

Explicit Examples of Rational and Calabi-Yau Threefolds with Primitive Automorphisms of Positive Entropy

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To Late Professor Kunihiko Kodaira

Abstract. We present the first explicit examples of a rational threefold and a Calabi-Yau threefold, admitting biregular automorphisms of positive entropy not preserving any dominant rational maps to lower positive dimensional varieties. These examples are also the first whose dynamical degrees are not Salem numbers. Crucial parts are the rationality of the quotient threefold of a certain 3-dimensional torus of product type and a numerical criterion of primitivity of birational automorphisms in terms of dynamical degrees.

1. Introduction

Unless stated otherwise, we work in the category of projective varieties defined over the complex number field \mathbf{C} . Our main result is Theorem 1.4 below. We first introduce some background for the study in this paper.

1.1. General Problem

Complex dynamics of biholomorphic automorphisms of compact Kähler surfaces is now fairly well-understood since Cantat [Ca99]. Especially, Bedford-Kim ([BK09], [BK12], [BK10]) and McMullen ([Mc02], [Mc07], [Mc11], [Mc13]) show very beautiful aspects of automorphisms of rational surfaces and K3 surfaces. A general theory for the dynamics of biholomorphic automorphisms of higher dimensions were developed by Dinh and

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Sibony [DS05-2] [DS10], however interesting examples are not yet found enough. Indeed, the following basic problem is completely open:

Problem 1.1. Find (many) examples of rational manifolds and Calabi-Yau manifolds admitting *primitive* biregular automorphisms of *positive entropy*.

REMARK 1.2. (1) Here a dominant selfmap f of a manifold M is *imprimitive* if f comes from lower dimensional manifolds, or more precisely, there are a dominant rational map $\varphi : M \cdots \rightarrow B$ with $0 < \dim B < \dim M$ and a dominant rational map $g : B \cdots \rightarrow B$ such that $\varphi \circ f = g \circ \varphi$. A dominant selfmap that is not imprimitive is *primitive*. This notion was introduced by Zhang [Zh09]. In [Zh09], it is also proved that if a projective threefold M admits a primitive birational automorphism of infinite order, then M is birationally equivalent to either a 3-dimensional complex torus, a weak Calabi-Yau threefold or a rationally connected threefold. Here a minimal threefold in the sense of the minimal model theory is called a weak Calabi-Yau threefold if the canonical divisor is numerically trivial and the irregularity is 0. Needless to say, rational threefolds (resp. smooth Calabi-Yau threefolds) are the most basic examples of rationally connected threefolds (resp. weak Calabi-Yau threefolds).

(2) *Entropy* is an important invariant that measures how fast two general points spread out under the action of $\langle f \rangle$. The original definition is a completely topological one, involving a metric $d(x, y)$ on the manifold and the induced metrics $d_n(x, y) = \max_{0 \leq j \leq n} d(f^j x, f^j y)$ (See e.g. Bowen [Bo73] for the case of continuous maps, and for the case of rational maps see Friedland [Fr91] and Guedj [Gu05].) When f is a regular endomorphism of a compact Kähler manifold, a fundamental theorem of Gromov [Gr03] and Yomdin [Yo87] says that the entropy of f is

$$h_{top}(f) = \log \max_{0 \leq k \leq \dim M} \lambda_k(f) ,$$

where $\lambda_k(f)$ is the k -th *dynamical degree*, i.e., the spectral radius of $f^*|H^{2k}(M, \mathbf{Z})$. In particular, if f is of positive entropy, then f is of infinite order. The theorem of Gromov and Yomdin was partially extended to the case of meromorphic maps in Dinh-Sibony [DS05-1] [DS04] (See also Section 3).

(3) If an automorphism of a compact Kähler manifold is imprimitive, then there are also some strong constraints on its dynamical degrees (Theorem 4.1). This is an important point of our construction of primitive automorphisms. For example, in dimension 3, its first and second dynamical degrees must be the same for imprimitive ones. In our construction in Theorem 1.4 (see also Proposition 4.4), the automorphisms have *different first and second dynamical degrees*, therefore it follows that they are primitive. Our theorem 4.1 is a simple but useful consequence of recent progress in complex dynamics ([DN11] and [DNT11]), which we shall review in Section 3. We also note that in dimension 3, if the first dynamical degrees are either 1 or a Salem number, then the first and second dynamical degrees of these examples must be the same ([OT13]).

The main aim of this note is to present first explicit examples of a rational threefold and a Calabi-Yau threefold in Problem 1.1.

1.2. Constructing rational threefolds with rich automorphisms

One of the essential parts of Theorem 1.4 is the rationality of the obtained threefold with rich automorphisms. To explain this, we recall two basic constructions of manifolds from a given manifold V :

(I) Take a finite successive blow up M of V along smooth centers.

(II) Take a nice resolution M of singularities of the quotient variety V/G by a finite subgroup $G \subset \text{Aut}(V)$.

In construction (I), it is clear that M is rational if $V = \mathbf{P}^3$ or $\mathbf{P}^2 \times \mathbf{P}^1$ or $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, and in this way, there are constructed some explicit, interesting examples of *birational* automorphisms of rational threefolds of infinite orders (Bedford-Kim [BK11][BK13], Perroni-Zhang [PZ11], Blanc [Bl12]). However, they are either only pseudo-automorphisms (i.e., birational self-maps isomorphic in codimension one) but not biregular, or imprimitive, or of null-entropy. Moreover, in all of these examples the first dynamical degrees are either 1 or a Salem number, and hence the first and second dynamical degrees of these examples must be the same (compare with Remark 1.2 (3)).

In fact, in dimension ≥ 3 , it is rather hard to construct *biregular* automorphisms following construction (I). For instance, the following very simple question by Professor Eric Bedford (at a conference in Paris in 2011), which was studied by the second author in [Tr12-1], still remains open:

QUESTION 1.3. Is there a smooth rational threefold W obtained by a successive blow-up of \mathbf{P}^3 along smooth centers such that W admits biregular automorphisms of positive entropy?

The results in [Tr12-1] suggest that the answer to Question 1.3 could be negative. So, at the moment, construction (I) may not be so promising to Problem 1.1.

In the second construction (II), M has many biregular automorphisms if V has many biregular automorphisms normalizing G . As we shall do in one specific case, we may have more chances to find primitive automorphisms of positive entropy among them, by using the method explained in Remark 1.2 (3). However, the rationality of V/G is highly non-trivial in this construction (even if V itself is rational!). There are now many general methods to see if M is uniruled, rationally connected or not ([MM86], [KMM92], [KL09]), and some useful methods to conclude M is *not* rational ([IM71], [CG72], [AM72], [Ko95], [Be12]). However, in dimension ≥ 3 , there is essentially no general method to conclude M is rational or unirational, and this is in general very hard ([Ko02], see also the excellent survey [Ko01]).

1.3. Main Result

Let

$$\omega := \frac{-1 + \sqrt{-3}}{2}$$

and

$$E_\omega := \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\omega)$$

be the elliptic curve of period ω . The elliptic curve E_ω is a very special one that is characterized as the unique elliptic curve admitting an automorphism c of order 3 such that $c^*\sigma = \omega\sigma$. Here σ is a non-zero holomorphic 1-form on the elliptic curve.

Let X be the canonical resolution of the quotient threefold

$$E_\omega \times E_\omega \times E_\omega / \langle \text{diag}(\omega, \omega, \omega) \rangle ,$$

i.e., the blow-up at the singular points of type $1/3(1, 1, 1)$ and Y be the canonical resolution of the quotient threefold

$$E_\omega \times E_\omega \times E_\omega / \langle \text{diag}(-\omega, -\omega, -\omega) \rangle ,$$

i.e., the blow-up at the singular points of type $1/2(1, 1, 1)$, $1/3(1, 1, 1)$, $1/6(1, 1, 1)$.

Y is then the canonical resolution of X/ι , where ι is the involution naturally induced by $\text{diag}(-1, -1, -1)$. It is well-known that X is a Calabi-Yau threefold with very special properties, e.g., it is rigid and plays a crucial role in the classification of fiber space structures on Calabi-Yau threefolds ([Be82], [OS01] and references therein). Our theorem shows that X has also a rich structure in the complex dynamical view. We remark that also for Calabi-Yau manifolds, it seems much harder to construct biregular automorphisms compared with birational selfmaps (See for instance [Og12], [CO12]). On the other hand, our new manifold Y seems so far caught no attention.

Now we state the main result:

THEOREM 1.4. (1) Y is rational, i.e., birationally equivalent to \mathbf{P}^3 .

(2) Both X and Y admit primitive biregular automorphisms of positive entropy whose first and second dynamical degrees are not the same (hence they are not Salem numbers).

Our X and Y are the first explicit examples of Calabi-Yau threefolds and rational threefolds which admit biregular, primitive automorphisms of positive entropy. The condition that the first and second dynamical degrees are not the same makes the results in [DS05-2][DS10] and Dinh-de Thelin [DT12] applicable to these examples, therefore showing that they have good dynamical properties. As we explained in the last subsection, the crucial point of the proof is the rationality of Y . Our method is quite elementary but tricky. We find a birational model of Y , which is an affine quintic hypersurface of a very simple form (Theorem 2.5), admitting a conic bundle structure with a rational section over the affine space \mathbf{C}^2 .

Our construction is originally motivated by the following closely related question asked by Kenji Ueno ([Ue75, Page 199]) in 1975 and asked again by Frédéric Campana ([Ca12]) in 2012:

QUESTION 1.5. Let $E_{\sqrt{-1}} := \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\sqrt{-1})$ be the elliptic curve of period $\sqrt{-1}$. Let Z be the canonical resolution ([Ue75, Page 199]) of the quotient threefold

$$E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}} / \langle \text{diag}(\sqrt{-1}, \sqrt{-1}, \sqrt{-1}) \rangle ,$$

i.e., the blow up at the singular points of type $1/2(1, 1, 1)$ and $1/4(1, 1, 1)$. Is this Z rational or unirational?

REMARK 1.6. (1) Campana himself showed that Z is rationally connected ([Ca12]).

(2) Using a method which is similar to, but slightly more involved than, that in this paper, in a joint work with Fabrizio Catanese, we showed that Z is unirational. This is done by finding an explicit affine quintic hypersurface, say Z' , birationally equivalent to Z ([COT13], September 2013).

(3) Using Brauer's group theory and the explicit equation of Z' in [COT13], Colliot-Thélène finally proved that Z' rational, whence so is Z ([CTh13], October 2013).

(4) However, since $E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}} / \langle \text{diag}(-1, -1, -1) \rangle$ is already a minimal threefold with 64 singular points of type $1/2(1, 1, 1)$, we can *not* construct a smooth Calabi-Yau threefold from $E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}}$ as in Theorem 1.4.

The remaining of this paper is organized as follows. In Section 2 we show that Y is rational. In Section 3 we will recall sufficient details on dynamical degrees and relative dynamical degrees for use in Section 4. In Section 4, we first prove our numerical criterion of primitivity of birational automorphisms (Theorem 4.1) by using the results in Section 3. Then we construct primitive automorphisms of positive entropy of both X and Y in a uniform manner.

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2. Rationality of Y

In this section, we shall show that Y in Theorem 1.4 is rational.

LEMMA 2.1. E_ω is the projective non-singular model, say \tilde{C} , of the affine curve

$$C : y^2 = x^3 - 1 .$$

Moreover, the complex multiplication $-\omega$ on E_ω is the extension \tilde{g} of the automorphism of C given by

$$g^*(x, y) \mapsto (\omega x, -y) .$$

PROOF. The projective curve $y^2t = x^3 - t^3$ is a smooth cubic in \mathbf{P}^2 with homogeneous coordinates $[t : x : y]$, hence it defines an elliptic curve \tilde{C} which is the compactification of C in \mathbf{P}^2 . The rational 1-form dx/y on C defines a regular 1-form σ of \tilde{C} . Since $g^*dx/y = -\omega dx/y$ and the point $O = [0 : 0 : 1]$ is a fixed point, the unique extension \tilde{g} of g defines an automorphism of \tilde{C} such that $\tilde{g}^*\sigma = -\omega\sigma$ with a fixed point O . Thus, identifying the origin of E_ω with the point O , we obtain $(E_\omega, -\omega) \simeq (\tilde{C}, \tilde{g})$. \square

Let $k \in \{1, 2, 3\}$ and let \mathbf{C}_k^2 be the affine plane with coordinates (x_k, y_k) , and

$$C_k : y_k^2 = x_k^3 - 1 , \quad g_k : (x_k, y_k) \mapsto (\omega x_k, -y_k) ,$$

$$V := C_1 \times C_2 \times C_3 , \quad g = g_1 \times g_2 \times g_3 .$$

By Lemma 2.1, our Y is birationally equivalent to the affine threefold $W := V/\langle g \rangle$.

LEMMA 2.2. *The affine coordinate ring $\mathbf{C}[W]$ of W is the subring*

$$R := \mathbf{C}[y_1^2, y_2^2, y_3^2, y_1y_2, y_2y_3, y_3y_1, x_1^i x_2^j x_3^k \ (0 \leq i, j, k \leq 2, i + j + k = 3)] ,$$

of the quotient ring

$$\mathbf{C}[y_1, y_2, y_3, x_1, x_2, x_3]/(y_1^2 - x_1^3 + 1, y_2^2 - x_2^3 + 1, y_3^2 - x_3^3 + 1) .$$

PROOF. Since $V = C_1 \times C_2 \times C_3$, the affine coordinate ring of V is $\mathbf{C}[y_1, x_1]/(y_1^2 - x_1^3 + 1) \otimes_{\mathbf{C}} \mathbf{C}[y_2, x_2]/(y_2^2 - x_2^3 + 1) \otimes_{\mathbf{C}} \mathbf{C}[y_3, x_3]/(y_3^2 - x_3^3 + 1)$.

This ring is naturally isomorphic to

$$\tilde{R} := \mathbf{C}[y_1, y_2, y_3, x_1, x_2, x_3]/(y_1^2 - x_1^3 + 1, y_2^2 - x_2^3 + 1, y_3^2 - x_3^3 + 1) .$$

Note that the subring $\mathbf{C}[y_k]$ of $\mathbf{C}[y_k, x_k]/(y_k^2 - x_k^3 + 1)$ is isomorphic to the polynomial ring of one variable and $\mathbf{C}[y_k, x_k]/(y_k^2 - x_k^3 + 1)$ is a free $\mathbf{C}[y_k]$ -module with free basis $\langle 1, x_k, x_k^2 \rangle$. Therefore the subring $\mathbf{C}[y_1, y_2, y_3]$ of \tilde{R} is isomorphic to the polynomial ring of the three variables and \tilde{R} is also a free $\mathbf{C}[y_1, y_2, y_3]$ -module with the free basis

$$x_1^i x_2^j x_3^k, \quad i, j, k \in \{0, 1, 2\} .$$

In particular, each element of \tilde{R} is uniquely expressed in the following form:

$$f := \sum_{(i,j,k) \in \{0,1,2\}^3} a_{ijk} x_1^i x_2^j x_3^k$$

where $a_{ijk} = a_{ijk}(y_1, y_2, y_3)$ are polynomials of y_1, y_2, y_3 . The action of g on \tilde{R} is given by $gy_k = -y_k$ and $gx_k = \omega x_k$ and the affine coordinate ring of $V/\langle g \rangle$ is isomorphic to the invariant ring \tilde{R}^g . Here and hereafter, we denote the action of g on \tilde{R} , which is g^* , simply by g .

Since $\langle g \rangle = \langle g^2, g^3 \rangle$, we have

$$\tilde{R}^g = (\tilde{R}^{g^2})^{g^3} .$$

Note that $g^2 y_m = y_m$ and $g^2 x_m = \omega^2 x_m$ ($m = 1, 2, 3$). Hence, any polynomial $a_{ijk}(y_1, y_2, y_3)$ are g^2 -invariants and therefore $f \in R = \tilde{R}^{g^2}$ if and only if $x_1^i x_2^j x_3^k$ are all g^2 -invariant (for $a_{i,j,k} \neq 0$). That is, $i + j + k$ is divisible by 3:

$$3 | (i + j + k) .$$

Note then that $i + j + k = 0, 3$ and 6 and the term of $i + j + k = 6$ is only $(x_1 x_2 x_3)^2$ by $0 \leq i, j, k \leq 2$. Hence

$$\tilde{R}^{g^2} = \mathbf{C}[y_1, y_2, y_3, x_1^i x_2^j x_3^k \ (0 \leq i, j, k \leq 2, \ i + j + k = 3)] .$$

Since $g^3 x_m = x_m$ and $g^3 y_m = -y_m$ ($m = 1, 2, 3$), it follows that any polynomial in x_1, x_2, x_3 is g^3 -invariant. Hence f is g^3 -invariant if and only if for any monomial $y_1^{i_1} y_2^{i_2} y_3^{i_3}$ appearing in a_{ijk} we have $i_1 + i_2 + i_3 =$ an even number. Therefore

$$(\tilde{R}^{g^2})^{g^3} = \mathbf{C}[y_m y_n \ (1 \leq m \leq n \leq 3), \ x_1^i x_2^j x_3^k \ (0 \leq i, j, k \leq 2, \ i + j + k = 3)] ,$$

as claimed. \square

Let $Q(\tilde{R})$ be the quotient field of \tilde{R} .

LEMMA 2.3. *The rational function field $\mathbf{C}(Y)$ of Y , hence the function field of W , is isomorphic to the following subfield of $Q(\tilde{R})$:*

$$\mathbf{C}\left(y_1^2, \frac{y_2}{y_1}, \frac{y_3}{y_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}\right).$$

PROOF. It is clear that the field above, say K , is a subfield of $\mathbf{C}(W)$ (e.g. $y_2/y_1 = (y_2^2)/(y_1y_2)$ and $x_2/x_1 = (x_2^2x_1)/(x_1^2x_2)$). It suffices to show that each generator of $\mathbf{C}[W]$ in Lemma 2.2 is in K . For y_my_n , we have

$$y_my_n = y_1^2 \frac{y_m}{y_1} \frac{y_n}{y_1} \in K.$$

For $x_1^i x_2^j x_3^k$ with $i + j + k = 3$, we have

$$x_1^i x_2^j x_3^k = \left(\frac{x_2}{x_1}\right)^i \left(\frac{x_3}{x_1}\right)^j (x_1)^3 = \left(\frac{x_2}{x_1}\right)^i \left(\frac{x_3}{x_1}\right)^j (y_1^2 + 1) \in K.$$

Hence the result follows. \square

LEMMA 2.4. *The rational function field $\mathbf{C}(W)$, hence the rational function field $\mathbf{C}(Y)$ of Y , is isomorphic to the following subfield of $Q(\tilde{R})$:*

$$\mathbf{C}(t, s, z, w),$$

with a single equation

$$(w^3 - 1)(t^2 - 1) = (z^3 - 1)(s^2 - 1).$$

More precisely, the subfield $\mathbf{C}(s, z, w)$ of $Q(\tilde{R})$ is isomorphic to the purely transcendental extension of \mathbf{C} of transcendental degree 3, and the field $\mathbf{C}(Y)$ is isomorphic to the quotient ring

$$\mathbf{C}(s, z, w)[T]/I,$$

where I is the principal ideal generated by

$$(w^3 - 1)(T^2 - 1) - (z^3 - 1)(s^2 - 1),$$

in the polynomial ring $\mathbf{C}(s, z, w)[T]$ over $\mathbf{C}(s, z, w)$.

PROOF. Set

$$(2.1) \quad u := y_1^2, \quad t := \frac{y_2}{y_1}, \quad s := \frac{y_3}{y_1}, \quad z := \frac{x_2}{x_1}, \quad w := \frac{x_3}{x_1}.$$

By Lemma 2.3, these 5 elements are generators of $\mathbf{C}(W) = \mathbf{C}(Y)$. Then

$$(2.2) \quad y_2 = ty_1, \quad y_3 = sy_1, \quad y_2^2 = t^2u, \quad y_3^2 = s^2u.$$

Hence

$$(2.3) \quad z^3 = \frac{x_2^3}{x_1^3} = \frac{y_2^2 + 1}{y_1^2 + 1} = \frac{t^2u + 1}{u + 1}, \quad w^3 = \frac{x_3^3}{x_1^3} = \frac{y_3^2 + 1}{y_1^2 + 1} = \frac{s^2u + 1}{u + 1}.$$

Note that $w^3 - 1 \neq 0$ in $\mathbf{C}(Y)$. Indeed, otherwise, $w = x_3/x_1$ would be constant, a contradiction to the freeness of $x_1^i x_2^j x_3^k$ ($i, j, k \in \{0, 1, 2\}$) over $\mathbf{C}[y_1, y_2, y_3]$. Similarly $z^3 - 1 \neq 0$ in $\mathbf{C}(Y)$. Then, solving each equation in (2.3) in the variable u , we obtain

$$u = \frac{z^3 - 1}{t^2 - z^3}, \quad u = \frac{w^3 - 1}{s^2 - w^3}$$

that is,

$$(2.4) \quad \frac{-1}{u} = \frac{z^3 - t^2}{z^3 - 1} = 1 - \frac{t^2 - 1}{z^3 - 1}$$

and

$$(2.5) \quad \frac{-1}{u} = \frac{w^3 - ts^2}{w^3 - 1} = 1 - \frac{s^2 - 1}{w^3 - 1}.$$

Hence by (2.4) and (2.5), we find that $\mathbf{C}(W) = \mathbf{C}(t, s, z, w)$. Here z, w, s, t satisfy a relation

$$\frac{t^2 - 1}{z^3 - 1} = \frac{s^2 - 1}{w^3 - 1},$$

which is the same as

$$(2.6) \quad (w^3 - 1)(t^2 - 1) = (z^3 - 1)(s^2 - 1).$$

Since W is a 3-dimensional projective variety, it follows from (2.6) that s, z, w are a transcendental basis of $\mathbf{C}(W)$ and t is algebraic over the field $\mathbf{C}(s, z, w)$. Therefore

$$\mathbf{C}(W) = \mathbf{C}(s, z, w)(t) = \mathbf{C}(s, z, w)[t] .$$

The equation (2.6), or more precisely, the polynomial

$$(2.7) \quad (w^3 - 1)(T^2 - 1) - (z^3 - 1)(s^2 - 1) ,$$

gives an algebraic equation of t over $\mathbf{C}(s, z, w)$. Rewrite (2.7) as

$$(w^3 - 1)(T^2 - C) ,$$

where

$$C = \frac{(z^3 - 1)(s^2 - 1) + (w^3 - 1)}{w^3 - 1} .$$

Recall that $\mathbf{C}(s, z, w)$ is a purely transcendental extension of \mathbf{C} of transcendental degree 3 as we already observed. We note that in the representation of C the numerator and denominator are relatively prime to each other, and the denominator $w^3 - 1$ is clearly not a square of an element in $\mathbf{C}(s, z, w)$. Therefore, it follows that $\sqrt{C} \notin \mathbf{C}(s, z, w)$. Hence the polynomial (2.7) is irreducible in $\mathbf{C}(s, z, w)[T]$. It follows that

$$\mathbf{C}(s, z, w)[t] = \mathbf{C}(s, z, w)[T]/I$$

as claimed. \square

THEOREM 2.5. *Y is birationally equivalent to the affine hypersurface Q defined by*

$$(w^3 - 1)(t^2 - 1) = (z^3 - 1)(s^2 - 1)$$

in the affine space \mathbf{A}^4 with affine coordinates (t, s, z, w) .

PROOF. Lemma 2.4 means that the rational function field of Y is isomorphic to the rational function field of the affine hypersurface Q . This implies the result. \square

The next proposition completes the proof of Theorem 1.4 (1):

PROPOSITION 2.6. *The hypersurface Q in Theorem 2.5 is rational.*

PROOF. The restriction π_Q of the natural projection to the last two factors

$$\pi : \mathbf{A}^4 \rightarrow \mathbf{A}^2, (t, s, z, w) \mapsto (z, w)$$

gives a conic bundle structure of Q over \mathbf{A}^2 with an obvious section $(s, t) = (1, 1)$. Let η be the generic point of \mathbf{A}^2 in the sense of scheme. Then the fiber $C_\eta := \pi_Q^{-1}(\eta)$ is the conic defined by

$$(w^3 - 1)(t^2 - 1) = (z^3 - 1)(s^2 - 1)$$

in the affine plane $\pi^{-1}(\eta) = \mathbf{A}_\eta^2$ with affine coordinates (s, t) over $\mathbf{C}(\mathbf{A}^2) = \mathbf{C}(z, w)$. Moreover, the point $(s, t) = (1, 1)$ is clearly a rational point of C_η also over $\mathbf{C}(\mathbf{A}^2)$. Hence C_η is birational to the affine line \mathbf{A}_η^1 over $\mathbf{C}(\mathbf{A}^2)$. For this, one may just consider the projection from the above rational point to \mathbf{A}_η^1 also over $\mathbf{C}(\mathbf{A}^2)$. Note that the generic point $\tilde{\eta}$ of Q is lying over η . Denote the residue field of $\tilde{\eta}$ (resp. of η) by $K(\tilde{\eta})$ (resp. $K(\eta)$). Then $K(\eta) = \mathbf{C}(\mathbf{A}^2) = \mathbf{C}(z, w)$, $K(\eta) \subset K(\tilde{\eta})$ and

$$\mathbf{C}(Q) = K(\tilde{\eta}) = K(\eta)(C_\eta) \simeq K(\eta)(\mathbf{A}_\eta^1) = \mathbf{C}(z, w)(\gamma) = \mathbf{C}(z, w, \gamma),$$

where γ is the affine coordinate function of \mathbf{A}_η^1 . Since $\dim Q = 3$, the rational functions z, w and γ are algebraically independent over \mathbf{C} . Hence $\mathbf{C}(Q)$ is purely transcendental over \mathbf{C} . This implies the result. \square

REMARK 2.7. Finding a "good" explicit birational affine hypersurface as above is also crucial in both [COT13] and [CTh13]. Only for the rationality of our W , we can prove more explicitly as follows. Put

$$\gamma := (s - 1)/(t - 1)$$

in $\mathbf{C}(W)$. Then $s = \gamma(t - 1) + 1$ and from $(w^3 - 1)(t^2 - 1) = (z^3 - 1)(s^2 - 1)$, we obtain a relation

$$(2.8) \quad (w^3 - 1)(t + 1) = (z^3 - 1)\gamma(\gamma(t - 1) + 2),$$

i.e.,

$$(2.9) \quad ((w^3 - 1) - \gamma^2(z^3 - 1))t = (z^3 - 1)\gamma(2 - \gamma) - (w^3 - 1).$$

Observe that

$$(w^3 - 1) - \gamma^2(z^3 - 1) = \frac{(w^3 - 1)(t - 1)^2 - (z^3 - 1)(s - 1)^2}{(t - 1)^2} \neq 0 .$$

Indeed, otherwise, comparing with the defining equation of W , we would obtain $s = t$ in $\mathbf{C}(W)$, a contradiction. Thus, we can solve t in terms of w, z, γ from (2.9). Hence $\mathbf{C}(W) = \mathbf{C}(t, s, z, w)$ is generated by w, z, γ over \mathbf{C} . Since W has dimension 3, this implies that W is rational.

REMARK 2.8. \tilde{C} in Lemma 2.1 is the elliptic curve defined over any field K of characteristic $\neq 2, 3$ and the automorphism \tilde{g} is defined over any field K containing the primitive third root of unity ω . The argument in this section shows that $V/\langle g \rangle$ is birationally equivalent to Q and it is rational, also over any field K containing ω and of characteristic $\neq 2, 3$.

REMARK 2.9. Consider the quotient variety of dimension $n \geq 2$:

$$V_n := E_\omega^n / \langle -\omega I_n \rangle .$$

Then V_n has isolated singular points of type $1/2(1, 1, \dots, 1)$, $1/3(1, 1, \dots, 1)$ and one isolated singular point O of type $1/6(1, 1, \dots, 1)$. From this, it is easy to see that $\mathcal{O}_{V_n}(6K_{V_n}) \simeq \mathcal{O}_{V_n}$, $h^1(\mathcal{O}_{V_n}) = 0$ and the singular point O is Kawamata log terminal but not canonical when $2 \leq n \leq 5$, canonical but not terminal when $n = 6$ and terminal (but not smooth) when $n \geq 7$. Also all other singular points are terminal when $n \geq 4$. (See [KMM87], [KM98] for terminologies and basic notions of minimal model theory.) So, when $n \geq 7$ (resp. when $n = 6$), V_n (resp. the blow up of V_6 at O) are minimal but singular Calabi-Yau varieties, in the sense of minimal model theory, and therefore, they are not even uniruled. On the other hand, the Kodaira dimension of the resolution of V_n ($2 \leq n \leq 5$) is $-\infty$. So, V_2 is rational by Castelnuovo's criterion (see eg. [BHPV04, Page 252]). Our result shows that V_3 is rational. It would be interesting to see if V_4, V_5 are rational, unirational or not.

3. Topological Entropy, Dynamical Degrees and Relative Dynamical Degrees

For the convenience of the readers, in this section we briefly summarize definitions and properties of dynamical degrees which will be used later. The

readers who are familiar with these may proceed directly to the next section. Our main references are [DS05-1], [DS05-2], [DN11] and [DNT11]. *In this section, X and Y are compact Kähler manifolds and they have nothing to do with X and Y in our main theorem.*

3.1. Topological entropy

Let $f : X \rightarrow X$ be a surjective holomorphic map. Then f is in particular continuous, and by the classical ergodic theory we can define the topological entropy of f . Topological entropy is an indication of the complexity of f . The following fundamental theorem is due to Gromov-Yomdin ([Yo87], [Gr03]):

THEOREM 3.1. *Let X be a compact Kähler manifold of dimension k and $f : X \rightarrow X$ a surjective holomorphic map. Then, $\lambda_p(f) = \text{sp}_p(f)$ and*

$$h_{\text{top}}(f) = \log \max_{0 \leq p \leq k} \lambda_p(f) = \log \max_{0 \leq p \leq k} \text{sp}_p(f) = \log \text{sp}(f) .$$

Here $\text{sp}_p(f)$ (resp. $\text{sp}(f)$) is the spectral radius of $f^*|_{H^{p,p}(X)}$ (resp. of $f^*|_{\bigoplus_{p=0}^{2k} H^k(M, \mathbf{Z})}$). The p -th dynamical degree $\lambda_p(f)$ is defined by (3.1) below.

3.2. Dynamical degrees

A meromorphic map $f : X \cdots \rightarrow Y$ between two complex manifolds is a holomorphic map $f|_U : U \rightarrow Y$ from a Zariski open dense set U of X into Y so that the closure of the graph of $f|_U$ is an analytic subvariety of $X \times Y$. We say that f is dominant if $f|_U(U)$ is dense in Y . Let \tilde{X} be a resolution of the indeterminacy of f and $p_i : \tilde{X} \rightarrow X$ ($i = 1, 2$) be the natural projections. For any smooth closed (p, p) -form α , we define the pullback $f^*(\alpha) = (p_1)_* p_2^*(\alpha)$ as a current, where p_2^* is the natural pullback as forms and $(p_1)_*$ is the natural pushforward as currents. The action f^* naturally descends to a linear action on $H^{p,p}(X)$. Note however that in general we do not have the compatibility of pullback maps: $(f^n)^*$ may be different from $(f^*)^n$ on $H^{p,p}(X)$. Russakovskii and Shiffman [RS97] (the case $X = \mathbb{P}^k$) and Dinh and Sibony [DS04, DS05-1] (the case of compact Kähler manifolds) defined the p -th dynamical degree $\lambda_p(f)$ as follows

$$(3.1) \quad \lambda_p(f) = \lim_{n \rightarrow \infty} \left(\int_X (f^n)^*(\omega^p) \wedge \omega^{k-p} \right)^{\frac{1}{n}} .$$

The existence of the limit in (3.1) is a non-trivial fact, based on a good regularization of a positive closed (p, p) current. Such a regularization on compact Kähler manifolds were given in [DS04, DS05-1].

Here are some simple properties of dynamical degrees ([DS05-1]):

$$\lambda_0(f) = 1, \lambda_k(f) = \deg f,$$

where $\deg f$ is the topological degree of f , and (log-concavity)

$$\lambda_{p-1}(f)\lambda_{p+1}(f) \leq \lambda_p(f)^2.$$

In particular, $\lambda_p(f) \geq 1$ for $0 \leq p \leq k = \dim X$. Dinh-Sibony ([DS05-1]) proved the following:

THEOREM 3.2. *Let X be a compact Kähler manifold of dimension k and $f : X \rightarrow X$ be a dominant meromorphic map. Then,*

$$h_{\text{top}}(f) \leq \log \max_{0 \leq p \leq k} \lambda_p(f).$$

Moreover, the p -th dynamical degrees $\lambda_p(f)$ are bimeromorphic invariants in the sense that

$$\lambda_p(f) := \lambda_p(\mu^{-1} \circ f \circ \mu).$$

for any bimeromorphic map $\mu : X' \cdots \rightarrow X$ from a compact Kähler manifold X' .

REMARK 3.3. (1) Let X be a compact complex variety such that there is a bimeromorphic map $\pi : \tilde{X} \cdots \rightarrow X$, here \tilde{X} is a compact Kähler manifold (such a variety X is usually said to be of Fujiki's class \mathcal{C}). Given $f : X \cdots \rightarrow X$ a dominant meromorphic map and

$$\tilde{f} := \pi^{-1} \circ f \circ \pi : \tilde{X} \cdots \rightarrow \tilde{X}$$

the induced map, we define

$$\lambda_p(f) := \lambda_p(\tilde{f}).$$

This is well-defined by the above Theorem. We will use this in the proof of Theorem 1.4 (2).

(2) Using Chow's moving lemma, we can also define dynamical degrees algebraically for a rational selfmap over an arbitrary algebraic closed field. The case of characteristic zero is treated in [Tr12-2]. The general case and applications to dynamics over a non-Archimedean field will be given in an ongoing joint project of C. Favre and the second author.

3.3. Relative dynamical degrees

If a meromorphic map preserves a meromorphic fibration, then there are some relations which must be satisfied by its dynamical degrees. Specifically, let

$$f : X \cdots \rightarrow X, \quad g : Y \cdots \rightarrow Y, \quad \pi : X \cdots \rightarrow Y$$

be dominant meromorphic maps such that $\pi \circ f = g \circ \pi$, where X and Y are compact Kähler manifolds of dimensions k and l with $k \geq l$. Let ω_X be a Kähler form on X and ω_Y a Kähler form on Y .

In the case where π is *holomorphic*, the relative dynamical degrees $\lambda_p(f|\pi)$ (here $0 \leq p \leq k - l$) is defined by:

$$\lambda_p(f|\pi) = \lim_{n \rightarrow \infty} \left(\int_X (f^n)^*(\omega_X^p) \wedge \pi^*(\omega_Y^l) \wedge \omega_X^{k-l-p} \right)^{\frac{1}{n}}.$$

Here $(f^n)^*(\omega_X^p)$ is defined as a current and $\pi^*(\omega_Y^l) \wedge \omega_X^{k-p-l}$ is defined as the usual smooth form, and therefore the integration above makes sense. This definition is due to Dinh and Nguyen [DN11]. They also proved that $\lambda_p(f|\pi)$ satisfy the log-concavity property and that they are bimeromorphic invariants in the sense that

$$\lambda_p(f|\pi) = \lambda_p(\tilde{f}|\tilde{\pi})$$

for two surjective holomorphic maps between compact Kähler manifolds

$$\pi : X \rightarrow Y, \quad \tilde{\pi} : \tilde{X} \rightarrow \tilde{Y}$$

being bimeromorphic in an obvious sense. Here \tilde{f}, \tilde{g} are the meromorphic selfmaps of \tilde{X} and \tilde{Y} induced by f and g . Note also that $\lambda_0(f|\pi) = 1$ and $\lambda_{k-l}(f|\pi)$ is the topological degree of $f|_{X_t} : X_t \cdots \rightarrow X_{g(t)}$ for generic fibers X_t ($t \in Y$). The log-concavity property then implies that $\lambda_p(f|\pi) \geq 1$ for any *meaningful* p , i.e., for all integers p such that $0 \leq p \leq k - l$.

Using the bimeromorphic invariance, we can also define $\lambda_p(f|\pi)$ when π is not necessarily holomorphic: First take any resolution

$$\mu : \tilde{X} \rightarrow X$$

of the indeterminacy of π by a compact Kähler manifold \tilde{X} . Then the maps

$$\tilde{f} := \mu^{-1} \circ f \circ \mu : \tilde{X} \cdots \rightarrow \tilde{X}, \quad g : Y \cdots \rightarrow Y, \quad \tilde{\pi} := \mu \circ \pi : \tilde{X} \rightarrow Y$$

are dominant meromorphic maps such that $\tilde{\pi} \circ \tilde{f} = g \circ \tilde{\pi}$ and $\tilde{\pi}$ is holomorphic. We define the relative dynamical degree by

$$\lambda_p(f|\pi) := \lambda_p(\tilde{f}|\tilde{\pi}) .$$

Well-definedness follows from the bimeromorphic invariance.

The following result was proved in [DNT11]. Some special cases were proved in [DN11] and [NZ09].

THEOREM 3.4. *Let*

$$f : X \cdots \rightarrow X , g : Y \cdots \rightarrow Y , \pi : X \cdots \rightarrow Y$$

be dominant meromorphic maps such that $\pi \circ f = g \circ \pi$, where X and Y are compact Kähler manifolds of dimensions k and l with $k \geq l$. Then for all $0 \leq p \leq k$:

$$(3.2) \quad \lambda_p(f) = \max_{\max\{0, p-k+l\} \leq j \leq \min\{p, l\}} \lambda_j(g) \lambda_{p-j}(f|\pi).$$

REMARK 3.5. (1) Let X, Y be compact complex varieties, not necessarily smooth, which are bimeromorphically equivalent to compact Kähler manifolds \tilde{X}, \tilde{Y} . Let

$$\tilde{f} : \tilde{X} \cdots \rightarrow \tilde{X} , \tilde{g} : \tilde{Y} \cdots \rightarrow \tilde{Y} , \tilde{\pi} : \tilde{X} \cdots \rightarrow \tilde{Y}$$

be the induced maps. We have well-defined relative dynamical degrees

$$\lambda_p(f|\pi) := \lambda_p(\tilde{f}|\tilde{\pi}) .$$

(2) If $\pi : X \cdots \rightarrow Y$ is a generically finite meromorphic map, then Theorem 3.4 implies that $\lambda_p(f) = \lambda_p(g)$ for every p (note that this case was proved earlier in [DN11]).

(3) Let X be a compact Kähler manifold, $\pi : X \rightarrow Y = X/G$ the quotient by a finite group, and $f : X \rightarrow X, g : Y \rightarrow Y$ automorphisms such that $\pi \circ f = g \circ \pi$. Let $\tilde{g} : \tilde{Y} \rightarrow \tilde{Y}$ be an equivariant resolution of singularities of $g : Y \rightarrow Y$ (which exists by results of Hironaka). Then, by the previous remark and Gromov-Yomdin’s theorem, we have $h_{top}(f) = h_{top}(\tilde{g})$.

4. Primitive Automorphisms of X and Y

In this section, we prove Theorem 1.4 (2). We obtain the following strong restriction of the dynamical degrees of imprimitive rational selfmaps of projective manifolds.

THEOREM 4.1. (1) *Let M be a smooth projective manifold and f a dominant meromorphic selfmap of M . If $\lambda_1(f) > \lambda_2(f)$ then f is primitive.*

(2) *Let M be a compact Kähler manifold of dimension 3 and $f : M \cdots \rightarrow M$ a bimeromorphic map. Assume that f is imprimitive. Then $\lambda_1(f) = \lambda_2(f)$. In other words, if $\lambda_1(f) \neq \lambda_2(f)$, then f has to be primitive.*

REMARK 4.2. This kind of criterion should be known to experts in complex dynamics. However, an explicit statement of the criterion with a complete proof (which relies on recent results in complex dynamics, see Section 3) seems not be available in the literature. This theorem is crucial in our main result Theorem 1.4 (2). Part (1) of Theorem 4.1 is in particular useful when considering maps in dimensions at least 4, please see Remark 4.6 for more details. Therefore, we present here such a statement and proof together with a brief, but hopefully comprehensible, account of needed results (see Section 3).

PROOF. We first prove (1). Assume that f is imprimitive. Then there are compact Kähler manifold B , a dominant meromorphic maps $\pi : M \cdots \rightarrow B$, $g : B \cdots \rightarrow B$ such that $\pi \circ f = g \circ \pi$ and $0 < \dim B < \dim M$. Note that

$$\lambda_0(f) = \lambda_0(g) = \lambda_0(f|\pi) = 1 ,$$

by definition. Then by Theorem 3.4, we have

$$\lambda_1(f) = \max\{\lambda_1(g), \lambda_1(f|\pi)\} ,$$

$$\lambda_2(f) = \max\{\lambda_2(f|\pi), \lambda_1(g)\lambda_1(f|\pi), \lambda_2(g)\} .$$

Here the maximum is taken among the meaningful ones. By $0 < \dim B < \dim M$, both terms $\lambda_1(g)$ and $\lambda_1(f|\pi)$ are meaningful, and hence they are greater than or equal to 1 (log-concavity). Therefore

$$\lambda_2(f) \geq \lambda_1(g)\lambda_1(f|\pi) \geq \max\{\lambda_1(g), \lambda_1(f|\pi)\} = \lambda_1(f) .$$

Hence if $\lambda_1(f) > \lambda_2(f)$ then f is primitive.

Now we prove (2). Assume that f is imprimitive. Then there are a compact Kähler manifold B , dominant meromorphic maps $\pi : M \cdots \rightarrow B$, $g : B \cdots \rightarrow B$ such that $\pi \circ f = g \circ \pi$ and $0 < \dim B < 3 = \dim M$. Then as in (1), we have

$$\lambda_1(f) = \max\{\lambda_1(g), \lambda_1(f|\pi)\} ,$$

$$\lambda_2(f) = \max\{\lambda_2(f|\pi), \lambda_1(g)\lambda_1(f|\pi), \lambda_2(g)\} .$$

Here again the maximum is taken among the meaningful terms. Moreover, since f , and hence g , are bimeromorphic, it follows that

$$\lambda_3(f) = \lambda_{\dim B}(g) = 1 .$$

First consider the case where $\dim B = 1$. Then, by Theorem 3.4, we also have

$$1 = \lambda_3(f) = \lambda_1(g)\lambda_2(f|\pi) .$$

Hence

$$\lambda_1(g) = \lambda_2(f|\pi) = 1 .$$

Therefore

$$\lambda_1(f) = \max\{\lambda_1(g), \lambda_1(f|\pi)\} = \max\{\lambda_1(f|\pi), 1\} ,$$

$$\lambda_2(f) = \max\{\lambda_2(f|\pi), \lambda_1(g)\lambda_1(f|\pi)\} = \max\{\lambda_1(f|\pi), 1\} .$$

Hence $\lambda_1(f) = \lambda_2(f)$.

Next consider the case where $\dim B = 2$. Then, by Theorem 3.4, we also have

$$1 = \lambda_3(f) = \lambda_2(g)\lambda_1(f|\pi) .$$

Hence

$$\lambda_2(g) = \lambda_1(f|\pi) = 1 .$$

Therefore

$$\lambda_1(f) = \max\{\lambda_1(g), \lambda_1(f|\pi)\} = \max\{\lambda_1(g), 1\} ,$$

$$\lambda_2(f) = \max\{\lambda_1(g)\lambda_1(f|\pi), \lambda_2(g)\} = \max\{\lambda_1(g), 1\} .$$

Hence $\lambda_1(f) = \lambda_2(f)$ as well. \square

Now we are ready to prove our main theorem (2).

Set $M := E_\omega \times E_\omega \times E_\omega$. Then we have a natural embedding of groups

$$\mathrm{GL}(3, \mathbf{Z}[\omega]) \subset \mathrm{Aut}(M) .$$

Moreover, since $\pm \mathrm{diag}(\omega, \omega, \omega)$ is in the center of $\mathrm{GL}(3, \mathbf{Z}[\omega])$, it follows that any $f \in \mathrm{GL}(3, \mathbf{Z}[\omega])$ naturally descends to the *biregular* automorphisms of X and Y . The fact that they are biregular follows from the construction and the universal property of blow-up. We denote them by f_X and f_Y respectively.

LEMMA 4.3. *Let a be any positive integer. Consider the automorphism $f := f_a$ of M given by the matrix*

$$P = P_a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 3a^2 & 0 \end{pmatrix} .$$

Then $d_2(f) > d_1(f) > 1$. Moreover $d_1(f)$ is not a Salem number.

Note here that a *Salem number* is a real algebraic integer $a > 1$ with Galois conjugates $a, 1/a$ such that all other Galois conjugates are on the unit circle $S^1 \subset \mathbf{C}$.

PROOF. Note that $P \in \mathrm{SL}(3, \mathbf{Z})$ so that P gives an automorphism f of M . Thus $\lambda_p(f)$ is the spectral radius of $f^*|H^{p,p}(M)$. The characteristic polynomial $\Phi(t)$ of the matrix P is

$$\Phi(x) = x^3 - 3a^2x + 1 .$$

Observe $\Phi'(x) = 3(x - a)(x + a)$, and

$$\Phi(-a) = 2a^2 + 1 > 0, \quad \Phi(0) = 1 > 0,$$

$$\Phi(1) = 2 - 3a^2 < 0, \quad \Phi(a) = -2a^2 + 1 < 0 .$$

Hence $\Phi(x)$ has three real roots α, β, γ such that

$$\alpha < -a < 0 < \beta < 1 < a < \gamma .$$

We have also

$$\alpha + \beta + \gamma = 0$$

by the shape of $\Phi(x)$. Hence

$$0 < |\beta| < 1 < |\gamma| < |\alpha| .$$

Observe that the characteristic polynomial of $f^*|H^1(M, \mathbf{Z})$ is $\Phi(x)^2$. Hence α, β, γ are the eigenvalues of $f^*|H^1(M, \mathbf{Z})$ each of which is of multiplicity 2. Since

$$H^{2k}(M, \mathbf{Z}) = \wedge^{2k} H^1(M, \mathbf{Z}) ,$$

it now follows that $\lambda_1(f) = \alpha^2$ and $\lambda_2(f) = \alpha^2\gamma^2$, therefore $\lambda_2(f) > \lambda_1(f) > 1$.

Since f is an automorphism in dimension 3 and $\lambda_1(f) \neq \lambda_2(f)$, by the results in [OT13] we have that $\lambda_1(f)$ is not a Salem number. We can also see this directly as follows. Note that $\Phi(x)$ is irreducible over \mathbf{Q} , or equivalently, over \mathbf{Z} . Indeed, otherwise, $\Phi(x)$ would have one of ± 1 as its root, a contradiction. Hence the three roots α, β and γ are Galois conjugate over \mathbf{Q} . Thus, so are $\alpha^2, \beta^2, \gamma^2$. Since $\alpha^2 > \gamma^2 > 1$, it follows that $\lambda_1(f) = \alpha^2$ is not a Salem number. \square

The next proposition completes the proof of Theorem 1.4 (2):

PROPOSITION 4.4. *The induced automorphisms f_X and f_Y are primitive and of positive entropy for $f := f_a$ in Lemma 4.3. Moreover, their first dynamical degrees are not Salem numbers.*

PROOF. By Remark 3.5, we have

$$\lambda_p(f_X) = \lambda_p(f) = d_p(f_Y) .$$

By Lemma 4.3, we have $\lambda_2(f) > \lambda_1(f)$. Hence

$$\lambda_2(f_X) > \lambda_1(f_X) > 1 , \lambda_2(f_Y) > \lambda_1(f_Y) > 1 .$$

Thus f_X and f_Y are primitive by Theorem 4.1. They are of positive entropy by $\lambda_1(f_X) > 1$ and $\lambda_1(f_Y) > 1$. By Lemma 4.3, $\lambda_1(f_X) = \lambda_1(f_Y) = \lambda_1(f)$ is not a Salem number as well. \square

REMARK 4.5. For the same reason, f in Lemma 4.3 is also a primitive automorphism of the 3-torus M of positive entropy.

REMARK 4.6. (1) We give only one explicit example of smooth rational threefold and Calabi-Yau threefold in the strict sense, admitting primitive automorphisms of positive entropy. However, we expect that there should be many such smooth rational threefolds and Calabi-Yau threefolds. It will be interesting to find more examples.

(2) It will be also very interesting to find rational manifolds and Calabi-Yau manifolds (in both weak and strict senses), of dimension ≥ 4 , admitting primitive automorphisms of positive entropy. In finding such examples, our theorem 4.1 (1) might be useful.

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