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Asymptotic Behaviors of Solutions to One-dimensional Tumor Invasion Model with Quasi-variational Structure

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Abstract. We consider a one-dimensional tumor invasion model of Chaplain–Anderson type with quasi-variational structure, which is originally proposed in [3]. One object is to show the existence of global-in-time solutions by using the limit procedure for suitable approximate solutions. The other is to consider the asymptotic behaviors of global-in-time solutions as time goes to ∞. Actually, we construct at least one global-in-time solution, which enables us to consider the convergence to a certain constant steady-state solution as time goes to ∞ whenever the initial data satisfy suitable conditions.

1. Introduction

In this paper, we consider the following one-dimensional tumor invasion model of Chaplain–Anderson type, which is a nonlinear system composed of two PDEs, one ODE and constraint conditions:

$$
\text{(P)} \begin{cases}
n_t = (d_1 n_x - \lambda(f) n f_x)_x + \mu_p n(1-n-f) - \mu_d n & \text{in } Q_T, \\
f_t = -amf & \text{in } Q_T, \\
m_t = d_2 m_{xx} + bn - cm & \text{in } Q_T, \\
n \geq 0, \quad f \geq 0, \quad n + f \leq \alpha & \text{in } Q_T \\
(d_1 n_x - \lambda(f) n f_x)(\pm L, t) = 0 & \text{for } t \in (0,T), \\
m_x(\pm L, t) = 0 & \text{for } t \in (0,T), \\
(n(0), f(0), m(0)) = (n_0, f_0, m_0) & \text{in } \Omega,
\end{cases}
$$

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where $T \in (0, \infty]$, $\Omega = (-L, L)$ for some $L > 0$ and $Q_T = \Omega \times (0, T)$; a, b, c, d_1 and d_2 are positive constants; $\alpha \geq 1$ is a constant; λ is a non-negative function on $\mathbb{R}$; μ_p and μ_d are non-negative functions on Q_∞; a triplet (n_0, f_0, m_0) is a prescribed initial datum.

In the original model proposed in [3], such constraint conditions as are described by the fourth condition of (P) are not imposed and make it difficult for us to analyze (P) mathematically. From the biological point of view, the unknown functions n, f and m indicate the distributions of tumor cells, the extracellular matrix denoted by ECM and the enzyme degrading ECM denoted by MDE, respectively. Especially, $n(x,t)$ and $f(x,t)$ indicate the local ratios of tumor cells and ECM at the position $x \in \Omega$ and the time $t \in [0, \infty)$, respectively. Actually, $n(x,t) = \alpha$ (resp. $n(x,t) = 0$) means that the position x is completely occupied with tumor cells (resp. there are not any tumor cells at x) at t. Similarly, $f(x,t) = \alpha$ (resp. $f(x,t) = 0$) means that ECM at x is completely healthy (resp. completely destroyed by the biochemical reaction with MDE) at t. Moreover, the value $\alpha - n - f$ means the ratio occupied with the other normal tissues or cells. The first equation describes the dynamics of tumor cells, which is composed of a random motility $d_1 n_{xx}$, a haptotaxis $-(\lambda(f) n f_x)_x$, a cell proliferation of logistic growth type $\mu_p n(1 - n - f)$ and a cell death $-\mu_d n$. The second one describes the kinetics of ECM, which is derived from the biochemical reaction between ECM and MDE with a velocity a. The third one describes the dynamics of MDE, which is composed of a space-uniform diffusion $d_2 m_{xx}$, a secretion from tumor cells bn and a natural decay $-cm$. In this paper, we omit the detail explanation of (P) and entrust it to [3].

Of course, many tumor invasion models are proposed from the biological point of view and analyzed mathematically in [1, 2, 4, 5, 6, 7, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] and their references. The common thing among all models is that the kinetic equation of tumor cells contains a chemotaxis term and/or a haptotaxis one. Furthermore, some mathematical results, which are obtained from the analysis of Keller-Segel system in [8, 26] and their references, are made use of to deal with the tumor invasion models.

In this paper, we concentrate our discussion on the models of Chaplain–Anderson type without and with constraint conditions. For the model without constraint, for example, in [14] G. Liţcanu and C. Morales-Rodrigo showed the existence and uniqueness of global-in-time solutions and consider their asymptotic behaviors as time goes to ∞ for the three-dimensional case. The main ideas in their argument, which will be also used in this paper, come from the properties of the Neumann heat semigroup $\{e^{td\Delta} \mid t \geq 0\}$ on $L^p(\Omega)$ for any $d > 0$ and $p \in [1, \infty]$; for any $t \geq 0$ and $\varphi \in L^p(\Omega)$ the function $y(t) = e^{td\Delta}\varphi$ is a unique solution to the system:

$$\begin{cases} y_t = d\Delta y & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial y}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ y(0) = \varphi & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and ν is the outer normal unit vector on $\partial\Omega$. The properties for the one-dimensional Neumann heat semigroup will be clearly stated in Lemma 2.1 in Section 2.

On the other hand, the models with constraint are dealt with in [9, 10, 11, 12], in which the homogeneous Dirichlet boundary condition is imposed for n instead of the fifth condition of (P) and the haptotaxis sensitivity $\lambda(f)$ is independent of f but a positive function on $[0, \infty)$. In [10, 12] the authors showed for the first time the existence of local-in-time and global-in-time solutions for the case that the coefficient d_1 of a random motility of tumor cells is given by a prescribed positive function $d_1(t)$ of the time variable t. In [9] R. Kano showed the existence of global-in-time solutions for the case that $d_1 = d_1(x, t)$ is a positive function on $\bar{\Omega} \times [0, \infty)$, where $\bar{\Omega} = [-L, L]$. Recently, in [11] R. Kano and A. Ito succeeded in showing the existence of global-in-time solutions for the case that $d_1 = d_1(x, t, f)$ especially depends upon the distribution f of ECM, which is one of the unknown functions in (P). The difficulty to analyze (P) mathematically comes from the constraint conditions $0 \leq n \leq \alpha - f$ in Q_∞. In order to overcome this difficulty, first of all we formally rewrite (P) into the single system of n by using suitable solution operators Λ_1 and Λ_2, which assign n to $f = \Lambda_1 n$ and $m = \Lambda_2 n$, respectively. Then, the constraint conditions are expressed by $0 \leq n \leq \alpha - \Lambda_1 n$ in Q_∞. It is important that the interval $[0, \alpha - f] = [0, \alpha - \Lambda_1 n]$, where the values are allowed to be taken by n, depends upon the unknown function n itself. Such structure is sometimes called *quasi-variational structure*. In order to deal with such systems, we mainly use the theory of quasi-variational inequalities established in [13]. Unfortunately, in general it seems quite difficult and impossible to show the uniqueness of solutions to the systems which have the quasi-variational structures. As a result, until now it is also difficult to deal with the asymptotic behaviors of global-in-time solutions as time goes to ∞ even if we succeed in showing their existences.

Throughout this paper, we denote by $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$ the norms of $L^p(\Omega)$ and $W^{m,p}(\Omega)$, respectively, where $W^{m,2}(\Omega) = H^m(\Omega)$. Furthermore, we assume that (A1)–(A6) are satisfied for the prescribed data:

(A1) λ is non-negative globally Lipschitz continuous on $\mathbb{R}$, whose Lipschitz constant is denoted by L_λ.
(A2) μ_p is continuous on $\Omega \times [0, \infty)$. Moreover, there exist $\mu_1 > 0$ and $\mu_2 > 0$ such that $\mu_1 \le \mu_p(x,t) \le \mu_2$ for all $(x,t) \in \Omega \times [0,\infty)$.
(A3) μ_d is continuous on $\Omega \times [0,\infty)$. Moreover, there exists $\mu_3 > 0$ such that $0 \le \mu_d(x,t) \le \mu_3$ for all $(x,t) \in \Omega \times [0,\infty)$.
(A4) $n_0 \in H^1(\Omega)$ and $0 \le n_0 \le \alpha$ in $\bar{\Omega}$.
(A5) $f_0 \in W^{1,\infty}(\Omega)$ and $0 \le f_0 \le \alpha - n_0$ in $\bar{\Omega}$.
(A6) $m_0 \in W^{1,\infty}(\Omega)$ and $m_0 \ge 0$ in $\bar{\Omega}$.

Under the above assumptions, at first we show the existence of non-negative global-in-time solutions to (P), which is clearly stated in Theorem 1.1.

THEOREM 1.1. *Assume that (A1)–(A6) are satisfied. Then, (P) has at least one non-negative global-in-time solution (n, f, m) satisfying the following properties for any $T > 0$:*
(1) $n \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega))$ and the first equation in (P) is satisfied in the following quasi-variational sense in $L^2(0,T; L^2(\Omega))$:

$$
\begin{aligned}
&\iint_{Q_T} n_t(n-\eta) + d_1 \iint_{Q_T} n_x(n_x - \eta_x) - \iint_{Q_T} \lambda(f) n f_x (n_x - \eta_x) \\
&\quad \le \iint_{Q_T} \mu_p n(1-n-f)(n-\eta) - \iint_{Q_T} \mu_d n(n-\eta) \qquad (1.1)
\end{aligned}
$$

for any $\eta \in L^2(0,T; H^1(\Omega))$ with $0 \le \eta \le \alpha - f$ a.e. in Q_T.

(2) f is given by the following expression:

$$
f(x,t) = f_0(x) \exp\left(-a \int_0^t m(x,s)\,ds\right) \quad \text{for all } (x,t) \in \bar{\Omega} \times [0,T]. \qquad (1.2)
$$

(3) $m \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T; W^{1,\infty}(\Omega)) \cap L^2(0,T; H^2(\Omega))$ and the

third equation in (P) is satisfied in the following variational sense in $L^2(\Omega)$*:*

$$\int_\Omega m_t(t)\zeta + d_2\int_\Omega m_x(t)\zeta_x + c\int_\Omega m(t)\zeta = b\int_\Omega n(t)\zeta \tag{1.3}$$
$$\text{for any } \zeta \in H^1(\Omega) \text{ and a.e. } t \in (0,T).$$

(4) The following constraint conditions are satisfied:

$$n \geq 0, \quad 0 \leq n + f \leq \alpha \quad \text{a.e. in } Q_T. \tag{1.4}$$

(5) $(n(0), f(0), m(0)) = (n_0, f_0, m_0)$ *in* $H^1(\Omega) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$.

Moreover, Theorem 1.2 below guarantees that there exists at least one non-negative global-in-time solution (n, f, m) to (P), which enables us to consider the asymptotic behavior as time goes to ∞.

THEOREM 1.2. *Assume that (A3)' instead of (A3), (A7)–(A9) are satisfied besides (A1)–(A6) except (A3):*
(A3)' $\mu_d \equiv 0$ *on* Q_∞.
(A7) There exists $n_* > 0$ *such that* $n_0(x) \geq n_*$ *for all* $x \in \bar{\Omega}$.
(A8) $n_0(x) \leq 1 - f_0(x)$ *for all* $x \in \bar{\Omega}$.
(A9) There exists $m_* > 0$ *such that* $m_0(x) \geq m_*$ *for all* $x \in \bar{\Omega}$.
Then, there exists a non-negative global-in-time solution (n, f, m) *such that*

$$(n(t), f(t), m(t)) \longrightarrow (1, 0, \tfrac{b}{c}) \quad \text{in } L^2(\Omega) \times C(\bar{\Omega}) \times L^2(\Omega) \quad \text{as} \quad t \to \infty,$$

where $(1, 0, b/c)$ *is a constant solution to the steady state system* $(P)_S$*:*

$$(P)_S \begin{cases} (d_1\bar{n}_x - \lambda(\bar{f})\bar{n}\bar{f}_x)_x + \bar{\mu}_p\bar{n}(1 - \bar{n} - \bar{f}) = 0 & \text{in } \Omega, \\ -a\bar{m}\bar{f} = 0 & \text{in } \Omega, \\ d_2\bar{m}_{xx} + b\bar{n} - c\bar{m} = 0 & \text{in } \Omega, \\ \bar{n} \geq 0, \quad \bar{f} \geq 0, \quad \bar{n} + \bar{f} \leq \alpha & \text{in } \Omega, \\ (d_1\bar{n}_x - \lambda(\bar{f})\bar{n}\bar{f}_x)(\pm L) = 0, & \\ \bar{m}_x(\pm L) = 0, & \end{cases}$$

where $\bar{\mu}_p \in C(\Omega)$ *is any function satisfying* $\mu_1 \leq \bar{\mu}_p \leq \mu_2$ *a.e. in* Ω.

Theorem 1.2 says that ECM in the region Ω is completely destroyed by the biochemical reaction with MDE and the tumor cells reach the saturated state in the whole Ω at time $t = \infty$ when all conditions below are satisfied:

(A2) Cell proliferation rate in the logistic term is positive.
(A3)' Cell death does not exist.
(A7) Tumor cells extends over the whole Ω initially.
(A8) $\mu_p n(1-n-f)$ really works as the proliferation of tumor cells at $t = 0$.
(A9) MDE extends over the the whole Ω initially.

But it does not give any information about the behaviors of tumor cells, ECM and MDE from their early stages.

Remark 1.1. In [6] K. Fujie et al. propose a new tumor invasion model, which is a modified one of Chaplain–Anderson type without constraint conditions and give an idea of the mathematical control method of a tumor invasion phenomenon by heat stress. They show the existence and uniqueness results of classical local-in-time solutions to their system. Moreover, in [7] K. Fujie et al. show the existence of classical global-in-time solutions and consider their asymptotic behaviors as time goes to ∞. They succeed in showing that any classical global-in-time solution always converges to a constant steady state solution as time goes to ∞ although the kinetic equation of tumor cells contains the chemotaxis effect for a substance, which comes from the biochemical reaction between ECM and MDE.

2. Local-in-time Solutions to Approximate Systems of (P)

In this section, for each $\varepsilon \in (0,1)$ we show the existence and uniqueness of local-in-time solutions to the following approximate system $(\mathrm{P})_\varepsilon$ of (P):

$$(\mathrm{P})_\varepsilon \begin{cases} n_t^\varepsilon = (d_1 n_x^\varepsilon - \lambda(f^\varepsilon) n^\varepsilon f_x^\varepsilon)_x - \beta_\varepsilon(f^\varepsilon; n^\varepsilon) + g(n^\varepsilon, f^\varepsilon) & \text{in } Q_\infty, \\ f_t^\varepsilon = -am^\varepsilon f^\varepsilon & \text{in } Q_\infty, \\ m_t^\varepsilon = d_2 m_{xx}^\varepsilon + bn^\varepsilon - cm^\varepsilon & \text{in } Q_\infty, \\ (d_1 n_x^\varepsilon - \lambda(f^\varepsilon) n^\varepsilon f_x^\varepsilon)(\pm L, t) = 0 & \text{for } t > 0, \\ m_x^\varepsilon(\pm L, t) = 0 & \text{for } t > 0, \\ (n^\varepsilon(0), f^\varepsilon(0), m^\varepsilon(0)) = (n_0, f_0, m_0) & \text{in } \Omega, \end{cases}$$

where $g(n, f) = \mu_p n(1 - n - f) - \mu_d n$ and for each $f \in \mathbb{R}$ the function $\beta_\varepsilon(f; \cdot)$ is increasing and globally Lipschitz continuous on $\mathbb{R}$ defined by

$$\beta_\varepsilon(f; r) := \begin{cases} \dfrac{r - \max\{0, \alpha - f\}}{\varepsilon} & \text{if } r > \max\{0, \alpha - f\}, \\ 0 & \text{if } 0 \le r \le \max\{0, \alpha - f\}, \\ \dfrac{r}{\varepsilon} & \text{if } r < 0, \end{cases}$$

whose Lipschitz constant is $1/\varepsilon$. Then, we see that $\beta_\varepsilon(f; \cdot)$ satisfies the following inequalities, which are used without notification below:

$$|\beta_\varepsilon(f; r)| \le \frac{|r|}{\varepsilon} \quad \text{for any } f, r \in \mathbb{R},$$

$$|\beta_\varepsilon(f_1; r) - \beta_\varepsilon(f_2; r)| \le \frac{|f_1 - f_2|}{\varepsilon} \quad \text{for any } f_1, f_2, r \in \mathbb{R}.$$

Moreover, $\hat{\beta}_\varepsilon(f; r) = \int_0^r \beta_\varepsilon(f; \sigma) d\sigma$ gives Moreau–Yosida's regularization of the indicator function on $[0, \max\{\alpha - f\}]$. In order to use the argument similar to those in [6, 14], we use the following change of variables:

$$w = nz, \quad z = \exp\left(-\frac{1}{d_1}\int_0^f \lambda(r) dr\right) > 0. \tag{2.1}$$

Then, the system $(\mathrm{P})_\varepsilon$ can be rewritten into the following system:

$$\begin{cases} w_t^\varepsilon = d_1 w_{xx}^\varepsilon + \lambda(f^\varepsilon) w_x^\varepsilon f_x^\varepsilon + g_\varepsilon(w^\varepsilon, f^\varepsilon, m^\varepsilon) & \text{in } Q_\infty, \\ f_t^\varepsilon = -a m^\varepsilon f^\varepsilon & \text{in } Q_\infty, \\ m_t^\varepsilon = d_2 m_{xx}^\varepsilon + b w^\varepsilon (z^\varepsilon)^{-1} - c m^\varepsilon & \text{in } Q_\infty, \\ w_x^\varepsilon(\pm L, t) = 0 & \text{for any } t > 0, \\ m_x^\varepsilon(\pm L, t) = 0 & \text{for any } t > 0, \\ (w^\varepsilon(0), f^\varepsilon(0), m^\varepsilon(0)) = (w_0, f_0, m_0) & \text{in } \Omega, \end{cases} \tag{2.2}$$

where $(w, n, f, z) = (w^\varepsilon, n^\varepsilon, f^\varepsilon, z^\varepsilon)$ in (2.1) and $g_\varepsilon(w, f, m)$ is given by

$$g_\varepsilon(w, f, m) = -z\beta_\varepsilon(f; wz^{-1}) + \mu_p w(1 - wz^{-1} - f) - \mu_d w + \frac{a\lambda(f) w m f}{d_1}.$$

Then, $w_0 \in H^1(\Omega)$ is easily seen from (A1), (A4) and (A5).

In order to show the existence of solutions (w, f, m) to (2.2), we use the following lemma, whose proof is omitted in this paper and entrusted to [26].

Lemma 2.1. *Let $1 \le p \le q \le \infty$. For each $d > 0$, the one-dimensional Neumann heat semigroup $\{e^{td\Delta}\}_{t\ge 0}$, which is given by the same definition for the three-dimensional case in Section 1, satisfies the following estimates:*
(1) There exists $C_1 = C_1(d, p, q) > 0$ such that

$$\|e^{td\Delta}\varphi\|_q \le C_1 t^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\varphi\|_p \quad \text{for any } t > 0 \text{ and } \varphi \in L^p(\Omega).$$

(2) There exists $C_2 = C_2(d, p, q) > 0$ such that

$$\left\|\left(e^{td\Delta}\varphi\right)_x\right\|_q \le C_2\left(1 + t^{-\frac{1}{2}\left(1+\frac{1}{p}-\frac{1}{q}\right)}\right)\|\varphi\|_p$$
$$\text{for any } t > 0 \text{ and } \varphi \in L^p(\Omega).$$

(3) For each $p \in [2, \infty]$ there exists $C_3 = C_3(d, p) > 0$ such that

$$\|e^{td\Delta}\varphi\|_{1,p} \le C_3\|\varphi\|_{1,p} \quad \text{for any } t > 0 \text{ and } \varphi \in W^{1,p}(\Omega).$$

Now we give the main theorem in this section, which guarantees the existence and uniqueness of non-negative local-in-time solutions to $(\mathrm{P})_\varepsilon$.

Theorem 2.1. *For each $\varepsilon \in (0, 1)$ there exists $T^\varepsilon_{max} \in (0, \infty]$ such that the approximate system $(P)_\varepsilon$ has one and only one non-negative solution $(n^\varepsilon, f^\varepsilon, m^\varepsilon)$ on $[0, T^\varepsilon_{max})$ satisfying the following properties for any $T \in (0, T^\varepsilon_{max})$:*
(1) $n^\varepsilon \in C([0, T]; H^1(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega))$ and the first equation in $(P)_\varepsilon$ is satisfied in the following variational sense in $L^2(\Omega)$:

$$\int_\Omega n^\varepsilon_t(t)\zeta + d_1\int_\Omega n^\varepsilon_x(t)\zeta_x - \int_\Omega \left(\lambda(f^\varepsilon)n^\varepsilon f^\varepsilon_x\right)(t)\zeta_x + \int_\Omega \beta_\varepsilon(f^\varepsilon(t); n^\varepsilon(t))\zeta$$
$$= \int_\Omega \mu_p(t)n^\varepsilon(t)(1 - n^\varepsilon(t) - f^\varepsilon(t))\zeta - \int_\Omega \mu_d(t)n^\varepsilon(t)\zeta$$
$$\text{for any } \zeta \in H^1(\Omega) \text{ and a.e. } t \in (0, T).$$

Moreover, the initial condition $n^\varepsilon(0) = n_0$ *is satisfied in* $H^1(\Omega)$.
(2) f^ε *is also expressed by (1.2), in which* $(f, m) = (f^\varepsilon, m^\varepsilon)$.
(3) $m^\varepsilon \in C([0, T]; W^{1,\infty}(\Omega))$ *and is expressed by the following variation-of-constants formula for all* $t \in [0, T]$:

$$m^\varepsilon(t) = e^{t(d_2\Delta - c)} m_0 + b \int_0^t e^{(t-s)(d_2\Delta - c)} n^\varepsilon(s) ds \quad in \quad L^\infty(\Omega).$$

Moreover, if $T^\varepsilon_{max} < \infty$, *then we have*

$$\|n^\varepsilon(t)\|_{1,2} + \|f^\varepsilon(t)\|_{1,\infty} + \|m^\varepsilon(t)\|_{1,\infty} \longrightarrow \infty \quad as \quad t \nearrow T^\varepsilon_{max}. \tag{2.3}$$

In order to show Theorem 2.1, we mainly deal with the approximate system (2.2) in the following argument. For any $\tau \in (0, 1)$ we consider a Banach space $X^\tau = C([0, \tau]; H^1(\Omega)) \times C([0, \tau]; W^{1,\infty}(\Omega)) \times C([0, \tau]; W^{1,\infty}(\Omega))$ with norm $\|(w, f, m)\|_{X^\tau} = \max_{0\le t\le\tau} (\|w(t)\|_{1,2} + \|f(t)\|_{1,\infty} + \|m(t)\|_{1,\infty})$, and for any $\rho > 0$ a closed ball $B^\tau(0, \rho)$ with center 0 and radius ρ.

First of all, we show the following lemma, which gives some estimates of z defined by (2.1) on $B^\tau(0, \rho)$.

Lemma 2.2. *There exists* $C_4(\rho) > 1$ *such that*

$$\max_{(x,t)\in\bar{Q}_\tau} \left\{|z_1(x, t)|, |z_1^{-1}(x, t)|\right\} \le C_4(\rho), \tag{2.4}$$

$$\min_{(x,t)\in\bar{Q}_\tau} \left\{|z_1(x, t)|,\ |z_1^{-1}(x, t)|\right\} \ge \frac{1}{C_4(\rho)}, \tag{2.5}$$

$$\max\left\{|(z_1 - z_2)(x, t)|,\ |(z_1^{-1} - z_2^{-1})(x, t)|\right\} \le C_4(\rho)|(f_1 - f_2)(x, t)| \quad \text{for all } (x, t) \in \bar{Q}_\tau = \bar{\Omega} \times [0, \tau] \tag{2.6}$$

for any $(w_i, f_i, m_i) \in B^\tau(0, \rho)$, $i = 1, 2$, *where* z_i *is given by (2.1).*

Proof. Let (w_i, f_i, m_i), $i = 1, 2$, be any elements in $B^\tau(0, \rho)$. First of all, we see from (A1) that the following inequalities are satisfied:

$$|\lambda(\sigma)| \le L_\lambda|\sigma| + \lambda(0) =: \bar{C}_4(\sigma) \quad \text{for all } \sigma \in \mathbb{R}, \tag{2.7}$$

$$\left|\int_0^\sigma \lambda(r)dr\right| \le \frac{L_\lambda\sigma^2}{2} + \lambda(0)|\sigma| =: \tilde{C}_4(\sigma) \quad \text{for all } \sigma \in \mathbb{R}. \tag{2.8}$$

By taking $C_4(\rho) = \exp(\tilde{C}_4(\rho)/d_1)$, we see from (2.1) and (2.8) that (2.4) and (2.5) are satisfied.

Next, we use (2.4), (2.7) and apply the mean value theorem for e^r, which is often used without notification in the following argument. Then, we have

$$\max\left\{|z_1 - z_2|,\, |z_1^{-1} - z_2^{-1}|\right\} \le \frac{\bar{C}_4(\rho)}{d_1} \exp\left(\frac{\tilde{C}_4(\rho)}{d_1}\right) \cdot |f_1 - f_2| \quad \text{on } \bar{Q}_\tau,$$

hence, by taking up $C_4(\rho)$ again as

$$C_4(\rho) = \left(1 + \frac{\bar{C}_4(\rho)}{d_1}\right) \exp\left(\frac{\tilde{C}_4(\rho)}{d_1}\right),$$

we see that (2.4)–(2.6) are satisfied. □

Now, we define a mapping Φ^ε on $B^\tau(0,\rho)$ by

$$(\Phi^\varepsilon(w,f,m))(t) = \begin{pmatrix} (\Phi_1^\varepsilon(w,f,m))(t) \\ (\Phi_2^\varepsilon(w,f,m))(t) \\ (\Phi_3^\varepsilon(w,f,m))(t) \end{pmatrix}$$

$$= \begin{pmatrix} e^{td_1\Delta}w_0 + \displaystyle\int_0^t e^{(t-s)d_1\Delta}\hat{g}_\varepsilon((w,f,m)(s))ds \\ f_0 - a\displaystyle\int_0^t (mf)(s)ds \\ e^{t(d_2\Delta - c)}m_0 + b\displaystyle\int_0^t e^{(t-s)(d_2\Delta-c)}(wz^{-1})(s)ds \end{pmatrix} \quad \text{for all } t \in [0,\tau],$$

where $\hat{g}_\varepsilon((w,f,m)(s)) = \lambda(f(s))w_x(s)f_x(s) + g_\varepsilon(w(s),f(s),m(s))$. Then, the mapping Φ^ε satisfies the following properties.

Lemma 2.3. *There exists $\rho_1 > 0$ such that for each $\varepsilon \in (0,1)$ there exists $\tau_1^\varepsilon \in (0,1)$ such that $\Phi^\varepsilon(B^\tau(0,\rho_1)) \subset B^\tau(0,\rho_1)$ for all $\tau \in (0,\tau_1^\varepsilon]$.*

Proof. From the property of the Neumann heat semigroup $\{e^{td\Delta}\}_{t\ge 0}$, we have $\Phi^\varepsilon(B^\tau(0,\rho)) \subset X^\tau$. Without notification we use the compact imbedding $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$, hence, there exists $C_5 > 0$ such that

$$\|\varphi\|_{C(\bar{\Omega})} \le C_5\|\varphi\|_{1,2} \quad \text{for all } \varphi \in H^1(\Omega).$$

By using Lemma 2.2, we have

$$\|z(t)\beta_\varepsilon(f(t);(wz^{-1})(t))\|_2 \le \frac{\rho C_4(\rho)^2}{\varepsilon}. \tag{2.9}$$

By the elemental calculation, we see from (2.9) and Lemma 2.2 again that there exists $C_6(\rho,\varepsilon) > 0$ such that

$$\max_{0\le t\le\tau} \|\hat{g}_\varepsilon(w(t),f(t),m(t))\|_2 \le C_6(\rho,\varepsilon). \tag{2.10}$$

Hence, we see from Lemma 2.1 with (2.10) that the following estimates are satisfied:

$$\|(\Phi_1^\varepsilon(w,f,m))(t)\|_{1,2} \le \tilde{C}_3\|w_0\|_{1,2} + \tilde{C}_{1,2}C_6(\rho,\varepsilon)\left(\tau + 2\sqrt{\tau}\right), \tag{2.11}$$

$$\|(\Phi_2^\varepsilon(w,f,m))(t)\|_{1,\infty} \le \|f_0\|_{1,\infty} + 3a\rho^2\tau, \tag{2.12}$$

$$\|(\Phi_3^\varepsilon(w,f,m))(t)\|_{1,\infty} \le \bar{C}_3\|m_0\|_{1,\infty} + b\rho\bar{C}_{1,2}C_4(\rho)C_5\left(\tau + 2\sqrt{\tau}\right), \tag{2.13}$$

where $\tilde{C}_{1,2} = C_1(d_1,2,2)+C_2(d_1,2,2)$, $\bar{C}_{1,2} = C_1(d_2,\infty,\infty)+C_2(d_2,\infty,\infty)$, $\tilde{C}_3 = C_3(d_1,2)$ and $\bar{C}_3 = C_3(d_2,\infty)$.

At last, by using (2.11)–(2.13), we have

$$\|\Phi^\varepsilon(w,f,m)\|_{X^\tau} \le C_7 + C_8(\rho,\varepsilon)\left(\tau + 2\sqrt{\tau}\right), \tag{2.14}$$

where

$$C_7 = \tilde{C}_3\|w_0\|_{1,2} + \|f_0\|_{1,\infty} + \bar{C}_3\|m_0\|_{1,\infty},$$
$$C_8(\rho,\varepsilon) = \tilde{C}_{1,2}C_6(\rho,\varepsilon) + 3a\rho^2 + b\rho\bar{C}_{1,2}C_4(\rho)C_5.$$

By fixing $\rho_1 > C_7$ and $\tau_1^\varepsilon > 0$ satisfying $C_8(\rho_1,\varepsilon)(\tau_1^\varepsilon + 2\sqrt{\tau_1^\varepsilon}) < \rho_1 - C_7$, then (2.14) implies that this lemma holds. □

Lemma 2.4. *For each $\varepsilon \in (0,1)$ there exists $\tau_2^\varepsilon \in (0,\tau_1^\varepsilon]$ such that Φ^ε is contraction on $B^{\tau_2^\varepsilon}(0,\rho_1)$, where τ_1^ε and ρ_1 are the same numbers that are obtained in Lemma 2.3.*

Proof. Throughout this proof, let $\tau \in (0,\tau_1^\varepsilon]$ be any number and put $(\bar{W},\bar{F},\bar{M}) = (w_1 - w_2, f_1 - f_2, m_1 - m_2)$. At first, we see from Lemma 2.2

that there exists $C_9(\rho_1,\varepsilon)>0$ such that

$$
\begin{aligned}
&\int_\Omega |z_1(t)\beta_\varepsilon(f_1(t);(w_1z_1^{-1})(t))-z_2(t)\beta_\varepsilon(f_2(t);(w_2z_2^{-1})(t))|^2\\
&\le \frac{4}{\varepsilon^2}\int_\Omega |(w_1z_1^{-1}(z_1-z_2))(t)|^2+\frac{4}{\varepsilon^2}\int_\Omega |(z_2z_1^{-1}\bar{w})(t)|^2\\
&\qquad +\frac{4}{\varepsilon^2}\int_\Omega |(z_2w_2(z_1^{-1}-z_2^{-1}))(t)|^2+\frac{4}{\varepsilon^2}\int_\Omega |z_2(t)|^2|\bar{f}(t)|^2\\
&\le C_9(\rho_1,\varepsilon)\left(\|\bar{W}(t)\|_{1,2}^2+\|\bar{F}(t)\|_{1,\infty}^2\right)\qquad \text{for all } t\in[0,\tau].
\end{aligned}
\tag{2.15}
$$

By the elemental calculation with (2.15), we see from that there exists $C_{10}(\rho_1,\varepsilon)>0$ such that

$$
\begin{aligned}
&\|\hat{g}_\varepsilon((w_1,f_1,m_1)(t))-\hat{g}_\varepsilon((w_2,f_2,m_2)(t))\|_2\\
&\le C_{10}(\rho_1,\varepsilon)\|(\bar{W},\bar{F},\bar{M})\|_{X^\tau}\quad \text{for all } t\in[0,\tau].
\end{aligned}
\tag{2.16}
$$

By using Lemma 2.1 and (2.16), for the mapping Φ_1^ε we have

$$
\begin{aligned}
&\|(\Phi_1^\varepsilon(w_1,f_1,m_1))(t)-(\Phi_1^\varepsilon(w_2,f_2,m_2))(t)\|_{1,2}\\
&\le \tilde{C}_{1,2}C_{10}(\rho_1,\varepsilon)\left(\tau+2\sqrt{\tau}\right)\|(\bar{W},\bar{F},\bar{M})\|_{X^\tau}\quad \text{for all } t\in[0,\tau].
\end{aligned}
\tag{2.17}
$$

By repeating the argument similar to Φ_1^ε for the mapping Φ_3^ε, we have

$$
\begin{aligned}
&\|(\Phi_3^\varepsilon(w_1,f_1,m_1))(t)-(\Phi_3^\varepsilon(w_2,f_2,m_2))(t)\|_{1,\infty}\\
&\le b\bar{C}_{1,2}\int_0^t\left(1+(t-s)^{-\frac{1}{2}}\right)\|(w_1z_1^{-1})(s)-(w_2z_2^{-1})(s)\|_\infty ds\\
&\le C_{11}(\rho_1)\left(\tau+2\sqrt{\tau}\right)\|(\bar{W},\bar{F},\bar{M})\|_{X^\tau}\qquad \text{for all } t\in[0,\tau],
\end{aligned}
\tag{2.18}
$$

where $C_{11}(\rho_1)=b(1+\rho_1)\bar{C}_{1,2}C_4(\rho_1)C_5$.

For the mapping Φ_2^ε we have

$$
\begin{aligned}
&\|(\Phi_2^\varepsilon(w_1,f_1,m_1))(t)-(\Phi_2^\varepsilon(w_2,f_2,m_2))(t)\|_{1,\infty}\\
&\le 2a\rho_1\tau\|(\bar{W},\bar{F},\bar{M})\|_{X^\tau}\qquad \text{for all } t\in[0,\tau].
\end{aligned}
\tag{2.19}
$$

At last, we see from (2.17)–(2.19) that the following estimate is satisfied:

$$
\begin{aligned}
&\|\Phi^\varepsilon(w_1,f_1,m_1)-\Phi^\varepsilon(w_2,f_2,m_2)\|_{X^\tau}\\
&\le \left(\tilde{C}_{1,2}C_{10}(\rho_1,\varepsilon)+C_{11}(\rho_1)+2a\rho_1\right)\left(\tau+2\sqrt{\tau}\right)\|(\bar{W},\bar{F},\bar{M})\|_{X^\tau}.
\end{aligned}
\tag{2.20}
$$

By taking $\bar{\tau}_2^\varepsilon$ satisfying

$$(\tilde{C}_{1,2}C_{10}(\rho_1,\varepsilon)+C_{11}(\rho_1)+2a\rho_1)\left(\bar{\tau}_2^\varepsilon+2\sqrt{\bar{\tau}_2^\varepsilon}\right)<1,$$

and putting $\tau_2^\varepsilon=\min\{\tau_1^\varepsilon,\bar{\tau}_2^\varepsilon\}$, (2.20) implies that this lemma holds. □

Now, we are in a position to give the proof of Theorem 2.1.

Proof of Existence Part. We define T^ε_{max} by

$$T^\varepsilon_{max}=\sup\{T>0\,|\,(\mathrm{P})_\varepsilon \text{ has a solution } (n^\varepsilon,f^\varepsilon,m^\varepsilon) \text{ on } [0,T]\}.$$

By using Lemmas 2.3 and 2.4, and applying Banach's fixed point theorem, we see that Φ^ε has a unique fixed point $(w^\varepsilon,f^\varepsilon,m^\varepsilon)\in B^{\tau_2^\varepsilon}(0,\rho_1)$. Because it is clear that $(w^\varepsilon,f^\varepsilon,m^\varepsilon)$ is a solution to (2.2) on $[0,\tau_2^\varepsilon]$, we see from (2.1) and Lemma 2.2 that $(w^\varepsilon(z^\varepsilon)^{-1},f^\varepsilon,m^\varepsilon)$ is also a solution to $(\mathrm{P})_\varepsilon$ on $[0,\tau_2^\varepsilon]$. Hence, we have $\tau_2^\varepsilon\in\{T>0\,|\,(\mathrm{P})_\varepsilon$ has a solution $(n^\varepsilon,f^\varepsilon,m^\varepsilon)$ on $[0,T]\}$, so, T^ε_{max} exists.

Moreover, from the definition of X^τ, which is the function space applied Banach's fixed point theorem, and the boundedness of $(n^\varepsilon,f^\varepsilon,m^\varepsilon)$, it is clear that (2.3) must be satisfied whenever $T^\varepsilon_{max}<\infty$. □

In the next proof, we let $(n^\varepsilon,f^\varepsilon,m^\varepsilon)$ a solution to $(\mathrm{P})_\varepsilon$ on $[0,T^\varepsilon_{max})$ and show their non-negativities by using the argument in [14, Theorem 3.2].

Proof of Non-negativities of Solutions. It is clear from (1.2) and (A5) that f^ε is non-negative on $\bar{\Omega}\times[0,T^\varepsilon_{max})$.

Next, we show the non-negativity of n^ε by using the same method that is given in [14]. We consider a function $H\in C^3(\mathbb{R})$ given by

$$H(r)=\begin{cases}0 & \text{if } r\in[0,\infty),\\ 8r^4 & \text{if } r\in[-\frac{1}{2},0),\\ -8r^4-32r^3-24r^2-8r-1 & \text{if } r\in[-1,-\frac{1}{2}),\\ 24r^2+24r+7 & \text{if } r\in(-\infty,-1).\end{cases}$$

Then, we see from the elemental calculation that $H(\cdot)$ satisfies

$$\begin{cases}0\le H'(r)r\le 12H(r) & \text{for all } r\in\mathbb{R},\\ 0\le H''(r)r^2\le 12H(r) & \text{for all } r\in\mathbb{R}.\end{cases}\tag{2.21}$$

Now, we define the non-negative function $\varphi^\varepsilon(\cdot)$ by

$$\varphi^\varepsilon(t) = \int_\Omega H(n^\varepsilon(t)) \quad \text{for all } t \in [0, T^\varepsilon_{max}).$$

Then, it is clear that $\varphi^\varepsilon \in W^{1,2}(0,\tau)$ for any $\tau \in (0, T^\varepsilon_{max})$ and $\varphi(0) = 0$, which comes from (A4). By integrating the first equation in $(\mathrm{P})_\varepsilon$ on Ω and using the non-negativity of f^ε with (2.21), we have

$$\begin{aligned} \left(\frac{d\varphi^\varepsilon}{dt}\right) &= \int_\Omega H'(n^\varepsilon) n^\varepsilon_t \\ &\leq -d_1 \int_\Omega H''(n^\varepsilon)|n^\varepsilon_x|^2 + \int_\Omega \lambda(f^\varepsilon) H''(n^\varepsilon) n^\varepsilon n^\varepsilon_x f^\varepsilon_x \\ &\quad - \int_\Omega H'(n^\varepsilon)\beta_\varepsilon(f^\varepsilon; n^\varepsilon) + \int_\Omega \mu_p H'(n^\varepsilon) n^\varepsilon (1 - n^\varepsilon) \\ &=: -d_1 \int_\Omega H''(n^\varepsilon)|n^\varepsilon_x|^2 + I_1 - I_2 + I_3 \quad \text{a.e. in } (0,\tau). \end{aligned} \tag{2.22}$$

Then, we see that the following estimates are satisfied for a.e. $t \in (0,\tau)$:

$$I_1(t) \leq d_1 \int_\Omega H''(n^\varepsilon(t))|n^\varepsilon_x(t)|^2 + \frac{3C_4(\|f^\varepsilon\|_\infty)^2 \|f^\varepsilon\|^2_{1,\infty}}{d_1} \cdot \varphi^\varepsilon(t), \tag{2.23}$$

$$I_3(t) \leq 12\mu_2 \left(1 + C_5 \|n^\varepsilon(t)\|_{1,2}\right) \varphi^\varepsilon(t). \tag{2.24}$$

Moreover, since both $H'(\cdot)$ and $\beta_\varepsilon(f;\cdot)$ are increasing on $\mathbb{R}$ satisfying $H'(0) = \beta_\varepsilon(f;0) = 0$, we have

$$I_2(t) \geq 0 \quad \text{for a.e. } t \in (0,\tau). \tag{2.25}$$

By substituting (2.23)–(2.25) into (2.22), we have

$$\left(\frac{d\varphi^\varepsilon}{dt}\right)(t) \leq \ell^\varepsilon(t)\varphi^\varepsilon(t) \quad \text{for a.e. } t \in (0,\tau), \tag{2.26}$$

where $\ell^\varepsilon \in C[0,\tau]$ is given by

$$\ell^\varepsilon(t) = \frac{3C_4(\|f^\varepsilon(t)\|_\infty)^2 \|f^\varepsilon(t)\|^2_{1,\infty}}{d_1} + 12\mu_2 \left(1 + C_5 \|n^\varepsilon(t)\|_{1,2}\right).$$

By applying Gronwall's lemma to (2.26), we have $\varphi^\varepsilon(t) = 0$, i.e., $n^\varepsilon(t) \geq 0$ on $\bar{\Omega}$ for all $t \in [0, \tau]$. Finally, by applying the maximum principle for parabolic PDEs to the third equation in $(\mathrm{P})_\varepsilon$, we have $m^\varepsilon \geq 0$ on $\bar{\Omega} \times [0, \tau]$. Since $\tau \in (0, T^\varepsilon_{max})$ is arbitrary, we see that n^ε and m^ε are non-negative on $\bar{\Omega} \times [0, T^\varepsilon_{max})$. □

Remark 2.1. From the non-negativity of m^ε on $\bar{\Omega} \times [0, T^\varepsilon_{max})$ and (1.2), we see that $0 \leq f^\varepsilon \leq \alpha$ on $\bar{\Omega} \times [0, T^\varepsilon_{max})$ and for each $x \in \bar{\Omega}$ the function $f(x, \cdot\,)$ is decreasing on $[0, T^\varepsilon_{max})$. These facts play important roles not only to show the uniqueness of local-in-time solutions but also the existence of global-in-time solutions to $(\mathrm{P})_\varepsilon$.

Finally, we show the uniqueness of local-in-time solutions to $(\mathrm{P})_\varepsilon$ and make the proof of Theorem 2.1 complete. We refer that the original idea of the uniqueness proof is given in [6, Theorem 1].

Proof of Uniqueness Part. Let $(n_i^\varepsilon, f_i^\varepsilon, m_i^\varepsilon)$, $i = 1, 2$, be two non-negative solutions to $(\mathrm{P})_\varepsilon$ on $[0, T^\varepsilon_{\max})$. Throughout this proof, we omit the index ε and put $(N, F, M) = (n_1 - n_2, f_1 - f_2, m_1 - m_2)$ as well as

$$\begin{cases} P = \lambda(f_1) n_1 (f_1)_x - \lambda(f_2) n_2 (f_2)_x, \\ Q = \beta_\varepsilon(f_1; n_1) - \beta_\varepsilon(f_2; n_2) \\ R = \mu_p(1 - n_1 - n_2 - f_1)N - \mu_p n_2 F - \mu_d N. \end{cases}$$

Then, we see that for any $\tau \in (0, T^\varepsilon_{max})$ the triplet (N, F, M) satisfies the following system (D):

$$(\mathrm{D}) \begin{cases} N_t = (d_1 N_x - P)_x - Q + R & \text{in } Q_\tau, \\ F_t = -a f_1 M - a m_2 F & \text{in } Q_\tau, \\ M_t = d_2 M_{xx} + bN - cM & \text{in } Q_\tau, \\ (d_1 N_x - P)(\pm L, t) = 0 & \text{for } t \in (0, \tau), \\ M_x(\pm L, t) = 0 & \text{for } t \in (0, \tau), \\ (N(0), F(0), M(0)) = (0, 0, 0) & \text{in } \Omega. \end{cases}$$

We note from (2.3) that the following estimate holds:

$$C_{12}(\tau) = \|(n_1, f_1, m_1)\|_{X^\tau} + \|(n_2, f_2, m_2)\|_{X^\tau} < \infty. \tag{2.27}$$

At first, we multiply the first equation in (D) by N and integrate its result on Ω. By using the increasing property of $\beta_\varepsilon(f;\cdot)$, we have

$$-\int_\Omega QN \le \int_\Omega |\beta_\varepsilon(f_1;n_2)-\beta_\varepsilon(f_2;n_2)||N| \le \frac{1}{2\varepsilon}\cdot\left(\|N\|_2^2+\|F\|_2^2\right). \tag{2.28}$$

By using Remark 2.1, we see that there exists $C_{13}(\tau)>0$ such that the following inequalities are satisfied a.e. in $(0,\tau)$:

$$\begin{aligned}
&\int_\Omega (d_1N_x-P)_xN \le \frac{1}{4d_1}\int_\Omega |P|^2\\
&\le \frac{3}{4d_1}\int_\Omega \{(L_\lambda|n_1(f_1)_xF|)^2+|\lambda(f_2)(f_1)_xN|^2+|\lambda(f_2)n_2F_x|^2\}\\
&\le C_{13}(\tau)\left(\|N\|_2^2+\|F\|_2^2+\|F_x\|_2^2\right)
\end{aligned} \tag{2.29}$$

as well as

$$\int_\Omega RN \le \mu_2\left(1+\frac{C_5C_{12}(\tau)}{2}\right)\left(\|N\|_2^2+\|F\|_2^2\right). \tag{2.30}$$

By using (2.28)–(2.30), we have

$$\begin{aligned}
&\frac{d}{dt}\|N(t)\|_2^2 \le C_{14}(\tau,\varepsilon)\left(\|N(t)\|_2^2+\|F(t)\|_2^2+\|F_x(t)\|_2^2\right)\\
&\qquad\qquad \text{for a.e. } t\in(0,\tau),
\end{aligned} \tag{2.31}$$

where

$$C_{14}(\tau,\varepsilon)=2C_{13}(\tau)+\mu_2(2+C_5C_{12}(\tau))+\frac{1}{\varepsilon}.$$

Moreover, by using (1.2) and Remark 2.1, we have

$$\begin{aligned}
|F_x(t)| &\le a|(f_0)_x|\exp\left(a\int_0^t|m_1(s)|ds+a\int_0^t|m_2(s)|ds\right)\int_0^t|M(s)|ds\\
&\quad +a|F(t)|\int_0^t|(m_1)_x(s)|ds+a|f_2(t)|\int_0^t|M_x(s)|ds\\
&\le a\|f_0\|_{1,\infty}e^{2a\tau C_{12}(\tau)}\int_0^t|M(s)|ds+a\tau C_{12}(\tau)|F(t)|+a\alpha\int_0^t|M_x(s)|ds
\end{aligned}$$

for a.e. in Ω, which gives the following estimate:

$$\|F_x(t)\|_2^2 \le C_{15}(\tau)\left(\|F(t)\|_2^2+J_1(t)\right) \quad \text{for all } t\in[0,\tau], \tag{2.32}$$

where

$$J_1(t) = c\int_0^t \|M(s)\|_2^2 ds + 2d_2\int_0^t \|M_x(s)\|_2^2 ds,$$

$$C_{15}(\tau) = 3a^2\tau^2 C_{12}(\tau)^2 + \frac{3a^2\tau e^{4a\tau C_{12}(\tau)}\|f_0\|_{1,\infty}^2}{c} + \frac{3a^2\alpha^2\tau}{2d_2}.$$

By substituting (2.32) into (2.31), we have

$$\frac{d}{dt}\|N(t)\|_2^2 \le C_{14}(\tau,\varepsilon)(1 + C_{15}(\tau))J(t) \quad \text{for a.e. } t \in (0,\tau), \tag{2.33}$$

where $J(\cdot)$ is a non-negative function on $[0,\tau]$ given by

$$J(t) = \|N(t)\|_2^2 + \|F(t)\|_2^2 + \|M(t)\|_2^2 + J_1(t).$$

Secondly, we multiply the second equation in (D) by F and integrate its result on Ω to derive

$$\frac{d}{dt}\|F(t)\|_2^2 \le a\alpha\left(\|F(t)\|_2^2 + \|M(t)\|_2^2\right) \quad \text{for a.e. } t \in (0,\tau). \tag{2.34}$$

Thirdly, by multiplying the third equation in (D) by M and integrating its result on Ω, we have

$$\frac{d}{dt}\left(\|M(t)\|_2^2 + J_1(t)\right) \le \frac{b^2}{c}\|N(t)\|_2^2 \quad \text{for a.e. } t \in (0,\tau). \tag{2.35}$$

By adding (2.33)–(2.35), we have

$$\left(\frac{dJ}{dt}\right)(t) \le C_{16}(\tau,\varepsilon)J(t) \quad \text{for a.e. } t \in (0,\tau), \tag{2.36}$$

where

$$C_{16}(\tau,\varepsilon) = C_{14}(\tau,\varepsilon)(C_{15}(\tau)+1) + a\alpha + \frac{b^2}{c}.$$

By applying Gronwall's lemma to (2.36), we have $J(t) = 0$ for all $t \in [0,\tau]$, i.e., $(n_1, f_1, m_1) = (n_2, f_2, m_2)$ in $(L^2(\Omega))^3$ on $[0,\tau]$. Since $\tau \in (0, T^{\varepsilon}_{max})$ is arbitrary, the solution $(n^{\varepsilon}, f^{\varepsilon}, m^{\varepsilon})$ to $(\mathrm{P})_{\varepsilon}$ on $[0, T^{\varepsilon}_{max})$ is unique. □

3. Global-in-time Solutions to Approximate Systems of (P)

In this section, we use the same notations in Section 2 and devote ourselves to show the existence of global-in-time solutions to $(\mathrm{P})_\varepsilon$. For this, we use the argument similar to that of [14, Section 4].

Theorem 3.1. *For each $\varepsilon \in (0,1)$ the system $(P)_\varepsilon$ has one and only one non-negative global-in-time solution $(n^\varepsilon, f^\varepsilon, m^\varepsilon)$, that is, $T^\varepsilon_{max} = \infty$.*

We fix any $\varepsilon \in (0,1)$ and prepare some boundedness of the local-in-time solution $(n^\varepsilon, f^\varepsilon, m^\varepsilon)$ to $(\mathrm{P})_\varepsilon$. First of all, we give L^1-boundedness of $(n^\varepsilon, m^\varepsilon)$ and L^∞-boundedness of f^ε in Lemma 3.1 below.

Lemma 3.1. *$(n^\varepsilon, f^\varepsilon, m^\varepsilon)$ satisfies the following estimates:*

$$\max_{0\le t<T^\varepsilon_{max}} \|n^\varepsilon(t)\|_1 \le 2L \max\left\{\alpha, \frac{\mu_2}{\mu_1}\right\}, \tag{3.1}$$

$$\max_{0\le t<T^\varepsilon_{max}} \|m^\varepsilon(t)\|_1 \le \|m_0\|_1 + \frac{2bL}{c} \max\left\{\alpha, \frac{\mu_2}{\mu_1}\right\}. \tag{3.2}$$

$$\max_{0\le t<T^\varepsilon_{max}} \|f^\varepsilon(t)\|_{C(\bar\Omega)} \le \alpha, \tag{3.3}$$

Proof. First of all, since m^ε is non-negative on $\bar\Omega \times [0, T^\varepsilon_{max})$, we see from (A5) and (1.2) that (3.3) holds. (cf. See Remark 2.1.)

Next, we integrate the first equation in $(\mathrm{P})_\varepsilon$ on Ω, and use the non-negativities of the functions $(n^\varepsilon, f^\varepsilon)$ and the increasing property of $\beta_\varepsilon(f;\cdot)$ with $\beta_\varepsilon(f;0)=0$. Then, we have

$$\frac{d}{dt}\|n^\varepsilon(t)\|_1 \le \mu_2\|n^\varepsilon(t)\|_1 - \frac{\mu_1}{2L}\|n^\varepsilon(t)\|_1^2 \quad \text{for a.e. } t \in (0, T^\varepsilon_{max}),$$

which implies that (3.1) is satisfied.

Finally, we integrate the third equation in $(\mathrm{P})_\varepsilon$ on Ω to have

$$\frac{d}{dt}\|m^\varepsilon(t)\|_1 + c\|m^\varepsilon(t)\|_1 \le b\|n^\varepsilon(t)\|_1 \quad \text{for a.e. } t \in (0, T^\varepsilon_{max}). \tag{3.4}$$

By applying Gronwall's lemma to (3.4) and using (3.1), we see that (3.2) is satisfied. □

From (2.1), Remark 2.1 and Lemma 3.1, the following corollary is immediately obtained.

COROLLARY 3.1. *z^ε and w^ε satisfy the following estimates:*

$$\max_{0\le t<T^\varepsilon_{max}} \|z^\varepsilon(t)\|_\infty \le 1, \tag{3.5}$$

$$\max_{0\le t<T^\varepsilon_{max}} \|(z^\varepsilon)^{-1}(t)\|_\infty \le C_4(\alpha), \tag{3.6}$$

$$\max_{0\le t<T^\varepsilon_{max}} \|w^\varepsilon(t)\|_1 \le 2L \max\left\{\alpha, \frac{\mu_2}{\mu_1}\right\}, \tag{3.7}$$

where $C_4(\alpha)$ is the same constant that is obtained in Lemma 2.2 with $\rho = \alpha$.

The next lemmas give us L^∞-boundedness of $(n^\varepsilon, m^\varepsilon)$. Although the original and essential proof is precisely shown [14, Proposition 4.2], we give their proofs to make the difference between ours and those in [14] clear.

LEMMA 3.2. *There exists $C_{17} > 0$ such that*

$$\max_{0\le t<T^\varepsilon_{max}} \|m^\varepsilon(t)\|_\infty \le C_{17}.$$

PROOF. By using (1) in Lemma 2.1 and (3.1), we have

$$\begin{aligned}\|m^\varepsilon(t)\|_\infty &\le \|m_0\|_\infty + b\hat{C}_1 \int_0^t e^{-c(t-s)}(t-s)^{-\frac{1}{2}}\|n^\varepsilon(s)\|_1\, ds \\ &\le \|m_0\|_\infty + \frac{2bL\hat{C}_1\Gamma(\frac{1}{2})}{\sqrt{c}} \max\left\{\alpha, \frac{\mu_2}{\mu_1}\right\},\end{aligned}$$

where $\hat{C}_1 = C_1(d_2, 1, \infty)$ and Γ denotes the Gamma function throughout this paper. This estimate implies that this lemma holds. □

LEMMA 3.3. *There exists $C_{18} > 0$ such that*

$$\max_{0\le t<T^\varepsilon_{max}} \|n^\varepsilon(t)\|_\infty \le C_{18}.$$

Proof. We omit the index ε of solutions $(w^\varepsilon, f^\varepsilon, m^\varepsilon)$ to (2.2) and z^ε throughout this proof. Let $p \geq 2$ be any number. At first, we multiply the first equation in (2.2) by $pw^{p-1}z^{-1}$ and integrate its result on Ω. By using the non-negativities of (w, f, m, z) and $z^{-1} \geq 1$ on $\bar{\Omega} \times [0, T^\varepsilon_{max})$, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega z^{-1}w^p \leq -\frac{4d_1(p-1)}{p}\cdot\|(w^{p/2})_x\|_2^2 + p\int_\Omega \mu_p z^{-1}w^p \\ &-p\int_\Omega w^{p-1}\beta_\varepsilon(f; wz^{-1}) + \frac{a(p-1)}{d_1}\int_\Omega z^{-1}\lambda(f)w^p mf. \end{aligned} \tag{3.8}$$

Since $\beta_\varepsilon(f;\cdot)$ is increasing on $\mathbb{R}$ with $\beta_\varepsilon(f;0) = 0$, we have

$$\int_\Omega w^{p-1}\beta_\varepsilon(f; wz^{-1}) \geq 0. \tag{3.9}$$

By using (3.3) and Corollary 3.1, we see that there exists $C_{19} > 0$ such that

$$\int_\Omega \mu_p z^{-1}w^p \leq C_{19}\|w\|_p^p, \tag{3.10}$$

$$\frac{a}{d_1}\int_\Omega z^{-1}\lambda(f)w^p mf \leq C_{19}\int_\Omega w^p m. \tag{3.11}$$

By substituting (3.9)–(3.11) into (3.8), for any $\delta \in (0,1)$ we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega z^{-1}w^p + \delta d_1\|w\|_p^p \leq -\frac{4d_1(p-1)}{p}\cdot\|(w^{p/2})_x\|_2^2 \\ &+(pC_{19} + \delta d_1)\|w\|_p^p + C_{19}(p-1)\int_\Omega w^p m. \end{aligned} \tag{3.12}$$

By using Young's inequality and Lemma 3.2, we have

$$\begin{aligned} &\int_\Omega w^p m \leq C_5^{3/2}\|w^{p/2}\|_{1,2}^{3/2}\|w^{p/2}\|_1^{1/2}\|m\|_2 \\ &\leq \frac{\delta d_1}{C_{19}(p-1)}\|w^{p/2}\|_{1,2}^2 + \frac{3^3L^2C_5^6C_{17}^4C_{19}^3(p-1)^3}{4^3\delta^3 d_1^3}\cdot\|w\|_{p/2}^p. \end{aligned} \tag{3.13}$$

By substituting (3.13) into (3.12), we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega z^{-1}w^p + \delta d_1\|w\|_p^p \leq d_1\left\{\delta - \frac{4(p-1)}{p}\right\}\cdot\|(w^{p/2})_x\|_2^2 \\ &+(pC_{19} + 2d_1\delta)\|w\|_p^p + \frac{3^3L^2C_5^6C_{17}^4C_{19}^4(p-1)^4}{4^3\delta^3 d_1^3}\cdot\|w\|_{p/2}^p. \end{aligned} \tag{3.14}$$

Next, we estimate the second term in the right-hand side of (3.14). By repeating the argument similar to (3.13), we have

$$\|w\|_p^p \le \delta\|w\|_p^p + \delta\|(w^{p/2})_x\|_2^2 + \frac{3^3L^2C_5^6}{4^3\delta^3}\cdot\|w\|_{p/2}^p,$$

hence,

$$\|w\|_p^p \le \frac{1}{1-\delta}\left\{\delta\|(w^{p/2})_x\|_2^2 + \frac{3^3L^2C_5^6}{4^3\delta^3}\cdot\|w\|_{p/2}^p\right\}. \tag{3.15}$$

We see from (3.14) and (3.15) that there exists $C_{20}(\delta, p) > 0$ such that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega z^{-1}w^p + \delta d_1\|w\|_p^p \le C_{20}(\delta,p)\|w\|_{p/2}^p \\ &+d_1\left\{\delta + \frac{\delta}{1-\delta}\left(\frac{pC_{19}}{d_1}+2\delta\right) - \frac{4(p-1)}{p}\right\}\|(w^{p/2})_x\|_2^2. \end{aligned} \tag{3.16}$$

By the elemental calculation, we have

$$\delta_p + \frac{\delta_p}{1-\delta_p}\left(\frac{pC_{19}}{d_1}+2\delta_p\right) < \frac{4(p-1)}{p}, \quad \delta_p = \frac{1}{C_{21}(p+1)}, \tag{3.17}$$

where $C_{21} = C_{19}/3d_1 + 7/6$.

Moreover, we see that there exists $C_{22} > 0$ such that $C_{22}(p+1)^7 \ge C_{20}(\delta_p, p)$ for all $p \ge 2$. Hence, by using (3.16), (3.17) and Corollary 3.1, we see that the following inequality is satisfied a.e. in $(0, T_{max}^\varepsilon)$:

$$\frac{d}{dt}\int_\Omega z^{-1}w^p + \frac{d_1}{C_4(\alpha)C_{21}(p+1)}\int_\Omega z^{-1}w^p \le C_{22}(p+1)^7\|w\|_{p/2}^p.$$

By applying Gronwall's lemma to the above inequality, we see that the following estimate is satisfied for all $t \in [0, T_{max}^\varepsilon)$:

$$\begin{aligned} &\int_\Omega (z^{-1}w^p)(t) \le \int_\Omega z_0^{-1}w_0^p + a(p)C_4(\alpha)\left(\max_{0\le t<T_{max}}\|w(t)\|_{p/2}\right)^p \\ &\le 2C_4(\alpha)\max\left\{2\alpha^pL,\, a(p)\left(\max_{0\le t<T_{max}}\|w(t)\|_{p/2}\right)^p\right\}, \end{aligned}$$

hence,

$$\|w(t)\|_p \le (2a(p)C_4(\alpha))^{\frac{1}{p}}\max\left\{\alpha(2L)^{\frac{1}{p}},\, \max_{0\le t<T_{max}}\|w(t)\|_{p/2}\right\}, \tag{3.18}$$

where

$$(p+1)^8 < a(p) = \frac{C_{21}C_{22}(p+1)^8}{d_1} < \frac{C_{21}C_{22}(2p)^8}{d_1}.$$

Then, by taking $p = 2$ in (3.18) and using Corollary 3.1 again, we have

$$\max_{0\le t<T^{\varepsilon}_{max}} \|w(t)\|_2 \le (2C_4(\alpha)a(2))^{\frac{1}{2}}(2L+1)\max\left\{\alpha, \frac{\mu_2}{\mu_1}\right\}. \tag{3.19}$$

By repeating the same operation in $p = 2$ for $p = 2^j$ inductively, we see that the following estimate is satisfied for all $t \in [0, T^{\varepsilon}_{max})$:

$$\begin{aligned}\|w(t)\|_{2^j} &\le \prod_{k=1}^{j} a(2^k)^{\frac{1}{2^k}} \cdot (2C_4(\alpha))^{\sum_{k=1}^{j}\frac{1}{2^j}} \cdot (2L+1)\max\left\{\alpha, \frac{\mu_2}{\mu_1}\right\} \\ &\le \frac{2^{9+8\sum_{j=1}^{\infty}\frac{j}{2^j}}(2L+1)C_4(\alpha)C_{21}C_{22}}{d_1}\max\left\{\alpha, \frac{\mu_2}{\mu_1}\right\}.\end{aligned} \tag{3.20}$$

By taking the limit $j \to \infty$ in (3.20), we see that the following estimate is satisfied for all $t \in [0, T^{\varepsilon}_{max})$:

$$\|w(t)\|_\infty \le \frac{2^{9+8\sum_{j=1}^{\infty}\frac{j}{2^j}}(2L+1)C_4(\alpha)C_{21}C_{22}}{d_1}\max\left\{\alpha, \frac{\mu_2}{\mu_1}\right\}. \tag{3.21}$$

Finally, we see from (2.1) and (3.21) that this lemma holds. □

In Lemma 3.4 below, we give $W^{1,\infty}$-boundedness of $(f^{\varepsilon}, m^{\varepsilon})$.

Lemma 3.4. *There exists $C_{23} > 0$ such that*

$$\max_{0\le t<T^{\varepsilon}_{max}} \|m^{\varepsilon}(t)\|_{1,\infty} \le C_{23}, \tag{3.22}$$

$$\max_{0\le t<T^{\varepsilon}_{max}} \|f^{\varepsilon}(t)\|_{1,\infty} \le C_{23}(1+T^{\varepsilon}_{max}). \tag{3.23}$$

Proof. By repeating the same argument in Lemma 2.3 and using Lemma 3.3, we see that the following estimate is satisfied for all $t \in [0, T^{\varepsilon}_{max})$:

$$\|m^{\varepsilon}(t)\|_{1,\infty} \le \bar{C}_3\|m_0\|_{1,\infty} + b\bar{C}_{1,2}C_{18}\left(\frac{1}{c} + \frac{\Gamma(\frac{1}{2})}{\sqrt{c}}\right) =: C_{24}, \tag{3.24}$$

which implies that (3.22) is satisfied.

By using (1.2), (3.24) and the non-negativity of m^ε on $[0, T^\varepsilon_{max})$, we have

$$\|f^\varepsilon(t)\|_{1,\infty} \le \|f_0\|_{1,\infty} + a\alpha \int_0^t \|m^\varepsilon_x(s)\|_\infty ds \le \|f_0\|_{1,\infty} + a\alpha C_{24} t$$

for all $t \in [0, T^\varepsilon_{max})$, which implies that (3.23) holds. □

Before showing H^1-boundedness of n^ε, we prepare the following lemma, which is one of the Gronwall's lemma. Although its proof is quite standard, we give it in this paper by way of caution.

LEMMA 3.5. *Let $\theta \in (0,1)$, $\tau > 0$, $\gamma(\tau) > 0$ be fixed numbers, $\phi(\cdot)$ be a non-negative continuous function on $[0,\tau)$ and $h(\cdot)$ be a positive continuous function on $(0,\infty)$. Assume that for any $M > \gamma(\tau)$ the following inequality is satisfied for all $t \in [0,\tau)$:*

$$\phi(t) \le h(M) + \gamma(\tau) \int_0^t e^{-M(t-s)} \left(1 + (t-s)^{-\theta}\right) \phi(s) ds. \tag{3.25}$$

Then, there exists $C_{25}(\tau) > 0$ such that

$$\max_{0 \le t < \tau} \phi(t) \le C_{25}(\tau).$$

PROOF. We put $\Phi(\tau) = \max_{0 \le t < \tau} \phi(t)$. At first, we see from (3.25) that the following inequalities are satisfied:

$$\phi(t) \le h(M) + \gamma(\tau) \int_0^t e^{-M(t-s)} \phi(s) ds + \frac{\gamma(\tau)\Gamma(1-\theta)}{M^{1-\theta}} \cdot \Phi(\tau), \tag{3.26}$$

hence,

$$e^{Mt}\phi(t) \le \left\{ h(M) + \frac{\gamma(\tau)\Gamma(1-\theta)}{M^{1-\theta}} \cdot \Phi(\tau) \right\} e^{Mt} + \gamma(\tau) \int_0^t e^{Ms} \phi(s) ds.$$

By applying Gronwall's lemma to the above inequality, we have

$$\int_0^t e^{Ms} \phi(s) ds \le \frac{1}{M - \gamma(\tau)} \left\{ h(M) + \frac{\gamma(\tau)\Gamma(1-\theta)}{M^{1-\theta}} \cdot \Phi(\tau) \right\} e^{Mt},$$

hence,

$$\int_0^t e^{-M(t-s)}\phi(s)ds \le \frac{1}{M-\gamma(\tau)}\left\{h(M) + \frac{\gamma(\tau)\Gamma(1-\theta)}{M^{1-\theta}}\cdot\Phi(\tau)\right\}. \tag{3.27}$$

By substituting (3.27) into (3.26), we have

$$\Phi(\tau) \le \frac{Mh(M)}{M-\gamma(\tau)} + \frac{\gamma(\tau)M\Gamma(1-\theta)}{(M-\gamma(\tau))M^{1-\theta}}\cdot\Phi(\tau). \tag{3.28}$$

We can choose $M_1 > \gamma(\tau)$ satisfying

$$M_2 = 1 - \frac{\gamma(\tau)M_1\Gamma(1-\theta)}{(M_1-\gamma(\tau))M_1^{1-\theta}} > 0,$$

because

$$\frac{\gamma(\tau)M\Gamma(1-\theta)}{(M-\gamma(\tau))M^{1-\theta}} \longrightarrow 0 \quad \text{as} \quad M \to \infty.$$

Hence, we see from (3.28) that the following boundedness is satisfied:

$$\Phi(\tau) \le \frac{M_1 h(M_1)}{M_2(M_1-\gamma(\tau))},$$

which directly implies that this lemma holds. □

Lemma 3.6. *There exists $C_{26}(\varepsilon) > 0$ such that*

$$\max_{0\le t<T^\varepsilon_{max}} \|n^\varepsilon(t)\|_{1,2} \le C_{26}(\varepsilon).$$

Proof. For any $M > \gamma(T^\varepsilon_{max})$, where $\gamma(T^\varepsilon_{max})$ is exactly determined in the following argument, we use the variation-of-constants formula of w^ε:

$$w^\varepsilon(t) = e^{t(d_1\Delta-M)}w_0 + \int_0^t e^{(t-s)(d_1\Delta-M)}\hat{g}_{\varepsilon,M}((w^\varepsilon, f^\varepsilon, m^\varepsilon)(s))ds,$$

where $\hat{g}_{\varepsilon,M}(w,f,m) = \hat{g}_\varepsilon(w,f,m) + Mw$. We see from Lemmas 3.1–3.4 that there exist $C_{27} > 0$ and $C_{28}(\varepsilon, M) > 0$ such that the following estimate is satisfied for all $t \in [0, T^\varepsilon_{max})$:

$$\|\hat{g}_{\varepsilon,M}((w^\varepsilon, f^\varepsilon, m^\varepsilon)(s))\|_2 \le C_{27}(1+T^\varepsilon_{max})\|w^\varepsilon(s)\|_{1,2} + C_{28}(\varepsilon, M). \tag{3.29}$$

By using (3.29) and Lemma 2.1, we have

$$\|w^\varepsilon(t)\|_{1,2} \leq h_\varepsilon(M) + \gamma(T^\varepsilon_{max}) \int_0^t e^{-M(t-s)} \left(1 + (t-s)^{-\frac{1}{2}}\right) \|w^\varepsilon(s)\|_{1,2}\, ds,$$

where

$$h_\varepsilon(M) = \tilde{C}_3 \|w_0\|_{1,2} + \tilde{C}_{1,2} C_{28}(\varepsilon, M) \left(\frac{1}{M} + \frac{\Gamma(\frac{1}{2})}{\sqrt{M}} \right),$$
$$\gamma(T^\varepsilon_{max}) = \tilde{C}_{1,2} C_{27} \left(1 + T^\varepsilon_{max}\right).$$

By applying Lemma 3.5, we see that there exists $C_{29}(\varepsilon) > 0$ such that

$$\max_{0 \leq t < T^\varepsilon_{max}} \|w^\varepsilon(t)\|_{1,2} \leq C_{29}(\varepsilon). \tag{3.30}$$

Moreover, we see from (2.1) and Lemmas 3.1–3.4 that the following estimate is satisfied for all $t \in [0, T^\varepsilon_{max})$:

$$\begin{aligned} \|n^\varepsilon_x(t)\|_2^2 &\leq 2 \int_\Omega |(w^\varepsilon_x (z^\varepsilon)^{-1})(t)|^2 + \frac{2}{d_1^2} \int_\Omega |(\lambda(f^\varepsilon) w^\varepsilon f^\varepsilon_x (z^\varepsilon)^{-1})(t)|^2 \\ &\leq 2C_4(\alpha)^2 \left\{ 1 + \frac{C_4(\alpha)^2 C_{23}^2 \left(1 + T^\varepsilon_{max}\right)^2}{d_1^2} \right\} \|w^\varepsilon(t)\|^2_{1,2}. \end{aligned}$$

Hence, we see that this lemma holds. □

Now, we are in a position to give the proof of Theorem 3.1.

Proof of Theorem 3.1. We assume $T^\varepsilon_{max} < \infty$ and let $(n^\varepsilon, f^\varepsilon, m^\varepsilon)$ a unique solution to $(\mathrm{P})_\varepsilon$ on $[0, T^\varepsilon_{max})$. Then, we see from Lemmas 3.4 and 3.6 that there exists $C_{30}(\varepsilon) > 0$ such that

$$\max_{0 \leq t < T^\varepsilon_{max}} \left(\|n^\varepsilon(t)\|_{1,2} + \|f^\varepsilon(t)\|_{1,\infty} + \|m^\varepsilon(t)\|_{1,\infty} \right) \leq C_{30}(\varepsilon),$$

which is in contradiction with (2.3). Hence, $T^\varepsilon_{max} = \infty$ must be holds. □

4. Existence of Global-in-time Solutions to (P)

In this section, we devote ourselves to show Theorem 1.1. For each $\varepsilon \in (0,1)$ we let $(n^\varepsilon, f^\varepsilon, m^\varepsilon)$ (resp. $(w^\varepsilon, f^\varepsilon, m^\varepsilon)$) a unique non-negative global-in-time solution to $(\mathrm{P})_\varepsilon$ (resp. (2.2)) and use the same notations in Sections 2 and 3. Then, we note that we have already had their uniform boundedness in Lemmas 3.1– 3.4 and Corollary 3.1, which are clearly stated in Lemma 4.1 below again.

Lemma 4.1. *There exists $C_{31} > 1$ such that*

$$\sup_{\varepsilon\in(0,1)}\left\{\sup_{t\geq 0}\left(\|n^\varepsilon(t)\|_\infty + \|m^\varepsilon(t)\|_{1,\infty}\right.\right.$$
$$\left.\left. + \|(z^\varepsilon)^{-1}(t)\|_\infty + \|w^\varepsilon(t)\|_\infty\right)\right\} \leq C_{31},$$
$$\sup_{\varepsilon\in(0,1)} \|f_x^\varepsilon(t)\|_\infty \leq C_{31}(1+t) \quad \text{for all } t \geq 0$$

as well as

$$\sup_{\varepsilon\in(0,1)}\left\{\sup_{t\geq 0}\|f^\varepsilon(t)\|_\infty\right\} \leq \alpha, \qquad \sup_{\varepsilon\in(0,1)}\left\{\sup_{t\geq 0}\|z^\varepsilon(t)\|_\infty\right\} \leq 1.$$

Since the boundedness of n^ε in $L^\infty(0,T;H^1(\Omega))$ obtained in Lemma 3.6 depends upon ε, first of all we derive the uniform boundedness of $\{n^\varepsilon\}_{\varepsilon\in(0,1)}$ in $W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$, which plays an important role to the limit procedure of the approximate solutions $(n^\varepsilon, f^\varepsilon, m^\varepsilon)$ to $(\mathrm{P})_\varepsilon$. For each $f \in \mathbb{R}$ we consider a non-negative function $\bar\beta_\varepsilon(f;\cdot)$ given by

$$\bar\beta_\varepsilon(f;r) = z^2\hat\beta_\varepsilon(f;rz^{-1}) \quad \text{for all } r \in \mathbb{R}.$$

Then, we have the following lemmas.

Lemma 4.2. *The following inequality holds:*

$$\frac{\partial}{\partial t}\bar\beta_\varepsilon(f^\varepsilon;w^\varepsilon) \leq w_t^\varepsilon z^\varepsilon \beta_\varepsilon(f^\varepsilon;w^\varepsilon(z^\varepsilon)^{-1}) \quad \text{a.e. in } Q_\infty.$$

Proof. For each $f \in [0,\alpha]$ the function $\bar\beta_\varepsilon(f;\cdot)$ is expressed by

$$\bar\beta_\varepsilon(f;r) = \begin{cases} \dfrac{(r+zf-\alpha z)^2}{2\varepsilon} & \text{if } r > \alpha z - zf, \\ 0 & \text{if } 0 \leq r \leq \alpha z - zf, \\ \dfrac{r^2}{2\varepsilon} & \text{if } r < 0. \end{cases}$$

Then, we see from Lemma 4.1 that the following equality is satisfied:

$$\frac{\partial}{\partial t}\bar{\beta}_\varepsilon(f^\varepsilon; w^\varepsilon) = w_t^\varepsilon z^\varepsilon \beta_\varepsilon(f^\varepsilon; w^\varepsilon (z^\varepsilon)^{-1}) + \ell_\varepsilon(f^\varepsilon; w^\varepsilon) \quad \text{a.e. in } Q_\infty, \tag{4.1}$$

where $\ell_\varepsilon(f; w)$ is given by

$$\ell_\varepsilon(f; w) = \begin{cases} \dfrac{(w + zf - \alpha z)\{(zf)_t - \alpha z_t\}}{\varepsilon} & \text{if } w > \alpha z - zf, \\ 0 & \text{if } w \le \alpha z - zf. \end{cases}$$

By using (2.1), the second equation in (2.2) and Lemma 4.1, we have

$$(z^\varepsilon f^\varepsilon)_t - \alpha z_t^\varepsilon = \frac{a z^\varepsilon \lambda(f^\varepsilon) m^\varepsilon f^\varepsilon (f^\varepsilon - \alpha)}{d_2} - a z^\varepsilon m^\varepsilon f^\varepsilon \le 0$$

for the case $w^\varepsilon > \alpha z^\varepsilon - z^\varepsilon f^\varepsilon$, hence,

$$\ell_\varepsilon(f^\varepsilon; w^\varepsilon) \le 0 \quad \text{a.e. in } Q_\infty. \tag{4.2}$$

We see from (4.1) and (4.2) that this lemma holds. □

Lemma 4.3. *For each $T > 0$ there exists $C_{32}(T) > 0$ such that*

$$\sup_{\varepsilon \in (0,1)} \|n_t^\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \sup_{\varepsilon \in (0,1)} \left\{ \sup_{0 \le t \le T} \|n^\varepsilon(t)\|_{1,2} \right\} + \sup_{\varepsilon \in (0,1)} \left\{ \sup_{0 \le t \le T} \int_\Omega \hat{\beta}_\varepsilon(f^\varepsilon(t); n^\varepsilon(t)) \right\} \le C_{32}(T).$$

Proof. We multiply the first equation in (2.2) by $w_t^\varepsilon(t)$ and integrate its result on Ω. By using the estimate, which is obtained in the proof of (3.23), and Lemmas 4.1 and 4.2, we see that there exist $C_{33} > 0$ and $C_{34} > 0$ such that

$$\begin{aligned} &\frac{1}{4}\|w_t^\varepsilon(t)\|_2^2 + \frac{d}{dt}\left(\frac{d_1}{2}\|w_x^\varepsilon(t)\|_2^2 + \int_\Omega \bar{\beta}_\varepsilon(f^\varepsilon(t); w^\varepsilon(t))\right) \\ &\le C_{33}(1+t)^2\|w_x^\varepsilon(t)\|_2^2 + C_{34} \quad \text{for a.e. } t \in (0,T). \end{aligned} \tag{4.3}$$

By using the non-negativity of $\bar{\beta}_\varepsilon(f;\cdot)$ with $\bar{\beta}_\varepsilon(f_0;w_0)=0$, which comes from (A4) and (A5), and applying Gronwall's lemma to (4.3), we have

$$\begin{aligned}&\frac{1}{4}\int_0^t \|w_s^\varepsilon(s)\|_2^2\,ds+\frac{d_1}{2}\|w_x^\varepsilon(t)\|_2^2+\int_\Omega \bar{\beta}_\varepsilon(f^\varepsilon(t);w^\varepsilon(t))\\ &\le \left(\frac{d_1}{2}\|w_0\|_{1,2}^2+C_{34}T\right)\exp\left(\frac{2C_{33}(1+T)^3}{3d_1}\right)\quad \text{for all } t\in[0,T].\end{aligned}\tag{4.4}$$

From Lemma 4.1 we have

$$\int_\Omega \hat{\beta}_\varepsilon(f^\varepsilon(t);n^\varepsilon(t))\le C_{31}^2\int_\Omega \bar{\beta}_\varepsilon(f^\varepsilon(t);w^\varepsilon(t)).\tag{4.5}$$

By using the similar estimate, which is obtained at the end of the proof of Lemma 3.6, and the following equality:

$$n_t^\varepsilon=\frac{w_t^\varepsilon}{z^\varepsilon}-\frac{a\lambda(f^\varepsilon)w^\varepsilon m^\varepsilon f^\varepsilon}{d_1 z^\varepsilon},$$

we see from (4.4), (4.5) and Lemma 4.1 that this lemma holds. □

From the standard argument of parabolic PDEs and Lemma 4.3, we see that the following uniform estimate for $\{m^\varepsilon\}_{\varepsilon\in(0,1)}$ is satisfied. Since its proof is quite standard, we omit it in this paper.

Lemma 4.4. *For each $T>0$ there exists $C_{35}(T)>0$ such that*

$$\sup_{\varepsilon\in(0,1)}\|m_t^\varepsilon\|_{L^2(0,T;L^2(\Omega))}+\sup_{\varepsilon\in(0,1)}\|m^\varepsilon\|_{L^2(0,T;H^2(\Omega))}\le C_{35}(T).$$

By using the uniform boundedness of approximate non-negative global-in-time solutions to $(\mathrm{P})_\varepsilon$, we can construct a non-negative global-in-time solution to (P). At first we give two propositions. One gives the existence result of the limit (n,f,m) of a suitable sequence of approximate solutions $(n^\varepsilon,f^\varepsilon,m^\varepsilon)$, of which each component converges each corresponding component in a suitable function space. The other shows that the limit (n,f,m) satisfies the constraint conditions.

Proposition 4.1. *There exist a sequence $\{\varepsilon_k\}$ and a triplet (n,f,m) of non-negative functions on Q_∞ such that*

$$\varepsilon_k\searrow 0\quad as\quad k\to\infty\tag{4.6}$$

and for any $T > 0$ the following convergences are satisfied as $k \to \infty$:

$$n^{\varepsilon_k} \longrightarrow n \quad in \begin{cases} C([0,T];L^2(\Omega)), \\ weakly\ in\ W^{1,2}(0,T;L^2(\Omega)), \\ *\text{-}weakly\ in\ L^\infty(0,T;H^1(\Omega)), \end{cases} \tag{4.7}$$

$$f^{\varepsilon_k} \longrightarrow f \quad in\ W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;H^1(\Omega)), \tag{4.8}$$

$$m^{\varepsilon_k} \longrightarrow m \quad in \begin{cases} C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \\ weakly\ in\ W^{1,2}(0,T;L^2(\Omega)), \\ weakly\ in\ L^2(0,T;H^2(\Omega)), \\ *\text{-}weakly\ in\ L^\infty(0,T;W^{1,\infty}(\Omega)). \end{cases} \tag{4.9}$$

Moreover, (n,f,m) satisfies the same estimates that are obtained in Lemmas 4.1–4.4.

Proof. At first, by using Lemmas 4.1, 4.3 and 4.4 with $T = 1$, we can take out a sequence $\{\varepsilon_{1,k}\} \subset (0,1)$ and a pair (n^1, m^1) satisfying the convergences (4.6), (4.7) and (4.9) as $k \to \infty$, in which (ε_k, n, m) is replaced by $(\varepsilon_{1,k}, n^1, m^1)$. By using Lemma 4.1 and the second equation in $(\mathrm{P})_\varepsilon$ with (1.2), we see that there exists $C_{36} > 0$ such that the following estimates are satisfied for any $t \in (0,1)$:

$$\|F^{1,k}(t)\|_2 \le C_{36}\sqrt{t}\|M^{1,k}\|_{L^2(0,t;L^2(\Omega))}, \tag{4.10}$$

$$\|F_x^{1,k}(t)\|_2 \le C_{36}\left(\sqrt{t}\|M^{1,k}\|_{L^2(0,t;H^1(\Omega))} + t\|F^{1,k}\|_{L^\infty(0,t;L^2(\Omega))}\right), \tag{4.11}$$

$$\|F_t^{1,k}(t)\|_2 \le C_{36}\left(\|M^{1,k}(t)\|_2 + \|F^{1,k}(t)\|_2\right), \tag{4.12}$$

$$\|F_{tx}^{1,k}(t)\|_2 \le C_{36}\left\{(1+t)\|M^{1,k}(t)\|_{1,2} + \|F^{1,k}(t)\|_{1,2}\right\}, \tag{4.13}$$

where $(F^{1,k}, M^{1,k}) = (f^{\varepsilon_{1,k}} - f^1, m^{\varepsilon_{1,k}} - m^1)$ and $(f,m) = (f^1, m^1)$ in (1.2). Hence, we see from (4.9)–(4.13) that (4.8) is satisfied, in which (f^{ε_k}, f) is replaced by $(f^{\varepsilon_{1,k}}, f^1)$.

Next, we assume that for any $i \in \mathbb{N}$ there exist a sequence $\{\varepsilon_{i,k}\}$ and a triplet (n^i, f^i, m^i) such that all convergences in (4.6)– (4.9) are satisfied, in which $(T, \varepsilon_k, n, f, m)$ is replaced by $(i, \varepsilon_{i,k}, n^i, f^i, m^i)$. We consider the case

$i+1$. By applying Lemmas 4.1, 4.3 and 4.4 with $T = i+1$ and repeating the argument similar to the case $T = 1$, we can choose a subsequence $\{\varepsilon_{i+1,k}\} \subset \{\varepsilon_{i,k}\}$ and a triplet $(n^{i+1}, f^{i+1}, m^{i+1})$ satisfying all convergences in (4.6)–(4.9), in which $(T, \varepsilon_k, n, f, m)$ is replaced by $(i+1, \varepsilon_{i+1,k}, n^{i+1}, f^{i+1}, m^{i+1})$.

Now, we define a sequence $\{\varepsilon_k\} = \{\varepsilon_{k,k}\}$ and a triplet (n, f, m) by $(*)$:

$$(*)\left(\begin{array}{l} \text{for each } T > 0 \text{ there exists } i_T \in \mathbb{N} \text{ such that } i_T - 1 < T \leq i_T \text{ and} \\ (n(t), f(t), m(t)) = (n^{i_T}(t), f^{i_T}(t), m^{i_T}(t)) \quad \text{for all } t \in [0, T]. \end{array}\right.$$

Then, $\{\varepsilon_k\}$ and (n, f, m) are desired ones in this lemma because it is clear from the constitution method of (n, f, m) that

$$(n^{i_1}(t), f^{i_1}(t), m^{i_1}(t)) = (n^{i_2}(t), f^{i_2}(t), m^{i_2}(t)) \quad \text{for any } t \in [0, i_1]$$

whenever $i_1 \leq i_2$ for any i_1, $i_2 \in \mathbb{N}$. □

Remark 4.1. We see from the uniform estimates of approximate solutions $(n^\varepsilon, f^\varepsilon, m^\varepsilon)$ to $(\mathrm{P})_\varepsilon$ obtained in Lemmas 4.1, 4.3 and 4.4 that for each $T > 0$ there exist a sequence $\{\varepsilon_{T,k}\} \subset (0, 1)$ and a triplet (n^T, f^T, m^T), which depend upon T, such that all convergences in Proposition 4.1 are satisfied. It is not clear that $(n^{T_1}(t), f^{T_1}(t), m^{T_1}(t)) = (n^{T_2}(t), f^{T_2}(t), m^{T_2}(t))$ for all $t \in [0, \min\{T_1, T_2\}]$ because it is difficult to show the uniqueness of solutions to (P) on $[0, \min\{T_1, T_2\}]$ in general by the quasi-variational structure of (P), which is made clear in Theorem 4.1 below.

Proposition 4.2. *Let (n, f, m) be the same triplet that is obtained in Proposition 4.1. Then, the following constraint conditions are satisfied:*

$$n \geq 0, \quad f \geq 0, \quad n + f \leq \alpha \quad \textit{on } \bar{Q}_\infty = \bar{\Omega} \times [0, \infty).$$

Proof. We fix any $T > 0$ and consider the same sequence $\{\varepsilon_k\}$ that is obtained in Proposition 4.1. From the convergences in Proposition 4.1 with Gagliardo–Nirenberg's inequality; there exists $C_{37} > 0$ such that

$$\|\varphi\|_{C(\bar{\Omega})} \leq C_{37} \|\varphi\|_{1,2}^{\frac{1}{2}} \|\varphi\|_2^{\frac{1}{2}} \quad \text{for any } \varphi \in H^1(\Omega),$$

without loss of generality we may assume that the following convergence is satisfied as $k \to \infty$:

$$(4.14) \qquad (n^{\varepsilon_k}, f^{\varepsilon_k}, m^{\varepsilon_k}) \longrightarrow (n, f, m) \quad \text{in } C(\bar{Q}_T).$$

Since we see from Lemma 4.1 and (4.14) that $n \geq 0$ and $f \geq 0$ on $\bar{Q}_T$, it is enough to show $n + f \leq \alpha$ on $\bar{Q}_T$. In order to do this, for each $f \in [0, \alpha]$ we consider the indicator function $\hat{\beta}(f; \cdot)$ on $[0, \alpha - f]$.

For the case that $(x, t) \in \bar{Q}_T$ satisfies $n(x,t) + f(x,t) \leq \alpha$, we have

$$(4.15) \qquad \hat{\beta}(f(x,t); n(x,t)) = 0 \leq \hat{\beta}_{\varepsilon_k}(f^{\varepsilon_k}(x,t); n^{\varepsilon_k}(x,t)) \quad \text{for all } k \in \mathbb{N}.$$

For the case that $(x, t) \in \bar{Q}_T$ satisfies $n(x,t) + f(x,t) > \alpha$, we take out and fix $r(x,t) > 0$ satisfying $n(x,t) + f(x,t) \geq \alpha + r(x,t)$. Then, we see from (4.14) that there exists $k(x,t) \in \mathbb{N}$ such that

$$n^{\varepsilon_k}(x,t) + f^{\varepsilon_k}(x,t) \geq \alpha + \frac{r(x,t)}{2} \quad \text{for all } k \geq k(x,t),$$

hence, from the definition of $\hat{\beta}_\varepsilon$

$$\hat{\beta}_{\varepsilon_k}(f^{\varepsilon_k}(x,t); n^{\varepsilon_k}(x,t)) \geq \frac{(r(x,t))^2}{8\varepsilon_k} \quad \text{for all } k \geq k(x,t),$$

which implies

$$(4.16) \qquad \lim_{k\to\infty} \hat{\beta}_{\varepsilon_k}(f^{\varepsilon_k}(x,t); n^{\varepsilon_k}(x,t)) = \infty = \hat{\beta}(f(x,t); n(x,t)).$$

We see from (4.15) and (4.16) that the following inequality is satisfied:

$$(4.17) \qquad \hat{\beta}(f; n) \leq \liminf_{k\to\infty} \hat{\beta}_{\varepsilon_k}(f^{\varepsilon_k}; n^{\varepsilon_k}) \quad \text{on } \bar{Q}_T.$$

By applying Fatou's lemma with (4.17) and using Lemma 4.3, we have

$$\iint_{Q_T} \hat{\beta}(f; n) \leq \liminf_{k\to\infty} \iint_{Q_T} \hat{\beta}_{\varepsilon_k}(f^{\varepsilon_k}; n^{\varepsilon_k}) \leq T C_{32}.$$

Since $\hat{\beta}(f; \cdot)$ is the indicator function on $[0, \alpha - f]$, we see that the following equality must hold:

$$\iint_{Q_T} \hat{\beta}(f; n) = 0,$$

which implies that $0 \le n \le \alpha - f$ a.e. in Q_T. Since $T > 0$ is arbitrary, we see from the compact imbedding $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$ that this lemma holds. □

Next, we give the approximation of a test function $\eta \in L^2(0,T;H^1(\Omega))$ satisfying the constraint condition $0 \le \eta \le \alpha - f$ a.e. in Q_T.

Lemma 4.5. *Let $\{\varepsilon_k\}$ and (n,f,m) be the same sequence and triplet that are obtained in Proposition 4.1. For any $T > 0$ and $\eta \in L^2(0,T;H^1(\Omega))$ with $0 \le \eta \le \alpha - f$ a.e. in Q_T there exists a sequence $\{\eta^k\} \subset L^2(0,T;H^1(\Omega))$ such that the following properties are satisfied:*

$$0 \le \eta^k \le \alpha - f^{\varepsilon_k} \quad a.e.\ in \quad Q_T \quad for\ all\ k \in \mathbb{N}, \tag{4.18}$$

$$\eta^k \longrightarrow \eta \quad in\ L^2(0,T;H^1(\Omega)) \quad as \quad k \to \infty. \tag{4.19}$$

Proof. For each $k \in \mathbb{N}$ we define $\eta^k = \max\{0,\ \eta + f - f^{\varepsilon_k}\}$. Then, it is clear that $\{\eta^k\} \subset L^2(0,T;H^1(\Omega))$ and satisfies (4.18). We put $Q_T^k = \{(x,t) \in Q_T \,|\, \eta(x,t) < f^{\varepsilon_k}(x,t) - f(x,t)\}$. Then, we have

$$\eta^k(x,t) - \eta(x,t) = \begin{cases} -\eta(x,t) & \text{if } (x,t) \in Q_T^k, \\ f(x,t) - f^{\varepsilon_k}(x,t) & \text{if } (x,t) \in Q_T \setminus Q_T^k. \end{cases}$$

Since it is clear that

$$0 \ge \eta^k(x,t) - \eta(x,t) = -\eta(x,t) > -f^{\varepsilon_k}(x,t) + f(x,t)$$

whenever $(x,t) \in Q_T^k$, we have

$$|\eta^k(x,t) - \eta(x,t)| \le |f^{\varepsilon_k}(x,t) - f(x,t)| \quad \text{for all } (x,t) \in Q_T. \tag{4.20}$$

From (4.8) and (4.20), we have

$$\eta^k \longrightarrow \eta \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \quad \text{as} \quad k \to \infty. \tag{4.21}$$

Moreover, we see from (4.8) that the following convergence holds:

$$f^{\varepsilon_k} \longrightarrow f \quad \text{in} \quad C([0,T];C(\bar{\Omega})) \quad \text{as} \quad k \to \infty. \tag{4.22}$$

From $\eta \geq 0$ a.e. in Q_T with (4.22), we have $|Q_T \setminus Q_T^k| \longrightarrow |Q_T|$, hence,

$$|Q_T^k| \longrightarrow 0 \quad \text{as} \quad k \to \infty, \tag{4.23}$$

besides

$$\begin{aligned} \iint_{Q_T} |\eta_x^k - \eta_x|^2 &= \iint_{Q_T^k} |\eta_x|^2 + \iint_{Q_T \setminus Q_T^k} |f_x^{\varepsilon_k} - f_x|^2 \\ &\leq \iint_{Q_T^k} |\eta_x|^2 + \|f^{\varepsilon_k} - f\|_{L^2(0,T;H^1(\Omega))}^2. \end{aligned} \tag{4.24}$$

From (4.8), (4.23) and (4.24), we have

$$\eta_x^k \longrightarrow \eta_x \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \quad \text{as} \quad k \to \infty. \tag{4.25}$$

Hence, we see from (4.21) and (4.25) that $\{\eta^k\}$ is a desired approximate sequence. □

Now, we are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. From Propositions 4.1 and 4.2 it is enough to show that the limit (n, f, m), which is obtained in Proposition 4.1, satisfies (1.1). In order to do this, let $\{\varepsilon_k\}$ be the same sequence that is obtained Proposition 4.1. We take any test function $\eta \in L^2(0,T;H^1(\Omega))$ satisfying the constraint conditions $0 \leq \eta \leq \alpha - f$ a.e. in Q_T and consider the sequence $\{\eta^k\}$ which is constructed in Lemma 4.5. We multiply the first equation in $(\mathrm{P})_{\varepsilon_k}$ by $n^{\varepsilon_k} - \eta^k$ and integrate its result on Q_T. Then, we have

$$\begin{aligned} &\iint_{Q_T} n_t^{\varepsilon_k}(n^{\varepsilon_k} - \eta^k) + d_1 \iint_{Q_T} n_x^{\varepsilon_k}(n_x^{\varepsilon_k} - \eta_x^k) \\ &- \iint_{Q_T} \lambda(f^{\varepsilon_k}) n^{\varepsilon_k} f_x^{\varepsilon_k}(n_x^{\varepsilon_k} - \eta_x^k) + \iint_{Q_T} \beta_{\varepsilon_k}(f^{\varepsilon_k}; n^{\varepsilon_k})(n^{\varepsilon_k} - \eta^k) \\ &= \iint_{Q_T} \mu_p n^{\varepsilon_k}(1 - n^{\varepsilon_k} - f^{\varepsilon_k})(n^{\varepsilon_k} - \eta^k) + \iint_{Q_T} \mu_d n^{\varepsilon_k}(n^{\varepsilon_k} - \eta^k). \end{aligned} \tag{4.26}$$

By using the convexity of $\hat{\beta}_{\varepsilon_k}(f^{\varepsilon_k}; \cdot)$ and (4.18), we have

$$\iint_{Q_T} \beta_{\varepsilon_k}(f^{\varepsilon_k}; n^{\varepsilon_k})(n^{\varepsilon_k} - \eta^k) \geq \iint_{Q_T} \hat{\beta}_{\varepsilon_k}(f^{\varepsilon_k}; n^{\varepsilon_k}). \tag{4.27}$$

By substituting (4.27) into (4.26), we have

$$
\begin{aligned}
&\iint_{Q_T} n_t^{\varepsilon_k}(n^{\varepsilon_k}-\eta^k)+d_1\iint_{Q_T}|n_x^{\varepsilon_k}|^2-d_1\iint_{Q_T} n_x^{\varepsilon_k}\eta_x^k\\
&-\iint_{Q_T}\lambda(f^{\varepsilon_k})n^{\varepsilon_k}f_x^{\varepsilon_k}(n_x^{\varepsilon_k}-\eta_x^k)+\iint_{Q_T}\hat{\beta}_{\varepsilon_k}(f^{\varepsilon_k};n^{\varepsilon_k})\\
\le &\iint_{Q_T}\mu_p n^{\varepsilon_k}(1-n^{\varepsilon_k}-f^{\varepsilon_k})(n^{\varepsilon_k}-\eta^k)+\iint_{Q_T}\mu_d n^{\varepsilon_k}(n^{\varepsilon_k}-\eta^k).
\end{aligned}
\tag{4.28}
$$

Since we see from Lemma 4.1 and Proposition 4.2 that there exists $C_{38}>0$ such that

$$
\begin{cases}
\|\lambda(f^{\varepsilon_k})n^{\varepsilon_k}f_x^{\varepsilon_k}-\lambda(f)nf_x\|_{L^2(0,T;L^2(\Omega))}\\
\le C_{38}(1+T)\left(\|n^{\varepsilon_k}-n\|_{L^2(0,T;L^2(\Omega))}+\|f^{\varepsilon_k}-f\|_{L^2(0,T;H^1(\Omega))}\right),\\
\|\mu_p n^{\varepsilon_k}(1-n^{\varepsilon_k}-f^{\varepsilon_k})-\mu_p n(1-n-f)\|_{L^2(0,T;L^2(\Omega))}\\
\le C_{38}\left(\|n^{\varepsilon_k}-n\|_{L^2(0,T;L^2(\Omega))}+\|f^{\varepsilon_k}-f\|_{L^2(0,T;L^2(\Omega))}\right),
\end{cases}
$$

we see from Proposition 4.1 that the following convergences hold as $k\to\infty$:

$$
\begin{cases}
\lambda(f^{\varepsilon_k})n^{\varepsilon_k}f_x^{\varepsilon_k}\longrightarrow\lambda(f)nf_x \quad \text{in } L^2(0,T;L^2(\Omega)),\\
\mu_p n^{\varepsilon_k}(1-n^{\varepsilon_k}-f^{\varepsilon_k})\longrightarrow\mu_p n(1-n-f) \quad \text{in } L^2(0,T;L^2(\Omega)).
\end{cases}
\tag{4.29}
$$

By taking $\liminf_{k\to\infty}$ in (4.28) and using Propositions 4.1 and 4.2 as well as (4.29), we see that (1.1) holds. □

In the rest of this section, we make the quasi-variational structure of (P) clear. For each $T>0$ and $v\in L^\infty(0,T;H^1(\Omega))$ satisfying the constraint conditions $0\le v\le\alpha$ a.e. in Q_T we prepare a functional $\Psi_v^T(\cdot)$ on $L^2(0,T;L^2(\Omega))$ by the following way. We denote by $\Lambda_2 v=m^v$ a unique solution to the system

$$
\begin{cases}
m_t^v=d_2m_{xx}^v+bv-cm^v & \text{in } Q_T,\\
m_x(\pm L,t)=0 & \text{for any } t>0,\\
m^v(0)=m_0 & \text{in } \Omega,
\end{cases}
$$

and $\Lambda_1 v$ by (1.2), in which $(f,m)=(\Lambda_1 v,\Lambda_2 v)$. Then, we note that the following lemma holds. Since its proof is quite standard and the similar results have already been obtained in Lemmas 3.1–3.3, we omit it here.

LEMMA 4.6. *The following boundedness are satisfied:*

$$0 \le \Lambda_1 v \le \alpha, \quad 0 \le \Lambda_2 v \le \|m_0\|_\infty + \frac{b\hat{C}_1 C_{18}}{c} \quad in\ \bar{Q}_T,$$

where $\hat{C}_1$ *and* C_{18} *are the same constants that are obtained in Lemmas 3.2 and 3.3, respectively.*

Moreover, there exists $C_{39}(T) > 0$ *such that*

$$\|\Lambda_1 v\|_{L^\infty(0,T;H^1(\Omega))} + \|\Lambda_2 v\|_{L^2(0,T;H^1(\Omega))} \le C_{39}(T).$$

Now, we define a functional $\Psi_v^T(\cdot)$ on $L^2(0,T;L^2(\Omega))$ by

$$\Psi_v^T(\eta) = \begin{cases} \dfrac{d_1}{2} \displaystyle\iint_{Q_T} |\eta_x|^2 - \iint_{Q_T} \lambda(\Lambda_1 v) v (\Lambda_1 v)_x \eta_x & \text{if } \eta \in D(\Psi_v^T) \\ \infty \quad \text{if } \eta \in L^2(0,T;L^2(\Omega)) \setminus D(\Psi_v^T), \end{cases}$$

where $D(\Psi_v^T) = \{\eta \in L^2(0,T;H^1(\Omega)) \,|\, 0 \le \eta \le \alpha - \Lambda_1 v \text{ a.e. in } Q_T\}$. Then, we have the following proposition, which gives the characterization of the subdifferential of the functional $\Psi_v^T(\cdot)$ on $L^2(0,T;L^2(\Omega))$.

PROPOSITION 4.3. *For each* $T > 0$ *and* $v \in L^\infty(0,T;H^1(\Omega))$ *satisfying the constraint conditions* $0 \le v \le \alpha$ *a.e. in* Q_T*, the functional* $\Psi_v^T(\cdot)$ *is proper, l.s.c. and convex on* $L^2(0,T;L^2(\Omega))$*.*

Moreover, $\xi^* \in \partial\Psi_v^T(\xi)$ *if and only if* $\xi \in L^2(0,T;H^1(\Omega))$ *and the following properties are satisfied:*

(1) $0 \le \xi \le \alpha - \Lambda_1 v$ *a.e. in* Q_T*,*

(2) The following inequality is satisfied:

$$\iint_{Q_T} \xi^*(\xi - \eta) \le d_1 \iint_{Q_T} \xi_x(\xi_x - \eta_x) - \iint_{Q_T} \lambda(\Lambda_1 v) v (\Lambda_1 v)_x (\xi_x - \eta_x)$$
$$\text{for any } \eta \in L^2(0,T;H^1(\Omega)) \text{ with } 0 \le \eta \le \alpha - \Lambda_1 v \text{ a.e. in } Q_T,$$

where $\partial\Psi_v^T(\cdot)$ *is the subdifferential of* $\Psi_v^T(\cdot)$ *on* $L^2(0,T;L^2(\Omega))$*.*

PROOF. It is clear that $\Psi_v^T(\cdot)$ is proper and convex on $L^2(0,T;L^2(\Omega))$. So, we show the lower semi-continuity of $\Psi_v^T(\cdot)$ on $L^2(0,T;L^2(\Omega))$. For any $r \in \mathbb{R}$, we put the level set $D_r = \{\eta \in L^2(0,T;L^2(\Omega)) \mid \Psi_v^T(\eta) \leq r\}$ and consider any sequence $\{\eta^k\} \subset D_r$ and any element $\eta \in L^2(0,T;L^2(\Omega))$ satisfying

$$\eta^k \longrightarrow \eta \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \quad \text{as} \quad k \to \infty. \tag{4.30}$$

Then, we see from Lemma 4.6 that the following estimate is satisfied:

$$\iint_{Q_T} |\eta_x^k|^2 \leq \frac{4r}{d_1} + \frac{4\alpha^2(L_\lambda \alpha + \lambda(0))^2 T C_{39}(T)^2}{d_1^2},$$

which implies that D_r is bounded in $L^2(0,T;H^1(\Omega))$. Hence, without loss of generality, we may assume that the following convergence is satisfied:

$$\eta^k \longrightarrow \eta \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \quad \text{as} \quad k \to \infty. \tag{4.31}$$

From (4.30) and (4.31), we have

$$\Psi_v^T(\eta) \leq \liminf_{k\to\infty} \Psi_v^T(\eta^k) \leq r,$$

so, $\eta \in D_r$. Thus, D_r is closed in $L^2(0,T;L^2(\Omega))$. Hence, we see that $\Psi_v^T(\cdot)$ is l.s.c. on $L^2(0,T;L^2(\Omega))$.

Next, we assume $\xi^* \in \partial\Psi_v^T(\xi)$. Then, we see from the definition of the subdifferential $\partial\Psi_v^T(\cdot)$ that $\xi \in D(\Psi_v^T)$, which implies that (1) holds, and

$$\iint_{Q_T} \xi^*(\eta - \xi) \leq \Psi_v^T(\eta) - \Psi_v^T(\xi) \quad \text{for any } \eta \in D(\Psi_v^T). \tag{4.32}$$

For any $\varepsilon \in (0,1)$ we substitute $\xi + \varepsilon(\eta - \xi)$ as η in (4.32) to have

$$\begin{aligned} \iint_{Q_T} \xi^*(\eta - \xi) &\leq d_1 \iint_{Q_T} \xi_x^*(\eta_x - \xi_x) + \frac{\varepsilon d_1}{2} \iint_{Q_T} |\eta_x - \xi_x|^2 \\ &- \iint_{Q_T} \lambda(\Lambda_v) v (\Lambda_1 v)_x (\eta_x - \xi_x) \qquad \text{for any } \eta \in D(\Psi_v^T). \end{aligned} \tag{4.33}$$

By taking the limit $\varepsilon \to 0$ in (4.33), we see that (2) holds.

Conversely, we assume that (1) and (2) are satisfied. Then, we see that (4.32) holds. Hence, $\xi^* \in \partial\Psi_v^T(\xi)$. □

By using Proposition 4.3, we make the quasi-variational structure of (P) clear in Theorem 4.1.

THEOREM 4.1. *Let (n, f, m) be a non-negative global-in-time solution to (P). Then, for each $T > 0$ the following relations are satisfied: $f = \Lambda_1 n$, $m = \Lambda_2 n$ and*

$$-n_t + \mu_p n(1 - n - \Lambda_1 n) - \mu_d n \in \partial\Psi_n^T(n).$$

PROOF. We see from (1) and (4) in Theorem 1.1 and Proposition 4.3 that this theorem holds. □

REMARK 4.2. Theorem 4.1 says that the kinetics of the distribution of tumor cells, denoted by n, is governed by the subdifferential of $\Psi_n^T(\cdot)$, which is regarded as the subgradient flow in $L^2(0, T; L^2(\Omega))$. It is quite important that the functional $\Psi_n^T(\cdot)$ depends upon n itself. Actually, $\Psi_n^T(\cdot)$ contains not only n but also $f = \Lambda_1 n$ in its definition, which is a non-local term generated by $m = \Lambda_2 n$ (cf. (1.2)) and plays a role like a memory effect of m, so, n itself. This quasi-variational structure is a characteristic in tumor invasion model of Chaplain–Anderson type with constraint conditions.

5. Asymptotic Behavior of a Global-in-time Solution to (P)

In this section, we consider the asymptotic behavior of the global-in-time solution as time goes to ∞. Throughout this section, we assume that (A1), (A2), (A3)', (A4)–(A9) are fulfilled and use the same notations in the previous sections. In the following argument, we let (n, f, m) the global-in-time solution to (P) constructed in Section 4. We begin with showing the positivity of n on $\Omega \times [0, \infty)$ by using the argument similar to that of [14, Lemma 6.1].

LEMMA 5.1. *There exists $C_{40} > 0$ such that $n \geq C_{40}$ on $\bar{Q}_\infty$.*

PROOF. Let $\{\varepsilon_k\}$ and $\{(n^{\varepsilon_k}, f^{\varepsilon_k}, m^{\varepsilon_k})\}$ be the same sequences that are obtained in Proposition 4.1. Then, we note that $(w^{\varepsilon_k}, f^{\varepsilon_k}, m^{\varepsilon_k})$ is a solution to (2.2). We fix any $T > 0$ and take

$$C_{40} = n_* \exp\left(-\frac{1}{d_1}\int_0^\alpha \lambda(r)dr\right).$$

We put $(w^{\varepsilon_k} - C_{40})_- = \max\{C_{40} - w^{\varepsilon_k}, 0\}$ and multiply the first equation in (2.2) by $(z^{\varepsilon_k})^{-1}(w^{\varepsilon_k} - C_{40})_-$. By integrating its result on Ω and using the non-negativities of $(w^{\varepsilon_k}, f^{\varepsilon_k}, m^{\varepsilon_k})$, we see that the following inequality is satisfied a.e. in $(0,T)$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (z^{\varepsilon_k})^{-1}|(w^{\varepsilon_k} - C_{40})_-|^2 \\
(5.1)\qquad \leq\ & \int_\Omega \beta_{\varepsilon_k}(f^{\varepsilon_k}; (w^{\varepsilon_k}(z^{\varepsilon_k})^{-1}))(w^{\varepsilon_k} - C_{40})_- \\
&- \int_\Omega \mu_p w^{\varepsilon_k}(z^{\varepsilon_k})^{-1}\left\{1 - (w^{\varepsilon_k}(z^{\varepsilon_k})^{-1}) - f^{\varepsilon_k}\right\}(w^{\varepsilon_k} - C_{40})_-.
\end{aligned}$$

Since for any $x \in \Omega$ the function $f^{\varepsilon_k}(x, \cdot)$ is decreasing on $[0,T]$, we see from (A8) and Lemma 3.1 that

$$(5.2)\qquad C_{40}(z^{\varepsilon_k})^{-1} \leq n_* \leq 1 - f^{\varepsilon_k} \quad \text{on } Q_T,$$

hence

$$(5.3)\qquad 1 - w^{\varepsilon_k}(z^{\varepsilon_k})^{-1} - f^{\varepsilon_k} > 0 \quad \text{on } \bigcup_{0\leq t\leq T} \Omega_k(t) \times \{t\},$$

where $\Omega_k(t) = \{x \in \Omega \,|\, w^{\varepsilon_k}(x,t) < C_{40}\}$ for all $t \in [0,T]$. Besides, since for any $(x,t) \in Q_T$ the function $\beta_{\varepsilon_k}(f^{\varepsilon_k}(x,t); \cdot)$ is increasing on $\mathbb{R}$, we see from (5.2) that the following inequality is satisfied:

$$(5.4)\qquad \beta_{\varepsilon_k}(f^{\varepsilon_k}; w^{\varepsilon_k}(z^{\varepsilon_k})^{-1})(w^{\varepsilon_k} - C_{40}) \geq 0 \quad \text{on } \bigcup_{0\leq t\leq T} \Omega_k(t) \times \{t\}.$$

We see from (5.1), (5.3) and (5.4) that the following inequality is satisfied:

$$\frac{d}{dt}\int_\Omega (z^{\varepsilon_k})^{-1}(t)|(w^{\varepsilon_k}(t) - C_{40})_-|^2 \leq 0 \quad \text{for a.e. } t \in (0,T).$$

Since $w_0 \geq C_{40}$ on $\bar{\Omega}$, we have $w^{\varepsilon_k}(t) \geq C_{40}$, hence, $n^{\varepsilon_k}(t) \geq C_{40}$ on $\bar{\Omega}$ for all $t \in [0,T]$. By taking the limit $k \to \infty$ and using (4.14) (cf. (4.7)), we have $n(t) \geq C_{40}$ on $\bar{\Omega}$ for all $t \in [0,T]$. Since $T > 0$ is arbitrary, we see that this lemma holds. □

Moreover, we can show the boundedness of m in the following lemma.

Lemma 5.2. *m satisfies the following estimates:*
(1) $m_* \le m \le \max\left\{\|m_0\|_\infty, \dfrac{b\alpha}{c}\right\}$ *on* $\bar{Q}_\infty$.
(2) There exists $C_{41} > 0$ *such that*

$$\iint_{Q_t} |m_x|^2 \le C_{41}(1+t) \quad \text{for all } t \ge 0.$$

Proof. By using (A9) and the comparison theorem for parabolic PDEs to the third equation in (P), it is clear that (1) holds. In order to show (2), we substitute $\zeta = m(t)$ into (1.3). Then, we see from Proposition 4.2 that

$$\frac{1}{2}\frac{d}{dt}\|m(t)\|_2^2 + d_2\|m_x(t)\|_2^2 \le \frac{b^2\alpha^2 L}{2c} \quad \text{for a.e. } t > 0,$$

which implies that (2) holds. □

By using (1.2) and Lemma 5.2, we see that the next lemma holds, which gives the decay estimate of f and the boundedness of f_x.

Lemma 5.3. *f satisfies the following estimates:*
(1) $\|f(t)\|_{C(\bar{\Omega})} \le e^{-am_* t}$ *for all* $t \ge 0$.
(2) There exists $C_{42} > 0$ *such that*

$$\|f_x(t)\|_2 \le C_{42}(1+t)e^{-am_* t} \quad \text{for all } t \ge 0.$$

Proof. We see from (1.2), Lemma 5.2 that (1) holds. Moreover, by using the following equality:

$$f_x(x,t) = (f_0)_x(x)\exp\left(-a\int_0^t m(x,s)ds\right) - af(x,t)\int_0^t m_x(x,s)ds,$$

which has already used in the previous sections, for example, the uniqueness part of Theorem 1.1, Lemma 3.4 and Proposition 4.1, we have

$$\|f_x(t)\|_2^2 \le 2e^{-2am_* t}\left(\|f_0\|_{1,2}^2 + a^2 t\iint_{Q_t} |m_x|^2\right) \quad \text{for all } t \ge 0. \tag{5.5}$$

Hence, we see from (5.5) and Lemma 5.2 that (b) holds. □

Finally, by using Lemma 5.3, we can show the asymptotic convergences of (n, m) as time goes to ∞ in the next lemmas.

LEMMA 5.4. *There exists $C_{43} > 0$ such that*

$$\int_0^\infty \|n(t) - 1\|_2^2 \, dt \le C_{43}.$$

PROOF. For any $T > 0$, we substitute $\eta = 1 - f$ into (1.1) to derive

$$\begin{aligned} &\iint_{Q_T} n_t(n + f - 1) + d_1 \iint_{Q_T} n_x(n_x + f_x) \\ &\le \iint_{Q_T} \lambda(f) n f_x(n_x + f_x) + \iint_{Q_T} \mu_p n(1 - n - f)(n + f - 1). \end{aligned} \tag{5.6}$$

By using the second equation in (P), and Lemmas 5.2 and 5.3, we see that there exist $C_{44} > 0$ and $C_{45} > 0$ such that

$$\begin{aligned} &\iint_{Q_T} n_t(n + f - 1) \\ &\ge \frac{1}{2}\|n(T) + f(T) - 1\|_2^2 - \frac{\mu_1 n_*}{2} \int_0^T \|n(t) + f(t) - 1\|_2^2 \, dt \\ &\qquad - \frac{a^2 L}{\mu_1 m_*} \left(\max\left\{ \|m_0\|_\infty, \frac{b\alpha}{c} \right\} \right)^2 \int_0^T \|f(t)\|_\infty^2 dt - L \\ &\ge \frac{1}{2}\|n(T) + f(T) - 1\|_2^2 - \frac{\mu_1 n_*}{2} \int_0^T \|n(t) + f(t) - 1\|_2^2 \, dt - C_{44}, \end{aligned} \tag{5.7}$$

$$\iint_{Q_T} \lambda(f) n f_x(n_x + f_x) \le \frac{d_1}{2} \int_0^T \|n_x(t)\|_2^2 \, dt + C_{45} \int_0^T \|f_x(t)\|_2^2 \, dt, \tag{5.8}$$

$$d_1 \iint_{Q_T} n_x(n_x + f_x) \ge \frac{d_1}{2} \int_0^T \|n_x(t)\|_2^2 dt - \frac{d_1}{2} \int_0^T \|f_x(t)\|_2^2 dt, \tag{5.9}$$

$$\iint_{Q_T} \mu_p n(1 - n - f)(n + f - 1) \le -\mu_1 n_* \int_0^T \|n(t) + f(t) - 1\|_2^2 \, dt. \tag{5.10}$$

By substituting (5.7)–(5.10) into (5.6) and using Lemma 5.3, we see that the following inequality is satisfied for any $T \in (0, \infty)$:

$$\|n(T) + f(T) - 1\|_2^2 + \mu_1 n_* \int_0^T \|n(t) + f(t) - 1\|_2^2 \, dt \leq C_{46}, \tag{5.11}$$

where

$$C_{46} = 2C_{44} + C_{42}^2(d_1 + 2C_{45}) \int_0^\infty (1+t)^2 e^{-2am_* t} dt.$$

By applying Gronwall's lemma, we have

$$\int_0^T \|n(t) + f(t) - 1\|_2^2 \, dt \leq \frac{C_{46}}{\mu_1 m_*} \quad \text{for all } T \geq 0.$$

Since $T > 0$ is arbitrary, we see from (1) of Lemma 5.3 that this lemma holds. □

LEMMA 5.5. *There exists $C_{47} > 0$ such that*

$$\int_0^\infty \left\| m(t) - \frac{b}{c} \right\|_2^2 dt \leq C_{47}.$$

PROOF. For any $T > 0$ we substitute $\zeta = m(t) - b/c$ into (1.3) to derive

$$\frac{1}{2}\frac{d}{dt}\left\| m(t) - \frac{b}{c} \right\|_2^2 + c \int_\Omega m(t) \left(m(t) - \frac{b}{c} \right) \leq b \int_\Omega n(t) \left(m(t) - \frac{b}{c} \right)$$

for a.e. $t \in (0, T)$, hence, from Lemma 5.4

$$\left\| m(t) - \frac{b}{c} \right\|_2^2 + c \int_0^t \left\| m(s) - \frac{b}{c} \right\|_2^2 ds \leq C_{48} \quad \text{for all } t \in [0, T], \tag{5.12}$$

where

$$C_{48} = \left\| m_0 - \frac{b}{c} \right\|_2^2 + \frac{b^2}{c} \int_0^\infty \|n(t) - 1\|_2^2 \, dt.$$

By applying Gronwall's lemma to (5.12), we have

$$\int_0^T \left\| m(t) - \frac{b}{c} \right\|_2^2 dt \leq \frac{C_{48}}{c}.$$

Since $T > 0$ is arbitrary, we see that this lemma holds. □

Theorem 1.2 is a direct consequence of Lemmas 5.3–5.5.

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