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Jeffery-Hamel's Flows in the Plane

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Abstract. We consider a radial steady flow of an incompressible viscous fluid which either converges or diverges in a two dimensional wedge domain. We prove the existence of a solution to the stationary Navier-Stokes equations for the restricted flux condition which depends only on the angle of the wedge domain.

1. Introduction

1.1. Jeffery-Hamel's flow

We consider a two-dimensional wedge domain ω between two rays that emanate from the same initial point at the origin, and form an angle of size 2α, $\alpha \in (0, \pi)$. We also assume that the x_1-axis is the angle bisector. Let Σ_L^α denote the intersection of ω with a circle of radius L centered at the origin.

$$\omega = \{(r, \theta) \in \mathbb{R}^2; r > 0, -\alpha < \theta < \alpha\},$$
$$\Sigma_L^\alpha = \{(r, \theta) \in \omega; r = L\},$$

where (r, θ) are the polar coordinates. See Figure 1.

In the wedge domain ω let us consider the steady radial flow of an incompressible viscous fluid which emerges from the origin or converges on the origin. The steady radial fluid motion is governed by the steady Navier-Stokes equations

$$-\nu\Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{0} \quad \text{in} \quad \omega, \tag{1.1}$$
$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \omega, \tag{1.2}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \partial\omega, \tag{1.3}$$
$$\boldsymbol{u} \to \boldsymbol{0} \quad \text{as} \quad r \to \infty \tag{1.4}$$

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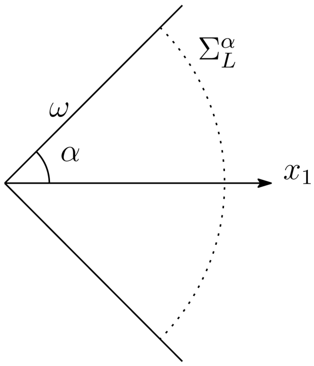

Fig. 1.

with the flux condition

$$\int_{\Sigma_L^\alpha} \boldsymbol{u} \cdot \boldsymbol{e}_r dS = \gamma, \ (\boldsymbol{e}_r = (\cos\theta, \sin\theta)), \tag{1.5}$$

where $\boldsymbol{u}$ and p denote the unknown velocity vector and the unknown pressure of the fluid, respectively, while ν is the given viscosity constant and γ is the given flux. Since the flow is radial, we assume

$$\boldsymbol{u} = \frac{\gamma g(\theta)}{r} \boldsymbol{e}_r, \tag{1.6}$$

where $g(\theta)$ is an unknown scalar function. Then the function $\boldsymbol{u}$ automatically satisfies (1.2). In this paper we prove that for a given flux γ there exists a solution $\boldsymbol{u}$ of the Navier-Stokes equations (1.1)-(1.4) of the form (1.6) satisfying the flux condition (1.5). Such a solution is usually called "Jeffery-Hamel's flow".

To obtain a solution of Jeffery-Hamel's flow, we find a function g satisfying

$$g'' + 4g + \frac{\gamma}{\nu} g^2 = \frac{\Phi}{\nu\gamma} \quad \text{on} \quad (-\alpha, \alpha) \tag{1.7}$$

with the boundary condition

$$g(\pm\alpha) = 0 \tag{1.8}$$

and the flux condition

$$\int_{-\alpha}^{\alpha} g(\theta)d\theta = 1. \tag{1.9}$$

We add the following symmetry condition

$$g(\theta) = g(-\theta) \quad (\theta \in (-\alpha, \alpha)), \tag{1.10}$$

where Φ is an arbitrary constant. In this paper, we add the symmetric condition (1.10) because the stable flows are symmetric.

L. Rosenhead [8] and L. D. Landau and E. M. Lifshitz [7] prove the existence of solutions of Jeffery-Hamel's flow which are given in terms of Jacobian elliptic functions and investigate the behavior of the solutions of Jeffery-Hamel's flow. G. P. Galdi, M. Padula, V. A. Solonnikov [6] obtain a unique solution of Jeffery-Hamel's flow for $\alpha = \frac{\pi}{2}$ under the restricted flux condition applying functional analysis. See Lemma 5.1 and Appendix of [6].

In this paper we succeed in proving the existence of the unique solution of Jeffery-Hamel's flow for any $\alpha \in (0, \pi)$ under the restricted flux condition flux in a wedge domain using a method similar to that of [6].

L. Rosenhead [8] and L. D. Landau and E. M. Lifshitz [7] show the behavior of the solution $\gamma g(\theta)$ of the ODE (1.7) with (1.9)-(1.10). The ratio $R = |\frac{\gamma}{\nu\rho}|$ is dimensionless and plays the role of the Reynolds number in this problem, where ρ is the density. For simplicity $\rho = 1$ in this paper. L. Rosenhead [8] and L. D. Landau and E. M. Lifshitz [7] show that if γ is negative then for any $\alpha \in (0, \frac{\pi}{2})$ and any R there exists a convergent symmetrical flow. In other words, the solution $\gamma g(\theta)$ is symmetric with respect to $\theta = 0$ and negative for any $\theta \in (-\alpha, \alpha)$. See Figure 2. L. Rosenhead [8] and L. D. Landau and E. M. Lifshitz [7] show that if γ is positive then for any $\alpha \in (0, \frac{\pi}{2})$ there exists $R_{\max} > 0$ such that for $R < R_{\max}$ a symmetrical flow, everywhere divergent, appears. In other words, the solution $\gamma g(\theta)$ is symmetric with respect to $\theta = 0$ and positive for any $\theta \in (-\alpha, \alpha)$. These two flow are stable. See Figure 3. It is shown that $R_{\max} \to 0$ as $\alpha \to \frac{1}{2}\pi$ and $R_{\max} \to \infty$ as $\alpha \to 0$. We have not been able to find the formula for the relation between γ_0 in this paper (see Definition 1.1) and the constant $R_{\max}$.

Note that, as R increases, the steady divergent flow of the kind described here becomes unstable soon after R exceeds $R_{\max}$. A symmetrical flow,

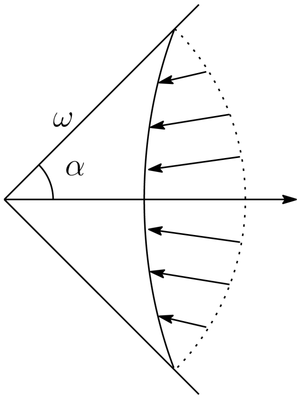

Fig. 2.

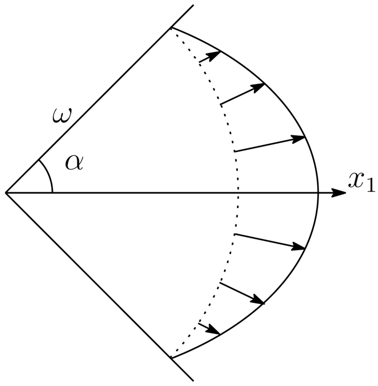

Fig. 3.

everywhere divergent, does not appear. The velocity of the flow has one maximum and one minimum. In other words, soon after R exceeds $R_{\max}$, the solution $\gamma g(\theta)$ is not symmetric with respect to $\theta = 0$ and is not always positive for all $\theta \in (-\alpha, \alpha)$. See Figure 4.

When R increases further, a symmetrical solution appears. The velocity becomes symmetric with respect to the x_1-axis and has two minima and one maximum. In other words, when R increases further, the solution $\gamma g(\theta)$ becomes symmetric with respect to $\theta = 0$ and is not always positive for any $\theta \in (0, \alpha)$. See Figure 5. As R goes to infinity, the number of alternating maxima or minima increases without limit.

But the total flux of both the flows of Figure 4 and Figure 5 is γ.

To the knowledge of the author, the uniqueness and existence of solutions of Jeffery-Hamel's problem have seldom been treated for the class of steady flows. In this paper, we prove the existence and uniqness of the solution of ODE (1.7) with (1.8)-(1.10) applying functional analysis under the restricted flux constant which depends only on the angle of size 2α. The result in this paper is different from the result in L. Rosenhead [8] and L. D. Landau and E. M. Lifshitz [7], because properties of the solutions and the Reynolds number are different from each other. See section 5. In this paper the result for $\alpha = \frac{\pi}{2}$ is better than that of G. P. Galdi, M. Padula, V. A. Solonnikov [6], because they do not use $\alpha = \frac{\pi}{2}$. See Remark 1.1.

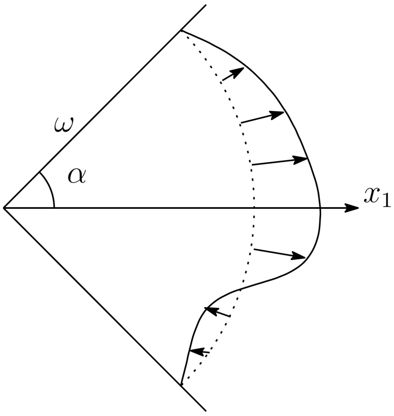

Fig. 4.

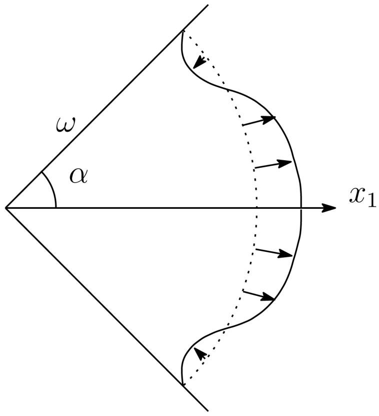

Fig. 5.

1.2. Function space

We introduce some function spaces.

$C[-\alpha,\alpha]$ is the set of all the continuous scalar functions on $[-\alpha,\alpha]$.

$$\begin{aligned}
C^S[-\alpha,\alpha] &:= \{b \in C[-\alpha,\alpha]; b(s) = b(-s)\ (s \in [-\alpha,\alpha])\}.\\
C_0[-\alpha,\alpha] &:= \{b \in C[-\alpha,\alpha]; b(\pm\alpha) = 0\}.\\
C_+[-\alpha,\alpha] &:= \{b \in C[-\alpha,\alpha]; b(s_1) \geq b(s_2) \text{ and}\\
&\qquad b(-s_1) \geq b(-s_2)\ (0 \leq s_1 < s_2 \leq \alpha)\}.\\
C_-[-\alpha,\alpha] &:= \{b \in C[-\alpha,\alpha]; b(s_1) \leq b(s_2) \text{ and}\\
&\qquad b(-s_1) \leq b(-s_2)\ (0 \leq s_1 < s_2 \leq \alpha)\}.\\
C_0^S[-\alpha,\alpha] &:= C^S[-\alpha,\alpha] \cap C_0[-\alpha,\alpha].\\
C_{0,+}^S[-\alpha,\alpha] &:= C^S[-\alpha,\alpha] \cap C_0[-\alpha,\alpha] \cap C_+[-\alpha,\alpha].\\
C_{0,-}^S[-\alpha,\alpha] &:= C^S[-\alpha,\alpha] \cap C_0[-\alpha,\alpha] \cap C_-[-\alpha,\alpha].
\end{aligned}$$

$C_+[-\alpha,\alpha]$, $C_-[-\alpha,\alpha]$, $C_{0,+}^S[-\alpha,\alpha]$, $C_{0,-}^S[-\alpha,\alpha]$ are closed sets but are not linear spaces. The norm $\|\cdot\|$ is the usual norm of $C[-\alpha,\alpha]$.

1.3. Results

To construct a unique solution of the ODE (1.7)-(1.10), we first define the following upper bound constant:

DEFINITION 1.1. Let

$$C(\alpha) := \begin{cases} 1-\cos 2\alpha+\sin 2\alpha\tan 2\alpha & \left(0<\alpha<\dfrac{1}{4}\pi\right) \\ 1-\cos 2\alpha+\dfrac{2-\sin 2\alpha}{|\cos 2\alpha|} & \left(\dfrac{1}{4}\pi<\alpha\le\dfrac{1}{2}\pi\right) \\ 3+\cos 2\alpha+\dfrac{2-\sin 2\alpha}{|\cos 2\alpha|} & \left(\dfrac{1}{2}\pi<\alpha<\dfrac{3}{4}\pi\right) \\ 3+\cos 2\alpha+\dfrac{4+\sin 2\alpha}{\cos 2\alpha} & \left(\dfrac{3}{4}\pi<\alpha<\pi\right), \end{cases}$$

$$K(\alpha) := \frac{\alpha(1-\cos 2\alpha)}{|\tan 2\alpha-2\alpha||\cos 2\alpha|}.$$

We set

$$\gamma_0(\alpha) := \frac{\nu}{C(\alpha)K(\alpha)(2\alpha K(\alpha)+1)}.$$

Our main theorem on the existence of a unique solution of Jeffery-Hamel's flow now reads.

THEOREM 1.1. *Let* $\alpha \in (0,\pi)\backslash\{\frac{1}{4}\pi, \alpha_0, \frac{3}{4}\pi\}$, *where* $2\alpha_0 = \tan 2\alpha_0$, *and* $\omega = \{(r,\theta); -\alpha<\theta<\alpha\}$. *We suppose that* $|\gamma| < \gamma_0(\alpha)$, *where the constant* $\gamma_0(\alpha)$ *is defined as in* Definition 1.1.

Then there exists a solution of the Navier-Stokes equations in ω *of the form*

$$\boldsymbol{u} = \frac{\gamma g(\theta)}{r}\boldsymbol{e}_r,\ p = \frac{2\nu\gamma g(\theta)-\frac{1}{2}\Phi}{r^2} + D \quad (D\in\mathbb{R}) \tag{1.11}$$

with

$$g \in C_0^S[-\alpha,\alpha]\cap C^\infty(-\alpha,\alpha), \quad \int_{-\alpha}^{\alpha} g(\theta)d\theta = 1, \quad |g(\theta)| \le 2K(\alpha).$$

Moreover the function g *is unique in the above class.*

COROLLARY 1.1. *Let* $\alpha \in (0, \frac{1}{4}\pi)$. *We suppose that* $0<\gamma<\gamma_0(\alpha)$. *Then we have* $g \in C_{0,+}^S[-\alpha,\alpha]\cap C^\infty(-\alpha,\alpha)$.

Remark 1.1. G. P. Galdi, M. Padula, V. A. Solonnikov [6] prove that for $\alpha = \frac{\pi}{2}$ if $|\gamma| < \frac{\nu}{36}$, there exists a solution of the Navier-Stokes equations with the above form (1.11). If $\alpha = \frac{\pi}{2}$, then $\gamma_0(\frac{\pi}{2}) = \frac{\pi}{24}\nu$. Therefore this result is better than that of [6].

Remark 1.2. We do not know whether the constant $\gamma_0(\alpha)$ is optimal.

Remark 1.3. We cannot obtain a solution of Jeffery-Hamel's flow for $\alpha = \frac{1}{4}\pi$, $\alpha = \alpha_0$, where $2\alpha_0 = \tan 2\alpha_0$, and $\alpha = \frac{3}{4}\pi$ by the method in this paper.

1.4. The properties of the upper bound constant

We state the following easy consequences of the Definition 1.1 without proof.

Proposition 1.1. *We have*

$$\begin{aligned}
\lim_{\alpha \to +0} \gamma_0(\alpha) &= \infty, \\
\lim_{\alpha \to +0} \alpha\gamma_0(\alpha) &= \frac{4\nu}{45}, \\
\lim_{\alpha \to \frac{1}{4}\pi} \gamma_0(\alpha) &= 0, \\
\lim_{\alpha \to \alpha_0} \gamma_0(\alpha) &= 0 \quad (\tan 2\alpha_0 = 2\alpha_0), \\
\lim_{\alpha \to \frac{3}{4}\pi} \gamma_0(\alpha) &= 0, \\
\lim_{\alpha \to \pi - 0} \gamma_0(\alpha) &= \infty, \\
\lim_{\alpha \to \pi - 0} \gamma_0(\alpha)(1 - \cos 2\alpha) &= \frac{\pi}{4}.
\end{aligned}$$

1.5. Derivation of ODE (1.7)

In this section we formulate the Navier-Stokes equation into the ODE (1.7).

The Navier-Stokes equations in polar coordinates are as follows.

$$u_r \frac{\partial u_r}{\partial r} + \frac{u_r}{r}\frac{\partial u_\theta}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right), \tag{1.12}$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} - \frac{u_r u_\theta}{r} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \nu \left(\nabla^2 u_\theta - \frac{u_\theta}{r} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right), \tag{1.13}$$

where

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$

We set

$$u_r = \frac{\gamma g(\theta)}{r}, \quad u_\theta = 0.$$

From (1.13) we obtain

$$\frac{\partial p}{\partial \theta}(r, \theta) = \frac{2\nu\gamma}{r^2} g'(\theta).$$

Therefore for a certain function $P_0(r)$ we have

$$p(r, \theta) = \frac{2\nu\gamma}{r^2} g(\theta) + P_0(r),$$

$$\frac{\partial p}{\partial r}(r, \theta) = -\frac{4\nu\gamma}{r^3} g(\theta) + P_0'(r). \tag{1.14}$$

Using (1.14) and (1.12), we deduce

$$\nu\gamma g'' + 4\nu\gamma g + \gamma^2 g^2 = P_0'(r) r^3 \quad \text{on} \quad (-\alpha, \alpha). \tag{1.15}$$

The left hand side of (1.15) depends only on θ and the right hand side of (1.15) depends only on r. Consequently, we obtain $P_0(r) = -\frac{\Phi}{2r^2}$ and the ODE

$$g'' + 4g + \frac{\gamma}{\nu} g^2 = \frac{\Phi}{\nu\gamma} \quad \text{on} \quad (-\alpha, \alpha).$$

2. Linear Equation and Its Properties

2.1. Linear equation

In this subsection for any $\alpha \in (0, \pi)$, we solve the linear problem

$$h'' + 4h = b(\theta) \quad \text{on} \quad (-\alpha, \alpha) \tag{2.1}$$

with

$$h(\theta) = h(-\theta), \tag{2.2}$$

$$h(\pm\alpha) = 0, \tag{2.3}$$

where $b \in C^S[-\alpha, \alpha]$.

The function

$$\begin{aligned} h^\alpha(\theta) = &\frac{1}{2}\int_\theta^\alpha \sin 2(s-\theta) b(s) ds \\ &- \frac{\sin 2(\alpha-\theta)}{2\cos 2\alpha}\int_0^\alpha \cos(2s) b(s) ds \quad (\theta \in (-\alpha, \alpha)) \end{aligned} \tag{2.4}$$

is the unique solution of (2.1) with (2.2) and (2.3). It is easy to see that $h^\alpha \in C_0^S[-\alpha, \alpha] \cap C^2(-\alpha, \alpha)$. We define an operator $\mathcal{L}^\alpha$ by

$$\mathcal{L}^\alpha[b] = h^\alpha \quad (b \in C^S[-\alpha, \alpha]).$$

Remark 2.1. For $\alpha = \frac{1}{4}\pi$ and $\alpha = \frac{3}{4}\pi$, we cannot obtain a solution of the linear problem (2.1) with (2.2) and (2.3).

We apply the linear operator $\mathcal{L}^\alpha$ to the nonlinear ODE in the section 3.

Let us consider the reason why the linear problem (2.1) with (2.2) and (2.3) is not solvable for $\alpha = \frac{1}{4}\pi$. The linear problem (2.1) with (2.2) and (2.3) is equivalent to the problem on the half interval

$$h'' + 4h = b(\theta) \quad \text{on} \quad (0, \frac{1}{4}\pi) \tag{2.5}$$

with

$$h'(0) = 0, \tag{2.6}$$

$$h(\frac{1}{4}\pi) = 0. \tag{2.7}$$

The function

$$h(\theta) = \frac{1}{2}\int_\theta^{\frac{1}{4}\pi} \sin 2(s-\theta) b(s) ds + C_1 \cos 2\theta + C_2 \sin 2\theta$$

is a general solution of the problem (2.5). We choose the constants C_1 and C_2 satisfying the initial value (2.6) and (2.7). Such a problem is equivalent to the following system of linear equations:

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ q \end{pmatrix}, \quad q = \int_0^{\frac{1}{4}\pi} \cos(2s) b(s) ds. \tag{2.8}$$

The constant q is not zero because a given function b is symmetric. The rank of this coefficient matrix is 1. The rank of the enlarged coefficient matrix is 2 because $q \neq 0$. Therefore this system of the linear equations (2.8) is not solvable on $\mathbb{R}^2$. This implies that the linear symmetric ODE (2.1) with (2.2) and (2.3) is not solvable for $\alpha = \frac{1}{4}\pi$. We have a similar result for $\alpha = \frac{3}{4}\pi$.

2.2. The properties of the operator $\mathcal{L}^\alpha$

In this subsection we discuss the properties of the operator $\mathcal{L}^\alpha$ for any $\alpha \in (0, \pi) \backslash \{\frac{1}{4}\pi, \frac{3}{4}\pi\}$.

LEMMA 2.1. *$\mathcal{L}^\alpha$ is a linear operator from $C^S[-\alpha, \alpha]$ to $C_0^S[-\alpha, \alpha]$. If $b \in C^S[-\alpha, \alpha]$, then*

$$\|\mathcal{L}^\alpha[b]\| \leq \frac{1}{4} \|b\| C(\alpha), \tag{2.9}$$

where the constant $C(\alpha)$ is defined in Definition 1.1.

PROOF. Let $b \in C^S[-\alpha, \alpha]$. We have

$$|\mathcal{L}^\alpha[b](\theta)| \leq \frac{1}{2} \|b\| \left(\int_0^\alpha |\sin 2s| ds + \frac{|\sin 2(\alpha - \theta)|}{2|\cos 2\alpha|} \int_0^\alpha |\cos 2s| ds \right). \tag{2.10}$$

It is easy to prove that

$$\int_0^\alpha |\sin 2s| ds = \begin{cases} \dfrac{1 - \cos 2\alpha}{2} & \left(0 < \alpha \leq \dfrac{1}{2}\pi\right) \\ \dfrac{3 + \cos 2\alpha}{2} & \left(\dfrac{1}{2}\pi < \alpha < \pi\right), \end{cases}$$

$$|\sin 2(\alpha - \theta)| \leq \begin{cases} \sin 2\alpha & \left(0 < \alpha < \dfrac{1}{4}\pi\right) \\ 1 & \left(\dfrac{1}{4}\pi < \alpha < \pi\right), \end{cases}$$

$$\int_0^\alpha |\cos 2s| ds = \begin{cases} \dfrac{\sin 2\alpha}{2} & \left(0 < \alpha < \dfrac{1}{4}\pi\right) \\ \dfrac{2 - \sin 2\alpha}{2} & \left(\dfrac{1}{4}\pi < \alpha < \dfrac{3}{4}\pi\right) \\ \dfrac{4 + \sin 2\alpha}{2} & \left(\dfrac{3}{4}\pi < \alpha < \pi\right). \end{cases} \quad \square$$

LEMMA 2.2. *If $b(\theta) = 1$, then*

$$\mathcal{L}^\alpha[1](\theta) = \frac{1}{4}\left(1 - \frac{\cos 2\theta}{\cos 2\alpha}\right), \tag{2.11}$$

$$\|\mathcal{L}^\alpha[1]\| = \frac{1}{4}\left(\frac{1 - \cos 2\alpha}{|\cos 2\alpha|}\right). \tag{2.12}$$

REMARK 2.2. If $b(\theta) = 1$, the right hand side of (2.9) is larger than that of (2.12).

We apply (2.9) and (2.12) to the estimates of the solution of the nonlinear ODE.

2.3. The properties of the operator $\mathcal{L}^{\alpha_1}$ for $\alpha_1 \in (0, \frac{1}{4}\pi)$

In this subsection we discuss the properties of the operator $\mathcal{L}^{\alpha_1}$ for $\alpha_1 \in (0, \frac{1}{4}\pi)$ in order to prove Corollary 1.1.

LEMMA 2.3. *Let $\alpha_1 \in (0, \frac{1}{4}\pi)$.*
Suppose that $b \in C^S_{0,+}[-\alpha_1, \alpha_1]$. Then $\mathcal{L}^{\alpha_1}[b] \in C^S_{0,-}[-\alpha_1, \alpha_1]$.
Suppose that $b \in C^S_{0,-}[-\alpha_1, \alpha_1]$. Then $\mathcal{L}^{\alpha_1}[b] \in C^S_{0,+}[-\alpha_1, \alpha_1]$.

PROOF. Let $b \in C^S_{0,+}[-\alpha_1, \alpha_1]$, $0 \le \theta \le \alpha_1$. Then we have

$$\mathcal{L}^{\alpha_1}[b](\theta) = \frac{1}{2}\int_\theta^{\alpha_1} \sin 2(s - \theta) b(s) ds - \frac{\sin 2(\alpha_1 - \theta)}{2\cos 2\alpha_1}\int_0^{\alpha_1} \cos(2s) b(s) ds.$$

We obtain

$$(\mathcal{L}^{\alpha_1}[b])'(\theta) = -\int_\theta^{\alpha_1} \cos 2(s - \theta) b(s) ds + \frac{\cos 2(\alpha_1 - \theta)}{\cos 2\alpha_1}\int_0^{\alpha_1} \cos(2s) b(s) ds$$

$$
\begin{aligned}
&\geq -\int_{\theta}^{\alpha_1} \cos 2(s-\theta) b(s) ds + \int_{0}^{\alpha_1} \cos(2s) b(s) ds \\
&= -\int_{\theta}^{\alpha_1} \cos 2(s-\theta) b(s) ds + \int_{\alpha_1-\theta}^{\alpha_1} \cos(2s) b(s) ds \\
&\quad + \int_{0}^{\alpha_1-\theta} \cos(2s) b(s) ds \\
&= -\int_{\theta}^{\alpha_1} \cos 2(s-\theta) b(s) ds + \int_{\alpha_1-\theta}^{\alpha_1} \cos(2s) b(s) ds \\
&\quad + \int_{\theta}^{\alpha_1} \cos 2(s-\theta) b(s-\theta) ds \\
&= \int_{\theta}^{\alpha_1} \cos 2(s-\theta)(b(s-\theta)-b(s)) ds + \int_{\alpha_1-\theta}^{\alpha_1} \cos(2s) b(s) ds \\
&\geq 0.
\end{aligned}
$$

Since $\mathcal{L}^{\alpha_1}[b](\alpha_1) = 0$, we have $\mathcal{L}^{\alpha_1}[b] \in C^S_{0,-}[-\alpha_1, \alpha_1]$. □

According to (2.11), the following lemma holds true.

Lemma 2.4. *Let* $\alpha_1 \in (0, \frac{1}{4}\pi)$ *and* $b(\theta) = 1$.
Then $\mathcal{L}^{\alpha_1}[1] \in C^S_{0,-}[-\alpha_1, \alpha_1]$.

Applying Lemma 2.3 and 2.4 to the nonlinear ODE, we prove corollary 1.1.

3. Formulation

In this section we rewrite the ODE (1.7)-(1.10) for any $\alpha \in (0, \pi) \backslash \{\frac{1}{4}\pi, \frac{3}{4}\pi\}$.

Applying $\mathcal{L}^{\alpha}$ to (1.7), then we have

$$
g = \frac{\Phi}{\nu\gamma}\mathcal{L}^{\alpha}[1] - \frac{\gamma}{\nu}\mathcal{L}^{\alpha}[g^2].
$$

The flux condition (1.9) implies

$$
1 = \frac{\Phi}{\nu\gamma}\int_{-\alpha}^{\alpha} \mathcal{L}^{\alpha}[1](\theta) d\theta - \frac{\gamma}{\nu}\int_{-\alpha}^{\alpha} \mathcal{L}^{\alpha}[g^2](\theta) d\theta.
$$

Therefore the constant Φ must satisfy

$$\Phi = \frac{\nu\gamma}{\frac{1}{2}\alpha - \frac{1}{4}\tan 2\alpha}\left(1 + \frac{\gamma}{\nu}\int_{-\alpha}^{\alpha} \mathcal{L}^{\alpha}[g^2](\theta)d\theta\right).$$

Hence a solution g of the ODE (1.7)-(1.10) exists if and only if $g \in C_0^S[-\alpha,\alpha]$ is a solution of the equation

$$(3.1) \qquad g = \frac{1}{\frac{1}{2}\alpha - \frac{1}{4}\tan 2\alpha}\left(1 + \frac{\gamma}{\nu}\int_{-\alpha}^{\alpha} \mathcal{L}^{\alpha}[g^2](\theta)d\theta\right)\mathcal{L}^{\alpha}[1] - \frac{\gamma}{\nu}\mathcal{L}^{\alpha}[g^2] \quad \text{on} \quad [-\alpha,\alpha].$$

In order to solve the equation (3.1), we consider the operator $\mathcal{J}^{\alpha}$ defined by

$$(3.2) \quad \mathcal{J}^{\alpha}[g] = \frac{1}{\frac{1}{2}\alpha - \frac{1}{4}\tan 2\alpha}\left(1 + \frac{\gamma}{\nu}\int_{-\alpha}^{\alpha} \mathcal{L}^{\alpha}[g^2](\theta)d\theta\right)\mathcal{L}^{\alpha}[1] - \frac{\gamma}{\nu}\mathcal{L}^{\alpha}[g^2],$$

where $g \in C_0^S[-\alpha,\alpha]$. It is easy to see that $\mathcal{J}^{\alpha}[g] \in C_0^S[-\alpha,\alpha]$ for any $g \in C_0^S[-\alpha,\alpha]$. We find a fixed point of the operator $\mathcal{J}^{\alpha}$ in $C_0^S[-\alpha,\alpha]$.

Remark 3.1. We cannot define the operator J^{α} for $\alpha = \frac{1}{4}\pi$, $\alpha = \alpha_0$, where $2\alpha_0 = \tan 2\alpha_0$, and $\alpha = \frac{3}{4}\pi$, because for $\alpha = \frac{1}{4}\pi$ and $\alpha = \frac{3}{4}\pi$ we cannot define the linear operator $\mathcal{L}^{\alpha}$ and for $\alpha = \alpha_0$ the denominator is zero.

4. The Proof

4.1. Proof of Theorem 1.1

In this subsection let us prove Theorem 1.1. In other words, we prove that for any $\alpha \in (0,\pi)\backslash\{\frac{1}{4}\pi, \alpha_0, \frac{3}{4}\pi\}$, where $2\alpha_0 = \tan 2\alpha_0$, and $|\gamma| < \gamma_0(\alpha)$ the operator $\mathcal{J}^{\alpha}$ is a contraction in a suitable ball of $C_0^S[-\alpha,\alpha]$.

For any $a > 0$, set

$$\mathcal{B}(0,a) := \{g \in C_0^S[-\alpha,\alpha]; \|g\| \leq a\}.$$

Lemma 4.1. *Let $\alpha \in (0,\pi)\backslash\{\frac{1}{4}\pi, \alpha_0, \frac{3}{4}\pi\}$, where $2\alpha_0 = \tan 2\alpha_0$, and $|\gamma| < \gamma_0(\alpha)$, where $\gamma_0(\alpha)$ is defined as in* Definition 1.1.

Then the operator $\mathcal{J}^\alpha$ is a contraction from $\mathcal{B}(0, 2K(\alpha))$ to $\mathcal{B}(0, 2K(\alpha))$, where $K(\alpha)$ is defined as in Definition 1.1.

Proof. Let $g \in \mathcal{B}(0, 2K(\alpha))$. Then we have

$$\begin{aligned}|\mathcal{J}^\alpha[g](\theta)| &\le K(\alpha)\left(1 + \frac{|\gamma|}{\nu}\int_{-\alpha}^{\alpha}\frac{1}{4}\|g\|^2 C(\alpha)d\theta\right) + \frac{|\gamma|}{\nu}\cdot\frac{1}{4}\|g\|^2 C(\alpha) \\ &\le K(\alpha)\left(1 + \frac{|\gamma|}{\nu}\cdot 2\alpha K(\alpha)^2 C(\alpha)\right) + \frac{|\gamma|}{\nu}\cdot K(\alpha)^2 C(\alpha) \\ &= K(\alpha) + \frac{|\gamma|}{\gamma_0(\alpha)}K(\alpha) \\ &< 2K(\alpha).\end{aligned}$$

This proves $\mathcal{J}^\alpha[g] \in \mathcal{B}(0, 2K(\alpha))$.

Let $g_1, g_2 \in \mathcal{B}(0, 2K(\alpha))$. Then we have

$$\begin{aligned}&|\mathcal{J}^\alpha[g_1](\theta) - \mathcal{J}^\alpha[g_2](\theta)| \\ &\le \frac{|\gamma|}{\nu}\cdot 2\alpha K(\alpha)\cdot\frac{1}{4}\|g_1 - g_2\|\|g_1 + g_2\|C(\alpha) \\ &\quad + \frac{|\gamma|}{\nu}\cdot\frac{1}{4}\|g_1 - g_2\|\|g_1 + g_2\|C(\alpha) \\ &\le \frac{|\gamma|}{\nu}(2\alpha K(\alpha) + 1)K(\alpha)C(\alpha)\|g_1 - g_2\| \\ &= \frac{|\gamma|}{\gamma_0(\alpha)}\|g_1 - g_2\|.\end{aligned}$$

This proves that $\mathcal{J}^\alpha$ is a contraction from $\mathcal{B}(0, 2K(\alpha))$ to $\mathcal{B}(0, 2K(\alpha))$. □

Therefore there exists a unique fixed point of the operator $\mathcal{J}^\alpha$ in $\mathcal{B}(0, 2K(\alpha))$ by the fixed point theorem for a contraction operator.

4.2. Proof of Corollary 1.1

In this subsection we prove Corollary 1.1. In other words, we prove that for any $\alpha_1 \in (0, \frac{1}{4}\pi)$ and $0 < \gamma < \gamma_0(\alpha_1)$ the operator $\mathcal{J}^{\alpha_1}$ is a contraction in a suitable ball of $C^S_{0,+}[-\alpha_1, \alpha_1]$.

For any $a > 0$, set

$$\mathcal{B}_+(0, a) := \{g \in C^S_{0,+}[-\alpha_1, \alpha_1]; \|g\| \le a\}.$$

Lemma 4.2. *Let* $\alpha_1 \in (0, \frac{1}{4}\pi)$ *and* $0 < \gamma < \gamma_0(\alpha_1)$.
Then $\mathcal{J}^{\alpha_1}$ *is an operator from* $\mathcal{B}_+(0, 2K(\alpha_1))$ *to* $\mathcal{B}_+(0, 2K(\alpha_1))$.

Proof. Since we know that, for any $g \in \mathcal{B}_+(0, 2K(\alpha_1))$, $\mathcal{L}^{\alpha_1}[g^2]$, $\mathcal{L}^{\alpha_1}[1] \in C^S_{0,-}[-\alpha_1, \alpha_1]$ and $\frac{1}{2}\alpha_1 - \frac{1}{4}\tan 2\alpha_1 < 0$, we prove that $1 + \frac{\gamma}{\nu}\int_{-\alpha_1}^{\alpha_1} \mathcal{L}^{\alpha_1}[g^2](\theta)d\theta$ is positive in order to obtain $\mathcal{J}^{\alpha_1}[g] \in C^S_{0,+}[-\alpha_1, \alpha_1]$. A simple calculation yields

$$\begin{aligned} 1 + \frac{\gamma}{\nu}\int_{-\alpha_1}^{\alpha_1} \mathcal{L}^{\alpha_1}[g^2](\theta)d\theta &\geq 1 - \frac{\gamma}{\nu}\cdot 2\alpha_1 K(\alpha_1)^2 C(\alpha_1) \\ &> 1 - \frac{\gamma}{\nu}\cdot(2\alpha_1 K(\alpha_1) + 1)K(\alpha_1)C(\alpha_1) \\ &= 1 - \frac{\gamma}{\gamma_0(\alpha_1)} \\ &> 0. \ \square \end{aligned}$$

Lemma 4.3. *Let* $\alpha_1 \in (0, \frac{1}{4}\pi)$ *and* $0 < \gamma < \gamma_0(\alpha_1)$.
Then the operator $\mathcal{J}^{\alpha}$ *is a contraction from* $\mathcal{B}_+(0, 2K(\alpha_1))$ *to* $\mathcal{B}_+(0, 2K(\alpha_1))$.

The proof of Lemma (4.3) is similar to lemma 4.1.

5. The Maximum Speed and the Reynolds Number

In this section we compare the result in this paper with the previous work [7] for the maximum speed and the Reynolds number, since we treat Jeffery-Hamel's flows by the different method from the previous works [7], [8].

Firstly, let us consider the maximum speed. Let g^α be the unique solution of the ODE. (1.7)-(1.10). The velocity of Jeffery-Hamel's flow is the form

$$\boldsymbol{u} = \frac{\gamma g^\alpha(\theta)}{r}\boldsymbol{e}_r.$$

It is easy to see that

$$|\boldsymbol{u}| \leq |\gamma|\frac{1 - \cos 2\alpha}{|\tan 2\alpha - 2\alpha||\cos 2\alpha|}\cdot\frac{1}{r} \quad (\alpha \in (0, \pi)\backslash\{\frac{1}{4}\pi, \alpha_0, \frac{3}{4}\pi\}),$$

where $2\alpha_0 = \tan 2\alpha_0$. The right-hand side is the upper bound of the speed at the point $(r, \theta) \in \omega$.

On the other hand, let v be a solution of Jeffery-Hamel's flow in L. D. Landau and E. M. Lifshitz [7], which is given in terms of Jacobian elliptic functions. Then we have

$$|v| \le \frac{|\gamma|}{\alpha} \cdot \frac{1}{r} \quad (\alpha \in (0, \frac{1}{2}\pi)).$$

Lastly, let us consider the Reynolds number. It is easy to see that the following constants $\overline{R}_1(\alpha_1)$, $\overline{R}_2(\alpha_2)$, $\overline{R}_3(\alpha_3)$, $\overline{R}_4(\alpha_4)$ are the Reynolds number in this paper.

$$\overline{R}_1(\alpha_1) := \frac{(\tan 2\alpha_1 - 2\alpha_1)\cos 2\alpha_1}{(1 - \cos 2\alpha_1 + \sin 2\alpha_1 \tan 2\alpha_1)\left(\frac{2\alpha_1(1-\cos 2\alpha_1)}{(\tan 2\alpha_1 - 2\alpha_1)\cos 2\alpha_1} + 1\right)(1 - \cos 2\alpha_1)},$$

$$\overline{R}_2(\alpha_2) := \frac{|2\alpha_2 - \tan 2\alpha_2||\cos 2\alpha_2|}{\left(1 - \cos 2\alpha_2 + \frac{2 - \sin 2\alpha_2}{|\cos 2\alpha_2|}\right)\left(\frac{2\alpha_2(1-\cos 2\alpha_2)}{(2\alpha_2 - \tan 2\alpha_2)|\cos 2\alpha_2|} + 1\right)(1 - \cos 2\alpha_2)},$$

$$\overline{R}_3(\alpha_3) := \frac{|2\alpha_3 - \tan 2\alpha_3||\cos 2\alpha_3|}{\left(3 + \cos 2\alpha_3 + \frac{2 - \sin 2\alpha_3}{|\cos 2\alpha_3|}\right)\left(\frac{2\alpha_3(1-\cos 2\alpha_3)}{|2\alpha_3 - \tan 2\alpha_3||\cos 2\alpha_3|} + 1\right)(1 - \cos 2\alpha_3)},$$

$$\overline{R}_4(\alpha_4) := \frac{(2\alpha_4 - \tan 2\alpha_4)\cos 2\alpha_4}{\left(3 + \cos 2\alpha_4 + \frac{4 + \sin 2\alpha_4}{\cos 2\alpha_4}\right)\left(\frac{2\alpha_4(1-\cos 2\alpha_4)}{(2\alpha_4 - \tan 2\alpha_4)\cos 2\alpha_4} + 1\right)(1 - \cos 2\alpha_4)},$$

where $\alpha_1 \in (0, \frac{1}{4}\pi)$, $\alpha_2 \in (\frac{1}{4}\pi, \frac{1}{2}\pi]$, $\alpha_3 \in (\frac{1}{2}\pi, \frac{3}{4}\pi)$, $\alpha_4 \in (\frac{3}{4}\pi, \pi)$.

On the other hand, the Reynolds number in L. D. Landau and E. M. Lifshitz [7] is

$$R_{\max} = -6\beta\frac{1-k^2}{1-2k^2} + \frac{12}{\sqrt{1-2k^2}}\int_0^{\frac{\pi}{2}}\sqrt{1 - k^2\sin^2 x}\,dx,$$
$$\beta = 2\sqrt{1-2k^2}\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2 x}\,dx,$$
$$k^2 = \frac{u_0}{1+2u_0},$$

where u_0 is the maximum value of the solutions of the ODE (1.7)-(1.10) which are given in the terms of Jacobian elliptic functions.

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