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On a Semilinear Strongly Degenerate Parabolic Equation in an Unbounded Domain

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Abstract. We study the existence and long-time behavior of solutions to a semilinear strongly degenerate parabolic equation on $\mathbb{R}^N$ under an arbitrary polynomial growth order of the nonlinearity. To overcome some significant difficulty caused by the lack of compactness of the embeddings, the existence of global attractors is proved by combining the tail estimates method and the asymptotic *a priori* estimate method.

1. Introduction

The understanding of asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to attack the problem for a dissipative dynamical system is to consider its global attractor. The existence of global attractors has been proved for a large class of nondegenerate partial differential equations [8, 21]. In the last few years, a number of papers are devoted to the study of asymptotic behavior of solutions to degenerate equations.

One of the classes of degenerate equations that has been studied widely in recent years is the class of equations involving an operator of Grushin type

$$G_\alpha u = \Delta_x u + |x|^{2\alpha} \Delta_y u, \ \alpha \geq 0.$$

This operator was first introduced by Grushin in [9]. Noting that $G_0 = \Delta$, the Laplacian operator, and G_α, when $\alpha > 0$, is not elliptic in domains intersecting with the surface $x = 0$. The long-time behavior of solutions to semilinear parabolic equations involving this operator has been studied recently in both autonomous and non-autonomous cases [2, 3, 4, 5, 7]. We also refer the reader to some recent results about the generalized Grushin

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operators [10, 11, 14, 15, 16, 24]. Very recently, Thuy and Tri [22] considered a strongly degenerate operator

$$P_{\alpha,\beta}u = \Delta_x u + \Delta_y u + |x|^{2\alpha}|y|^{2\beta}\Delta_z u,\ \alpha,\beta \geq 0,$$

which is degenerate on two intersecting surfaces $x = 0$ and $y = 0$, and established some compact embedding theorems for weighted Sobolev spaces associated to the operator in bounded domains. Then using the theory of critical values of nonlinear functionals in Banach spaces [1], they also proved the existence of nontrivial solutions to Dirichlet problem for the associated elliptic equations.

In this paper we consider the following semilinear strongly degenerate parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - P_{\alpha,\beta}u + \lambda u + f(X,u) = g(X), \\ \qquad X = (x,y,z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3} = \mathbb{R}^N, t > 0, \\ u(X,0) = u_0(X),\ \ X \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $u_0 \in L^2(\mathbb{R}^N)$, the nonlinearity f and the external force g satisfy the following conditions:

(F) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$f(X,u)u \geq \alpha_1|u|^p - C_1(X), \tag{1.2}$$

$$|f(X,u)| \leq \alpha_2|u|^{p-1} + C_2(X), \tag{1.3}$$

$$\frac{\partial f}{\partial u}(X,u) \geq -\alpha_3, \tag{1.4}$$

for some $p \geq 2$, where $\alpha_1, \alpha_2, \alpha_3$ are positive constants, $C_1(\cdot) \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and $C_2(\cdot) \in L^{p'}(\mathbb{R}^N)$ with $1/p + 1/p' = 1$, are nonnegative functions. Denote $F(X,s) = \int_0^s f(X,\tau)d\tau$. Then we assume that F satisfies

$$-C_4(X) + \alpha_4|u|^p \leq F(X,u) \leq \alpha_5|u|^p + C_3(X), \tag{1.5}$$

where α_4, α_5 are positive constants, and $C_3(\cdot), C_4(\cdot) \in L^1(\mathbb{R}^N)$ are nonnegative functions;

(G) $g \in L^2(\mathbb{R}^N)$.

In order to study problem (1.1), we use the weighted Sobolev space introduced in [22]. Assume that $\Omega \subset \mathbb{R}^N$, we define the space $\mathcal{S}^1(\Omega)$ consisting of all functions u such that

$$\|u\|^2_{\mathcal{S}^1(\Omega)} := \int_\Omega \Big(|u|^2 + |\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2\Big)dX < +\infty.$$

Then the space $\mathcal{S}^1(\Omega)$ is a Hilbert space with the inner product

$$(u,v)_{\mathcal{S}^1(\Omega)} = \int\limits_{\mathbb{R}^N} \Big(uv + \nabla_x u \nabla_x v + \nabla_y u \nabla_y v + |x|^{2\alpha}|y|^{2\beta}\nabla_z u \nabla_z v\Big)dX.$$

It is noticed that if Ω is a bounded domain, then the embedding $\mathcal{S}^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact (see [22]). However, this property is no longer true for unbounded domains. Here the natural energy space for problem (1.1) involves the space $\mathcal{S}^1(\mathbb{R}^N)$ and its dual space $\mathcal{S}^{-1}(\mathbb{R}^N)$. We also use the space $\mathcal{S}^2(\mathbb{R}^N)$, which was used before in more general situations in [11, 19], consisting of all functions u such that

$$\|u\|^2_{\mathcal{S}^2(\mathbb{R}^N)} : = \int_\Omega \Big(|u|^2 + |P_{\alpha,\beta}u|^2\Big)dX < +\infty.$$

One can check that the embedding $\mathcal{S}^2(\mathbb{R}^N) \hookrightarrow \mathcal{S}^1(\mathbb{R}^N)$ is continuous.

The main aim of this paper is to prove the existence of a global attractor in the space $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ for the semigroup generated by problem (1.1). First, we use the Galerkin method to prove the global existence of a weak solution and then construct the continuous semigroup associated to problem (1.1). Next, we use *a priori* estimates to show the existence of a bounded absorbing set in $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ for the semigroup. In the case of bounded domains, since the embedding $\mathcal{S}^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, this immediately implies the asymptotic compactness of the semigroup in $L^2(\Omega)$, and therefore the existence of a global attractor in $L^2(\Omega)$ (see [23] for more details). Here because the embedding is no longer compact, the proof of the asymptotic compactness in $L^2(\mathbb{R}^N)$ is much more involved. To do this, we exploit the tail estimates method introduced in [25], and as a result, we obtain the existence of a global attractor in $L^2(\mathbb{R}^N)$. When proving the existence of global attractors in $L^p(\mathbb{R}^N)$ and in $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, to overcome the difficulty arising due to the lack of embedding results, we use the asymptotic *a priori* estimate method initiated in [13, 26]. The main new feature of the paper is that we are able to prove the existence of global attractors for a class of semilinear strongly degenerate parabolic equations in unbounded domains.

It is noticed that the results obtained in the paper are also true for problem (1.1) in an arbitrary (bounded or unbounded) domain Ω in $\mathbb{R}^N$, not necessary the whole space $\mathbb{R}^N$, with the homogeneous Dirichlet boundary condition. Then, instead of $\mathcal{S}^1(\mathbb{R}^N)$ and $\mathcal{S}^2(\mathbb{R}^N)$, we use the spaces $\mathcal{S}_0^1(\Omega)$ and $\mathcal{S}_0^2(\Omega)$, defined as the completions of $C_0^\infty(\Omega)$ in the corresponding norms. For the existence, continuity and long-time behavior of strong solutions to this problem in bounded domains, we refer the reader to a very recent work [6]. These results can be also extended to the non-autonomous case, i.e. when the external force g may depend on time t, by using the theory of uniform/pullback attractors (see e.g. [2, 4, 7] for the case of Grushin operator). In particular, when $\alpha = \beta = 0$, our results recover/extend some existing ones for semilinear nondegenerate parabolic equations.

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of a weak solution to problem (1.1) by using the Galerkin method. In Section 3, we show the existence of global attractors in various function spaces for the semigroup generated by problem (1.1) by exploiting and combining the tail estimates method and the asymptotic *a priori* estimate method.

2. Existence and Uniqueness of Weak Solutions

We first give the definition of a weak solution.

Definition 2.1. A function $u : (0, +\infty) \to \mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ is said to be a weak solution of (1.1) if $u \in L^2(0, T; \mathcal{S}^1(\mathbb{R}^N)) \cap L^p(0, T; L^p(\mathbb{R}^N)) \cap L^\infty(0, T; L^2(\mathbb{R}^N))$ for all $T > 0$, and

$$\begin{aligned}
(u(t), v)_{L^2(\mathbb{R}^N)} &+ \int_0^t \int_{\mathbb{R}^N} (\nabla_x u \nabla_x v + \nabla_y u \nabla_y v + |x|^{2\alpha}|y|^{2\beta}\nabla_z u \nabla_z v) dX ds \\
&+ \lambda \int_0^t (u, v)_{L^2(\mathbb{R}^N)} dt + \int_0^t \int_{\mathbb{R}^N} f(X, u) v dX ds \\
&= (u_0, v)_{L^2(\mathbb{R}^N)} + \int_0^t (g, v)_{L^2(\mathbb{R}^N)} ds
\end{aligned}$$

for all $v \in \mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ and all $t > 0$.

We now prove the following theorem.

THEOREM 2.1. *Let* $(\mathbf{F}) - (\mathbf{G})$ *hold. Then, for any* $u_0 \in L^2(\mathbb{R}^N)$ *given, problem* (1.1) *has a unique weak solution* u. *Moreover, the mapping* $u_0 \mapsto u(t)$ *is continuous on* $L^2(\mathbb{R}^N)$.

PROOF. *i) Existence.* For each $m \geq 1$, we denote

$$\Omega_m = \{X \in \mathbb{R}^N : |X|_{\mathbb{R}^N} < m\},$$

where $|.|_{\mathbb{R}^N}$ denotes the Euclidean norm in $\mathbb{R}^N$. For each integer $n \geq 1$, we denote by

$$u_n(t) = \sum_{j=1}^{n} \gamma_{nj}(t)\omega_j$$

a solution of

$$\begin{aligned} &\frac{d}{dt}(u_n(t), \omega_j) - (P_{\alpha,\beta} u_n(t), \omega_j) + \lambda(u_n(t), \omega_j) + (f(X, u_n(t)), \omega_j) \\ &\qquad = (g, \omega_j), t > 0, \\ &(u_n(0), \omega_j) = (u_0, \omega_j), \quad j = 1, 2, \ldots, n, \end{aligned} \tag{2.1}$$

where $\{\omega_j : j \geq 1\} \subset \mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ is a Hilbert basis of $L^2(\mathbb{R}^N)$ such that $\operatorname{span}\{\omega_j : j \geq 1\}$ is dense in $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$.

Multiplying the first equation in (2.1) by $\gamma_{nj}(t)$, taking the sum from 1 to n, and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\mathbb{R}^N)} &+ \int_{\mathbb{R}^N} (|\nabla_x u_n|^2 + |\nabla_y u_n|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u_n|^2) dX \\ &+ \lambda\|u_n\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} f(X, u_n) u_n dX = \int_{\mathbb{R}^N} g u_n dX. \end{aligned}$$

Using (1.4) and the Cauchy inequality, we get

$$\begin{aligned} &\frac{d}{dt}\|u_n\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N} (|\nabla_x u_n|^2 + |\nabla_y u_n|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u_n|^2) dX \\ &+\lambda\|u_n\|^2_{L^2(\mathbb{R}^N)} + 2\alpha_1 \int_{\mathbb{R}^N} |u_n|^p dX \leq C + \frac{1}{\lambda}\|g\|^2_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Integrating from 0 to t, $0 < t \leq T$, we obtain

$$\|u_n(t)\|^2_{L^2(\mathbb{R}^N)} + 2\int_0^t \int_{\mathbb{R}^N} (|\nabla_x u_n|^2 + |\nabla_y u_n|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u_n|^2) dX ds$$
$$+\lambda \int_0^t \|u_n\|^2_{L^2(\mathbb{R}^N)} ds + 2\alpha_1 \int_0^t \int_{\mathbb{R}^N} |u_n|^p dX ds \leq \|u_0\|^2_{L^2(\mathbb{R}^N)}$$
$$+CT + \frac{1}{\lambda} T \|g\|^2_{L^2(\mathbb{R}^N)}.$$

Hence it deduces that

$$\{u_n\} \text{ is bounded in } L^2(0,T;\mathcal{S}^1(\mathbb{R}^N)) \cap L^p(0,T;L^p(\mathbb{R}^N)) \cap L^\infty(0,T;L^2(\mathbb{R}^N)). \tag{2.2}$$

Using (1.3), one can check that

$$\{f(X,u_n)\} \text{ is bounded in } L^{p'}(0,T;L^{p'}(\mathbb{R}^N))$$

for all $T > 0$. Then, there exists a subsequence $\{u_\mu\}$ such that

$$u_\mu \rightharpoonup^* u \text{ weakly-star in } L^\infty(0,T;L^2(\mathbb{R}^N)),$$
$$u_\mu \rightharpoonup u \text{ in } L^p(0,T;L^p(\mathbb{R}^N)),$$
$$u_\mu \rightharpoonup u \text{ in } L^2(0,T;\mathcal{S}^1(\mathbb{R}^N)), \tag{2.3}$$
$$f(X,u_\mu) \rightharpoonup \chi \text{ in } L^{p'}(0,T;L^{p'}(\mathbb{R}^N)), \tag{2.4}$$

for all $T > 0$. Hence, (2.3) implies that

$$-P_{\alpha,\beta}u_\mu + \lambda u_\mu \rightharpoonup -P_{\alpha,\beta}u + \lambda u \text{ in } L^2(0,T;\mathcal{S}^{-1}(\mathbb{R}^N)).$$

Now, to prove that $\chi(t) = f(.,u(t))$, we argue similarly to [18]. Arguing in a similar way as in [18, p. 75], we get

$$\lim_{a\to 0} \sup_\mu \int_0^{T-a} \|u_\mu(t+a) - u_\mu(t)\|^2_{L^2(\mathbb{R}^N)} dt = 0. \tag{2.5}$$

Let $\phi \in C^1([0,+\infty))$ be a function such that

$$0 \leq \phi(s) \leq 1,$$
$$\phi(s) = 1 \quad \forall s \in [0,1],$$

$$\phi(s) = 0 \quad \forall s \geq 2.$$

For each μ and $m \geq 1$, we define

$$v_{\mu,m}(X,t) = \phi\Big(\frac{|X|^2_{\mathbb{R}^N}}{m^2}\Big)u_\mu(t), \quad \forall X \in \Omega_{2m}, \forall \mu, \forall m \geq 1. \tag{2.6}$$

We obtain from (2.2) that, for all $m \geq 1$, the sequence $\{v_{\mu,m}\}_{\mu\geq 1}$ is bounded in $L^\infty(0,T;L^2(\Omega_{2m})) \cap L^p(0,T;L^p(\Omega_{2m})) \cap L^2(0,T;\mathcal{S}^1(\Omega_{2m}))$, for all $T > 0$. In particular, it follows that

$$\lim_{a\to 0} \sup_\mu \Big(\int_0^a \|v_{\mu,m}(X,t)\|^2_{L^2(\Omega_{2m})} dt + \int_{T-a}^T \|v_{\mu,m}(X,t)\|^2_{L^2(\Omega_{2m})} dt \Big) = 0.$$

On the other hand, from (2.5) we deduce that for all $m \geq 1$,

$$\lim_{a\to 0} \sup_\mu \Big(\int_0^{T-a} \|v_{\mu,m}(X,t+a) - v_{\mu,m}(X,t)\|^2_{L^2(\Omega_{2m})} dt \Big) = 0.$$

Moreover, since Ω_{2m} is a bounded set, then $\mathcal{S}^1(\Omega_{2m})$ is included in $L^2(\Omega_{2m})$ with compact injection [22]. Then, by Theorem 13.3 and Remark 13.1 in [20], we obtain that

$$\{v_{\mu,m}\}_{\mu\geq 1} \text{ is relatively compact in } L^2(0,T;L^2(\Omega_{2m})),$$

and thus, taking into account that $v_{\mu,m}(X,t) = u_\mu(X,t)$ for all $X \in \Omega_m$, we deduce that, in particular, for all $m \geq 1$,

$$\{u_\mu|_{\Omega_m}\} \text{ is pre-compact in } L^2(0,T;L^2(\Omega_m)). \tag{2.7}$$

Hence, by a diagonal proceduce, one can conclude from (2.7) and (2.3) that there exists a subsequence $\{u^\mu_\mu\}_{\mu\geq 1} \subset \{u_\mu\}_{\mu\geq 1}$ such that

$$u^\mu_\mu \to u \text{ in } \Omega_m \times (0,+\infty) \text{ as } n \to \infty, \forall m \geq 1.$$

Then, as $f(\cdot,\cdot)$ is continuous,

$$f(X,u^\mu_\mu) \to f(X,u) \text{ a.e. in } \Omega_m \times (0,+\infty),$$

and as $\{f(X,u^\mu_\mu)\}$ is bounded in $L^{p'}(\Omega_m \times (0,T))$, by Lemma 1.3 in [12, Chapter 1], we obtain

$$f(X,u^\mu_\mu) \rightharpoonup f(X,u) \text{ in } L^{p'}(0,T;L^{p'}(\Omega_m)).$$

By the uniqueness of the weak limit, we have

$$\chi = f(X,u) \text{ a.e. in } \Omega_m \times (0,T) \quad \forall T>0, \forall m \geq 1,$$

and thus, taking into account that $\cup_{m=1}^{\infty}\Omega_m = \mathbb{R}^N$, we obtain

$$\chi = f(X,u) \text{ a.e. in } \mathbb{R}^N \times (0,+\infty). \tag{2.8}$$

Then, (2.8) and (2.4) yield that

$$f(X,u_\mu) \rightharpoonup f(X,u) \text{ in } L^{p'}(0,T;L^{p'}(\mathbb{R}^N)) \quad \forall T>0. \tag{2.9}$$

Hence, it is standard matter to show that u is a weak solution to problem (1.1).

(ii) Uniqueness and continuous dependence. Let $u_0, v_0 \in L^2(\mathbb{R}^N)$. Denote by u, v two corresponding solutions of problem (1.1) with initial data u_0, v_0. Then $w = u - v$ satisfies

$$\begin{cases} w_t - P_{\alpha,\beta}w + \lambda w + f(X,u) - f(X,v) = 0, \\ w(0) = u_0 - v_0. \end{cases}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\mathbb{R}^N)} + C\|w\|^2_{\mathcal{S}^1(\mathbb{R}^N)} + \int_{\mathbb{R}^N}(u-v)(f(X,u)-f(X,v))dx = 0,$$

for a.e. $t \in [0,T]$. Using the condition (1.4), we have

$$\frac{d}{dt}\|w\|^2_{L^2(\mathbb{R}^N)} + C\|w\|^2_{\mathcal{S}^1(\mathbb{R}^N)} \leq 2\alpha_3\|w\|^2_{L^2(\mathbb{R}^N)}, \quad \text{for a.e. } t\in[0,T].$$

Applying the Gronwall inequality, we obtain

$$\|w(t)\|^2_{L^2(\mathbb{R}^N)} \leq \|w(0)\|^2_{L^2(\mathbb{R}^N)}e^{2\alpha_3 t}.$$

This implies the uniqueness (if $u_0 = v_0$) and the continuous dependence of solutions. □

3. Existence of Global Attractors

Thanks to Theorem 2.1, we can define a continuous semigroup

$$S(t): L^2(\mathbb{R}^N) \to \mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N),$$

where $S(t)u_0 := u(t)$ is the unique weak solution of (1.1) subject to u_0 as initial datum.

3.1. Existence of bounded absorbing sets

For the sake of brevity, in the following lemmas, we give some formal caculations, the rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [17].

We first prove the existence of an absorbing set for $S(t)$ in $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$.

Lemma 3.1. *Suppose* $(\mathbf{F}) - (\mathbf{G})$ *hold. Then the semigroup* $S(t)$ *generated by* (1.1) *has a bounded absorbing set in* $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$*, that is, there exists a positive constant* ρ*, such that for every bounded subset* B *in* $L^2(\mathbb{R}^N)$*, there is a number* $T = T(B) > 0$*, such that for all* $t \geq T$*,* $u_0 \in B$*, we have*

$$\|u(t)\|^2_{\mathcal{S}^1(\mathbb{R}^N)} + \|u(t)\|^p_{L^p(\mathbb{R}^N)} \leq \rho.$$

Proof. Taking the inner product of (1.1) with u in $L^2(\mathbb{R}^N)$ we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N}(|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2)dX \\ + \lambda\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} f(X,u)udX = (g,u)_{L^2(\mathbb{R})}.\end{aligned} \tag{3.1}$$

Using (1.2), we have

$$\int_{\mathbb{R}^N} f(X,u)udX \geq \alpha_1 \int_{\mathbb{R}^N} |u|^p dX - \int_{\mathbb{R}^N} C_1(X)dX. \tag{3.2}$$

By the Cauchy inequality, the right-hand side of (3.1) is estimated as follows

$$|(g,u)_{L^2(\mathbb{R})}| \leq \|g\|_{L^2(\mathbb{R}^N)}\|u\|_{L^2(\mathbb{R}^N)} \leq \frac{\lambda}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2\lambda}\|g\|^2_{L^2(\mathbb{R}^N)}. \tag{3.3}$$

It follows from (3.1) - (3.3) that

$$\begin{aligned}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N}(|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2)dX \\ + \lambda\|u\|^2_{L^2(\mathbb{R}^N)} + 2\alpha_1\int_{\mathbb{R}^N}|u|^p dX \leq C + \frac{1}{\lambda}\|g\|^2_{L^2(\mathbb{R}^N)}.\end{aligned} \tag{3.4}$$

Hence, in particular, we have

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^N)} \leq -\lambda\|u(t)\|^2 + C + \frac{1}{\lambda}\|g\|^2_{L^2(\mathbb{R}^N)}. \tag{3.5}$$

Using the Gronwall inequality, we obtain

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)} \le e^{-\lambda t}\|u_0\|^2_{L^2(\mathbb{R}^N)} + \Big(\frac{C}{\lambda} + \frac{1}{\lambda^2}\|g\|^2_{L^2(\mathbb{R}^N)}\Big)(1-e^{-\lambda t}). \tag{3.6}$$

From (3.6) we deduce the existence of a bounded absorbing set in $L^2(\mathbb{R}^N)$: There are a constant R and a time $t_0(\|u_0\|_{L^2(\mathbb{R}^N)})$ such that for the solution $u(t) = S(t)u_0$,

$$\|u(t)\|_{L^2(\mathbb{R}^N)} \le R \quad \text{for all} \quad t \ge t_0(\|u_0\|_{L^2(\mathbb{R}^N)}).$$

Integrating (3.4) on $(t, t+1), t \ge t_0(\|u_0\|_{L^2(\mathbb{R}^N)})$, and using (1.5), we find that

$$\begin{aligned}\int_t^{t+1} \Big(\int_{\mathbb{R}^N} (|\nabla_x u(s)|^2 + |\nabla_y u(s)|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u(s)|^2)dX \\ + \lambda\|u(s)\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N} F(X,u(s))dX\Big)ds \\ \le C\Big(\|u(t)\|^2_{L^2(\mathbb{R}^N)} + 1 + \|g\|^2_{L^2(\mathbb{R}^N)}\Big) \\ \le C\Big(R^2 + 1 + \|g\|^2_{L^2(\mathbb{R}^N)}\Big).\end{aligned} \tag{3.7}$$

Multiplying (1.1) by $u_t(s)$ and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned}\|u_t(s)\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\frac{d}{ds}\Big(\int_{\mathbb{R}^N} (|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2)dX \\ + \lambda\|u(s)\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N} F(X,u(s))dX\Big) \\ = \int_{\mathbb{R}^N} g u_t(s) dX \le \frac{1}{2}\|g\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\|u_t(s)\|^2_{L^2(\mathbb{R}^N)}.\end{aligned} \tag{3.8}$$

Hence,

$$\begin{aligned}\frac{d}{ds}\Big(\int_{\mathbb{R}^N} (|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2)dX + \lambda\|u(s)\|^2_{L^2(\mathbb{R}^N)} \\ + 2\int_{\mathbb{R}^N} F(X,u(s))dX\Big) \le \|g\|^2_{L^2(\mathbb{R}^N)}.\end{aligned} \tag{3.9}$$

Combining (3.7), (3.9), and using the uniform Gronwall inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N} (|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2)dX + \lambda\|u(t)\|^2_{L^2(\mathbb{R}^N)} \\ + 2\int_{\mathbb{R}^N} F(X,u(t))dX \le C\Big(R^2 + 1 + \|g\|^2_{L^2(\mathbb{R}^N)}\Big).\end{aligned} \tag{3.10}$$

Using (1.5) once again, we finish the proof. □

We now derive uniform estimates of the derivative of solutions in time.

LEMMA 3.2. *Suppose* (**F**) – (**G**) *hold. Then for every bounded subset* B *in* $L^2(\mathbb{R}^N)$, *there exists a constant* $T = T(B) > 0$ *such that*

$$\|u_t(s)\|^2_{L^2(\mathbb{R}^N)} \leq \rho_1 \text{ for all } u_0 \in B, \text{ and } s \geq T,$$

where $u_t(s) = \frac{d}{dt}(S(t)u_0)|_{t=s}$ *and* ρ_1 *is a positive constant independent of* B.

PROOF. By differentiating (1.1) in time and denoting $v = u_t$, we get

$$\frac{\partial v}{\partial t} - P_{\alpha,\beta}v + \lambda v + \frac{\partial f}{\partial u}(x,u)v = 0.$$

Taking the inner product of the above equality with v in $L^2(\mathbb{R}^N)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\Big(|\nabla_x v|^2 + |\nabla_y v|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z v|^2\Big)dX \\ +\lambda\|v\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\frac{\partial f}{\partial u}(x,u)|v|^2 dX = 0.\end{aligned} \tag{3.11}$$

By (1.4), it follows from (3.11) that

$$\frac{d}{dt}\|v\|^2_{L^2(\mathbb{R}^N)} \leq 2\alpha_3\|v\|^2_{L^2(\mathbb{R}^N)}. \tag{3.12}$$

On the other hand, integrating (3.8) from t to $t+1$ and using (3.10), we obtain

$$\int_t^{t+1}\|u_t(s)\|^2_{L^2(\mathbb{R}^N)}ds \leq C(\rho, \|g\|^2_{L^2(\mathbb{R}^N)}) \tag{3.13}$$

as t large enough. Combining (3.12) with (3.13), and using the uniform Gronwall inequality, we have

$$\|u_t(s)\|^2_{L^2(\mathbb{R}^N)} \leq C(\rho, \|g\|^2_{L^2(\mathbb{R}^N)}).$$

The proof is complete. □

We now show the existence of a bounded absorbing set in $\mathcal{S}^2(\mathbb{R}^N)$.

Lemma 3.3. *The semigroup* $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set in* $\mathcal{S}^2(\mathbb{R}^N)$*, i.e., there exists a constant* $\rho_2>0$ *such that for any bounded subset* $B\subset L^2(\mathbb{R}^N)$*, there is a* $T_B>0$ *such that*

$$\|P_{\alpha,\beta}u(t))\|^2_{L^2(\mathbb{R}^N)}+\|u(t)\|^2_{L^2(\mathbb{R}^N)}\leq \rho_2,\ \textit{for any } t\geq T_B, u_0\in B.$$

Proof. Taking the L^2-inner product of (1.1) with $-P_{\alpha,\beta}u+\lambda u$, we have

$$\begin{aligned}&\|P_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)}+\lambda^2\|u\|^2_{L^2(\mathbb{R}^N)}+\lambda\int_{\mathbb{R}^N}f(X,u)udX\\ \leq\ & 2\lambda\int_{\mathbb{R}^N}uP_{\alpha,\beta}udX-\int_{\mathbb{R}^N}u_t\Big(-P_{\alpha,\beta}u+\lambda u\Big)dX\\ &+\int_{\mathbb{R}^N}f(X,u)P_{\alpha,\beta}udX+\int_{\mathbb{R}^N}g\Big(-P_{\alpha,\beta}u+\lambda u\Big)dX.\end{aligned}$$

Using (1.2) and integrating by parts the third term on the right-hand side, we have

$$\begin{aligned}&\|P_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)}+\lambda^2\|u\|^2_{L^2(\mathbb{R}^N)}+\lambda\alpha_1\int_{\mathbb{R}^N}|u|^pdX\\ \leq\ & 2\lambda\int_{\mathbb{R}^N}uP_{\alpha,\beta}udX-\int_{\mathbb{R}^N}u_t\Big(-P_{\alpha,\beta}u+\lambda u\Big)dX\\ &-\int_{\mathbb{R}^N}f'_u(X,u)\Big(|\nabla_xu|^2+|\nabla_yu|^2+|x|^{2\alpha}|y|^{2\beta}|\nabla_zu|^2\Big)dX\\ &+\int_{\mathbb{R}^N}g\Big(-P_{\alpha,\beta}u+\lambda u\Big)dX+\lambda\int_{\mathbb{R}^N}C_1(X)dX.\end{aligned}$$

By the Cauchy inequality and assumption (1.4), in particular, we have

$$\begin{aligned}&\|P_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)}+\|u\|^2_{L^2(\mathbb{R}^N)}\\ \leq\ & C(1+\|u_t\|^2_{L^2(\mathbb{R}^N)}+\|u\|^2_{\mathcal{S}^1(\mathbb{R}^N)}+\|u\|^p_{L^p(\mathbb{R}^N)}+\|g\|^2_{L^2(\mathbb{R}^N)}).\end{aligned}$$

Hence, from Lemmas 3.1 and 3.2 there exists $\rho_2>0$ such that

$$\|P_{\alpha,\beta}u(t)\|^2_{L^2(\mathbb{R}^N)}+\|u(t)\|^2_{L^2(\mathbb{R}^N)}\leq\rho_2$$

for all t large enough. This completes the proof. □

3.2. Existence of a global attractor in $L^2(\mathbb{R}^N)$

LEMMA 3.4. *Suppose* (**F**) − (**G**) *hold. Then for any* $\eta > 0$ *and any bounded subset* $B \subset L^2(\mathbb{R}^N)$*, there exist* $T = T(\eta, B) > 0$ *and* $K = K(\eta, B) > 0$ *such that for all* $t \geq T$ *and* $k \geq K$*,*

$$\int_{|X|\geq k} |u(X,t)|^2 dX \leq \eta,$$

where u *is the weak solution of* (1.1) *subject to the initial condition* $u(0) = u_0 \in B$.

PROOF. Let θ be a smooth function satisfying $0 \leq \theta(s) \leq 1$ for $s \in \mathbb{R}^+$, and

$$\theta(s) = 0, \text{ for } 0 \leq s \leq 1; \quad \theta(s) = 1 \text{ for } s \geq 2.$$

Then there exists a constant C such that $|\theta'(s)| \leq C$ for all $s \in \mathbb{R}^+$. Taking the inner product of (1.1) with $\theta(\frac{|X|^2}{k^2})u$ in $L^2(\mathbb{R}^N)$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \theta(\frac{|X|^2}{k^2})|u|^2 dX - \int_{\mathbb{R}^N} \theta(\frac{|X|^2}{k^2}) u P_{\alpha,\beta} u dX \\ + \lambda \int_{\mathbb{R}^N} \theta(\frac{|X|^2}{k^2})|u|^2 dX + \int_{\mathbb{R}^N} \theta(\frac{|X|^2}{k^2}) f(X,u) u dX \\ = \int_{\mathbb{R}^N} \theta(\frac{|X|^2}{k^2}) g(X) u(X,t) dX.\end{aligned} \tag{3.14}$$

For the right-hand side of (3.14) we find that

$$\begin{aligned}&\int_{\mathbb{R}^N} \theta(\frac{|X|^2}{k^2}) g(X) u(X,t) dX \\ &= \int_{|X|\geq k} \theta(\frac{|X|^2}{k^2}) g(X) u(X,t) dX \\ &\leq \frac{\lambda}{2}\int_{|X|\geq k} \theta^2(\frac{|X|^2}{k^2})|u|^2 dX + \frac{1}{2\lambda}\int_{|X|\geq k} |g(X)|^2 dX \\ &\leq \frac{\lambda}{2}\int_{\mathbb{R}^N} \theta^2(\frac{|X|^2}{k^2})|u|^2 dX + \frac{1}{2\lambda}\int_{|X|\geq k} |g(X)|^2 dX.\end{aligned} \tag{3.15}$$

We estimate the last term of the left-hand side of (3.14) as follows

$$\begin{aligned}&\int_{\mathbb{R}^N}\theta(\frac{|X|^2}{k^2})f(X,u)udX\\ &\geq \alpha_1\int_{\mathbb{R}^N}\theta(\frac{|X|^2}{k^2})|u|^pdX-\int_{\mathbb{R}^N}\theta(\frac{|X|^2}{k^2})C_1(X)dX\\ &\geq -\int_{|X|\geq k}C_1(X)dX.\end{aligned}\tag{3.16}$$

For the second term on the left-hand side of (3.14), we have

$$\Big|-\int_{\mathbb{R}^N}\theta(\frac{|X|^2}{k^2})uP_{\alpha,\beta}udX\Big|\leq\frac{1}{2}\int_{\mathbb{R}^N}\Big(|u|^2+|P_{\alpha,\beta}u|^2\Big)dX.\tag{3.17}$$

It follows from (3.14)-(3.17) that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^N}\theta(\frac{|X|^2}{k^2})|u|^2dX+\lambda\int_{\mathbb{R}^N}\theta(\frac{|X|^2}{k^2})|u|^2dX\\ &\leq 2\int_{|X|\geq k}|C_1(X)|dX+\frac{1}{\lambda}\int_{|X|\geq k}|g(X)|^2dX\\ &+\int_{\mathbb{R}^N}(|u|^2+|P_{\alpha,\beta}u|^2)dX.\end{aligned}\tag{3.18}$$

Multiplying (3.18) by $e^{\lambda t}$ and then integrating over (T_0,t), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\theta(\frac{|X|^2}{k^2})|u(t)|^2dX\leq e^{-\lambda t}\int_{\mathbb{R}^N}\theta(\frac{|X|^2}{k^2})|u(T_0)|^2dX\\ &+2e^{-\lambda t}\int_{T_0}^t\int_{|X|\geq k}e^{\lambda\xi}|C_1(X)|dXd\xi\\ &+\frac{1}{\lambda}e^{-\lambda t}\int_{T_0}^t e^{\lambda\xi}\int_{|X|\geq k}|g(X)|^2dXd\xi\\ &+e^{-\lambda t}\int_{T_0}^t e^{\lambda\xi}\Big(\int_{k\leq|X|\leq\sqrt{2}k}(|u|^2+|P_{\alpha,\beta}u|^2)dXd\xi\\ &\leq e^{-\lambda t}\|u(T_0)\|^2_{L^2(\mathbb{R}^N)}+\frac{2}{\lambda}\int_{|X|\geq k}|C_1(X)|dX\\ &+\frac{1}{\lambda^2}\int_{|X|\geq k}|g(X)|^2dX\\ &+e^{-\lambda t}\int_{T_0}^t e^{\lambda\xi}\Big(\int_{k\leq|X|\leq\sqrt{2}k}(|u|^2+|P_{\alpha,\beta}u|^2)dXd\xi.\end{aligned}\tag{3.19}$$

Noting that for given $\eta > 0$, there is $T_1 = T_1(\eta) > 0$ such that for all $t \geq T_1$,

$$e^{-\lambda t}\|u(T_0)\|^2_{L^2(\mathbb{R}^N)} \leq \frac{\eta}{4}. \tag{3.20}$$

Since $C_1(\cdot) \in L^1(\mathbb{R}^N)$, there exists $K_1 = K_1(\eta) > K_0$ such that for all $k \geq K_1$,

$$\frac{2}{\lambda}\int_{|X|\geq k} |C_1(X)|dX \leq \frac{\eta}{4}. \tag{3.21}$$

On the other hand, since $g \in L^2(\mathbb{R}^N)$, there is $K_2 = K_2(\eta) > K_1$ such that for all $k \geq K_2$,

$$\frac{1}{\lambda^2}\int_{|X|\geq k} |g(X)|^2 dX \leq \frac{\eta}{4}. \tag{3.22}$$

For the last term on the right-hand side of (3.19), it follows from Lemma 3.3 that there is $T_2 > 0$ such that for all $\xi \geq T_2$,

$$\int_{\mathbb{R}^N} (|u(\xi)|^2 + |P_{\alpha,\beta}u(\xi)|^2)dX \leq \rho_2.$$

Therefore, there is $K_3 = K_3(\eta) > K_2$ such that for all $k \geq K_3$ and $t \geq T_2$,

$$e^{-\lambda t}\int_{T_0}^{t} e^{\lambda\xi}\Big(\int_{\mathbb{R}^N}(|u|^2 + |P_{\alpha,\beta}u|^2)dX\Big)d\xi \leq \frac{\eta}{4}. \tag{3.23}$$

Let $T = \max\{T_0, T_1, T_2\}$. Then by (3.19) - (3.23) we find that for all $k \geq K_3$ and $t \geq T$,

$$\int_{\mathbb{R}^N} \theta(\frac{|X|^2}{k^2})|u(t)|^2 dX \leq \eta,$$

and hence for all $k \geq K_3$ and $t \geq T$,

$$\int_{|X|\geq\sqrt{2}k} |u(t)|^2 dX \leq \int_{\mathbb{R}^N} \theta(\frac{|X|^2}{k^2})|u(t)|^2 dX \leq \eta,$$

which completes the proof. □

Now, we show the asymptotic compactness of $S(t)$ in $L^2(\mathbb{R}^N)$.

Lemma 3.5. *Suppose* **(F)** - **(G)** *hold. Then* $S(t)$ *is asymptotically compact in* $L^2(\mathbb{R}^N)$*, that is, for any bounded sequence* $\{x_n\}_{n=1}^{\infty} \subset L^2(\mathbb{R}^N)$ *and any sequence* $t_n \geq 0, t_n \to \infty$*,* $\{S(t_n)x_n\}_{n=1}^{\infty}$ *has a convergent subsequence with respect to the topology of* $L^2(\mathbb{R}^N)$*.*

Proof. We use the uniform estimates on the tails of solutions to establish the precompactness of $\{u_n(t_n) := S(t_n)x_n\}$, that is, we prove that for every $\eta > 0$, the sequence $\{u_n(t_n)\}$ has a finite covering of balls of radii less than η. Given $K > 0$, denote

$$\Omega_K = \{X : |X| \leq K\} \text{ and } \Omega_K^c = \{X : |X| > K\}.$$

Then by Lemma 3.4, for the given $\eta > 0$, there exist $K = K(\eta) > 0$ and $T = T(\eta) > 0$ such that for $t \geq T$,

$$\|u_n(t)\|_{L^2(\Omega_K^c)} \leq \eta.$$

Since $t_n \to \infty$, there is $N_1 = N_1(\eta) > 0$ such that $t_n \geq T$ for all $n \geq N_1$, and hence we obtain that, for all $n \geq N_1$,

$$\|u_n(t_n)\|_{L^2(\Omega_K^c)} \leq \eta. \tag{3.24}$$

By Lemma 3.1, there exist $C > 0$ and $N_2 > 0$ such that for all $n \geq N_2$,

$$\|u_n(t_n)\|_{\mathcal{S}^1(\Omega_K)} \leq C. \tag{3.25}$$

Since the compactness of the embedding $\mathcal{S}^1(\Omega_K) \hookrightarrow L^2(\Omega_K)$ (see [22]), the sequence $\{u_n(t_n)\}$ is precompact in $L^2(\Omega_K)$. Therefore, for the given $\eta > 0$, $\{u_n(t_n)\}$ has a finite covering in $L^2(\Omega_K)$ of balls of radii less than η, which along with (3.24) shows that $\{u_n(t_n)\}$ has a finite covering in $L^2(\mathbb{R}^N)$ of balls of radii less than η, and thus $\{u_n(t_n)\}$ is precompact in $L^2(\mathbb{R}^N)$. □

We are now ready to prove the existence of a global attractor for $S(t)$ in $L^2(\mathbb{R}^N)$.

Theorem 3.1. *Suppose* **(F)** - **(G)** *hold. Then the semigroup* $S(t)$ *generated by problem* (1.1) *has a global attractor* $\mathcal{A}_{L^2}$ *in* $L^2(\mathbb{R}^N)$*.*

Proof. Denote

$$B = \Big\{u : \|u\|_{L^2(\mathbb{R}^N)} \leq R\Big\},$$

where R is the positive constant in the proof of Lemma 3.1. Then B is a bounded absorbing set for $S(t)$ in $L^2(\mathbb{R}^N)$. In addition, $S(t)$ is asymptotically compact in $L^2(\mathbb{R}^N)$ since Lemma 3.5. Thus, we get the conclusion. □

3.3. Existence of a global attractor in $L^p(\mathbb{R}^N)$

First, from Lemma 3.1, one can see that $S(t)$ maps compact subsets of $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ to bounded subsets of $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. Hence, by Theorem 3.2 in [26], we see that $S(t)$ is norm-to-weak continuous on $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$.

To obtain the existence of a global attractor in $L^p(\mathbb{R}^N)$, we need the following lemma, whose proof is very similar to the proof of Corollary 5.7 in [26], so we omit it here.

Lemma 3.6. *Let $\{S(t)\}_{t\geq 0}$ be a norm-to-weak continuous semigroup on $L^p(\mathbb{R}^N)$, and be continuous or weak continuous on $L^2(\mathbb{R}^N)$, and have a global attractor in $L^2(\mathbb{R}^N)$. Then $\{S(t)\}_{t\geq 0}$ has a global attractor in $L^p(\mathbb{R}^N)$ if and only if*

(i) *$\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $L^p(\mathbb{R}^N)$;*

(ii) *for any $\epsilon > 0$ and any bounded subset B of $L^p(\mathbb{R}^N)$, there exist positive constants $M = M(\epsilon, B)$ and $T = T(\epsilon, B)$ such that*

$$\int_{\mathbb{R}^N(|S(t)u_0| \geq M)} |S(t)u_0|^p dX < \epsilon, \tag{3.26}$$

for any $u_0 \in B$ and $t \geq T$.

Theorem 3.2. *Assume* **(F)** - **(G)** *hold. Then the semigroup $S(t)$ generated by problem* (1.1) *has a global attractor $\mathcal{A}_{L^p}$ in $L^p(\mathbb{R}^N)$.*

Proof. We only need to show that $\{S(t)\}$ satisfies the condition (ii) in Lemma 3.6. Taking M large enough such that $\alpha_1|u|^{p-1} \leq f(X, u)$ in

$$\mathbb{R}^N(u \geq M) := \{X \in \mathbb{R}^N : u(X,t) \geq M\},$$

and denote

$$(u - M)_+ = \begin{cases} u - M, & u \geq M \\ 0, & u \leq M. \end{cases}$$

First, for any fixed $\epsilon > 0$, there exists $\delta > 0$ such that for any $e \subset \mathbb{R}^N$ with $m(e) \leq \delta$, we have

$$\int_e |g|^2 dX < \epsilon. \tag{3.27}$$

In $\mathbb{R}^N(u \geq M)$ we see that

$$\begin{aligned} g(u-M)_+^{p-1} &\leq \frac{\alpha_1}{2}(u-M)_+^{2p-2} + \frac{1}{2\alpha_1}|g|^2 \\ &\leq \frac{\alpha_1}{2}(u-M)_+^{p-1}|u|^{p-1} + \frac{1}{2\alpha_1}|g|^2, \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} f(X,u)(u-M)_+^{p-1} &\geq \alpha_1|u|^{p-1}(u-M)_+^{p-1} \\ &\geq \frac{\alpha_1}{2}(u-M)_+^{p-1}|u|^{p-1} + \frac{\alpha_1 M^{p-2}}{2}(u-M)_+^p. \end{aligned} \tag{3.29}$$

Multiplying equation (1.1) by $|(u-M)_+|^{p-1}$ and using (3.28), (3.29), we deduce that

$$\begin{aligned} &\frac{2}{p}\frac{d}{dt}\|(u-M)_+\|^p_{L^p(\mathbb{R}^N(u\geq M))} \\ &+ 2(p-1)\int_{\mathbb{R}^N(u\geq M)} (|\nabla_x(u-M)_+|^2 + |\nabla_y(u-M)_+|^2 \\ &+ |x|^{2\alpha}|y|^{2\beta}|\nabla_z(u-M)_+|^2)||(u-M)_+|^{p-2})dX \\ &+ \lambda\int_{\mathbb{R}^N(u\geq M)} |(u-M)_+|^p dX + \alpha_1 M^{p-2}\int_{\mathbb{R}^N(u\geq M)} |(u-M)_+|^p dX \\ &\leq \frac{1}{\alpha_1}\int_{\mathbb{R}^N(u\geq M)} |g|^2 dX. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{d}{dt}\|(u-M)_+\|^p_{L^p(\mathbb{R}^N(u\geq M))} + CM^{p-2}\|(u-M)_+\|^p_{L^p(\mathbb{R}^N(u\geq M))} \\ &\leq C\|g\|^2_{L^2(\mathbb{R}^N(u\geq M))}. \end{aligned}$$

By the Gronwall inequality, we have for all $M \geq M_1$ and $t \geq T_1$,

$$\int_{\mathbb{R}^N(u\geq M)} |(u-M)_+|^p dX \leq \varepsilon. \tag{3.30}$$

Repeating the same step above, just taking $(u+M)_-$ instead of $(u-M)_+$, where

$$(u+M)_- = \begin{cases} u+M, & u \leq -M, \\ 0, & u \geq -M, \end{cases}$$

we deduce that there exist $M_2 > 0$ and $T_2 > 0$ such that for any $t > T_2$ and any $M \geq M_2$, we have

$$\int_{\mathbb{R}^N(u\leq -M)} |(u+M)_-|^p dX \leq \varepsilon. \tag{3.31}$$

Let $M_0 = \max\{M_1, M_2\}$ and $T = \max\{T_1, T_2\}$, we obtain

$$\int_{\mathbb{R}^N(|u|\geq M)} (|u|-M)^p dX \leq \varepsilon \qquad \text{for } t \geq T \text{ and } M \geq M_0.$$

Using (3.30) and (3.31), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N(|u|\geq 2M)} |u|^p dX \\
&= \int_{\mathbb{R}^N(|u|\geq 2M)} ((|u|-M)+M)^p dX \\
&\leq 2^p\Big(\int_{\mathbb{R}^N(|u|\geq 2M)} (|u|-M)^p dX + \int_{\mathbb{R}^N(|u|\geq 2M)} M^p dX\Big) \\
&\leq 2^p\Big(\int_{\mathbb{R}^N(|u|\geq 2M)} (|u|-M)^p dX + \int_{\mathbb{R}^N(|u|\geq 2M)} (|u|-M)^p dX\Big) \\
&\leq 2^{p+1}\varepsilon.
\end{aligned} \tag{3.32}$$

This completes the proof. □

3.4. Existence of a global attractor in $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$

LEMMA 3.7. *Suppose* (**F**) – (**G**) *hold. Then the semigroup* $S(t)$ *is asymptotically compact in* $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$.

PROOF. Let B be a bounded subset in $L^2(\mathbb{R}^N)$, we will show that for any $\{u_{0n}\} \subset B$ and $t_n \to \infty$, $\{u_n(t_n)\} := \{S(t_n)u_{0n}\}$ is precompact in $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. Thanks to Theorem 3.2, we only need to show that the sequence $\{u_n(t_n)\}$ is precompact in $\mathcal{S}^1(\mathbb{R}^N)$. By Lemma 3.5, we can assume that $\{u_n(t_n)\}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$. For any $n, m \geq 1$, it follows from (1.1) that

$$\begin{aligned} &-P_{\alpha,\beta}(u_n(t_n) - u_m(t_m))) + \lambda(u_n(t_n) - u_m(t_m)) \\ &\qquad + f(X, u_n(t_n)) - f(X, u_m(t_m)) = -\frac{d}{dt}u_n(t_n) + \frac{d}{dt}u_m(t_m). \end{aligned} \tag{3.33}$$

Multiplying (3.33) by $u_n(t_n) - u_m(t_m)$ and using (1.4) we get

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla_x(u_n(t_n) - u_m(t_m))|^2 + |\nabla_y(u_n(t_n) - u_m(t_m))|^2 \\ &+ |x|^{2\alpha}|y|^{2\beta}|\nabla_z(u_n(t_n) - u_m(t_m))|^2)dX \\ &+ \lambda\|u_n(t_n) - u_m(t_m)\|^2_{L^2(\mathbb{R}^N)} \\ &\leq \|u_{nt}(t_n) - u_{mt}(t_m)\|_{L^2(\mathbb{R}^N)}\|u_n(t_n) - u_m(t_m)\|_{L^2(\mathbb{R}^N)} \\ &+ \alpha_3\|u_n(t_n) - u_m(t_m)\|^2_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{3.34}$$

By Lemma 3.2, for any bounded subset B in $L^2(\mathbb{R}^N)$, there exists $T = T(B)$ such that for all $t_n \geq T$,

$$\|u_{nt}(t_n)\|_{L^2(\mathbb{R}^N)} \leq C,$$

which along with (3.34) shows that, for all $n, m \geq N$,

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla_x(u_n(t_n) - u_m(t_m))|^2 + |\nabla_y(u_n(t_n) - u_m(t_m))|^2 \\ &+ |x|^{2\alpha}|y|^{2\beta}|\nabla_z(u_n(t_n) - u_m(t_m))|^2)dX \\ &+ \lambda\|u_n(t_n) - u_m(t_m)\|^2_{L^2(\mathbb{R}^N)} \\ &\leq 2C\|u_n(t_n) - u_m(t_m)\|_{L^2(\mathbb{R}^N)} + \alpha_3\|u_n(t_n) - u_m(t_m)\|^2_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{3.35}$$

Hence, it implies that $\{u_n(t_n)\}$ is a Cauchy sequence in $\mathcal{S}^1(\mathbb{R}^N)$. □

THEOREM 3.3. *Suppose* $(\mathbf{F}) - (\mathbf{G})$ *hold. Then the semigroup* $S(t)$ *generated by problem* (1.1) *has a global attractor* $\mathcal{A}_{\mathcal{S}^1 \cap L^p}$ *in* $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$.

PROOF. By Lemma 3.1, there exists a bounded absorbing set for $S(t)$ in $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. In addition, $S(t)$ is asymptotically compact in $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ since Lemma 3.7. Thus, there exists a global attractor for $S(t)$ in $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. □

REMARK 3.1. The global attractors $\mathcal{A}_{L^2}$, $\mathcal{A}_{L^p}$ and $\mathcal{A}_{\mathcal{S}^1 \cap L^p}$ obtained in Theorems 3.1, 3.2 and 3.3 are of course the same object and will be denoted by $\mathcal{A}$. In particular, $\mathcal{A}$ is a compact connected set in $\mathcal{S}^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$.

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