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Hypersurfaces with Constant Anisotropic Mean Curvatures

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Abstract. In this note, we apply the evolution method to present another proof of the anisotropic version of Heinze-Karcher inequality for hypersurfaces in the Euclidean space, from which the Alexandrov type theorem follows from a standard argument via the Minkowski formula.

1. Introduction

The classical Alexandrov theorem is one of the most remarkable results which states that any closed embedded constant mean curvature hypersurface in the Euclidean space is a round sphere. There are different methods to prove it, for instance, Alexandrov reflection ([1]), application of Reilly's formula ([21, 22]), Montiel-Ros' integration ([18]), a spinorial Reilly-type inequality ([10]), etc. It can also be generalized to many other ambient manifolds or hypersurfaces with constant higher order mean curvatures ([21], [17], [22], [18], [14], [7] and references therein). Recently, S. Brendle ([3]) proved an Alexandrov type theorem in certain warped product manifolds, including deSitter-Schwarzschild and Reissner-Nordstorm manifolds. His proof is based on evolution equations, which seems to have generality.

On the other hand, as a natural generalization of surfaces with constant mean curvature, extensive research has been devoted to studying surfaces with constant anisotropic mean curvature in the Euclidean space in the fields of analysis, geometry and material sciences (cf. [23], [8], [2], [9], [5], [20], [6], [24], [15, 16], [11, 12] and the references therein). Let $F : \mathbb{S}^n \to \mathbb{R}^+$ be a smooth positive function defined on the unit sphere which satisfies the following convexity condition:

$$A_F := (D^2F + F1)_x > 0, \quad \forall x \in \mathbb{S}^n,$$

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where D^2F denotes the Hessian of F on $\mathbb{S}^n$, 1 denotes the identity on $T_x\mathbb{S}^n$ and >0 means that the matrix is positive definite. Now let $x:\Sigma\to\mathbb{R}^{n+1}$ be a smooth immersion of a closed orientable hypersurface and $\nu:\Sigma\to\mathbb{S}^n$ denote its Gauss map. Then the anisotropic surface energy of x is defined as follows:

$$\mathcal{F}(x)=\int_\Sigma F(\nu)dA.$$

Notice that if $F\equiv 1$, then $\mathcal{F}(x)$ is the usual area functional of x. The algebraic $(n+1)$-volume enclosed by Σ is given by

$$V=\frac{1}{n+1}\int_\Sigma \langle x,\nu\rangle dA.$$

It is very interesting to study the critical points of $\mathcal{F}$ for volume-preserving variations. The Euler-Lagrange equation for this constrained variational problem is

$$H_F:=-\mathrm{div}_\Sigma DF+nHF=constant, \tag{1.1}$$

where $H:=-\frac{1}{n}\mathrm{tr}d\nu$ is the mean curvature of x. Thus H_F is called the *anisotropic mean curvature* of x. Notice that if $F\equiv 1$ then H_F is nothing but nH.

Among all hypersurfaces with constant anisotropic mean curvature, there is one class of special hypersurfaces which are the generalization of the unit spheres. Consider the map

$$\begin{aligned}\varphi\colon \mathbb{S}^n&\to\mathbb{R}^{n+1}\\ x&\mapsto DF_x+F(x)x,\end{aligned}$$

where DF is the gradient of F on $\mathbb{S}^n$. We call $W_F=\varphi(S^n)$ the *Wulff shape* of F or $\mathcal{F}$. Under the convexity condition of F, W_F is a smooth convex hypersurface and $\mathcal{F}$ is called a *parametric elliptic functional.* When $F\equiv 1$, the Wulff shape is the unit sphere.

Observe that

$$H_F=-\mathrm{tr}d(\varphi\circ\nu),$$

so one can call

$$S_F:=-d(\varphi\circ\nu)=-A_F\circ d\nu$$

the *anisotropic Weingarten operator* of x. Let $S := -d\nu$ be the classical Weingarten operator. Remark that in general S_F is not symmetric, but it still has real eigenvalues $\lambda_1, \cdots, \lambda_n$, which are called *anisotropic principal curvatures*. Similar to the classical hypersurfaces theory, we have the following characterization for the *anisotropic umbilical hypersurfaces* in $\mathbb{R}^{n+1}$:

LEMMA 1.1 (See [11, 12]). *Let $x : \Sigma^n \to \mathbb{R}^{n+1}$ be an immersed closed hypersurface. If $\lambda_1 = \lambda_2 = \cdots = \lambda_n \neq 0$ holds everywhere on Σ, then Σ is the Wulff shape, up to translations and hometheties.*

Let σ_r be the elementary symmetric functions of the anisotropic principal curvatures $\lambda_1, \cdots, \lambda_n$, i.e., $\sigma_r := \sum_{i_1<\cdots<i_r} \lambda_{i_1} \cdots \lambda_{i_r}$ for $1 \leq r \leq n$. Set $\sigma_0 = 1$. Then the r-th anisotropic mean curvature H_r is defined by $H_r = \sigma_r / C_n^r$, where $C_n^r = \frac{n!}{r!(n-r)!}$. In particular, $H_1 = H_F/n$.

We have proved the following Alexandrov type theorem in [13]:

THEOREM 1.1. *Let Σ be a closed oriented hypersurface embedded in the Euclidean space $\mathbb{R}^{n+1}$. If H_r is constant for some $r = 1, \cdots, n$, then Σ is the Wulff shape, up to translations and hometheties.*

In this note, we will apply the evolution method introduced by Brendle [3] to derive another proof of the Heinze-Karcher inequality (Theorem 3.1), from which Alexandrov type Theorem 1.1 follows from a standard argument via the Minkowski formula. In Section 2, we first recall hypersurfaces theory in the Euclidean space in terms of moving frames and then we prove three fundamental equations for an immersed oriented hypersurface in $\mathbb{R}^{n+1}$ related to its anisotropic mean curvature. In Section 3, we use one of the fundamental equations obtained in Section 2 and employ the evolution method introduced by Brendle ([3]) to show the Heintz-Karcher type inequality. Then Theorem 1.1 follows from the standard argument.

2. Preliminaries and Basic Equations

For the convenience of the reader, we firstly recall the basic facts related to anisotropic mean curvature of a hypersurface in terms of moving frames. See more details in [12].

Let $x : \Sigma \to \mathbb{R}^{n+1}$ be a smooth oriented hypersurface with its Gauss map $\nu : \Sigma \to \mathbb{S}^n$. Let $\{E_1, \cdots, E_n\}$ be a local orthonormal frame on $\mathbb{S}^n$, then $\{e_1 := E_1 \circ \nu, \cdots, e_n := E_n \circ \nu\}$ is a local orthonormal frame of Σ and $e_1, \cdots, e_n, e_{n+1} = \nu$ is a local orthonormal frame on $\mathbb{R}^{n+1}$ along x. Denote the dual frames of E_i and e_i by θ_i and ω_i, respectively, and the corresponding connection forms by θ_{ij} and ω_{ij}.

Throughout this paper, we agree on the range of indices: $1 \leq i, j, \cdots \leq n$ and $1 \leq A, B, \cdots \leq n+1$. Recall that the structure equations of $y : \mathbb{S}^n \to \mathbb{R}^{n+1}$ are as follows:

$$\begin{aligned} &dy = \sum_i \theta_i E_i, \quad dE_i = \sum_j \theta_{ij} E_j - \theta_i y, \\ &d\theta_i = \sum_j \theta_{ij} \wedge \theta_j, \quad d\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj} = -\theta_i \wedge \theta_j, \end{aligned} \tag{2.1}$$

where $\theta_{ij} + \theta_{ji} = 0$. Let $F \in C^\infty(\mathbb{S}^n)$ be a smooth function defined on $\mathbb{S}^n$. With respect to the dual frame field $\theta_1, \cdots, \theta_n$ of $\mathbb{S}^n$ chosen above, the exterior derivative, the second covariant derivative and the third covariant derivative of F are defined by

$$\begin{aligned} &dF = \sum_i F_i \theta_i, \\ &\sum_j F_{ij} \theta_j = dF_i + \sum_j F_j \theta_{ji}, \\ &\sum_k F_{ijk} \theta_k = dF_{ij} + \sum_k F_{ik} \theta_{kj} + \sum_k F_{kj} \theta_{ki}, \end{aligned} \tag{2.2}$$

respectively. It follows from (2.1) and Ricci identity that

$$F_{ijk} - F_{ikj} = F_j \delta_{ik} - F_k \delta_{ij},$$

which implies that $(F_{ij} + F\delta_{ij})_{,k} = (F_{ik} + F\delta_{ik})_{,j}$. Denote the coefficients of A_F by $A_{ij} = F_{ij} + F\delta_{ij}$, then we have

$$A_{ij,k} = A_{ik,j},$$

where

$$\sum_k A_{ij,k} \theta_k = dA_{ij} + A_{ik} \theta_{kj} + A_{kj} \theta_{ki}.$$

The structure equations of $x : \Sigma \to \mathbb{R}^{n+1}$ are given by

$$dx = \sum_i \omega_i e_i,$$
$$de_i = \sum_j \omega_{ij} e_j + \sum_j h_{ij}\omega_j \nu, \quad d\nu = -\sum_{ij} h_{ij}\omega_j e_i,$$
$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad d\omega_{ij} = \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} R_{ijkl}\omega_k \wedge \omega_l,$$

where $\omega_{in+1} = \sum_j h_{ij}\omega_j$ and $h_{ij} = h_{ji}$. Making use of (2.1), we get

$$de_i = d(E_i \circ \nu) = \nu^* dE_i = \sum_j \nu^*\theta_{ij} e_j - \nu^*\theta_i \nu,$$

thus we have

$$\omega_{ij} = \nu^*\theta_{ij}, \qquad \nu^*\theta_i = -\sum_j h_{ij}\omega_j. \tag{2.3}$$

Let $f \in C^\infty(\Sigma)$ be a smooth function defined on Σ. With respect to the dual frame field $\omega_1, \cdots, \omega_n$ of Σ defined above, the exterior derivative and the second covariant derivative of f are given by

$$df = \sum_i f_i \omega_i, \qquad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j \omega_{ji}.$$

Particularly, considering x and ν as smooth functions on Σ, we have

$$x_i = e_i, \quad x_{ij} = h_{ij}\nu, \tag{2.4}$$
$$\nu_i = -\sum_j h_{ij} e_j, \quad \nu_{ij} = -\sum_k h_{ikj} e_k - \sum_k h_{ik} h_{kj} \nu, \tag{2.5}$$

where h_{ijk} is defined by $\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}$.

For a smooth positive function $F : \mathbb{S}^n \to \mathbb{R}^+$ defined on $\mathbb{S}^n$, $F \circ \nu$ is a function on Σ. We define the covariant derivatives of $(F \circ \nu)$ and $A_{ij} \circ \nu$ by

$$\begin{aligned} & d(F \circ \nu) = \sum_i (F \circ \nu)_i \omega_i, \\ & \sum_j (F_i \circ \nu)_j \omega_j = d(F_i \circ \nu) + \sum_j (F_j \circ \nu)\omega_{ji}, \\ & \sum_k (A_{ij} \circ \nu)_k \omega_k = d(A_{ij} \circ \nu) + \sum_k (A_{kj} \circ \nu)\omega_{ki} + \sum_k (A_{ik} \circ \nu)\omega_{kj}. \end{aligned} \tag{2.6}$$

Taking the pull-back ν^* on both sides of equations (2.2) and using (2.3) and (2.6), we get ([12])

$$(F\circ\nu)_i = -\sum_j h_{ij}(F_j\circ\nu), \tag{2.7}$$

$$(F_i\circ\nu)_j = -\sum_k h_{jk}(F_{ik}\circ\nu), \tag{2.8}$$

$$(A_{ij}\circ\nu)_k = -\sum_p (A_{ijp}\circ\nu)h_{pk}. \tag{2.9}$$

Denote $S_F e_j = \sum_i s_{ij}e_i$, then $s_{ij} = \sum_l (A_{il}\circ\nu)h_{lj}$. Thus we have

$$s_{ijk} = -\sum_{lp}(A_{ilp}\circ\nu)h_{pk}h_{lj} + \sum_l (A_{il}\circ\nu)h_{ljk}, \tag{2.10}$$

and

$$s_{ijk} = s_{ikj}. \tag{2.11}$$

Let $p := \langle x,\nu\rangle$ denote the support function. The following identities were already derived in [24] in the case when the anisotropic mean curvature is constant and (2.13) has been obtained in [6]. (2.12) will play an important role in the proof of Theorem 3.1.

PROPOSITION 2.1.

$$\Delta_F(F\circ\nu) + \mathrm{tr}(A_F S^2)(F\circ\nu) + \langle\nabla H_F, DF|_\nu\rangle = \mathrm{tr}(S_F^2), \tag{2.12}$$

$$\Delta_F\nu + \mathrm{tr}(A_F S^2)\nu + \nabla H_F = 0, \tag{2.13}$$

$$\Delta_F p + \mathrm{tr}(A_F S^2)p + H_F + \langle x, \nabla H_F\rangle = 0, \tag{2.14}$$

where $\Delta_F f := \mathrm{div}(A_F\nabla f)$, ∇f *denotes the gradient of* f *with respect to the induced metric on* Σ, $\mathrm{tr}(S_F^2) = \sum_{i,j} s_{ij}s_{ji}$ *and* $\mathrm{tr}(S^2) = \sum_{i,j} h_{ij}^2$.

PROOF. Making use of (2.9), (2.7), (2.8), (2.10) and (2.11), we get

$$\Delta_F(F\circ\nu) = \sum_{k,i}((A_{ik}\circ\nu)(F\circ\nu)_k)_i$$

$$
\begin{aligned}
&= \sum_{j,i,k,p} (A_{ikp} \circ \nu) h_{pi} h_{kj} (F_j \circ \nu) - \sum_{k,i,j} (A_{ik} \circ \nu) h_{kji} (F_j \circ \nu) \\
&\qquad - \sum_{k,i,j} (A_{ik} \circ \nu) h_{kj} (F_j \circ \nu)_i \\
&= \sum_j \Big(\sum_{i,k,p} (A_{ikp} \circ \nu) h_{pi} h_{kj} - \sum_{i,k} (A_{ik} \circ \nu) h_{kji} \Big) (F_j \circ \nu) \\
&\qquad + \sum_{k,i,j,p} (A_{ik} \circ \nu) h_{kj} h_{ip} (F_{jp} \circ \nu) \\
&= -\sum_j s_{iji} (F_j \circ \nu) + \sum_{k,i,j,p} (A_{ik} \circ \nu) h_{kj} h_{ip} (A_{jp} \circ \nu - (F \circ \nu) \delta_{jp}) \\
&= -\sum_j s_{iij} (F_j \circ \nu) + \sum_{i,j} s_{ij} s_{ji} - \sum_{i,j,k} (A_{ik} \circ \nu) h_{kj} h_{ij} (F \circ \nu) \\
&= -\sum_j (H_F)_j (F_j \circ \nu) + \mathrm{tr}(S_F^2) - \mathrm{tr}(A_F S^2) F \circ \nu.
\end{aligned}
$$

This proves (2.12).

By (2.9), (2.5), (2.10) and direct calculations, we get

$$
\begin{aligned}
\Delta_F \nu &= \sum_{k,i} ((A_{ik} \circ \nu) \nu_k)_i \\
&= \sum_{i,k,p,l} (A_{ikp} \circ \nu) h_{pi} h_{kl} e_l - \sum_{i,k,p} (A_{ik} \circ \nu)(h_{kip} e_p + h_{ip} h_{pk} \nu) \\
&= -\sum_l s_{iil} e_l - \sum_{i,k,p} (A_{ik} \circ \nu) h_{kp} h_{pi} \nu \\
&= -\sum_l (H_F)_l e_l - \mathrm{tr}(A_F S^2) \nu,
\end{aligned}
$$

which immediately verifies (2.13).

It follows from $dp = \langle x, d\nu \rangle = \langle x, \nu_i \omega_i \rangle$ that $p_i = \langle x, \nu_i \rangle$. Using (2.4) and (2.5), we get

$$
\begin{aligned}
p_{ij} &= \langle x, \nu_i \rangle_j = \langle e_j, -h_{ik} e_k \rangle + \langle x, -h_{ijk} e_k - h_{ik} h_{kj} \nu \rangle \\
&= -h_{ij} - h_{ijk} \langle x, e_k \rangle - h_{ik} h_{kj} p.
\end{aligned}
$$

Together with (2.9) and (2.10), it yields

$$\begin{aligned}\Delta_F p &= \sum_{i,k}((A_{ik}\circ\nu)p_k)_i \\ &= \sum_{i,k,q,l}(A_{ikq}\circ\nu)h_{qi}h_{kl}\langle x,e_l\rangle \\ &\qquad -\sum_{i,k,q}(A_{ik}\circ\nu)(h_{ki}+h_{kiq}\langle x,e_q\rangle+h_{kq}h_{qi}p) \\ &= -\sum_l s_{iil}\langle x,e_l\rangle - H_F - \operatorname{tr}(A_F S^2)p \\ &= -\langle x,\nabla H_F\rangle - H_F - \operatorname{tr}(A_F S^2)p,\end{aligned}$$

which completes the proof of (2.14). □

The following Minkowski formula and its higher order version were obtained in [11] and [12].

PROPOSITION 2.2. *Let Σ be a closed orientable hypersurface immersed in $\mathbb{R}^{n+1}$. Then*

$$\int_\Sigma (nF\circ\nu + H_F\langle x,\nu\rangle)dA = 0.$$

More generally,

$$\int_\Sigma (H_r F\circ\nu + H_{r+1}\langle x,\nu\rangle)dA = 0 \tag{2.15}$$

for $r=0,1,\cdots,n-1$.

3. Heintze-Karcher Type Inequality

Consider a closed orientable hypersurface Σ embedded in $\mathbb{R}^{n+1}$. Denote by ν the inner unit normal vector field to Σ. Assume that the anisotropic mean curvature H_F with respect to the inner normal ν is everywhere positive on Σ. Suppose that there exists a domain $\Omega\subset\mathbb{R}^{n+1}$ such that $\partial\Omega=\Sigma$.

Given a smooth positive F with convexity condition, we can associate the dual norm $F^*:\mathbb{R}^{n+1}\to\mathbb{R}$ defined by ([19], [13])

$$F^*(x)=\sup\{\frac{\langle x,z\rangle}{F(z)}|z\in\mathbb{S}^n\}.$$

Then we can define the F-distance function $d_F : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ to be $d_F(x, y) = F^*(y - x)$. Note that in general $d_F(x, y) \neq d_F(y, x)$ and when $F \equiv 1$, d_F is the Euclidean distance function d.

For each $p \in \Omega$, let $\rho_F(p) = d_F(\Sigma, p)$ be the F-distance of p from Σ. Then the level set of ρ_F can be expressed as

$$\begin{aligned} \Sigma_t &:= \{p \in \Omega | \rho_F(p) = t\} \\ &= \{\Phi(x, t) = x + t\nu_F(x) | x \in \Sigma\} \end{aligned}$$

for t is small enough. Remark that for fixed t,

$$d\Phi(x, t) = (I - tS_F) \circ dx.$$

Compactness of Σ guarantees that there exists $\epsilon > 0$, such that $\Phi(x, t)$ remains an immersed hypersurface if $|t| < \epsilon$. Moreover, up to parallel translations, ν is still the unit normal vector field of $\Phi(x, t)$. Define $c : \Sigma \to (0, +\infty)$ to be the F-cut function of Σ satisfying $c(x)$ is the greatest t such that $d_F(\Sigma, x + t\nu_F(x)) = t$ (See [13]). The point $\Phi(x, t)$ on Σ_t satisfies

$$\frac{\partial}{\partial t}\Phi(x, t) = \nu_F =: f\nu + \xi, \tag{3.1}$$

and Σ_t will disappear at the time $T = max_\Sigma c(x)$, where f is a smooth function defined on $\Sigma \times I \subset \Sigma \times \mathbb{R}$ and ξ is tangent to Σ_t. Remark that the anisotropic normal $\nu_F = \phi \circ \nu = F(\nu)\nu + DF|_\nu$. So $f = F(\nu)$ and $\xi = DF|_\nu$ in (3.1).

Proposition 3.1. *Under the flow* (3.1)*, we have the following evolution equations:*

$$\begin{aligned} &\frac{\partial}{\partial t} dA_t = (\mathrm{div}\xi - nHf)dA_t, \\ &\frac{\partial \nu}{\partial t} = -\nabla f + d\nu(\xi), \\ &\frac{\partial}{\partial t} F \circ \nu_t = \langle DF(\nu_t), -\nabla f + d\nu_t(\xi)\rangle, \\ &\frac{\partial}{\partial t} H_F = \Delta_F f + \mathrm{tr}(A_F S^2)f + \langle \nabla H_F, \xi\rangle. \end{aligned}$$

PROOF. In fact, the first three equations are classical results and the last one follows from Lemma 2.1 of [12] or (4.20) in [6]. □

Define

$$Q(t) := n\int_{\Sigma_t} \frac{F\circ\nu_t}{H_F} dA_t.$$

From the identities in Proposition 3.1, we have

$$\begin{aligned}\frac{1}{n}Q'(t) &= \int_{\Sigma_t}\{f\mathrm{div}(\frac{1}{H_F}DF|_{\nu_t}) - \mathrm{div}(\frac{f}{H_F}DF|_{\nu_t}) + \mathrm{div}(\frac{F\circ\nu_t}{H_F}\xi) \\ &\quad -\frac{f}{H_F}nH(F\circ\nu_t) - \frac{F\circ\nu_t}{H_F^2}(\Delta_F f + \mathrm{tr}(A_F S^2)f)\}dA_t.\end{aligned}$$

Thus by the divergence theorem and the definition of H_F (1.1) we get

$$\begin{aligned}\frac{1}{n}Q'(t) &= \int_{\Sigma_t}\{f\langle\nabla\frac{1}{H_F}, DF\circ\nu_t\rangle + \frac{f}{H_F}[\mathrm{div}(DF\circ\nu_t) - nHF\circ\nu_t] \\ &\quad -\frac{F\circ\nu_t}{H_F^2}[\Delta_F f + \mathrm{tr}(A_F S^2)f]\}dA_t \\ &= \int_{\Sigma_t}\{-\frac{f}{H_F^2}\langle\nabla H_F, DF\circ\nu_t\rangle \\ &\quad -f - \frac{F\circ\nu_t}{H_F^2}[\Delta_F f + \mathrm{tr}(A_F S^2)f]\}dA_t.\end{aligned}$$

Now taking into account that $f = F\circ\nu_t$ and (2.12), we get

$$\begin{aligned}\frac{1}{n}Q'(t) &= \int_{\Sigma_t}\{-\frac{F\circ\nu_t}{H_F^2}[\langle\nabla H_F, DF\circ\nu_t\rangle + \Delta_F(F\circ\nu_t) \\ &\quad +\mathrm{tr}(A_F S^2)F\circ\nu_t] - F\circ\nu_t\}dA_t \\ &= -\int_{\Sigma_t}(\frac{\mathrm{tr}(S_F^2)}{H_F^2}+1)F\circ\nu_t dA_t \\ &\le -(1+\frac{1}{n})\int_{\Sigma_t}F\circ\nu_t dA_t < 0,\end{aligned}$$

where we have used $\frac{\mathrm{tr}(S_F^2)}{H_F^2} \ge \frac{1}{n}$ and the equal sign holds if and only if $S_F = \frac{H_F}{n}\mathrm{Id}$. This shows that $Q(t)$ is monotone decreasing.

For $0 < \tau < T$,

$$\begin{aligned} Q(0) - Q(\tau) &= -\int_0^\tau Q'(t)dt \geq (n+1)\int_0^\tau \int_{\Sigma_t} F \circ \nu_t dA dt \\ &= (n+1)\int_{\Omega \cap \{\rho_F \leq \tau\}} dV, \end{aligned}$$

where the last equality follows from the co-area formula. Let $\tau \to T$. Then we obtain the following Heintze-Karcher type integral inequality that one can find also in [13] where it was proved using the ideas of [18].

THEOREM 3.1. *Let $x : \Sigma \to \mathbb{R}^{n+1}$ be a closed hypersurface embedded into the Euclidean space. If the anisotropic mean curvature H_F with respect to the inner normal ν is everywhere positive on Σ, then we have*

$$n \int_\Sigma \frac{F \circ \nu}{H_F} dA \geq (n+1)V(\Omega), \tag{3.2}$$

where $V(\Omega)$ is the volume of the compact domain Ω determined by Σ. Moreover, the equality holds if and only if Σ is anisotropic umbilical.

Once we have Minkowski formula (2.15), Heintze-Karcher type inequality (3.2) and the characterization for the anisotropic umbilical hypersurfaces in $\mathbb{R}^{n+1}$ (Lemma 1.1), it is straightforward to prove the Alexandrov type theorem 1.1 by the standard argument ([18], [13]).

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