J. Math. Sci. Univ. Tokyo
19 (2012), 559–586.

On the Sheaf of Laplace Hyperfunctions with Holomorphic Parameters

By Naofumi Honda and Kohei Umeta

Abstract. We give a vanishing theorem of cohomology groups on a pseudoconvex open subset for holomorphic functions with exponential growth at infinity. As an application, we construct the sheaf of Laplace hyperfunctions and that with holomorphic parameters, and we also study several properties of these sheaves.

1. Introduction

The theory of Laplace hyperfunctions has been established by H. Komatsu ([6] - [12]) as a framework of operational calculus. In the paper [6], he introduced the space of Laplace hyperfunctions with support in $[a, +\infty]$ ($a \in \mathbb{R} \sqcup \{+\infty\}$) and constructed the Laplace transformation of hyperfunctions. Using this machinery, he had succeeded in giving a rigid interpretation for operational calculus without any growth condition.

It is highly desirable to localize the notion of Laplace hyperfunctions, in other words, to obtain the sheaf whose sections with support in $[a, \infty]$ give the space of Laplace hyperfunctions introduced by H. Komatsu. In this paper, we construct the sheaf of Laplace hyperfunctions and that with holomorphic parameters by establishing the vanishing theorem of cohomology groups on a pseudoconvex open subset for holomorphic functions with exponential growth at infinity. The vanishing theorem established here not only plays an important role in construction for the sheaf of Laplace hyperfunctions but also has its own interest, for which we briefly explain from now.

Let $\mathbb{D}^{2n}$ be the radial compactification $\mathbb{C}^n \sqcup S^{2n-1}$ of $\mathbb{C}^n$, and set $\hat{X} := \mathbb{D}^{2n} \times \mathbb{C}^m$ ($n \geq 1$ and $m \geq 0$), on which the sheaf $\mathcal{O}_{\hat{X}}^{\exp}$ of holomorphic

2010 *Mathematics Subject Classification*. Primary 32A45; Secondary 44A10.
Key words: Laplace hyperfunctions, Laplace transform, holomorphic functions of exponential type, sheaves, cohomology.
This work is supported in part by JSPS Grant-in-Aid No. 23540178.

functions of exponential type is defined. We refer the reader to Section 2. for the details on these notions. Let $\Omega := U \times V$ be an open subset of product type in $\hat{X}$, and assume that $U \cap \mathbb{C}^n$ and V are pseudoconvex in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Then, for the case of $n = 1$, the vanishing theorem holds as is expected, that is, we have $H^k(\Omega, \mathcal{O}_X^{\exp}) = 0$ for $k \neq 0$. However, if n is grater than one, the vanishing theorem does not hold anymore as Example 3.17 shows. To overcome this difficulty, we introduce the new notion that Ω is regular at ∞. Roughly speaking, the regularity condition specifies behavior of the boundary $\partial\Omega$ that sufficiently many points in $\partial\Omega$ are accumulated at ∞. And we can show, with the regularity condition for Ω at ∞, the vanishing theorem in the higher dimensional case.

As an application of the vanishing theorem, we have the purely 1-codimensionality of $\overline{N}$ with respect to the sheaf of holomorphic functions of exponential type on $\mathbb{D}^2 \times \mathbb{C}^m$, where $\overline{N}$ is the closure of $N = \mathbb{R} \times \mathbb{C}^m$ in $\mathbb{D}^2 \times \mathbb{C}^m$. Hence we can construct the sheaf $\mathcal{BO}_N^{\exp}$ of Laplace hyperfunctions with holomorphic parameters as the local cohomology group of the sheaf of holomorphic functions of exponential type with support in $\overline{N}$. We also show that the sheaf $\mathcal{BO}_N^{\exp}$ is flabby with respect to the variable of $\overline{\mathbb{R}}$, and has a unique continuation property with respect to holomorphic parameters.

The plan of the paper is as follows. In Section 2, the definition of Laplace hyperfunctions with compact support and several fundamental theorems established by H. Komatsu are reviewed. In Section 3, we show the vanishing theorem on a pseudoconvex open subset for holomorphic functions of exponential type. We first define the sheaf $\mathcal{O}_X^{\exp}$ of holomorphic functions of exponential type on $\hat{X}$. We also introduce the regularity condition at infinity for an open subset. Then we review L^2-estimates for the $\overline{\partial}$ operator obtained by L. Hörmander [3] which is a main tool for the proof of the vanishing theorem in subsection 3.1. The proof of the vanishing theorem is given in subsection 3.2. The fundamental ideas and techniques were already established in the papers T. Kawai [5] and S. Saburi [16] which treated several vanishing theorems for holomorphic functions with infra-exponential growth. Hence, by their methods, the problem can be reduced to that of a construction of a family of suitable plurisubharmonic functions on Ω, and here, we essentially use the condition that Ω is regular at infinity. In Section 4, we prove the pure-codimensionality of $\overline{N}$ with respect to the sheaf $\mathcal{O}_X^{\exp}$. We define the sheaf $\mathcal{BO}_N^{\exp}$ of Laplace hyperfunctions with holomorphic parameters and the sheaf $\mathcal{B}_{\mathbb{R}}^{\exp}$ of Laplace hyperfunctions. In Section 5, we study the vanishing theorem on an open subset in $\hat{X} = \mathbb{D}^2 \times \mathbb{C}^m$ which is not necessarily regular at infinity. We also establish the theorems for the flabbiness and the unique continuation property of $\mathcal{BO}_N^{\exp}$.

At the end of the introduction, the authors would like to express their sincere gratitude to Professor Hikosaburo Komatsu for the valuable lectures and advises in Hokkaido University.

2. Laplace Hyperfunctions

We briefly recall the definition of Laplace hyperfunctions with support in $[a, \infty]$ $(a \in \mathbb{R} \sqcup \{+\infty\})$ and several fundamental theorems established by H.Komatsu ([6] - [12]).

Definition 2.1 ([7]). We denote by $\mathbb{D}^2$ the radial compactification $\mathbb{C} \sqcup S^1\infty$ of $\mathbb{C}$. The topology of $\mathbb{D}^2$ is defined in the following way. A fundamental system of neighborhoods of $\xi\infty \in S^1\infty$ consists of all the sets given by

$$\{z \in \mathbb{C};\, z/|z| \in \Gamma,\, |z| > r\} \sqcup \{w\infty; w \in \Gamma\} \tag{1}$$

for a neighborhood Γ of ξ in S^1 and $r > 0$.

Let $\mathcal{O}_{\mathbb{C}}$ denote the sheaf of holomorphic functions on $\mathbb{C}$.

Definition 2.2 ([6]). Let U be an open subset in $\mathbb{D}^2$. The set $\mathcal{O}_{\mathbb{C}}^{\exp}(U)$ of holomorphic functions of exponential type on U consists of a holomorphic function $F(z)$ on $U \cap \mathbb{C}$ which satisfies, for any compact set K in U,

$$|F(z)| \le C_K e^{H_K|z|}, \qquad (z \in K \cap \mathbb{C}) \tag{2}$$

with some positive constants C_K and H_K. We denote by $\mathcal{O}_{\mathbb{C}}^{\exp}$ the associated sheaf on $\mathbb{D}^2$ of the presheaf $\{\mathcal{O}_{\mathbb{C}}^{\exp}(U)\}_U$.

Note that, if $U \cap S^1\infty = \emptyset$, then the growth condition (2) is always satisfied, and hence we have $\mathcal{O}_{\mathbb{C}}^{\exp}|_{\mathbb{C}} = \mathcal{O}_{\mathbb{C}}$.

Definition 2.3 ([6]). Let $-\infty < a \le \infty$. The space $\mathcal{B}_{[a,\infty]}^{\exp}$ of Laplace hyperfunctions with support in $[a, \infty]$ is the quotient space

$$\mathcal{O}_{\mathbb{C}}^{\exp}(\mathbb{D}^2 \setminus [a, \infty])/\mathcal{O}_{\mathbb{C}}^{\exp}(\mathbb{D}^2). \tag{3}$$

The class $f(x) = [F] \in \mathcal{B}^{\exp}_{[a,\infty]}$ of an $F(z) \in \mathcal{O}^{\exp}_{\mathbb{C}}(\mathbb{D}^2 \setminus [a, \infty])$ can be considered as a boundary value of $F(z)$, and we sometimes denote it by

$$f(x) = F(x + i0) - F(x - i0). \tag{4}$$

THEOREM 2.4 ([10]). *We have the natural isomorphism*

$$\mathcal{B}^{\exp}_{[a,\infty]} \cong \mathcal{O}^{\exp}_{\mathbb{C}}(V \setminus [a, \infty])/\mathcal{O}^{\exp}_{\mathbb{C}}(V) \tag{5}$$

for any open neighborhood V of $[a, \infty]$ in $\mathbb{D}^2$.

Remember that the space $\mathcal{B}_{[a,\infty)}$ of ordinary hyperfunctions with support in $[a, \infty)$ is defined by

$$\mathcal{B}_{[a,\infty)} := \mathcal{O}_{\mathbb{C}}(\mathbb{C} \setminus [a, \infty))/\mathcal{O}_{\mathbb{C}}(\mathbb{C}). \tag{6}$$

Hence the restrictions $\mathcal{O}^{\exp}_{\mathbb{C}}(\mathbb{D}^2 \setminus [a, \infty]) \to \mathcal{O}_{\mathbb{C}}(\mathbb{C} \setminus [a, \infty))$ and $\mathcal{O}^{\exp}_{\mathbb{C}}(\mathbb{D}^2) \to \mathcal{O}_{\mathbb{C}}(\mathbb{C})$ induce the canonical morphism $\rho : \mathcal{B}^{\exp}_{[a,\infty]} \to \mathcal{B}_{[a,\infty)}$, for which we have the followings.

THEOREM 2.5 ([6]). *The morphism $\rho : \mathcal{B}^{\exp}_{[a,\infty]} \to \mathcal{B}_{[a,\infty)}$ is surjective.*

Since every ordinary hyperfunction with support in $[a, \infty)$ can be extended to a Laplace hyperfunction by the above theorem, we have

$$\mathcal{B}_{[a,\infty)} \cong \mathcal{B}^{\exp}_{[a,\infty]}/\mathcal{B}^{\exp}_{\{\infty\}}. \tag{7}$$

DEFINITION 2.6 ([6]). The Laplace transform $\widehat{f}(\lambda)$ of a Laplace hyperfunction $f(x) = [F] \in \mathcal{B}^{\exp}_{[a,\infty]}$ is defined by the integral

$$\widehat{f}(\lambda) := \int_C e^{-\lambda z} F(z) dz, \tag{8}$$

where the path C of the integration is composed of a ray from $e^{i\alpha}\infty(-\pi/2 < \alpha < 0)$ to a point $c < a$ and a ray from c to $e^{i\beta}\infty(0 < \beta < \pi/2)$.

It follows from Pólya's theorem ([15]) that the Laplace transform with origin at $c \in \mathbb{C}$

$$\widehat{m}_c(\lambda) = \int_c^{\infty} e^{-\lambda z} m(z) dz \tag{9}$$

of an $m(z) \in \mathcal{O}_{\mathbb{C}}^{\exp}(\mathbb{D}^2)$ is a holomorphic function outside a convex compact set. Hence the Laplace transform $\widehat{f}(\lambda)$ does not depend on a choice of F.

THEOREM 2.7 ([6]). *The Laplace transformation $\mathcal{L}$ is an isomorphism of linear spaces*

$$\mathcal{L} \,:\, \mathcal{B}^{\exp}_{[a,\infty]} \longrightarrow \mathcal{L}\mathcal{B}^{\exp}_{[a,\infty]}, \tag{10}$$

where $\mathcal{L}\mathcal{B}^{\exp}_{[a,\infty]}$ is the space of all holomorphic functions $\widehat{f}(\lambda)$ of exponential type defined on a neighborhood Ω of the semi-circle $\{e^{i\theta}; |\theta| < \pi/2\}$ in $\mathbb{D}^2$ which satisfies

$$\overline{\lim_{\rho\to\infty}} \frac{\log|\widehat{f}(\rho e^{i\theta})|}{\rho} \le -a\cos\theta, \quad |\theta| < \pi/2. \tag{11}$$

For $\widehat{f}(\lambda) \in \mathcal{L}\mathcal{B}^{\exp}_{[a,\infty]}$, the inverse image $\mathcal{L}^{-1}\widehat{f}$ is given by

$$\left[\frac{1}{2\pi\sqrt{-1}} \int_{\Lambda}^{\infty} e^{\lambda z} \widehat{f}(\lambda) d\lambda\right] \in \mathcal{B}^{\exp}_{[a,\infty]}, \tag{12}$$

where Λ is a fixed point in $\Omega \cap \mathbb{C}$ and the path of the integration is taken in $\Omega \cap \mathbb{C}$.

3. The Vanishing Theorem for Holomorphic Functions of Exponential Type

The purpose of the section is to establish the vanishing theorem for cohomology groups on a pseudoconvex open subset with coefficients in the sheaf of holomorphic functions with exponential growth at infinity. We first introduce several notions which are needed later.

Let $n \in \mathbb{N}$ and m be a non-negative integer. We first introduce *the radial compactification* $\mathbb{D}^{2n}$ of $\mathbb{C}^n$, on which the sheaf of holomorphic functions of exponential type is defined. The set $\mathbb{D}^{2n}$ is the disjoint union of $\mathbb{C}^n$ and the real $(2n-1)$-dimensional unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$. Let D be a closed unit ball in $\mathbb{C}^n$ which is considered as a real $2n$-dimensional topological manifold with the boundary S^{2n-1}, and let $\rho : D \to \mathbb{D}^{2n}$ be the bijection defined by

$$\rho(z) = \begin{cases} \dfrac{z}{1-|z|} \in \mathbb{C}^n, & (z \in D^\circ) \\ z \in S^{2n-1}, & (z \in \partial D) \end{cases}.$$

Then $\mathbb{D}^{2n}$ is equipped with the topology so that ρ becomes a topological isomorphism. Note that any closed subset in $\mathbb{D}^{2n}$ is compact.

Let $X := \mathbb{C}^{n+m}$ and $\hat{X}$ be the partial radial compactification $\mathbb{D}^{2n} \times \mathbb{C}^m$ of $\mathbb{C}^{n+m}$. We denote by X_∞ the closed subset $\hat{X} \setminus X$ in $\hat{X}$, and we also denote by $p_1 : \hat{X} = \mathbb{D}^{2n} \times \mathbb{C}^m \to \mathbb{D}^{2n}$ (resp. $p_2 : \hat{X} = \mathbb{D}^{2n} \times \mathbb{C}^m \to \mathbb{C}^m$) the canonical projection to the first (resp. second) space. A family of fundamental neighborhoods of $(z_0, w_0) \in X \subset \hat{X}$ consists of

$$B_\epsilon(z_0, w_0) := \{(z, w) \in X;\ |z - z_0| < \epsilon,\ |w - w_0| < \epsilon\} \tag{13}$$

for $\epsilon > 0$, and that of $(z_0, w_0) \in X_\infty$ consists of a product of an open cone and an open ball

$$G_r(\Gamma, w_0) := \left(\left\{z \in \mathbb{C}^n;\ |z| > r,\ \frac{z}{|z|} \in \Gamma\right\} \cup \Gamma\right) \times \left\{w \in \mathbb{C}^m;\ |w - w_0| < \frac{1}{r}\right\}, \tag{14}$$

where $r > 0$ and Γ runs through open neighborhoods of z_0 in S^{2n-1}.

Let K be a subset in $\hat{X}$. Then K is compact if and only if K is closed and $p_2(K)$ is bounded in $\mathbb{C}^m$. The following lemma is easily proved.

LEMMA 3.1. *For an open subset $\Omega \subset \hat{X}$, there exists an exhausting family $\{\Omega_k\}_{k \in \mathbb{N}}$ of Ω satisfying the conditions below.*

(1) *Ω_k is an open subset of Ω, and the union of Ω_k is equal to Ω.*

(2) *$\overline{\Omega_k}$ is a compact set and $\overline{\Omega_k} \subset \Omega_{k+1}$ $(k = 1, 2, \dots)$.*

(3) *Each Ω_k is a finite union of open subsets given by either* (13) *or* (14).

We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X.

Definition 3.2. Let Ω be an open subset in $\hat{X}$. The set $\mathcal{O}_X^{\exp}(\Omega)$ of holomorphic functions of exponential type on Ω consists of a holomorphic function $f(z,w)$ on $\Omega \cap X$ which satisfies, for any compact set K in Ω,

$$|f(z,w)| \le C_K e^{H_K|z|}, \qquad ((z,w) \in K \cap X) \tag{15}$$

with some positive constants C_K and H_K. We denote by $\mathcal{O}_X^{\exp}$ the associated sheaf on $\hat{X}$ of the presheaf $\{\mathcal{O}_X^{\exp}(\Omega)\}_\Omega$.

Let $\{\Omega_k\}_k$ be an exhausting family of Ω satisfying the conditions given in Lemma 3.1. Then $f \in \mathcal{O}_X^{\exp}(\Omega)$ if and only if the estimate (15) holds with $K = \overline{\Omega_k}$ $(k = 1, 2, \dots)$. In particular, if $\Omega \subset X$, then each $\overline{\Omega_k}$ is bounded in X and the estimate (15) is always satisfied, which implies $\mathcal{O}_X^{\exp}(\Omega) = \mathcal{O}_X(\Omega)$. Hence we have $\mathcal{O}_X^{\exp}\big|_X = \mathcal{O}_X$.

Before stating the main result in our paper, we introduce some notations which are needed for the main theorem. Let A be a subset in $\hat{X}$. We define the set $\mathrm{clos}^1_\infty(A) \subset X_\infty$ as follows. A point $(z,\, w) \in X_\infty$ belongs to $\mathrm{clos}^1_\infty(A)$ if and only if there exist points $\{(z_k,\, w_k)\}_{k\in\mathbb{N}}$ in $A \cap X$ that satisfy

$$(z_k,\, w_k) \to (z,\, w) \text{ in } \hat{X} \text{ and } \frac{|z_{k+1}|}{|z_k|} \to 1 \quad (k \to \infty). \tag{16}$$

Set

$$N^1_\infty(A) := X_\infty \setminus \mathrm{clos}^1_\infty(X \setminus A). \tag{17}$$

We give some properties of $N^1_\infty(A)$.

Lemma 3.3.

(1) *For subsets $A_1,\, A_2,\, \dots,\, A_\ell$ in $\hat{X}$, we have*

$$N^1_\infty(A_1 \cap \dots \cap A_\ell) = N^1_\infty(A_1) \cap \dots \cap N^1_\infty(A_\ell).$$

(2) *If U is an open subset in $\hat{X}$, then $N^1_\infty(U) \supset U \cap X_\infty$ holds.*

DEFINITION 3.4. Let U be an open subset in $\hat{X}$. We say that U is regular at ∞ if $N^1_\infty(U) = U \cap X_\infty$ is satisfied.

Note that, by Lemma 3.3, a finite intersection of open subsets which are regular at ∞ is again regular at ∞. We give a sufficient condition for which an open subset becomes regular at ∞. Let A be a subset in $\hat{X}$, and we set

$$N^L_\infty(A) := \left\{(\zeta,\, w) \in X_\infty;\ (\zeta,\, w) \in \overline{(\mathbb{R}_+\zeta \times \{w\}) \cap A}\right\} \subset X_\infty, \tag{18}$$

where $\mathbb{R}_+\zeta$ is the real half line in $\mathbb{C}^n$ with direction ζ and the closure is taken in $\hat{X}$. For subsets $A_1,\, A_2,\, \ldots,\, A_\ell$ in $\hat{X}$, we have

$$N^L_\infty(A_1 \cup \cdots \cup A_\ell) = N^L_\infty(A_1) \cup \cdots \cup N^L_\infty(A_\ell). \tag{19}$$

LEMMA 3.5. *Let U be an open subset in $\hat{X}$. If $N^L_\infty(U) = U \cap X_\infty$ holds, then U is regular at ∞.*

PROOF. By noticing $N^L_\infty(A) \supset N^1_\infty(A)$ for any subset A in $\hat{X}$, we have

$$N^L_\infty(U) \supset N^1_\infty(U) \supset U \cap X_\infty.$$

The lemma follows from this. □

A finite union of open subsets which satisfy the condition given in the above lemma is also regular at ∞ by (19). We give some examples of open subsets which are regular at ∞.

Example 3.6.

(1) Let U be the open set $G_r(\Gamma, 0) \cup \tilde{U}$ where $\tilde{U}$ is a bounded open subset in X and the cone $G_r(\Gamma,\, 0)$ was defined by (14) with $r > 0$ and Γ being an open subset in S^{2n-1}. Then U is regular at ∞ as we have $N^L_\infty(U) = U \cap X_\infty$. In particular, $\mathbb{D}^2$ and $\mathbb{D}^2 \setminus [a, +\infty]$ $(a \in [-\infty, \infty))$ are regular at ∞.

(2) For the set $U := \mathbb{D}^2 \setminus \{1, 2, 3, 4, \ldots, +\infty\}$ we have $N^1_\infty(U) = S^1\infty \setminus \{+\infty\}$, and hence U is regular at ∞. However $U := \mathbb{D}^2 \setminus \{1, 2, 4, 8, 16, \ldots, +\infty\}$ is not regular because of $N^1_\infty(U) = S^1\infty$. Note that we have $N^L_\infty(U) = S^1\infty$ for the both cases.

For a subset A in X, we denote by $\operatorname{dist}(p, A)$ the distance between a point p and A, i.e.,

$$\operatorname{dist}(p,\, A) := \inf_{q \in A} |p - q|.$$

For convenience, we set $\operatorname{dist}(p,\, A) = +\infty$ if A is empty. We also define, for $q = (z, w) \in X$,

$$\operatorname{dist}_{\mathbb{D}^{2n}}(q,\, A) := \operatorname{dist}(q,\, A \cap p_2^{-1}(p_2(q))) = \inf_{(\zeta,\, w) \in A} |z - \zeta|.$$

Now we give the main theorem. Let Ω be an open subset in $\hat{X}$, and set

(20)

$$\psi(p) := \min\left\{\frac{1}{2},\ \frac{\operatorname{dist}_{\mathbb{D}^{2n}}(p,\, X \setminus \Omega)}{1 + |z|}\right\}, \qquad (p = (z, w) \in X),$$

$$\Omega_\epsilon := \left\{p = (z, w) \in \Omega \cap X;\, \operatorname{dist}(p,\, X \setminus \Omega) > \epsilon,\, |w| < \frac{1}{\epsilon}\right\}, \quad (\epsilon > 0).$$

Note that $\psi(p)$ is lower semicontinuous (i.e., $\{p \in X;\, \psi(p) > c\}$ is open for $c \in \mathbb{R}$) and continuous with respect to the variables z, however, it is not necessarily continuous with respect to the variables w. Furthermore, if $p_1((X \setminus \Omega) \cap p_2^{-1}(w_0))$ $(w_0 \in \mathbb{C}^m)$ is a bounded subset in $\mathbb{C}^n$, then $\psi(z,\, w_0)$ is identically equal to $\frac{1}{2}$ for a sufficiently large z. Hence the values of $\psi(z,\, w)$ for a large z are independent of the shape of Ω in a bounded region.

Theorem 3.7. *Assume the following conditions* 1. *and* 2.

1. $\Omega \cap X$ *is pseudoconvex in* X *and* Ω *is regular at* ∞.
2. *At a point in* $\Omega \cap X$ *sufficiently close to* $z = \infty$ *the* $\psi(z, w)$ *is continuous and uniformly continuous with respect to the variables* w, *that is, for any* $\epsilon > 0$, *there exist* $\delta_\epsilon > 0$ *and* $R_\epsilon > 0$ *for which* $\psi(z, w)$ *is continuous on* $\Omega_{\epsilon,\, R_\epsilon} := \Omega_\epsilon \cap \{|z| > R_\epsilon\}$ *and it satisfies*

$$\left|\psi(z,\, w) - \psi(z,\, w')\right| < \epsilon, \qquad ((z,\, w),\, (z,\, w') \in \Omega_{\epsilon,\, R_\epsilon},\ |w - w'| < \delta_\epsilon).$$

Then we have

$$H^k(\Omega,\, \mathscr{O}_X^{\exp}) = 0, \qquad (k \neq 0). \tag{21}$$

As a corollary, we have the following.

COROLLARY 3.8. *Let U (resp. W) be an open subset in $\mathbb{D}^{2n}$ (resp. $\mathbb{C}^m$). If $U \cap \mathbb{C}^n$ and W are pseudoconvex in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively and if U is regular at ∞ in $\mathbb{D}^{2n}$, then we have (21) for $\Omega := U \times W$.*

The corollary immediately follows from the theorem as the condition 2. in the theorem is automatically satisfied for a product of open sets. We now give an example.

Example 3.9. Assume $n = m = 1$, i.e., $X = \mathbb{C}_z \times \mathbb{C}_w$ and $\hat{X} = \mathbb{D}^2 \times \mathbb{C}_w$. Let $f : X \to \mathbb{C}$ be the holomorphic map defined by $f(z, w) = zw$. Set

$$\tilde{\Omega} := \{\zeta \in \mathbb{C};\ |\zeta| < 1\} \cup \{\zeta \in \mathbb{C};\ |\arg \zeta| < 1\} \subset \mathbb{C},$$
$$\Omega := \left(\overline{f^{-1}(\tilde{\Omega})}\right)^{\circ} \subset \hat{X}.$$

Here the closure and the interior are taken in $\hat{X}$. To understand the shape of Ω clearly, the intersection of Ω and the complex line $\{(z, w) \in \hat{X};\ w = w_0\}$ for $w_0 \in \mathbb{C}_w$ is described below.

$$p_1(\Omega \cap p_2^{-1}(w_0)) = \begin{cases} \left(\overline{\dfrac{1}{w_0}\tilde{\Omega}}\right)^{\circ} \subset \mathbb{D}^2, & (w_0 \neq 0), \\ \mathbb{C} \subset \mathbb{D}^2, & (w_0 = 0). \end{cases}$$

Then Ω satisfies all the conditions of the theorem, and hence, we have $H^k(\Omega, \mathcal{O}_X^{\exp}) = 0$ $(k \neq 0)$.

The subsequent subsections are devoted to the proof of Theorem 3.7.

3.1. L^2-estimates for the $\overline{\partial}$ operator

We briefly review the result obtained by L. Hörmander [3] which is a main tool for the proof of Theorem 3.7.

DEFINITION 3.10 ([3]). A function u defined in an open set $\Omega \subset \mathbb{C}^n$ with values in $[-\infty, +\infty)$ is called plurisubharmonic if

(a) u is semicontinuous from above.

(b) For arbitrary z and $w \in \mathbb{C}^n$, the function $\tau \longmapsto u(z + \tau w)$ is subharmonic in the part of $\mathbb{C}$ where it is defined.

Let Ω be an open subset in $\mathbb{C}^n$.

Definition 3.11 ([3]). We say that Ω is pseudoconvex if there exists a continuous plurisubharmonic function u in Ω such that

$$\Omega_c = \{z \in \Omega;\, u(z) < c\} \subset\subset \Omega \tag{22}$$

for every $c \in \mathbb{R}$. Here $\Omega_c \subset\subset \Omega$ implies that Ω_c is relatively compact in Ω.

Let φ be a continuous function on Ω. We denote by $L^2_\varphi(\Omega)$ the set of square integrable functions with respect to the measure $e^{-\varphi} d\lambda$ where $d\lambda$ is the Lebesgue measure on $\mathbb{R}^{2n} = \mathbb{C}^n$. Let $L^2_{\mathrm{loc}}(\Omega)$ designate the set of locally square integrable functions. Clearly every function in $L^2_{\mathrm{loc}}(\Omega)$ belongs to $L^2_\varphi(\Omega)$ for some φ. By $L^{2,\,(p,\,q)}_\varphi$ (resp. $L^{2,\,(p,\,q)}_{\mathrm{loc}}$) we denote the set of $(p,\, q)$-forms with coefficients in $L^2_\varphi(\Omega)$ (resp. $L^2_{\mathrm{loc}}(\Omega)$).

Theorem 3.12 (Theorem 4.4.2. [3]). *Let Ω be a pseudoconvex open set in $\mathbb{C}^n$ and φ any plurisubharmonic function in Ω. For every $g \in L^{2,\,(p,\,q+1)}_\varphi(\Omega)$ with $\overline{\partial} g = 0$ there is a solution $u \in L^{2,\,(p,\,q)}_{\mathrm{loc}}(\Omega)$ of the equation $\overline{\partial} u = g$ such that*

$$\int_\Omega |u|^2 e^{-\varphi}(1 + |z|^2)^{-2} d\lambda \le \int_\Omega |g|^2 e^{-\varphi} d\lambda. \tag{23}$$

3.2. The proof of Theorem 3.7

Before going into the proof for the main theorem, we need to prepare several lemmas. Let Ω be an open subset in $\hat{X}$. We denote by $\mathcal{G}(\Omega)$ the set of real valued continuous functions $\varphi(z, w)$ on $\Omega \cap X$ that satisfy, for any compact set K in Ω,

$$\varphi(z, w) \le \alpha_K + \beta_K |z|, \qquad ((z, w) \in K \cap X) \tag{24}$$

with some positive constants α_K and β_K. The following easy lemma is needed later.

LEMMA 3.13. *Let Ω be an open subset in $\hat{X}$ and $\{\Omega_\lambda\}_{\lambda\in\Lambda}$ a locally finite open covering of Ω. For any family $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ ($\varphi_\lambda \in \mathcal{G}(\Omega_\lambda)$), there exists $\varphi \in \mathcal{G}(\Omega)$ satisfying*

$$\inf_{\{\lambda;\,(z,w)\in\Omega_\lambda\}} \varphi_\lambda(z,w) \le \varphi(z,w), \qquad ((z,w)\in\Omega\cap X).$$

Clearly $\mathcal{G}(\Omega)$ is a directed set with respect to the partial order $f \le g \iff f(z,w) \le g(z,w)$ for $(z,w)\in\Omega\cap X$.

LEMMA 3.14. *Let Ω be an open subset in $\hat{X}$. Assume that Ω satisfies the conditions* 1. *and* 2. *given in Theorem* 3.7. *Then the subset in $\mathcal{G}(\Omega)$ that consists of a plurisubharmonic function on $\Omega\cap X$ is cofinal in $\mathcal{G}(\Omega)$.*

PROOF. We first take an increasing convex continuous function $\kappa(t)$ ($t\ge 0$) that satisfies

$$2(j+2)\left(1+R_{\frac{1}{j+2}}\right) \le \kappa(j), \qquad (j=0,1,2,\dots).$$

Here the constant R_ϵ was determined in the condition 2. and we may assume

$$1 \le R_1 \le R_{\frac{1}{2}} \le R_{\frac{1}{3}} \le \dots .$$

Set, for $p=(z,w)\in\Omega\cap X$,

$$\tau(p) := \max\left\{\frac{1}{\psi(p)},\ \kappa\left(\max\left\{\frac{1}{\operatorname{dist}(p,\,X\setminus\Omega)},\ |w|\right\}\right)\right\}, \tag{25}$$

where $\psi(p)$ was given by (20). It follows from the condition 2. that $\psi(p)$ is continuous on the open subset $T := \bigcup_{j\in\mathbb{N}}\left(\Omega_{\frac{1}{j}} \cap \left\{|z|>R_{\frac{1}{j}}\right\}\right)$. Since, for $p=(z,w)\in\left(\Omega_{\frac{1}{j+2}}\setminus\Omega_{\frac{1}{j}}\right)\cap\left\{|z|\le 2R_{\frac{1}{j+2}}\right\}$ ($j=0,1,2,\dots$) where we set $\Omega_{\frac{1}{0}} := \emptyset$, we have the estimate

$$\begin{aligned}\frac{1+|z|}{\operatorname{dist}_{\mathbb{D}^{2n}}(p,\,X\setminus\Omega)} &\le \frac{1+|z|}{\operatorname{dist}(p,\,X\setminus\Omega)} \\ &\le (j+2)\left(1+2R_{\frac{1}{j+2}}\right) \\ &\le \kappa\left(\max\left\{\frac{1}{\operatorname{dist}(p,\,X\setminus\Omega)},\ |w|\right\}\right),\end{aligned}$$

we obtain

$$\frac{1}{\psi(p)} \leq \kappa\left(\max\left\{\frac{1}{\operatorname{dist}(p,\, X \setminus \Omega)},\ |w|\right\}\right)$$

in some open neighborhood of $(\Omega \cap X) \setminus T$. Hence we conclude that $\tau(p)$ is continuous on $\Omega \cap X$.

We define, for $j \in \mathbb{N}$,

$$\begin{aligned} Z_j &:= \{p \in \Omega \cap X;\ \tau(p) \leq j\} \subset X, \\ K_j &:= \overline{Z_j} \subset \hat{X}, \end{aligned} \tag{26}$$

where the closure is taken in $\hat{X}$. Note that K_j is a compact set in $\hat{X}$ as it is closed and $p_2(K_j)$ is bounded, and that τ is plurisubharmonic on $\Omega \cap X$ by the facts that $\log(1+|z|)$, $-\log \operatorname{dist}(p,\, X\setminus\Omega)$ and $-\log \operatorname{dist}_{\mathbb{D}^{2n}}(p,\, X\setminus\Omega)$ are plurisubharmonic. The last fact is shown in the following way. Set $\rho_k(p) := |z|+k|w|$ for $p=(z,w) \in X$, and define $d_k(p,\, A) := \inf_{q\in A} \rho_k(p-q)$ for a closed subset $A \subset X$. Then $\{d_k(p,\, A)\}_{k\in\mathbb{N}}$ is an increasing sequence of functions of p and we have $\lim_{k\to\infty} d_k(p,\, A) = \operatorname{dist}_{\mathbb{D}^{2n}}(p,\, A)$. Hence $\{-\log d_k(p,\, X \setminus \Omega)\}_{k\in\mathbb{N}}$ is a decreasing sequence of plurisubharmonic functions on $\Omega \cap X$, and $-\log \operatorname{dist}_{\mathbb{D}^{2n}}(p,\, X \setminus \Omega)$ becomes plurisubharmonic.

Lemma 3.15. *The sets $\{K_j\}_{j\in\mathbb{N}}$ satisfy the following conditions.*

(1) *K_j is a compact subset in Ω.*

(2) *$K_j \subset K_{j+1}$ and $\cup_{j\in\mathbb{N}} K_j^\circ = \Omega$.*

Proof. As K_j is compact in $\hat{X}$, the condition (1) in the lemma follows if we show $K_j \subset \Omega$. Clearly $K_j \cap X \subset \Omega \cap X$ holds. It suffices to prove $K_j \cap X_\infty \subset \Omega \cap X_\infty$.

Let $p_\infty = (z_\infty,\, w_\infty) \in K_j \cap X_\infty$. Then we can find points $(z_k, w_k) \in Z_j$ $(k = 1, 2, \dots)$ with $(z_k, w_k) \to (z_\infty, w_\infty)$ $(k \to \infty)$ in $\hat{X}$. If we could prove $p_\infty \in N^1_\infty(\Omega)$, then, as Ω is regular at ∞, we have $p_\infty \in \Omega \cap X_\infty$, from which $K_j \cap X_\infty \subset \Omega \cap X_\infty$ follows.

Let us prove $p_\infty \in N^1_\infty(\Omega)$. Suppose $p_\infty \in \operatorname{clos}^1_\infty(X\setminus\Omega)$. Then there exist points $\{(\zeta_k,\, \nu_k)\}_{k\in\mathbb{N}}$ in $X \setminus \Omega$ satisfying $(\zeta_k,\, \nu_k) \to p_\infty$ in $\hat{X}$ and $\dfrac{|\zeta_{k+1}|}{|\zeta_k|} \to 1$ $(k \to \infty)$.

We may assume $|z_k| \geq |\zeta_1|$ $(k = 1, 2, \dots)$. Then there exists $\theta : \mathbb{N} \to \mathbb{N}$ which satisfies $\theta(k) \to \infty$ $(k \to \infty)$ and

$$|\zeta_{\theta(k)}| \leq |z_k| \leq |\zeta_{\theta(k)+1}|, \qquad (k = 1, 2, \dots).$$

As $(\zeta_k,\, \nu_k) \to p_\infty$ and $(z_k,\, w_k) \to p_\infty$ in $\hat{X}$, we have

$$\begin{aligned} &|w_k - w_\infty| \leq \epsilon_k, \qquad |\nu_k - w_\infty| \leq \epsilon_k, \\ &\big|z_k - |z_k| z_\infty\big| \leq \epsilon_k |z_k|, \qquad \big|\zeta_k - |\zeta_k| z_\infty\big| \leq \epsilon_k |\zeta_k| \end{aligned}$$

for $\epsilon_k > 0$ with $\epsilon_k \to 0$ $(k \to \infty)$. Hence we obtain

$$\begin{aligned} |z_k - \zeta_{\theta(k)}| &\leq \epsilon_k |z_k| + \epsilon_{\theta(k)} |\zeta_{\theta(k)}| + \big||z_k| - |\zeta_{\theta(k)}|\big| \\ &\leq \left(\epsilon_k + \epsilon_{\theta(k)} + \left|1 - \frac{|\zeta_{\theta(k)}|}{|z_k|}\right|\right) |z_k|. \end{aligned}$$

By noticing

$$\frac{|\zeta_{\theta(k)}|}{|\zeta_{\theta(k)+1}|} \leq \frac{|\zeta_{\theta(k)}|}{|z_k|} \leq 1 \ \text{ and } \ \frac{|\zeta_{k+1}|}{|\zeta_k|} \to 1,$$

we have $\dfrac{|\zeta_{\theta(k)}|}{|z_k|} \to 1$ $(k \to \infty)$, from which we get

$$\frac{|z_k - \zeta_{\theta(k)}|}{|z_k|} \to 0, \qquad (k \to \infty). \tag{27}$$

On the other hand, by noticing $w_k - \nu_{\theta(k)} \to 0$ $(k \to \infty)$ and $(z_k,\, w_k) \in Z_j$, we have

$$\begin{aligned} (z_k,\, \nu_{\theta(k)}) \in &\left\{ p = (z, w) \in \Omega \cap X;\ \kappa\left(\max\left\{\frac{1}{\operatorname{dist}(p,\, X \setminus \Omega)}, |w|\right\}\right) < j+1 \right\} \\ &= \Omega_{\frac{1}{\kappa^{-1}(j+1)}}, \end{aligned}$$

for sufficiently large k. Then, applying the condition 2. in Theorem 3.7 to the points $(z_k,\, w_k) \in Z_j$ and $(z_k,\, \nu_{\theta(k)})$, we obtain $(z_k,\, \nu_{\theta(k)}) \in Z_{j+1}$ for large k's. Hence, for such a k, we have

$$\frac{|z_k - \zeta_{\theta(k)}|}{|z_k| + 1} \geq \frac{1}{j+1} > 0, \tag{28}$$

by the definition of Z_{j+1} and the facts $(z_k,\, \nu_{\theta(k)}) \in Z_{j+1}$ and $(\zeta_{\theta(k)},\, \nu_{\theta(k)}) \in X \setminus \Omega$. This contradicts (27), and we have $p_\infty \notin \mathrm{clos}^1_\infty(X \setminus \Omega)$, which implies $p_\infty \in N^1_\infty(\Omega)$.

Finally we show the condition (2) in the lemma. Let $p = (z_\infty,\, w_\infty)$ be a point in $\Omega \cap X_\infty$. Then p has an open neighborhood $G_R(\Gamma,\, w_\infty)$ defined by (14) which is relatively compact in Ω. Such a $G_R(\Gamma,\, w_\infty)$ is contained in some K_j because there exists another $G_{R'}(\Gamma',\, w_\infty)$ which satisfies $G_R(\Gamma,\, w_\infty) \subset\subset G_{R'}(\Gamma',\, w_\infty) \subset \Omega$ with $R' < R$ and $\Gamma \subset\subset \Gamma' \subset S^{2n-1}$. Therefore we conclude $p \in K_j^\circ$, which implies $\Omega \subset \cup_j K_j^\circ$. Hence $\Omega = \cup_j K_j^\circ$ follows from the fact $K_j \subset \Omega$.

The proof has been completed. □

We continue the proof of Lemma 3.14. Let $g \in \mathcal{G}(\Omega)$. Now we show that there exists a plurisubharmonic function $f \in \mathcal{G}(\Omega)$ with $g \le f$. As $g \in \mathcal{G}(\Omega)$ and K_j is compact in Ω, there exist positive constants $\{\alpha_j\}$ and $\{\beta_j\}$ satisfying

$$g(z,w) \le \alpha_j + \beta_j|z| \ \text{ for } \ (z,w) \in K_j \cap X, \qquad (j = 1, 2, \dots).$$

We can take a continuous increasing function $\Phi(t)$ on $\{t \ge 0\}$ which is convex and satisfies

$$\log\max\{\alpha_j, \beta_j\} \le \Phi(j-1), \qquad (j = 1, 2, \dots).$$

Then, for $(z,w) \in (K_j \setminus K_{j-1}) \cap X$ $(j \ge 1)$ where we set $K_0 = \emptyset$ by convention, we have

$$\begin{aligned} g(z,w) &\le \alpha_j + \beta_j|z| \le \max\{\alpha_j,\, \beta_j\}(1+|z|) \\ &= \exp\left(\log\max\{\alpha_j,\, \beta_j\} + \log(1+|z|)\right) \\ &\le \exp\left(\Phi(j-1) + \log(1+|z|)\right) \\ &\le \exp\left(\Phi(\tau(z,w)) + \log(1+|z|)\right) = e^{\Phi(\tau(z,w))}(1+|z|). \end{aligned} \tag{29}$$

Set $f := e^{\Phi(\tau(z,w))}(1+|z|)$. This f satisfies all the required conditions. As a matter of fact, as a compact set K in Ω is contained in some K_j and $\Phi(\tau(z,w))$ is bounded by $\Phi(j)$ in $K_j \cap X$, f belongs to $\mathcal{G}(\Omega)$. The estimate (29) implies $g(z,w) \le f(z,w)$ for $(z,w) \in \Omega \cap X$ by $\cup K_j \setminus K_{j-1} = \Omega$. Since $\Phi(\tau(z,w))$ and $\log(1+|z|)$ are plurisubharmonic, f is also plurisubharmonic.

The proof has been completed. □

Let Ω be an open subset in $\hat{X}$. We denote by $L^2_{\mathcal{G}}(\Omega)$ the set of locally square integrable functions f on $\Omega \cap X$ satisfying

$$\int_{\Omega\cap X} |f(z,w)|^2 e^{-\varphi(z,w)} d\lambda < +\infty \tag{30}$$

for some $\varphi \in \mathcal{G}(\Omega)$, and $L^{2,(p,q)}_{\mathcal{G}}(\Omega)$ designates the set of (p,q)-forms on $\Omega\cap X$ with coefficients in $L^2_{\mathcal{G}}(\Omega)$. Set

$$\tilde{L}^{2,(p,q)}_{\mathcal{G}}(\Omega) := \left\{ f \in L^{2,(p,q)}_{\mathcal{G}}(\Omega);\ \overline{\partial} f \in L^{2,(p,q+1)}_{\mathcal{G}}(\Omega) \right\}.$$

Since any open covering has a countable open subcovering that is locally finite, it follows from Lemma 3.13 that the presheaf $\left\{L^2_{\mathcal{G}}(\Omega)\right\}_\Omega$ (resp. $\left\{L^{2,(p,q)}_{\mathcal{G}}(\Omega)\right\}_\Omega$ and $\left\{\tilde{L}^{2,(p,q)}_{\mathcal{G}}(\Omega)\right\}_\Omega$) forms a sheaf on $\hat{X}$. We denote it by $\mathcal{L}^2_{\mathcal{G}}$ (resp. $\mathcal{L}^{2,(p,q)}_{\mathcal{G}}$ and $\tilde{\mathcal{L}}^{2,(p,q)}_{\mathcal{G}}$). Note that these sheaves are soft.

LEMMA 3.16. *Let Ω be an open subset in $\hat{X}$ and $f \in \mathcal{O}_X(\Omega\cap X)$. Then $f \in \mathcal{O}^{\exp}_X(\Omega)$ if and only if $f \in \tilde{\mathcal{L}}^2_{\mathcal{G}}(\Omega)$.*

PROOF. We first show that $f \in \mathcal{O}^{\exp}_X(\Omega)$ implies $f \in \tilde{\mathcal{L}}^2_{\mathcal{G}}(\Omega)$. Set

$$\tilde{\varphi}(z,w) := 2\log\max\{1, |f(z,w)|\}, \qquad ((z,w) \in \Omega \cap X).$$

Then we have $\tilde{\varphi} \in \mathcal{G}(\Omega)$ and $|f(z,w)|^2 e^{-\tilde{\varphi}(z,w)} \le 1$ for $(z,w) \in \Omega \cap X$. Therefore $\varphi(z,w) := \tilde{\varphi}(z,w) + |z| + |w|$ also belongs to $\mathcal{G}(\Omega)$, for which the estimate (30) holds.

Let us show converse implication. Let $\{\Omega_j\}_{j\in\mathbb{N}}$ be an exhausting family of Ω given in Lemma 3.1. Set $K_j := \overline{\Omega_j} \subset \hat{X}$. Then, for each j, there exist $r > 0$ and $j' \ge j$ which satisfy

$$(K_j \cap X) + B_r(0,0) \subset K_{j'} \cap X.$$

Here $B_r(z,w)$ denotes the open ball in X with radius of r and center at (z,w). As f is holomorphic, we get

$$\begin{aligned} |f(z_0,w_0)| &\le \kappa \int_{B_r(z_0,w_0)} |f(z,w)|^2 d\lambda \\ &\le \kappa \sup_{(z,w)\in B_r(z_0,w_0)} e^{\varphi(z,w)} \int_{\Omega\cap X} |f(z,w)|^2 e^{-\varphi(z,w)} d\lambda, \end{aligned}$$

where $\kappa > 0$ is a constant which depends only on r and $n+m$, and $\varphi \in \mathcal{G}(\Omega)$ is chosen as the estimate (30) is satisfied. Hence we have, for a positive constant C,

$$|f(z_0, w_0)| \le C \sup_{(z,w)\in B_r(z_0,w_0)} e^{\varphi(z,w)},$$

if $B_r(z_0, w_0) \subset \Omega \cap X$ is satisfied. Since $B_r(z_0, w_0) \subset K_{j'}$ for $(z_0, w_0) \in K_j \cap X$ and $\varphi(z, w) \le \alpha+\beta|z|$ holds in $K_{j'} \cap X$ with some positive constants α and β, we have

$$|f(z, w)| \le C' e^{\alpha|z|}, \qquad (z, w) \in K_j \cap X.$$

This completes the proof. □

Now we give the proof of Theorem 3.7.

Proof. Let us consider the $\overline{\partial}$ complex $\mathcal{L}$ as

$$0 \to \tilde{\mathcal{L}}_{\mathcal{G}}^{2,(0,0)}(\Omega) \xrightarrow{\overline{\partial_0}} \tilde{\mathcal{L}}_{\mathcal{G}}^{2,(0,1)}(\Omega) \xrightarrow{\overline{\partial_1}} \dots \xrightarrow{\overline{\partial}} \tilde{\mathcal{L}}_{\mathcal{G}}^{2,(0,n+m)}(\Omega) \to 0. \tag{31}$$

By Lemma 3.14, we may assume that $\varphi \in \mathcal{G}(\Omega)$ which appears in the estimate (30) is always plurisubharmonic. Hence we can apply Theorem 3.12 to $\mathcal{L}$ and obtain $H^k(\mathcal{L}) = 0$ $(k \ne 0)$. Moreover $H^0(\mathcal{L}) = \operatorname{Ker} \overline{\partial_0} = \mathcal{O}_X^{\exp}(\Omega)$ follows from Lemma 3.16.

A point $(z, w) \in \hat{X}$ has a family $\{\Omega_j\}_{j\in\mathbb{N}}$ of fundamental neighborhoods of product type, for which $\Omega_j \cap X$ is pseudoconvex and Ω_j is regular at ∞. Therefore, by replacing Ω with Ω_j in (31) and taking its inductive limit, we know that the complex of sheaves

$$0 \to \tilde{\mathcal{L}}_{\mathcal{G}}^{2,(0,0)} \xrightarrow{\overline{\partial_0}} \tilde{\mathcal{L}}_{\mathcal{G}}^{2,(0,1)} \xrightarrow{\overline{\partial_1}} \dots \xrightarrow{\overline{\partial}} \tilde{\mathcal{L}}_{\mathcal{G}}^{2,(0,n+m)} \to 0 \tag{32}$$

is a soft resolution of $\mathcal{O}_X^{\exp}$. Hence we have $H^k(\Omega, \mathcal{O}_X^{\exp}) = H^k(\mathcal{L})$ and $H^k(\Omega, \mathcal{O}_X^{\exp}) = 0$ $(k \ne 0)$ follows. This completes the proof. □

In Section 4, we show that, if $n = 1$, the vanishing theorem still holds for an open subset $\Omega = U \times W \subset \mathbb{D}^2 \times \mathbb{C}^m$ of product type without the regularity of U at ∞. However, if n is greater than one, one cannot expect the vanishing theorem anymore without the regularity condition of Ω at ∞ as the following example shows.

Example 3.17. Assume $n = 2$ and $m = 0$, i.e., $X = \mathbb{C}^2_{(z_1,z_2)}$ and $\hat{X} = \mathbb{D}^4$. Set

$$U := \left\{(z_1,\, z_2) \in X;\, |\arg(z_1)| < \frac{\pi}{4},\, |z_2| < |z_1|\right\},$$
$$\Omega := \left(\overline{U}\right)^\circ \setminus \{p_\infty\} \subset \hat{X},$$

where p_∞ denotes the point $(1, 0, 0, 0)$ in $S^3 \subset \mathbb{D}^4$. Note that $\Omega \cap X = U$ is pseudoconvex in X, while Ω is not regular at ∞. In this case, we have $H^1(\Omega,\, \mathcal{O}^{\exp}_X) \neq 0$ which is shown below, and the vanishing theorem does not hold for Ω.

Let $Y = \mathbb{C}^1_z \times \mathbb{C}^1_w$ and $\hat{Y} = \mathbb{D}^2 \times \mathbb{C}^1$, and let us consider the holomorphic map $f : X \setminus \{z_1 = 0\} \to Y$ defined by $f(z_1,\, z_2) = \left(z_1,\, \dfrac{z_2}{z_1}\right)$. Set

$$\tilde{U} := \left\{(z,\, w) \in Y;\, |\arg(z)| < \frac{\pi}{4},\, |w| < 1\right\},$$
$$\tilde{\Omega} := \left(\overline{\tilde{U}}\right)^\circ \setminus (\{q_\infty\} \times \{0\}) \subset \hat{Y}.$$

Here $q_\infty = (1, 0) \in S^1 \subset \mathbb{D}^2$. Note that $\tilde{\Omega}$ is an open subset of non-product type in $\mathbb{D}^2 \times \mathbb{C}$. As f gives a biholomorphic map between U and $\tilde{U}$ which extends to a continuous isomorphism between Ω and $\tilde{\Omega}$, we have

$$H^k(\Omega,\, \mathcal{O}^{\exp}_X) = H^k(\tilde{\Omega},\, \mathcal{O}^{\exp}_Y), \qquad (k \in \mathbb{N}).$$

Hence it suffices to prove $H^1(\tilde{\Omega},\, \mathcal{O}^{\exp}_Y) \neq 0$. Set

$$V := \left(\overline{\left\{z \in \mathbb{C};\, |\arg(z)| < \frac{\pi}{4}\right\}}\right)^\circ \subset \mathbb{D}^2, \qquad W := \{w \in \mathbb{C};\, |w| < 1\}.$$

Noticing $((V \setminus \{q_\infty\}) \times W) \cup (V \times (W \setminus \{0\})) = \tilde{\Omega}$, we have the long exact sequence

$$\begin{aligned}(33)\quad & \mathcal{O}^{\exp}_Y((V \setminus \{q_\infty\}) \times W) \oplus \mathcal{O}^{\exp}_Y(V \times (W \setminus \{0\})) \\ & \qquad \xrightarrow{\iota} \mathcal{O}^{\exp}_Y((V \setminus \{q_\infty\}) \times (W \setminus \{0\})) \to H^1(\tilde{\Omega},\, \mathcal{O}^{\exp}_Y).\end{aligned}$$

Suppose $H^1(\tilde{\Omega},\, \mathcal{O}^{\exp}_Y) = 0$. Then ι becomes surjective. It is well known that there exists a holomorphic function $g(z)$ in $\mathcal{O}^{\exp}_{\mathbb{C}}(V \setminus \{q_\infty\})$ which does not belong to $\mathcal{O}^{\exp}_{\mathbb{C}}(V)$ (for existence of such a holomorphic function, see [14]).

Set $h(z, w) := \dfrac{g(z)}{w}$. Then $h(z, w)$ belongs to $\mathcal{O}_Y^{\exp}((V \setminus \{q_\infty\}) \times (W \setminus \{0\}))$. As ι is surjective, there exist $h_1(z, w) \in \mathcal{O}_Y^{\exp}((V \setminus \{q_\infty\}) \times W)$ and $h_2(z, w) \in \mathcal{O}_Y^{\exp}(V \times (W \setminus \{0\}))$ satisfying $h = h_1 + h_2$. Clearly we have

$$\begin{aligned} 2\pi\sqrt{-1}g(z) = \int_C h(z, w)dw &= \int_C (h_1(z, w) + h_2(z, w))dw \\ &= \int_C h_2(z, w)dw, \end{aligned}$$

where C is a small circle turning around the origin in W. Since $\displaystyle\int_C h_2(z, w)dw$ belongs to $\mathcal{O}_{\mathbb{C}}^{\exp}(V)$, we get $g(z) \in \mathcal{O}_{\mathbb{C}}^{\exp}(V)$, which contradicts the choice of $g(z)$, i.e., $g(z) \notin \mathcal{O}_{\mathbb{C}}^{\exp}(V)$. Therefore we have obtained the conclusion $H^1(\Omega, \mathcal{O}_X^{\exp}) = H^1(\tilde{\Omega}, \mathcal{O}_Y^{\exp}) \neq 0$.

4. Laplace Hyperfunctions with Holomorphic Parameters

As an application of Theorem 3.7 established in the previous section, we construct cohomologically the sheaf $\mathcal{B}_{\mathbb{R}}^{\exp}$ of Laplace hyperfunctions and the sheaf $\mathcal{BO}_N^{\exp}$ of Laplace hyperfunctions with holomorphic parameters.

Let $N = \mathbb{R} \times \mathbb{C}^m (m \geq 0)$, and let $\overline{N} = \overline{\mathbb{R}} \times \mathbb{C}^m$ be the closure of N in $\hat{X} = \mathbb{D}^2 \times \mathbb{C}^m$. Then we have the following theorem.

Theorem 4.1. *The closed set* $\overline{N}$ *is purely* 1*-codimensional with respect to the sheaf* $\mathcal{O}_X^{\exp}$*,i.e.,*

$$\mathscr{H}_{\overline{N}}^k(\mathcal{O}_X^{\exp}) = 0, \quad (k \neq 1). \tag{34}$$

Here $\mathscr{H}_{\overline{N}}^k(\mathcal{O}_X^{\exp})$ *is the k-th derived sheaf of* $\mathcal{O}_X^{\exp}$ *with support in* $\overline{N}$*.*

Proof. Let $p = (x, w) \in \overline{N}$. As $N \subset \mathbb{C}^{m+1}$ is purely 1-codimensional with respect to the sheaf $\mathcal{O}_{\mathbb{C}^{m+1}}$ of holomorphic functions on $\mathbb{C}^{m+1}$, it is sufficient to prove the theorem at $p = (+\infty, w) \in \overline{N}$. Note that we have

$$\mathscr{H}_{\overline{N}}^k(\mathcal{O}_X^{\exp})_p = \varinjlim_{U \times T \ni p} H^k_{(\overline{\mathbb{R}} \cap U) \times T}(U \times T, \mathcal{O}_X^{\exp}), \tag{35}$$

where $U \times T$ runs through open neighborhoods of p in $\hat{X}$. Let us consider the long exact sequence

(36)
$$\begin{aligned}
0 &\to H^0_{(\overline{\mathbb{R}}\cap U)\times T}(U \times T, \mathcal{O}_X^{\exp}) \to H^0(U \times T, \mathcal{O}_X^{\exp}) \to H^0((U \setminus \overline{\mathbb{R}}) \times T, \mathcal{O}_X^{\exp}) \\
&\to H^1_{(\overline{\mathbb{R}}\cap U)\times T}(U \times T, \mathcal{O}_X^{\exp}) \to H^1(U \times T, \mathcal{O}_X^{\exp}) \to H^1((U \setminus \overline{\mathbb{R}}) \times T, \mathcal{O}_X^{\exp}) \\
&\to H^2_{(\overline{\mathbb{R}}\cap U)\times T}(U \times T, \mathcal{O}_X^{\exp}) \to H^2(U \times T, \mathcal{O}_X^{\exp}) \to H^2((U \setminus \overline{\mathbb{R}}) \times T, \mathcal{O}_X^{\exp}) \\
&\to \cdots .
\end{aligned}$$

As a domain in $\mathbb{C}$ is always pseudoconvex, it follows from Theorem 3.7 that, if T is pseudoconvex, we get

$$H^k(U \times T, \mathcal{O}_X^{\exp}) = H^k((U \setminus \overline{\mathbb{R}}) \times T, \mathcal{O}_X^{\exp}) = 0 \quad (k \geq 1). \tag{37}$$

Set $V_\varepsilon := \{z \in \mathbb{C}\ ;\ |\arg z| < \varepsilon,\ |z| > 1/\varepsilon\}$ and $U_\varepsilon := (\overline{V_\varepsilon})^\circ \subset \mathbb{D}^2$ for $\varepsilon > 0$. Then $\{U_\varepsilon\}_{\varepsilon>0}$ is a fundamental system of neighborhoods of $x = +\infty$. Let $\{T_\varepsilon\}_{\varepsilon>0}$ be a fundamental system of neighborhoods of $w \in \mathbb{C}^m$. By replacing U (resp. T) in the long exact sequence (36) with U_ε (resp. T_ε) and taking its inductive limit, we obtain

(38)
$$\begin{aligned}
&0 \to \mathscr{H}^0_{\overline{N}}(\mathcal{O}_X^{\exp})_p \to (\mathcal{O}_X^{\exp})_p \to \varinjlim_{\varepsilon\downarrow 0} \mathcal{O}_X^{\exp}((U_\varepsilon \setminus \overline{\mathbb{R}}) \times T_\varepsilon) \to \mathscr{H}^1_{\overline{N}}(\mathcal{O}_X^{\exp})_p \to 0, \\
&\mathscr{H}^k_{\overline{N}}(\mathcal{O}_X^{\exp})_p = 0 \quad (k \geq 2).
\end{aligned}$$

Clearly the morphism $(\mathcal{O}_X^{\exp})_p \to \varinjlim_{\varepsilon\downarrow 0} \mathcal{O}_X^{\exp}((U_\varepsilon \setminus \overline{\mathbb{R}}) \times T_\varepsilon)$ is injective, from which we also have $\mathscr{H}^0_{\overline{N}}(\mathcal{O}_X^{\exp}) = 0$. This completes the proof. □

As a particular case, we have the following corollary.

COROLLARY 4.2. *$\overline{\mathbb{R}}$ is purely 1-codimensional with respect to the sheaf $\mathcal{O}_{\mathbb{C}}^{\exp}$, that is,*

$$\mathscr{H}^k_{\overline{\mathbb{R}}}(\mathcal{O}_{\mathbb{C}}^{\exp}) = 0 \quad (k \neq 1). \tag{39}$$

DEFINITION 4.3. The sheaf $\mathcal{BO}_N^{\exp}$ of Laplace hyperfunctions of one variable with holomorphic parameters is defined by

$$\mathcal{BO}_N^{\exp} := \mathscr{H}^1_{\overline{N}}(\mathcal{O}_X^{\exp}) \underset{\mathbb{Z}_{\overline{N}}}{\otimes} \omega_{\overline{N}}, \tag{40}$$

where $\mathbb{Z}_{\overline{N}}$ denotes the constant sheaf on $\overline{N}$ having stalk $\mathbb{Z}$ and $\omega_{\overline{N}}$ denotes the orientation sheaf $\mathscr{H}^1_{\overline{N}}(\mathbb{Z}_{\hat{X}})$ on $\overline{N}$.

The global sections of the sheaf $\mathcal{BO}_N^{\exp}$ can be written in terms of cohomology groups by Theorem 4.1. For an open set $\Omega \subset \overline{\mathbb{R}}$ and a pseudoconvex open subset $T \subset \mathbb{C}^m$, by taking a complex neighborhood V of Ω in $\mathbb{D}^2$, we have

$$\mathcal{BO}_N^{\exp}(\Omega \times T) = H^1_{\Omega\times T}(V \times T, \mathcal{O}_X^{\exp}) = \frac{\mathcal{O}_X^{\exp}((V \setminus \Omega) \times T)}{\mathcal{O}_X^{\exp}(V \times T)}. \tag{41}$$

Note that the above representation does not depend on a choice of the complex neighborhood V.

Definition 4.4. We define the sheaf $\mathcal{B}_{\mathbb{R}}^{\exp}$of Laplace hyperfunctions of one variable on $\overline{\mathbb{R}}$ by

$$\mathcal{B}_{\mathbb{R}}^{\exp} := \mathscr{H}^1_{\overline{\mathbb{R}}}(\mathcal{O}_{\mathbb{C}}^{\exp}) \underset{\mathbb{Z}_{\overline{\mathbb{R}}}}{\otimes} \omega_{\overline{\mathbb{R}}}, \tag{42}$$

where $\mathbb{Z}_{\overline{\mathbb{R}}}$ denotes the constant sheaf on $\overline{\mathbb{R}}$ having stalk $\mathbb{Z}$ and $\omega_{\overline{\mathbb{R}}}$ denotes the orientation sheaf $\mathscr{H}^1_{\overline{\mathbb{R}}}(\mathbb{Z}_{\hat{X}})$ on $\overline{\mathbb{R}}$.

The restriction of $\mathcal{B}_{\mathbb{R}}^{\exp}$ to $\mathbb{R}$ is isomorphic to the sheaf $\mathcal{B}_{\mathbb{R}}$ of ordinary hyperfunctions because of $\mathcal{O}_{\mathbb{C}}^{\exp}|_{\mathbb{C}} = \mathcal{O}_{\mathbb{C}}$. By Corollary 4.2 we have

$$\Gamma_{[a,\,\infty]}(\overline{\mathbb{R}},\, \mathcal{B}_{\mathbb{R}}^{\exp}) = \frac{\mathcal{O}_{\mathbb{C}}^{\exp}(\mathbb{D}^2 \setminus [a,\, \infty])}{\mathcal{O}_{\mathbb{C}}^{\exp}(\mathbb{D}^2)}. \tag{43}$$

Hence the set $\mathcal{B}^{\exp}_{[a,\,\infty]}$ defined by H. Komatsu coincides with $\Gamma_{[a,\infty]}(\overline{\mathbb{R}},\, \mathcal{B}_{\mathbb{R}}^{\exp})$ in our framework.

5. Several Properties of $\mathcal{BO}_N^{\exp}$

We study several properties for the sheaf $\mathcal{BO}_N^{\exp}$ constructed in the previous section. We first establish the vanishing theorem on an open subset which is not necessarily regular at ∞ in the case of $n = 1$, i.e., $\hat{X} = \mathbb{D}^2 \times \mathbb{C}^m$. This is a key to show the flabbiness of $\mathcal{BO}_N^{\exp}$.

PROPOSITION 5.1. *Let F be a closed subset in S^1 and W a pseudoconvex open subset in $\mathbb{C}^m$. Then we have*

$$H^k(U \times W, \mathcal{O}_X^{\exp}) = 0, \qquad (k \neq 0),$$

where we set $U := \mathbb{D}^2 \setminus F$.

PROOF. If $F = S^1$ or $F = \emptyset$, then the proposition clearly holds. Therefore we assume $F \neq S^1$ and $F \neq \emptyset$. Set, for $m \in \mathbb{N}$,

$$\Gamma_{\geq m} := \left\{ z \in \mathbb{C};\ \frac{z}{|z|} \in F,\ |z| \geq m \right\} \sqcup F.$$

Since $U_m := \mathbb{D}^2 \setminus \Gamma_{\geq m}$ is regular at ∞, by Theorem 3.7, we have

$$H^k(U_m \times W, \mathcal{O}_X^{\exp}) = 0, \qquad (k \neq 0).$$

Then, as $U = \cup_m U_m$, it follows from Proposition 1.4.2 [4] that

$$H^k(U \times W, \mathcal{O}_X^{\exp}) = 0, \qquad (k \geq 2).$$

Therefore it suffices to show $H^1(U \times W, \mathcal{O}_X^{\exp}) = 0$, which is equivalent to saying that $H^2_{F\times W}(\mathbb{D}^2 \times W, \mathcal{O}_X^{\exp}) = 0$ because of $H^k(\mathbb{D}^2 \times W, \mathcal{O}_X^{\exp}) = 0$ $(k \neq 0)$ and the long exact sequence

$$(44) \quad \to H^1(\mathbb{D}^2 \times W, \mathcal{O}_X^{\exp}) \to H^1(U \times W, \mathcal{O}_X^{\exp}) \\ \to H^2_{F\times W}(\mathbb{D}^2 \times W, \mathcal{O}_X^{\exp}) \to H^2(\mathbb{D}^2 \times W, \mathcal{O}_X^{\exp}) \to .$$

Let us show $H^2_{F\times W}(\mathbb{D}^2 \times W, \mathcal{O}_X^{\exp}) = 0$. Set

$$F_1 := \left\{ z \in \mathbb{C};\ \frac{z}{|z|} \in F,\ |z| \in \{2,4,6,\dots\} \right\} \sqcup F,$$
$$F_2 := \left\{ z \in \mathbb{C};\ \frac{z}{|z|} \in F,\ |z| \in \{1,3,5,\dots\} \right\} \sqcup F,$$

and set $\check{\mathbb{D}}^2 := \mathbb{D}^2 \setminus \{0\}$. As $\check{\mathbb{D}}^2 \setminus F_j$ $(j = 1,2)$ and $\check{\mathbb{D}}^2 \setminus (F_1 \cup F_2)$ are regular at ∞, we have

$$(45) \quad \begin{aligned} &H^k((\check{\mathbb{D}}^2 \setminus F_j) \times W, \mathcal{O}_X^{\exp}) = 0, \\ &H^k((\check{\mathbb{D}}^2 \setminus (F_1 \cup F_2)) \times W, \mathcal{O}_X^{\exp}) = 0,\ (k \neq 0). \end{aligned}$$

This implies, in particular,

$$H^2_{(F_1\cup F_2)\times W}(\check{\mathbb{D}}^2\times W,\,\mathcal{O}_X^{\exp})=0,\ \ H^2_{F_j\times W}(\check{\mathbb{D}}^2\times W,\,\mathcal{O}_X^{\exp})=0,\ \ (j=1,2),$$

by the long exact sequence of cohomology groups. Hence, by noticing $F_1\cap F_2=F$, we have the long exact sequence

$$\begin{aligned}(46)\quad &\to \bigoplus_{j=1,2} H^1_{F_j\times W}(\check{\mathbb{D}}^2\times W,\,\mathcal{O}_X^{\exp}) \xrightarrow{\iota} H^1_{(F_1\cup F_2)\times W}(\check{\mathbb{D}}^2\times W,\,\mathcal{O}_X^{\exp})\\ &\qquad\qquad \to H^2_{F\times W}(\check{\mathbb{D}}^2\times W,\,\mathcal{O}_X^{\exp})\to 0.\end{aligned}$$

If we could prove that the morphism ι is surjective, then we obtain

$$H^2_{F\times W}(\mathbb{D}^2\times W,\,\mathcal{O}_X^{\exp})=H^2_{F\times W}(\check{\mathbb{D}}^2\times W,\,\mathcal{O}_X^{\exp})=0,$$

and the proposition follows. Hence we show the surjectivity of ι. Since we have the exact sequences

$$\begin{aligned}&\mathcal{O}_X^{\exp}((\check{\mathbb{D}}^2\setminus F_j)\times W)\to H^1_{F_j\times W}(\check{\mathbb{D}}^2\times W,\,\mathcal{O}_X^{\exp})\to 0,\qquad (j=1,2),\\ &\mathcal{O}_X^{\exp}((\check{\mathbb{D}}^2\setminus (F_1\cup F_2))\times W)\to H^1_{(F_1\cup F_2)\times W}(\check{\mathbb{D}}^2\times W,\,\mathcal{O}_X^{\exp})\to 0,\end{aligned}$$

it suffices to show that the morphism

$$\mathcal{O}_X^{\exp}((\check{\mathbb{D}}^2\setminus F_1)\times W)\oplus\mathcal{O}_X^{\exp}((\check{\mathbb{D}}^2\setminus F_2)\times W)\xrightarrow{\tilde{\iota}}\mathcal{O}_X^{\exp}((\check{\mathbb{D}}^2\setminus (F_1\cup F_2))\times W)$$

is surjective. This is done by the usual argument with the Runge approximation theorem as follows. Let $G(z,w)\in\mathcal{O}_X^{\exp}((\check{\mathbb{D}}^2\setminus(F_1\cup F_2))\times W)$. We define, for $k\in\mathbb{N}$,

$$g_k(z,w):=\frac{1}{2\pi\sqrt{-1}}\int_{C_k}\frac{G(\zeta,w)}{\zeta-z}d\zeta$$

with C_k being two circles $\{\zeta\in\mathbb{C};|\zeta|=k\pm\epsilon\}$. Here we take $\epsilon>0$ sufficiently small so that the point z is located outside the set $\{\zeta\in\mathbb{C};\ k-\epsilon\le|\zeta|\le k+\epsilon\}$, and the orientation of the outer (resp. inner) circle is anti-clockwise (resp. clockwise). Set

$$\Gamma_k=\left\{z\in\mathbb{C};\ \frac{z}{|z|}\in F,\ |z|=k\right\}.$$

Then $g_k(z,w)$ is holomorphic on $(\mathbb{P}^1\setminus\Gamma_k)\times W$ and $G(z,w)-g_k(z,w)$ is holomorphic near $\Gamma_k\times W$.

Let $\{L_j\}_{j\in\mathbb{N}}$ be an exhausting family of W for which each L_j is compact and holomorphically convex in W. Set

$$Z_j := \left\{z \in \mathbb{C};\ |z| < \frac{1}{j+1}\right\} \cup \left\{z \in \mathbb{C};\ |z| > j - \frac{1}{2},\ \operatorname{dist}(z, \Gamma_{\geq 1}) < j^{-1}|z|\right\}$$

and $K_j := (\mathbb{C} \setminus Z_j) \times L_j$ $(j \in \mathbb{N})$.

As $(\mathbb{P}^1 \setminus Z_2) \times L_2$ is compact and holomorphically convex in $(\mathbb{P}^1 \setminus (\{0\} \cup \Gamma_4)) \times W$, it follows from the Runge approximation theorem that there exists $h_2(z,w) \in \mathcal{O}((\mathbb{P}^1 \setminus (\{0\} \cup \Gamma_4)) \times W)$ satisfying

$$|g_2(z,w) - h_2(z,w)| < \frac{1}{2^2}, \qquad ((z,w) \in K_2).$$

Then applying the argument above to $g_4(z,w) + h_2(z,w) \in \mathcal{O}((\mathbb{P}^1 \setminus (\{0\} \cup \Gamma_4)) \times W)$, we can find $h_4(z,w) \in \mathcal{O}((\mathbb{P}^1 \setminus (\{0\} \cup \Gamma_6)) \times W)$ with

$$|(g_4(z,w) + h_2(z,w)) - h_4(z,w)| < \frac{1}{2^4}, \qquad ((z,w) \in K_4).$$

Hence we obtain a family $\{h_{2k}(z,w)\}_{k\in\mathbb{N}}$ of holomorphic functions that satisfy $h_{2k}(z,w) \in \mathcal{O}((\mathbb{P}^1 \setminus (\{0\} \cup \Gamma_{2k+2})) \times W)$ and

$$|(g_{2k}(z,w) + h_{2k-2}(z,w)) - h_{2k}(z,w)| < \frac{1}{2^{2k}}, \qquad ((z,w) \in K_{2k})$$

for $k \in \mathbb{N}$ where we set $h_0 = 0$. Define

$$G_1(z,w) := \sum_{k=1}^{\infty}((g_{2k}(z,w) + h_{2k-2}(z,w)) - h_{2k}(z,w)).$$

Then we have $G_1(z,w) \in \mathcal{O}_X^{\exp}((\check{\mathbb{D}}^2 \setminus F_1) \times W)$ and $G - G_1$ is holomorphic near $F_1 \times W$. Hence we have $G = G_1 + (G - G_1)$ with $G_1 \in \mathcal{O}_X^{\exp}((\check{\mathbb{D}}^2 \setminus F_1) \times W)$ and $G - G_1 \in \mathcal{O}_X^{\exp}((\check{\mathbb{D}}^2 \setminus F_2) \times W)$, from which we have obtained that the morphism $\tilde{\iota}$ is surjective. This completes the proof. □

As an immediate consequence of the proposition, we have the following theorem.

THEOREM 5.2. *Let U be an open subset in $\mathbb{D}^2$, and W a pseudoconvex open subset in $\mathbb{C}^m$. Then we have*

$$H^k(U \times W, \mathcal{O}_X^{\exp}) = 0, \qquad (k \neq 0).$$

PROOF. Set $Y := \mathbb{C}^1 \times W \subset X$. Let us consider the long exact sequence of cohomology groups

$$(47) \quad \to H^k((U \times W) \cup Y, \mathcal{O}_X^{\mathrm{exp}}) \to H^k(U \times W, \mathcal{O}_X^{\mathrm{exp}}) \oplus H^k(Y, \mathcal{O}_X^{\mathrm{exp}}) \\ \to H^k((U \times W) \cap Y, \mathcal{O}_X^{\mathrm{exp}}) \to .$$

It follows from Proposition 5.1 that we have $H^k((U \times W) \cup Y, \mathcal{O}_X^{\mathrm{exp}}) = 0$ $(k \neq 0)$. By noticing $\mathcal{O}_X^{\mathrm{exp}}|_X = \mathcal{O}_X$, we obtain

$$H^k((U \times W) \cap Y, \mathcal{O}_X^{\mathrm{exp}}) = H^k((U \times W) \cap Y, \mathcal{O}_X) = 0, \qquad (k \neq 0).$$

Hence $H^k(U \times W, \mathcal{O}_X^{\mathrm{exp}}) = 0$ $(k \neq 0)$ follows from the above long exact sequence. □

Let $N = \mathbb{R} \times \mathbb{C}^m$ $(m \geq 0)$ and let $\overline{N} = \overline{\mathbb{R}} \times \mathbb{C}^m$ be the closure of N in $\hat{X} = \mathbb{D}^2 \times \mathbb{C}^m$. Now we establish the theorems for the flabbiness and the unique continuation property of $\mathcal{BO}_N^{\mathrm{exp}}$.

THEOREM 5.3. *Let Ω_1 and Ω_2 be open subsets in $\overline{\mathbb{R}}$ with $\Omega_1 \subset \Omega_2$, and W a pseudoconvex open subset in $\mathbb{C}^m$. Then the restriction $\mathcal{BO}_N^{\mathrm{exp}}(\Omega_2 \times W) \to \mathcal{BO}_N^{\mathrm{exp}}(\Omega_1 \times W)$ is surjective.*

PROOF. Set $\hat{Y} := \mathbb{D}^2 \times W \subset \hat{X}$. We may assume $\Omega_2 = \overline{\mathbb{R}}$, and hence, it suffices to show, for a closed subset $F \subset \overline{\mathbb{R}}$,

$$(48) \qquad H^1_{F \times W}(\overline{\mathbb{R}} \times W, \mathcal{BO}_N^{\mathrm{exp}}) = H^2_{F \times W}(\hat{Y}, \mathcal{O}_X^{\mathrm{exp}}) = 0.$$

Since we have the long exact sequence

$$\to H^1(\hat{Y} \setminus (F \times W), \mathcal{O}_X^{\mathrm{exp}}) \to H^2_{F \times W}(\hat{Y}, \mathcal{O}_X^{\mathrm{exp}}) \to H^2(\hat{Y}, \mathcal{O}_X^{\mathrm{exp}}) \to,$$

(48) follows from Theorem 5.2. □

COROLLARY 5.4 ([6]). *The sheaf $\mathcal{B}_{\mathbb{R}}^{\mathrm{exp}}$ of Laplace hyperfunctions is flabby.*

The following theorem shows that the sheaf $\mathcal{BO}_N^{\mathrm{exp}}$ has a unique continuation property with respect to holomorphic parameters.

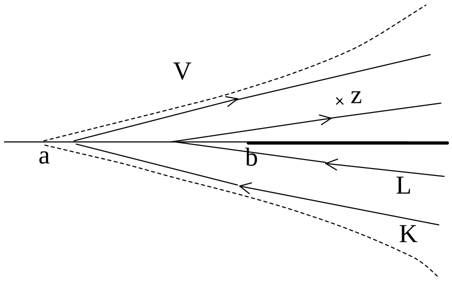

Fig. 1.

THEOREM 5.5. *Let W_1 and W_2 be non-empty connected open subsets in $\mathbb{C}^m$ with $W_1 \subset W_2$ and Ω an open subset in $\overline{\mathbb{R}}$. Then the restriction $\mathcal{BO}_N^{\exp}(\Omega \times W_2) \longrightarrow \mathcal{BO}_N^{\exp}(\Omega \times W_1)$ is injective.*

PROOF. We may assume $\Omega = (a, \infty]$ in $\overline{\mathbb{R}}$, and W_1 and W_2 are pseudoconvex. Let $f \in \mathcal{BO}_N^{\exp}(\Omega \times W_2)$ represented by $F(z, w) \in \mathcal{O}_X^{\exp}((V \setminus \Omega) \times W_2)$ for a complex neighborhood V of Ω. Suppose that f satisfies $f|_{\Omega \times W_1} = 0$. Then there exists a $G(z, w) \in \mathcal{O}_X^{\exp}(V \times W_1)$ with $F = G$ on $V \times W_1$. It follows from the unique continuation property of the ordinary hyperfunction with holomorphic parameters that the support of f is contained in $[b, \infty] \times W_2$ with $a < b \le \infty$. Now we take an arbitrary point $(z, w) \in ((V \setminus [b, \infty]) \cap \mathbb{C}) \times W_2$ and closed sectors K and L in V as Figure 1. Let Z be a relatively compact open subset in W_2 satisfying $W_1 \cap Z \neq \emptyset$, and we assume $w \in Z$. By Cauchy's integral formula we have

(49)
$$F(z, w) = \frac{e^{Az}}{2\pi\sqrt{-1}} \int_{\partial L} \frac{F(\lambda, w)e^{-A\lambda}}{\lambda - z} d\lambda - \frac{e^{Az}}{2\pi\sqrt{-1}} \int_{\partial K} \frac{F(\lambda, w)e^{-A\lambda}}{\lambda - z} d\lambda$$

for a sufficiently large A. Let $F_j(z, w)$ $(j = 0, 1)$ be the functions given by the integrals on the right hand side of (49) corresponding to ∂L and ∂K in that order. Note that, if the point w belongs to W_1, $F(\lambda, w) = G(\lambda, w)$ holds. Then, as $e^{-A\lambda}$ compensates the exponential growth of $G(\lambda, w)$ at infinity, by deforming the contour ∂L, we have

$$(50) \quad F_0(z, w) = \frac{e^{Az}}{2\pi\sqrt{-1}} \int_{b+i0}^{\infty+i0} \frac{G(\lambda, w)e^{-A\lambda}}{\lambda - z} d\lambda - \frac{e^{Az}}{2\pi\sqrt{-1}} \int_{b-i0}^{\infty-i0} \frac{G(\lambda, w)e^{-A\lambda}}{\lambda - z} d\lambda = 0,$$

for w belonging to a relatively compact open subset in $W_1 \cap Z$. This gives $F_0(z, w) \equiv 0$ for $w \in W_2$ by the unique continuation property of a holomorphic function, and we obtain $F(z, w) = -F_1(z, w)$ which is a holomorphic function of exponential type on a neighborhood of $\Omega \times W_2$. Hence we have $[F] \equiv 0$ on $\Omega \times W_2$. □

References

[1] Bromwich, T. J. I'A., Normal coordinates in dynamical systems, Proc. London Math. Soc. **15** (1916), 401–448.

[2] Honda, N. and T. Murata, The cohomology vanishing theorem for the Cauchy-Riemann complex of tempered ultra-distributions, and Gevrey growth, in preprint.

[3] Hörmander, L., *An Introduction to Complex Analysis in Several Variables*, 3rd edition, North-Holland, Amsterdam (1990), 89.

[4] Kashiwara, M. and P. Schapira, Microlocal study of sheaves, Astérisque **128** (1985).

[5] Kawai, T., On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients, J. Fac. Sci. Univ. Tokyo **17** (1970), 467–517.

[6] Komatsu, H., Laplace transforms of hyperfunctions: a new foundation of the Heaviside calculus, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), 805–820.

[7] Komatsu, H., Laplace transforms of hyperfunctions: another foundation of the Heaviside operational calculus, Generalized Functions, Convergence Structures, and Their Applications (Proc. Internat. Conf., Dubrovnik, 1987; B. Stanković, editor), Plenum Press, New York (1988), 57–70.

[8] Komatsu, H., Operational calculus, hyperfunctions and ultradistributions, Algebraic Analysis (M. Sato Sixtieth Birthday Vols.), Vol. I, Academic Press, New York (1988), 357–372.

[9] Komatsu, H., Operational calculus and semi-groups of operators, Functional Analysis and Related Topics (Proc. Internat. Conf. in Memory of K. Yoshida, Kyoto, 1991), Lecture Notes in Math., vol. 1540, Springer-Verlag, Berlin (1993), 213–234.

[10] Komatsu, H., Multipliers for Laplace hyperfunctions - A Justification of Heaviside's rules, Proceedings of the Steklov Institute of Mathematics **203** (1994), 323–333.
[11] Komatsu, H., Solution of differential equations by means of Laplace hyperfunctions, in Structure of Solutions of Differential Equations, World Scientific, Singapore, (1996), 227–252.
[12] Komatsu, H., An introduction to ultra-distributions, Iwanami *Kiso Sûgaku* (1978) in Japanese.
[13] Mikusiński, J., Operational Calculus, Pergamon Press, 1959, 2nd Ed., (1983).
[14] Morimoto, M. and K. Yoshino, Some examples of analytic functionals with carrier at the infinity, Proc. Japan Acad. **56** (1980), 357–361.
[15] G. Pólya, Untersuchungen über Lücken und Singularitäten von Pntenzreihen, Math. Z. **29** (1929), 549–640.
[16] Saburi, Y., Vanishing theorem of cohomology groups with values in the sheaves $\mathcal{O}_{\mathrm{inc},\,\phi}$ and $\mathcal{O}_{\mathrm{dec}}$, Tokyo J. Math. **5** (1982), 225–248.

(Received December 22, 2011)
(Revised October 1, 2012)

Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: honda@math.sci.hokudai.ac.jp
k-umeta@math.sci.hokudai.ac.jp