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Examples of Vanishing Gromov–Witten–Welschinger Invariants

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In memory of Professor Kunihiko Kodaira

The aim of this note is to give examples of real enumerative problems without real solutions. Methods to define and compute real enumerative invariants in dimensions 2 and 3 were developed by [Wel05] and partially extended to higher dimensions in [GZ13]. The number of real solutions was determined or estimated in many cases; see [IKS05, Wel10, IKS13b, FK13, IKS13a, GZ13, GZ14]. Computations using tropical methods are given in [BM07, FM10, ABLdM11, BP13, Bru14].

The number of rational curves of degree d defined over $\mathbb{R}$ in $\mathbb{P}^2$ that pass through a given real set of $3d-1$ points is almost always nonzero; see the extensive tables in [ABLdM11]. By contrast, the predicted number of rational curves of degree d in $\mathbb{P}^3$, defined over $\mathbb{R}$, that pass through a given real set of $2d$ points is always 0 for d even; see [Wel05, 2.4] or [GZ13, Cor.1.4]. These formulas count curves with signs, thus it is not clear that there are no such curves when the predicted number is 0. For $d=4$ Mikhalkin found configurations of 8 points in $\mathbb{P}^3$ without any degree 4 real curve of genus 0 through them; see [Wel05, 2.6].

The main aim of this note is to show that, for any even d, there are certain types of configurations of $2d$ points in general position with no rational curves of degree d defined over $\mathbb{R}$ passing through them; see Theorem 10. We cover all numerical possibilities where some of the $2d$ points are real. (We have no such examples for d conjugate point pairs for $d>4$.)

The key to the examples is the study of a degenerate situation when all the points lie on a degree 4 elliptic curve in $\mathbb{P}^3$. It turns out that, in this case, all the curves in question lie on some quadric surface; these in turn

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can be understood by studying the torsion points on the elliptic curve. The geometry of the situation is summarized in Proposition 3.

In the first non-trivial case, when $d = 4$, the point-octet P_8 always lies on a degree 4 elliptic curve. In Section 2 we describe all possibilities. There are 4 topologically distinct types if P_8 consists of 4 conjugate pairs of complex points and 11 topologically distinct types if P_8 consists of 8 real points. Already for $d = 6$ the method is unlikely to cover all cases.

More examples of enumerative problems without real solutions are given in Section 4. In most cases we study what happens when all the constraints lie on a quadric hypersurface. This is similar to the method used in [Wel07] for surfaces. Our examples account for all configurations involving rational curves of degree ≤ 3 in $\mathbb{P}^3$. We also get two infinite series: lines in $\mathbb{P}^{4n-1}$ that meet 4 linear subspaces of dimension $2n-1$ (Example 12) and rational curves of degree $2d+1$ in $\mathbb{P}^3$ that pass through $4d$ points and meet 4 lines (Example 18); the latter pointed out by Zinger.

1. Degree 4 Elliptic Curves in $\mathbb{P}^3$

1 (Eight points in $\mathbb{P}^3$). Let P_8 be 8 general points in $\mathbb{P}^3$. The space of quadrics in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^9$, hence P_8 lies on a unique pencil of quadrics $|Q_\lambda|$ whose base locus is a degree 4 elliptic curve E_4.

Let $C_4 \subset \mathbb{P}^3$ be any irreducible, degree 4 curve through P_8. Working over an algebraically closed field, any point of $C_4 \setminus P_8$ is contained in some $Q = Q_\lambda$. Then C_4 and Q meet in ≥ 9 points, hence $C_4 \subset Q$. A smooth quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and linear equivalence classes are classified by their bidegree. (See Definition 2 for our conventions for marked quadrics and bidegree.) Since C_4 has degree 4, it has bidegree either $(2,2)$ or $(1,3)$ (up to changing the marking of Q). In the first case, $C_4 \sim 2H_Q$ where H_Q is the hyperplane class. Except when $C_4 = E_4$, this implies that $P_8 = (C_4 \cap E_4) \sim 2H_Q|_{E_4}$ This is not the case for general P_8. Indeed, fix 7 of the points and move the eighth in E_4. Then $[P_8] \in \operatorname{Pic}(E_4)$ varies but $2H_Q \in \operatorname{Pic}(E_4)$ stays fixed.

If the quadric Q is singular then $2C_4 \sim 4H_Q$. As before this would give $2P_8 \sim 4H_Q|_{E_4}$, which is again not the case for general P_8. Thus the quadric Q is smooth if it contains a curve $C_4 \neq E_4$ and, for a suitable choice of the marking, C_4 has bidegree $(1,3)$ on Q. The marking of Q gives a line bundle $L_2 := \pi_1^*\mathcal{O}_{\mathbb{P}^1}(1)|_{E_4}$ of degree 2 on E_4. Conversely, a degree 2 line bundle L_2

on E_4 determines a quadric surface that is swept out by the lines $\langle p_1, p_2\rangle$ where $p_1, p_2 \in E_4$ are points such that $\mathcal{O}_{E_4}(p_1+p_2) \cong L_2$.

Set $H_4 := H_Q|_{E_4}$; this is the hyperplane class on E_4, independent of the choice of Q. The other coordinate projection $\pi_2 : Q \to \mathbb{P}^1$ corresponds to $H_4 - L_2$ and so

$$P_8 = (E_4 \cdot C_4) \sim 3L_2 + (H_4 - L_2) = H_4 + 2L_2. \tag{1.1}$$

Thus a marked quadric $Q \in |Q_\lambda|$ that contains a curve C_4 corresponds to a solution of the equation

$$2L_2 \sim P_8 - H_4 \quad \text{where} \quad L_2 \in \operatorname{Pic}(E_4). \tag{1.2}$$

Over an algebraically closed field (of characteristic $\neq 2$) there are 4 solutions $L_2^1, \dots, L_2^4$ of the equation (1.2); any two solutions differ by a 2-torsion point of $\operatorname{Pic}(E_4)$. We get 4 different quadric surfaces $Q^1, \dots, Q^4 \in |Q_\lambda|$.

We claim that on each quadric Q^i there is a unique curve C_4^i that passes through P_8. To see this note that a curve of bidegree $(1,3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ can be viewed as the graph of a degree 3 map

$$\mathbb{P}^1 \to \mathbb{P}^1 \quad \text{given by} \quad (s{:}t) \mapsto \bigl(g_1(s,t){:}g_2(s,t)\bigr),$$

where g_1, g_2 are cubic forms. Passing through any given point gives a linear equation on the 8 coefficients of g_1, g_2. Thus passing through 7 points $p_1, \dots, p_7$ of P_8 gives a unique pair g_1, g_2, up to scalar.

A nice feature is that passing through the 8th point p_8 on E_4 comes for free. Indeed, if the resulting curve passes through an 8th point $q \in E_4$, then

$$p_1 + \cdots + p_7 + q = (E_4 \cdot C_4) \sim 3L_2 + (H_4 - L_2) \sim p_1 + \cdots + p_7 + p_8$$

shows that $q \sim p_8$ hence $q = p_8$ since E_4 is elliptic. Thus we get 4 curves $C_4^i \subset Q^i$ for $i = 1, \dots, 4$.

Furthermore, a curve C_4^i is defined over a subfield $k \Leftrightarrow$ the corresponding quadric surface Q^i is defined over $k \Leftrightarrow$ the corresponding $[L_2^i] \in \operatorname{Pic}(E_4)$ is a k-point. (Note that even for $k = \mathbb{R}$ it can happen that a k-point of $\operatorname{Pic}(E_4)$ does not correspond to an actual line bundle on E_4 that is defined over k. In this case necessarily $E_4(k) = \emptyset$; see (5.3).)

For $d > 4$, general point sets $P_{2d} \subset \mathbb{P}^3$ do not lie on a degree 4 elliptic curve. However, it turns out that point sets P_{2d} that do lie on a degree 4 elliptic curve can be studied the same way. The equation (1.2) is replaced by a similar one. A new complication that arises is that on a given quadric surface Q^i there are usually many curves C_d^{ij}.

Definition 2. A *marked quadric surface* $Q \subset \mathbb{P}^3$ is a smooth quadric plus a choice of a coordinate projection $\pi_1 : Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. The other coordinate projection is denoted by π_2. This choice is equivalent to fixing an isomorphism $\operatorname{Pic}(Q) \cong \mathbb{Z}^2$ such that both $(1, 0)$ and $(0, 1)$ correspond to lines.

We say that a curve B on Q has type (a, b) if $\mathcal{O}_Q(B) \cong \pi_1^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(b)$.

Proposition 3. *Let k be an algebraically closed field and $|Q_\lambda|$ a pencil of quadrics in $\mathbb{P}^3$ whose base locus is an elliptic curve E. Let $P_{2d} \subset E$ be a set of $2d$ general points and C_d a connected curve of degree d through P_{2d} not containing E.*

Then C_d is irreducible and is contained in one of the quadrics $Q = Q_\lambda$. Furthermore, such marked quadrics $Q = Q(a, L_2)$ correspond to

(1) *a choice of $0 < a < d/2$ and*

(2) *a solution of $(d - 2a)L_2 \sim P_{2d} - aH_4$ where $L_2 \in \operatorname{Pic}_2(E)$ and H_4 is the hyperplane class on E.*

For a fixed marked quadric $Q = Q(a, L_2)$ there can be many such curves C_d but they all have type $(a, d - a)$ in $\operatorname{Pic}(Q)$.

Proof. Assume first that C_d is irreducible. Any point of $C_d \setminus P_{2d}$ is contained in some $Q = Q_\lambda$. Then C_d and Q meet in $\geq 2d + 1$ points, hence $C_d \subset Q$. Thus C_d has type $(a, d - a)$ for some $0 < a \leq d/2$ and suitable choice of the marking.

The choice of Q plus the projection $Q \to \mathbb{P}^1$ corresponds to a line bundle $L_2 \in \operatorname{Pic}_2(E)$. The other projection $Q \to \mathbb{P}^1$ corresponds to $H_4 - L_2$ and so

$$P_{2d} = (E \cdot C_d) \sim (d - a)L_2 + a(H_4 - L_2) = aH_4 + (d - 2a)L_2. \tag{3.3}$$

If $a = d/2$ then this gives $P_{2d} \sim aH_4$, which is not the case for general P_{2d}. Thus the case $a = d/2$ is excluded and the rest of (1–2) is clear.

In order to complete the proof, we need to exclude the reducible cases. Let now $\sum C^i$ be a degree d, connected but possibly reducible curve passing through P_{2d}. If one of the C^i passes through more than $2\deg C^i$ points of P_{2d} then it has $> 2\deg C^i$ intersection with every Q_λ, thus $C^i = E$. Otherwise, every C^i passes through exactly $2\deg C^i$ points $P^i_{2d} \subset P_{2d}$, thus it is a curve as described above. Since the union of the P^i_{2d} is P_{2d}, the P^i_{2d} are disjoint from each other, hence the different C^i pass through different points of P_{2d}.

If two curves C^i, C^j are contained in different quadric surfaces $Q^i \neq Q^j$, then the curve is disconnected since $C^i \cap C^j \subset Q^i \cap Q^j = E_4$ but the different C^i pass through different points. Thus $\sum C^i$ is contained in a single quadric surface Q.

If C^i is of type $(a^i, d^i - a^i)$ then we get subsets $P^i \subset P_{2d}$ of degree $2d^i$ such that

$$P^i \sim (d^i - 2a^i)L_2 + a^i H_4. \tag{3.4}$$

Together with (3.3) we get

$$(d-2a)P^i \sim (d^i - 2a^i)P_{2d} + \big(a^i(d-2a) - a(d^i - 2a^i)\big)H_4. \tag{3.5}$$

If k is not an algebraic closure of a finite field, then we can choose the points p_i such that $p_1, \dots, p_{2d}$ and H_4 generate a free subgroup of rank $2d+1$ in $\operatorname{Pic}(E_4)$, thus (3.5) is impossible unless $P^i = P_{2d}$. Thus there are no reducible curves C_d for a very general choice of the points P_{2d}. The conclusion then holds in a Zariski open set of E^{2d}. (If k is an algebraic closure of a finite field then first we find P_{2d} over the algebraic closure of $k(x)$ and then note that the above Zariski open set of E^{2d} must contain k-points.) □

Remark 4. The above method reduces the computation of GW-invariants on $\mathbb{P}^3$ (with 0-dimensional constraints) to GW-invariants on $\mathbb{P}^1 \times \mathbb{P}^1$. It would be interesting to work this out in detail.

This should work over the reals as well, but there is a subtle point that I find confusing. Following the method gives that, on the quadric surface Q we need to find curves of degree d passing through P_{2d}. However, the space of rational curves of degree d on Q has dimension $2d-1$, thus GW-theory counts the number of curves that pass through a subset $P_{2d-1} \subset P_{2d}$. As we noted at the end of Paragraph 1, these curves then automatically pass through the last point of P_{2d} as well. However, in the case when we start with conjugate point pairs, we can not choose P_{2d-1} to be real. Nonetheless, the answer has a real structure.

I also have not computed the normal bundle of the resulting curves; this also effects the count.

2. Real Point-Octets in $\mathbb{P}^3$

Starting with a real point-octet in $\mathbb{P}^3$, we analyze the number and types of real curves C_4^i in (6) to get a complete list of possibilities.

5 (Picard group of a real elliptic curve). Let E be a real elliptic curve. The Picard group of the corresponding complex elliptic curve is denoted by $\mathrm{Pic} = \mathrm{Pic}(E)$. Let $\mathrm{Pic}_r \subset \mathrm{Pic}$ denote the set of complex line bundles of degree r.

The set of real points of Pic is denoted by $\mathrm{Pic}(\mathbb{R})$. Corresponding to the 3 different topological types of $E(\mathbb{R})$, there are 3 different descriptions for $\mathrm{Pic}(\mathbb{R})$.

(5.1) $E(\mathbb{R}) \sim \mathbb{S}^1$. In this case $\mathrm{Pic}^0_r(\mathbb{R}) := \mathrm{Pic}_r(\mathbb{R}) \sim \mathbb{S}^1$ for every r.

(5.2) $E(\mathbb{R}) \sim \mathbb{S}^1 \amalg \mathbb{S}^1$. In this case $\mathrm{Pic}_r(\mathbb{R}) \sim \mathbb{S}^1 \amalg \mathbb{S}^1$ for every r.

If r is even, then one of these components, denoted by $\mathrm{Pic}^0_r(\mathbb{R})$ consists of line bundles that are topologically trivial on both components of $E(\mathbb{R})$. We call this component and the line bundles in it *even.* The other component, denoted by $\mathrm{Pic}^1_r(\mathbb{R})$ consists of line bundles that are topologically nontrivial on both components of $E(\mathbb{R})$. We call this component and the line bundles in it *odd.*

For any complex line bundle L, the tensor product $L \otimes \bar{L}$ is real and even. Restricting to $\mathrm{Pic}(\mathbb{R})$ this is the same as multiplication by 2, denoted by m_2. Thus the image of $m_2 : \mathrm{Pic}(\mathbb{R}) \to \mathrm{Pic}(\mathbb{R})$ is the union of the even components.

(5.3) $E(\mathbb{R}) = \emptyset$. In this case $\mathrm{Pic}_r(\mathbb{R}) = \emptyset$ for odd r and $\mathrm{Pic}_r(\mathbb{R}) \sim \mathbb{S}^1 \amalg \mathbb{S}^1$ for even r.

If r is even, then one of these components, denoted by $\mathrm{Pic}^0_r(\mathbb{R})$ consists of real line bundles. (For $r = 2$ these correspond to degree 2 maps to $\mathbb{P}^1$.) We call this component and the line bundles in it *even.* The other component, denoted by $\mathrm{Pic}^*_r(\mathbb{R})$ consists of points that do not correspond to real line bundles. (For $r = 2$ these correspond to degree 2 maps to the "empty" conic $\tilde{\mathbb{P}}^1 := (x_0^2 + x_1^2 + x_2^2 = 0)$.) We call this component *twisted.* As before, the image of $m_2 : \mathrm{Pic}(\mathbb{R}) \to \mathrm{Pic}(\mathbb{R})$ is the union of the even components.

6 (Counting quartic curves through 8 points in $\mathbb{P}^3$). Let $P_8 \subset \mathbb{P}^3$ be a real set of 8 points in general position, that is, P_8 is made up of real points and complex conjugate point pairs.

If P_8 is real then so is E_4. The answer to our problem of counting real curves C_4 of degree 4 passing through P_8 is determined by the topological type of $E_4(\mathbb{R})$ and the positions of $H_4 \in \mathrm{Pic}_4(\mathbb{R})$ and $[P_8] \in \mathrm{Pic}_8(\mathbb{R})$.

There are 3 possibilities for $E_4(\mathbb{R})$.

(6.1) $E_4(\mathbb{R}) = \mathbb{S}^1$. Then necessarily $H_4 \in \mathrm{Pic}_4^0(\mathbb{R})$ and $[P_8] \in \mathrm{Pic}_8^0(\mathbb{R})$. Thus $P_8 - H_4 \in \mathrm{Pic}_4^0(\mathbb{R})$ and $2L_2 \sim P_8 - H_4$ has 2 real solutions. This gives 2 real curves C_4^1, C_4^2.

(6.2) $E_4(\mathbb{R}) = \mathbb{S}^1 \amalg \mathbb{S}^1$. There are 4 sub-cases.

a) H_4 is odd P_8 is even. Then $P_8 - H_4$ is odd, thus $2L_2 \sim P_8 - H_4$ has no real solutions. There are no degree 4 rational curves C_4 defined over $\mathbb{R}$.

b) H_4 is even P_8 is odd. Again $P_8 - H_4$ is odd and no real curves.

c) H_4 and P_8 are both even. Then $P_8 - H_4$ is even, 4 real curves.

d) H_4 and P_8 are both odd. Then $P_8 - H_4$ is even, 4 real curves.

(Note that a) and c) can happen if there are no real points in P_8 and also if all points of P_8 are real while P_8 must have at least 2 real points in cases b) and d).)

(6.3) $E_4(\mathbb{R}) = \emptyset$. Then P_8 has no real points hence $[P_8] \in \mathrm{Pic}_8^0(\mathbb{R})$ and also $H_4 \in \mathrm{Pic}_4^0(\mathbb{R})$. Thus $P_8 - H_4$ is even and $2L_2 \sim P_8 - H_4$ has 4 solutions in $\mathrm{Pic}_2(\mathbb{R})$.

Note that 2 of these solutions correspond to real line bundles, giving quadrics $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and curves $C_4 \cong \mathbb{P}^1$. The other 2 solutions do not correspond to real line bundles. These give "empty" quadrics $Q \cong (x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0)$ and "empty" curves $C_4 \cong (x_0^2 + x_1^2 + x_2^2 = 0)$.

Thus we get 2 real curves $C_4 \cong \mathbb{P}^1$ and 2 real curves $C_4 \cong \tilde{\mathbb{P}}^1 := (x_0^2 + x_1^2 + x_2^2 = 0)$.

Remark 7. There is a twisted form of (6.3) when E_4 is inside the twisted projective space $\tilde{\mathbb{P}}^3 := S^3\tilde{\mathbb{P}}^1$. Necessarily $E_4(\mathbb{R}) = \emptyset$. Then H_4 sits in $\mathrm{Pic}_4^*(\mathbb{R})$ but P_8 is in $\mathrm{Pic}_8^0(\mathbb{R})$. Thus $2L_2 \sim P_8 - H_4$ has no real solutions. This is, however, not surprising since there are no even degree rational curves in $\tilde{\mathbb{P}}^3$.

3. Even Degree Rational Curves in $\mathbb{P}^3$

Of the cases studied in Section 2, only one yields a simple answer for $d > 4$.

Proposition 8. *Let $E \subset \mathbb{P}^3$ be a real elliptic curve of degree 4 such that $E(\mathbb{R}) = \mathbb{S}^1 \amalg \mathbb{S}^1$ and $H_4 = \mathcal{O}_{\mathbb{P}^3}(1)|_E$ is even. (Equivalently, both components of $E(\mathbb{R})$ are trivial in $\pi_1(\mathbb{RP}^3)$.) Let $P_{4d} \subset E$ be a general real subset that has an odd number of points on both components of $E(\mathbb{R})$. Let C_{2d} be a connected real curve of degree $\leq 2d$ that contains P_{4d}. Then $E \subset C_{2d}$.*

Proof. Assume that $E \not\subset C_{2d}$. By Proposition 3, C_{2d} is geometrically irreducible and it is contained in a quadric $Q(a, L_2)$ as in (3.2). Since C_{2d} is real, so is the quadric $Q(a, L_2)$. By (3.2), such quadrics correspond to the solutions of the equation

$$(2d - 2a)L_2 \sim P_{4d} - aH_4. \tag{8.1}$$

Here P_{4d} is odd and H_4 is even, thus $P_{4d} - aH_4$ is odd. As we noted in (5.2), $P_{4d} - aH_4$ is not an even multiple of a real point of $\operatorname{Pic}(E)$. $\square$

Remark 9. Once we have established that such a curve C_{2d} must lie on a quadric surface, a simple topological argument also shows that (8.1) has no solutions, not even for homology classes with $\mathbb{Z}/2$-coefficients.

Under the natural homeomorphism $Q(\mathbb{R}) \sim \mathbb{S}^1 \times \mathbb{S}^1$ the homology class of each component of $E_i \subset E(\mathbb{R})$ is either $(0, 0)$ or $(1, 1)$ in $H_1\big(Q(\mathbb{R}), \mathbb{Z}/2\big)$. Since C_{2d} has even degree, the homology class of $C_{2d}(\mathbb{R})$ is again either $(0, 0)$ or $(1, 1)$. Thus $E_i \cap C_{2d}(\mathbb{R})$ is always even.

Theorem 10. *Let $E \subset \mathbb{P}^3$ and $P_{4d} \subset E$ be as in Proposition 8. Then there is a (semialgebraic) open subset $[P_{4d}] \in U \subset (S^{4d}\mathbb{P}^3)(\mathbb{R})$ such that if R_{4d} is a real set of $4d$ points in $\mathbb{P}^3$ and $[R_{4d}] \in U$, then there is no connected real curve of degree $\leq 2d$ with geometrically rational irreducible components that contains R_{4d}.*

Proof. Note that $4d$-element subsets of $\mathbb{P}^3$ are parametrized by the points of the symmetric power $S^{4d}\mathbb{P}^3$ and real $4d$-element subsets correspond to the real points of the symmetric power $(S^{4d}\mathbb{P}^3)(\mathbb{R})$ (which is not the same as the symmetric power of the real points $S^{4d}(\mathbb{RP}^3)$).

Let $W \subset (S^{4d}\mathbb{P}^3)(\mathbb{R})$ denote the set of points $[R_{4d}]$ such that there is a connected real curve of degree $\le 2d$ with geometrically rational components that contains R_{4d}. By (8) we know that $[P_{4d}] \notin W$. We claim that W is a closed semialgebraic subset of $(S^{4d}\mathbb{P}^3)(\mathbb{R})$. If this holds then we can take $U := (S^{4d}\mathbb{P}^3)(\mathbb{R}) \setminus W$.

The claim about W follows a standard argument. Let $\mathrm{Chow}_e(\mathbb{P}^3)$ denote the Chow variety of curves (or 1-cycles) of degree $\le e$ in $\mathbb{P}^3$ [Kol96, Sec.I.3]. Let $\mathrm{RatCycles}_e \subset \mathrm{Chow}_e(\mathbb{P}^3)$ denote the subset corresponding to curves that are connected with rational irreducible components. $\mathrm{RatCycles}_e$ is a Zariski closed subset by [Kol96, II.2.2].

Next consider $(S^{4d}\mathbb{P}^3) \times \mathrm{RatCycles}_{2d} \times \mathbb{P}^3$ with coordinate projections π_i. Let $\mathcal{U}_{4d} \subset (S^{4d}\mathbb{P}^3) \times \mathbb{P}^3$ be the universal family of $4d$-element subsets of $\mathbb{P}^3$ and $\mathcal{C}_{2d} \subset \mathrm{RatCycles}_{2d} \times \mathbb{P}^3$ the universal family of 1-cycles [Kol96, I.3.21].

Let $\mathcal{Y}_{2d} \subset (S^{4d}\mathbb{P}^3) \times \mathrm{RatCycles}_{2d}$ be the set of pairs $\big([R_{4d}], C_{2d}\big)$ satisfying $R_{4d} \subset C_{2d}$. Then $\mathcal{Y}_{2d}$ is a Zariski closed subset since it is the complement of the Zariski open subset

$$(\pi_1 \times \pi_2)\Big((\pi_1 \times \pi_3)^{-1}\big(\mathcal{U}_{2d}\big) \setminus (\pi_2 \times \pi_3)^{-1}\big(\mathcal{C}_{4d}\big)\Big).$$

Finally $W = \pi_1\big(\mathcal{Y}_{2d}(\mathbb{R})\big) \subset (S^{4d}\mathbb{P}^3)(\mathbb{R})$ is a closed semialgebraic subset by the Tarski–Seidenberg theorem; cf. [BCR98, 2.2.1]. □

4. Other Examples with Linear Constraints

We get more examples without real solutions by studying what happens when we choose the linear constraints to lie on a quadric hypersurface.

11 (Linear subspaces on quadric hypersurfaces). [HP47, Book IV, Sec.XIII.4]

Let $Q^{2n} \subset \mathbb{P}^{2n+1}$ be a smooth quadric hypersurface over $\mathbb{C}$. It contains 2 families of n-dimensional linear subspaces and two subspaces L_1^n, L_2^n belong to the same family iff

$$\dim\big(L_1^n \cap L_2^n\big) \equiv n \mod 2$$

(where the empty set has dimension -1). If n is odd then two general linear subspaces in the same family are disjoint from each other.

We are especially interested in the "empty" real quadric

$$Q_E^{2n} := (x_0^2 + \cdots + x_{2n+1}^2 = 0) \subset \mathbb{P}^{2n+1}, \tag{11.1}$$

which contains the conjugate pair of n-dimensional linear subspaces

$$L_\pm := (x_0 \pm \sqrt{-1}x_1 = \cdots = x_{2n} \pm \sqrt{-1}x_{2n+1} = 0)$$

which are disjoint from each other. Thus, if n is odd then they are members of the same family. Therefore both families are defined over $\mathbb{R}$. If n is even, then $L_\pm$ are members of different families, hence the two families are conjugate.

Example 12 (Lines in $\mathbb{P}^{4n-1}$). For every $n \geq 1$ there are generic configurations

$$L_1^{2n-1}, \bar{L}_1^{2n-1}, L_2^{2n-1}, \bar{L}_2^{2n-1} \subset \mathbb{P}^{4n-1}$$

such that no real line intersects all 4 subspaces.

Proof. Let $Q_E^{4n-2} \subset \mathbb{P}^{4n-1}$ be the empty quadric (11.1). Take first a special configuration where $L_1^{2n-1}, \bar{L}_1^{2n-1}, L_2^{2n-1}, \bar{L}_2^{2n-1} \subset Q_E^{4n-2}$ are disjoint members of the same family.

We claim that there is no real line L that intersects all 4 subspaces. Indeed, any line that intersects all 4 subspaces has 4 points in common with the quadric Q_E^{4n-2}, thus it is contained in Q_E^{4n-2}. However, Q_E^{4n-2} has no real points hence the line can not be real.

Since a limit of real lines is a real line, any small perturbation of the configuration has the required property. □

Remark 13 (Lines in $\mathbb{P}^{4n+1}$). It has been known classically that in general there are $r+1$ complex lines intersecting 4 linear subspaces $L_1^r, \dots, L_4^r \subset \mathbb{P}^{2r+1}$. Thus if $r = 2n$ is even and $L_1^r, \dots, L_4^r$ is a real configuration then there is at least 1 real line meeting all 4 subspaces.

Example 14 (Lines in $\mathbb{P}^3$). Another interesting degenerate situation is given by taking the lines to lie on a cubic surface in $\mathbb{P}^3$.

Take a real set of 4 points $p_1, \dots, p_4$ in $\mathbb{P}^2$ plus two more points p_5, p_6. By blowing up the 6 points, we get a cubic surface $S_3 \subset \mathbb{P}^3$; the first 4 points give 4 lines $L_1, \ldots, L_4 \subset S_3$. Any line meeting these 4 meets the cubic in 4 points, thus it is contained in it.

Thus the 2 lines meeting $L_1, \ldots, L_4$ are obtained as the birational transforms of the conics through the points $p_1, \ldots, p_4, p_5$ resp. $p_1, \ldots, p_4, p_6$. If p_5, p_6 are real, we get real lines. If they are complex conjugates, we get complex conjugate lines.

Note that, unlike in (12), here we can choose 4 or 2 of the lines to be real.

Example 15 (Conics in $\mathbb{P}^{4n-1}$). For every $n \geq 1$ there are generic configurations

$$L_1^{2n-1}, \bar{L}_1^{2n-1}, L_2^{2n-1}, \bar{L}_2^{2n-1}, L_3^{2n-1}, \bar{L}_3^{2n-1} \subset \mathbb{P}^{4n-1}$$

such that no real conic with real points intersects all 6 subspaces. (Note that these constraints define a 2-dimensional moduli space.)

Proof. Take first a special configuration where all 6 subspaces are disjoint members of the same family on $Q_E^{4n-2} \subset \mathbb{P}^{4n-1}$.

A conic C that intersects all 6 subspaces has 6 points in common with the quadric Q_E^{4n-2}. If C is irreducible (over $\mathbb{R}$) then it is contained in Q_E^{4n-2}. Since Q_E^{4n-2} has no real points, C is an empty conic.

If C is reducible (over $\mathbb{R}$) then its irreducible components are real lines and at least one of them must be contained in Q_E^{4n-2}, which is impossible.

Since a limit of real conics with real points is a real conic with real points, any small perturbation of the configuration has the required property. □

We can add either a pair of real subspaces L_4^{4n-3}, L_5^{4n-3} or a single real subspace L^{4n-4} to the constraints in (15) to get a vanishing GWW-invariant, but the example does not show what happens if we add a conjugate pair of subspaces $L_4^{4n-3}, \bar{L}_4^{4n-3}$. In $\mathbb{P}^3$ a different example excludes all real conics.

Example 16 (Conics in $\mathbb{P}^3$). The space of conics in $\mathbb{P}^3$ has dimension 8. Thus, working with conjugate pairs of linear constraints, we get a GWW-invariant in the following cases

(1) $p_1, \bar{p}_1, p_2, \bar{p}_2$,

(2) $p_1, \bar{p}_1, L_1, \bar{L}_1, L_2, \bar{L}_2$,

(3) $L_1, \bar{L}_1, \dots, L_4, \bar{L}_4$,

Every conic lies in a unique plane and 4 general points of $\mathbb{P}^3$ do not lie in a plane. Thus there are no conics through 4 general pints.

In the remaining 2 cases there are always complex conics, but we claim that there are generic configurations such that no real conic intersects all of the constraints.

Case 2. $p_1, \bar{p}_1, L_1, \bar{L}_1, L_2, \bar{L}_2 \subset \mathbb{P}^3$.

We start with the construction over $\mathbb{C}$. Two points p_1, p_2 determine a line and projecting from it gives $\pi_1 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$. Given two points p_1, p_2 and two lines L_1, L_2, there is a 1-dimensional family of quadrics passing through them; this gives $\pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$. The product of these gives a map

$$\pi := \pi_1 \times \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

The fibers of π are the conics that pass through p_1, p_2 and intersect L_1, L_2.

Given any other line L_3, it intersects the fibers of π_1 (which are planes) in 1 point and the fibers of π_2 (which are quadrics) in 2 points. Thus $\pi(L) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a curve of bidegree $(2,1)$. Two curves of bidegree $(2,1)$ intersect in 4 points, giving 4 conics that pass through 2 points and intersect 4 lines.

If $p_1 \cup p_2$ and $L_1 \cup L_2$ are real, then π is defined over $\mathbb{R}$. In $\mathbb{P}^1 \times \mathbb{P}^1$ it is easy to write down examples of two curves of bidegree $(2,1)$ (real or conjugate pairs) that have no real intersections points.

Case 3. $L_1, \bar{L}_1, \dots, L_4, \bar{L}_4 \subset \mathbb{P}^3$.

Again we start with the construction over $\mathbb{C}$. A line ℓ in $\mathbb{P}^3$ determines a projection $\pi(\ell) : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$. If $C \subset \mathbb{P}^3$ is a smooth conic then $\pi(\ell)|_C : C \to \mathbb{P}^1$ has degree 2 if C is disjoint from ℓ and degree 1 or 0 if C intersects ℓ.

Using this for a pair of disjoint lines $\ell_1, \ell_2 \subset \mathbb{P}^3$ we get a map

$$\pi := \pi(\ell_1) \times \pi(\ell_2) : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

The fibers of π are lines that connect a point of ℓ_1 to a point of ℓ_2. Thinking of $\mathbb{P}^1 \times \mathbb{P}^1$ as a quadric surface in $\mathbb{P}^3$, the resulting map

$$\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$

is given by the quadratic forms on $\mathbb{P}^3$ that vanish on both lines.

Choose 8 lines $L_1, \dots, L_8$ to be fibers of π in general position. Then a conic C that intersects all 8 lines corresponds to a rational curve of bidegree $(2,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ passing through 8 general points. However, the space of rational curves of bidegree $(2,2)$ has dimension 7, thus there are no such curves through 8 general points. Thus we conclude that if a degree 2 curve $C \subset \mathbb{P}^3$ meets all 8 lines then either ℓ_1 or ℓ_2 is an irreducible component of C.

To get an example over $\mathbb{R}$, use the above construction for a conjugate pair of disjoint lines $\ell, \bar{\ell} \subset \mathbb{P}^3$. Choose linear forms α, β such that $\ell = (\alpha = \beta = 0)$. Then $\bar{\ell} = (\bar{\alpha} = \bar{\beta} = 0)$ and the space of quadratic forms that vanish on both $\ell, \bar{\ell}$ is spanned by $\alpha\bar{\alpha}, \beta\bar{\beta}, \alpha\bar{\beta}, \beta\bar{\alpha}$. These satisfy the obvious equation

$$(\alpha\bar{\alpha})(\beta\bar{\beta}) = (\alpha\bar{\beta})(\beta\bar{\alpha}).$$

To get a real basis, we change to

$$\langle \alpha\bar{\alpha} + \beta\bar{\beta}, \alpha\bar{\alpha} - \beta\bar{\beta}, \alpha\bar{\beta} + \beta\bar{\alpha}, \sqrt{-1}(\alpha\bar{\beta} - \beta\bar{\alpha}).$$

These satisfy the equation

$$(\alpha\bar{\alpha} + \beta\bar{\beta})^2 - (\alpha\bar{\alpha} - \beta\bar{\beta})^2 = (\alpha\bar{\beta} + \beta\bar{\alpha})^2 + \big(\sqrt{-1}(\alpha\bar{\beta} - \beta\bar{\alpha})\big)^2.$$

Thus π can be thought of as a map

$$\pi := \pi(\ell) \times \pi(\bar{\ell}) : \mathbb{P}^3 \dashrightarrow Q,$$

where Q is isomorphic to the "sphere" $(x^2 + y^2 + z^2 = t^2)$.

Now choose $L_1, \bar{L}_1, \dots, L_4, \bar{L}_4$ to be fibers of π in general position. By the above considerations, a degree 2 curve C meets all 8 lines iff either ℓ or $\bar{\ell}$ is an irreducible component of C. The only real degree 2 curve with this property is $\ell + \bar{\ell}$. This is, however, geometrically disconnected and not a limit of conics.

Example 17 (Cubics in $\mathbb{P}^3$). The space of rational cubics in $\mathbb{P}^3$ has dimension 12. Thus, working with conjugate pairs of linear constraints, we get a GWW-invariant in the following cases

(1) $p_1, \bar{p}_1, p_2, \bar{p}_2, p_3, \bar{p}_3,$

(2) $p_1, \bar{p}_1, p_2, \bar{p}_2, L_1, \bar{L}_1, L_2, \bar{L}_2$,

(3) $p_1, \bar{p}_1, L_1, \bar{L}_1, \dots, L_4, \bar{L}_4$,

(4) $L_1, \bar{L}_1, \dots, L_6, \bar{L}_6$.

It has been classically known that there is a unique rational normal curve through 6 general points in $\mathbb{P}^3$. This curve is real whenever the 6 points form a real set.

We claim that in the remaining cases there are generic configurations such that no degree 3 rational curve defined over $\mathbb{R}$ intersects all of the constraints.

Again we work on the empty quadric $Q^2_E := (x_0^2 + \cdots + x_3^2 = 0) \subset \mathbb{P}^3$ and choose special configurations as follows.

(2') We choose $L_1, \bar{L}_1, L_2, \bar{L}_2$ to be disjoint members in one family of lines and the points in general position on Q^2_E.

(3') We choose $L_1, \bar{L}_1, L_2, \bar{L}_2, L_3, \bar{L}_3$ to be disjoint members in one family of lines, $L_4, \bar{L}_4$ are chosen from the other family and the points in general position on Q^2_E.

(4') We choose $L_1, \bar{L}_1, \dots, L_4, \bar{L}_4$ to be disjoint members in one family of lines and the remaining lines to be disjoint members of the other family.

In all of these cases, the following 2 properties hold

(a) there are at least 8 disjoint constraints and

(b) for both coordinate projections $\pi_i : Q^2_E \to Q^1_E$ at least 4 fibers contain a constraint.

Let B be a real cubic curve that meets all the constraints. By (a), B and Q^2_E have at least 8 points in common, thus at least 1 of the irreducible components of B is contained in Q^2_E. Since Q^2_E does not contain odd degree real curves, B decomposes as $C + L$ where C is a degree 2 curve contained in Q^2_E and L is a real line.

There are 2 possibilities for C.

(i) C is a smooth conic. Note that $C+L$ can not be written as the image of a geometrically connected real curve of arithmetic genus 0 since we would need to resolve 1 of the nodes, but they form a conjugate pair. Thus $C+L$ can not be obtained as a limit of real cubics of geometric genus 0.

(ii) C is a conjugate pair of lines. Then $Q^2_E \cap (C+L) = C$ but (b) shows that C can not meet all the constraints.

Thus, after a general perturbation we get no real curves.

The following generalization of (17.2) was pointed out by Zinger.

Example 18 (Odd degree rational curves in $\mathbb{P}^3$). For every odd $d \geq 1$ there are generic configurations

$$p_1, \bar{p}_1, \dots, p_{d-1}, \bar{p}_{d-1}, L_1, \bar{L}_1, L_2, \bar{L}_2 \subset \mathbb{P}^3$$

such that no degree d rational curve defined over $\mathbb{R}$ intersects all of the constraints.

Set $P_{2d-2} := \{p_1, \bar{p}_1, \dots, p_{d-1}, \bar{p}_{d-1}\}$. As in (17.2') we choose $L_1, \bar{L}_1, L_2, \bar{L}_2 \subset Q^2_E$ to be disjoint members in one family of lines and the points in general position on Q^2_E. Let C_d be a real curve of degree d with geometrically rational irreducible components that intersects all of the constraints.

Note that C_d and Q^2_E have at least $(2d-2)+4$ points in common, thus at least 1 irreducible component of C_d is contained in Q^2_E. Every real curve contained in Q^2_E has even degree, thus C_d can not be contained in Q^2_E. We can thus write $C_d = C_{2e} + C_{d-2e}$ where C_{2e} is contained in Q^2_E and none of the irreducible components of C_{d-2e} is contained in Q^2_E. The subscripts indicate the degree.

For general choice of the points, C_{2e} can pass through at most a $4e-2$ element subset P_{4e-2} of P_{2d-2} and C_{d-2e} can pass through at most a $2d-4e$ element subset P_{2d-4e} of P_{2d-2}. Thus P_{2d-2} is a disjoint union $P_{4e-2} \cup P_{2d-4e}$. Note further that $C_{d-2e} \cap Q^2_E = P_{2d-4e}$ which is disjoint from C_{2e}. Thus $C_d = C_{2e} + C_{d-2e}$ is disconnected and it is not a limit of geometrically connected real curves.

[Wel07] shows that the GWW-invariants for curves of any degree in $\mathbb{P}^2$ give the optimal value if the constraints (with at most 1 exception) lie near an empty conic. The results of [Wel07] apply to many other surfaces as well. The next examples illustrate this approach by considering the two known vanishing GWW-invariants for curves in $\mathbb{P}^2$; see the tables in [ABLdM11].

Example 19 (Cubics in $\mathbb{P}^2$). There are generic configurations $p_1, \bar{p}_1, \dots, p_4, \bar{p}_4 \in \mathbb{P}^2$ such that no degree 3 rational curve defined over $\mathbb{R}$ passes through all 8 points.

PROOF. First choose all 8 points on the empty conic $Q^1_E := (x^2+y^2+z^2=0)$. Any cubic that contains the 8 points is of the form $Q^1_E + L$ where L is a line. This leads to a contradiction as in (17). □

Example 20 (Quartics in $\mathbb{P}^2$). There are generic configurations $p_1, \bar{p}_1, \dots, p_5, \bar{p}_5 \in \mathbb{P}^2$ such that there is no degree 4 map $\mathbb{P}^1 \to \mathbb{P}^2$ defined over $\mathbb{R}$ whose image passes through all 10 points. (Note that these constraints define a 1-dimensional moduli space.)

PROOF. First choose all 10 points on the empty conic $Q^1_E := (x^2 + y^2 + z^2 = 0)$. Then any quartic that contains the 10 points is of the form $Q^1_E + Q'$ where Q' is a conic.

A quick case analysis shows that $Q^1_E + Q'$ can be written as the image of a real curve of arithmetic genus 0 only when Q' is a conjugate pair of lines $L + \bar{L}$. (In this case we can remove the singular point of Q' and one each of the intersections $Q^1_E \cap L, Q^1_E \cap \bar{L}$.)

Thus $Q^1_E + Q'$ can only be obtained as a limit of real quartics of geometric genus 0 if their normalization has no real points. □

REMARK 21. A degeneration argument as in (20) fails to work for higher degree curves. The curve $2Q^1_E + L$ is the image of a genus 0 curve consisting of L and a conjugate pair of complex conics, both mapping isomorphically to Q^1_E.

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