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*Another Direct Proof of Oka's Theorem (Oka IX)**

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Abstract. In 1953 K. Oka IX solved in first and in a final form Levi's problem (Hartogs' inverse problem) for domains or Riemann domains over $\mathbf{C}^n$ of arbitrary dimension. Later on a number of the proofs were given; cf. e.g., Docquier-Grauert's paper in 1960, R. Narasimhan's paper in 1961/62, Gunning-Rossi's book, and Hörmander's book (in which the holomorphic separability is pre-assumed in the definition of Riemann domains and thus the assumption is stronger than the one in the present paper). Here we will give another direct elementary proof of Oka's Theorem, relying only on Grauert's finiteness theorem by the *induction on the dimension* and the *jets over Riemann domains*; here we do *not* use even Behnke-Stein's theorem on the Steiness of an open Riemann surface. Hopefully, the proof is the easiest.

1. Introduction

In 1953 K. Oka [10] IX solved in first and in a final form Levi's problem (Hartogs' inverse problem) for domains or Riemann domains over $\mathbf{C}^n$ of arbitrary dimension (cf. below for notation):

Theorem 1.1 (Oka [10] IX, ('43)/'53[1]). *Let $\pi : X \to \mathbf{C}^n$ be a Riemann domain, and let $\delta_{\mathrm{P}\Delta}(x, \partial X)$ denote the boundary distance function with respect to a polydisc $\mathrm{P}\Delta$. If $-\log \delta_{\mathrm{P}\Delta}(x, \partial X)$ is plurisubharmonic, then X is Stein.*

Besides Oka's original proof there are known a number of the proofs in generalized forms; e.g., Docquier-Grauert [2], Narasimhan [8], Gunning-Rossi [6], and Hörmander [7] (in which the holomorphic separability is pre-

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[1]It is now possible to confirm that Oka IX published in 1953 was written in French from his report in Japanese dated 1943 and addressed to Teiji Takagi. Cf. the introductions of Oka IX and VIII, and also Oka VI published in 1942; see http://www.lib.nara-wu.ac.jp/oka/index_eng.html.

assumed in the definition of Riemann domains and thus the assumption is stronger than the one in the present paper).

Here we will give another direct elementary proof of Oka's Theorem 1.1 by making use of the followings in an essential way, and it is new in this sense (see the proof of Lemma 3.2).

(i) The induction on the dimension $n = \dim X$.

(ii) The jets over X.

(iii) Grauert's Finiteness Theorem 2.10 over a strongly pseudoconvex domain Ω of a complex manifold applied not only for the structure sheaf $\mathcal{O}_\Omega$, but also for a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_\Omega$ (cf. Narasimhan [8], Docquier-Grauert [2], Gunning-Rossi [6]).

The others are the vanishing of higher cohomologies of coherent sheaves on polydiscs and on Stein manifolds, and a sort of ϵ-δ arguments, to say, a content presented in Chap. 2 of Hörmander [7] (see, e.g. the proof of Lemma 3.7). Thus, the proof is elementary, self-contained and hopefully simplest.

To be precise we give the exact definitions of notions we will use.

DEFINITION 1.2 (Stein manifold). A connected complex manifold M with the second countability axiom is called a *Stein manifold* if it satisfies the following three conditions. Here, $\mathcal{O}(M)$ denotes the set of all holomorphic functions on M.

(i) (Holomorphic separability) For distinct two points $x, y \in M$ there exists an element$f \in \mathcal{O}(M)$ such that $f(x) \neq f(y)$.

(ii) (Holomorphic local coordinates) For an arbitrary point $x \in M$ there are n $(= \dim M)$ elements $f_j \in \mathcal{O}(M)$A $1 \leq j \leq n$ such that $(f_j)_{1 \leq j \leq n}$ gives rise to a holomorphic local coordinate system in a neighborhood of x.

(iii) (Holomorphic convexity) For a compact subset $K \Subset M$ its holomorphic convex hull

$$\hat{K}_M = \{x \in M; |f(x)| \leq \max_K |f|, \ {}^\forall f \in \mathcal{O}(M)\}$$

is also compact in M.

N.B. In a number of references the definition of Stein manifolds consists of the above (iii) and the following *K-completeness* due to Grauert [3]:

(K) "For every point $x \in M$ there exist finitely many $f_j \in \mathcal{O}(M)$, $1 \leq j \leq l$ such that all $f_j(x) = 0$ and x is isolated in the analytic subset $\{f_j = 0; 1 \leq j \leq l\}$."

In fact, they are equivalent: it is trivial that the present definition 1.2 implies the above (K), but the converse is *not* trivial at all (cf. Grauert [55], and Andreotti-Narasimhan [1] Introduction).

Let X be a complex manifold and let $\pi : X \to \mathbf{C}^n$ be a holomorphic map.

DEFINITION 1.3 (Riemann domain). $\pi : X \to \mathbf{C}^n$ or simply X is called a *Riemann domain* if the following properties are satisfied:

(i) X is connected.

(ii) For every point $x \in X$ there are neighborhoods $U \ni x$ in X and $V \ni \pi(x)$ in $\mathbf{C}^n$ such that the restriction $\pi|_U : U \to V$ is biholomorphic.

N.B. (i) A Riemann domain X is metrizable and hence X satisfies the second countability axiom.

(ii) In the above definition we do *not* assume the holomorphic separability for a Riemann domain.

A Riemann domain $\hat{\pi} : \hat{X} \to \mathbf{C}^n$ is called a *holomorphic extension* of a Riemann domain $\pi : X \to \mathbf{C}^n$ if there is a holomorphic injection $\iota : X \to \hat{X}$ satisfying

(i) $\pi = \hat{\pi} \circ \iota$;

(ii) every holomorphic function $f \in \mathcal{O}(X)$ is analytically continued to an element $\hat{f} \in \mathcal{O}(\hat{X})$.

A Riemann domain X is called a *domain of holomorphy* if there exists no holomorphic extension of X other than X itself.

In this paper X denotes always a Riemann domain. We take a polydisc $\mathrm{P}\Delta = \mathrm{P}\Delta(0; r_0)$ $(r_0 = (r_{0j}))$ with center at the origin $0 \in \mathbf{C}^n$. Then by definition there are $\rho > 0$ and a neighborhood $U_\rho(x) \ni x$ for every $x \in X$ such that

$$\pi|_{U_\rho(x)} : U_\rho(x) \to \pi(x) + \rho \mathrm{P}\Delta$$

is biholomorphic. The supremum of such $\rho > 0$

$$\delta_{\mathrm{P}\Delta}(x, \partial X) = \sup\{\rho > 0; {}^\exists U_\rho(x)\} \leq \infty$$

is called the *boundary distance function* of X to the relative boundary.

If $\delta_{\mathrm{P}\Delta}(x, \partial X) = \infty$, then π is a holomorphic isomorphism, and thus there is nothing to discuss more. Henceforth we assume $\delta_{\mathrm{P}\Delta}(x, \partial X) < \infty$ in what follows.

For a subdomain $\Omega \subset X$ we define similarly

$$\delta_{\mathrm{P}\Delta}(x, \partial\Omega) = \sup\{\rho > 0; {}^\exists U_\rho(x) \subset \Omega\}.$$

The boundary distance functions $\delta_{\mathrm{P}\Delta}(x, \partial X)$ and $\delta_{\mathrm{P}\Delta}(x, \partial\Omega)$ are continuous with Lipschitz' condition. For a subset set $A \subset X$ (resp. $A \subset \Omega$) we set

$$\delta_{\mathrm{P}\Delta}(A, \partial X) = \inf_{x \in A} \delta_{\mathrm{P}\Delta}(x, \partial X)$$
$$(\text{resp. } \delta_{\mathrm{P}\Delta}(A, \partial\Omega) = \inf_{x \in A} \delta_{\mathrm{P}\Delta}(x, \partial\Omega)).$$

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2. Preliminaries

Here we list up the lemmas and theorems we will use.

Lemma 2.1. *Let $\pi : X \to \mathbf{C}^n$ be a domain of holomorphy, let $K \Subset X$ be a compact subset, and let $f \in \mathcal{O}(X)$. If*

$$\delta_{\mathrm{P}\Delta}(x, \partial X) \geq |f(x)|, \quad x \in K,$$

then

$$\delta_{\mathrm{P}\Delta}(x, \partial X) \geq |f(x)|, \quad x \in \hat{K}_X.$$

In particular, taking f to be constant we have

$$\delta_{\mathrm{P}\Delta}(K, \partial X) = \delta_{\mathrm{P}\Delta}(\hat{K}_X, \partial X). \tag{2.2}$$

The proof is the same as in the case of univalent domains. This lemma implies the following as well:

THEOREM 2.3. *If X is a domain of holomorphy, then $-\log \delta_{\mathrm{P}\Delta}(x, \partial X)$ is plurisubharmonic.*

DEFINITION 2.4. In general, a complex manifold M is said to be *pseudoconvex* if M carries a continuous plurisubharmonic exhaustion function.

The following is not trivial, but elementary due to Oka [10] IX (cf. Nishino [9], p. 350):

LEMMA 2.5. *If $-\log \delta_{\mathrm{P}\Delta}(x, \partial X)$ is plurisubharmonic (for one fixed* $\mathrm{P}\Delta$*), then X is pseudoconvex.*

THEOREM 2.6 (Oka's Fundamental Theorem, I∼II, VII). *Let* $\mathrm{P}\Delta(0; r)$ *be an arbitrary polydisc, and let $\mathcal{I} \subset \mathcal{O}_\Omega^N$ be a coherent sheaf of submodules. Then*

$$H^q(\mathrm{P}\Delta(0; r), \mathcal{I}) = 0, \qquad q \geq 1.$$

This theorem over polydiscs together with Oka's Jôkûiko[2] leads to the following:

THEOREM 2.7 (Oka-Cartan). *Let M be a Stein manifold, and let $\mathcal{S} \to M$ be a coherent sheaf. Then*

$$H^q(M, \mathcal{S}) = 0, \qquad q \geq 1.$$

[2]A direct English translation may be "transformation to the upper space". It is a method to imbed the domain under consideration into a higher dimensional polydisc $\mathrm{P}\Delta$, to extend the analytic objects over $\mathrm{P}\Delta$, and to solve the problem over $\mathrm{P}\Delta$ by the simplicity of the space $\mathrm{P}\Delta$. This method was developed by K. Oka [10] I∼III and was a very key to solve Cousin Problems I and II.

Lemma 2.8.

(i) *Let* $\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset X$ *be a series of subdomains. Assume that* Ω_3 *is Stein. If*

$$\delta_{\mathrm{P}\Delta}(\partial\Omega_1, \partial\Omega_3) > \max_{x \in \partial\Omega_2} \delta_{\mathrm{P}\Delta}(x, \partial\Omega_3),$$

then there is an $\mathcal{O}(\Omega_3)$*-analytic polyhedron* P *such that*

$$\Omega_1 \Subset P \Subset \Omega_2.$$

(ii) *An arbitrary holomorphic function* $f \in \mathcal{O}(P)$ *can be approximated uniformly on compact subsets by elements of* $\mathcal{O}(\Omega_3)$*; that is,* (P, Ω_3) *is a Runge pair.*

Proof. (i) The assumption and (2.2) imply that $\widehat{(\bar{\Omega}_1)}_{\Omega_3} \Subset \Omega_2$, and hence such P exists.

(ii) By Theorem 2.6 we can apply Oka's Jôkûiko to reduce the domain to a polydisc, and the statement is proved. □

Let $\Omega \Subset M$ be a relatively compact domain.

Definition 2.9. Ω is said to be *strongly pseudoconvex* if there are a neighborhood $U(\subset M)$ of the boundary $\partial\Omega$ of Ω, and a real valued C^2 function $\phi : U \to \mathbf{R}$ satisfying the conditions

(i) $\{x \in U : \phi(x) < 0\} = \Omega \cap U$,
(ii) $i\partial\bar{\partial}\phi(x) > 0$ $(x \in U)$.

Theorem 2.10 (Grauert [4], [5]). *Let* $\Omega \Subset M$ *be a strongly pseudoconvex domain. Let* $\mathcal{F}$ *be a coherent sheaf defined over a neighborhood of the closure* $\bar{\Omega}$*. Then we have*

$$\dim H^q(\Omega, \mathcal{F}) < \infty, \quad q \geq 1.$$

We will use this theorem for the structure sheaf and an ideal sheaf of a closed complex submanifold. In the first, we apply this for $\mathcal{F} = \mathcal{O}_M$ to deduce

THEOREM 2.11. *Let Ω be as in Theorem 2.10. Then Ω is holomorphically convex.*

N.B. The above described was the circumstance just after Grauert [4] ('58), or Docquier-Grauert [2] ('60) and Narasimhan [8] ('61/'62).

3. A Proof of Oka's Theorem 1.1

By Lemma 2.5 it suffices to show the following for the proof.

THEOREM 3.1. *A pseudoconvex Riemann domain is Stein.*

The following lemma is our key in the proof.

LEMMA 3.2. *If $\Omega \Subset X$ is a strongly pseudoconvex domain, then Ω is Stein.*

PROOF. We use the induction on the dimension $n \geq 1$.

(a) $n = 1$: In this case Ω is an open Riemann surface and hence by Behnke-Stein's Theorem it is Stein. For the completeness we show this with the preparation in §2. The holomorphic convexity is finished by Theorem 2.11. The holomorphic local coordinates follow just from the definition of Riemann domain. It is remaining to show the holomorphic separability.

Take two distinct points $a, b \in \Omega$. If $\pi(a) \neq \pi(b)$, the proof is done. Suppose that $\pi(a) = \pi(b)$. By a translation of $\mathbf{C}$ we may assume that $\pi(a) = \pi(b) = 0 \in \mathbf{C}$. Let $U_0 \ni a$ be a neighborhood such that $U_0 \not\ni b$ and $\pi|_{U_0} : U_0 \to \Delta(0; \delta)$ with $\delta > 0$ is biholomorphic. Put $U_1 = \Omega \setminus \{a\}$. Then $\mathcal{U} = \{U_0, U_1\}$ is an open covering of Ω. For each $k \in \mathbf{N}$ we set

$$\gamma_k(x) = \frac{1}{\pi(x)^k}, \quad x \in U_0 \cap U_1.$$

Then γ_k defines an element of $H^1(\mathcal{U}, \mathcal{O}_\Omega)$. It is noted that $H^1(\mathcal{U}, \mathcal{O}_\Omega) \hookrightarrow H^1(\Omega, \mathcal{O}_\Omega)$ is injective. By Theorem 2.10 there is a non-trivial linear relation

$$\sum_{k=1}^{h} c_k \gamma_k = 0, \quad c_k \in \mathbf{C},\ c_h \neq 0.$$

Therefore there are elements $f_j \in \mathcal{O}(U_j), j = 0, 1$ such that

$$f_1(x) - f_0(x) = \sum_{k=1}^{h} c_k \frac{1}{\pi(x)^k}, \quad x \in U_0 \cap U_1.$$

Thus we obtain a meromorphic function in Ω with a pole only at a,

$$F = f_1 = f_0 + \sum_{k=1}^{h} c_k \frac{1}{\pi^k}.$$

From the construction we get

$$\begin{aligned} \pi(x)^h F(x) &\in \mathcal{O}(\Omega), \\ \pi(a)^h F(a) &= c_h \neq 0, \\ \pi(b)^h F(b) &= 0. \end{aligned}$$

Therefore a and b are separated by an element of $\mathcal{O}(\Omega)$.

(b) We assume the assertion holds in $\dim X = n - 1$. Let $\dim X = n \geq 2$. By the definition of Riemann domain it is sufficient to prove the holomorphic convexity and the holomorphic separability; the first is finished by Theorem 2.11, and the latter remains to be shown.

(1) We take arbitrary distinct points $a, b \in \Omega$. As in (a) we may assume that $\pi(a) = \pi(b) = 0$. Taking a hyperplane $L = \{z_n = 0\}$, we consider the restriction

$$\pi_{X'} : X' = \pi^{-1} L \to L.$$

Since $L \cong \mathbf{C}^{n-1}$ (biholomorphic), every connected component X'' of X' is $(n-1)$ dimensional Riemann domain. Put

$$\Omega' = X' \cap \Omega.$$

Then every connected component Ω'' of Ω' is a strongly pseudoconvex domain of a connected component of X'. By the induction hypothesis Ω'' is Stein.

(2) Let $\mathfrak{m}\langle a \rangle \subset \mathcal{O}_{X',a}$ be the maximal ideal of the local ring $\mathcal{O}_{X',a}$ and let $\mathfrak{m}^k\langle a \rangle$ denote its k-th power. Set

$$\mathfrak{m}^k\langle a, b \rangle = \mathfrak{m}^k\langle a \rangle \otimes \mathfrak{m}^k\langle b \rangle \subset \mathcal{O}_{X'}.$$

This is a coherent ideal sheaf of $\mathscr{O}_{X'}$.

Since every connected component of Ω' is Stein, Theorem 2.7 implies the existence of $g_k \in \mathscr{O}(\Omega')$ for each $k \in \mathbf{N}$ such that

$$
\begin{aligned}
&\underline{g_k}_a \equiv 0 \quad (\mathrm{mod}\ \mathfrak{m}^{k-1}\langle a,b\rangle_a),\\
&\underline{g_k}_a \not\equiv 0 \quad (\mathrm{mod}\ \mathfrak{m}^{k}\langle a,b\rangle_a),\\
&\underline{g_k}_b \equiv 0 \quad (\mathrm{mod}\ \mathfrak{m}^{k}\langle a,b\rangle_b),
\end{aligned}
\tag{3.3}
$$

where $\underline{g_k}_a$ stands for a germ of g_k at a.

(3) Let $\mathscr{I}$ be the ideal sheaf of the analytic subset $X' \subset X$. By Oka's Second Coherence Theorem ([10] VII, VIII) $\mathscr{I}$ is coherent.[3] Restricting this to Ω, we have a short exact sequence:

$$0 \to \mathscr{I} \to \mathscr{O}_\Omega \to \mathscr{O}_{\Omega'} \to 0.$$

This implies the following exact sequence,

$$\mathscr{O}(\Omega) \to \mathscr{O}(\Omega') \xrightarrow{\delta} H^1(\Omega, \mathscr{I}). \tag{3.4}$$

We write g_k for the restriction of g_k to Ω' by the same letter. We have that $\{\delta(g_k)\}_{k\in\mathbf{N}} \subset H^1(\Omega,\mathscr{I})$. By Theorem 2.10 $H^1(\Omega,\mathscr{I})$ is finite dimensional,

[3]There seems to be a confusion in the historical comprehension of the development of the "coherence theorems". In Oka VII and VIII K. Oka proved *three fundamental coherence theorems*. Firstly in Oka VII which was received in 1948 and published in 1950, he proved the coherence of the structure sheaf $\mathscr{O}_{\mathbf{C}^n}$ on $\mathbf{C}^n$ (Oka's First Coherence Theorem), and he was writing in *two places* that in the forthcoming paper he would deal with the coherence of ideal sheaves of analytic subsets, "*idéaux géométriques de domaines indéterminés*" he termed, and that one would see it to hold without any assumption; see 1) the last six lines of the paper at p. 27, and 2) the last two lines of p. 7 to the line just before §3 of p. 8. There he wrote that there are two cases for which the coherence problem are solvable, the first is that of $\mathscr{O}_{\mathbf{C}^n}$ dealt with in VII, and the second is that of the ideal sheaf of an analytic subsets (Oka's Second Coherence Theorem), of which proof appeared in Oka VIII in 1951, while H. Cartan's proof appeared in 1950 in the same volume as Oka VII, to which the theorem is attributed in most references.

For this many refer only to the first point 1), but never to the second point 2) so far by the knowledge of the present author, where K. Oka was writing more detailed descriptions what should be done for the Second Coherence Theorem. In VIII he wrote its proof and moreover proved the coherence of normalizations (Oka's Third Coherence Theorem). For a convenience we give a complete list of of K. Oka's paper at the end of the references, which is not very long but hard to find a complete correct one.

and thus there is a non-trivial linear relation

$$\sum_{k=k_0}^{N} c_k \delta(g_k) = 0, \quad c_k \in \mathbf{C},\ N < \infty.$$

We may assume that $c_{k_0} \neq 0$. It follows from (3.4) that there is an element $f \in \mathcal{O}(\Omega)$ such that

$$f|_{\Omega'} = \sum_{k=k_0}^{N} c_k g_k.$$

We use $\pi = (z_1, \ldots, z_n)$ as a holomorphic local coordinate system in a sufficiently small neighborhood of $a \in \Omega$, $z' = (z_1, \ldots, z_{n-1})$. Then we get

$$f(z) = \sum_{k=k_0}^{N} c_k g_k(z') + h(z) \cdot z_n, \tag{3.5}$$

where $h(z)$ is a holomorphic function in a neighborhood of a. It follows from (3.3) that there is a partial differentiation of order k_0 in z'

$$\mathrm{D} = \frac{\partial^{k_0}}{\partial z_1^{\alpha_1} \cdots \partial z_{n-1}^{\alpha_{n-1}}}, \quad \sum_{j=1}^{n-1} \alpha_j = k_0$$

such that

$$\begin{aligned} \mathrm{D}g_{k_0}(a) &\neq 0, \\ \mathrm{D}g_k(a) &= 0, \quad k > k_0, \\ \mathrm{D}g_k(b) &= 0, \quad k \geq k_0. \end{aligned} \tag{3.6}$$

The definition of D and (3.5) imply that

$$\mathrm{D}f(z) = \sum_{k=k_0}^{N} c_k \mathrm{D}g_k(z') + (\mathrm{D}h(z)) \cdot z_n.$$

Since $z_n = 0$ at a and b, (3.6) leads to

$$\mathrm{D}f(a) \neq 0, \quad \mathrm{D}f(b) = 0.$$

Since $\mathrm{D}f \in \mathcal{O}(\Omega)$, the holomorphic separability of Ω was proved. □

Under the assumption we take a plurisubharmonic exhaustion function $\phi : X \to \mathbf{R}$. We set

$$X_c = \{x \in X; \phi(x) < c\}, \quad c \in \mathbf{R}.$$

For X being Stein it suffices to prove the followings:

LEMMA 3.7.

(i) *X_c is Stein for an arbitrary $c \in \mathbf{R}$:*

(ii) *For every pair of $c < b$, (X_c, X_b) is a Runge pair.*

PROOF. (i) Let $K \Subset X_c$ be a compact subset. We put

$$\eta = \delta_{\mathrm{P}\Delta}(K, \partial X_c)\,(> 0).$$

We take $b > c$ so that

$$\max_{x \in \partial X_c} \delta_{\mathrm{P}\Delta}(x, \partial X_b) < \eta. \tag{3.8}$$

Since $\|\pi(x)\|^2$ is strongly plurisubharmonic everywhere and ϕ is plurisubharmonic, there exists a strongly pseudoconvex domain Ω such that

$$X_c \Subset \Omega \Subset X_b.$$

By Lemma 3.2 Ω is Stein. Therefore conditions (i) and (ii) of Definition 1.2 are satisfied, and there remains (iii) (holomorphic convexity) to be shown.

CLAIM 3.9. $\hat{K}_{X_c} \Subset X_c$.

PROOF. The application of (2.2) to $K \Subset \Omega$ yields

$$\delta_{\mathrm{P}\Delta}(\hat{K}_\Omega, \partial\Omega) = \delta_{\mathrm{P}\Delta}(K, \partial\Omega) > \eta.$$

On the other hand, from (3.8) it follows that

$$\max_{x \in \partial X_c} \delta_{\mathrm{P}\Delta}(x, \partial\Omega) < \eta.$$

The above two equations imply

$$\hat{K}_{X_c} \subset \hat{K}_\Omega \Subset X_c\,. \tag{3.10}$$

(ii) We use the same notation as in (i).

(1) We now know that all X_c ($c \in \mathbf{R}$) are Stein. Therefore, replacing Ω by X_b in the above arguments in (i), we see that

$$\hat{K}_{X_c} \subset \hat{K}_{X_b} \Subset X_c \Subset X_b\,. \tag{3.11}$$

Claim 3.12. $\hat{K}_{X_c} = \hat{K}_{X_b}$.

Proof. By (3.11) we can take an $\mathcal{O}(X_b)$-analytic polyhedron P such that

$$\hat{K}_{X_c} \subset \hat{K}_{X_b} \Subset P \Subset X_c \Subset X_b\,.$$

If there is a point $\zeta \in \hat{K}_{X_b} \setminus \hat{K}_{X_c}$, then there is some $g \in \mathcal{O}(X_c)$ such that

$$\max_K |g| < |g(\zeta)|.$$

By Lemma 2.8 (ii) g can be approximated uniformly on $\hat{K}_{X_b}$ by an element of $\mathcal{O}(X_b)$. Hence there is a holomorphic function $f \in \mathcal{O}(X_b)$ such that

$$\max_K |f| < |f(\zeta)|.$$

This is absurd.

(2) It follows from Claim 3.12 that

$$\hat{K}_{X_c} = \hat{K}_{X_t}, \quad c \leq {}^{\forall} t \leq b. \tag{3.13}$$

We set

$$E = \{t \geq c\,; \hat{K}_{X_t} = \hat{K}_{X_c}\} \subset [c, \infty).$$

By definition $t \in E$ implies $[c, t] \subset E$. The result of (1) shows that E is an open subset of $[c, \infty)$.

(3) We put $a = \sup E$.

Claim 3.14. $a = \infty$; i.e., $E = [c, \infty)$.

Proof. Suppose that $a < \infty$. From the definition we obtain

$$K_1 = \hat{K}_{X_c} = \hat{K}_{X_t}, \quad c \leq {}^{\forall} t < a.$$

Letting $t < a$ sufficiently close to a, we have

$$\delta_{\mathrm{P}\Delta}(K_1, \partial X_a) > \max_{x \in \partial X_t} \delta_{\mathrm{P}\Delta}(x, \partial X_a).$$

Because X_a is Stein,

$$\delta_{\mathrm{P}\Delta}(\hat{K}_{1X_a}, \partial X_a) = \delta_{\mathrm{P}\Delta}(K_1, \partial X_a) > \max_{x \in \partial X_t} \delta_{\mathrm{P}\Delta}(x, \partial X_a).$$

Thus, $\hat{K}_{1X_a} \Subset X_t$ follows. One gets

$$\hat{K}_{X_t} \subset \hat{K}_{X_a} \subset \hat{K}_{1X_a} \Subset X_t \Subset X_a.$$

In the same way as in (1) we see that $\hat{K}_{X_t} = \hat{K}_{X_a}$. Therefore, $a \in E$. Since E is open, there exists a number $a' \in E$ with $a' > a$. This contradicts to the choice of a.

(4) It follows from (2) that for arbitrary $c < b$ and a compact subset $K \Subset X_c$,

$$\hat{K}_{X_c} = \hat{K}_{X_b}.$$

Therefore, Oka's Jôkûiko and Theorem 2.6 imply that (X_c, X_b) is a Runge pair. □

References

[1] Andreotti, A. and R. Narasimhan, Oka's Heftungslemma and the Levi problem for complex spaces, Trans. Amer. Math. Soc. **111** (1964), 345–366.

[2] Docquier, F. and H. Grauert, Levisches Problem und Rundescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten, Math. Ann. **140** (1960), 94–123.

[3] Grauert, H., Charakterisierung der holomorph vollständigen komplexen Räume, Math. Ann. **129** (1955), 233–259.

[4] Grauert, H., On Levi's problem and the imbedding of real-analytic manifolds, Ann. Math. **68** (1958), 460–472.

[5] Grauert, H., Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann. **146** (1962), 331–368.

[6] Gunning, R. C. and H. Rossi, Analytic Functions of Several Complex Variables, Prentice-Hall Inc., Englewood Cliffs, N. J., 1965.

[7] Hörmander, L., Introduction to Complex Analysis in Several Variables, Third Edition, North-Holland, 1989.

[8] Narasimhan, R., The Levi problem for complex spaces, Math. Ann. **142** (1961), 355–365; ibid. II, Math. Ann. **146** (1962), 195–216.

[9] Nishino, T., Function Theory in Several Complex Variables, Transl. Math. Mono. Volume 193, Amer. Math. Soc., Providence, Rhode Island, 2001.

[10] Oka, K.[4], Sur les fonctions analytiques de plusieurs variables:

I–Domaines convexes par rapport aux fonctions rationnelles,
J. Sci. Hiroshima Univ. Ser. A 6 (1936), 245–255 *[Rec. 1 mai 1936]*.

II–Domaines d'holomorphie,
J. Sci. Hiroshima Univ. Ser. A 7 (1937), 115–130 *[Rec. 10 déc 1936]*.

III–Deuxieme problème de Cousin,
J. Sci. Hiroshima Univ. 9 (1939), 7–19 *[Rec. 20 jan 1938]*.

IV–Domaines d'holomorphie et domaines rationnellement convexes,
Jpn. J. Math. 17 (1941), 517–521 *[Rec. 27 mar 1940]*.

V–L'intégrale de Cauchy,
Jpn. J. Math. 17 (1941), 523–531 *[Rec. 27 mar 1940]*.

VI–Domaines pseudoconvexes.
Tôhoku Math. J. 49 (1942(+43)), 15–52 *[Rec. 25 oct 1941]*.

VII–Sur quelques notions arithmétiques,
Bull. Soc. Math. France 78 (1950), 1–27 *[Rec. 15 oct 1948]*.

VIII–Lemme fondamental,
J. Math. Soc. Japan 3 (1951) No. 1, 204–214; No. 2, 259–278
[Rec. 15 mar 1951].

IX–Domaines finis sans point critique intérieur,
Jpn. J. Math. 23 (1953), 97–155 *[Rec. 20 oct 1953]*.

X–Une mode nouvelle engendrant les domaines pseudoconvexes,
Jpn. J. Math. 32 (1962), 1–12 *[Rec. 20 sep 1962]*.

[34] Note sur les familles de fonctions multiformes etc.,
J. Sci. Hiroshima Univ. 4 (1934), p.93-98 *[Rec. 20 jan 1934]*.

[41] Sur les domaines pseudoconvexes,
Proc. of the Imperial Academy, Tokyo, (1941) 7–10
[Comm. 13 jan 1941].

[49] Note sur les fonctions analytiques de plusieurs variables,
Kōdai Math. Sem. Rep., (1949). no. 5–6, 15–18 *[Rec. 19 déc 1949]*.

[4]It is now rather difficult to find a complete, correct list of K. Oka's papers. The most referred volume of Kiyoshi Oka's works may be "Kiyoshi Oka, Collected Papers, Springer-Verlag, 1984", which unfortunately lacks the fundamental records of the **received dates** of all papers. And there are a bibliographically incorrect record and a lack of a volume number in this collected volume. Here are the correct complete data of his all published articles presented at one place for the sake of convenience.

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