, $S_{m+e} = S \otimes \mathbb{Z}/p^{m+e}\mathbb{Z}$, $T_{m+e} = T \otimes \mathbb{Z}/p^{m+e}\mathbb{Z}$, the right top arrow and the right bottom arrow are

canonical closed immersions, the left top arrow and the left bottom arrow are smooth. Under this situation, we can define the inverse image functor

$$f^* : \overline{D}^{(-m)}(Y_{m+e}/T)_n \longrightarrow \overline{D}^{(-m)}(X_{m+e}/S)_n \tag{2.24}$$

in the same way as the inverse image functor

$$f^* : D^{(-m)}(Y_{n'}/T)_n \longrightarrow D^{(-m)}(X_{n'}/S)_n \quad (n' \geq n) \tag{2.25}$$

defined before. Also, when $n' \geq \max(m+n, m+e)$ and the diagram (2.23) is liftable to the diagram

$$\begin{array}{ccccc} X_{n'} & \longrightarrow & S_{n'} & \xrightarrow{\subseteq} & S \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ Y_{n'} & \longrightarrow & T_{n'} & \xrightarrow{\subseteq} & T \end{array} \tag{2.26}$$

(where $S_{n'} = S \otimes \mathbb{Z}/p^{n'}\mathbb{Z}, T_{n'} = T \otimes \mathbb{Z}/p^{n'}\mathbb{Z}$, and the left top arrow and the left bottom arrow are smooth), we have the equality $R \circ (2.24) = (2.25) \circ R$. Also, when $e = 1$, we have the inverse image functor

$$f^{*,\mathrm{qn}} : \overline{D}^{(-m)}(Y_{m+e}/T)^{\mathrm{qn}}_n \longrightarrow \overline{D}^{(-m)}(X_{m+e}/S)^{\mathrm{qn}}_n \tag{2.27}$$

in the same way as the inverse image functor

$$f^{*,\mathrm{qn}} : D^{(-m)}(Y_{n'}/T)^{\mathrm{qn}}_n \longrightarrow D^{(-m)}(X_{n'}/S)^{\mathrm{qn}}_n \quad (n' \geq n) \tag{2.28}$$

defined before, and when $n' \geq \max(m+n, m+e)$ and the diagram (2.23) is liftable to the diagram (2.26), $R \circ (2.27) = (2.28) \circ R$. Hence we have the following corollary of Proposition 2.28, which is the first main result in this subsection.

COROLLARY 2.29.

(1) *Let the notations be as in Proposition 2.28*(1). *Then the category* $\mathrm{MIC}^{(m)}(X_n) = D^{(-m)}(X_{n'}/S)_n$ (*where* $X_n := X_{n'} \otimes \mathbb{Z}/p^n\mathbb{Z}$) *depends only on* $X_{m+e} \longrightarrow S_{m+e} \hookrightarrow S$ *and is functorial with respect to this diagram.*

(2) *Let the notations be as in Proposition 2.28*(2)*. Then the category* $\mathrm{MIC}^{(m)}(X_n)^{\mathrm{qn}} = D^{(-m)}(X_{n'}/S)_n^{\mathrm{qn}}$ *(where* $X_n := X_{n'} \otimes \mathbb{Z}/p^n\mathbb{Z}$*) depends only on the diagram* $X_{m+1} \longrightarrow S_{m+1} \hookrightarrow S$ *and is functorial with respect to this diagram.*

REMARK 2.30. Let S, n' be as above and put $S_j := S \otimes \mathbb{Z}/p^j\mathbb{Z}$ for $j \in \mathbb{N}$. Note that the above corollary does not imply that the category $\mathrm{MIC}^{(m)}(X_n)$ depends only on $X_{m+e} \longrightarrow S_{m+e} \hookrightarrow S$ for any smooth morphism $X_n \longrightarrow S_n$: The above corollary is applicable only for the smooth morphism $X_n \longrightarrow S_n$ which is liftable to a smooth morphism $X_{n'} \longrightarrow S_{n'}$.

Next we discuss the crystalline property of the level raising inverse image functor. To do so, we need the following proposition.

PROPOSITION 2.31. *Let the notations be as in* $\mathbf{Hyp}(\infty, \infty, \infty)$*. Let* $m, e \in \mathbb{N}, \geq 1$*, let* $\mathcal{E}$ *be a left* $\mathcal{D}^{(-m)}_{X^{(1)}/S}$*-module and assume one of the following*:

(a) $p \geq 3, e = 1$ *or* $p = 2, e = 2$.

(b) $\mathcal{E}$ *is quasi-nilpotent and* $e = 1$.

Suppose that we have another morphism $F' : X \longrightarrow X^{(1)}$ *over* S *lifting the morphism* F_{X_1/S_1} *which coincides with* F *modulo* p^{m+e}*. Then the isomorphism* $\overline{\tau}_{F,F'} : F'^*\mathcal{E} \xrightarrow{=} F^*\mathcal{E}$ *defined in Proposition 2.24 is actually* $\mathcal{D}^{(-m+1)}_{X/S}$*-linear.*

PROOF. Let $t_i, t_i', \tau_i, \tau_i'$ be as in the proof of Proposition 2.14. Then we can write $F^*(t_i') = t_i^p + pa_i, F'^*(t_i') = t_i^p + pa_i + p^{m+e}b_i$ for some $a_i, b_i \in \mathcal{O}_X$, and we see by the same calcuation as in the proof of Proposition 2.14 that there exist elements $\sigma_i \in I := \mathrm{Ker}(\mathcal{O}_{X^2} \to \mathcal{O}_X), \sigma_i' \in \mathcal{O}_{X^2}$ such that $(F \times F')^*(\tau_i') = \tau_i^p + p\sigma_i + p^{m+e}\sigma_i'$. So, for $m \geq 2$, the image of this element in $\mathcal{O}_{T_{X,(-m+1)}}$ belongs to $p^m\mathcal{O}_{T_{X,(-m+1)}}$ and in the case $m = 1$, the image of this element in $\mathcal{O}_{P_{X,(0)}}$ belongs to $p\mathcal{O}_{P_{X,(0)}}$. Therefore, in both cases, the image of this element in $\mathcal{O}_{P_{X,(-m+1)}}$ belongs to $p^m\mathcal{O}_{P_{X,(-m+1)}}$.

Now let us consider the morphism $h' := (F \times F, F' \times F') : X^2 \longrightarrow (X^{(1)})^4$. Then we have the commutative diagram

$$\begin{array}{ccc} X_{m+e} & \xrightarrow{F_{m+e}} & X^{(1)} \\ \downarrow & & \downarrow \\ X^2 & \xrightarrow{h'} & (X^{(1)})^4, \end{array} \tag{2.29}$$

where F_{m+e} is the composite $X_{m+e} \hookrightarrow X \xrightarrow{F} X^{(1)}$, which is also written as the composite $X_{m+e} \hookrightarrow X \xrightarrow{F'} X^{(1)}$. Let us denote the q-th projection $(X^{(1)})^4 \longrightarrow X^{(1)}$ by π_q $(0 \leq q \leq 3)$, $(q, q+1)$-th projection $(X^{(1)})^4 \longrightarrow (X^{(1)})^2$ by $\pi_{q,q+1}$ $(0 \leq q \leq 2)$ and put $\tau'_{i,q} := \pi^*_{q+1} t'_i - \pi^*_q t'_i = \pi^*_{q,q+1}(\tau'_i)$. Then $\mathrm{Ker}(\mathcal{O}_{(X^{(1)})^4} \longrightarrow \mathcal{O}_{X^{(1)}})$ is generated by $\tau'_{i,q}$'s $(1 \leq i \leq d, 0 \leq q \leq 3)$. If we denote the i-th projection $X^2 \longrightarrow X$ by $p_i\,(i = 0, 1)$, we have $h'^*(\tau_{i,0}) = h'^*(\pi^*_1 t'_i - \pi^*_0 t'_i) = p^*_0 F^* t'_i - p^*_0 F^* t'_i = 0$ and by similar reason, we also have $h'^*(\tau_{i,2}) = 0$. Also, we have

$$h'^*(\tau_{i,1}) = h'^*(\pi^*_2 t'_i - \pi^*_1 t'_i) = p^*_1 F'^* t'_i - p^*_0 F^* t'_i = (F \times F')^*(\tau'_i)$$

and the image of this element in $\mathcal{O}_{P_{X,(-m+1)}}$ belongs to $p^m \mathcal{O}_{P_{X,(-m+1)}}$. Hence, by the universality of formal blow-up, the morphism $P_{X,(-m+1)} \longrightarrow X^2 \xrightarrow{h'} (X^{(1)})^4$ factors as

$$P_{X,(-m+1)} \xrightarrow{h''} T_{X^{(1)},(-m)}(3) \longrightarrow (X^{(1)})^4.$$

Furthermore, since $\mathcal{O}_{T_{X^{(1)},(-m)}(3)}$ is locally topologically generated by the elements in $\mathcal{O}_{X^{(1)}}$ and the elements of the form $\tau'_{i,q}/p^m$, the commutative diagram (2.29) induces the commutative diagram

$$\begin{array}{ccc} X_e & \xrightarrow{F_e} & X^{(1)} \\ \downarrow & & \downarrow \\ P_{X,(-m+1)} & \xrightarrow{h''} & T_{X^{(1)},(-m)}(3), \end{array}$$

where $X_e = X \otimes \mathbb{Z}/p^e\mathbb{Z}$ and F_e is the composite $X_e \hookrightarrow X_{m+e} \xrightarrow{F_{m+e}} X^{(1)}$. Noting that the defining ideal of the closed immersion $X_e \hookrightarrow P_{X,(-m+1)}$

admits a PD-structure canonically, we see that the above diagram gives rise to the morphism $h : P^l_{X,(-m+1)} \longrightarrow P_{Y,(-m)}(3)$ for any $l \in \mathbb{N}$. Then, in the case (b), we can prove the commutativity of the diagram (2.17) on $P^l_{X,(-m+1)}$ by using the morphism h, in the same way as the proof of Proposition 2.24. In the case (a), we see that the morphism h factors as $P^l_{X,(-m+1)} \xrightarrow{h_s} P^s_{Y,(-m)}(3) \longrightarrow P_{Y,(-m)}(3)$ for some $s \in \mathbb{N}, s \geq l$ because the the PD-structure on the ideal of the defining ideal of $X_e \hookrightarrow P^l_{X,(-m+1)}$ is topologically PD-nilpotent, and then we can prove the commutativity of the diagram (2.17) on $P^l_{X,(-m+1)}$ by using h_s. So we are done. □

Again by the argument in [2, 2.1.6] (see also [2, 2.2.6]), we have the following immediate corollary of Proposition 2.31 (we omit the proof):

COROLLARY 2.32. *Let the notations be as in* $\mathbf{Hyp}(\infty, \infty, \infty)$.

(1) *Let us put* $e = 1$ *if* $p \geq 3$ *and* $e = 2$ *if* $p = 2$. *Then the level raising inverse image functor*

$$F^* : (\text{left } \mathcal{D}^{(-m)}_{X^{(1)}/S}\text{-modules}) \longrightarrow (\text{left } \mathcal{D}^{(-m+1)}_{X/S}\text{-modules}),$$

which is equal to the level raising inverse image functor $F^* : \mathrm{MIC}^{(m)}(X^{(1)}) \longrightarrow \mathrm{MIC}^{(-m+1)}(X)$, *depends only on* $F_{m+e} := F \bmod p^{m+e}$.

(2) *The level raising inverse image functor*

$$F^{*,\mathrm{qn}} : (\text{quasi-nilpotent left } \mathcal{D}^{(-m)}_{X^{(1)}/S}\text{-modules}) \longrightarrow (\text{quasi-nilpitent left } \mathcal{D}^{(-m+1)}_{X/S}\text{-modules}),$$

which is equal to the level raising inverse image functor $F^{*,\mathrm{qn}} : \mathrm{MIC}^{(m)}(X^{(1)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}^{(-m+1)}(X)^{\mathrm{qn}}$, *depends only on* $F_{m+1} := F \bmod p^{m+1}$.

Next we consider the case of p^n-torsion objects. Let $m \in \mathbb{N}, \geq 1$, assume that we are in the situation of $\mathbf{Hyp}(\infty, m+e, m+e)$ with $e = 1$ if $p \geq 3$ and $e = 2$ if $p = 2$, and let us take $n \geq 1$. Then we can define the level raising inverse image functor

$$F^*_{m+e} : \overline{D}^{(-m)}(X^{(1)}_{m+e}/T)_n \longrightarrow \overline{D}^{(-m+1)}(X_{m+e}/S)_n \tag{2.30}$$

in the same way as the level raising inverse image functor

$$F^*_{n'} : D^{(-m)}(X^{(1)}_{n'}/T)_n \longrightarrow D^{(-m+1)}(X_{n'}/S)_n \quad (n' \geq n+1) \tag{2.31}$$

defined before. Also, when $n' \geq \max(m+n, m+e)$ and when we are in the situation of $\mathbf{Hyp}(\infty, n', n')$, we have the equality $R \circ (2.30) = (2.31) \circ R$. Also, when $e = 1$, we have the inverse image functor

$$F^{*,\mathrm{qn}}_{m+e} : \overline{D}^{(-m)}(X^{(1)}_{m+e}/T)^{\mathrm{qn}}_n \longrightarrow \overline{D}^{(-m+1)}(X_{m+e}/S)^{\mathrm{qn}}_n \tag{2.32}$$

in the same way as the inverse image functor

$$F^{*,\mathrm{qn}}_{n'} : D^{(-m)}(X^{(1)}_{n'}/T)^{\mathrm{qn}}_n \longrightarrow D^{(-m)}(X_{n'}/S)^{\mathrm{qn}}_n \quad (n' \geq n+1) \tag{2.33}$$

defined before, and when $n' \geq \max(m+n, m+e)$ and when we are in the situation of $\mathbf{Hyp}(\infty, n', n')$, we have the equality $R \circ (2.32) = (2.33) \circ R$. Hence we have the following corollary, which is the second main result in this subsection.

COROLLARY 2.33.

(1) *Let $m \geq 1$ and let us put $e = 1$ if $p \geq 3$, $e = 2$ if $p = 2$. Then, under $\mathbf{Hyp}(\infty, n', n')$ with $n' \geq \max(m+n, m+e)$, the level raising inverse image functor*

$$F^*_{n'} : D^{(-m)}(X^{(1)}_{n'}/S)_n \longrightarrow D^{(-m+1)}(X_{n'}/S)_n$$

(which is equal to the level raising inverse image functor $F^*_{n+1} : \mathrm{MIC}^{(m)}(X^{(1)}_n/S) \longrightarrow \mathrm{MIC}^{(m-1)}(X_n/S)$) depends only on $F_{m+e} = F_{n'} \bmod p^{m+e}$.

(2) *Let $m \geq 1$. Then, under $\mathbf{Hyp}(\infty, n', n')$ with $n' \geq m+n$, the level raising inverse image functor*

$$F^{*,\mathrm{qn}}_{n'} : D^{(-m)}(X^{(1)}_{n'}/S)^{\mathrm{qn}}_n \longrightarrow D^{(-m+1)}(X_{n'}/S)^{\mathrm{qn}}_n$$

(which is equal to the level raising inverse image functor $F^{*,\mathrm{qn}}_{n+1} : \mathrm{MIC}^{(m)}(X^{(1)}_n/S)^{\mathrm{qn}} \longrightarrow \mathrm{MIC}^{(m-1)}(X_n/S)^{\mathrm{qn}}$) depends only on $F_{m+1} = F_{n'} \bmod p^{m+1}$.

REMARK 2.34. Let m, e, n' be as above. Note that the above corollary does not imply that, under the situation $\mathbf{Hyp}(\infty, n+1, n+1)$, the level raising inverse image functor

$$F_{n+1}^* : \mathrm{MIC}^{(m)}(X_n^{(1)}/S) \longrightarrow \mathrm{MIC}^{(m-1)}(X_n/S))$$
$$(\text{resp. } F_{n+1}^{*,\mathrm{qn}} : \mathrm{MIC}^{(m)}(X_n^{(1)}/S)^{\mathrm{qn}} \longrightarrow \mathrm{MIC}^{(m-1)}(X_n/S)^{\mathrm{qn}})$$

depends only on F_{m+e} (resp. F_{m+1}): The above corollary is applicable only in the situation $\mathbf{Hyp}(\infty, n', n')$.

3. Frobenius Descent to the Level Minus One

In this section, we prove that the level raising inverse image functor for relative Frobenius gives an equivalence between the category of modules with quasi-nilpotent integrable p-connection and the category of modules with quasi-nilpotent integrable connection. In terms of $\mathcal{D}$-modules, this is an equivalence of the category of quasi-nilpotent left $\mathcal{D}$-modules of level -1 and the the category of quasi-nilpotent left $\mathcal{D}$-modules of level 0. So we can say this result as 'the Frobenius descent to the level -1'. The method of the proof is similar to the proof of Frobenius descent due to Berthelot [2].

The main result in this section is the following:

THEOREM 3.1 (Frobenius descent to the level minus one). *Assume that we are in the situation of* $\mathbf{Hyp}(\infty, \infty, \infty)$. *Then the level raising inverse image functor*

$$F^* : \mathrm{MIC}^{(1)}(X^{(1)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}(X)^{\mathrm{qn}}$$

is an equivalence of categories.

We have the following immediate corollaries:

COROLLARY 3.2.

(1) *Assume that we are in the situation of* $\mathbf{Hyp}(\infty, n+1, n+1)$. *Then the level raising inverse image functors*

$$F_{n+1}^* : \mathrm{MIC}^{(1)}(X_n^{(1)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}(X_n)^{\mathrm{qn}}$$
$$F_{n+1}^* : D^{(1)}(X_{n+1}^{(1)})_n^{\mathrm{qn}} \longrightarrow D^{(0)}(X_{n+1})_n^{\mathrm{qn}}$$

are equivalences of categories.

(2) *Assume that we are in the situation of* $\mathbf{Hyp}(\infty,2,2)$. *Then, for* $n \in \mathbb{N}$, *the level raising inverse image functor*

$$F_2^* : \overline{D}^{(1)}(X_2^{(1)})_n^{\mathrm{qn}} \longrightarrow \overline{D}^{(0)}(X_2)_n^{\mathrm{qn}}$$

is an equivalence of categories.

PROOF. Since all the categories appearing in the statement satisfy the descent property for the Zariski topology, we may work Zariski locally. Then we can assume that we are in the situation of $\mathbf{Hyp}(\infty,\infty,\infty)$, and in this case, the level raising inverse image functors are interpreted as the p^n-torsion part of the level raising inverse image functor

$$F^* : \mathrm{MIC}^{(1)}(X^{(1)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}(X)^{\mathrm{qn}}$$

in Theorem 3.1. So the corollary follows from Theorem 3.1. □

Note that this gives a possible answer to Question 1.18. We prove several lemmas to prove Theorem 3.1.

LEMMA 3.3. *Let the notations be as in* $\mathbf{Hyp}(\infty,\infty,\infty)$ *and assume that the relative dimension of* $f : X \longrightarrow S$ *is equal to* d. *Then the morphism* $\Phi : P_{X/S,(0)} \longrightarrow P_{X^{(1)}/S,(-1)}$ *defined in Proposition 2.14 is a finite flat morphism of degree* p^{2d}.

PROOF. It suffices to prove that the morphism $P_{X/S,(0)} \longrightarrow X \times_{X^{(1)}} P_{X^{(1)}/S,(-1)}$ induced by Φ is a finite flat morphism of degree p^d. To show this, we may work locally. So we can take a local coordinate $t_1,...,t_d$ of X over S and a local coordinate $t'_1,...,t'_d$ of $X^{(1)}$ over S such that $F^*(t'_i) = t_i^p + pa_i$ for some $a_i \in \mathcal{O}_X$ $(1 \leq i \leq d)$. Let us put $\tau_i := 1 \otimes t_i - t_i \otimes 1 \in \mathcal{O}_{X^2}, \tau'_i := 1 \otimes t'_i - t'_i \otimes 1 \in \mathcal{O}_{(X^{(1)})^2}$. The homomorphism of sheaves corresponding to the morphism $P_{X/S,(0)} \longrightarrow X \times_{X^{(1)}} P_{X^{(1)}/S,(-1)}$ has the form

$$\mathcal{O}_X\langle \tau'_i/p\rangle_{1\leq i\leq d} \longrightarrow \mathcal{O}_X\langle \tau_i\rangle_{1\leq i\leq d}. \tag{3.1}$$

Since the morphism $F^2 : X^2 \longrightarrow (X^{(1)})^2$ sends τ'_i to

$$\begin{aligned} 1 \otimes (t_i^p + pa_i) - (t_i^p + pa_i) \otimes 1 &= (\tau_i + t_i \otimes 1)^p - t_i^p \otimes 1 + p(1 \otimes a_i - a_i \otimes 1) \\ &= \tau_i^p + \sum_{k=1}^{p-1} \binom{p}{k} t_i^{p-k} \tau_i^k + p(1 \otimes a_i - a_i \otimes 1) \end{aligned}$$

and the morphism Φ is induced by F^2, it follows that τ_i'/p is sent by the morphism (3.1) to the element $\sum_{k=1}^{p-1} p^{-1}\binom{p}{k} t_i^{p-k}\tau_i^k + \tau_i^{[p]} + \sum_{k\in\mathbb{N}^d, k\neq 0} \partial^{\langle k\rangle_0}(a_i)\tau^{[k]}$. For $l \in \mathbb{N}$, let us put $I_l := \{k = (k_i)_i \in \mathbb{N}^d \,|\, k \neq 0, \forall i, k_i < p^{l+1}\}$. Then, since we have $\partial^{\langle k\rangle_0}(a_i) \in k!\mathcal{O}_X \subseteq p\mathcal{O}_X$ for $k \in \mathbb{N}^d \setminus (I_0 \cup \{0\})$, we see that τ_i'/p is sent by the morphism (3.1) to the element of the form $\tau_i^{[p]} + \sum_{k\in I_0} u_{i,k}\tau^k + pv_i$ for some $u_{i,k} \in \mathcal{O}_X, v_i \in \mathcal{O}_X\langle\tau_i\rangle_{1\leq i\leq d}$. Hence, for $l \in \mathbb{N}$, $(\tau_i'/p)^{[p^l]}$ is sent by the morphism (3.1) to an element of the form $\tau_i^{[p^{l+1}]} + \sum_{k\in I_l} u_{i,l,k}\tau^{[k]} + pv_{i,l}$ for some $u_{i,l,k} \in \mathcal{O}_X, v_{i,l} \in \mathcal{O}_X\langle\tau_i\rangle_{1\leq i\leq d}$.

To prove the lemma, we may assume that X is affine and it suffices to prove that the morphism (3.1) modulo p is finite flat of degree p^d. Let us put $A := \mathcal{O}_X/p\mathcal{O}_X$. Then the morphism (3.1) modulo p has the form

$$A[x_{i,l}]_{1\leq i\leq d, l\geq 0}/(x_{i,l}^p)_{i,l} \longrightarrow A[y_{i,l}]_{1\leq i\leq d, l\geq 0}/(y_{i,l}^p)_{i,l} \tag{3.2}$$

($x_{i,l}$ corresponds to the element $(\tau_i'/p)^{[p^l]} \bmod p$ and $y_{i,l}$ corresponds to the element $\tau^{[p^l]} \bmod p$) and $x_{i,l}$ is sent to an element of the form $y_{i,l+1} + a_{i,l}$, where $a_{i,l}$ is an element in $A[y_{i,l'}]_{1\leq i\leq d, 0\leq l'\leq l}/(y_{i,l'}^p)_{i,l'}$ with no degree 0 part. (Here the degree is taken with respect to $y_{i,k}$'s.) Let us denote the degree 1 part of $a_{i,l}$ by $b_{i,l}$. Let us consider the A-algebra homomorphism

$$\alpha : A[z_{i,l}]_{1\leq i\leq d, l\geq -1}/(z_{i,l}^p)_{i,l} \longrightarrow A[y_{i,l}]_{1\leq i\leq d, l\geq 0}/(y_{i,l}^p)_{i,l}$$

defined by $\alpha(z_{i,-1}) = y_{i,0}, \alpha(z_{i,l}) = y_{i,l+1} + b_{i,l}\,(l \geq 0)$. Let us define the A-algebra homomorphism

$$\beta : A[y_{i,l}]_{1\leq i\leq d, l\geq 0}/(y_{i,l}^p)_{i,l} \longrightarrow A[z_{i,l}]_{1\leq i\leq d, l\geq -1}/(z_{i,l}^p)_{i,l}$$

of the converse direction inductively, in the following way: First, let us define $\beta(y_{i,0}) := z_{i,-1}$. When we defined $\beta(y_{i,l'})$ for $0 \leq l' \leq l$, we can define $\beta(b_{i,l})$ since $b_{i,l}$ is a linear form in $y_{i,l'}$'s for $1 \leq i \leq d, 0 \leq l' \leq l$. Then we define $\beta(y_{i,l+1}) := z_{i,l} - \beta(b_{i,l})$. Then β is well-defined, and it is easy to see that α and β are the inverse of each other. Hence β is an isomorphism, and so it suffices to prove that the composite $\beta \circ (3.2)$ is finite flat of degree p^d. Notice that we can factorize $\beta \circ (3.2)$ as

$$\begin{aligned} A[x_{i,l}]_{1\leq i\leq d, l\geq 0}/(x_{i,l}^p)_{i,l} &\xrightarrow{\subset} A[x_{i,l}]_{1\leq i\leq d, l\geq -1}/(x_{i,l}^p)_{i,l} \\ &\longrightarrow A[z_{i,l}]_{1\leq i\leq d, l\geq -1}/(z_{i,l}^p)_{i,l} \end{aligned} \tag{3.3}$$

by introducing new variables $x_{i,-1}\,(1 \leq i \leq d)$ and by sending them to $z_{i,-1}\,(1 \leq i \leq d)$. Also, noting the fact that β sends $y_{i,l}$ to linear forms in $z_{i,l'}$'s $(-1 \leq l' \leq l)$, we see that $x_{i,l}\,(l \geq 0)$ is sent by $\beta \circ (3.2)$ (thus by the second homomorphism of (3.3)) to an element of the form $z_{i,l} + c_{i,l}$, where $c_{i,l}$ is an element in $A[z_{i,l'}]_{1 \leq i \leq d, -1 \leq l' \leq l}/(z_{i,l'}^p)_{i,l'}$ whose degree 0 part and degree 1 part are zero. From this expression, we see easily that the second homomorphism in (3.3) is an isomorphism. On the other hand, it is clear that the first homomorphism in (3.3) is finite flat of degree p^d. So we see that $\beta \circ (3.2)$ is finite flat of degree p^d and so the proof of the lemma is finished. □

LEMMA 3.4. *Let the notations be as in* $\mathbf{Hyp}(\infty, \infty, \infty)$. *Then the defining ideal of the diagonal morphism* $X \hookrightarrow X \times_{X^{(1)}} X$ *admits a unique PD-structure. So the closed immersion* $X \times_{X^{(1)}} X \hookrightarrow X \times_S X$ *induces the PD-morphism* $X \times_{X^{(1)}} X \longrightarrow P_{X,(0)}$.

PROOF. We have the equality $\mathcal{O}_{X \times_{X^{(1)}} X} = \mathcal{O}_{X \times_S X}/J$ (where J is the ideal topologically generated by the elements $1 \otimes F^*(y) - F^*(y) \otimes 1\,(y \in \mathcal{O}_{X^{(1)}})$), and the kernel I of $\mathcal{O}_{X \times_{X^{(1)}} X} \longrightarrow \mathcal{O}_X$ is topologically generated by the elements $1 \otimes x - x \otimes 1\,(x \in \mathcal{O}_X)$. For $x \in \mathcal{O}_X$, let us take a lift $x' \in \mathcal{O}_{X^{(1)}}$ of $1 \otimes x \in \mathcal{O}_{X_1^{(1)}}$ and put $F^*(x') = x^p + pz$. Then we have

$$\begin{aligned}
0 &= 1 \otimes F^*(x') - F^*(x') \otimes 1 \\
&= 1 \otimes x^p - x^p \otimes 1 + p(1 \otimes z - z \otimes 1) \\
&= ((1 \otimes x - x \otimes 1) + x \otimes 1)^p - x^p \otimes 1 + p(1 \otimes z - z \otimes 1) \\
&= (1 \otimes x - x \otimes 1)^p \\
&\quad + \sum_{i=1}^{p-1} \binom{p}{i} (x^{p-i} \otimes 1)(1 \otimes x - x \otimes 1)^i + p(1 \otimes z - z \otimes 1)
\end{aligned}$$

in $\mathcal{O}_{X \times_{X^{(1)}} X}$. So, in $\mathcal{O}_{X \times_{X^{(1)}} X}$, $(1 \otimes x - x \otimes 1)^p$ has the form

$$pa(1 \otimes x - x \otimes 1) + p(1 \otimes z - z \otimes 1) \tag{3.4}$$

for some $a \in \mathcal{O}_{X \times_{X^{(1)}} X}$ and $z \in \mathcal{O}_X$.

We should prove that the ideal I admits a unique PD-structure. Since $\mathcal{O}_{X \times_{X^{(1)}} X}$ is a flat $\mathbb{Z}_p$-algebra, it suffices to prove that, for any $x \in \mathcal{O}_X$ and

$k \in \mathbb{N}$, $(1 \otimes x - x \otimes 1)^k \in k!\mathcal{O}_{X \times_{X^{(1)}} X}$. First we prove it for $k = p^l$ ($l \in \mathbb{N}$), by induction on l. In the case $l = 0$, it is trivially true. In general, we have

$$\begin{aligned}
&(1 \otimes x - x \otimes 1)^{p^l} \\
&= ((1 \otimes x - x \otimes 1)^p)^{p^{l-1}} \\
&= (pa(1 \otimes x - x \otimes 1) + p(1 \otimes z - z \otimes 1))^{p^{l-1}} \qquad \text{(by (3.4))} \\
&= \sum_{i=0}^{p^{l-1}} p^{p^{l-1}} \binom{p^{l-1}}{i} a^{p^{l-1}-i}(1 \otimes x - x \otimes 1)^{p^{l-1}-i}(1 \otimes z - z \otimes 1)^i,
\end{aligned}$$

and by induction hypothesis, the i-th term is contained in

$$p^{p^{l-1}} \binom{p^{l-1}}{i} (p^{l-1} - i)!i!\mathcal{O}_{X \times_{X^{(1)}} X} = p^{p^{l-1}}(p^{l-1})!\mathcal{O}_{X \times_{X^{(1)}} X} = p^l!\mathcal{O}_{X \times_{X^{(1)}} X}.$$

So we have $(1 \otimes x - x \otimes 1)^{p^l} \in p^l!\mathcal{O}_{X \times_{X^{(1)}} X}$, as desired. For $k \in \mathbb{N}$ with $k \neq p^l$ ($\forall l \in \mathbb{N}$), let us take the maximal integer l' such that k is divisible by $p^{l'}$. Then we have $(1 \otimes x - x \otimes 1)^k = (1 \otimes x - x \otimes 1)^{p^{l'}} (1 \otimes x - x \otimes 1)^{k-p^{l'}}$, and it is contained in $p^{l'}!(k-p^{l'})!\mathcal{O}_{X \times_{X^{(1)}} X} = k!\mathcal{O}_{X \times_{X^{(1)}} X}$ by the result for $p^{l'}$ and the induction hypothesis. So we are done. □

Remark 3.5. By the same argument, we see the following: For $d \in \mathbb{N}$, the kernel I_1 of the homomorphism $\mathcal{O}_{X \times_{X^{(1)}} X}\langle y_1, ..., y_d\rangle \longrightarrow \mathcal{O}_X\langle y_1, ..., y_d\rangle$ (induced by the diagonal morphism) admits a unique PD-structure. Moreover, if we put

$$\begin{aligned}
I_2 &:= \mathrm{Ker}(\mathcal{O}_{X \times_{X^{(1)}} X}\langle y_1, ..., y_d\rangle \longrightarrow \mathcal{O}_{X \times_{X^{(1)}} X}), \\
I &:= I_1 + I_2 = \mathrm{Ker}(\mathcal{O}_{X \times_{X^{(1)}} X}\langle y_1, ..., y_d\rangle \longrightarrow \mathcal{O}_X),
\end{aligned}$$

we see that I admits a unique PD-structure compatible with that on I_1 and that on I_2, since, for $x \in I_1 \cap I_2$, we have $n!x^{[n]} = x^n$ for both PD-structures.

Remark 3.6. If we put $X(r)$ to be the $(r+1)$-fold fiber product of X over $X^{(1)}$, we see in the same way as the proof given above that the defining ideal of the diagonal morphism $X \hookrightarrow X(r)$ admits a unique PD-structure. So the closed immersion $X(r) \hookrightarrow X^{r+1}$ induces the PD-morphism $X(r) \longrightarrow P_{X,(0)}(r)$.

LEMMA 3.7. *Let the situation be as in* $\mathbf{Hyp}(\infty,\infty,\infty)$. *Then the defining ideal of the diagonal closed immersion* $X \hookrightarrow P_{X,(0)} \times_{P_{X^{(1)},(-1)}} P_{X,(0)}$ *admits a unique PD-structure.*

PROOF. The uniqueness follows from the fact that $P_{X,(0)} \times_{P_{X^{(1)},(-1)}} P_{X,(0)}$, being flat over $P_{X,(0)}$ by Lemma 3.3, is flat over $\mathbb{Z}_p$. Let us prove the existence of the desired PD-structure. Since we have the uniqueness, we may work locally. So we have the isomorphisms

$$\begin{aligned}\mathcal{O}_{P_{X,(0)}\times_{P_{X^{(1)},(-1)}}P_{X,(0)}} &\cong \mathcal{O}_X\langle\tau_i\rangle_i \otimes_{\mathcal{O}_{X^{(1)}}\langle\tau_i'/p\rangle_i} \mathcal{O}_X\langle\tau_i\rangle_i \\ &\cong (\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{O}_X)\langle\tau_{i,0},\tau_{i,1}\rangle_i/I\end{aligned} \tag{3.5}$$

(where $\tau_{i,0} := \tau_i \otimes 1, \tau_{i,1} := 1 \otimes \tau_i$), where I is the closed ideal topologically generated by

$$\Phi_1^*(\tau_i'/p)^{[k]} - \Phi_0^*(\tau_i'/p)^{[k]} \quad (1 \le i \le d, k \in \mathbb{N}), \tag{3.6}$$

and $\Phi_j^*\,(j = 0, 1)$ is the homomorphism

$$\begin{aligned}\mathcal{O}_{X^{(1)}}\langle\tau_i'/p\rangle_i = \mathcal{P}_{X^{(1)},(-1)} &\xrightarrow{\Phi^*} \mathcal{P}_{X,(0)} \\ &\overset{j\text{-th incl.}}{\longrightarrow} \mathcal{P}_{X,(0)} \otimes_{\mathcal{O}_X} \mathcal{P}_{X,(0)} \\ &= (\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{O}_X)\langle\tau_{i,0},\tau_{i,1}\rangle_i.\end{aligned}$$

First, let us note that, by Remark 3.5, the ideal

$$\begin{aligned}&\mathrm{Ker}((\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{O}_X)\langle\tau_{i,0},\tau_{i,1}\rangle_i \longrightarrow \mathcal{O}_X) \\ =&\,\mathrm{Ker}((\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{O}_X)\langle\tau_{i,0},\tau_{i,1}\rangle_i \longrightarrow \mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{O}_X) \\ &+ \mathrm{Ker}((\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{O}_X)\langle\tau_{i,0},\tau_{i,1}\rangle_i \longrightarrow \mathcal{O}_X\langle\tau_{i,0},\tau_{i,1}\rangle_i)\end{aligned}$$

admits the PD-structure compatible with the canonical PD-structure on the ideal $\mathrm{Ker}((\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{O}_X)\langle\tau_{i,0},\tau_{i,1}\rangle_i \longrightarrow \mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{O}_X)$. So it suffices to prove that the ideal I is a PD-subideal of $\mathrm{Ker}((\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{O}_X)\langle\tau_{i,0},\tau_{i,1}\rangle_i \longrightarrow \mathcal{O}_X)$ to prove the lemma. If we denote the element (3.6) simply by $a - b$, it suffices to prove that $(a-b)^{[l]} \in I$ for any $l \ge 1$. If we take the maximal integer m such that l is divisible by p^m, we have $(a-b)^{[l]} = \binom{l}{p^m}(a -$

$b)^{[l-p^m]}(a-b)^{[p^m]}$. So it suffices to prove that $(a-b)^{[p^m]} \in I$ for $m \geq 0$. To prove it, it suffices to prove the following claim: For $m \geq 0$, there exist elements $c_j \in \mathcal{O}_X\langle \tau_{i,0}, \tau_{i,1}\rangle_i \, (0 \leq j \leq m)$ with $(a-b)^{[p^m]} = \sum_{j=0}^{m} c_j(a^{[p^j]} - b^{[p^j]})$, because $a^{[p^j]} - b^{[p^j]} \in I$. We prove this claim by induction on m. When $m = 0$, the claim is trivially true. Assume that the claim is true for $m-1$ and put $(a-b)^{[p^{m-1}]} = \sum_{j=0}^{m-1} c_j(a^{[p^j]} - b^{[p^j]})$. Then we have

$$\begin{aligned}(a-b)^{[p^m]} &= \frac{(p^{m-1}!)^p p!}{p^m!}((a-b)^{[p^{m-1}]})^{[p]} \\ &= \frac{(p^{m-1}!)^p p!}{p^m!}(\sum_{j=0}^{m-1} c_j(a^{[p^j]} - b^{[p^j]}))^{[p]} \\ &= \frac{(p^{m-1}!)^p p!}{p^m!}\left(\sum_{j=0}^{m-1} c_j^p(a^{[p^j]} - b^{[p^j]})^{[p]} + A\right)\end{aligned}$$

for some A of the form $\sum_{j=0}^{m-1} d_j(a^{[p^j]} - b^{[p^j]})$. Moreover, we have

$$(a^{[p^j]} - b^{[p^j]})^{[p]} = \frac{p^{j+1}!}{(p^j!)^p p!}(a^{[p^{j+1}]} - b^{[p^{j+1}]}) + \sum_{s=1}^{p-1} \frac{1}{s!(p-s)!}(a^{[p^j]})^{p-s}(-b^{[p^j]})^s$$

in the case $p \geq 3$, and it is easy to see that the second term on the right hand side is a multiple of $a^{[p^j]} - b^{[p^j]}$. Hence the proof is finished in the case $p \geq 3$. The case $p = 2$ follows from the equality

$$\begin{aligned}(a^{[2^j]} - b^{[2^j]})^{[2]} &= ua^{[2^{j+1}]} - a^{[2^j]}b^{[2^j]} + ub^{[2^{j+1}]} \\ &= u(a^{[2^{j+1}]} - b^{[2^{j+1}]}) + 2ub^{[2^{j+1}]} - a^{[2^j]}b^{[2^j]} \\ &= u(a^{[2^{j+1}]} - b^{[2^{j+1}]}) - b^{[2^j]}(a^{[2^j]} - b^{[2^j]}),\end{aligned}$$

where $u = \dfrac{2^{j+1}!}{(2^j)!2}$. So we are done. □

Now we are ready to prove Theorem 3.1. The proof is similar to that of [2, 2.3.6].

Proof of Theorem 3.1. In the proof, we freely regard an object in $\mathrm{MIC}^{(1)}(X^{(1)})^{\mathrm{qn}}$ (resp. $\mathrm{MIC}(X)^{\mathrm{qn}}$) as a quasi-nilpotent left $\mathcal{D}^{(-1)}_{X^{(1)}/S}$-module (resp. $\mathcal{D}^{(0)}_{X/S}$-module) or a p-power torsion module with (-1)-HPD-stratification on $X^{(1)}$ (resp. 0-HPD-stratification on X).

Since $F : X \longrightarrow X^{(1)}$ is finite flat, the functor F^* is faithful. Let us prove that F^* is full. Let $\Phi : P_{X,(0)} \longrightarrow P_{X^{(1)},(-1)}$ be the morphism defined in Proposition 2.14, let $u : X \times_{X^{(1)}} X \longrightarrow P_{X,(0)}$ be the morphism defined in Lemma 3.4 and let $\overline{p}_j : X \times_{X^{(1)}} X \longrightarrow X\,(j = 0, 1)$ be the j-th projection. Let us take an object $(\mathcal{E}', \epsilon') \in \mathrm{MIC}^{(1)}(X^{(1)})^{\mathrm{qn}}$ and let us put $(\mathcal{E}, \epsilon) := F^*(\mathcal{E}', \epsilon') = (F^*\mathcal{E}', \Phi^*\epsilon') \in \mathrm{MIC}(X)^{\mathrm{qn}}$. Then $u^*\epsilon = u^*\Phi^*\epsilon'$ is an isomorphism $\overline{p}_1^*\mathcal{E} \overset{=}{\longrightarrow} \overline{p}_0^*\mathcal{E}$, and by using Remarks 2.15 and 3.6, we see that it satisfies the cocycle condition on $X \times_{X^{(1)}} X \times_{X^{(1)}} X$. So $(\mathcal{E}, u^*\epsilon)$ is a descent data on X relative to $X^{(1)}$. If we take a local coordinate $t_1, ..., t_d$ of X over S and a local coordinate $t_1, ..., t_d$ of $X^{(1)}$ over S with $F^*(t_i') = t_i^p + pa_i\,(1 \leq i \leq d)$ and if we put $\tau_i := 1 \otimes t_i - t_i \otimes 1 \in \mathcal{O}_{X^2}, \tau_i' := 1 \otimes t_i' - t_i' \otimes 1 \in \mathcal{O}_{(X^{(1)})^2}$, the PD-homomorphism of sheaves

$$(\Phi \circ u)^* : \mathcal{O}_{P_{X^{(1)},(-1)}} \longrightarrow \mathcal{O}_{P_{X,(0)}} \longrightarrow \mathcal{O}_{X \times_{X^{(1)}} X}$$

associated to $\Phi \circ u$ sends τ_i'/p as

$$\begin{aligned}
\tau_i'/p &\mapsto \tau_i^{[p]} + p^{-1} \sum_{j=1}^{p-1} \binom{p}{j} t_i^{p-j} \tau_i^j + (1 \otimes a_i - a_i \otimes 1) \\
&\mapsto -p^{-1} \sum_{j=1}^{p-1} \binom{p}{j} t_i^{p-j} \tau_i^j - (1 \otimes a_i - a_i \otimes 1) \\
&\qquad + p^{-1} \sum_{j=1}^{p-1} \binom{p}{j} t_i^{p-j} \tau_i^j + (1 \otimes a_i - a_i \otimes 1) = 0.
\end{aligned}$$

So $\Phi \circ u$ factors through $X^{(1)}$. Hence $u^*\epsilon$ is the pull-back of the identity map on $\mathcal{E}'$ by $X \times_{X^{(1)}} X \longrightarrow X^{(1)}$, that is, the descent data $(\mathcal{E}, u^*\epsilon)$ is equal to the one coming canonically from $\mathcal{E}'$.

Now let us take $(\mathcal{E}', \epsilon'), (\mathcal{F}', \eta') \in \mathrm{MIC}^{(1)}(X_n^{(1)})^{\mathrm{qn}}$, put $(\mathcal{E}, \epsilon) := F^*(\mathcal{E}', \epsilon'), (\mathcal{F}, \eta) := F^*(\mathcal{F}', \eta) \in \mathrm{MIC}(X_n)^{\mathrm{qn}}$ and assume that we are given a morphism $\varphi : (\mathcal{E}, \epsilon) \longrightarrow (\mathcal{F}, \eta)$. For an open subscheme U of X, let us put $U^{(1)} := U \times_X X^{(1)}$, $A := \Gamma(U, \mathcal{O}_X), A' := \Gamma(U^{(1)}, \mathcal{O}_{X^{(1)}}), E := \Gamma(U, \mathcal{E}), E' := \Gamma(U^{(1)}, \mathcal{E}'), F := \Gamma(U, \mathcal{F}), F' := \Gamma(U^{(1)}, \mathcal{F}')$. Then we have $A \otimes_{A'} E' = E, A \otimes_{A'} F' = F$ and by the argument in the previous paragraph, E, F are naturally endowed with the descent data relative to A' coming canonically from E', F' and the morphism $\Gamma(U, \varphi) : E \longrightarrow F$ is a

morphism of descent data. Hence it descends to a morphism $\psi_U : E' \longrightarrow F'$. By letting U vary, we see that $\{\psi_U\}_U$ defines a morphism $\psi : \mathcal{E}' \longrightarrow \mathcal{F}'$ with $F^*(\psi) = \varphi$. To prove that ψ induces a morphism $(\mathcal{E}', \epsilon') \longrightarrow (\mathcal{F}', \eta')$, we should prove the compatibility of ψ with ϵ', η'. Since Φ is finite flat, it suffices to prove the compatibility of $F^*\psi = \varphi$ with $\Phi^*\epsilon' = \epsilon, \Phi^*\eta' = \eta$ and it follows from definition. So ψ is a morphism in $\mathrm{MIC}^{(1)}(X^{(1)})^{\mathrm{qn}}$ with $F^*\psi = \varphi$ and so the functor F^* is full, as desired.

We prove that the functor F^* is essentially surjective. Let us take $(\mathcal{E}, \epsilon) \in \mathrm{MIC}(X)^{\mathrm{qn}}$. Then, as we saw above, $u^*\epsilon$ defines a descent data on $\mathcal{E}$ relative to $X^{(1)}$. Hence, for any open subscheme U of X and A, A', E as above, $u^*\epsilon$ defines a descent data on the A-module E relative to A'. Hence it descends to a A'-module E' satisfying $A \otimes_{A'} E' = E$, since A is finite flat over A'. Next, let U be an open subscheme of X, $U = \bigcup_i U_i$ be an open covering and put $U_{ij} := U_i \cap U_j$. Let E_i, E_{ij} (resp. E'_i, E'_{ij}) be the module E (resp. E') in the case $U = U_i, U = U_{ij}$ respectively. Then we have the exact sequence

$$0 \longrightarrow E \longrightarrow \prod_i E_i \longrightarrow \prod_{i,j} E_{ij},$$

and it implies the exactness of the sequence

$$0 \longrightarrow E' \longrightarrow \prod_i E'_i \longrightarrow \prod_{i,j} E'_{ij}.$$

Hence, by letting U vary, E''s induce a sheaf of $\mathcal{O}_{X^{(1)}}$-module $\mathcal{E}'$ with $F^*\mathcal{E}' = \mathcal{E}$.

Let $p_j : P_{X,(0)} \longrightarrow X, p'_j : P_{X^{(1)},(-1)} \longrightarrow X^{(1)}\,(j = 0, 1)$ be the morphisms induced by j-th projection. Then $\epsilon : p_1^*\mathcal{E} \longrightarrow p_0^*\mathcal{E}$ is rewritten as $\epsilon : \Phi^* p'^*_1\mathcal{E}' \longrightarrow \Phi^* p'^*_0\mathcal{E}'$. We prove that ϵ descents to a morphism $\epsilon' : p'^*_1\mathcal{E}' \longrightarrow p'^*_0\mathcal{E}'$. Let $\pi_{ij} : P_{X,(0)} \times_{P_{X^{(1)},(-1)}} P_{X,(0)} \longrightarrow P_{X,(0)}\,((i,j) = (0,1),(2,3))$ be the morphism induced by the (i,j)-th projection $X^4 \longrightarrow X^2$ and let $\rho_j\,(j = 0, 1)$ be the descent data on $p_j^*\mathcal{E}$ coming from $p'^*_j\mathcal{E}'$. Then, to see the existence of ϵ', it suffices to prove the commutativity of the following diagram, since Φ is finite flat:

$$\begin{array}{ccc} \pi_{23}^*(p_1^*\mathcal{E}) & \xrightarrow{\rho_1} & \pi_{01}^*(p_1^*\mathcal{E}) \\ {\scriptstyle \pi_{23}^*\epsilon}\downarrow & & {\scriptstyle \pi_{01}^*\epsilon}\downarrow \\ \pi_{23}^*(p_0^*\mathcal{E}) & \xrightarrow{\rho_0} & \pi_{01}^*(p_0^*\mathcal{E}). \end{array} \tag{3.7}$$

Note that, by Lemma 3.7, the morphism $\widetilde{P} := P_{X,(0)} \times_{P_{X^{(1)},(-1)}} P_{X,(0)} \longrightarrow X^4$ induced by two $P_{X,(0)} \longrightarrow X^2$'s induces the morphism $v : \widetilde{P} \longrightarrow P_{X,(0)}(3)$. Let $q_{ij} : P_{X,(0)}(3) \longrightarrow P_{X,(0)}$ $(0 \leq i < j \leq 3)$ be the morphism induced by the (i,j)-th projection $X^4 \longrightarrow X^2$. By definition, ρ_0 is equal to the pull-back by $p_0 \times p_0 : \widetilde{P} \longrightarrow X \times_{X^{(1)}} X$ of the descent data on $\mathcal{E}$ relative to $X^{(1)}$ coming from $\mathcal{E}'$, and it is equal to the pull-back of ϵ by

$$u \circ (p_0 \times p_0) = q_{02} \circ v : \widetilde{P} \longrightarrow P_{X,(0)}.$$

So we have $\rho_0 = (p_0 \times p_0)^* u^* \epsilon = v^* q_{02}^* \epsilon$. We see the equality $\rho_1 = v^* q_{13}^* \epsilon$ in the same way. On the other hand, we have $\pi_{23}^* \epsilon = v^* q_{23} \epsilon$, $\pi_{01}^* \epsilon = v^* q_{01} \epsilon$ by definition. Hence the commutativity of the diagram (3.7) follows from the cocycle condition for ϵ. So we have proved the existence of the morphism $\epsilon' : p_1'^* \mathcal{E}' \longrightarrow p_0'^* \mathcal{E}'$.

ϵ' is an isomorphism because so is ϵ. Also, the cocycle condition for ϵ' is reduced to that for ϵ because Φ is finite flat. Therefore, ϵ' is a (-1)-HPD-stratification and so $(\mathcal{E}', \epsilon')$ forms an object in $\mathrm{MIC}^{(1)}(X^{(1)})^{\mathrm{qn}}$ with $F^*(\mathcal{E}', \epsilon') = (\mathcal{E}, \epsilon)$. Hence we have shown the essential surjectivity of the functor F^* and so the proof of the theorem is now finished. □

REMARK 3.8. Assume that we are in the situation of $\mathbf{Hyp}(\infty, 2, 2)$. Also, let $\iota : \mathrm{HIG}(X_1^{(1)})^{\mathrm{qn}} \longrightarrow \mathrm{HIG}(X_1^{(1)})^{\mathrm{qn}}$ be the functor $(\mathcal{E}, \theta) \mapsto (\mathcal{E}, -\theta)$. Then, by Corollary 3.2(1), the functor

$$F_2^* \circ \iota : \mathrm{HIG}(X_1^{(1)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}(X_1)^{\mathrm{qn}}$$

is an equivalence. In view of Remark 1.12, this reproves [9, 2.11] and a special case of [4, 5.8] (the case $m = 0$ in the notation there). So we can regard Theorem 3.1 as a generalization of their results in some sense. Note however that our result is slightly weaker than their results in the sense that we need the existence of a flat p-adic formal scheme S with $S \otimes \mathbb{Z}/p\mathbb{Z} = S_1$.

4. A Comparison of de Rham Cohomologies

In this section, we prove a comparison theorem between the de Rham cohomology of certain modules with p^m-connection on p-adic formal schemes and the de Rham cohomology of the pull-back of them by the level raising

inverse image functor associated to certain lift of Frobenius. As an application, we prove the equivalence between the $\mathbb{Q}$-linearization of the category of nilpotent modules with integrable p^m-connection and the $\mathbb{Q}$-linearization of the category of nilpotent modules with integrable p^{m-1}-connection, under the existence of a nice lift of Frobenius.

Let $X \longrightarrow S$ be a smooth morphism between p-adic formal schemes flat over $\mathbb{Z}_p$ and let $m \in \mathbb{N}$. We define the category $\mathrm{MIC}^{(m)}(X)^{\mathrm{qn}}_{\bullet}$ as the category of projective systems $(\mathcal{E}_\bullet, \nabla_\bullet)_{\bullet\in\mathbb{N}}$ in $\mathrm{MIC}^{(m)}(X)$ with $(\mathcal{E}_n, \nabla_n) \in \mathrm{MIC}^{(m)}(X)^{\mathrm{qn}}_n$ such that $(\mathcal{E}_{n+1}, \nabla_{n+1}) \longrightarrow (\mathcal{E}_n, \nabla_n)$ induces the isomorphism $(\mathcal{E}_{n+1}, \nabla_{n+1}) \otimes \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{=} (\mathcal{E}_n, \nabla_n)$ for any $n \in \mathbb{N}$. If we put $(\mathcal{E}, \nabla) := \varprojlim_n (\mathcal{E}_n, \nabla_n) \in \mathrm{MIC}^{(m)}(X)$, we have $(\mathcal{E}, \nabla) \otimes \mathbb{Z}/p^n\mathbb{Z} = (\mathcal{E}_n, \nabla_n)$ when each $\mathcal{E}_n$ is quasi-coherent, by [1, 3.3.1]. We define several nilpotent properties for objects in $\mathrm{MIC}^{(m)}(X)^{\mathrm{qn}}_{\bullet}$ which are stronger than quasi-nilpotence as follows:

Definition 4.1. Let $X \longrightarrow S, m$ be as above.

(1) For a smooth scheme Y over $S_n := S \otimes \mathbb{Z}/p^n\mathbb{Z}$, an object $(\mathcal{E}, \nabla)$ in $\mathrm{MIC}^{(m)}(Y)^{\mathrm{qn}}$ is called f-constant (resp. lf-constant) if $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_Y$-module flat over $\mathbb{Z}/p^n\mathbb{Z}$ (resp. a locally free $\mathcal{O}_Y$-module of finite rank) and it is generated as $\mathcal{O}_Y$-module by elements e with $\nabla(e) = 0$. For $l \in \mathbb{N}$, It is called f-nilpotent of length $\leq l$ (resp. lf-nilpotent of length $\leq l$) if it can be written as an iterated extension of length $\leq l$ by f-constant (resp. lf-constant) objects.

(2) An object $(\mathcal{E}_\bullet, \nabla_\bullet)$ in $\mathrm{MIC}^{(m)}(X)^{\mathrm{qn}}_{\bullet}$ is called f-nilpotent (resp. lf-nilpotent) if there exists some $l \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$, there exists some etale surjective morphism $Y_n \longrightarrow X_n := X \otimes \mathbb{Z}/p^n\mathbb{Z}$ such that $(\mathcal{E}_n, \nabla_n)|_{Y_n} \in \mathrm{MIC}^{(m)}(Y_n)^{\mathrm{qn}}$ is f-nilpotent of length $\leq l$ (resp. lf-nilpotent of length $\leq l$).

(3) An object $(\mathcal{E}_\bullet, \nabla_\bullet)$ in $\mathrm{MIC}^{(m)}(X)^{\mathrm{qn}}_{\bullet}$ is called nilpotent if it can be written as an iterated extension by the object $(\mathcal{O}_{X_\bullet}, p^m d)$.

We denote the full subcategory of $\mathrm{MIC}^{(m)}(X)^{\mathrm{qn}}_{\bullet}$ consisting of f-nilpotent (resp. lf-nilpotent, nilpotent) objects by $\mathrm{MIC}^{(m)}(X)^{\mathrm{fn}}_{\bullet}$ (resp. $\mathrm{MIC}^{(m)}(X)^{\mathrm{lfn}}_{\bullet}$, $\mathrm{MIC}^{(m-1)}(X)^{\mathrm{n}}_{\bullet}$).

Definition 4.2. Let the notations be as in Definition 4.1. We call an object $(\mathcal{E}, \nabla)$ in $\mathrm{MIC}^{(m)}(X)$ f-nilpotent (resp. lf-nilpotent, nilpotent)

if there exists an object $(\mathcal{E}_n, \nabla_n)_n$ in $\mathrm{MIC}^{(m)}(X)^{\mathrm{fn}}_{\bullet}$ (resp. $\mathrm{MIC}^{(m)}(X)^{\mathrm{lfn}}_{\bullet}$, $\mathrm{MIC}^{(m)}(X)^{\mathrm{n}}_{\bullet}$) with $(\mathcal{E}, \nabla) := \varprojlim_n (\mathcal{E}_n, \nabla_n) \in \mathrm{MIC}^{(m)}(X)$, and denote the category of f-nilpotent (resp. lf-nilpotent, nilpotent) objects in $\mathrm{MIC}^{(m)}(X)$ by $\mathrm{MIC}^{(m)}(X)^{\mathrm{fn}}$ (resp. $\mathrm{MIC}^{(m)}(X)^{\mathrm{lfn}}$, $\mathrm{MIC}^{(m)}(X)^{\mathrm{n}}$). Since each $\mathcal{E}_n$ is quasi-coherent for any $(\mathcal{E}_n, \nabla_n)_n \in \mathrm{MIC}^{(m)}(X)^{\mathrm{fn}}_{\bullet}$, the functor $(\mathcal{E}_n, \nabla_n) \mapsto (\mathcal{E}, \nabla) := \varprojlim_n (\mathcal{E}_n, \nabla_n)$ induces the equivalence

$$\mathrm{MIC}^{(m)}(X)^{\mathrm{fn}}_{\bullet} \xrightarrow{=} \mathrm{MIC}^{(m)}(X)^{\mathrm{fn}},$$

$$(\text{resp. } \mathrm{MIC}^{(m)}(X)^{\mathrm{lfn}}_{\bullet} \xrightarrow{=} \mathrm{MIC}^{(m)}(X)^{\mathrm{lfn}}, \quad \mathrm{MIC}^{(m)}(X)^{\mathrm{n}}_{\bullet} \xrightarrow{=} \mathrm{MIC}^{(m)}(X)^{\mathrm{n}}).$$

(The inverse is given by $(\mathcal{E}, \nabla) \mapsto ((\mathcal{E}, \nabla) \otimes \mathbb{Z}/p^n\mathbb{Z})_n$.)

Note that we have implications

$$\text{nilpotent} \Longrightarrow \text{lf-nilpotent} \Longrightarrow \text{f-nilpotent}.$$

Recall that an object $(\mathcal{E}, \nabla)$ in $\mathrm{MIC}^{(m)}(X)$ induces morphisms

$$\nabla_k : \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^k_{X/S} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{k+1}_{X/S}$$

and they form a complex

$$0 \longrightarrow \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \xrightarrow{\nabla_1} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_{X/S} \xrightarrow{\nabla_2} \cdots,$$

which we call the de Rham complex of $(\mathcal{E}, \nabla)$. We denote the cohomology sheaf of this complex by $\mathcal{H}^i(\mathcal{E}, \nabla)$ and the hypercohomology of it on X by $H^i(X, (\mathcal{E}, \nabla))$.

To state the main result in this section, we give the following definition to fix the situation.

DEFINITION 4.3. In this definition, $\mathbb{G}_{m,T}$ denotes the scheme $\operatorname{Spec} T[t^{\pm 1}]$ for a scheme T and the p-adic formal scheme $\operatorname{Spf} T\{t^{\pm 1}\}$ for a p-adic formal scheme T. For $a, b, c, c' \in \mathbb{N} \cup \{\infty\}$ with $a \geq b \geq c \geq c'$, we mean by $\mathbf{Hyp}(a, b, c, c')$ the following hypothesis: Let $S_j\,(j \leq a), f_j : X_j \longrightarrow S_j\,(j \leq b), f_j^{(1)} : X_j^{(1)} \longrightarrow S_j\,(j \leq b), F_j : X_j \longrightarrow X_j^{(1)}\,(j \leq c)$ be as in $\mathbf{Hyp}(a, b, c)$. Also, we assume that there exists a Cartesian diagram

$$\begin{array}{ccc} X_{c'} & \xrightarrow{F_{c'}} & X^{(1)}_{c'} \\ \downarrow & & \downarrow \\ \mathbb{G}^d_{m,S_{c'}} & \xrightarrow{p} & \mathbb{G}^d_{m,S_{c'}} \end{array} \tag{4.1}$$

for some $d \in \mathbb{N}$ Zariski locally, where the vertical arrows are etale and the map p is the morphism over S which sends the coordinates to the p-th power of them.

When $a = \infty$ (resp. $b = \infty$, $c = \infty$), we denote S_a (resp. $f_b : X_b \longrightarrow S_b$ and $f_b^{(1)} : X_b^{(1)} \longrightarrow S_b$, $F_c : X_c \longrightarrow X_c^{(1)}$) simply by S (resp. $f : X \longrightarrow S$ and $f^{(1)} : X^{(1)} \longrightarrow S$, $F : X \longrightarrow X^{(1)}$). In the local situation where there exists the diagram (4.1), let $t_1, ..., t_d$ (resp. $t'_1, ..., t'_d$) be the local coordinate on X (resp. $X^{(1)}$) induced from the canonical coordinate of $\mathbb{G}_{m,S}$ via the left vertical arrow (resp. the right vertical arrow) in (4.1) and let us put $\tau_i := 1 \otimes t_i - t_i \otimes 1 \in \mathcal{O}_{X^2}, \tau'_i := 1 \otimes t'_i - t'_i \otimes 1 \in \mathcal{O}_{(X^{(1)})^2}$.

Roughly speaking, $\mathbf{Hyp}(a, b, c, c')$ means that S_1 is lifted to a scheme S_a flat over $\mathbb{Z}/p^a\mathbb{Z}$, $f_1 : X_1 \longrightarrow S_1$ and $f_1^{(1)} : X_1^{(1)} \longrightarrow S_1$ is lifted to morphisms over $S_b = S_a \otimes \mathbb{Z}/p^b\mathbb{Z}$, the relative Frobenius $F_{X_1/S_1} : X_1 \longrightarrow X_1^{(1)}$ is lifted to a morphism over $S_c = S_a \otimes \mathbb{Z}/p^c\mathbb{Z}$ and nicely lifted to a morphism over $S_{c'} = S_a \otimes \mathbb{Z}/p^{c'}\mathbb{Z}$.

Then the main result in this section is stated in the following way.

THEOREM 4.4. *Let the notations be as in* $\mathbf{Hyp}(\infty, \infty, \infty, \infty)$ *and let* $(\mathcal{E}, \nabla)$ *be an object in* $\mathrm{MIC}^{(m)}(X^{(1)})^{\mathrm{fn}}$. *Then the level raising inverse image functor* $F^* : \mathrm{MIC}^{(m)}(X^{(1)}) \longrightarrow \mathrm{MIC}^{(m-1)}(X)$ *induces the natural isomorphisms*

$$\mathcal{H}^0(\mathcal{E}, \nabla) \xrightarrow{=} \mathcal{H}^0(F^*(\mathcal{E}, \nabla)).$$
$$\mathcal{H}^i(\mathcal{E}, \nabla) \otimes \mathbb{Q} \xrightarrow{=} \mathcal{H}^i(F^*(\mathcal{E}, \nabla)) \otimes \mathbb{Q} \quad (i \in \mathbb{N}).$$

More precisely, F^* *induces the natural injective morphism*

$$\mathcal{H}^i(\mathcal{E}, \nabla) \longrightarrow \mathcal{H}^i(F^*(\mathcal{E}, \nabla)) \quad (i \in \mathbb{N})$$

which is isomorphic when $i = 0$ *and whose cokernel is killed by* p^c, *where* c *is a natural number which depends only on* m, i *and the length of lf-nilpotence of* $(\mathcal{E}, \nabla)$.

Before the proof, we give several corollaries.

COROLLARY 4.5. *Let the notations be as in* $\mathbf{Hyp}(\infty, \infty, \infty, \infty)$ *and let* $(\mathcal{E}, \nabla)$ *be an object in* $\mathrm{MIC}^{(m)}(X^{(1)})^{\mathrm{fn}}$. *Then the level raising inverse image*

functor $F^*: \mathrm{MIC}^{(m)}(X^{(1)}) \longrightarrow \mathrm{MIC}^{(m-1)}(X)$ *induces the isomorphisms*

$$H^0(X^{(1)}, (\mathcal{E}, \nabla)) \xrightarrow{=} H^0(X, F^*(\mathcal{E}, \nabla)).$$
$$H^i(X^{(1)}, (\mathcal{E}, \nabla)) \otimes \mathbb{Q} \xrightarrow{=} H^i(X, F^*(\mathcal{E}, \nabla)) \otimes \mathbb{Q} \quad (i \in \mathbb{N}).$$

More precisely, F^* *induces the homomorphism*

$$H^i(X^{(1)}, (\mathcal{E}, \nabla)) \longrightarrow H^i(X, F^*(\mathcal{E}, \nabla)) \quad (i \in \mathbb{N})$$

which is isomorphic when $i = 0$ *and whose kernel and cokernel are killed by* p^c*, where* c *is a natural number which depends only on* m, i *and the length of lf-nilpotence of* $(\mathcal{E}, \nabla)$*.*

PROOF. This follows from Theorem 4.4 and the spectral sequences

$$E_2^{s,t} = H^s(X^{(1)}, \mathcal{H}^t(\mathcal{E}, \nabla)) \Longrightarrow H^{s+t}(X^{(1)}, (\mathcal{E}, \nabla)),$$
$$E_2^{s,t} = H^s(X, \mathcal{H}^t(F^*(\mathcal{E}, \nabla))) \Longrightarrow H^{s+t}(X, F^*(\mathcal{E}, \nabla)). \square$$

To give two more corollaries, we introduce some more categories. Recall that, under $\mathbf{Hyp}(\infty, \infty, m+1)$, we have an equivalence

$$\mathrm{MIC}^{(m-1)}(X)_n^{\mathrm{qn}} \xrightarrow{=} \mathrm{MIC}^{(m-1)}(X_{n+m-1})_n^{\mathrm{qn}} \xrightarrow{=} D^{(m-1)}(X_{n+m-1}/S)_n^{\mathrm{qn}}$$
$$\xrightarrow{=} \overline{D}^{(m-1)}(X_{n+m-1}/S)_n^{\mathrm{qn}} \xrightarrow{=} \overline{D}^{(m-1)}(X_{m+1}/S)_n^{\mathrm{qn}}$$

for $n \in \mathbb{N}$. Let us define the category $\overline{D}^{(m-1)}(X_{m+1}/S)_\bullet^{\mathrm{qn}}$ as the category of projective systems $(\mathcal{E}_\bullet)_{\bullet \in \mathbb{N}}$ in $\overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{qn}} := \bigcup_n \overline{D}^{(m-1)}(X_{m+1}/S)_n^{\mathrm{qn}}$ with $\mathcal{E}_n \in \overline{D}^{(m-1)}(X_{m+1}/S)_n^{\mathrm{qn}}$ such that $\mathcal{E}_{n+1} \longrightarrow \mathcal{E}_n$ induces the isomorphism $\mathcal{E}_{n+1} \otimes \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{=} \mathcal{E}_n$ for any $n \in \mathbb{N}$. Then the above equivalence induces the equivalence

$$\mathrm{MIC}^{(m-1)}(X)_\bullet^{\mathrm{qn}} \xrightarrow{=} \overline{D}^{(m-1)}(X_{m+1}/S)_\bullet^{\mathrm{qn}}. \tag{4.2}$$

Then we define the category $\overline{D}^{(m-1)}(X_{m+1}/S)_\bullet^{\mathrm{lfn}}, \overline{D}^{(m-1)}(X_{m+1}/S)_\bullet^{\mathrm{n}}$ by the essential image of $\mathrm{MIC}^{(m-1)}(X)_\bullet^{\mathrm{lfn}}, \mathrm{MIC}^{(m-1)}(X)_\bullet^{\mathrm{n}}$ by the equivalence (4.2), respectively. Also, we can define the category $\overline{D}^{(m)}(X_{m+1}^{(1)}/S)_\bullet^{\mathrm{qn}}$, the equivalence

$$\mathrm{MIC}^{(m)}(X^{(1)})_\bullet^{\mathrm{qn}} \xrightarrow{=} \overline{D}^{(m)}(X_{m+1}^{(1)}/S)_\bullet^{\mathrm{qn}},$$

and the categories $\overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{lfn}}_{\bullet}, \overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{n}}_{\bullet}$ in the same way. Note that we have the level raising inverse image functor

$$F^*_{m+1} : \overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{qn}}_{\bullet} \longrightarrow \overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{qn}}_{\bullet}$$

induced by the ones $F^*_{m+1} : \overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{qn}}_{n} \longrightarrow \overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{qn}}_{n}$ for $n \in \mathbb{N}$.

Then we have the following corollary:

Corollary 4.6.

(1) *Let the sitation be as in* $\mathbf{Hyp}(\infty, \infty, m+1, m+1)$. *Then the level raising inverse image functor*

$$F^*_{m+1} : \overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{qn}}_{\bullet} \longrightarrow \overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{qn}}_{\bullet}$$

induces the fully faithful functor

$$F^*_{m+1} : \overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{lfn}}_{\bullet} \longrightarrow \overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{lfn}}_{\bullet}, \tag{4.3}$$

and via the induced fully faithful functor

$$\overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{lfn}}_{\bullet,\mathbb{Q}} \longrightarrow \overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{lfn}}_{\bullet,\mathbb{Q}} \tag{4.4}$$

between $\mathbb{Q}$*-linealized categories,* $\overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{lfn}}_{\bullet,\mathbb{Q}}$ *is a thick full subcategory of* $\overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{lfn}}_{\bullet,\mathbb{Q}}$.

(2) *Let the sitation be as in* $\mathbf{Hyp}(\infty, \infty, \infty, m+1)$. *Then the level raising inverse image functor* $F^* : \mathrm{MIC}(X^{(1)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}^{(m-1)}(X)^{\mathrm{qn}}$ *induces the fully faithful functor*

$$F^* : \mathrm{MIC}^{(m)}(X^{(1)})^{\mathrm{lfn}} \longrightarrow \mathrm{MIC}^{(m-1)}(X)^{\mathrm{lfn}}, \tag{4.5}$$

and via the induced fully faithful functor

$$\mathrm{MIC}(X^{(1)})^{\mathrm{lfn}}_{\mathbb{Q}} \longrightarrow \mathrm{MIC}^{(m-1)}(X)^{\mathrm{lfn}}_{\mathbb{Q}} \tag{4.6}$$

between $\mathbb{Q}$*-linealized categories,* $\mathrm{MIC}^{(m)}(X^{(1)})^{\mathrm{lfn}}_{\mathbb{Q}}$ *is a thick full subcategory of* $\mathrm{MIC}^{(m-1)}(X)^{\mathrm{lfn}}_{\mathbb{Q}}$.

PROOF. First we prove (2) under $\mathbf{Hyp}(\infty,\infty,\infty,\infty)$. Since the source and the target in (4.5) and (4.6) are rigid tensor categories and F^* is a tensor functor, it suffices to prove that, for $(\mathcal{E},\nabla) \in \mathrm{MIC}^{(m)}(X^{(1)})^{\mathrm{lfn}}$, the level raising inverse image functor induces the isomorphisms

$$H^0(X^{(1)},(\mathcal{E},\nabla)) \xrightarrow{=} H^0(X,F^*(\mathcal{E},\nabla)),$$
$$H^1(X^{(1)},(\mathcal{E},\nabla))\otimes\mathbb{Q} \xrightarrow{=} H^1(X,F^*(\mathcal{E},\nabla))\otimes\mathbb{Q},$$

and it follows from Corollary 4.5.

Next we prove (1). First we work locally and assume the existence of $F : X \longrightarrow X^{(1)}$ which lifts F_{m+1} and assume that we are in the situation of $\mathbf{Hyp}(\infty,\infty,\infty,\infty)$ with this F. Then the functor (4.3) is identical with the functor (4.5) and the latter is fully faithful by the argument in the previous paragraph. Hence the former is also fully faithful.

Since the categories appearing in (4.3) satisfies the descent property for Zariski coverings, we can deduce from the argument in the previous paragraph that the functor (4.3) is fully faithful globally, under $\mathbf{Hyp}(\infty,\infty,m+1,m+1)$. Let us prove that $\overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{lfn}}_{\bullet,\mathbb{Q}}$ is a thick full subcategory of $\overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{lfn}}_{\bullet,\mathbb{Q}}$ via (4.4) globally, under $\mathbf{Hyp}(\infty,\infty,m+1,m+1)$. Let us take an exact sequence

$$0 \longrightarrow F^*_{m+1}\mathcal{E}' \xrightarrow{f} \mathcal{E} \xrightarrow{g} F^*_{m+1}\mathcal{E}'' \longrightarrow 0 \tag{4.7}$$

in $\overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{lfn}}_{\bullet,\mathbb{Q}} = \mathrm{MIC}^{(m-1)}(X)^{\mathrm{lfn}}_{\mathbb{Q}} \subseteq \mathrm{MIC}^{(m-1)}(X)_{\mathbb{Q}}$ with $\mathcal{E}',\mathcal{E}''$ contained in $\overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{lfn}}_{\bullet}$. It suffices to prove that $\mathcal{E}$ is in the essential image of the functor (4.4). We may assume that f,g are morphisms in $\mathrm{MIC}^{(m-1)}(X)$ and that $g\circ f = 0$ in $\mathrm{MIC}^{(m-1)}(X)$. Then we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^*_{m+1}\mathcal{E}' & \xrightarrow{f} & \mathcal{E} & \xrightarrow{g} & F^*_{m+1}\mathcal{E}'' & \longrightarrow & 0 \\ & & \alpha\downarrow & & \| & & \beta\uparrow & & \\ 0 & \longrightarrow & \mathrm{Ker}g & \longrightarrow & \mathcal{E} & \longrightarrow & \mathrm{Im}g & \longrightarrow & 0 \end{array} \tag{4.8}$$

in $\mathrm{MIC}^{(m-1)}(X)$. Since α,β are isomorphisms in $\mathrm{MIC}^{(m-1)}(X)_{\mathbb{Q}}$, we have some $a\in\mathbb{N}$ and morphisms

$$\alpha' : \mathrm{Ker}g \longrightarrow F^*_{m+1}\mathcal{E}', \quad \beta' : F^*_{m+1}\mathcal{E}'' \longrightarrow \mathrm{Im}g$$

such that $\alpha \circ \alpha' = \alpha' \circ \alpha = p^a$, $\beta \circ \beta' = \beta' \circ \beta = p^a$. If we push the lower horizontal line in (4.8) by α' and pull it back by β', we obtain the exact sequence of the form

$$0 \longrightarrow F_{m+1}^* \mathcal{E}' \longrightarrow \mathcal{E}_1 \longrightarrow F_{m+1}^* \mathcal{E}'' \longrightarrow 0$$

in $\mathrm{MIC}^{(m-1)}(X)^{\mathrm{lfn}}$ with $\mathcal{E}_1$ isomorphic to $\mathcal{E}$ in the category $\mathrm{MIC}^{(m-1)}(X)_{\mathbb{Q}}$. So, by replacing $\mathcal{E}$ by $\mathcal{E}_1$, we may assume that the exact sequence (4.7) is the one in the category $\mathrm{MIC}^{(m-1)}(X)^{\mathrm{lfn}}$. Now let us take a Zariski covering $X = \bigcup_\alpha X_\alpha$ by open subschemes such that, if we denote the corresponding open covering of $X^{(1)}$ by $X^{(1)} = \bigcup_\alpha X_\alpha^{(1)}$, there exists a lift $F_\alpha : X_\alpha \longrightarrow X_\alpha^{(1)}$ of F_{m+1} for each α with which we are in the situation of $\mathbf{Hyp}(\infty, \infty, \infty, \infty)$ (for X_α). Then, since the functor $F_{m+1}^*|_{X_\alpha^{(1)}}$ is equal to $F_\alpha^* : \mathrm{MIC}^{(m)}(X_\alpha^{(1)})^{\mathrm{lfn}} \longrightarrow \mathrm{MIC}^{(m-1)}(X_\alpha)^{\mathrm{lfn}}$, we have the homomorphism

$$H^1(X_\alpha^{(1)}, \mathcal{H}\mathrm{om}(\mathcal{E}'', \mathcal{E}')) \longrightarrow H^1(X_\alpha, \mathcal{H}\mathrm{om}(F_{m+1}^* \mathcal{E}'', F_{m+1}^* \mathcal{E}'))$$

whose kernel and cokernel are killed by p^c, where c is a natural number independent of α. So, by multiplying the extension class $[\mathcal{E}]$ of the exact sequence (4.7) by p^c, we may assume that there exists an exact sequence

$$0 \longrightarrow \mathcal{E}'|_{X_\alpha^{(1)}} \longrightarrow \mathcal{F}_\alpha \longrightarrow \mathcal{E}''|_{X_\alpha^{(1)}} \longrightarrow 0 \tag{4.9}$$

in $\mathrm{MIC}^{(m)}(X_\alpha^{(1)})^{\mathrm{lfn}}$ for each α such that $F_{m+1}^*(4.9)$ is isomorphic to $(4.7)|_{X_\alpha}$. In particular, we have the isomorphism $i_\alpha : F_{m+1}^* \mathcal{F}_\alpha \xrightarrow{=} \mathcal{E}|_{X_\alpha}$. So, if we put $i_{\alpha\beta} := i_\beta^{-1} \circ i_\alpha : F_{m+1}^* \mathcal{F}_\alpha \longrightarrow F_{m+1}^* \mathcal{F}_\beta$ on $X_\alpha \cap X_\beta$, it satisfies the cocycle condition. Then, since F_{m+1}^* is fully faithful, we see that there exists an object $\mathcal{F}$ in $\overline{D}^{(m-1)}(X_{m+1}^{(1)}/S)_\bullet^{\mathrm{lfn}} = \mathrm{MIC}^{(m-1)}(X^{(1)})^{\mathrm{lfn}}$ with $F_{m+1}^* \mathcal{F} = \mathcal{E}$. Hence $\mathcal{E}$ is in the essential image of the functor (4.4) and so we have shown that $\overline{D}^{(m)}(X_{m+1}^{(1)}/S)_{\bullet,\mathbb{Q}}^{\mathrm{lfn}}$ is a thick full subcategory of $\overline{D}^{(m-1)}(X_{m+1}/S)_{\bullet,\mathbb{Q}}^{\mathrm{lfn}}$. So the proof of (1) is finished.

Finally, since the functors (4.5), (4.6) are identified with (4.3), (4.4), the assertion (2) in general case is an immediate consequence of the assertion (1). So the proof of corollary is finished. □

Corollary 4.7. *If we are in the situation of* $\mathbf{Hyp}(\infty, \infty, m+1, m+1)$*, the level raising inverse image functor* $F_{m+1}^* : \overline{D}^{(m)}(X_{m+1}^{(1)}/S)_\bullet^{\mathrm{qn}} \longrightarrow$

$\overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{qn}}_{\bullet}$ *induces the fully faithful functor*

$$F^*_{m+1} : \overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{n}}_{\bullet} \longrightarrow \overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{n}}_{\bullet}, \tag{4.10}$$

giving the equivalence

$$\overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{n}}_{\bullet,\mathbb{Q}} \xrightarrow{=} \overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{n}}_{\bullet,\mathbb{Q}} \tag{4.11}$$

between $\mathbb{Q}$-linearized categories. Also, if we are in the situation of $\mathbf{Hyp}(\infty,\infty,\infty,m+1)$ *with X quasi-compact, the level raising inverse image functor $F^* : \mathrm{MIC}(X^{(1)})^{\mathrm{qn}} \longrightarrow \mathrm{MIC}^{(m-1)}(X)^{\mathrm{qn}}$ induces the fully faithful functor*

$$F^* : \mathrm{MIC}^{(m)}(X^{(1)})^{\mathrm{n}} \longrightarrow \mathrm{MIC}^{(m-1)}(X)^{\mathrm{n}}, \tag{4.12}$$

giving the equivalence

$$\mathrm{MIC}(X^{(1)})^{\mathrm{n}}_{\mathbb{Q}} \xrightarrow{=} \mathrm{MIC}^{(m-1)}(X)^{\mathrm{n}}_{\mathbb{Q}} \tag{4.13}$$

between $\mathbb{Q}$-linealized categories.

PROOF. The full faithfulness of (4.10) and (4.12) follows from that of (4.3) and (4.5). Also, by the same argument as the proof of Corollary 4.6, we see that $\overline{D}^{(m)}(X^{(1)}_{m+1}/S)^{\mathrm{n}}_{\bullet,\mathbb{Q}}$ (resp. $\mathrm{MIC}(X^{(1)})^{\mathrm{n}}_{\mathbb{Q}}$) is a thick full subcategory of the category $\overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{n}}_{\bullet,\mathbb{Q}}$ (resp. $\mathrm{MIC}^{(m-1)}(X)^{\mathrm{n}}_{\mathbb{Q}}$) via the functor (4.11) (resp. (4.13)). Since any object in the categories $\overline{D}^{(m-1)}(X_{m+1}/S)^{\mathrm{n}}_{\bullet,\mathbb{Q}}$, $\mathrm{MIC}^{(m-1)}(X)^{\mathrm{n}}_{\mathbb{Q}}$ is written as an iterated extension of trivial objects, thickness above implies the equivalence of the functors (4.11), (4.13). □

Note that Corollaries 4.6, 4.7 give possible answers to Question 1.18.

We give a proof of Theorem 4.4.

PROOF OF THEOREM 4.4. We have the homomorphism of complexes from the de Rham complex of $(\mathcal{E},\nabla)$ to that of $F^*(\mathcal{E},\nabla)$ induced by the level raising inverse image functor. So, to prove the theorem, we may work locally. So we may assume that the Cartesian diagram (4.1) exists globally. Then we have $F^*\mathcal{E} = \oplus_a t^a\mathcal{E}$, where a runs through the set $I := \{a = (a_i)_i \in \mathbb{N}^d \,|\, 0 \leq$

$a_i \leq p-1\}$. If we express $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_{X^{(1)}}} \Omega^1_{X^{(1)}/S} = \oplus_{i=1}^d \mathcal{E} t'^{-1}_i dt'_i$ by $\nabla(e) = \sum_i \nabla_i(e) t'^{-1}_i dt'_i$, $F^*\nabla : F^*\mathcal{E} \longrightarrow F^*\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} = \oplus_{i=1}^d F^*\mathcal{E} t_i^{-1} dt_i$ has the property $F^*\nabla(e) = \sum_i \nabla_i(e) t_i^{-1} dt_i$. (Here we wrote the element $F^*(e) \in F^*\mathcal{E}$ simply by e.) Hence we have, for $a = (a_i)_i \in I$,

$$\begin{aligned} F^*\nabla(t^a e) &= t^a F^*\nabla(e) + p^{m-1} \sum_i a_i t^a e t_i^{-1} dt_i \\ &= t^a \sum_i \{(1 \otimes \nabla_i) + p^{m-1} a_i\}(e) t_i^{-1} dt_i. \end{aligned} \tag{4.14}$$

For $a = (a_i)_i \in I$, let $\theta_a : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_{X^{(1)}}} \Omega^1_{X^{(1)}/S} = \oplus_{i=1}^d \mathcal{E} t'^{-1}_i dt'_i$ be the linear map $e \mapsto \sum_i a_i e t'^{-1}_i dt'_i$. Then, by (4.14), we have $F^*(\mathcal{E}, \nabla) = (F^*\mathcal{E}, F^*\nabla) = \oplus_{a \in I}(\mathcal{E}, \nabla + p^{m-1}\theta_a)$ via the identification of $\oplus_{i=1}^d F^*\mathcal{E} t_i^{-1} dt_i$ and $\oplus_{a \in I} \oplus_{i=1}^d \mathcal{E} t'^{-1}_i dt'_i$ on the target. Hence we have

$$\mathcal{H}^i(F^*(\mathcal{E}, \nabla)) = \oplus_{a \in I} \mathcal{H}^i(\mathcal{E}, \nabla + p^{m-1}\theta_a). \tag{4.15}$$

Let us consider the following claim:

Claim 1. $p^{4l(m-1)(i+1)} \mathcal{H}^i(\mathcal{E}, \nabla + p^{m-1}\theta_a) = 0$ for all $i \in \mathbb{N}$ and $a \in I, \neq 0$.

First we prove that the claim 1 implies the theorem. We see easily that the claim 1 and the equality (4.15) imply the injective morphism $\mathcal{H}^i(\mathcal{E}, \nabla) \longrightarrow \mathcal{H}^i(F^*(\mathcal{E}, \nabla))$ whose cokernel is killed by $p^{4l(m-1)(i+1)}$. Also, since $\mathcal{E}_n := \mathcal{E} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$ for each n, $\mathcal{E}$ is flat over $\mathbb{Z}_p$ and hence the both hand sides of (4.15) are flat over $\mathbb{Z}_p$ when $i = 0$. So the claim implies the equalities $\mathcal{H}^0(\mathcal{E}, \nabla + p^{m-1}\theta_a) = 0$ for $a \in I, \neq 0$, and this and (4.15) imply the isomorphism $\mathcal{H}^0(\mathcal{E}, \nabla) \xrightarrow{=} \mathcal{H}^0(F^*(\mathcal{E}, \nabla))$, as desired. So it suffices to prove the claim 1.

In the following, we denote the map $\theta_a \otimes \mathbb{Z}/p^n\mathbb{Z} : \mathcal{E}_n \longrightarrow \mathcal{E}_n \otimes_{\mathcal{O}_{X^{(1)}}} \Omega^1_{X^{(1)}/S}$ also by θ_a, by abuse of notation. Let us take $l \in \mathbb{N}$ and etale surjective morphism $Y_n \longrightarrow X_n^{(1)}$ $(n \in \mathbb{N})$ such that $(\mathcal{E}_n, \nabla_n)|_{Y_n}$ is f-nilpotent of length $\leq l$. For an open subset $U \subseteq X^{(1)}$, let $(\mathcal{E}, \nabla + p^{m-1}\theta_a)(U)$, $(\mathcal{E}_n, \nabla_n + p^{m-1}\theta_a)(U)$ be the complex $\Gamma(U, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S})$, $\Gamma(U, \mathcal{E}_n \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S})$ induced from the de Rham complex of $(\mathcal{E}, \nabla + p^{m-1}\theta_a)$, $(\mathcal{E}_n, \nabla_n + p^{m-1}\theta_a)$ respectively and let $H^i((\mathcal{E}, \nabla + p^{m-1}\theta_a)(U))$, $H^i((\mathcal{E}_n, \nabla_n + p^{m-1}\theta_a)(U))$ be

the i-th cohomology of this complex. To prove claim 1, first prove the following claim:

CLAIM 2. For $n \in \mathbb{N}$ and an open affine formal subscheme $U \subseteq X^{(1)}$, we have $p^{2l(m-1)(i+1)}H^i((\mathcal{E}_n, \nabla + p^{m-1}\theta_a)(U)) = 0$ for all $i \in \mathbb{N}$ and $a \in I, \neq 0$.

We prove claim 2. Let us put $n' := n + m - 1$ and let $Y \longrightarrow X^{(1)}$ be an etale morphism lifting $Y_{n'} \longrightarrow X^{(1)}_{n'}$. Take an affine etale Čech hypercovering $U_\bullet \longrightarrow U$ such that $U_0 \longrightarrow U$ is a refinement of $Y \times_{X^{(1)}} U \longrightarrow U$. Then, since E_n is quasi-coherent, we have

$$H^i((\mathcal{E}_n, \nabla + p^{m-1}\theta_a)(U)) = H^i((\mathcal{E}_n, \nabla + p^{m-1}\theta_a)(U_\bullet))$$

(where the right hand side denotes the i-th cohomology of the double complex $(\mathcal{E}_n, \nabla + p^{m-1}\theta_a)(U_\bullet)$) and so we have the spectral sequence

$$E_2^{s,t} := H^t((\mathcal{E}_n, \nabla + p^{m-1}\theta_a)(U_s)) \Longrightarrow H^{s+t}((\mathcal{E}_n, \nabla + p^{m-1}\theta_a)(U)).$$

Hence it suffices to prove $p^{2l(m-1)}H^i((\mathcal{E}_n, \nabla + p^{m-1}\theta_a)(U_s)) = 0$ for all s. By assumption, $(\mathcal{E}_{n'}, \nabla_{n'})|_{U_s}$ is written as an iterated extension of f-constant objects $(\mathcal{E}_{n',j}, \nabla_{n',j})\,(1 \leq j \leq l)$ in $\mathrm{MIC}^{(m)}(U_s \otimes \mathbb{Z}/p^{n'}\mathbb{Z})$. Then $(\mathcal{E}_{n'}, \nabla_{n'} + p^{m-1}\theta_a)|_{U_s}$ is written as an iterated extension of $(\mathcal{E}_{n',j}, \nabla_{n',j} + p^{m-1}\theta_a)\,(1 \leq j \leq l)$, and $(\mathcal{E}_n, \nabla_n + p^{m-1}\theta_a)|_{U_s}$ is written an iterated extension of $(\mathcal{E}_{n,j}, \nabla_{n,j} + p^{m-1}\theta_a) := (\mathcal{E}_{n',j}, \nabla_{n',j} + p^{m-1}\theta_a) \otimes \mathbb{Z}/p^n\mathbb{Z}\,(1 \leq j \leq l)$. So it suffices to prove

$$p^{2(m-1)}H^i((\mathcal{E}_{n,j}, \nabla_{n,j} + p^{m-1}\theta_a)(U_s)) = 0.$$

Since $(\mathcal{E}_{n',j}, \nabla_{n',j})$ is f-constant, we have $\nabla_{n',j}(\mathcal{E}_{n',j}) \subseteq p^m\mathcal{E}_{n',j} \otimes \Omega^1_{X^{(1)}/S}$. Hence we can factorize $\nabla_{n',j}$ as

$$\mathcal{E}_{n',j} \overset{\overline{\nabla}_{n',j}}{\longrightarrow} \mathcal{E}_{n,j} \otimes \Omega^1_{X^{(1)}/S} \overset{\underline{p}^{m-1}}{\longrightarrow} \mathcal{E}_{n',j} \otimes \Omega^1_{X^{(1)}/S},$$

where $\underline{p}^{m-1}$ is the map induced from the multiplication by p^{m-1} on $\mathcal{E}_{n',j} \otimes \Omega^1_{X^{(1)}/S}$. Let $\overline{\overline{\nabla}}_j : \mathcal{E}_{n,j} \longrightarrow \mathcal{E}_{n,j} \otimes \Omega^1_{X^{(1)}/S}$ be $\overline{\nabla}_{n',j} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Then it is a

f-nilpotent p-connection and we have the commutative diagram

$$\begin{array}{ccccc} \mathcal{E}_{n,j} & \xrightarrow{p^{m-1}} & \mathcal{E}_{n,j} & \xrightarrow{=} & \mathcal{E}_{n,j} \\ {\scriptstyle \nabla_{n,j}+p^{m-1}\theta_a}\downarrow & & {\scriptstyle \overline{\overline{\nabla}}_j+\theta_a}\downarrow & & {\scriptstyle \nabla_{n,j}+p^{m-1}\theta_a}\downarrow \\ \mathcal{E}_{n,j}\otimes\Omega^1_{X^{(1)}/S} & \xrightarrow{=} & \mathcal{E}_{n,j}\otimes\Omega^1_{X^{(1)}/S} & \xrightarrow{p^{m-1}} & \mathcal{E}_{n,j}\otimes\Omega^1_{X^{(1)}/S}. \end{array} \tag{4.16}$$

For $k \in \mathbb{N}$, let us denote the homomorphism $\mathcal{E}_{n,j} \otimes \Omega^k_{X^{(1)}/S} \longrightarrow \mathcal{E}_{n,j} \otimes \Omega^{k+1}_{X^{(1)}/S}$ induced by $\nabla_{n,j} + p^{m-1}\theta_a$ (resp. $\overline{\overline{\nabla}}_j + \theta_a$) by $(\nabla_{n,j} + p^{m-1}\theta_a)_k$ (resp. $(\overline{\overline{\nabla}}_j + \theta_a)_k$). Then the commutativity of (4.16) implies that of the following diagram:

$$\begin{array}{ccccc} \mathcal{E}_{n,j}\otimes\Omega^{i-1}_{X^{(1)}/S} & \xrightarrow{p^{2(m-1)}} & \mathcal{E}_{n,j}\otimes\Omega^{i-1}_{X^{(1)}/S} & \xrightarrow{=} & \mathcal{E}_{n,j}\otimes\Omega^{i-1}_{X^{(1)}/S} \\ {\scriptstyle (\nabla_{n,j}+p^{m-1}\theta_a)_{i-1}}\downarrow & & {\scriptstyle (\overline{\overline{\nabla}}_j+\theta_a)_{i-1}}\downarrow & & {\scriptstyle (\nabla_{n,j}+p^{m-1}\theta_a)_{i-1}}\downarrow \\ \mathcal{E}_{n,j}\otimes\Omega^{i}_{X^{(1)}/S} & \xrightarrow{p^{m-1}} & \mathcal{E}_{n,j}\otimes\Omega^{i}_{X^{(1)}/S} & \xrightarrow{p^{m-1}} & \mathcal{E}_{n,j}\otimes\Omega^{i}_{X^{(1)}/S} \\ {\scriptstyle (\nabla_{n,j}+p^{m-1}\theta_a)_{i}}\downarrow & & {\scriptstyle (\overline{\overline{\nabla}}_j+\theta_a)_{i}}\downarrow & & {\scriptstyle (\nabla_{n,j}+p^{m-1}\theta_a)_{i}}\downarrow \\ \mathcal{E}_{n,j}\otimes\Omega^{i+1}_{X^{(1)}/S} & \xrightarrow{=} & \mathcal{E}_{n,j}\otimes\Omega^{i+1}_{X^{(1)}/S} & \xrightarrow{p^{2(m-1)}} & \mathcal{E}_{n,j}\otimes\Omega^{i+1}_{X^{(1)}/S}. \end{array} \tag{4.17}$$

From the commutative diagram (4.17), we see that it suffices to prove the equality $H^i((\mathcal{E}_{n,j}, \overline{\overline{\nabla}}_j + \theta_a)(U_s)) = 0$ to prove the claim 2. Since $(\mathcal{E}_{n,j}, \overline{\overline{\nabla}}_j)$ is a f-constant p-connection, we see that $\overline{\overline{\nabla}}(\mathcal{E}_{n,j}) \subseteq p\mathcal{E}_{n,j} \otimes \Omega^1_{X^{(1)}/S}$. So we can write $(\mathcal{E}_{n,j}, \overline{\overline{\nabla}}_j + \theta_a)$ as an iterated extension by the Higgs module $(\mathcal{E}_{1,j}, \theta_a)$ (where $\mathcal{E}_{1,j} = \mathcal{E}_{n,j} \otimes \mathbb{Z}/p\mathbb{Z}$), and by [8, 2.2.2], we have $H^i((\mathcal{E}_{1,j}, \theta_a)(U_s)) = 0$ for any $i \in \mathbb{N}$. Hence we have $H^i((\mathcal{E}_{n,j}, \overline{\overline{\nabla}}_j + \theta_a)(U_s)) = 0\,(i \in \mathbb{N})$ and so the proof of claim 2 is finished.

Finally we prove the claim 1. For any open affine $U \subseteq X^{(1)}$, we have the exact sequence

$$\begin{aligned} 0 \longrightarrow & \varprojlim{}^1 H^{i-1}((\mathcal{E}_n, \nabla_n + p^{m-1}\theta_a)(U)) \longrightarrow H^i((\mathcal{E}, \nabla + p^{m-1}\theta_a)(U)) \\ & \longrightarrow \varprojlim{}_n H^i((\mathcal{E}_n, \nabla_n + p^{m-1}\theta_a)(U)) \longrightarrow 0 \end{aligned}$$

and by claim 1, the first and the third term are killed by $p^{2l(m-1)(i+1)}$. Hence $H^i((\mathcal{E}, \nabla + p^{m-1}\theta_a)(U))$ is killed by $p^{4l(m-1)(i+1)}$ and so we have

$p^{4l(m-1)(i+1)}\mathcal{H}^i(\mathcal{E}, \nabla + p^{m-1}\theta_a) = 0$, as desired. So the proof of the theorem is now finished. □

5. Modules with Integrable p^m-Witt-Connection

In this section, we define the notion of modules with integrable p^m-Witt-connection. We also define the level raising inverse image functor from the categories of modules with integrable p^m-Witt-connection to that of p^{m-1}-Witt-connection, and prove that it induces the equivalence of categories between $\mathbb{Q}$-linearized categories when restricted to nilpotent objects. This is the Witt analogue of the equivalence (4.13) in Corollary 4.7. The categories defined in this section is more complicated than the categories defined in Section 1, but the equivalance proven in this section has the advantage that we need no assumpion on liftability of Frobenius.

Throughout this section, we fix a perfect scheme S_1 of characteristic $p > 0$.

DEFINITION 5.1. Let us assume given a smooth morphism $X_1 \longrightarrow S_1$. Let $W\mathcal{O}_{X_1} = \varprojlim_n W_n\mathcal{O}_{X_1}$ be the ring of Witt vectors of $\mathcal{O}_{X_1}$ and let $W\Omega^\bullet_{X_1} = \varprojlim_n W_n\Omega^\bullet_{X_1}$ be the de Rham-Witt complex of X_1. Then a p^m-Witt-connection on a $W\mathcal{O}_{X_1}$-module $\mathcal{E}$ is an additive map $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{W\mathcal{O}_{X_1}} W\Omega^1_{X_1}$ satisfying $\nabla(fe) = f\nabla(e) + p^m e \otimes df$ for $e \in \mathcal{E}, f \in W\mathcal{O}_{X_1}$. We call a p^m-Witt-connection also as a Witt-connection of level $-m$.

When we are given a $W\mathcal{O}_{X_1}$-module with p^m-Witt-connection $(\mathcal{E}, \nabla)$, we can define the additive map $\nabla_k : \mathcal{E} \otimes_{W\mathcal{O}_{X_1}} W\Omega^k_{X_1} \longrightarrow \mathcal{E} \otimes_{W\mathcal{O}_{X_1}} W\Omega^{k+1}_{X_1}$ which is characterized by $\nabla_k(e \otimes \omega) = \nabla(e) \wedge \omega + p^m e \otimes d\omega$.

DEFINITION 5.2. With the notation above, we call $(\mathcal{E}, \nabla)$ integrable if we have $\nabla_1 \circ \nabla = 0$. We denote the category of $W\mathcal{O}_{X_1}$-modules with integrable p^m-Witt-connection by $\mathrm{MIWC}^{(m)}(X_1)$.

It is easy to see that the category $\mathrm{MIWC}^{(m)}(X_1)$ is functorial with respect to X_1. We define the notion of nilpotence as follows:

DEFINITION 5.3. An object $(\mathcal{E}, \nabla)$ in $\mathrm{MIWC}^{(m)}(X_1)$ is called nilpotent if it can be written as an iterated extension by the object $(W\mathcal{O}_{X_1}, p^m d)$. We

denote the full subcategory of $\mathrm{MIWC}^{(m)}(X_1)$ consisting of nilpotent objects by $\mathrm{MIWC}^{(m)}(X_1)^{\mathrm{n}}$.

Let $F_{S_1} : S_1 \longrightarrow S_1$ be the Frobenius endomorphism and let $X_1^{(1)} := S_1 \times_{F,S_1} X_1$. Then we define the homomorphism of differential graded algebras

$$(W\Omega^{\bullet}_{X_1^{(1)}}, p^m d) \longrightarrow (W\Omega^{\bullet}_{X_1}, p^{m-1}d) \tag{5.1}$$

by the composite

$$\begin{aligned}(W\Omega^{\bullet}_{X_1^{(1)}}, p^m d) &\xrightarrow{=} \varprojlim_n (W_n\mathcal{O}_{S_1} \otimes_{W_{n+1}\mathcal{O}_{S_1}} W_{n+1}\Omega^{\bullet}_{X_1^{(1)}}, \mathrm{id} \otimes p^m d)\\ &\xrightarrow{=} \varprojlim_n (W_n\mathcal{O}_{S_1} \otimes_{F,W_{n+1}\mathcal{O}_{S_1}} W_{n+1}\Omega^{\bullet}_{X_1}, \mathrm{id} \otimes p^m d)\\ &\xrightarrow{\mathrm{id}\otimes F} (W_n\Omega^{\bullet}_{X_1}, p^{m-1}d)\end{aligned}$$

(where F is the Frobenius in de Rham-Witt complexes). We denote the homomorphism (5.1) also by F. Also, for a $W\mathcal{O}_{X_1^{(1)}}$-module $\mathcal{E}$ and a local section $e \in \mathcal{E}$, denote $W\mathcal{O}_{X_1} \otimes_{F,W\mathcal{O}_{X_1^{(1)}}} \mathcal{E}$ and $1 \otimes e \in W\mathcal{O}_{X_1} \otimes_{F,W\mathcal{O}_{X_1^{(1)}}} \mathcal{E}$ simply by $F_*\mathcal{E}, F_*(e)$. Then, for $(\mathcal{E}, \nabla) \in \mathrm{MIWC}^{(m)}(X_1^{(1)})$, we define $F_*(\mathcal{E}, \nabla) \in \mathrm{MIWC}^{(m-1)}(X_1)$ by $F_*(\mathcal{E}, \nabla) := (F_*\mathcal{E}, F_*\nabla)$, where $F_*\mathcal{E}$ is as above and $F_*\nabla : F_*\mathcal{E} \longrightarrow F_*\mathcal{E} \otimes_{W\mathcal{O}_{X_1}} W\Omega^1_{X_1}$ is the map defined by $F_*\nabla(fF_*(e)) = fF_*(\nabla(e)) + p^{m-1}F_*(e)df$ for $e \in \mathcal{E}, f \in W\mathcal{O}_{X_1}$. (The integrability of $F_*(\mathcal{E}, \nabla)$ follows from the well-known formula $dF = pFd$ for de Rham-Witt complex.) So we have the functor

$$F_* : \mathrm{MIWC}^{(m)}(X_1^{(1)}) \longrightarrow \mathrm{MIWC}^{(m-1)}(X_1),$$

and it is easy to see that it induces the functor

$$F_* : \mathrm{MIWC}^{(m)}(X_1^{(1)})^{\mathrm{n}} \longrightarrow \mathrm{MIWC}^{(m-1)}(X_1)^{\mathrm{n}}. \tag{5.2}$$

An object $(\mathcal{E}, \nabla)$ in $\mathrm{MIWC}^{(m-1)}(X_1)$ (resp. $\mathrm{MIWC}^{(m)}(X_1^{(1)})$) induces the complex

$$\begin{aligned}&0 \longrightarrow \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{W\mathcal{O}_{X_1}} W\Omega^1_{X_1} \xrightarrow{\nabla_1} \mathcal{E} \otimes_{W\mathcal{O}_{X_1}} W\Omega^2_{X_1} \xrightarrow{\nabla_2} \cdots\\ &(\text{resp. } 0 \longrightarrow \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{W\mathcal{O}_{X_1^{(1)}}} W\Omega^1_{X_1^{(1)}} \xrightarrow{\nabla_1} \mathcal{E} \otimes_{W\mathcal{O}_{X_1^{(1)}}} W\Omega^2_{X_1^{(1)}} \xrightarrow{\nabla_2} \cdots),\end{aligned}$$

which is called the de Rham complex of (E, ∇). We denote the cohomology sheaf of this complex by $\mathcal{H}^i(\mathcal{E}, \nabla)$ and the hypercohomology of it on X_1 (resp. $X_1^{(1)}$) by $H^i(X_1, (\mathcal{E}, \nabla))$ (resp. $H^i(X_1^{(1)}, (\mathcal{E}, \nabla))$).

Then we have the following theorem, which is the Witt version of Theorem 4.4:

THEOREM 5.4. *Let the notations be as above and let* $(\mathcal{E}, \nabla) \in \mathrm{MIWC}^{(m)}(X_1^{(1)})^{\mathrm{n}}$. *Then the level raising inverse image functor* (5.2) *induces the natural isomorphisms*

$$\mathcal{H}^0(\mathcal{E}, \nabla) \xrightarrow{=} \mathcal{H}^0(F_*(\mathcal{E}, \nabla)).$$
$$\mathcal{H}^i(\mathcal{E}, \nabla) \otimes \mathbb{Q} \xrightarrow{=} \mathcal{H}^i(F_*(\mathcal{E}, \nabla)) \otimes \mathbb{Q} \quad (i \in \mathbb{N}).$$

We have the following corollaries, which we can prove in the same way as Corollaries 4.5, 4.7. (So we omit the proof.)

COROLLARY 5.5. *Let* $S_1, X_1, X_1^{(1)}$ *be as above and let* $(\mathcal{E}, \nabla) \in \mathrm{MIWC}^{(m)}(X_1^{(1)})^{\mathrm{n}}$. *Then the level raising inverse image functor* (5.2) *induces the isomorphisms*

$$H^0(X_1^{(1)}, (\mathcal{E}, \nabla)) \xrightarrow{=} H^0(X_1, F_*(\mathcal{E}, \nabla)).$$
$$H^i(X_1^{(1)}, (\mathcal{E}, \nabla)) \otimes \mathbb{Q} \xrightarrow{=} H^i(X_1, F_*(\mathcal{E}, \nabla)) \otimes \mathbb{Q} \quad (i \in \mathbb{N}).$$

COROLLARY 5.6. *Let* $S_1, X_1, X_1^{(1)}$ *be as above. Then the level raising inverse image functor* (5.2) *is a fully faithful functor which gives the equivalence*

$$\mathrm{MIWC}(X_1^{(1)})^{\mathrm{n}}_{\mathbb{Q}} \xrightarrow{=} \mathrm{MIWC}^{(m-1)}(X_1)^{\mathrm{n}}_{\mathbb{Q}} \tag{5.3}$$

between $\mathbb{Q}$*-linealized categories.*

PROOF OF THEOREM 5.4. Since we may work locally, we may assume that $S_1 = \operatorname{Spec} R, X_1 = \operatorname{Spec} B_1$ are affine. (Hence $W_n S = \operatorname{Spec} W_n(R)$.)

Also, we may assume that the homomorphism $R \longrightarrow B_1$ corresponding to $X_1 \longrightarrow S_1$ factors as

$$R \longrightarrow A_1 := R[T_i]_{1\le i\le d} \longrightarrow A_1' := R[T_i^{\pm 1}]_{1\le i\le d} \longrightarrow B_1, \tag{5.4}$$

where the last map is etale. Recall [6, 3.2] that a Frobenius lift of a smooth R-algebra C_1 is a projective system of data $(C_n, \phi_n, \delta_n)_{n\ge 1}$ (with C_1 the same as the given one), where

- $C_n\,(n \ge 1)$ is a smooth lifting over $W_n(R)$ of C_1 satisfying $W_n(R) \otimes_{W_{n+1}(R)} C_{n+1} \xrightarrow{=} C_n$.
- $\phi_n\,(n > 1)$ is a map $C_n \longrightarrow C_{n-1}$ over $F : W_n(R) \longrightarrow W_{n-1}(R)$ which is compatible with the Frobenius morphism $C_1 \longrightarrow C_1$.
- $\delta_n\,(n \ge 1)$ is a map $C_n \longrightarrow W_n(C_1)$ such that the diagram

$$\begin{array}{ccc} C_n & \xrightarrow{\delta_n} & W_n(C_1) \\ {\scriptstyle \phi_n}\downarrow & & \downarrow{\scriptstyle F} \\ C_{n-1} & \xrightarrow{\delta_{n-1}} & W_{n-1}(C_1) \end{array}$$

  is commutative.

Now let us define the Frobenius lift $(A_n, \phi_n, \delta_n)_n, (A_n', \phi_n', \delta_n')$ of A_1, A_1' by

$$\begin{aligned} A_n &:= W_n(R)[T_i]_{1\le i\le d}, \qquad \phi_n(T_i) = T_i^p, \qquad \delta_n(T_i) = [T_i], \\ A_n' &:= W_n(R)[T_i^{\pm 1}]_{1\le i\le d}, \qquad \phi_n(T_i^{\pm 1}) = T_i^{\pm p}, \qquad \delta_n(T_i^{\pm 1}) = [T_i^{\pm 1}], \end{aligned}$$

where $[-]$ denotes the Teichmüller lift. Then we have a map $(f_n)_n : (A_n, \phi_n, \delta_n)_n \longrightarrow (A_n', \phi_n', \delta_n')_n$ over the map $A_1 \longrightarrow A_1'$ in (5.4) such that the diagrams

$$\begin{array}{ccc} A_n' & \xrightarrow{\phi_n'} & A_{n-1}' \\ {\scriptstyle f_n}\uparrow & & \uparrow{\scriptstyle f_{n-1}} \\ A_n & \xrightarrow{\phi_n} & A_{n-1}, \end{array} \qquad \begin{array}{ccc} A_n' & \xrightarrow{\delta_n'} & W_n(A_1') \\ {\scriptstyle f_n}\uparrow & & \uparrow{\scriptstyle W_n(f_1)} \\ A_n & \xrightarrow{\delta_n} & W_n(A_1) \end{array} \tag{5.5}$$

are coCartesian. Moreover, by the proof of [6, 3.2], there exists a Frobenius lift $(B_n, \psi_n, \epsilon_n)_n$ of B_1 and a map $(g_n)_n : (A'_n, \phi'_n, \delta'_n)_n \longrightarrow (B_n, \psi_n, \epsilon_n)_n$ over the map $A'_1 \longrightarrow B_1$ in (5.4) such that the diagrams

$$\begin{array}{ccc} B_n & \xrightarrow{\psi'_n} & B_{n-1} \\ {\scriptstyle g_n}\uparrow & & \uparrow{\scriptstyle g_{n-1}} \\ A'_n & \xrightarrow{\phi'_n} & A'_{n-1}, \end{array} \qquad \begin{array}{ccc} B_n & \xrightarrow{\epsilon_n} & W_n(B_1) \\ {\scriptstyle g_n}\uparrow & & \uparrow{\scriptstyle W_n(g_1)} \\ A'_n & \xrightarrow{\delta'_n} & W_n(A'_1) \end{array} \tag{5.6}$$

are coCartesian. Let us define the Frobenius lift $(A_n^{(1)}, \phi_n^{(1)}, \delta_n^{(1)})_n$ of $A_1^{(1)} := R \otimes_{F,R} A_1$ by

$$A_n^{(1)} := W_n(R) \otimes_{F, W_n(R)} A_n, \qquad \phi_n^{(1)} := F \otimes \phi_n, \qquad \delta_n^{(1)} := \mathrm{id} \otimes \delta_n,$$

and define the Frobenius lift $(A'^{(1)}_n, \phi'^{(1)}_n, \delta'^{(1)}_n)_n$ of $A'^{(1)}_1 := R \otimes_{F,R} A'_1$ and the Frobenius lift $(B_n^{(1)}, \psi_n^{(1)}, \epsilon_n^{(1)})_n$ of $B_1^{(1)} := R \otimes_{F,R} B_1$ in the same way.

Let us put $B := \varprojlim_n B_n, B^{(1)} := \varprojlim_n B_n^{(1)}$. Recall that we have the map $F : (W\Omega^\bullet_{B_1^{(1)}}, p^m d) \longrightarrow (W\Omega^\bullet_{B_1}, p^{m-1} d)$ of differential graded algebras defined in (5.1). On the other hand, we have the map $\overline{F}^* : (\Omega^\bullet_{B^{(1)}}, p^m d) \longrightarrow (\Omega^\bullet_B, p^{m-1} d)$ of differential graded algebras such that the composite

$$\Omega^i_{B^{(1)}} \xrightarrow{\overline{F}^*} \Omega^i_B \xrightarrow{p^i} \Omega^i_B$$

is the one defined by the homomorphism $\psi : B^{(1)} \longrightarrow B$ induced by ψ_n's. (This ψ is a lift of relative Frobenius morphism $B_1^{(1)} \longrightarrow B_1$, and the map $\overline{F}^*$ for $\bullet = 1$ is the same as the one defined in Section 1.) Furthermore, the morphisms $\epsilon_n, \epsilon_n^{(1)}$ $(n \in \mathbb{N})$ induce the homomorphisms

$$\begin{aligned} &\epsilon_n : (\Omega^\bullet_{B_n}, p^{m-1} d) \longrightarrow (W_n\Omega^\bullet_{B_1}, p^{m-1} d), \\ &\epsilon := \varprojlim_n \epsilon_n : (\Omega^\bullet_B, p^{m-1} d) \longrightarrow (W\Omega^\bullet_{B_1}, p^{m-1} d), \\ &\epsilon_n^{(1)} : (\Omega^\bullet_{B_n^{(1)}}, p^m d) \longrightarrow (W_n\Omega^\bullet_{B_1^{(1)}}, p^m d), \\ &\epsilon^{(1)} := \varprojlim_n \epsilon_n^{(1)} : (\Omega^\bullet_{B^{(1)}}, p^m d) \longrightarrow (W\Omega^\bullet_{B_1^{(1)}}, p^m d), \end{aligned}$$

and the well-known formulas $FV = p, FdV = d, Fd[x] = [x]^{p-1} d[x]$ of de

Rham-Witt complexes imply the commutativity of the following diagram:

$$\begin{array}{ccc} (W\Omega^{\bullet}_{B_1^{(1)}}, p^m d) & \xrightarrow{F} & (W\Omega^{\bullet}_{B_1}, p^{m-1}d) \\ {\scriptstyle \epsilon^{(1)}}\uparrow & & \uparrow{\scriptstyle \epsilon} \\ (\Omega^{\bullet}_{B^{(1)}}, p^m d) & \xrightarrow{\overline{F}^*} & (\Omega^{\bullet}_{B}, p^{m-1}d). \end{array} \tag{5.7}$$

To prove the theorem, it suffices to prove that the homomorphisms

$$H^0(F) : H^0(W\Omega^{\bullet}_{B_1^{(1)}}, p^m d) \longrightarrow H^0(W\Omega^{\bullet}_{B_1}, p^{m-1}d), \tag{5.8}$$

$$\begin{aligned} H^i(F) \otimes \mathbb{Q} : H^i(W\Omega^{\bullet}_{B_1^{(1)}}, p^m d) \otimes \mathbb{Q} \\ \longrightarrow H^i(W\Omega^{\bullet}_{B_1}, p^{m-1}d) \otimes \mathbb{Q} \quad (i \in \mathbb{N}) \end{aligned} \tag{5.9}$$

induced by F are isomorphisms. We prove this by using the commutative diagram (5.7). First, by Theorem 4.4, the homomorphisms

$$H^0(\overline{F}^*) : H^0(\Omega^{\bullet}_{B^{(1)}}, p^m d) \longrightarrow H^0(\Omega^{\bullet}_{B}, p^{m-1}d), \tag{5.10}$$

$$H^i(\overline{F}^*) \otimes \mathbb{Q} : H^i(\Omega^{\bullet}_{B^{(1)}}, p^m d) \otimes \mathbb{Q} \longrightarrow H^i(\Omega^{\bullet}_{B}, p^{m-1}d) \otimes \mathbb{Q} \quad (i \in \mathbb{N}) \tag{5.11}$$

induced by $\overline{F}^*$ are isomorphisms. (**Hyp**$(\infty, \infty, \infty, \infty)$ is satisfied because of the left cocartesian diagram in (5.6).) Next we prove that the homomorphisms

$$H^0(\epsilon) : H^0(\Omega^{\bullet}_{B}, p^{m-1}d) \longrightarrow H^0(W\Omega^{\bullet}_{B_1}, p^{m-1}d), \tag{5.12}$$

$$\begin{aligned} H^i(\epsilon) \otimes \mathbb{Q} : H^i(\Omega^{\bullet}_{B}, p^{m-1}d) \otimes \mathbb{Q} \\ \longrightarrow H^i(W\Omega^{\bullet}_{B_1}, p^{m-1}d) \otimes \mathbb{Q} \quad (i \in \mathbb{N}) \end{aligned} \tag{5.13}$$

induced by ϵ are isomorphisms. To prove this, we follow the argument in [6, Theorem 3.5]: First, let us note the commutative diagram appeared there

$$\begin{array}{ccc} (\Omega^{\bullet}_{B_n}, d) & \xleftarrow{=} & (B_{2n} \otimes_{A_{2n}, \phi^n} \Omega^{\bullet}_{A_n}, \mathrm{id} \otimes d) \\ {\scriptstyle \epsilon_n}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes \delta_n} \\ (W_n\Omega^{\bullet}_{B_1}, d) & \xleftarrow{=} & (B_{2n} \otimes_{A_{2n}, \phi^n} W_n\Omega_{A_1}, \mathrm{id} \otimes d), \end{array} \tag{5.14}$$

where we denoted the maps induced from ϵ_n, δ_n by the same latters abusively, and $\phi^n : A_{2n} \longrightarrow A_n$ is the map $\phi_{n+1} \circ \cdots \circ \phi_{2n}$. Also, in the proof of

[6, Theorem 3.5], it is shown that we have the decomposition $(W_n\Omega^\bullet_{A_1}, d) = (C^\bullet_{\mathrm{int}}, d) \oplus (C^\bullet_{\mathrm{frac}}, d)$ such that $\delta_n : (\Omega^\bullet_{A_n}, d) \longrightarrow (W_n\Omega_{A_1}, d)$ induces the isomorphism of complexes $(\Omega^\bullet_{A_n}, d) \xrightarrow{=} (C^\bullet_{\mathrm{int}}, d)$ and that the complex $(C^\bullet_{\mathrm{frac}}, d)$ is acyclic. Hence $\delta_n : (\Omega^\bullet_{A_n}, d) \longrightarrow (W_n\Omega_{A_1}, d)$ is a quasi-isomorphism and so is $\epsilon_n : (\Omega^\bullet_{B_n}, d) \longrightarrow (W_n\Omega^\bullet_{B_1}, d)$ by (5.14). Hence $\epsilon := \varprojlim_n \epsilon_n : (\Omega^\bullet_B, d) \longrightarrow (W\Omega^\bullet_{B_1}, d)$ is also a quasi-isomorphism. Since $\Omega^i_B, W\Omega^i_{B_1}$'s $(i = 0, 1)$ are p-torsion free, we have $H^0(\Omega^\bullet_B, p^{m-1}d) = H^0(\Omega^\bullet_B, d), H^0(W\Omega^\bullet_{B_1}, p^{m-1}d) = H^0(W\Omega^\bullet_{B_1}, d)$ and so the map (5.12) is an isomorphism. Also, using the commutative diagrams

$$\begin{array}{ccccc}
\Omega^{i-1}_B & \xrightarrow{=} & \Omega^{i-1}_B & \xrightarrow{p^{2(m-1)}} & \Omega^{i-1}_B \\
{\scriptstyle d}\downarrow & & {\scriptstyle p^{m-1}d}\downarrow & & {\scriptstyle d}\downarrow \\
\Omega^i_B & \xrightarrow{p^{m-1}} & \Omega^i_B & \xrightarrow{p^{m-1}} & \Omega^i_B \\
{\scriptstyle d}\downarrow & & {\scriptstyle p^{m-1}d}\downarrow & & {\scriptstyle d}\downarrow \\
\Omega^{i+1}_B & \xrightarrow{p^{2(m-1)}} & \Omega^{i+1}_B & \xrightarrow{=} & \Omega^{i+1}_B,
\end{array}
\qquad
\begin{array}{ccccc}
W\Omega^{i-1}_{B_1} & \xrightarrow{=} & W\Omega^{i-1}_{B_1} & \xrightarrow{p^{2(m-1)}} & W\Omega^{i-1}_{B_1} \\
{\scriptstyle d}\downarrow & & {\scriptstyle p^{m-1}d}\downarrow & & {\scriptstyle d}\downarrow \\
W\Omega^i_{B_1} & \xrightarrow{p^{m-1}} & W\Omega^i_{B_1} & \xrightarrow{p^{m-1}} & W\Omega^i_{B_1} \\
{\scriptstyle d}\downarrow & & {\scriptstyle p^{m-1}d}\downarrow & & {\scriptstyle d}\downarrow \\
W\Omega^{i+1}_{B_1} & \xrightarrow{p^{2(m-1)}} & W\Omega^{i+1}_{B_1} & \xrightarrow{=} & W\Omega^{i+1}_{B_1},
\end{array}$$

$$\begin{array}{ccccc}
\Omega^{i-1}_B & \xrightarrow{p^{2(m-1)}} & \Omega^{i-1}_B & \xrightarrow{\epsilon} & W\Omega^{i-1}_{B_1} & \xrightarrow{=} & W\Omega^{i-1}_{B_1} \\
{\scriptstyle p^{m-1}d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle p^{m-1}d}\downarrow \\
\Omega^i_B & \xrightarrow{p^{m-1}} & \Omega^i_B & \xrightarrow{\epsilon} & W\Omega^i_{B_1} & \xrightarrow{p^{m-1}} & W\Omega^i_{B_1} \\
{\scriptstyle p^{m-1}d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle p^{m-1}d}\downarrow \\
\Omega^{i+1}_B & \xrightarrow{=} & \Omega^{i+1}_B & \xrightarrow{\epsilon} & W\Omega^{i+1}_{B_1} & \xrightarrow{p^{2(m-1)}} & W\Omega^{i+1}_{B_1},
\end{array}$$

we can show that the kernel and the cokernel of the homomorphism

$$p^{2(m-1)}H^i(\epsilon) : H^i(\Omega^\bullet_B, p^{m-1}d) \longrightarrow H^i(W\Omega^\bullet_{B_1}, p^{m-1}d)$$

are killed by $p^{4(m-1)}$. Hence (5.13) is also an isomorphism.

In the same way as above, we can also prove that the homomorphisms

$$H^0(\epsilon^{(1)}) : H^0(\Omega^\bullet_{B^{(1)}}, p^m d) \longrightarrow H^0(W\Omega^\bullet_{B^{(1)}_1}, p^m d), \tag{5.15}$$

$$\begin{aligned} H^i(\epsilon^{(1)}) \otimes \mathbb{Q} : H^i(\Omega^\bullet_{B^{(1)}}, p^m d) \otimes \mathbb{Q} \\ \longrightarrow H^i(W\Omega^\bullet_{B^{(1)}_1}, p^m d) \otimes \mathbb{Q} \quad (i \in \mathbb{N}) \end{aligned} \tag{5.16}$$

induced by $\epsilon^{(1)}$ are isomorphisms. Then, by (5.10), (5.11), (5.12), (5.13), (5.15), (5.16) and the diagram (5.7), we can conclude that the homomorphisms (5.8), (5.9) are isomorphisms, as desired. □

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