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Existence of Solutions to the Heat Convection Equations in a Time-dependent Domain with Mixed Boundary Conditions

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Abstract. In this paper we are concerned with the initial boundary value problems of the heat convection equations in a time-dependent domain with mixed boundary conditions involving the total pressure of fluid. We obtain the existence of a weak solution to the problem. By a transformation of unknown functions and a penalty method we connect the problem to an elliptic operator equation for functions defined in the time-dependent domain. Owing to the transformation we do not need to assume that the given data are small enough. This method is also valid for the Navier-Stokes equations with the nonstandard boundary conditions.

1. Introduction

There are vast literatures for the initial boundary or periodic problems of the Navier-Stokes equations in time-dependent domains and various methods have been used for those problems.

Using a diffeomorphism conserving volumes, many researchers transform the time-dependent domains into the time-independent domains and the systems into perturbed one in time-independent domains (cf. [4]∼[7], [11], [18], [24], [30], [31], [36] and references therein).

There are also some papers that transform the systems to the differential inclusions with subdifferential operators on expanded spatial domains which include the time-dependent domains. For example, in [29] the original

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equation is reduced to an abstract equation in an appropriate Hilbert space, which can be regarded as a perturbed equation of an equation generated by a time dependent subdifferential operator. Then the desired solutions are constructed by the successive approximation method.

Penalty methods are also used for these problems. In [12] a kind of penalty method was introduced by which the problem is reduced to one in a time-independent domain. As a kind of penalty method, the elliptic regularization was used to study the Navier-Stokes equations in regions with moving boundaries(cf. [22], [23], [33]∼[35]).

Except [11] which dealt with Neumann type boundary condition, all papers mentioned above dealt with initial boundary value or periodic problems with Dirichlet boundary conditions. In [26] 2-D time periodic Navier-Stokes equations in a time-dependent domain conserving volume with a slip boundary condition was studied, where the equation of rotation was used.

On the other hand, in [32] the Navier-Stokes problem in a time-dependent domain with a mixed boundary condition is considered. In [32] the part of boundary for homogeneous Dirichlet condition is cylindrical and the boundary condition on the other part of boundary is such a special one that guarantees the existence of a solution to the elliptic operator equation obtained by the penalty method.

The Stokes and the Navier-Stokes equations in time-dependent domains with boundary conditions involving the pressure were studied in [20] by the elliptic regularization. But, the Navier-Stokes equations in a domain decreasing along the time was studied under homogeneous boundary condition for the velocity of fluid, which means that a, b, $c = 0$ in (2.1) bellow (see Theorem 4.1 in [20]). To avoid a monotone contraction condition on the time-variation of the domain used in [20], a similar (really more simple) problem was studied in [3]. It was assumed that boundary values must be small enough and every connected component of surface for the pressure boundary condition is included in one of the coordinates planes.

Initial boundary value or periodic problems of the heat convection equations in time-dependent domains were also studied (cf. [15], [17] and references therein). However, there seems to be less literature devoted to the heat convection equations on time-dependent domains.

In [27], [28] the initial boundary value and periodic problems of the heat convection equations in time-dependent domains with Dirichlet boundary condition were studied by the penalty method used in [12](weak solutions) and the differential inclusions with a subdifferential operators on an expanded spatial domain which includes time-dependent domains (strong solutions). And [21] dealt with heat convection equations in time-dependent domains with homogeneous Dirichlet boundary condition by a penalty method similar to [12]. Also, in [15]$\sim$[17] initial boundary value and periodic problems of the heat convection equations in time-dependent domains with Dirichlet boundary condition were studied by the differential inclusions with time dependent subdifferential operators.

In the case of time-independent domains the initial boundary value and periodic problems of the heat convection equations with homogeneous Dirichlet boundary condition for the velocity of fluid and mixture of non-homogeneous Dirichlet and Neumann conditions for the temperature was studied (cf. [25]).

In this paper we are concerned with the existence of weak solutions to the initial boundary value problems of the heat convection equations in a time-dependent domain with nonstandard boundary conditions involving the total pressure (Bernoulli's pressure) of fluid. In our case time-variation of the domain is similar to one in [20], but more general because in our case on some subsegments of finite time interval the domain is expanded. Unlike [3] we do not assume that boundary values must be small enough. For the temperature function we are concerned with mixed boundary conditions which may include inhomogeneous Dirichlet, Neumann and Robin conditions together. Due to such nonstandard boundary conditions it is difficult to reduce the problem to one in time-independent domains. Even though, in the case of time-independent domains, a lateral subsurface for one type of boundary condition may vary with respect to the time variable t, and so it is difficult to reduce the problem with boundary conditions on cylindrical surfaces expressed by products of parts of boundary of a spatial domain and time interval.

To obtain the existence of solutions, by changes of unknown functions and a penalty method we connect the problem to an elliptic operator equation for functions defined on the time-dependent domain. Using such a transformation of unknown functions, we get coercivity of the operator in the penalty problem without assuming that the given data are small enough and the sign of coefficient in Robin boundary condition for the temperature is positive.

This paper consists of 5 sections. In Section 2 notation, the problem, the definition of weak solution and the main result(Theorem 2.2) are stated. In Section 3 changes of unknown functions are considered. Thus, we come to the new problem 3.1 equivalent to the original one. In Section 4 we study an auxiliary penalized problem obtained by the elliptic regularization. In Section 5 by showing mainly the precompactness of solutions to the auxiliary problem, we prove the main result. The piece monotone condition of domain along the time was used only for the proof of the precompactness of solutions to the auxiliary problem.

Let us end this section by a remark that all arguments in this paper are valid for the Navier-Stokes equations with the nonstandard boundary conditions.

A practical model of our mathematical problem is a pump with a to-and-fro moving piston under consideration of heat exchange.

2. Problem and Main Result

Let $\Omega(t)$ be bounded connected domains of $R^l,\ l=2,3$, with Lipschitz boundary, $Q=\bigcup_{t\in(0,T)}\Omega(t)\times\{t\},\ 0<T<\infty$, $[0,T]=\bigcup_{i=0}^{\delta}[t_i,t_{i+1}],\ t_i<t_{i+1}$, $Q_i=\bigcup_{t\in(t_i,t_{i+1})}\Omega(t)\times\{t\},\ \Sigma=\bigcup_{t\in(0,T)}\partial\Omega(t)\times\{t\}$, $\Sigma_1,\ \Sigma_2,\ \Sigma_3$ be open subsets of Σ such that $\bar\Sigma_1\cup\bar\Sigma_2\cup\bar\Sigma_3=\Sigma$ and $\Sigma_i(t)=\Sigma_i\cap\bar\Omega(t)$. Let $n(x,t)$ be unit outward normal on $\partial\Omega(t)$ for fixed t, $e_l=(0,0,1)$ for $l=3$ and $e_l=(0,1)$ for $l=2$.

We consider the following initial boundary value problem of the Boussinesq approximation of heat convection

$$
(2.1)\quad
\begin{cases}
\dfrac{\partial v}{\partial t}-\nu\Delta v+(v,\nabla)v+\nabla p+\mu(x,t)\theta e_l=f_1,\\
\operatorname{div} v=0,\\
\dfrac{\partial \theta}{\partial t}-\Delta\theta+v\cdot\nabla\theta=f_2,\\
v\mid_{\Sigma_1}=a(x,t),\quad v\times n\mid_{\Sigma_2}=b(x,t)\times n,\quad (p+\dfrac{1}{2}|v|^2)\mid_{\Sigma_2}=p_0(x,t),\\
v\cdot n\mid_{\Sigma_3}=c(x,t),\ (\nabla\times v)\times n|_{\Sigma_3}=\varphi,\\
\theta\mid_{\Sigma_2\cup\Sigma_3}=d(x,t),\quad \left(k(x,t)\theta+\dfrac{\partial\theta}{\partial n}\right)\mid_{\Sigma_1}=e(x,t),\\
v(0)=v_0,\ \theta(0)=\theta_0,
\end{cases}
$$

where v denotes the velocity, p pressure and θ temperature. We will use the following notations. If X is a Banach space, then $\mathbf{X} = X^l$. Let $H^1(Q) = W_2^1(Q)$, and so $\mathbf{H}^1(Q) = H^1(Q)^l$. An inner product in the space $\mathbf{L}_2(Q)$ or $L_2(Q)$ is denoted by $(\cdot\,,\cdot)_Q$; and $\langle\cdot\,,\cdot\rangle_Y$ means the duality product between a space defined on Y and its dual space. Also, $(\cdot\,,\cdot)_{\Gamma_i}$ is an inner product in the $\mathbf{L}_2(\Gamma_i)$ and $\langle\cdot\,,\cdot\rangle_{\Gamma_i}$ means the duality product between $\mathbf{H}^{\frac{1}{2}}(\Gamma_i)$ and $\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)$. The inner product between a and b in R^l is denoted by $a \cdot b$. For functions of (x,t), the operators Δ, *div* and ∇ are with respect to the spatial variable x. Let $|v|^2_{\Omega(t)} = \int\limits_{\Omega(t)} |v|^2\,dx$ and $\|v\|^2_{\Omega(t)} = \int\limits_{\Omega(t)} |\nabla v|^2\,dx$ for $v \in \mathbf{H}^1(\Omega(t))$ or $v \in H^1(\Omega(t))$.

For function v defined on Q define $\alpha(v)$ by

$$\alpha(v) = \left(\int_0^T \|v\|^2_{\Omega(t)}\,dt\right)^{\frac{1}{2}}$$

whenever the integral make sense. Let

$$\begin{aligned}
&\Lambda(Q) = \{v \in \mathbf{C}^2(\bar{Q}) : \operatorname{div} v = 0,\ v\,|_{\Sigma_1} = 0,\quad v \times n\,|_{\Sigma_2} = 0,\ v \cdot n\,|_{\Sigma_3} = 0\},\\
&H(Q) = \{\text{the completion of } \Lambda(Q) \text{ in } \mathbf{L}_2(Q)\},\\
&\mathbf{V}(Q) = \{\text{the completion of } \Lambda(Q) \text{ under the norm}\,\alpha(v)\},\\
&\mathbf{W}(Q) = \{\text{the completion of } \Lambda(Q) \text{ in the space}\,\mathbf{H}^1(Q)\},\\
&H(\Omega(s)) = \{v \in \mathbf{L}_2(\Omega(s)) : \operatorname{div} v = 0\}.
\end{aligned}$$

Let

$$\begin{aligned}
&D(Q) = \{y : y \in C^2(\bar{Q}),\ y|_{\Sigma_2 \cup \Sigma_3} = 0\},\\
&X(Q) = \{\text{the completion of } D(Q) \text{ under the norm}\,\alpha(y)\},\\
&Y(Q) = \{\text{the completion of } D(Q) \text{ in the space}\,H^1(Q)\}.
\end{aligned}$$

Note that due to 1) in Assumption 2.2 bellow, $\alpha(v)$ is a norm, respectively, in $\Lambda(Q)$ and $D(Q)$.

For the domain Q and its boundary we assume the followings.

Assumption 2.1 (cf. $(A1)$ in [20]). *$\partial\Omega(t)$ is Lipschitz continuous for $0 < t < T$.*

ASSUMPTION 2.2 (cf. ($A2$) in [20]). *1)* $\Sigma_i \cap \Sigma_j = \emptyset,\ i \neq j,\ i,j = 1,2,3;$ $\Sigma_1(t) \neq \emptyset\ \forall t \in (0,T)$ *and* $\Sigma_2(t) \cup \Sigma_3(t) \neq \emptyset\ \forall t \in (0,T)$.
2) There exists a domain $\Omega \subset R^l$ *with Lipschitz boundary such that* $\Omega(t) \subset \Omega$ *for every* $t \in [0,T]$ *and* $(\Sigma_2 \cup \Sigma_3) \cap (\partial\Omega \times (0,T)) = \Sigma_2 \cup \Sigma_3$.
3) For every subinterval (t_i, t_{i+1}) *of* $[0,T] = \bigcup_{i=0}^{\delta}[t_i, t_{i+1}]$ *one of the following two conditions is satisfied:*
(3a)

$$\Omega(s) \subset \Omega(t), \quad \mu(\Omega(t) \setminus \Omega(s)) \leq \omega(s-t) \quad \text{for } t_i \leq t < s \leq t_{i+1}$$

or
(3b)

$$\Omega(t) \subset \Omega(s) \quad \mu(\Omega(s) \setminus \Omega(t)) \leq \omega(s-t) \quad \text{for } t_i \leq t < s \leq t_{i+1},$$

where $\mu(\sigma)$ *denotes the measure of set* σ *and the function* $\omega : R^+ \to R^+$ *is such that* $\omega(h) \to 0$ *as* $h \to 0$.
4) If $(x,t) \in \Sigma_1$, *then*

$$\begin{aligned} &\text{in the case (3a)}: \quad (x,s) \in \Sigma_1 \text{ or } (x,s) \notin \bar{Q} \quad \text{for } t_i < t < s < t_{i+1}, \\ &\text{in the case (3b)}: \quad (x,s) \in \Sigma_1 \text{ or } (x,s) \notin \bar{Q} \quad \text{for } t_1 < s < t < t_{i+1}. \end{aligned}$$

5) For $i = 2,3$

$$\begin{aligned} &\Sigma_i(s) \subset \Sigma_i(t) \quad \text{for } t_i < t < s < t_{i+1} \text{ in the case of (3a)}, \\ &\Sigma_i(t) \subset \Sigma_i(s) \quad \text{for } t_i < t < s < t_{i+1} \text{ in the case of (3b)}. \end{aligned}$$

6) There exists a constant $k_0 \geq 0$ *such that*

$$\Big(\frac{1}{2} + \|k(x,t)\|_{L_\infty(\Sigma_1)}\Big) \int_{\Sigma_1} \rho^2\, d\sigma \leq \frac{1}{4} \int_0^T \|\nabla\rho\|^2_{\Omega(t)}\, dt + k_0 \int_0^T |\rho|^2_{\Omega(t)}\, dt \qquad \forall \rho \in Y(Q).$$

REMARK 2.1. For any $t \in (0,T)$ there exists $k_0(t)$ such that

$$\Big(\frac{1}{2} + \|k(x,t)\|_{L_\infty(\Sigma_1)}\Big) \int_{\Sigma_1(t)} \rho^2\, d\sigma \leq \frac{1}{4}\|\nabla\rho\|^2_{\Omega(t)} + k_0(t)|\rho|^2_{\Omega(t)} \quad \forall \rho \in Y(Q)$$

(cf. Theorem 1.6.6 in [8] or (1), p. 258 in [10]). In the case of (3a) if $k(x,t) \geq 0$, then 6) is not necessary (see (4.9)).

REMARK 2.2. If we assume $\bar{\Sigma}_2(t) \cap \bar{\Sigma}_3(t) = \emptyset$ as [9], then 5) of Assumption 2.2 follows from 2)-4) of Assumption 2.2.

ASSUMPTION 2.3 (cf. ($A3$) in [20]). *There exists $c_1 > 0$ such that for every $v \in \mathbf{V}(Q)$ and $t \in (0,T)$*

$$|\nabla \times v(\cdot,t)\,|^2_{\Omega(t)} \geq c_1 \|v(\cdot,t)\|^2_{\mathbf{H}^1(\Omega(t))}.$$

REMARK 2.3. Under $\bar{\Sigma}_2(t) \cap \bar{\Sigma}_3(t) = \emptyset$ and 1) in Assumption 2.2, for the case of 3-D domain with $\partial\Omega \in C^{1.1}$ or convex polyhedron the inequality in Assumption 2.3 with $c_1(t)$ depending on t holds (cf. Proposition 1.1 in [2] or Lemma 1.4 in [9]). For domains with $\partial\Omega \in C^2$ without assuming $\bar{\Sigma}_2 \cap \bar{\Sigma}_3 = \emptyset$ the assertion above is valid(cf. Lemma 2 in [14]). When $\Sigma_3 = \emptyset$ for some 3-D domains with $\partial\Omega \in C^{0.1}$ also the assertion above is valid(cf. A.2 in [9]).

For the given data we assume the followings.

ASSUMPTION 2.4. (cf. (1.8) in [20]). *There exists a function $U \in \mathbf{H}^1(Q) \cap \mathbf{L}_\infty(Q)$ such that*
$div U(\cdot,t) = 0$, $U|_{\Sigma_1} = a(x,t)$, $U \times n|_{\Sigma_2} = b(x,t) \times n$, $U \cdot n|_{\Sigma_3} = c(x,t)$.

ASSUMPTION 2.5. *1) There exists a function $G(Q) \in H^1(Q) \cap L_\infty(Q)$ such that $G|_{\Sigma_2 \cup \Sigma_3} = d(x,t)$.*
2) $v_0 - U(x,0) \in H(\Omega(0))$, $\theta_0 - G(x,0) \in L_2(\Omega(0))$,
3) $f_1 \in \mathbf{L}_2(Q)$, $f_2 \in L_2(Q)$,
4) $p_0(x,t) \in L_2(0,T;H^{-1/2}(\Sigma_2(t)))$, $\varphi \in \mathbf{L}_2(0,T;H^{-1/2}(\Sigma_3(t)))$,
5) $k(x,t) \in L_\infty(\Sigma_1)$, $\mu(x,t) \in L_\infty(Q)$ and $e(x,t) \in L_2(0,T;H^{-1/2}(\Sigma_1(t)))$.

Let $r(x,t)$ be unit outward normal on the boundary Σ and $(\hat{r,t})$ be the angle between r and the positive direction of t-axis. For $v \in C^2(\bar{Q})$, $u \in$

$\Lambda(Q)$ in view of (2.1) we have

$$\int_Q \frac{\partial v}{\partial t} u\,dxdt = (v(x,T), u(x,T))_{\Omega(T)} - (v_0, u(x,0))_{\Omega(0)} - \int_Q v \frac{\partial u}{\partial t}\,dxdt, \tag{2.2}$$

where the fact that $\int_{\Sigma_2 \cup \Sigma_3} vu \cos(\hat{r,t})\,d\sigma = 0$ by 2) of Assumption 2.2 was used.

Using the facts that

$$-\Delta v = \operatorname{rot}\operatorname{rot} v - \operatorname{grad}(\operatorname{div} v),$$
$$(\operatorname{rot} v, u) - (v, \operatorname{rot} u) = -(v \times n, u)_{\partial\Omega},$$

for $v \in \{v : v \in \mathbf{C}^\infty(\bar{Q}), \operatorname{div} v = 0\}$, $p \in H^1(Q)$ and $u \in \Lambda(Q)$, we get

$$\begin{aligned}
-\nu(\Delta v, u)_Q &= \nu(\operatorname{rot} v, \operatorname{rot} u)_Q - \nu(\operatorname{rot} v \times n, u)_{\Sigma_2 \cup \Sigma_3} \\
&= \nu(\operatorname{rot} v, \operatorname{rot} u)_Q - \nu(\operatorname{rot} v \times n, u)_{\Sigma_2} - \nu(\operatorname{rot} v \times n, u)_{\Sigma_3} \\
&= \nu(\operatorname{rot} v, \operatorname{rot} u)_Q - \nu(\operatorname{rot} v \times n, u)_{\Sigma_3}, \\
(\nabla p, u) &= < p, u \cdot n >_{\Sigma_2} .
\end{aligned}$$

Thus, taking

$$(v, \nabla)v = \operatorname{rot} v \times v + \frac{1}{2}\operatorname{grad}|v|^2$$

into account, under the boundary condition of (2.1) we have

$$\begin{aligned}
&\int_0^T (-\nu\Delta v + (v,\nabla)v + \nabla p, u)_{\Omega(t)}\,dt \\
&\quad = \int_0^T (-\nu\Delta v + \operatorname{rot} v \times v + \frac{1}{2}\operatorname{grad}|v|^2 + \nabla p, u)_{\Omega(t)}\,dt \\
&\quad = \nu(\operatorname{rot} v, \operatorname{rot} u)_Q - \nu(\operatorname{rot} v \times n, u)_{\Sigma_3} \\
&\quad\quad + \int_0^T (\operatorname{rot} v \times v, u)_{\Omega(t)}\,dt + (p_0(x,t), u \cdot n)_{\Sigma_2}.
\end{aligned} \tag{2.3}$$

Similarly, when $\theta \in C^2(\bar{Q})$, $y \in D(Q)$, in view of (2.1) we have

$$\int_Q \frac{\partial \theta}{\partial t} y\,dxdt = (\theta(x,T), y(x,T))_{\Omega(T)} - (\theta_0, y(x,0))_{\Omega(0)} + \int_{\Sigma_1} \theta y \cos(\hat{r,t})\,d\sigma - \int_Q \theta \frac{\partial y}{\partial t}\,dxdt, \tag{2.4}$$

$$\int_Q (-\Delta\theta + v\cdot\nabla\theta)y\,dxdt = \int_Q (\nabla\theta\cdot\nabla y + v\cdot\nabla\theta y)\,dxdt + \int_{\Sigma_1} k(x,t)\theta y\,d\sigma - \int_{\Sigma_1} e(x,t)y\,d\sigma. \tag{2.5}$$

Thus, in view of (2.2)-(2.5), we introduce the following definition.

Definition 2.1. A function (v,θ) is called a solution to (2.1) if (v,θ) satisfies the following

$$\begin{aligned}
&v - U \in \mathbf{V}(Q),\ \theta - G \in X(Q);\\
&-\int_Q v\frac{\partial u}{\partial t}\,dxdt - \int_Q \theta\frac{\partial y}{\partial t}\,dxdt + \nu\int_0^T (\mathrm{rot}\,v, \mathrm{rot}\,u)_{\Omega(t)}\,dt\\
&\quad + \int_0^T (\mathrm{rot}\,v\times v, u)_{\Omega(t)}\,dt + \int_0^T (\mu(x,t)\theta e_l, u)_{\Omega(t)}\,dt\\
&\quad + \int_Q (\nabla\theta\cdot\nabla y + v\cdot\nabla\theta y)\,dxdt + \int_{\Sigma_1}[\theta y\cos(\hat{r,t}) + k(x,t)\theta y]\,d\sigma\\
&= \nu\int_0^T \langle\varphi, u\rangle_{\Sigma_3(t)}\,dt - \int_0^T \langle p_0(x,t), u\cdot n\rangle_{\Sigma_2(t)}\,dt + \int_0^T \langle e, y\rangle_{\Sigma_1(t)}\,dt\\
&\quad + (v_0, u(x,0))_{\Omega(0)} + (\theta_0, y(x,0))_{\Omega(0)} + \int_Q f_1 u\,dxdt + \int_Q f_2 y\,dxdt\\
&\qquad\qquad \forall u\in\Lambda(Q),\ \forall y\in D(Q) \text{ with } (u(x,T)=0,\ y(x,T)=0).
\end{aligned}$$

Now let us state the main result of this paper.

Theorem 2.2. *Under Assumptions 2.1∼2.5, there exists a solution* (v,θ) *to problem* (2.1) *and*

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|(v,\theta)\|_{\boldsymbol{L}_2(\Omega(t))^{l+1}} \le c.$$

Remark 2.4. Since $\int_0^T \langle p_0(x,t), u\cdot n\rangle_{\Sigma_2(t)}\,dx = \int_0^T \langle p_0(x,t)+c(t), u\cdot n\rangle_{\Sigma_2(t)}\,dx \quad \forall u\in\Lambda(Q)$, p is determined up to a constant with respect to x.

Since $H^{\frac{1}{2}}(\Sigma_i(t)) = H_0^{\frac{1}{2}}(\Sigma_i(t))$ (cf. Theorem 11.1, ch. 1 in [19]), $\langle\varphi, u\rangle_{\Sigma_3(t)}$ and $\langle p_0(x,t), u\cdot n\rangle_{\Sigma_2(t)}$ make sense.

3. Changes of Unknown Functions

Having in mind Assumption 2.4 and 1) of Assumption 2.5 and putting $v = \tilde{v} + U$, $\theta = \tilde{\theta} + G$, from Definition 2.1 we get

$$
\begin{aligned}
(3.1)\quad & \tilde{v} \in \mathbf{V}(Q),\ \tilde{\theta} \in X(Q); \\
& -\int_Q \tilde{v}\frac{\partial u}{\partial t}\,dxdt - \int_Q \tilde{\theta}\frac{\partial y}{\partial t}\,dxdt + \nu\int_0^T (\operatorname{rot}\tilde{v}, \operatorname{rot} u)_{\Omega(t)}\,dt \\
&\quad + \int_0^T (\operatorname{rot}\tilde{v}\times\tilde{v}, u)_{\Omega(t)}\,dt + \int_0^T (\operatorname{rot}\tilde{v}\times U + \operatorname{rot} U\times\tilde{v}, u)_{\Omega(t)}\,dt \\
&\quad + \int_0^T (\mu(x,t)\tilde{\theta}e_l, u)_{\Omega(t)}\,dt + \int_Q \nabla\tilde{\theta}\cdot\nabla y\,dxdt \\
&\quad + \int_Q \big(\tilde{v}\cdot\nabla\tilde{\theta} + U\cdot\nabla\tilde{\theta} + \tilde{v}\cdot\nabla G\big)y\,dxdt \\
&\quad + \int_{\Sigma_1} [\tilde{\theta}y\cos(\hat{r,t}) + k(x,t)\tilde{\theta}y]\,d\sigma \\
&= \int_Q U\frac{\partial u}{\partial t}\,dxdt + \int_Q G\frac{\partial y}{\partial t}\,dxdt - \nu\int_0^T (\operatorname{rot} U, \operatorname{rot} u)_{\Omega(t)}\,dt \\
&\quad - \int_0^T (\operatorname{rot} U\times U, u)_{\Omega(t)}\,dt - \int_0^T (\mu(x,t)Ge_l, u)_{\Omega(t)}\,dt \\
&\quad - \int_Q \nabla G\cdot\nabla y\,dxdt - \int_Q (U\cdot\nabla G)\,y\,dxdt \\
&\quad - \int_{\Sigma_1} [Gy\cos(\hat{r,t}) + k(x,t)Gy]\,d\sigma + \nu\int_0^T \langle\varphi, u\rangle_{\Sigma_3(t)}\,dt \\
&\quad - \int_0^T \langle p_0(x,t), u\cdot n\rangle_{\Sigma_2(t)}\,dt + \int_0^T \langle e, y\rangle_{\Sigma_1(t)}\,dt \\
&\quad + (v_0, u(x,0))_{\Omega(0)} + (\theta_0, y(x,0))_{\Omega(0)} + \int_Q f_1 u\,dxdt + \int_Q f_2 y\,dxdt \\
&\qquad\qquad \forall u \in \Lambda(Q),\ \forall y \in D(Q) \text{ with } u(x,T) = 0,\ y(x,T) = 0.
\end{aligned}
$$

In (3.1) let us make again changes of the unknown functions by $w = e^{k_1 t}\tilde{v}, \tau = e^{k_1 t}\tilde{\theta}$, where k_1 is a constants to be determined in Theorem 4.1 later. Then we have

$$
-\int_Q \tilde{v}\frac{\partial u}{\partial t}\,dxdt = -\int_Q e^{-k_1 t}w\frac{\partial u}{\partial t}\,dxdt = -\int_Q w\frac{\partial\tilde{u}}{\partial t}\,dxdt - k_1\int_Q w\tilde{u}\,dxdt,
$$

$$-\int_Q \tilde{\theta}\frac{\partial y}{\partial t}\,dxdt = -\int_Q e^{-k_1 t}\tau\frac{\partial y}{\partial t}\,dxdt = -\int_Q \tau\frac{\partial \tilde{y}}{\partial t}\,dxdt - k_1\int_Q \tau\tilde{y}\,dxdt,$$

$$\int_Q U\frac{\partial u}{\partial t}\,dxdt = \int_Q Ue^{k_1 t}e^{-k_1 t}\frac{\partial u}{\partial t}\,dxdt = \int_Q Ue^{k_1 t}(\frac{\partial \tilde{u}}{\partial t} + k_1\tilde{u})\,dxdt,$$

$$\int_Q G\frac{\partial y}{\partial t}\,dxdt = \int_Q Ge^{k_1 t}e^{-k_1 t}\frac{\partial y}{\partial t}\,dxdt = \int_Q Ge^{k_1 t}(\frac{\partial \tilde{y}}{\partial t} + k_1\tilde{y})\,dxdt,$$

$$\int_Q (\tilde{v}\cdot\nabla G)y\,dxdt = -\int_0^T (\mathrm{div}\,(\tilde{v}y), G)_{\Omega(t)}\,dt = -\int_Q \tilde{v}\cdot\nabla yG\,dxdt$$
$$= -\int_Q w\cdot\nabla\tilde{y}G\,dxdt,$$

where $\tilde{u} = e^{-k_1 t}u,\ \tilde{y} = e^{-k_1 t}y$.

Substituting these in (3.1), we know that the problem to find a solution to (2.1) in the sense of Definition 2.1 is equivalent to the following problem.

Problem 3.1. Find $(w,\tau) \in \mathbf{V}(Q)\times X(Q)$ such that

$$\begin{aligned}
(3.2)\quad & -\int_Q w\frac{\partial \tilde{u}}{\partial t}\,dxdt - \int_Q \tau\frac{\partial \tilde{y}}{\partial t}\,dxdt + \nu\int_0^T (\mathrm{rot}\,w, \mathrm{rot}\,\tilde{u})_{\Omega(t)}\,dt \\
& + \int_0^T e^{-k_1 t}(\mathrm{rot}\,w\times w, \tilde{u})_{\Omega(t)}\,dt \\
& + \int_0^T (\mathrm{rot}\,w\times U + \mathrm{rot}\,U\times w, \tilde{u})_{\Omega(t)}\,dt \\
& + \int_0^T (\mu(x,t)\tau e_l, \tilde{u})_{\Omega(t)}\,dt + \int_Q \nabla\tau\cdot\nabla\tilde{y}\,dxdt \\
& + \int_Q \Big(e^{-k_1 t}w\cdot\nabla\tau + U\cdot\nabla\tau\Big)\,\tilde{y}\,dxdt \\
& - \int_Q w\cdot\nabla\tilde{y}G\,dxdt - k_1\int_0^T [(w,\tilde{u})_{\Omega(t)} + (\tau,\tilde{y})_{\Omega(t)}]\,dt \\
& + \int_{\Sigma_1} [\tau\tilde{y}\cos(\hat{r,t}) + k(x,t)\tau\tilde{y}]\,d\sigma \\
& = \int_Q \bar{U}(\frac{\partial \tilde{u}}{\partial t} + k_1\tilde{u})\,dxdt + \int_Q \bar{G}(\frac{\partial \tilde{y}}{\partial t} + k_1\tilde{y})\,dxdt \\
& - \nu\int_0^T (\mathrm{rot}\,\bar{U}, \mathrm{rot}\,\tilde{u})_{\Omega(t)}\,dt - \int_0^T (\mathrm{rot}\,\bar{U}\times U, \tilde{u})_{\Omega(t)}\,dt
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^T (\mu(x,t)\bar{G}e_l, \tilde{u})_{\Omega(t)}\,dt - \int_Q \nabla\bar{G}\cdot\nabla\tilde{y}\,dxdt \\
&- \int_Q \big(\bar{U}\cdot\nabla G\big)\,\tilde{y}\,dxdt - \int_{\Sigma_1} [\bar{G}\tilde{y}\cos(\hat{r,t}) + k(x,t)\bar{G}\tilde{y}]\,d\sigma \\
&+ \nu\int_0^T \langle\bar{\varphi},\tilde{u}\rangle_{\Sigma_3(t)}\,dt - \int_0^T \langle\bar{p}_0(x,t), \tilde{u}\cdot n\rangle_{\Sigma_2(t)}\,dt + \int_0^T \langle\bar{e},\tilde{y}\rangle_{\Sigma_1(t)}\,dt \\
&+ (v_0, u(x,0))_{\Omega(0)} + (\theta_0, y(x,0))_{\Omega(0)} + \int_{Q_i}\bar{f}_1\tilde{u}\,dxdt + \int_Q \bar{f}_2\tilde{y}\,dxdt \\
&\qquad\qquad \forall\tilde{u}\in\Lambda(Q),\ \forall\tilde{y}\in D(Q) \text{ with } \tilde{u}(x,T)=0,\ \tilde{y}(x,T)=0,
\end{aligned}
$$

where $\bar{\gamma} = e^{k_1 t}\gamma$ for any γ.

Therefore, for proof of Theorem 2.2 it is enough to prove that the existence of a solution (w,τ) to Problem 3.1 and

$$
\operatorname*{ess\,sup}_{t\in(0,T)} \|(w,\tau)\|_{\mathbf{L}_2(\Omega(t))^{l+1}} \le c. \tag{3.3}
$$

To this end, first in the next section we will consider an auxiliary problem by the elliptic regularization.

4. Auxiliary Problem

Let $\mathbf{W}(Q_i)$ and $Y(Q_i)$ be, respectively, restrictions of $\mathbf{W}(Q)$ and $Y(Q)$ on Q_i; and let Σ_{1i} be the subsurface on (t_i, t_{i+1}) of Σ_1.

Let m be positive integers and $w_{t_i}\in H(\Omega(t_i))$, $\tau_{t_i}\in L_2(\Omega(t_i))$. Our main purpose in this section is to find functions $w^m\in\mathbf{W}(Q_i)$, $\tau^m\in Y(Q_i)$ satisfying the following

$$
\begin{aligned}
&\int_{Q_i}\Big(\frac{1}{m}\frac{\partial w^m}{\partial t} - w^m\Big)\cdot\frac{\partial u}{\partial t}\,dxdt + \int_{Q_i}\Big(\frac{1}{m}\frac{\partial \tau^m}{\partial t} - \tau^m\Big)\frac{\partial y}{\partial t}\,dxdt \\
&\quad + \nu\int_{t_i}^{t_{i+1}} (\operatorname{rot} w^m, \operatorname{rot} u)_{\Omega(t)}\,dt \\
&\quad + \int_{t_i}^{t_{i+1}} \Big(e^{-k_1 t}\operatorname{rot} w^m\times w^m \\
&\qquad\qquad + \operatorname{rot} w^m\times U + \operatorname{rot} U\times w^m, u\Big)_{\Omega(t)}\,dt
\end{aligned} \tag{4.1}
$$

$$
\begin{aligned}
&+ \int_{t_i}^{t_{i+1}} (\mu(x,t)\tau^m e_l, u)_{\Omega(t)}\,dt + \int_{Q_i} (\nabla\tau^m\cdot\nabla y)\,dxdt \\
&+ \int_{Q_i} (e^{-k_1 t} w^m\cdot\nabla\tau^m y + U\cdot\nabla\tau^m y)\,dxdt \\
&- \int_{Q_i} (w^m\cdot\nabla y)G\,dxdt - k_1\int_{t_i}^{t_{i+1}} (w^m, u)_{\Omega(t)}\,dt \\
&- k_1\int_{t_i}^{t_{i+1}} (\tau^m, y)_{\Omega(t)}\,dt + \int_{\Sigma_{1i}} [\cos(\hat{r,t}) + k(x,t)]\tau^m y\,d\sigma \\
&+ (w^m(t_{i+1}), u(t_{i+1}))_{\Omega(t_{i+1})} + (\tau^m(t_{i+1}), y(t_{i+1}))_{\Omega(t_{i+1})} \\
= &\int_{Q_i} \bar U\Big(\frac{\partial u}{\partial t} + k_1 u\Big)\,dxdt + \int_{Q_i} \bar G\Big(\frac{\partial y}{\partial t} + k_1 y\Big)\,dxdt \\
&- \nu\int_{t_i}^{t_{i+1}} (\operatorname{rot}\bar U, \operatorname{rot} u)_{\Omega(t)}\,dt - \int_{t_i}^{t_{i+1}} (\operatorname{rot}\bar U\times U, u)_{\Omega(t)}\,dt \\
&- \int_{t_i}^{t_{i+1}} (\mu(x,t)\bar G e_l, u)_{\Omega(t)}\,dt - \int_{Q_i} \nabla\bar G\cdot\nabla y\,dxdt \\
&- \int_{Q_i} \big(\bar U\cdot\nabla G\big)\,y\,dxdt - \int_{\Sigma_{1i}} [\cos(\hat{r,t}) + k(x,t)]\bar G y\,d\sigma \\
&+ \nu\int_{t_i}^{t_{i+1}} \langle\bar\varphi, u\rangle_{\Sigma_3(t)}\,dt - \int_{t_i}^{t_{i+1}} \langle\bar p_0(x,t), u\cdot n\rangle_{\Sigma_2(t)}\,dt \\
&+ \int_{t_i}^{t_{i+1}} \langle\bar e, y\rangle_{\Sigma_1(t)}\,dt + (w_{t_i}, u(x,t_i))_{\Omega(t_i)} + (\tau_{t_i}, y(x,t_i))_{\Omega(t_i)} \\
&+ \int_{Q_i} \bar f_1 u\,dxdt + \int_{Q_i} \bar f_2 y\,dxdt \\
&\qquad\qquad\qquad\qquad \forall u\in\mathbf{W}(Q_i),\ \forall y\in Y(Q_i).
\end{aligned}
$$

For (4.1) we have the following result on the existence and uniqueness of a solution.

THEOREM 4.1. *Under Assumptions 2.1∼2.5, for some k_1, which is taken for* (4.13)*, independent of m and i, there exists a unique solution to problem* (4.1)*.*

PROOF. Define an operator A_{mi} from $\mathbf{W}(Q_i)\times Y(Q_i)$ into its dual space by

$$
\begin{aligned}
(4.2)\qquad \big\langle A_{mi}(z,\rho),(u,y)\big\rangle &= \int_{Q_i}\frac{1}{m}\frac{\partial z}{\partial t}\frac{\partial u}{\partial t}\,dxdt+\int_{Q_i}\frac{1}{m}\frac{\partial \rho}{\partial t}\frac{\partial y}{\partial t}\,dxdt\\
&-\int_{Q_i} z\frac{\partial u}{\partial t}\,dxdt-\int_{Q_i}\rho\frac{\partial y}{\partial t}\,dxdt+\nu\int_{t_i}^{t_{i+1}}(\mathrm{rot}\,z,\mathrm{rot}\,u)_{\Omega(t)}\,dt\\
&+\int_{t_i}^{t_{i+1}}\Big(e^{-k_1t}\mathrm{rot}\,z\times z+\mathrm{rot}\,z\times U+\mathrm{rot}\,U\times z,u\Big)_{\Omega(t)}\,dt\\
&+\int_{t_i}^{t_{i+1}}(\mu(x,t)\rho e_l,u)_{\Omega(t)}\,dt\\
&+\int_{Q_i}(\nabla\rho\cdot\nabla y+e^{-k_1t}z\cdot\nabla\rho y+U\cdot\nabla\rho y+z\cdot\nabla yG)\,dxdt\\
&-k_1\int_{t_i}^{t_{i+1}}(z,u)_{\Omega(t)}\,dt-k_1\int_{t_i}^{t_{i+1}}(\rho,y)_{\Omega(t)}\,dt\\
&+\int_{\Sigma_{1i}}[\rho y\cos(\widehat{r,t})+k(x,t)\rho y]\,d\sigma\\
&+(z(t_{i+1}),u(t_{i+1}))_{\Omega(t_{i+1})}+(\rho(t_{i+1}),y(t_{i+1}))_{\Omega(t_{i+1})}\\
&\qquad\forall(z,\rho),\,(u,y)\in\mathbf{W}(Q_i)\times Y(Q_i),
\end{aligned}
$$

where $\langle\cdot\,,\cdot\rangle$ means the duality product between $\mathbf{W}(Q_i)\times Y(Q_i)$ and its dual space.

And also define an element $L_i\in\mathbf{W}(Q_i)^*\times Y(Q_i)^*$ by

$$
\begin{aligned}
(4.3)\qquad \langle L_i,(u,y)\rangle &= \int_{Q_i}\bar U(\frac{\partial u}{\partial t}+k_1u)\,dxdt+\int_{Q_i}\bar G(\frac{\partial y}{\partial t}+k_1y)\,dxdt\\
&-\nu\int_{t_i}^{t_{i+1}}(\mathrm{rot}\,\bar U,\mathrm{rot}\,u)_{\Omega(t)}\,dt-\int_{t_i}^{t_{i+1}}(\mathrm{rot}\,\bar U\times U,u)_{\Omega(t)}\,dt\\
&-\int_{t_i}^{t_{i+1}}(\mu(x,t)\bar G e_l,u)_{\Omega(t)}\,dt-\int_{Q_i}\nabla\bar G\cdot\nabla y\,dxdt\\
&-\int_{Q_i}\big(\bar U\cdot\nabla G\big)\,y\,dxdt-\int_{\Sigma_{1i}}[\cos(\widehat{r,t})+k(x,t)]\bar G y\,d\sigma\\
&+\nu\int_{t_i}^{t_{i+1}}\langle\bar\varphi,u\rangle_{\Sigma_3(t)}\,dt-\int_{t_i}^{t_{i+1}}\langle\bar p_0(x,t),u\cdot n\rangle_{\Sigma_2(t)}\,dt
\end{aligned}
$$

$$+ \int_{t_i}^{t_{i+1}} \langle \overline{e}, y \rangle_{\Sigma_1(t)}\, dt + (w_{t_i}, u(x,t_i))_{\Omega(t_i)} + (\tau_{t_i}, y(x,t_i))_{\Omega(t_i)}$$
$$+ \int_{Q_i} \bar{f}_1 u\, dxdt + \int_{Q_i} \bar{f}_2 y\, dxdt$$
$$\forall (u,y) \in \mathbf{W}(Q_i) \times Y(Q_i).$$

Now, let us consider the existence of a solution to the following problem

$$A_{mi}(w^m, \tau^m) = L_i, \tag{4.4}$$

which is equivalent to the existence of a solution to the auxiliary problem (4.1).

For all $(z,\rho) \in \mathbf{W}(Q_i) \times Y(Q_i)$, we have

$$\begin{aligned} &\langle A_{mi}(z,\rho), (z,\rho) \rangle \\ &= \int_{Q_i} \frac{1}{m} \left| \frac{\partial z}{\partial t} \right|^2 dxdt + \int_{Q_i} \frac{1}{m} \left| \frac{\partial \rho}{\partial t} \right|^2 dxdt - \int_{Q_i} z \frac{\partial z}{\partial t}\, dxdt \\ &- \int_{Q_i} \rho \frac{\partial \rho}{\partial t}\, dxdt + \nu \int_{t_i}^{t_{i+1}} |\mathrm{rot}\, z|^2_{\Omega(t)}\, dt \\ &+ \int_{t_i}^{t_{i+1}} (\mathrm{rot}\, z \times U + \mathrm{rot}\, U \times z + \mu(x,t)\rho e_l, z)_{\Omega(t)}\, dt \\ &+ \int_{Q_i} \Big(|\nabla \rho|^2 + e^{-k_1 t} z \cdot \nabla \rho \rho + U \cdot \nabla \rho \rho + z \cdot \nabla \rho G \Big)\, dxdt \\ &- k_1 \int_{Q_i} \left(|z|^2 + |\rho|^2 \right)\, dxdt + \int_{\Sigma_{1i}} [\cos(\hat{r,t}) + k(x,t)] |\rho|^2\, d\sigma \\ &+ |z(x,t_{i+1})|^2_{\Omega(t_{i+1})} + |\rho(x,t_{i+1})|^2_{\Omega(t_{i+1})}, \end{aligned} \tag{4.5}$$

where $(\mathrm{rot}\, z \times z, z) = 0$ was used.

Integrating by parts we get

$$- \int_{Q_i} z \frac{\partial z}{\partial t}\, dxdt = \frac{1}{2} \Big(|z(\cdot,t_i)|^2_{\Omega(t_i)} - |z(\cdot,t_{i+1})|^2_{\Omega(t_{i+1})} \Big) \quad \text{for } z \in \mathbf{W}(Q_i), \tag{4.6}$$

where the definition of $\mathbf{W}(Q_i)$ and the fact that $\int_{\Sigma_2 \cup \Sigma_3} z^2 \cos(\hat{r,t})\, d\sigma = 0$ by 2) of Assumption 2.2 were used.

In the same way using the definition of $Y(Q_i)$, we get

$$
\begin{aligned}
(4.7)\qquad & -\int_{Q_i} \rho \frac{\partial \rho}{\partial t}\, dxdt \\
&= \frac{1}{2}\left(|\rho(\cdot, t_i)|^2_{\Omega(t_i)} - |\rho(\cdot, t_{i+1})|^2_{\Omega(t_{i+1})} - \int_{\Sigma_{1i}} |\rho|^2 \cos(\hat{r,t})\, d\sigma \right) \\
&\qquad\qquad \text{for } \rho \in Y(Q_i).
\end{aligned}
$$

On the other hand, we have

$$
(z \cdot \nabla\rho, \rho)_{\Omega(t)} = \int_{\Sigma_2(t)\cup\Sigma_3(t)} z \cdot n\rho^2\, d\sigma - (\mathrm{div}\,(z\rho), \rho)_{\Omega(t)} = -(z \cdot \nabla\rho, \rho)_{\Omega(t)},
$$

and so $(z \cdot \nabla\rho, \rho)_{\Omega(t)} = 0$. Therefore,

$$
(4.8)\qquad \int_{Q_i} e^{-k_1 t} z \cdot \nabla\rho\rho\, dxdt = 0.
$$

By (4.6)∼(4.8), from (4.5) we have

$$
\begin{aligned}
(4.9)\quad & \langle A_{mi}(z,\rho), (z,\rho)\rangle \\
&= \int_{Q_i} \frac{1}{m}\left|\frac{\partial z}{\partial t}\right|^2 dxdt + \int_{Q_i} \frac{1}{m}\left|\frac{\partial \rho}{\partial t}\right|^2 dxdt + \nu \int_{t_i}^{t_{i+1}} |\mathrm{rot}\, z|^2_{\Omega(t)}\, dt \\
&+ \int_{t_i}^{t_{i+1}} (\mathrm{rot}\, z \times U, z)_{\Omega(t)}\, dt + \int_{t_i}^{t_{i+1}} (\mu(x,t)\rho e_l, z)_{\Omega(t)}\, dt \\
&+ \int_{Q_i} |\nabla\rho|^2\, dxdt + \int_{Q_i} (U \cdot \nabla\rho\rho + z \cdot \nabla\rho G)\, dxdt \\
&- k_1 \int_{Q_i} \left(|z|^2 + |\rho|^2\right) dxdt + \int_{\Sigma_{1i}} \left[\frac{1}{2}\cos(\hat{r,t}) + k(x,t)\right] |\rho|^2\, d\sigma \\
&+ \frac{1}{2}\Big(|z(x,t_i)|^2_{\Omega(t_i)} + |\rho(x,t_i)|^2_{\Omega(t_i)} \\
&\qquad + |z(x,t_{i+1})|^2_{\Omega(t_{i+1})} + |\rho(x,t_{i+1})|^2_{\Omega(t_{i+1})} \Big) \\
&\qquad\qquad\qquad \forall (z,\rho) \in \mathbf{W}(Q_i) \times Y(Q_i),
\end{aligned}
$$

where $(\mathrm{rot}\, U \times z, z) = 0$ was used.

By Assumption 2.3

$$
(4.10)\qquad \int_0^T |\mathrm{rot}\, z|^2_{\Omega(t)}\, dt \ge c_1 \int_0^T \|z\|^2_{\Omega(t)}\, dt, \qquad c_1 > 0.
$$

By Young's inequality, Assumption 2.4 and 1) of Assumption 2.5

$$\left|\int_{t_i}^{t_{i+1}} (\operatorname{rot} z \times U, z)_{\Omega(t)}\, dt\right| \leq \frac{\nu c_1}{2}\int_{t_i}^{t_{i+1}} \|z\|^2_{\Omega(t)}\, dt + c_2 \int_{t_i}^{t_{i+1}} |z|^2_{\Omega(t)}\, dt. \tag{4.11}$$

Also, we get

$$\left|\int_{t_i}^{t_{i+1}} (\mu(x,t)\rho e_l, z)_{\Omega(t)}\, dt + \int_{Q_i} (U\cdot\nabla\rho\rho + z\cdot\nabla\rho G)\, dxdt\right| \leq \frac{1}{2}\int_{t_i}^{t_{i+1}} |\nabla\rho|^2_{\Omega(t)}\, dt + c_3\int_{t_i}^{t_{i+1}} |\rho|^2_{\Omega(t)}\, dt + c_4\int_{t_i}^{t_{i+1}} |z|^2_{\Omega(t)}\, dt. \tag{4.12}$$

Having in mind 6) of Assumption 2.2 and taking $-k_1$ large enough in (4.9) independently of m and i, from (4.9)∼(4.12) we have, therefore

$$\langle A_{mi}(z,\rho),(z,\rho)\rangle \geq c_5\left(\|z\|^2_{\mathbf{W}(Q_i)} + \|\rho\|^2_{Y(Q_i)}\right), \quad \exists c_5 > 0,\ \forall (z,\rho)\in \mathbf{W}(Q_i)\times Y(Q_i), \tag{4.13}$$

where c_5 depends on m.

Now, let us prove that

(4.14) if $(z_k,\rho_k) \rightharpoonup (z,\rho)$ weakly in $\mathbf{W}(Q_i)\times Y(Q_i)$ as $k\to\infty$, then $\langle A_{mi}(z_k,\rho_k),(u,y)\rangle \to \langle A_{mi}(z,\rho),(u,y)\rangle$ $\forall (u,y)\in \mathbf{W}(Q_i)\times Y(Q_i)$.

In the same way as Lemma 3.2 in [20] we can prove that

$$\int_{t_i}^{t_{i+1}} e^{-k_1 t} (\operatorname{rot} z_k \times z_k, u)_{\Omega(t)}\, dt \to \int_{t_i}^{t_{i+1}} e^{-k_1 t} (\operatorname{rot} z \times z, u)_{\Omega(t)}\, dt \quad \forall u\in \mathbf{W}(Q_i) \text{ as } k\to\infty. \tag{4.15}$$

Similarly to Lemma 3.2 in [20], we can prove that

$$\int_{Q_i} e^{-k_1 t} z_k\cdot\nabla\rho_k y\, dxdt \to \int_{Q_i} e^{-k_1 t} z\cdot\nabla\rho y\, dxdt \quad \forall y\in Y(Q_i) \text{ as } k\to\infty. \tag{4.16}$$

Indeed,

$$\int_{Q_i} e^{-k_1 t} z_k \cdot \nabla\rho_k y\, dxdt - \int_{Q_i} e^{-k_1 t} z \cdot \nabla\rho y\, dxdt = \int_{Q_i} e^{-k_1 t}(z_k - z) \cdot \nabla\rho_k y\, dxdt + \int_{Q_i} e^{-k_1 t} z \cdot \nabla(\rho_k - \rho) y\, dxdt. \tag{4.17}$$

By the imbedding of $H^1(Q_i)$ in $L_4(Q_i)$ we have $e^{-k_1 t} zy \in \mathbf{L}_2(Q_i)$, and so the second integral in the right-hand side of (4.17) converges to zero when $k \to \infty$. Let us consider the first integral in the right-hand side of (4.17). For any $\varepsilon \geq 0$ we can choose $y_\varepsilon \in D(Q_i)$ such that $\|y - y_\varepsilon\|_{Y(Q_i)} \leq \varepsilon$. Then,

$$\begin{aligned}\int_{Q_i} e^{-k_1 t}(z_k - z) \cdot \nabla\rho_k y\, dxdt &= \int_{Q_i} e^{-k_1 t}(z_k - z) \cdot \nabla\rho_k y_\varepsilon\, dxdt \\ &\quad + \int_{Q_i} e^{-k_1 t}(z_k - z) \cdot \nabla\rho_k (y - y_\varepsilon)\, dxdt.\end{aligned} \tag{4.18}$$

Since $z_k \to z$ strongly in $\mathbf{L}_2(Q_i)$ as $k \to \infty$,

$$\begin{aligned}\left|\int_{Q_i} e^{-k_1 t}(z_k - z) \cdot \nabla\rho_k y_\varepsilon\, dxdt\right| &\leq C\|z_k - z\|_{\mathbf{L}_2(Q_i)} \|\nabla\rho_k\|_{\mathbf{L}_2(Q_i)} \|y_\varepsilon\|_{L_\infty(Q_i)} \\ &\leq C\|z_k - z\|_{\mathbf{L}_2(Q_i)} \|\nabla\rho_k\|_{\mathbf{L}_2(Q_i)} \|y_\varepsilon\|_{D(Q_i)} \to 0 \text{ as } k \to \infty.\end{aligned} \tag{4.19}$$

Also, since $z_k - z$ and ρ_k are bounded, respectively, in $\mathbf{W}(Q_i)$ and $Y(Q_i)$, we have

$$\begin{aligned}\int_{Q_i} e^{-k_1 t}(z_k - z) \cdot \nabla\rho_k (y - y_\varepsilon)\, dxdt &\leq C\|z_k - z\|_{\mathbf{L}_4(Q_i)} \|\nabla\rho_k\|_{\mathbf{L}_2(Q_i)} \|y - y_\varepsilon\|_{L_4(Q_i)} \\ &\leq C\|z_k - z\|_{\mathbf{W}(Q_i)} \|\rho_k\|_{Y(Q_i)} \|y - y_\varepsilon\|_{Y(Q_i)} \leq C\varepsilon.\end{aligned} \tag{4.20}$$

From (4.18)-(4.20), we know that the first integral in the right-hand side of (4.17) goes to zero when $k \to \infty$, and so we get (4.16).

It is easy to check that other terms in $\langle A_m(z_k, \rho_k), (u, y)\rangle$ converse when $k \to \infty$. This fact together with (4.15), (4.16) implies (4.14).

By (4.13) and (4.14), there exists a solution to (4.4) (cf. Theorem 1.2, ch. IV in [13]), and therefore (4.1) has a solution. □

THEOREM 4.2. *If ($w^m \in \mathbf{W}(Q_i)$, $\tau^m \in Y(Q_i)$) are solutions to problem* (4.1) *with the k_1 being chosen under Assumptions 2.2∼2.5, then*

$$\int_{Q_i} \frac{1}{m}\frac{\partial w^m}{\partial t}\frac{\partial u}{\partial t}\,dxdt + \int_{Q_i} \frac{1}{m}\frac{\partial \tau^m}{\partial t}\frac{\partial y}{\partial t}\,dxdt \to 0 \quad \text{as } m\to\infty \qquad \forall (u,y)\in \mathbf{W}(Q_i)\times Y(Q_i)) \tag{4.21}$$

and there exists a subsequence $\{(w^k,\tau^k)\}$ and $\{(w^k(x,t_{i+1}),\tau^k(x,t_{i+1}))\}$ such that $(w^k,\tau^k) \rightharpoonup (w,\tau)$ and $(w^k(x,t_{i+1}),\tau^k(x,t_{i+1})) \rightharpoonup (w_{t_{i+1}}(x), \tau_{t_{i+1}}(x))$ weakly, respectively, in $\mathbf{V}(Q_i)\times X(Q_i)$ and $H(\Omega(t_{i+1}))\times L_2(\Omega(t_{i+1}))$ as $k\to\infty$.

PROOF. By (4.4), (4.9)∼(4.12), we have

$$\begin{aligned}
&\int_{Q_i} \frac{1}{m}\left|\frac{\partial w^m}{\partial t}\right|^2 dxdt + \int_{Q_i} \frac{1}{m}\left|\frac{\partial \tau^m}{\partial t}\right|^2 dxdt \\
&+ \frac{\nu c_1}{2}\int_{t_i}^{t_{i+1}} \|w^m\|^2_{\mathbf{H}^1(\Omega(t))}\,dt + \frac{1}{4}\int_{Q_i} |\nabla \tau^m|^2\,dxdt \\
&+ \frac{1}{2}\Big(|w^m(x,t_i)|^2_{\Omega(t_i)} + |\tau^m(x,t_i)|^2_{\Omega(t_i)} \\
&\qquad + |w^m(x,t_{i+1})|^2_{\Omega(t_{i+1})} + |\tau^m(x,t_{i+1})|^2_{\Omega(t_{i+1})}\Big) \\
&\le \langle L_i,(w^m,\tau^m)\rangle .
\end{aligned} \tag{4.22}$$

On the other hand,

$$\begin{aligned}
&\int_{Q_i} \bar U \frac{\partial w^m}{\partial t}\,dxdt + \int_{Q_i} \bar G \frac{\partial \tau^m}{\partial t}\,dxdt \\
&= (\bar U, w^m(x,t_{i+1}))_{\Omega(t_{i+1})} + (\bar G, \tau^m(x,t_{i+1}))_{\Omega(t_{i+1})} \\
&\quad - (\bar U, w^m(x,t_i))_{\Omega(t_i)} - (\bar G, \tau^m(x,t_i))_{\Omega(t_i)} \\
&\quad + \int_{\Sigma_{1i}} \bar G \tau^m \cos(\hat{r,t})\,d\sigma - \int_{Q_i} \frac{\partial \bar U}{\partial t} w^m\,dxdt - \int_{Q_i} \frac{\partial \bar G}{\partial t}\tau^m\,dxdt \\
&\qquad \forall (w^m,\tau^m)\in \mathbf{W}(Q_i)\times Y(Q_i),
\end{aligned} \tag{4.23}$$

where 2) of Assumption 2.2 was used.

Taking (4.23) into account in (4.3) and applying Young's inequality to the right-hand side of (4.22), we have

$$
\begin{aligned}
(4.24)\qquad & \int_{Q_i} \frac{1}{m}\left|\frac{\partial w^m}{\partial t}\right|^2 dxdt + \int_{Q_i} \frac{1}{m}\left|\frac{\partial \tau^m}{\partial t}\right|^2 dxdt \\
&+ \frac{\nu c_1}{4}\int_{t_i}^{t_{i+1}} \|w^m\|^2_{\mathbf{H}^1(\Omega(t))}\, dt \\
&+ \frac{1}{8}\int_{Q_i} |\nabla \tau^m|^2\, dxdt + \frac{1}{4}\Big(|w^m(x,t_i)|^2_{\Omega(t_i)} + |\tau^m(x,t_i)|^2_{\Omega(t_i)}\Big) \\
&+ \frac{1}{4}\Big(|w^m(x,t_{i+1})|^2_{\Omega(t_{i+1})} + |\tau^m(x,t_{i+1})|^2_{\Omega(t_{i+1})}\Big) \le c,
\end{aligned}
$$

where c is independent of (m,i) and depends on $\bar{\varphi}, \bar{p}_0(x,t), \bar{e}(x,t),\ \tilde{v}_0, \tilde{\theta}_0, \bar{f}_1, \bar{f}_2$ and k_1.

Using $\int_{Q_i} \frac{1}{m}\left|\frac{\partial w^m}{\partial t}\right|^2 dxdt + \int_{Q_i} \frac{1}{m}\left|\frac{\partial \tau^m}{\partial t}\right|^2 dxdt \le c$, we can claim that

$$
\begin{aligned}
(4.25)\qquad & \int_{Q_i} \frac{1}{m}\frac{\partial w^m}{\partial t}\frac{\partial u}{\partial t}\, dxdt + \int_{Q_i} \frac{1}{m}\frac{\partial \tau^m}{\partial t}\frac{\partial y}{\partial t}\, dxdt \to 0 \ \text{ as } m \to \infty \\
&\qquad\qquad \forall (u,y) \in \mathbf{W}(Q_i) \times Y(Q_i)).
\end{aligned}
$$

Indeed, by Hölder's inequality

$$
\begin{aligned}
&\left|\int_{Q_i} \frac{1}{m}\frac{\partial w^m}{\partial t}\frac{\partial u}{\partial t}\, dxdt + \int_{Q_i} \frac{1}{m}\frac{\partial \tau^m}{\partial t}\frac{\partial y}{\partial t}\, dxdt\right| \\
&\le \frac{1}{\sqrt{m}}\bigg[\Big(\int_{Q_i}\Big|\frac{1}{\sqrt{m}}\frac{\partial w^m}{\partial t}\Big|^2 dxdt \cdot \int_{Q_i}\Big|\frac{\partial u}{\partial t}\Big|^2 dxdt\Big)^{\frac{1}{2}} \\
&\qquad\qquad + \Big(\int_{Q_i}\Big|\frac{1}{\sqrt{m}}\frac{\partial \tau^m}{\partial t}\Big|^2 dxdt \cdot \int_{Q_i}\Big|\frac{\partial y}{\partial t}\Big|^2 dxdt\Big)^{\frac{1}{2}}\bigg],
\end{aligned}
$$

which shows (4.25). By virtue of (4.24), the sets $\{(w^m(Q_i), \tau^m(Q_i))\}$ and $\{(w^m(\Omega(t_{i+1})), \tau^m(\Omega(t_{i+1})))\}$ are bounded, respectively, in the spaces $\mathbf{V}(Q_i) \times X(Q_i)$ and $H(\Omega(t_{i+1})) \times L_2(\Omega(t_{i+1}))$, and we come to the second conclusion. □

5. Proof of Theorem 2.2

In this section, we will give proof of Theorem 2.2. To do this, we need the following result.

LEMMA 5.1. *If* ($w^m \in \mathbf{W}(Q_i)$, $\tau^m \in Y(Q_i)$) *are solutions to problem* (4.1) *under Assumption 2.2∼2.5, then* $\{w^m\}$ *and* $\{\tau^m\}$ *are precompact, respectively, in* $\mathbf{L}_2(Q_i)$ *and* $L_2(Q_i)$ *for any fixed* i.

We postpone the proof of Lemma 5.1 and give the proof of Theorem 2.2 first.

PROOF OF THEOREM 2.2. Let $\{(w^k(Q_0), \tau^k(Q_0))\}$ is the sequence defined on Q_0 guaranteed by Theorem 4.2 with $(w_0, \tau_0) = (v_0, \theta_0)$. Using (w_{t_1}, τ_{t_1}) guaranteed in Theorem 4.2 we get $\{(w^k(Q_1), \tau^k(Q_1))\}$ on Q_1 and step by step we can get such sequences on all Q_i, $i = 0 \sim \delta$. By Lemma 5.1, we can choose its subsequence $\{(w^k(Q_i), \tau^k(Q_i))\}$, which is expressed with the same index for simplicity, such that $(w^k(Q_i), \tau^k(Q_i)) \to (w, \tau) \in V(Q_i) \times X(Q_i)$ strongly in $\mathbf{L}_2(Q_i) \times L_2(Q_i)$.

First, let us prove that

$$
\begin{aligned}
(5.1)\qquad &\int_{t_i}^{t_{i+1}} e^{-k_1 t}\left(\operatorname{rot} w^k \times w^k, u\right)_{\Omega(t)} dt \\
&\quad\to \int_{t_i}^{t_{i+1}} e^{-k_1 t}\left(\operatorname{rot} w \times w, u\right)_{\Omega(t)} dt \\
&\quad\forall u \in \Lambda(Q_i) \text{ as } k \to \infty.
\end{aligned}
$$

We can write

$$
\begin{aligned}
(5.2)\qquad &\int_{t_i}^{t_{i+1}} e^{-k_1 t}\left(\operatorname{rot} w^k \times w^k, u\right)_{\Omega(t)} dt \\
&\quad - \int_{t_i}^{t_{i+1}} e^{-k_1 t}\left(\operatorname{rot} w \times w, u\right)_{\Omega(t)} dt \\
&\quad = \int_{t_i}^{t_{i+1}} e^{-k_1 t}\left(\operatorname{rot} w^k \times (w^k - w), u\right)_{\Omega(t)} dt \\
&\quad + \int_{t_i}^{t_{i+1}} e^{-k_1 t}\left(\operatorname{rot} (w^k - w) \times w, u\right)_{\Omega(t)} dt.
\end{aligned}
$$

Since $\operatorname{rot} w^k$ is bounded in $\mathbf{L}_2(Q_i)$, $w^k \to w$ in $\mathbf{L}_2(Q_i)$ and $u \in L_\infty$, the first integral in the right hand side of (5.2) converges to zero as $k \to \infty$. Meanwhile, since $e^{-k_1 t} w u \in \mathbf{L}_2(Q_i)$ and $w_k \rightharpoonup w$ weakly in $\mathbf{V}(Q_i)$, the second integral in the right-hand side of (5.2) converges to zero either. Thus, (5.1) follows.

Using

$$\begin{aligned}\int_{Q_i} e^{-k_1 t} w^k \cdot \nabla \tau^k y \, dxdt - \int_{Q_i} e^{-k_1 t} w \cdot \nabla \tau y \, dxdt \\ = \int_{Q_i} e^{-k_1 t} (w^k - w) \cdot \nabla \tau^k y \, dxdt + \int_{Q_i} e^{-k_1 t} w \cdot \nabla (\tau^k - \tau) y \, dxdt,\end{aligned}$$

in the same way as above we can prove

$$\int_{Q_i} e^{-k_1 t} w^k \cdot \nabla \tau^k y \, dxdt \to \int_{Q_i} e^{-k_1 t} w \cdot \nabla \tau y \, dxdt \quad \forall y \in D(Q_i) \text{ as } k \to \infty. \tag{5.3}$$

It is easy to verify the convergence of other terms in (4.1) when $k \to \infty$. Passing to the limit as $k \to \infty$ in (4.1) with k instead of m and adding the results for i, by Theorem 4.2, (5.1), (5.3) we have (3.2). The estimate (3.3) can be obtained in the same way as proof of (4.2) in [20]. □

Now, we turn to the proof of Lemma 5.1

Proof of Lemma 5.1. We will follow the method in [20] and [32], to prove Lemma 5.1. For simplicity, we will not distinguish constants in estimate; and t_i, t_{i+1} and Q_i are, respectively, expressed by 0, T and Q. Thus, $w(x,t_i)$, $w(x,t_{i+1})$, $\tau(x,t_i)$, $\tau(x,t_{i+1})$ are, respectively, expressed by $w(x,0)$, $w(x,T)$, $\tau(x,0)$, $\tau(x,T)$.

Putting $\bar{w}^m(x,t) = 0$ on $(\Omega \times (-T,2T)) \setminus Q$, let us make $\bar{w}^m(x,t)$ an extension of w^m, where Ω is the domain in 2) of Assumption 2.2. Then, by Assumption 2.2, $\bar{w}^m(x,t) \in \mathbf{H}^1(\Omega \times (0,T))$.

Also, there exists a bounded linear extension operator from $H^1(Q)$ to $H^1(\Omega \times (0,T))$. Therefore, making $\bar{\tau}^m(x,t) \in H^1(\Omega \times (0,T))$ such an extension of $\tau^m(x,t)$ onto $\Omega \times (0,T)$ and putting $\bar{\tau}^m(x,t) = 0$ on $\Omega \times (-T,0)$ and $\Omega \times (T,2T)$ we make $\bar{\tau}^m(x,t)$ an extension of $\tau^m(x,t)$ onto $\Omega \times (-T,2T)$.

Thus, by virtue of (4.24) we get

$$\begin{aligned}&\int_{\Omega \times (0,T)} \frac{1}{m} \left| \frac{\partial \bar{w}^m}{\partial t} \right|^2 dxdt \le c, \quad \int_0^T \|\bar{w}^m\|_\Omega^2 \, dt \le c, \\ &|\bar{w}^m(x,0)|_\Omega \le c, \quad |\bar{w}^m(x,T)|_\Omega \le c,\end{aligned} \tag{5.4}$$

$$
(5.5) \qquad \int_{\Omega\times(0,T)} \frac{1}{m}\left|\frac{\partial\bar{\tau}^m}{\partial t}\right|^2 dxdt \le c, \quad \int_0^T \|\bar{\tau}^m\|_\Omega^2 \, dt \le c,
$$
$$
|\bar{\tau}^m(x,0)|_\Omega \le c, \quad |\bar{\tau}^m(x,T)|_\Omega \le c.
$$

First, let us consider the case (3a) in Assumption 2.2.
For $0 < h < T$ let

$$
w_h^m(x,t) = \frac{1}{h}\int_t^{t+h} \bar{w}^m(x,s)\,ds, \quad \tau_h^m(x,t) = -\frac{1}{h}\int_{t-h}^t \bar{\tau}^m(x,s)\,ds.
$$

Then, by Assumption 2.2 $w_h^m|_Q \in W(Q)$, $\tau_h^m|_Q \in Y(Q)$. Also we have that

$$
\begin{aligned}
\frac{\partial w_h^m(x,t)}{\partial t} &= \frac{1}{h}\left(\bar{w}^m(x,t+h) - \bar{w}^m(x,t)\right),\\
\frac{\partial \tau_h^m(x,t)}{\partial t} &= -\frac{1}{h}\left(\bar{\tau}^m(x,t) - \bar{\tau}^m(x,t-h)\right).
\end{aligned}
$$

Taking (4.23) into account and replacing (u,y) by $(w_h^m|_Q,\, \tau_h^m|_Q)$ in (4.1), we have

$$
\begin{aligned}
(5.6) \qquad & \int_Q \frac{1}{m}\frac{\partial w^m(x,t)}{\partial t}\frac{1}{h}\left[\bar{w}^m(x,t+h) - \bar{w}^m(x,t)\right] dxdt\\
&- \int_Q w^m \cdot \frac{1}{h}\left[\bar{w}^m(x,t+h) - \bar{w}^m(x,t)\right] dxdt\\
&- \int_Q \frac{1}{m}\frac{\partial \tau^m}{\partial t}\frac{1}{h}\left[\bar{\tau}^m(x,t) - \bar{\tau}^m(x,t-h)\right] dxdt\\
&+ \int_Q \tau^m \cdot \frac{1}{h}\left[\bar{\tau}^m(x,t) - \bar{\tau}^m(x,t-h)\right] dxdt\\
&+ \nu\int_0^T (\operatorname{rot} w^m, \operatorname{rot} w_h^m)_{\Omega(t)}\, dt\\
&+ \int_0^T \Big(e^{-k_1 t}\operatorname{rot} w^m \times w^m + \operatorname{rot} w^m \times U\\
&\qquad\qquad + \operatorname{rot} U \times w^m, w_h^m\Big)_{\Omega(t)}\, dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T (\mu(x,t)\tau^m e_l, w_h^m)_{\Omega(t)}\,dt \\
&+\int_Q (\nabla\tau^m\cdot\nabla\tau_h^m + e^{-k_1 t} w^m\cdot\nabla\tau^m\tau_h^m \\
&\qquad\qquad + U\cdot\nabla\tau^m\tau_h^m + w^m\cdot\nabla G\tau_h^m)\,dxdt \\
&- k_1\int_0^T (w^m, w_h^m)_{\Omega(t)}\,dt - k_1\int_0^T(\tau^m,\tau_h^m)_{\Omega(t)}\,dt \\
&+\int_{\Sigma_1}[\cos(\hat{r,t}) + k(x,t)]\tau^m\tau_h^m\,d\sigma + (w^m(T), w_h^m(T))_{\Omega(T)} \\
&+(\tau^m(T),\tau_h^m(T))_{\Omega(T)} \\
= {}&(\bar U, w_h^m(x,T))_{\Omega(T)} + (\bar G,\tau_h^m(x,T))_{\Omega(T)} - (\bar U, w_h^m(x,0))_{\Omega(0)} \\
&-(\bar G,\tau_h^m(x,0))_{\Omega(0)} + \int_{\Sigma_1}\bar G\tau_h^m\cos(\hat{r,t})\,d\sigma - \int_Q\frac{\partial\bar U}{\partial t}w_h^m\,dxdt \\
&-\int_Q\frac{\partial\bar G}{\partial t}\tau_h^m\,dxdt + k_1\int_Q\bar U w_h^m\,dxdt + k_1\int_Q\bar G\tau_h^m\,dxdt \\
&-\nu\int_0^T(\operatorname{rot}\bar U,\operatorname{rot}w_h^m)_{\Omega(t)}\,dt - \int_0^T(\operatorname{rot}\bar U\times U, w_h^m)_{\Omega(t)}\,dt \\
&-\int_0^T(\mu(x,t)\bar G e_l, w_h^m)_{\Omega(t)}\,dt - \int_Q\nabla\bar G\cdot\nabla\tau_h^m\,dxdt \\
&-\int_Q\left(\bar U\cdot\nabla G\right)\tau_h^m\,dxdt - \int_{\Sigma_1}[\cos(\hat{r,t}) + k(x,t)]\bar G\tau_h^m\,d\sigma \\
&+\nu\int_0^T\langle\bar\varphi, w_h^m\rangle_{\Sigma_3(t)}\,dt - \int_0^T\langle\bar p_0, w_h^m\cdot n\rangle_{\Sigma_2(t)}\,dt \\
&+\int_0^T\langle\bar e,\tau_h^m\rangle_{\Sigma_1(t)}\,dt + (w(x,0), w_h^m(0))_{\Omega(0)} \\
&+(\tau(x,0),\tau_h^m(0))_{\Omega(0)} + \int_Q\bar f_1 w_h^m\,dxdt + \int_Q\bar f_2\tau_h^m\,dxdt.
\end{aligned}
$$

Assuming $\bar w(x,t)\in\mathbf{C}^1(\bar\Omega\times[0,T])$, we estimate

$$
I_1\equiv\frac{1}{m}\int\limits_{\Omega\times(0,T)}\left|\frac{1}{h}\left[\bar w(x,t+h)-\bar w(x,t)\right]\right|^2\,dxdt.
$$

Applying Hölder's inequality, we have

$$(5.7)\quad \begin{aligned} I_1 &= \frac{1}{m}\int_{\Omega\times(0,T-h)} \frac{1}{h^2}\Big|\int_t^{t+h} \frac{\partial}{\partial s}\bar{w}(x,s)\,ds\Big|^2 dxdt \\ &\qquad + \frac{1}{m}\int_{\Omega\times(T-h,T)} \frac{1}{h^2}\Big[\bar{w}(x,T)+\int_T^t \frac{\partial}{\partial s}\bar{w}(x,s)\,ds\Big]^2 dxdt \\ &\le \frac{1}{m}\int_{\Omega\times(0,T-h)} \frac{1}{h}\int_t^{t+h}\Big|\frac{\partial \bar{w}(x,s)}{\partial s}\Big|^2 ds\,dxdt + \frac{1}{m}\frac{2}{h}|\bar{w}(x,T)|_\Omega^2 \\ &\qquad + \frac{1}{m}\frac{2}{h^2}\int_{\Omega\times(T-h,T)}\int_t^T \Big|\frac{\partial}{\partial s}\bar{w}(x,s)\Big|^2 ds\cdot h\,dxdt \\ &\le \frac{1}{h}\frac{1}{m}\Big[(T-h)\int_{\Omega\times(0,T)}\Big|\frac{\partial}{\partial t}\bar{w}(x,t)\Big|^2 dxdt + 2|\bar{w}(x,T)|_\Omega^2 \\ &\qquad + 2h\int_{\Omega\times(0,T)}\Big|\frac{\partial}{\partial t}\bar{w}(x,t)\Big|^2 dxdt\Big] \\ &\le \frac{1}{h}\frac{1}{m}\Big[(T+h)\int_{\Omega\times(0,T)}\Big|\frac{\partial}{\partial t}\bar{w}(x,t)\Big|^2 dxdt + 2|\bar{w}(x,T)|_\Omega^2\Big]. \end{aligned}$$

Since $\mathbf{C}^1(\bar{\Omega}\times[0,T])$ is dense in $\mathbf{H}^1(\Omega\times(0,T))$, by (4.24), (5.4) and (5.7) for any $\bar{w}^m \in \mathbf{H}^1(\Omega\times(0,T))$ we have

$$(5.8)\qquad \Big\{\frac{1}{m}\int_{\Omega\times(0,T)}\Big|\frac{1}{h}\left[\bar{w}^m(x,t)-\bar{w}^m(x,t-h)\right]\Big|^2 dxdt\Big\}^{\frac{1}{2}} \le \frac{c}{\sqrt{h}}\,.$$

By Hölder's inequality, from (4.24) and (5.8) we get

$$(5.9)\qquad \begin{aligned} &\Big|\int_Q \frac{1}{m}\frac{\partial w^m(x,t)}{\partial t}\cdot\frac{1}{h}\left[\bar{w}^m(x,t)-\bar{w}^m(x,t-h)\right]dxdt\Big| \\ &\le \Big(\int_Q \frac{1}{m}\Big|\frac{\partial w^m}{\partial t}\Big|^2 dxdt\Big)^{\frac{1}{2}} \\ &\quad\times\Big(\frac{1}{m}\int_Q\Big|\frac{1}{h}\left[\bar{w}^m(x,t)-\bar{w}^m(x,t-h)\right]\Big|^2 dxdt\Big)^{\frac{1}{2}} \\ &\le \frac{c}{\sqrt{h}}\,. \end{aligned}$$

Also, for $\bar{\tau}(x,t)\in C^1(\bar{\Omega}\times[0,T])$ estimating

$$I_2 \equiv \frac{1}{m}\int_{\Omega\times(0,T)}\Big|\frac{1}{h}\left[\bar{\tau}(x,t)-\bar{\tau}(x,t-h)\right]\Big|^2 dxdt,$$

we have

$$I_2 \le \frac{1}{h}\frac{1}{m}\Big\{(T+h)\int_{\Omega\times(0,T)}\Big|\frac{\partial}{\partial t}\bar\tau(x,t)\Big|^2\,dxdt + 2|\bar\tau(x,0)|^2_\Omega\Big\}. \tag{5.10}$$

Since $C^1(\bar\Omega\times[0,T])$ is dense in $H^1(\Omega\times(0,T))$, by (4.24), (5.5) and (5.10) for any $\bar\tau^m\in H^1(\Omega\times(0,T))$ we have

$$\Big\{\frac{1}{m}\int_{\Omega\times(0,T)}\Big|\frac{1}{h}\left[\bar\tau^m(x,t)-\bar\tau^m(x,t-h)\right]\Big|^2\,dxdt\Big\}^{\frac12} \le \frac{c}{\sqrt{h}}\,. \tag{5.11}$$

Thus, by Hölder's inequality, from (4.24) and (5.11) we have

$$\begin{aligned}&\Big|\int_Q \frac{1}{m}\frac{\partial\tau^m}{\partial t}\frac{1}{h}\left[\bar\tau^m(x,t)-\bar\tau^m(x,t-h)\right]\,dxdt\Big|\\ &\le \Big(\int_Q\frac{1}{m}\Big|\frac{\partial\tau^m}{\partial t}\Big|^2\,dxdt\Big)^{\frac12}\\ &\quad\times\Big(\frac{1}{m}\int_Q\Big|\frac{1}{h}\left[\bar\tau^m(x,t)-\bar\tau^m(x,t-h)\right]\Big|^2\,dxdt\Big)^{\frac12}\\ &\le \frac{c}{\sqrt{h}}\,.\end{aligned} \tag{5.12}$$

Using (5.4), we have

$$\begin{aligned}&\Big|\nu\int_0^T(\operatorname{rot} w^m,\operatorname{rot} w_h^m)_{\Omega(t)}\,dt\Big|\\ &\quad\le c\int_0^T\|w^m\|_{\Omega(t)}\Big\|\frac{1}{h}\int_t^{t+h}\bar w^m(x,s)\,ds\Big\|_{\Omega(t)}\,dt\\ &\quad\le c\int_0^T\|w^m\|_{\Omega(t)}\frac{1}{\sqrt{h}}\Big|\int_t^{t+h}\|\bar w^m(x,s)\|^2_{\Omega(s)}\,ds\Big|^{\frac12}\,dt\le c/\sqrt{h}.\end{aligned} \tag{5.13}$$

Now, let us estimate $\int_0^T\left(e^{-k_1t}\operatorname{rot} w^m\times w^m, w_h^m\right)_{\Omega(t)}\,dt$.
Since

$$\begin{aligned}\int_t^{t+h}\|w^m\|_{L_3(\Omega(s))}\,ds &\le \sqrt{h}\Big(\int_t^{t+h}\|w^m\|^2_{L_3(\Omega(s))}\,ds\Big)^{\frac12}\\ &\le c\sqrt{h}\|\bar w^m\|^2_{L_2(0,T;\mathbf{H}^1(\Omega))},\end{aligned}$$

by Hölder's inequality and (5.4),

$$
\begin{aligned}
(5.14)\quad & \Big|\int_0^T \Big(e^{-k_1 t}\mathrm{rot}\, w^m \times w^m, w_h^m\Big)_{\Omega(t)}\, dt\Big| \\
&\le \frac{c}{h}\int_0^T \Big[|\mathrm{rot}\, w^m|_{\Omega(t)} \cdot \|w^m\|_{L_6(\Omega(t))} \cdot \int_t^{t+h} \|w^m\|_{L_3(\Omega(s))}\, ds\Big] dt \\
&\le \frac{c}{\sqrt{h}}\int_0^T \|w^m\|^2_{\Omega(t)}\, dt \le \frac{c}{\sqrt{h}}\,.
\end{aligned}
$$

In the same way, we get

$$
(5.15)\qquad \Big|\int_0^T ((U,\nabla)w^m + (w^m,\nabla)U, w_h^m)_{\Omega(t)}\, dt\Big| \le c/\sqrt{h}\,,
$$

$$
(5.16)\qquad
\begin{aligned}
& \Big|\int_0^T (\mu(x,t)\tau^m e_l, w_h^m)_{\Omega(t)}\, dt\Big| \\
&\quad \le c\int_0^T |\tau^m|_{\Omega(t)} \frac{1}{\sqrt{h}}\Big|\int_t^{t+h} |\bar{w}^m(x,s)|^2_{\Omega(s)}\, ds\Big|^{\frac{1}{2}}\, dt \\
&\quad \le c/\sqrt{h}\,, \\
& \Big|\int_Q (\nabla\tau^m \cdot \nabla\tau_h^m + e^{-k_1 t} w^m \cdot \nabla\tau^m \tau_h^m \\
&\quad + U\cdot\nabla\tau^m\tau_h^m + w^m \cdot \nabla G \tau_h^m)\, dxdt\Big| \\
&\quad \le c/\sqrt{h}\,,
\end{aligned}
$$

$$
(5.17)\qquad \Big| - k_1\int_0^T (w^m, w_h^m)_{\Omega(t)}\, dt - k_1\int_0^T (\tau^m, \tau_h^m)_{\Omega(t)}\, dt\Big| \le c/\sqrt{h}\,.
$$

By Assumption 2.1, $\left|\frac{1}{\sin(\hat{\nu,t})}\right| \ge \delta > 0$ on Σ_1. Taking this and the trace theorem into account, we get

$$
\begin{aligned}
(5.18)\quad & \Big|\int_{\Sigma_1} [\cos(\hat{r,t}) + k(x,t)]\tau^m\tau_h^m\, d\sigma\Big| \\
&= \Big|\int_{\Sigma_1} [\cos(\hat{r,t}) + k(x,t)]\frac{1}{\sin(\hat{r,t})}\tau^m\tau_h^m\, dxdt\Big|
\end{aligned}
$$

$$\le c\int_0^T \|\tau^m\|_{\Omega(t)} \frac{1}{\sqrt{|h|}}\Big|\int_{t-h}^t \|\tau_h^m(x,s)\|_\Omega^2\,ds\Big|^{\frac12}\,dt \le c/\sqrt{|h|}\,.$$

Also, we have that

$$\begin{aligned}
&\big|(w^m(x,T), w_h^m(x,T))_{\Omega(T)}\big| = 0,\\
&\big|(\tau^m(x,T), \tau_h^m(x,T))_{\Omega(T)}\big| \le \frac{c}{\sqrt{h}}\Big|\int_{T-h}^T |\bar\tau^m(x,s)|_{\Omega(t)}^2\,ds\Big|^{\frac12}\,dt\\
&\qquad\qquad\qquad\qquad\qquad\quad \le c/\sqrt{h}\,,\\
&\big|(\bar U, w_h^m(x,T))_{\Omega(T)}\big| \le \frac{c}{\sqrt{h}}, \quad \big|(\bar G, \tau_h^m(x,T))_{\Omega(T)}\big| \le \frac{c}{\sqrt{h}}\,,\\
&\big|(\bar U, w_h^m(x,t_i))_{\Omega(0)}\big| \le \frac{c}{\sqrt{h}}, \quad \big|(\bar G, \tau_h^m(x,0))_{\Omega(0)}\big| \le \frac{c}{\sqrt{h}}\,,
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
&\Big| -\int_Q \frac{\partial \bar U}{\partial t} w_h^m\,dxdt - \int_Q \frac{\partial \bar G}{\partial t}\tau_h^m\,dxdt\Big| \le \frac{c}{\sqrt{h}}\,,\\
&\Big| k_1\int_Q \bar U w_h^m\,dxdt + k_1\int_Q \bar G \tau_h^m\,dxdt\Big| \le \frac{c}{\sqrt{h}}\,,\\
&\Big| -\nu\int_0^T (\operatorname{rot}\bar U, \operatorname{rot} w_h^m)_{\Omega(t)}\,dt - \int_0^T (\mu(x,t)\bar G e_l, w_h^m)_{\Omega(t)}\,dt\Big|\\
&\qquad \le \frac{c}{\sqrt{h}}\,,\\
&\Big| -\int_Q \nabla\bar G\cdot\nabla\tau_h^m\,dxdt\Big| \le \frac{c}{\sqrt{h}}\,.
\end{aligned} \tag{5.20}$$

In the same way as (5.14) we get

$$\begin{aligned}
&\Big|\int_0^T (\operatorname{rot}\bar U\times U, w_h^m)_{\Omega(t)}\,dt\Big| \le \frac{c}{\sqrt{h}}\,,\\
&\Big|\int_Q \big(\bar U\cdot\nabla G\big)\,\tau_h^m\,dxdt\Big| \le \frac{c}{\sqrt{h}}\,.
\end{aligned} \tag{5.21}$$

And also,

$$
\begin{aligned}
&\Big|\int_Q \bar f_1 w_h^m\,dxdt+\int_Q \bar f_2\tau_h^m\,dxdt\Big|\\
&\qquad\le c\int_0^T\Big[\frac{1}{\sqrt h}\Big|\int_{t-h}^t|\bar w^m(x,s)|^2_{\Omega(t)}\,ds\Big|^{\frac12}\\
&\qquad\qquad\qquad+\frac{1}{\sqrt h}\Big|\int_{t-h}^t|\bar\tau^m(x,s)|^2_{\Omega(t)}\,ds\Big|^{\frac12}\Big]\,dt\\
&\qquad\le c/\sqrt h\,,\\
&\Big|\nu\int_0^T\langle\bar\varphi,w_h^m\rangle_{\Sigma_3(t)}\,dt-\int_0^T\langle\bar p_0,w_h^m\cdot n\rangle_{\Sigma_2(t)}\,dt\\
&\qquad\qquad\qquad+\int_0^T\langle\bar e,\tau_h^m\rangle_{\Sigma_1(t)}\,dt\Big|^{\frac12}\,dt\\
&\qquad\le c/\sqrt h\,,\\
&\big|(w(0),w_h^m(0))_{\Omega(0)}+(\tau(0),\tau_h^m(0))_{\Omega(0)}\big|^{\frac12}\le c/\sqrt h\,.
\end{aligned}
\tag{5.22}
$$

Let us estimate

$$
\begin{aligned}
I_3&\equiv-\int_Q w^m\cdot\frac1h\,[\bar w^m(x,t+h)-\bar w^m(x,t)]\,dxdt,\\
I_4&\equiv\int_Q\tau^m\cdot\frac1h\,[\bar\tau^m(x,t)-\bar\tau^m(x,t-h)]\,dxdt.
\end{aligned}
$$

Setting $\Omega(t)=\Omega(T)$ for $t>T$ and using $-ab=-\frac12[(a+b)^2-a^2-b^2]$, we have

$$
\begin{aligned}
I_3&=-\frac1h\int_0^T(\bar w^m(t),\bar w^m(x,t+h)-\bar w^m(x,t))_{\Omega(t)}\,dt\\
&=\frac{1}{2h}\int_0^T|w^m(x,t)|^2_{\Omega(t)}\,dt-\frac{1}{2h}\int_0^T|\bar w^m(x,t+h)|^2_{\Omega(t)}\,dt\\
&\quad+\frac{1}{2h}\int_0^T|\bar w^m(x,t+h)-\bar w^m(x,t)|^2_{\Omega(t)}\,dt\\
&=\frac{1}{2h}\int_0^T|w^m(x,t)|^2_{\Omega(t)}\,dt-\frac{1}{2h}\int_h^T|\bar w^m(x,t)|^2_{\Omega(t-h)}\,dt
\end{aligned}
\tag{5.23}
$$

$$
\begin{aligned}
&+\frac{1}{2h}\int_0^T |\bar{w}^m(x,t+h)-\bar{w}^m(x,t)|^2_{\Omega(t)}\,dt\\
&=\frac{1}{2h}\int_0^T |w^m(x,t)|^2_{\Omega(t)}\,dt-\frac{1}{2h}\int_h^T |w^m(x,t)|^2_{\Omega(t)}\,dt\\
&\quad-\frac{1}{2h}\int_h^T |\bar{w}^m(x,t)|^2_{\Omega(t-h)\setminus\Omega(t)}\,dt\\
&\quad+\frac{1}{2h}\int_0^T |\bar{w}^m(x,t+h)-\bar{w}^m(x,t)|^2_{\Omega(t)}\,dt\\
&\geq\frac{1}{2h}\int_0^T |\bar{w}^m(x,t+h)-\bar{w}^m(x,t)|^2_{\Omega(t)}\,dt,
\end{aligned}
$$

where the fact that $\Omega(t)\subset\Omega(t-h)$ and $\bar{w}^m(x,t)=0$ on $\Omega(t-h)\setminus\Omega(t)$ were used.

Setting $\Omega(t)=\Omega(0)$ for $t<0$ and using $ab=\frac{1}{2}[a^2+b^2-(a-b)^2]$, we have the following estimate.

$$
\begin{aligned}
I_4&=\frac{1}{h}\int_0^T (\bar{\tau}^m(t),\bar{\tau}^m(x,t)-\bar{\tau}^m(x,t-h))_{\Omega(t)}\,dt\\
&=\frac{1}{h}\int_{-h}^{T-h} (\bar{\tau}^m(t+h),\bar{\tau}^m(x,t+h)-\bar{\tau}^m(x,t))_{\Omega(t+h)}\,dt\\
&=-\frac{1}{2h}\int_{-h}^{T-h} |\bar{\tau}^m(x,t)|^2_{\Omega(t+h)}\,dt+\frac{1}{2h}\int_0^T |\bar{\tau}^m(x,t)|^2_{\Omega(t)}\,dt\\
&\qquad\qquad+\frac{1}{2h}\int_{-h}^{T-h} |\bar{\tau}^m(x,t+h)-\bar{\tau}^m(x,t)|^2_{\Omega(t+h)}\,dt\\
&\geq\frac{1}{2h}\int_{T-h}^T |\bar{\tau}^m(x,t)|^2_{\Omega(t)}\,dt\\
&\qquad\qquad+\frac{1}{2h}\int_{-h}^{T-h} |\bar{\tau}^m(x,t+h)-\bar{\tau}^m(x,t)|^2_{\Omega(t)}\,dt\\
&\quad-\frac{1}{2h}\int_{-h}^{T-h} |\bar{\tau}^m(x,t+h)-\bar{\tau}^m(x,t)|^2_{\Omega(t)\setminus\Omega(t+h)}\,dt,
\end{aligned}
\tag{5.24}
$$

where the fact that $\Omega(t+h)\subset\Omega(t)$ was used. Applying Hölder's inequality and using that $H^1(\Omega) \hookrightarrow L_6(\Omega)$, (5.5) and Assumption 2.2, we have

$$
\begin{aligned}
(5.25)\quad & \Big| \frac{1}{2h} \int_{-h}^{T-h} \big|\bar{\tau}^m(x,t+h) - \bar{\tau}^m(x,t)\big|^2_{\Omega(t)\setminus\Omega(t+h)}\, dt \Big| \\
&\le \frac{1}{2h} \int_{-h}^{T-h} \Big[\int_{\Omega(t)\setminus\Omega(t+h)} |\bar{\tau}^m(x,t+h) - \bar{\tau}^m(x,t)|^6 \, dx \Big]^{\frac{1}{3}} \times \\
&\qquad \times \big[mes\,(\Omega(t) \setminus \Omega(t+h))\big]^{\frac{2}{3}} \, dt \\
&\le \frac{c}{2h} \int_0^T \|\bar{\tau}^m(x,t)\|^2_{H^1(\Omega)} \, dt \cdot \omega(h)^{\frac{2}{3}} \le \frac{c}{2h} \cdot \omega(h)^{\frac{2}{3}},
\end{aligned}
$$

where $\omega(\cdot)$ is one in 3) of Assumption 2.2. Substituting (5.25) in the right-hand side of (5.24), we have

$$
(5.26)\qquad I_4 \ge -\frac{c}{2h} \cdot \omega(h)^{\frac{2}{3}} + \frac{1}{2h} \int_0^T |\bar{\tau}^m(x,t+h) - \bar{\tau}^m(x,t)|^2_{\Omega(t)} \, dt.
$$

Formula (5.6), (5.9), (5.12)∼(5.23) and (5.26) imply

$$
\begin{aligned}
(5.27)\qquad & \int_0^T |\bar{w}^m(x,t+h) - \bar{w}^m(x,t)|^2_{\Omega(t)} \, dt \\
&\qquad + \int_0^T |\bar{\tau}^m(x,t+h) - \bar{\tau}^m(x,t)|^2_{\Omega(t)} \, dt \\
&\le c[\sqrt{h} + \omega(h)^{\frac{2}{3}}] \quad \text{for } T > h > 0.
\end{aligned}
$$

In the same way we will get

$$
\begin{aligned}
(5.28)\qquad & \int_0^T |\bar{w}^m(x,t-h) - \bar{w}^m(x,t)|^2_{\Omega(t)} \, dt \\
&\qquad + \int_0^T |\bar{\tau}^m(x,t-h) - \bar{\tau}^m(x,t)|^2_{\Omega(t)} \, dt \\
&\le c[\sqrt{h} + \omega(h)^{\frac{2}{3}}] \quad \text{for } T > h > 0.
\end{aligned}
$$

To this end, we will get other estimates for I_3 and I_4.

$$
\begin{aligned}
(5.29)\qquad I_3 &= -\frac{1}{h}\int_0^T (\bar{w}^m(t), \bar{w}^m(x,t+h) - \bar{w}^m(x,t))_{\Omega(t)}\, dt \\
&= \frac{1}{2h}\int_0^T |w^m(x,t)|^2_{\Omega(t)}\, dt - \frac{1}{2h}\int_0^T |\bar{w}^m(x,t+h)|^2_{\Omega(t)}\, dt \\
&\quad + \frac{1}{2h}\int_0^T |\bar{w}^m(x,t+h) - \bar{w}^m(x,t)|^2_{\Omega(t)}\, dt \\
&= \frac{1}{2h}\int_0^h |w^m(x,t)|^2_{\Omega(t)}\, dt + \frac{1}{2h}\int_h^T |w^m(x,t)|^2_{\Omega(t)}\, dt \\
&\quad - \frac{1}{2h}\int_h^{T+h} |\bar{w}^m(x,t)|^2_{\Omega(t-h)}\, dt \\
&\quad + \frac{1}{2h}\int_h^{T+h} |\bar{w}^m(x,t) - \bar{w}^m(x,t-h)|^2_{\Omega(t-h)}\, dt \\
&= \frac{1}{2h}\int_0^h |w^m(x,t)|^2_{\Omega(t)}\, dt \\
&\quad + \frac{1}{2h}\int_h^{T+h} |\bar{w}^m(x,t) - \bar{w}^m(x,t-h)|^2_{\Omega(t)}\, dt \\
&\quad + \frac{1}{2h}\int_h^{T+h} |\bar{w}^m(x,t) - \bar{w}^m(x,t-h)|^2_{\Omega(t-h)\setminus\Omega(t)}\, dt \\
&\geq \frac{1}{2h}\int_0^T |\bar{w}^m(x,t) - \bar{w}^m(x,t-h)|^2_{\Omega(t)}\, dt,
\end{aligned}
$$

where the fact that $\Omega(t) \subset \Omega(t-h)$ and $\bar{w}^m(x,t) = 0$ on $\Omega(t-h) \setminus \Omega(t)$ were used.

Setting $\Omega(t) = \Omega(0)$ for $t < 0$ and using $ab = \frac{1}{2}[a^2 + b^2 - (a-b)^2]$, we have the following estimate.

$$
\begin{aligned}
(5.30)\quad I_4 &= \frac{1}{h}\int_0^T (\bar{\tau}^m(t), \bar{\tau}^m(x,t) - \bar{\tau}^m(x,t-h))_{\Omega(t)}\, dt \\
&= \frac{1}{2h}\int_0^T |\bar{\tau}^m(x,t)|^2_{\Omega(t)}\, dt + \frac{1}{2h}\int_0^T |\bar{\tau}^m(x,t) - \bar{\tau}^m(x,t-h)|^2_{\Omega(t)}\, dt \\
&\qquad - \frac{1}{2h}\int_0^T |\bar{\tau}^m(x,t-h)|^2_{\Omega(t)}\, dt
\end{aligned}
$$

$$\geq \frac{1}{2h}\int_0^T |\bar\tau^m(x,t)|^2_{\Omega(t)}\,dt - \frac{1}{2h}\int_{-h}^{T-h} |\bar\tau^m(x,t)|^2_{\Omega(t+h)}\,dt$$
$$+ \frac{1}{2h}\int_0^T |\bar\tau^m(x,t) - \bar\tau^m(x,t-h)|^2_{\Omega(t)}\,dt$$
$$\geq \frac{1}{2h}\int_0^T |\bar\tau^m(x,t)|^2_{\Omega(t)}\,dt - \frac{1}{2h}\int_0^T |\bar\tau^m(x,t)|^2_{\Omega(t)}\,dt$$
$$+ \frac{1}{2h}\int_0^T |\bar\tau^m(x,t)|^2_{\Omega(t)\setminus\Omega(t+h)}\,dt$$
$$+ \frac{1}{2h}\int_0^T |\bar\tau^m(x,t) - \bar\tau^m(x,t-h)|^2_{\Omega(t)}\,dt$$
$$= \frac{1}{2h}\int_0^T |\bar\tau^m(x,t)|^2_{\Omega(t)\setminus\Omega(t+h)}\,dt$$
$$+ \frac{1}{2h}\int_0^T |\bar\tau^m(x,t) - \bar\tau^m(x,t-h)|^2_{\Omega(t)}\,dt,$$

where the fact that $\Omega(t+h) \subset \Omega(t)$ was used. Estimating the first term in the right side of (5.30) as (5.25), we get

$$I_4 \geq -\frac{c}{2h}\cdot\omega(h)^{\frac{2}{3}} + \frac{1}{2h}\int_0^T |\bar\tau^m(x,t) - \bar\tau^m(x,t-h)|^2_{\Omega(t)}\,dt. \tag{5.31}$$

Formula (5.6), (5.9), (5.12)∼(5.22), (5.29) and (5.31) imply (5.28).

Now, let us consider the case (3b) in Assumption 2.2. Define

$$w_h^m(x,t) = -\frac{1}{h}\int_{t-h}^t \bar w^m(x,s)\,ds,$$
$$\tau_h^m(x,t) = \frac{1}{h}\int_t^{t+h} \bar\tau^m(x,s)\,ds \text{ if } 0 < h < T.$$

Then, by Assumption 2.2 $w_h^m|_Q \in W(Q)$, $\tau_h^m|_Q \in Y(Q)$. Now, in the same way above, we get (5.27), (5.28).

Next, let

$$\tilde w^m(x,t) = \begin{cases} w^m(x,t) & \text{if } (x,\,t) \in Q \\ 0 & \text{if } (x,\,t) \in R^{l+1}\setminus Q, \end{cases}$$

$$\tilde \tau^m(x,t) = \begin{cases} \tau^m(x,t) & \text{if } (x,\,t) \in Q, \\ 0 & \text{if } (x,\,t) \in R^{l+1}\setminus Q. \end{cases}$$

Then, when $0 < |h| < T$, for any cases in 3) of Assumption 2.2 we have

$$
\begin{aligned}
&\int_0^T |\bar\tau^m(x,t+h) - \tilde\tau^m(x,t+h)|^2_{\Omega(t)}\, dt \\
&\quad \le \int_0^T |\bar\tau^m(x,t+h) - \tilde\tau^m(x,t+h)|^2_{\Omega(t)\setminus\Omega(t+h)}\, dt \\
&\qquad + \int_0^T |\bar\tau^m(x,t+h) - \tilde\tau^m(x,t+h)|^2_{\Omega(t+h)\setminus\Omega(t)}\, dt \\
&\quad \le c\int_0^T \|\bar\tau^m(x,t+h)\|^2_{\Omega}\, dt\cdot \omega(|h|)^{\frac{2}{3}} \le c\,\omega(|h|)^{\frac{2}{3}}, \\
&\bar w^m(x,t) = \tilde w^m(x,t) \quad \text{on } Q.
\end{aligned}
\tag{5.32}
$$

From (5.27)∼(5.32) we have

$$
\begin{aligned}
&\int_0^T |\tilde w^m(x,t+h) - w^m(x,t)|^2_{\Omega(t)}\, dt \\
&\qquad + \int_0^T |\tilde\tau^m(x,t+h) - \tau^m(x,t)|^2_{\Omega(t)}\, dt \\
&\qquad \le c[\sqrt{|h|} + \omega(|h|)^{\frac{2}{3}}] \quad \text{for } 0 < |h| < T.
\end{aligned}
\tag{5.33}
$$

Let $\tilde h \in R^l$ and

$$
\Omega_j(t) = \{x \in \Omega(t) : \operatorname{dist}(x, \partial\Omega(t)) > 2/j\} \quad \text{for } j = 1, 2, 3, \cdots.
$$

Then, by (5.4) and (5.5)

$$
\begin{aligned}
&\int_{\Omega(t)\setminus\Omega_j(t)} |w^m(x,t)|^2\, dx \\
&\quad \le \Big[\int_{\Omega(t)\setminus\Omega_j(t)} |w^m(x,t)|^6\, dx\Big]^{\frac{1}{3}} \cdot [mes\,(\Omega(t)\setminus\Omega_j(t))\,]^{\frac{2}{3}}\, dt \\
&\quad \le c\|w^m(x,t)\|^2_{\Omega(t)}\cdot (1/j)^{\frac{2}{3}} \le c\,(1/j)^{\frac{2}{3}}, \\
&\int_{\Omega(t)\setminus\Omega_j(t)} |\tau^m(x,t)|^2\, dx \le c\,(1/j)^{\frac{2}{3}},
\end{aligned}
\tag{5.34}
$$

where c depends only on Q. Now, if $|\tilde h| < 1/j$, then $x + s\tilde h \in \Omega_{2j}(t)$ provided

$x \in \Omega_j(t)$ and $s \in [0,1]$. For $w^m \in \mathbf{C}^\infty(\overline{\Omega}(t))$,

$$\begin{aligned}\int_{\Omega_j(t)} &|w^m(x+\tilde{h},t) - w^m(x,t)|^2\,dx \\ &\le \int_{\Omega_j(t)} \left[\int_0^1 \left|\frac{d}{ds} w^m(x+s\tilde{h},t)\right| ds\right]^2 dx \\ &\le \int_{\Omega_j(t)} dx \left[\int_0^1 \left|\nabla w^m(x+s\tilde{h},t)\right| \cdot |\tilde{h}|\,ds\right]^2 \\ &\le |\tilde{h}|^2 \int_0^1 \int_{\Omega_j(t)} \left|\nabla w^m(x+s\tilde{h},t)\right|^2 dxds \\ &\le |\tilde{h}|^2 \int_{\Omega_{2j}(t)} |\nabla w^m|^2\,dx \le (1/j)^2 \|w^m\|^2_{\Omega(t)}.\end{aligned} \tag{5.35}$$

Since $\mathbf{C}^\infty(\overline{\Omega}(t))$ is dense in $\mathbf{H}^1(\Omega(t))$, (5.35) is valid for any $w^m(t) \in \mathbf{H}^1(\Omega(t))$. By (5.4), (5.34) and (5.35) we have

$$\int_Q \left|\tilde{w}^m(x+\tilde{h},t) - \tilde{w}^m(x,t)\right|^2 dxdt \le c(1/j)^{2/3} \quad \text{if } |\tilde{h}| < 1/j, \tag{5.36}$$

where c is independent of m. Similarly, we have

$$\int_Q \left|\tilde{\tau}^m(x+\tilde{h},t) - \tilde{\tau}^m(x,t)\right|^2 dxdt \le c(1/j)^{2/3} \quad \text{if } |\tilde{h}| < 1/j. \tag{5.37}$$

Formulas (5.33), (5.36) and (5.37) imply that

$$\begin{aligned}&\int_Q \left|\tilde{w}^m(x+\tilde{h},t+h) - \tilde{w}^m(x,t)\right|^2 dxdt \to 0, \\ &\int_Q \left|\tilde{\tau}^m(x+\tilde{h},t+h) - \tilde{\tau}^m(x,t)\right|^2 dxdt \to 0 \\ &\qquad \text{uniformly with respect to } m \text{ as } (\tilde{h},h) \to 0 \text{ in } R^{l+1}.\end{aligned} \tag{5.38}$$

From (5.27), (5.28) and (5.34) we get

$$\begin{aligned}\forall\varepsilon,\ \exists Q_\varepsilon \text{ such that } \bar{Q}_\varepsilon \subset Q;\ &\int_{Q\setminus Q_\varepsilon} |w^m(x,t)|^2\,dxdt < \varepsilon, \\ &\int_{Q\setminus Q_\varepsilon} |\tau^m(x,t)|^2\,dxdt < \varepsilon.\end{aligned} \tag{5.39}$$

By (5.38) and (5.39) we know that the sets $\{w^m\}$ and $\{\tau^m\}$ are, respectively, precompact in $\mathbf{L}_2(Q)$ and $L_2(Q)$ (cf. Theorem 2.32 in [1]). □

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