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Two Semi-Continuity Results for the Algebraic Dimension of Compact Complex Manifolds

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En hommage à la mémoire du professeur Kunihiko Kodaira

Abstract. Using some relative codimension 1 cycle-space method, we give, following the ideas of D. Popovici [P.13], semicontinuity results for the algebraic dimension in a family a compact complex manifolds parametrized by a reduced complex space.

Contents

1. Statement of Results

This Note is inspired by Dan Popovici article [P.13]. We show here that using relative codimension 1 cycle-space, we may give another proof of his theorem 1.2 and 1.4 and also obtain from this alternative proof much more general results. In fact, very few is new in this approach, because we simply use the ideas in [P.13] and combine them with the classical tools

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introduced in [B.75], [B.78] and [C.80]; see also [B-M.14]. This gives a more geometric view on the use of the existence of a strongly Gauduchon form (see definition below) and of the existence of a relative Gauduchon metric on a holomorphic family of compact complex manifolds using a geometric approach to (relative) algebraic reduction in term of codimension 1 (relative) cycles.

We shall prove the following generalizations of Theorem 1.2 and 1.4 of [P.13] using the definition :

Definition 1.0.1. A proper surjective holomorphic map $\pi : \mathcal{X} \to S$ between two irreducible complex spaces will be a **holomorphic family of compact complex connected manifolds of dimension n parametrized by S** when each fiber of π is a compact complex connected manifold of dimension n and when this family is locally $\mathscr{C}^\infty-$trivial on S.

Note that the local $\mathscr{C}^\infty-$triviality is automatic for S smooth assuming that each fiber is reduced (or that π is a submersion).

So we may consider that we have a fix compact connected $\mathscr{C}^\infty-$manifold X and a holomorphic family of complex structures on X parametrized by S.

Theorem 1.0.2. *Let $\pi : \mathcal{X} \to S$ be a holomorphic family of compact complex connected manifolds of dimension n parametrized by an irreducible complex space S. Let s_0 in S such that the manifold X_{s_0} admits a (smooth) sG-form. Then there exists an open neighbourhood S_0 of s_0, a countable union Σ of closed irreducible analytic subsets in S_0 with no interior point and a non negative integer a such that*

(i) *For any $s \in S_0$ we have $a(X_s) \geq a$.*

(ii) *For any $s \in S_0 \setminus \Sigma$ we have $a(X_s) = a$.*

Note that the proof contains the construction of a smooth relative sG-form on $\pi^{-1}(S_0)$ where S_0 is an open set containing s_0 in S, and this implies the S_0-properness of the connected components of the space $\mathcal{C}_{n-1}(\pi_{|S_0})$ of relative $(n-1)-$cycles of $\pi_{|S_0}$.

In our second theorem we shall assume that a smooth relative sG-form exists on an dense Zariski open set S'; this is true if for each $s \in S'$ the manifold X_s admits a smooth sG-form (see the final remark of section 3).

THEOREM 1.0.3. *Let* $\pi : \mathcal{X} \to S$ *be a holomorphic family of compact complex connected manifolds of dimension* n *parametrized by an irreducible complex space* S. *Assume that there exists a dense Zariski open set* S' *in* S *such that for each* s *in* S' *the manifold* X_s *satisfies the* $\partial\bar{\partial}-$*lemma*[1] *and such that there exists a (smooth) relative sG-form for the family* $\pi_{|S'} : \mathcal{X}_{|S'} \to S'$. *Assume also that the function* $s \mapsto h^{0,1}(s)$ *is constant on* S.

Then if $a := \inf_{s\in S'}[a(X_s)]$ *we have* $a(X_s) \geq a$ *for each* $s \in S$.

SOME REMARKS.

1. When the map π is weakly kählerian in the sense of [C.81], which implies that X_s is in the class $\mathscr{C}$ of A. Fujiki[2] for all $s \in S$, then the semi-continuity result of the algebraic dimension (in the sense of theorem 1.0.2) of the fibers of π is proved in [C.81].

   But the weak kählerian assumption for π is rather strong as it implies the class $\mathscr{C}$ property for **all** $X_s, s \in S$ and also properness of the irreducible components of the relative cycles spaces for any dimension of the cycles. Here we only consider the properness for relative codimension 1 cycles and no kähler type assumption on X_s for $s \in S \setminus S'$.

2. In the absolute case, this compactness for irreducible components of the codimension 1 cycle space is always true for any compact complex space (see [C.82]) but this is far to be true for smaller cycles of positive dimension.

3. In the second theorem, the existence of a smooth relative sG-form on $\pi^{-1}(S')$ is only used to obtain the properness of the relative cycle-space $\mathcal{C}_{n-1}(\pi_{|S'})$.

   As the $\partial\bar{\partial}-$lemma is satisfied by any complex manifold in the class $\mathscr{C}$ of Fujiki (see [V.86]), the assumption that $\pi_{|S'}$ is weakly kähler in the sense of [C.81] gives a simple weak version of this result.

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[1]See for instance [Va.86].

[2]Note that it is proved in [Va.89] that a compact complex manifold is in the class $\mathscr{C}$ if and only if it is bimeromorphic to a compact Kähler manifold.

4. In both cases the key point is to produce a $(n-1, n-1)$ smooth positive $S-$relative form in $\mathcal{X}$ such that its integral on a dense set of members of an analytic family of relative $(n-1)-$cycles giving for general $s \in S'$ an algebraic reduction for X_s, is locally bounded near s_0. In the first case this is an easy consequence of the d_S-closedness of a relative sG-form ; in the second case, it uses the $\partial_S\bar{\partial}_S-$closedness of a relative G-form combined with a nice argument of D. Popovici [P.13] using the constancy of the numbers $h^{0,1}(s)$ near s_0 and the $\partial\bar{\partial}-$lemma.

5. Using theorem 1.0.2 we know that in the situation of theorem 1.0.3 there exists a general subset $S'' \subset S'$ such that $a(X_s) = a$ for all $s \in S''$. As we prove in fact that the constancy of $h^{0,1}(s)$ implies that the closure of any irreducible component of $\mathcal{C}_{n-1}(\pi_{|S'})$ is a $S-$proper irreducible component of $\mathcal{C}_{n-1}(\pi)$, the complement of S'' in S is also a countable union of closed analytic subsets in S with no interior points.

6. Of course, in the case where X_s is Moishezon for s general in S', our results imply that all fibers are Moishezon !

We thank the referee who suggested that the 1-dimensional parameter space case can be extended with little work into a statement for general parameter spaces, and also helped to improve the English style.

2. Algebraic Reduction

2.1. Absolute case

For a compact irreducible complex space X of dimension n the irreducible components of the complex space $\mathcal{C}_{n-1}(X)$ are compact Moishezon spaces. As it is difficult to find an explicit proof of this result, (despite the fact that it has been well-known to the experts for more than 30 years, see [C.82]), we will give a proof in the appendix.

Note that for codimension > 1 cycles the irreducible components of the cycle-space are not compact in general.

For any compact irreducible analytic subset Γ of dimension $\gamma \geq 1$ in $\mathcal{C}_{n-1}(X)$ there exists a "natural" meromorphic map

$$K_\Gamma : X \dashrightarrow \mathcal{C}_{\gamma-1}(\Gamma)$$

called the **Kodaira map** of Γ. This means that there exists a (natural) proper (thanks to [B.78]) modification $\tau_\Gamma : X_\Gamma \to X$ and a holomorphic map (also denote K_Γ)

$$K_\Gamma : X_\Gamma \to \mathcal{C}_{\gamma-1}(\Gamma)$$

associating to the generic point x of X the $(\gamma - 1)-$cycle in Γ which is the subset of Γ parametrizing the $(n-1)-$cycles containing x.

We shall denote its image by Q_Γ. Note that Q_Γ is always a compact irreducible Moishezon space. We shall recall some more details on this construction in the relative case in the next section.

The algebraic dimension $a(X)$ of X is the maximum of the dimension of the Q_Γ when Γ is an irreducible component of $\mathcal{C}_{n-1}(X)$. In fact, for a given X there exists a Γ such its Kodaira map is an algebraic reduction (see [C.80] or [C.81]).

LEMMA 2.1.1. *For any compact irreducible analytic subset* Γ *in* $\mathcal{C}_{n-1}(X)$*, the algebraic dimension of* X *is at least the dimension of* Q_Γ*.*

PROOF. If the generic member of Γ is not irreducible (but reduced), there exists a proper finite map $g : \tilde{\Gamma} \to \Gamma$ and an analytic family of $(n-1)-$cycles parametrized by $\tilde{\Gamma}$, with generic irreducible member, obtained by Stein factorization of the projection $G_\Gamma \to \Gamma$ of the graph G_Γ of the tautological family of $(n-1)-$cycles parametrized by Γ. If Γ' is the image of $\tilde{\Gamma}$ by the corresponding classifying map, we have a direct image $g_* : \mathcal{C}_{\gamma-1}(\tilde{\Gamma}) \to \mathcal{C}_{\gamma-1}(\Gamma)$ which is proper and finite and induces a generically finite surjective meromorphic map

$$Q_{\Gamma'} \dashrightarrow Q_\Gamma.$$

So the dimension of Q_Γ is again bounded by the algebraic dimension of X.

In the case of non reduced generic member in the family parametrized by Γ we may consider the pull back of the set theoretic graph $|G_\Gamma|$ on the normalisation Γ_1 of Γ which is the graph of an analytic family of $(n-1)-$cycles (see [B.75] or [B-M.14]) and replace Γ by the image Γ' of Γ_1 in $\mathcal{C}_{n-1}(X)$ by the corresponding classifying map. Then it is easy to see that Q_Γ and $Q_{\Gamma'}$ are bimeromorphic, so again $\dim Q_\Gamma$ is at most the algebraic dimension of X. $\square$

2.2. The relative case

We consider now a proper surjective $n-$equidimensional holomorphic map

$$\pi : \mathcal{X} \to S$$

between two irreducible complex spaces $\mathcal{X}$ and S, such that the generic fiber is irreducible[3]. Let $\dim \mathcal{X} = n + \sigma$, and assume that $\dim S := \sigma \geq 1$. Consider then an irreducible $S-$**proper** analytic subset $\Gamma \subset \mathcal{C}_{n-1}(\pi)$ of dimension γ of the space of $S-$relative $(n-1)-$cycles in $\mathcal{X}$. We say that Γ is a **good filling** for $\pi : \mathcal{X} \to S$ if the following conditions are satisfied

i) The generic member of the tautological family parametrized by Γ is irreducible.

ii) The graph $G_\Gamma \subset \Gamma \times_S \mathcal{X}$ of the tautological family parametrized by Γ projects surjectively on $\mathcal{X}$ by the second projection.

Then it is clear that G_Γ is irreducible, proper on S and has dimension $\gamma + n - 1$. Note that condition ii) implies that $\gamma + n - 1 \geq n + \sigma$, so that $\gamma \geq \sigma + 1$.

Lemma 2.2.1. *Let Γ be a good filling for $\pi : \mathcal{X} \to S$ and let $Y_\Gamma \subset \mathcal{X}$ the set of points x in $\mathcal{X}$ such that $\dim pr^{-1}(x) \geq \gamma - \sigma$, where $pr : G_\Gamma \to \mathcal{X}$ is the projection. Define*

$$T_\Gamma := \{s \in S \ / \ X_s \subset Y_\Gamma\}.$$

Then T_Γ is a closed analytic subset in S with codimension at least equal to 2. In particular, for $\dim S = 1$, the set T_Γ is empty for all good filling Γ.

Proof. The fact that T_Γ is a closed analytic subset is classical. Let T be an irreducible component of T_Γ of dimension d. Then $\pi^{-1}(T)$ has dimension $d+n$ and $pr^{-1}(\pi^{-1}(T))$ has dimension at least $d+n+\gamma-\sigma$. But if $d \geq \sigma - 1$ we have $d + n + \gamma - \sigma \geq n + \gamma - 1 = \dim G_\Gamma$. As G_Γ is irreducible, this implies the equality. In the case $d = \sigma - 1$ then G_Γ cannot be surjective on $\mathcal{X}$. If $T = S$ the dimension of $pr^{-1}(\pi^{-1}(T))$

[3]Using a Stein reduction of π, this assumption is not restrictive.

would be strictly bigger that the dimension of G_Γ. So we have $d \leq \sigma - 2$. $\square$

Then, for each ($S-$proper) good filling Γ we have a proper $S-$modification[4] $\tau_\Gamma : \mathcal{X}_\Gamma \to \mathcal{X}$ and a holomorphic $S-$map

$$K_{\Gamma/S} : \mathcal{X}_\Gamma \to \mathcal{C}_{\gamma-s-1}(\Gamma/S)$$

obtained by composition of the "fiber-map" of $pr : G_\Gamma \to \mathcal{X}$ and the $S-$relative direct-image map of relative $(\gamma - \sigma - 1)-$cycles via the $S-$map $p : G_\Gamma \to \Gamma$ (see [B.75] or [B-M.14]).

LEMMA 2.2.2. *Assume that* Γ *is* $S-$*proper and is a good filling for* π*. The* $S-$*map* $K_{\Gamma/S}$ *is proper, and so the image* Q_Γ *of* $K_{\Gamma/S}$ *is proper over* S.

PROOF. Let K be a compact set in $\mathcal{C}_{\gamma-s-1}(\Gamma/S)$ and let L be its projection on S. Then any relative cycle in K is contained in $p^{-1}(L)$ where $p : \Gamma \to S$ is the projection (which is assumed to be proper). Then the pull-back of the compact set $p^{-1}(L)$ on G_Γ is a compact set M and also its image $pr(M)$ is compact. Now $K_{\Gamma/S}^{-1}(K) \subset pr(M)$ is compact. The properness of Q_Γ on S is then easy. $\square$

As for each $s \in S \setminus T_\Gamma$ the set $\Gamma(s)$ is a finite union of irreducible compact analytic subsets in $\mathcal{C}_{n-1}(X_s)$ and has dimension at least equal to 1, the compact analytic set $Q_\Gamma(s)$ is a positive dimensional compact Moishezon space for each $s \in S \setminus T_\Gamma$.

Assume now that for a given Γ which is a $S-$proper good filling for π, there exists a dense subset D_Γ in $S \setminus T_\Gamma$ such that for each $s \in D_\Gamma$, the set $\Gamma(s)$ gives an algebraic reduction of X_s. Then we have, for such $s \in D_\Gamma$

$$\dim Q_\Gamma(s) = a(X_s).$$

This implies, by semi-continuity of the dimension of the fibers of the projection $Q_\Gamma \to S$, using the result at the end of paragraph 2.1, that for **each** s in $S \setminus T_\Gamma$, we have

$$\dim a(X_s) \geq \dim Q_\Gamma(s) \geq \inf_{s \in D_\Gamma} \dim Q_\Gamma(s) = \inf_{s \in D_\Gamma} a(X_s).$$

[4]It induces a proper modification on **each** $X_s, s \in S \setminus T_\Gamma$.

This proves the following lemma.

Lemma 2.2.3. *Assume that :*

i) *the irreducible analytic subset* Γ *of* $\mathcal{C}_{n-1}(\pi)$ *is a* $S-$*proper good filling for* π.

ii) *there exists a dense subset* D_Γ *such that*

$$\inf_{s\in D_\Gamma}[\dim Q_\Gamma(s)] = a.$$

Then for **each** $s \in S \setminus T_\Gamma$ *the inequality* $a(X_s) \geq a$ *holds.*

Note that for $\dim S = 1$ the set T_Γ is always empty, so the conclusion $a(X_s) \geq a$ holds for each $s \in S$.

3. A Sufficient Condition for Properness of the Connected Components of the Space of Relative Codimension 1 Cycles

We begin by a definition which is equivalent to the notion of a "strongly Gauduchon metric" introduced in [P.13].

Definition 3.0.4. Let X be a reduced complex space of pure dimension n. A $2n-2$ smooth form ω on X will be called a **sG-form** on X when it satisfies:

i) The form ω is $d-$closed.

ii) The $(n-1, n-1)$ part of ω is positive definite on X.

Recall that the strict positivity above means that in a local embedding in an open set U of some $\mathbb{C}^N$, the $(n-1, n-1)$ part of ω may be induced on X by a smooth, strictly positive in the sense of Lelong $(n-1, n-1)-$form on U.

We also need the relative version of this notion :

Definition 3.0.5. Let $\pi : \mathcal{X} \to S$ be a surjective proper $n-$equidimensional morphism of reduced complex spaces. A smooth $S-$relative

$2n-2$ form $\omega_{/S}$ will be called a $S-$**relative sG-form** for π if it induces a sG-form on each $X_s := \pi^{-1}(s)$.

The existence of a $S-$relative sG-form for a proper $n-$equidimensional surjective map $\pi : \mathcal{X} \to S$ implies a relative version of the compactness of the irreducible components of the space $\mathcal{C}_{n-1}(X)$ of compact $(n-1)-$cycles of a compact pure $n-$dimensional complex space X.

PROPOSITION 3.0.6. *Let $\pi : \mathcal{X} \to S$ be a surjective proper $n-$equidimensional morphism of reduced complex spaces admitting a $S-$relative sG-form ω. Then each connected component of the $S-$relative cycle space $\mathcal{C}_{n-1}(\pi)$ is proper over S.*

PROOF. For a $(n-1)-$relative cycle C of π define $F(C) := \int_C \omega$. This is a continuous function of C (see [B.75] or [B-M.14]) and on any given compact set K in S, this function is bigger than ϵ_K times the volume of C for a continuous hermitian metric defined on $\mathcal{X}$, because in the integration, only the $(n-1, n-1)$ part of $\omega_{/S}$ is relevant. As the function F is locally constant on the fibers of the projection $\mathcal{C}_{n-1}(\pi) \to S$ thanks to the $d_{/S}-$closeness of $\omega_{/S}$, this implies the properness of the projection $\mathcal{C}_{n-1}(\pi) \to S$ thanks to E. Bishop's theorem (for instance, see [B-M.14]). $\square$

LEMMA 3.0.7 [see [P.13]]. *Let $\pi : \mathcal{X} \to S$ be a proper holomorphic family of compact connected complex manifolds of dimension n parametrized by an irreducible complex space S. Assume that for a point $s_0 \in S$, the manifold $X_{s_0} := \pi^{-1}(s_0)$ has a sG-form ω_0. Then we can find a small open neighbourhood S' of s_0 in S and a relative sG-form ω on $\pi^{-1}(S')$ inducing ω_0 on X_{s_0}.*

PROOF. Thanks the local $\mathscr{C}^\infty-$triviality of π, there exists an open neighbourhood S_1 of s_0 in S and a $\mathscr{C}^\infty$ trivialisation $\theta : \pi^{-1}(S_1) \to S_1 \times X_{s_0}$ of the fibration π inducing the identity on X_{s_0}. Define $\omega := \theta^*(\omega_0)$. As ω_0 is $d-$closed, so is ω. We shall consider ω as a relative $d-$closed form. It induces ω_0 on X_0. As the complex structure of X_s varies continuously with $s \in S_1$, the $(n-1, n-1)$ part of the relative form ω varies continuously. As it is positive definite at $s = s_0$

there exists an open neighbourhood $S' \subset S_1$ of s_0 where it stays positive definite on each fiber ; so ω induces a relative sG-form on $\pi^{-1}(S')$. $\square$

In the situation of the previous lemma the proposition 3.0.6 gives that the connected components of $\mathcal{C}_{n-1}(\pi')$ are proper over S'.

Remark. Using the previous lemma and a smooth partition of unity on S, if we assume that each X_s admits a smooth sG-form, we can construct a smooth global $S-$relative sG-form on $\mathcal{X}$.

4. Proof of Theorem 1.0.2

The following proposition is an easy generalization of a result of [C.81].

PROPOSITION 4.0.8. *Let $\pi : \mathcal{X} \to S$ a proper surjective holomorphic $n-$equidimensional map between two irreducible complex spaces. Assume that any irreducible component of the complex space $\mathcal{C}_{n-1}(\pi)$ is proper over S. Then there exists a countable union Σ of closed irreducible analytic subsets with no interior point in S and a non negative integer a such that*

(i) *For any $s \in S \setminus \Sigma$ we have $a(X_s) = a$.*

(ii) *For any $s \in S$ we have $a(X_s) \geq a$.*

PROOF. Denote by A the set of irreducible components Γ of $\mathcal{C}_{n-1}(\pi)$ with generic irreducible cycle such that the projection $\Gamma \to S$ is not surjective, and define S_Γ as the image in S of such a Γ. Denote by B the set of irreducible components Γ of $\mathcal{C}_{n-1}(\pi)$ with generic irreducible cycle such that the projection $\Gamma \to S$ is surjective. Then the corresponding map $Q_\Gamma \to S$ is proper and surjective.

Note that, as $\mathcal{C}_{n-1}(\pi)$ has a countable set of irreducible components, the sets A and B are countable.

For $\Gamma \in B$ let $V_\Gamma \subset S$ be the closed analytic subset of S corresponding to big fibers[5] of the projection $Q_\Gamma \to S$. Then define :

$$\Sigma := \big(\cup_{\Gamma \in A} S_\Gamma\big) \cup \big(\cup_{\Gamma \in B} T_\Gamma\big) \cup \big(\cup_{\Gamma \in B} V_\Gamma\big)$$

[5]So $s \in V_\Gamma$ if and only if the dimension of $Q_\Gamma(s)$ is strictly bigger than $\dim Q_\Gamma - \sigma$.

and let $a := \inf_{s\in S\setminus\Sigma}[a(X_s)]$.

Remark that if $a = 0$ for each $\Gamma \in B$ the projection $Q_\Gamma \to S$ is generically finite. So conditions i) and ii) are clearly satisfied. So we may assume $a \geq 1$.

Choose a point $s_0 \in S \setminus \Sigma$. There exists a good filling $\Gamma_0 \subset \mathcal{C}_{n-1}(X_{s_0})$ for X_{s_0} giving an algebraic reduction for X_{s_0} and let Γ be an irreducible component of $\mathcal{C}_{n-1}(\pi)$ containing Γ_0. Then Γ is in B because $s_0 \notin \Sigma$, and the fiber at s_0 of Q_Γ has dimension $a(X_{s_0})$ because s_0 is not in T_Γ. Then any fiber of Q_Γ has dimension at least $a(X_{s_0})$ because the point s_0 is not in V_Γ by definition. Condition i) follows as well as condition ii) for all points in $S \setminus T_\Gamma$ by the semi-continuity of the dimension of the fibers of Q_Γ and results of the section 2.2.

This prove the theorem in the case $\dim S = 1$ because the set T_Γ is empty in this case.

To obtain ii) for a point $s \in T_\Gamma$, choose an open neighbourhood S' of this point in S an a closed analytic irreducible curve C in S' containing this point and not contained in Σ. This is possible thanks to the corollary 4.0.11 below. Now the previous argument applies to the holomorphic family $\tilde{\pi} : \mathcal{X}_{|\pi^{-1}(C)} \to C$ because we are in the case of a 1-dimensional parameter space. $\square$

LEMMA 4.0.9. *Let U be an open polydisc with center 0 in $\mathbb{C}^n$ and let S be a closed analytic subset in U with no interior point in U. Then the set A of the points in $\mathbb{P}_{n-1}$ such that the corresponding line Δ in $\mathbb{C}^n$ is such that $\Delta \cap U \subset S$ is a closed analytic subset with no interior point in $\mathbb{P}_{n-1}$.*

PROOF. As $\mathbb{P}_{n-1}$ parametrizes an analytic family of $1-$cycles in U the condition $\Delta \cap U \subset S$ is a closed analytic condition on $\Delta \in \mathbb{P}_{n-1}$. The fact that A has no interior point in $\mathbb{P}_{n-1}$ is obvious from the assumption $S \neq U$. $\square$

COROLLARY 4.0.10. *Let U be an open polydisc with center 0 in $\mathbb{C}^n$ and let $(S_\nu)_{\nu\in\mathbb{N}}$ be a sequence of closed analytic subsets in U with no interior point in U. Then there exists a line $\Delta \in \mathbb{P}_{n-1}$ such that $\Delta \cap U$ is not contained in $\cup_{\nu\in\mathbb{N}} S_\nu$.*

Proof. This is an easy consequence of the previous lemma and Baire's theorem apply to $\mathbb{P}_{n-1}$. □

Corollary 4.0.11. *Let S be a reduced complex space of pure dimension n and let $(T_\nu)_{\nu\in\mathbb{N}}$ be a sequence of closed analytic subsets in S with no interior point in S. For any $s_0 \in S$ there exists an open neighbourhood V of s_0 in S and a closed analytic irreducible curve C in V containing s_0 and not contained in $\cup_{\nu\in\mathbb{N}} T_\nu$.*

Proof. Choose V such that there exists a proper finite surjective map $\pi : V \to U$ where U is an open polydisc with center 0 in $\mathbb{C}^n$ and $\pi(s_0) = 0$. Apply now the previous corollary to the sequence $(S_\nu)_{\nu\in\mathbb{N}}$ where $S_\nu := \pi(T_\nu)$. So we find a line Δ containing 0 such $\Delta \cap U \not\subset \cup_{\nu\in\mathbb{N}} S_\nu$. Now any irreducible component C of the curve $\pi^{-1}(\Delta)$ containing s_0 satisfies $C \not\subset \cup_{\nu\in\mathbb{N}} T_\nu$. □

5. Proof of Theorem 1.0.3

Remark that the statement is empty if $a = 0$ so we may assume $a \geq 1$. Recall that in our situation there exists, thanks to [G.77], a smooth relative Gauduchon metric on $\mathcal{X}$. So there exists a smooth family γ_s of positive definite $(1,1)-$forms on $X_s, s \in S$, with the condition : $\partial_s\bar{\partial}_s\ \gamma_s^{\wedge(n-1)} = 0$ for each $s \in S$. Thanks to theorem 1.0.2, we can choose also a $S'-$proper good filling Γ' for the family $\pi' := \pi_{|S'} : \mathcal{X}_{|S'} \to S'$, such that for general $s \in S'$ we have $\dim Q_{\Gamma'}(s) = a(X_s)$. Let $\alpha \in H^2(X, \mathbb{Z})$ be the fundamental class of the relative $(n-1)-$cycles in the family parametrized by Γ'. Note $[C_s]$ the integration current on X_s of a member of the family Γ' contained in X_s for $s \in S'$. We shall prove that, as $h^{0,1}(s)$ is independent of $s \in S$, there exists, for each $s_0 \in S$ a constant $C > 0$ such that for $s \in S$ near enough s_0 the estimate

$$\langle\, [C_s], \gamma_s^{\wedge n-1}\,\rangle \ \leq C\ < +\infty$$

holds, where $\gamma_s^{\wedge(n-1)}$ is the $(n-1)-$th exterior power of a relative Gauduchon metric on $\mathcal{X}$.

This will be enough to conclude that the closure Γ of Γ' in $\mathcal{C}_{n-1}(\pi)$ is $S-$proper irreducible component of $\mathcal{C}_{n-1}(\pi)$, thanks to Bishop's theorem, and will finish the proof, because the semi-continuity of the dimension of the fibers of the projection $Q_\Gamma \to S$ allows to conclude, first on $S \setminus T_\Gamma$ and then for every $s \in S$ as in the proof of theorem 1.0.2.

Let $\tilde{\omega}_0$ be a smooth $d-$closed real $2-$form on X which is a de Rham representative of the class $\alpha \in H^2(X, \mathbb{Z})$. Fix a point $s_0 \in S$. Using the local $\mathscr{C}^\infty-$triviality of π we can find an open neighbourhood S_0 of s_0 in S and a smooth $d-$closed 2-form $\tilde{\omega}$ on $\pi^{-1}(S_0)$ inducing $\tilde{\omega}_0$ on X_{s_0}. Put $S'_0 := S_0 \cap S'$. For each $s \in S'_0$ there exists a real $1-$current β_s on X_s such that $[C_s] = \tilde{\omega} + d_s\beta_s$. As β_s is real, we have $\beta_s = \beta_s^{1,0} + \overline{\beta_s^{1,0}}$.

Note that type consideration shows that $\beta_s^{0,1} := \overline{\beta_s^{1,0}}$ is a solution of the equation

$$\bar{\partial}_s \beta_s^{0,1} = -\tilde{\omega}_s^{0,2}.$$

So following [P.13], we shall define $\tilde{\beta}_s^{0,1}$ as the (unique) solution of this equation with minimal L^2 norm defined by the Gauduchon metric γ_s on X_s for $s \in S'_0$. Defining $\tilde{\beta}_s = \overline{\tilde{\beta}_s^{0,1}} + \tilde{\beta}_s^{0,1}$ we have now that the real current

$$[C_s] - \tilde{\omega}_s - d_s(\tilde{\beta}_s)$$

is d_s-exact and of type $(1,1)$ on X_s. Then, as the complex compact manifold X_s satisfies the $\partial_s\bar{\partial}_s-$lemma for $s \in S'_0$, there exists a $0-$current φ_s on X_s such that

$$[C_s] = \tilde{\omega}_s + d_s(\tilde{\beta}_s) + i\partial_s\bar{\partial}_s\varphi_s \qquad \forall s \in S'.$$

Now, as $\partial_s\bar{\partial}_s\ \gamma_s^{\wedge(n-1)} = 0$, an easy consequence is that

$$\langle [C_s], \gamma_s^{\wedge n-1}\rangle\ = \langle \tilde{\omega}_s, \gamma_s^{\wedge n-1}\rangle + \langle d_s(\tilde{\beta}_s), \gamma_s^{\wedge n-1}\rangle \quad \forall s \in S'.$$

As the function $s \mapsto < \tilde{\omega}_s, \gamma_s^{\wedge n-1} >$ is continuous on S_0, to bound the left handside, it is enough to bound near s_0 on the dense set S'_0, the term

$$< d(\tilde{\beta}_s), \gamma_s^{\wedge n-1} >= \pm < \tilde{\beta}_s, d\gamma_s^{\wedge n-1} > .$$

Then it is enough to follow the argument concluding the proof of the proposition 3.1 in [P.13], showing that under the assumption that $h^{0,1}(s)$ is constant, $s \mapsto \tilde{\beta}_s$ depends continuously of $s \in S_0$ to conclude. $\square$

As it is not easy to find a reference for $\partial\bar{\partial}-$lemma for currents, we give a proof for the convenience of the reader of what we used in the proof above.

Lemma 5.0.12. *Let X be a compact connected complex manifold of dimension n and assume that X satisfies the property of the $\partial\bar\partial-$lemma*[6] *. Let T be a $d-$exact $(1,1)-$current on X. Then there exists a $(0,0)-$current Θ on X such that $\partial\bar\partial\,\Theta = T$.*

Proof. As the vector space :

$$coker\big[\partial\bar\partial : \mathscr{C}^{\infty,(n-1,n-1)}(X) \to \mathscr{C}^{\infty,(n,n)}(X)\big]$$

is finite dimensional, it is enough, thanks to the Hahn-Banach theorem, to show that for any $\partial\bar\partial-$closed smooth $(n-1,n-1)-$form ψ we have $\langle T,\psi\rangle = 0$.

As $d\psi$ is in $ker\partial \cap ker\bar\partial \cap Im\, d$ we may write $d\psi = \partial\bar\partial\eta$ where η is a smooth form. Then the form $\psi - \bar\partial\eta$ is $d-$closed. As $\bar\partial T = 0$ and T is $d-$exact we obtain

$$\langle T,\psi\rangle = \langle T,\psi - \bar\partial\eta\rangle + \langle T,\bar\partial\eta\rangle = 0,$$

concluding the proof. □

6. Appendix

We give here a proof of the following (classical) statement (compare with [F.82]) :

Theorem 6.0.13. *Let X be a compact irreducible complex space of dimension n. Then the irreducible components of the space $\mathcal{C}_{n-1}(X)$ of $(n-1)-$cycles in X are compact and Moishezon.*

Proof. Using Hironaka's desingularization theorem it is not restrictive to assume that X is a compact connected manifold of dimension n and that we have a holomorphic surjective map $r : X \to P$ where P is a projective manifold of dimension a where a is the algebraic dimension of X. This comes from the fact that by a proper modification of X we add only finitely many effective irreducible $(n-1)-$cycles, and then, for any irreducible component Γ of the space of $(n-1)-$cycle of our initial X,

[6] We define here this property as the equality $ker\partial\cap ker\bar\partial\cap Im\, d = Im\,\partial\bar\partial$ for smooth forms on X.

there exists an irreducible component $\tilde{\Gamma}$ of the space of $(n-1)-$cycles of the smooth modification of X such that the direct image of cycles gives a modification $\tilde{\Gamma} \to \Gamma$.

Recall now that the number of non polar effective irreducible divisors in X is bounded thanks to [C.82]. Let now Γ be an irreducible component of $\mathcal{C}_{n-1}(X)$ of positive dimension with generic member irreducible[7]. Then each member of the corresponding family is polar and so has a $(a-1)-$dimensional image in P. This means that the image of the graph $G_\Gamma \subset \Gamma \times X$ by the map $\mathrm{id}_\Gamma \times r$ is proper and $(a-1)-$equidimensional on Γ. Up to replace Γ by its normalization, we have the graph of an analytic family of $(a-1)-$cycles in P. As Γ is irreducible and P projective, the volume of these cycles in P is uniformly bounded.

Now using any continuous hermitian metric on X, we have a uniform bound for the volume of the generic fibers of r thanks to [B.78]. Then a Fubini type argument implies that the volume of the generic member of our initial family, which is bounded by the volume of the pull-back by r of its image by r is uniformely bounded. Now Bishop's theorem implies that Γ is relatively compact in $\mathcal{C}_{n-1}(X)$, and as it is closed, it is compact. To conclude we have to remember that the normalization of Γ dominates a compact analytic subspace of an irreducible component of $\mathcal{C}_{a-1}(P)$ and that this map is generically finite, because the pull-back by r of an irreducible effective divisor in P contains only finitely many irreducible effective divisors in X. So Γ is a compact irreducible Moishezon space. □

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[7] It is enough to treat this case for this result.

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