The Space of (Contact) Anosov Flows on 3-Manifolds

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Abstract. The first half of this paper concerns the topology of the space $\mathcal{A}(M)$ of (not necessarily contact) Anosov vector fields on the unit tangent bundle M of closed oriented hyperbolic surfaces Σ. We show that there are countably infinite connected components of $\mathcal{A}(M)$, each of which is not simply connected. In the second part, we study contact Anosov flows. We show in particular that the time changes of contact Anosov flows form a C^1-open subset of the space of the Anosov flows which leave a particular C^∞ volume form invariant, if the ambient manifold is a rational homology sphere.

1. Introduction

The main purpose of this paper is to study the topology of the space of contact Anosov vector fields on 3-manifolds. But before going to that subject, we first consider the space $\mathcal{A}(M)$ of (not necessarily contact) Anosov vector fields on the unit tangent bundle M of a closed oriented hyperbolic surface Σ.

The results we obtain concerning $\mathcal{A}(M)$ are elementary and easy to show. However the author cannot find it in the literature, which makes him to record these fundamental facts. Denote by $\mathcal{L}(M)$ the space of nonvanishing C^∞ vector fields on M. There is one distiguished connected component $\mathcal{L}_0(M)$ of $\mathcal{L}(M)$.

THEOREM 1.1. *The space $\mathcal{A}(M)$ is contained in $\mathcal{L}_0(M)$.*

THEOREM 1.2. *The space $\mathcal{A}(M)$ has countably infinite connected components, each of which is not simply connected.*

After we determine the mapping class group of M in Section 2, we prove these results in Section 3.

Sections 4 and 5 are devoted to the study of contact Anosov flows. In section 4, we determine which time change of a contact Anosov flow is again contact Anosov. Especially we show that if the ambient manifold N is a rational homology sphere, such a time change is obtained by a conjugation by an orbit preserving C^∞ diffeomorphism.

In section 5, we study the space of contact Anosov flows. Let Ω be a C^∞ volume form on a closed oriented manifold N. Denote by $\mathcal{A}_\Omega(N)$ the space of the Ω-preserving Anosov vector fields. The main result is the following.

THEOREM 1.3. *If N is a rational homology sphere, the subset formed by time changes of contact Anosov flows is C^1-open in $\mathcal{A}_\Omega(N)$.*

In [FH2], plenty of examples of contact Anosov flows are constructed on various manifolds including hyperbolic 3-manifolds. Theorem 1.3 can also serve as producing new examples which are C^1-near to classical examples.

2.   The Mapping Class Group of M

Let Σ be a closed oriented surface of genus ≥ 2. Fix a Riemannian metric m_0 of curvature -1. Let $\pi : M = T^1\Sigma \to \Sigma$ be the unit tangent bundle with respect to m_0. The purpose of this section is to determine the mapping class group $\mathrm{MCG}(M)$ of M, which is, by definition, the quotient of the group of all the C^∞ diffeomorphisms of M by the identity component.

Denote by $\mathcal{H}$ the plane field of M consisting of horizontal vectors with respect to the Levi-Civita connection. The principal S^1 action on M is denoted by V^t, $0 \leq t \leq 2\pi$, whose infinitesimal generator is the *vertical vector field* V. The map V^t leaves $\mathcal{H}$ invarinat. The *standard geodesic vector field* is a horizontal vector field X such that $\pi_* X_{(x,v)} = v$, where $v \in T^1_x(\Sigma)$ and $x \in \Sigma$. It generates the *standard geodesic flow* $\{X^t\}$.

Notation 2.1.   In this paper, the flow generated by a vector field A is denoted by $\{A^t\}$.

The three vector fields V, X and $Y = V^{\pi/2}_* X$ span a Lie subalgebra, isomorphic to the Lie algebra of $\mathrm{PSL}(2,\mathbb{R})$. On the topological aspect, the following fundamental fact is a consequence of the classification of S^1 bundles over surfaces. See for example [O].

PROPOSITION 2.2. *Any C^∞ free S^1 action on M is conjugate to $\{V^t\}$ by a C^∞ diffeomorphism isotopic to the identity.*

Given $[f] \in \mathrm{MCG}(M)$, there is a representative f which commutes with the S^1-action $\{V^t\}$. Such f induces a diffeomorphism of Σ. Thus we get a homomorphism

$$\pi_* : \mathrm{MCG}(M) \to \mathrm{MCG}^\diamond(\Sigma),$$

where $\mathrm{MCG}^\diamond(\Sigma)$ is the generalized mapping class group of Σ consisting of orientation preserving or reversing classes.

Conversely, given $[g] \in \mathrm{MCG}^\diamond(\Sigma)$, the derivative dg yields a class $[g_*] \in \mathrm{MCG}(M)$, where $g_* : M \to M$ is defined from dg just by normalizing the image vector. This yields a cross section

$$s : \mathrm{MCG}^\diamond(\Sigma) \to \mathrm{MCG}(M).$$

Notice that $s([g]) = [g_*]$ is always orientation preserving regardless of the orientation property of $[g]$.

Now let K be the kernel of π_*. Any element of K can be represented by a diffeomorphism f of M which preserves the fibers of the S^1 action $\{V^t\}$, i. e. a diffeomorphism which covers the identity of Σ. Restricted to each fiber, f must be orientation preserving. For, otherwise the fixed point set of f (two points set for each fiber) would yield a multi cross section of $\pi : M \to \Sigma$, contradicting the fact that π is a nontrivial S^1 bundle. Each class of K can be represented by a diffeomorphism f which is a rigid rotation $V^{\rho(x)}$ on each fiber $\pi^{-1}(x)$, where $\rho : \Sigma \to S^1$ is a C^∞ function. This yields an identification $K \cong [\Sigma, S^1] \cong H^1(\Sigma, \mathbb{Z})$ and therefore we get:

PROPOSITION 2.3. *There is an isomorphism*

(2.1) $$\mathrm{MCG}(M) \cong H^1(\Sigma; \mathbb{Z}) \rtimes \mathrm{MCG}^\diamond(\Sigma).$$

REMARK 2.4. There is no orientation reversing homeomorphism of M.

3. The Space $\mathcal{A}(M)$

The vector fields V, X and Y, as well as $-V$, all belong to the same component of the space $\mathcal{L}(M)$ of the nonvanishing vector fields of M. Denote it

by $\mathcal{L}_0(M)$ and call it the *untwisted component*. Notice that the components of $\mathcal{L}(M)$ is in one to one correspondence with the set $[M, S^2]$.

The differential of a diffeomorphism f yields a homeomorphism df : $\mathcal{L}(M) \to \mathcal{L}(M)$.

PROPOSITION 3.1. *For any diffeomorphism f of M, we have $df(\mathcal{L}_0) = \mathcal{L}_0$.*

PROOF.   This follows from the fact that each class of $\mathrm{MCG}(M)$ has a representative which maps V to a nonzero function multiple of V. $\square$

Let us denote by $\mathcal{A}(M)$ the subset of $\mathcal{L}(M)$ consisting of Anosov vector fields.

THEOREM 3.2. *The space $\mathcal{A}(M)$ of the Anosov vector fields is contained in the untwisted component $\mathcal{L}_0(M)$.*

PROOF.   In way of showing the global structural stability theorem for Anosov flows on the manifold M, E. Ghys [G] proved that for any Anosov flow $\{A^t\}$, the weak stable foliation can be made transverse to the S^1 fibers after the conjugation by a diffeomorphism f. Each class of $\mathrm{MCG}(M)$ has a representative which leaves the orbit foliation of the S^1 action invariant. This implies that the conjugacy f can be chosen to be isotopic to the identity. That is, one may assume that the vector field A which generates $\{A^t\}$ is tangent to a foliation transverse to V. Then clearly the vector field $(1-s)A + sV$, $0 \le s \le 1$ is nonvanishing, and A is homotopic to V. $\square$

Now given any negatively curved Riemannian metric m of Σ, the unit tangent bundle with respect to m can be identified with M just by changing the length, and the geodesic flow $\{A^t\}$ of m can be viewed as a flow on M in the following way. Given $p \in M$, a unit tangent vector of Σ with respect to m_0, change the length of p so that the modified vector p' is a unit vector with respect to m. Consider a geodesic curve γ with respect to m whose innitial velocity vector is p'. Consider the vector $\gamma'(t)$ and change its length to obtain $q \in M$. Then $A^t(p) = q$.

Let us denote by $\mathcal{A}_0(M)$ the connected component of $\mathcal{A}(M)$ which contains the standard geodesic vector field X.

PROPOSITION 3.3. *The geodesic vector field of any negatively curved Riemannian metric on Σ belongs to $\mathcal{A}_0(M)$.*

PROOF. This follows from the fact that the space of negatively curved Riemannian metrics is connected. $\square$

Now for any diffeomorphism f of M, we have $df(\mathcal{A}(M)) = \mathcal{A}(M)$.

PROPOSITION 3.4. *For any element $[f]$ of $\mathrm{MCG}(M)$ which belongs to $\mathrm{MCG}^\diamond(\Sigma)$ in the decomposition (2.1), we have $df(\mathcal{A}_0(M)) = \mathcal{A}_0(M)$.*

PROOF. We only need to show that for any diffeomorphism g of Σ, the induced diffeomorphism g_* of M carries the standard geodesic vector field X to an element of $\mathcal{A}_0(M)$, i. e. $d(g_*)X \in \mathcal{A}_0(M)$. But this follows immediately from Proposition 3.3, since $d(g_*)X$ is the geodesic vector field of the Riemannian metric $(g^{-1})^* m_0$. $\square$

The action of $H^1(\Sigma, \mathbb{Z})$ in the decomposition (2.1) on $\mathcal{A}(M)$ is quite different. To study this we need the following lemma.

LEMMA 3.5. *Let $\{A^t\}$ be an arbitrary Anosov flow on M. For any essential oriented closed curve c of Σ, there is a unique periodic orbit γ of $\{A^t\}$ such that $\pi(\gamma)$ is homotopic to c.*

PROOF. This is true for the standard geodesic flow $\{X^t\}$. On the other hand any Anosov flow $\{A^t\}$ is flow equivalent[1] to $\{X^t\}$ by a homeomorphism h [G]. Finally the homeomorphism h can be isotoped to a diffeomorphism h' which preserve the orbit foliation of the S^1-action, by an isotopy h_t, $0 \leq t \leq 1$, where $h_0 = h'$ and $h_1 = h$. Clearly the lemma holds for $\{h_0 X^t h_0^{-1}\}$. Therefore by the continuity of the family of the topological flows, it also holds for $\{h_1 X^t h_1^{-1}\}$. Now the latter is flow equivalent to $\{A^t\}$, completing the proof of the lemma. $\square$

The next proposition shows the first half of Theorem 1.2.

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[1]This means that h carries any orbit of $\{A^t\}$ onto an orbit of $\{X^t\}$ in a way to preserve the time orientation of the flows.

PROPOSITION 3.6. *For any nonzero element $a \in H^1(\Sigma, \mathbb{Z})$, the class $[f]$ of $\mathrm{MCG}(M)$ which corresponds to a in (2.1) satisfies $df(\mathcal{A}_0(M)) \cap \mathcal{A}_0(M) = \emptyset$.*

PROOF.   We need only to show that the flow $\{fX^tf^{-1}\}$ is not isotopic to the flow $\{X^t\}$. Choose a simple closed curve c in Σ such that $\langle a, c \rangle \neq 0$. The periodic orbit γ in Lemma 3.5 for the flow $\{X^t\}$ is obtained as follows. Homotope c to a simple closed geodesic l. Then γ is the horizontal lift of l.

Next consider the periodic orbit γ' corresponding to c for the flow $\{fX^tf^{-1}\}$. For a convenient choice of f from the class, γ' is the image of γ by a nontrivial Dehn twist on the torus $\pi^{-1}(l)$. Therefore γ' is not homotopic to γ. This shows that the flow $\{fX^tf^{-1}\}$ is not isotopic to the flow $\{X^t\}$. $\square$

Let us show the last part of Theorem 1.2. Let $\mathcal{A}_*$ be an arbitrary component of $\mathcal{A}(M)$. Choose $\{A^t\}$ from $\mathcal{A}_*$ and consider the loop $\{V^sA^tV^{-s}\}$, $0 \leq s \leq 2\pi$, in $\mathcal{A}_*$. Assume for contradiction that this loop is contractible. Choose a periodic orbit $\gamma(t)$, $0 \leq t \leq T$ of $\{A^t\}$ such that $\pi(\gamma)$ is homotopic to a simple closed curve on Σ. Then the (possibly singular) torus $\{V^s\gamma(t)V^{-s} \mid 0 \leq t \leq T, \ 0 \leq s \leq 2\pi\}$ is homotopic to an essential torus. Especially it is π_1-injective. This contradicts that the above loop is contractible.

REMARK 3.7.   We suspect that the union of $df(\mathcal{A}_0(M))$, $[f]$ from $H^1(M, \mathbb{Z})$, is the whole of $\mathcal{A}(M)$, and that $\mathcal{A}_0(M)$ is homotopy equivalent to the circle. The analogous statement for the Anosov diffeomorphisms on the two torus can be found in [FG]. Their method is an application of the thermodynamical formalism. But for flows on 3-manifolds, it seems quite difficult to deform the Lyapunov exponent although a potential tool is available in [A].

4.   Contact Anosov Flows and Their Time Changes

Let N be a closed oriented C^∞ 3-manifold.

DEFINITION 4.1.   An Anosov flow $\{A^t\}$ (resp. Anosov vector field A) on N is said to be *contact* if it is the Reeb flow (resp. Reeb vector field) of some contact form τ.

If A is contact Anosov, then it leaves the volume form $\tau \wedge d\tau$ invariant. The C^∞ plane field $\mathrm{Ker}(\tau)$ is invariant by A^t. On the other hand the sum $E^{uu} \oplus E^{ss}$ of the strong stable and unstable bundles is the only A^t-invariant plane field transverse to A. Therefore we have

$$\mathrm{Ker}(\tau) = E^{uu} \oplus E^{ss},$$

and the contact form τ is uniquely determined by the Anosov vector field A. It is known [FH1] that if A is a volume preserving Anosov flow and if $E^{uu} \oplus E^{ss}$ is Lipschitz continuous, then $E^{uu} \oplus E^{ss}$ is in fact C^∞, and the flow A is contact Anosov. The contact structure $\mathrm{Ker}(\tau)$ of a contact Anosov flow is shown to be tight using a result of [H]. Contact Anosov flows exhibit strong ergodicity properties [L1, T1, T2]. The geodesic flow of a negatively curved surface is a typical example of contact Anosov flows. In fact it was the only known example before [FH2].

Before going to the study of time changes of contact Anosov flows, let us recall a well known fact about the invariant volume of an Anosov flow, which follows from the ergodicity and the Livšic homological theorem [L2].

THEOREM 4.2. *If an Anosov flow on N is volume preserving, then the invariant volume is C^∞ and unique up to a positive constant multiple.*

If A is an Anosov vector field and ϕ is a positive C^∞ function, then ϕA is called a *time change* of A. It is also an Anosov vector field. If A leaves the volume form Ω invariant, then ϕA leaves the volume form $\phi^{-1}\Omega$ invariant.

The purpose of this section is to study what kind of time change of a contact Anosov flow A is again contact, and the main result is Proposition 4.5 below. But before going there, we need some fundamental facts.

PROPOSITION 4.3. *If A is a suspension Anosov vector field, then any time change of A cannot be contact Anosov.*

PROOF. Since A is a suspension, there is a closed 1-form α such that $\alpha(A) = 1$. If there is no A-invariant volume form, then any time change of A does not admit an invariant volume, and it cannot be contact Anosov. So assume A admits a C^∞ volume form Ω. Suppose for contradiction that

ϕA is contact for a contact form τ. Then by Theorem 4.2, $\tau \wedge d\tau = c\phi^{-1}\Omega$ for some constant $c \neq 0$. Therefore

$$(4.1) \qquad d\tau = \iota_{\phi A}(\tau \wedge d\tau) = \iota_{\phi A}(c\phi^{-1}\Omega) = c\,\iota_A\Omega.$$

On the other hand, since

$$\mathcal{L}_A(\alpha \wedge (\iota_A\Omega)) = d\iota_A(\alpha \wedge (\iota_A\Omega)) = d\iota_A\Omega = 0,$$

and $\alpha \wedge (\iota_A\Omega)$ is nonvanishing, Theorem 4.2 shows,

$$(4.2) \qquad \alpha \wedge (\iota_A\Omega) = c'\Omega,$$

for some $c' \neq 0$. But α and $\iota_A\Omega$ are both closed, which says that $\iota_A\Omega$ cannot be null cohomologous, contradicting (4.1). $\square$

PROPOSITION 4.4. *If A is contact Anosov with a positive contact form, then any time change of A cannot be contact with a negative contact form.*

PROOF. Assume that A is contact Anosov with a contact form τ, i. e. $\tau(A) = 1$ and $\iota_A d\tau = 0$. Also assume that there are a positive function ϕ and a contact form τ' such that $\tau'(\phi A) = 1$, $\iota_A d\tau' = 0$, and $\tau' \wedge d\tau' = -c\phi^{-1}\tau \wedge d\tau$ for some constant $c > 0$. Then we have

$$d\tau' = \iota_{\phi A}(\tau' \wedge d\tau') = -c\,\iota_{\phi A}(\phi^{-1}\tau \wedge d\tau) = -c\,\iota_A(\tau \wedge d\tau) = -c\,d\tau,$$

and hence

$$\tau' = -c\tau + \omega,$$

for some closed 1-form ω.

Now for any asymptotic cycle Γ of A, we have

$$\langle \omega, \Gamma \rangle \geq \min(\tau'(A) + c\tau(A)) = \min(\phi^{-1}) + c > 0.$$

This implies [S] that A has a global cross section, contradicting Proposition 4.3. $\square$

The following is the main result of this section.

PROPOSITION 4.5. *A time change ϕA of a contact Anosov vector field A is again contact Anosov if and only if $\phi^{-1} = \omega(A) + c$ for a closed 1-form ω and a constant $c > 0$.*

PROOF. Let A (resp. ϕA) be a contact Anosov vector field with the contact form τ (resp. τ'). Then by Proposition 4.4 and its proof, we have $\tau' = c\tau + \omega$ for some closed 1-form ω and $c > 0$. Evaluating on ϕA, we get $\phi^{-1} = c + \omega(A)$.

The converse can be shown just by reversing the argument. $\square$

When the manifold N is a rational homology sphere, the above criterion becomes more transparant. Notice that there are cocompact lattices Γ of $\mathrm{PSL}(2, \mathbb{R})$ such that the quotient spaces $\Gamma \backslash \mathrm{PSL}(2, \mathbb{R})$ are rational homology spheres. They all admit contact Anosov flows. As before, let A be a contact Anosov vector field on a closed oriented 3-manifold N.

PROPOSITION 4.6. *Assume N is a rational homology sphere. A time change $B = \phi A$ is contact Anosov if and only if for some $c > 0$, the flow $\{B^{ct}\}$ is conjugate to $\{A^t\}$ by an orbit preserving C^∞ diffeomorphism.*

PROOF. For a time change $B = \phi A$ of A, there is a C^∞ map $a : \mathbb{R} \times N \to \mathbb{R}$ such that

$$(4.3) \qquad B^{a(t,p)}(p) = A^t(p), \quad \forall t \in \mathbb{R}, \ p \in N.$$

The function a is a cocycle over $\{A^t\}$, that is,

$$(4.4) \qquad a(t + s, p) = a(s, A^t(p)) + a(t, p).$$

Define a function $\alpha : N \to \mathbb{R}$ by $\alpha(p) = \frac{\partial}{\partial t} a(t, p)|_{t=0}$. By (4.4), we have

$$\frac{\partial}{\partial t} a(t, p) = \alpha(A^t(p)).$$

This implies

$$(4.5) \qquad a(t, p) = \int_0^t \alpha(A^s(p)) ds.$$

Now the if part of the theorem is obvious. So assume B is also contact. Then by differentiating (4.3), we get $\alpha = \phi^{-1}$. Thus Proposition 4.5 implies that $\alpha = \omega(A) + c$. Since N is a rational homology sphere, there is a C^∞ function ψ such that $\omega = d\psi$, and thus

$$\alpha = A(\psi) + c.$$

Then (4.5) implies

$$a(t, p) = \psi(A^t(p)) - \psi(p) + ct.$$

Define a map $f : N \to N$ by $f(p) = B^{-\psi(p)}(p)$. Then

$$(4.6) \qquad f(A^t(p)) = B^{a(t,p)-\psi(A^t(p))}(p) = B^{ct-\psi(p)}(p) = B^{ct}(f(p)).$$

The equation (4.6) implies that the map f is a C^∞ diffeomorphism, showing that $\{B^{ct}\}$ is conjugate to $\{A^t\}$ by f. $\square$

5. Perturbations of a Contact Anosov Flow

Let A be a contact Anosov flow on a closed oriented 3-manifold N, with the contact form τ. Then $\Omega = \tau \wedge d\tau$ is an A-invariant volume form. Let us denote by $\mathcal{X}_\Omega(N)$ (resp. $\mathcal{A}_\Omega(N)$) the space of Ω-preserving vector fields (resp. Ω-preserving Anosov vector fields) on N. Then $\mathcal{A}_\Omega(N)$ is a C^1-open subset of the linear space $\mathcal{X}_\Omega(N)$. For any $B \in \mathcal{X}_\Omega(N)$ small in the C^1 topology, the flow $A + B$ is again an Ω preserving Anosov flow, i. e. $A + B \in \mathcal{A}_\Omega(N)$. In this section we ask which $A + B$ can be a time change of a contact Anosov flow.

Assume $\phi(A + B)$ is contact Anosov for some positive function ϕ, with the contact form τ'. Then by Theorem 4.2, we have $\tau' \wedge d\tau' = c\phi^{-1}\Omega$ for some $c > 0$. Thus

$$\iota_{A+B}\Omega = \iota_{\phi(A+B)}\phi^{-1}\Omega = c^{-1}\iota_{\phi(A+B)}(\tau' \wedge d\tau') = c^{-1}d\tau', \quad \text{and}$$

$$\iota_B\Omega = \iota_{A+B}\Omega - \iota_A\Omega = c^{-1}d\tau' - d\tau,$$

showing that $\iota_B\Omega$ is an exact 2-form. On the other hand B belongs to $\mathcal{X}_\Omega(N)$ if and only if $\iota_B\Omega$ is closed. Since the correspodence $B \leftrightarrow \iota_B\Omega$ is bijective, this show the following.

PROPOSITION 5.1. *The subset consisting of time changes of contact Anosov flows is contained in a subspace of codimension equal to* $\dim H^2(N; \mathbb{R})$ *in a neighbourhood of A in $\mathcal{A}_\Omega(N)$.*

From now on let us assume that N is a rational homology sphere and show Theorem 1.3. Notice that the validity of Theorem 1.3 does not change

if one changes Ω by a positive function multiple. Therefore it suffices to assume that $A \in \mathcal{A}_\Omega(N)$ is a contact Anosov vector field with the contact form τ such that $\Omega = \tau \wedge d\tau$ and to show that for any C^1-small $B \in \mathcal{X}_\Omega(N)$, $A + B$ is a time change of a contact Anosov vector field.

Now the 2-form $\iota_B \Omega$ is closed since B is Ω-preserving, and exact since N is a rational homology sphere. Choose a 1-form β such that $d\beta = \iota_B \Omega$. Then we have $d\tau + d\beta = \iota_{A'} \Omega$, and hence

$$(5.1) \qquad \iota_{A'}(d\tau + d\beta) = 0.$$

Our goal is to show that for C^1-small B, there is a 1-form τ' such that

$$(5.2) \qquad d\tau' = d\tau + d\beta \ \text{ and } \tau'(A') > 0.$$

For, then the equation (5.1), together with the fact that $\iota_{A'}\Omega$ is nonvanishing, shows that the form τ' is contact, and the time change $\tau'(A')^{-1}A'$ is contact Anosov.

Since $\tau(A) = 1$, we have

$$(5.3) \qquad \int_\gamma \tau = \mathrm{per}(\gamma) \quad \text{for any periodic orbit } \gamma \text{ of } A,$$

where $\mathrm{per}(\gamma)$ denotes the period of γ.

Let us show that for any $\epsilon > 0$, if B is sufficiently C^1-small,

$$(5.4) \qquad \int_{\gamma'} (\tau + \beta) > (1 - 3\epsilon)\mathrm{per}(\gamma') \quad \text{for any periodic orbit } \gamma' \text{ of } A'.$$

First of all if we choose B so that $\|B\| \, \|\tau\| < \epsilon$, then we have

$$\tau(A') = \tau(A) + \tau(B) > 1 - \epsilon,$$

and therefore

$$\int_{\gamma'} \tau > (1 - \epsilon)\mathrm{per}(\gamma') \quad \text{for any periodic orbit } \gamma' \text{ of } A'.$$

So what we need is to show that

$$(5.5) \qquad |\int_{\gamma'} \beta| < 2\epsilon \, \mathrm{per}(\gamma') \quad \text{for any periodic orbit } \gamma' \text{ of } A'.$$

Now the C^1-norm of A' is bounded, regardless of the choice of A' from a C^1-neighbourhood $\mathcal{U}$ of A. Choose a triangulation T of N by small geodesic simplices. If we choose T fine enough compared with the above C^1-norm, then the orbits of A' look like straight lines in a close-up.

Then there is a constant $C > 0$ depending only on $\mathcal{U}$ and T with the following property: for any periodic orbit γ' of A', there is a simplicial loop γ'_T and an annulus $\mathbb{A}$ such that

$$(5.6) \qquad \partial\mathbb{A} = \gamma' \cup (-\gamma'_T) \text{ and } \mathrm{Area}(\mathbb{A}) \leq C\,\mathrm{per}(\gamma'), \quad \text{and}$$

$$(5.7) \qquad C^{-1}\|\gamma'_T\|_1 \leq \mathrm{per}(\gamma') \leq C\,\|\gamma'_T\|_1,$$

where $\|\cdot\|_1$ denotes the l^1 norm of the real coefficient chain group of the triangulation T. To see this, first replace γ' by another simple closed curve γ'' with C^2 norm smaller than twice the C^2 norm of γ' such that

$$2\,\mathrm{mesh}(T) \leq d(\gamma'(t), \gamma''(t)) \leq 3\,\mathrm{mesh}(T).$$

The curve γ'' also looks like a straight line in the close-up. Define γ'_T to be a simplicial loop contained in $\mathrm{mesh}(T)$-neighbourhood of γ'' with the minimum l^1 norm. Then (5.6) and (5.7) are satisfied.

Now if B, and hence $d\beta = \iota_B\Omega$, is small enough, we have

$$(5.8) \qquad |\int_{\gamma'} \beta - \int_{\gamma'_T} \beta| = |\int_{\mathbb{A}} d\beta| \leq \mathrm{Area}(\mathbb{A})\,\|d\beta\| < \epsilon\,\mathrm{per}(\gamma').$$

Now the boundary operator $\partial_2 : C_2(T) \to B_1(T)$ admits a cross section $\sigma : B_1(T) \to C_2(T)$. The mapping norm $\|\sigma\|_1$ of σ is finite since $B_1(T)$ is finite dimensional. Thus if B is small enough, then

$$(5.9) \qquad |\int_{\gamma'_T} \beta| = |\int_{\sigma(\gamma'_T)} d\beta| \leq \|\sigma\|_1\,\|\gamma'_T\|_1\,\|d\beta\| < \epsilon\,\mathrm{per}(\gamma'),$$

where the last inequality follows from (5.7). Now (5.8) and (5.9) imply the desired inequality (5.5). The proof of (5.4) is complete.

Finally let us show that (5.4) implies (5.2). For any periodic orbit γ' of A', there is associated an A'-invariant measure $\delta_{\gamma'}$ supported on γ'. It is well known, easy to show by the specification property of Anosov flows,

that the set of measures $\delta_{\gamma'}$ is dense in the set of the ergodic probability measures. Thus (5.4) implies that

$$(5.10) \qquad \langle \mu, (\tau + \beta)(A') \rangle \geq 1 - 3\epsilon$$

for any A'-invariant probability measure μ.

Then we have

$$(5.11) \qquad t^{-1} \int_0^t (\tau + \beta)(A') \circ (A')^t dt > 1 - 4\epsilon$$

for any large t. For, otherwise one can construct an A'-invariant probability measure violating (5.10).

If we put

$$\tau' = t^{-1} \int_0^t ((A')^t)^*(\tau + \beta)dt,$$

the left hand side of (5.11) coincides with $\tau'(A')$. On the other hand, we have

$$d\tau' = t^{-1} \int_0^t ((A')^t)^*(d\tau + d\beta)dt = d\tau + d\beta,$$

because

$$\mathcal{L}_{A'}(d\tau + d\beta) = d\, \iota_{A'}(d\tau + d\beta) = d\iota_{A'}\iota_{A'}\Omega = 0.$$

This shows (5.2), as is required.

Theorem 1.3 can be generalized as follows.

COROLLARY 5.2. *For an arbitrary closed 3-manifold N and a C^∞ volume form Ω, there is a C^1-neighbourhood $\mathcal{U}$ of 0 in $\mathcal{X}_\Omega(N)$ such that if A is a contact Anosov vector field, and $B \in \mathcal{U}$ satisfies that $\iota_B \Omega$ is exact, then $A + B$ is a time change of a contact Anosov vector field.*

Let Σ be a closed oriented surface endowed with a Riemannian metric m of varying negative curvature K, and let $\pi : M \to \Sigma$ be the unit tangent bundle with respect to m. Denote the vertical vector field by V, the geodesic vector field by X, and $Y = V_*^{\pi/2}X$. They satify:

$$(5.12) \qquad [V, X] = Y, \quad [V, Y] = -X, \quad [X, Y] = K \circ \pi\, V.$$

The 1-forms ξ, η and θ dual to X, Y and V satisfy:

(5.13) $d\xi = \theta \wedge \eta, \;\; d\eta = -\theta \wedge \xi, \;\; d\theta = -K \circ \pi \, \xi \wedge \eta.$

The volume form $\Omega = \xi \wedge \eta \wedge \theta$ is left invariant by the three vector fields V, X and Y.

G. P. Paternain [P] considers what is called the *magnetic vector field*

$$A_\lambda = X + \lambda V$$

for a costant λ, and shows the following.

THEOREM 5.3. **(G. P. Paternain)** *For $|\lambda|$ small and nonzero, the vector field A_λ is not contact, unless K is constant.*

Let us consider more generally the vector field

$$A_\phi = X + \phi \circ \pi \, V$$

for a C^∞ function $\phi : \Sigma \to \mathbb{R}$.
Now we have

(5.14) $\mathcal{L}_{A_\phi} \Omega = d \, \iota_{A_\phi} \Omega = d(\phi \circ \pi \, \xi \wedge \eta) = 0,$

where the last equality follows from $V(\phi \circ \pi) = 0$. Thus the vector field A_ϕ leaves the volume form Ω invariant, and it is Anosov for C^1-small ϕ. Applying Corollary 5.2, we get:

PROPOSITION 5.4. *For any negatively curved metric m and a C^1-small function $\phi : \Sigma \to \mathbb{R}$, the vector field A_ϕ is a time change of a contact Anosov vector field.*

PROOF.   What we need to show is that the closed 2-form $\phi \circ \pi \, \xi \wedge \eta$ is exact, which is an easy consequence of the fact that $H_2(M, \mathbb{Z})$ is generated by vertical tori and that

$$\iota_V(\phi \circ \pi \, \xi \wedge \eta) = 0. \;\; \square$$

The contact forms which appear in Proposition 5.4 are C^1-perturbations of the contact form ξ and is positive. On the other hand, the connection form θ is negative and tight by a result of [H]. Compare Remark 2.4.

QUESTION 5.5. Is there a contact Anosov flow on M whose contact form is θ?

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