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Large Deviation Principle for the Pinned Motion of Random Walks

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Abstract. The large deviation principle is proved for the long time asymptotic of empirical measures associated with the pinned motions of random walks on the square lattice. Random walks are not reversible Markov chains in general, and thus nice property such as the Gaussian bounds on the transition probabilities, which was one of the key tools for proving the large deviations for periodic and reversible Markov chains in [1], are no longer available. For this reason the spectral radius of transition probabilities of random walk comes into play. With the help of Salvatori's theorem, a sufficient condition is given so that the spectral radius is held to be equal to 1 by certain gauge transform of the transition probabilities, and then the large deviation will be proved under the condition.

1. Introduction

In this paper we will prove the large deviation principle for the long time asymptotic of empirical measures associated with the pinned motions of random walks on the square lattice.

In [1], the large deviation principle has been discussed for the pinned motions of periodic and reversible Markov chains on the square lattice. We have shown that the corresponding rate function is associated with a new Markov chain constructed from the original one through harmonic transform based on a principal eigenfunction for the generator of the chain. Thus the generalized principal eigenfunction has played the special role there. Another key point in [1] has been that the transition probabilities of a reversible Markov chain on $\mathbb{Z}^d$ with an uniform invariant measure have Gaussian upper and lower bounds and that such a property is preserved by a kind of gauge transform. This offers the nice pointwise asymptotic of the transition probabilities as time goes to infinity, and based on this fact essentially, the uniform large deviation principle for the original chains has been established.

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In this paper we will be concerned with i.i.d. random walks on the square lattice where the reversibility cannot generally be expected any more and thus nice properties such as Gaussian estimates are no longer available. So we will compensate the lack of the Gaussian estimate with the estimate on the spectral radius of the transition probabilities of the walk. However a difficulty arises in this approach since, due to the lack of reversibility, we do not have the invariance of the spectral radius under the gauge transform, namely, there are random walks with spectral radii 1 for which the transformed walks have spectral radii strictly less than 1. Thus in order to obtain uniform large deviations as in [1], we must first clarify under what conditions the invariance under the gauge transform is guaranteed. With the help of Salvatori's Theorem (See [6]), a sufficient condition for the invariance of spectral radius will be given. Once this is done, under the condition we are able to show the large deviation principle with some modification of the arguments in [1].

In concluding the introduction we point out that this type of large deviations have been also carried out for covering diffusions and random walks on homogeneous trees [3, 4] by one of the authors.

2. Notations and Statements of Main Results

Let $\{\xi_k\}$, $k = 1, 2, \cdots$ be i.i.d. random variables on $\mathbb{Z}^d$ and let $\{X_n\}$, $n = 0, 1, 2, \cdots$ be the random walk on $\mathbb{Z}^d$ given by $X_n = X_0 + \sum_{k=1}^n \xi_k$. We denote by $(p_z)_{z\in\mathbb{Z}^d}$ the probability law of $\{\xi_k\}$. In other words, the Markov chain $\{X_n\}$ has the one-step transition probability $p(x, y) = p_{y-x}$. We assume the following conditions on the transition probabilities $\{p(x, y)\}_{x,y\in\mathbb{Z}^d}$ of the random walk $\{X_n\}$:

(A.1) There is a positive constant r_0 such that

$$\left\{z \in \mathbb{Z}^d; p_z \neq 0\right\} \subseteq B^1(0, r_0),$$

where $B^1(x, r_0) = \left\{y; \|x - y\| < r_0\right\}$, $\quad \|x\| = \sum_{i=1}^d |x_i|$.

(A.2) $\{X_n\}$ is irreducible and aperiodic as a Markov chain on $\mathbb{Z}^d$.

Under these assumptions let

$$\phi(\lambda) = \log Z_\lambda = \log\left(\sum_{z\in\mathbb{Z}^d} e^{\lambda\cdot z} p_z\right), \ \lambda \in \mathbb{R}^d,$$

and

$$h(x) = \sup\{x \cdot \lambda - \phi(\lambda),\ \lambda \in \mathbb{R}^d\}, \quad x \in \mathbb{R}^d.$$

Then, $\nabla h(x) = \lambda$ if and only if

$$\nabla \phi(\lambda) = \frac{1}{Z_\lambda} \sum_{z \in \mathbb{Z}^d} z e^{\lambda \cdot z} p_z = x,$$

and thus the new probability measure $(p_z^0)_{z \in \mathbb{Z}^d}$ defined by

$$p_z^0 = \frac{1}{Z_{\lambda_0}} e^{\lambda_0 \cdot z} p_z \tag{1}$$

with the choice of $\lambda_0 = \nabla h(0)$ satisfies that

$$\sum_{z \in \mathbb{Z}^d} z p_z^0 = 0.$$

We assume another condition on the transition probabilities $\{p(x,y)\}_{x,y \in \mathbb{Z}^d}$:

(A.3) Either the following (i) or (ii) is satisfied:

(i) The random walk $\{X_n\}$ with the transition probabilities given by $p(x,y) = p_{y-x}$ is a reversible Markov chain.

(ii) There is a $m_0 \in \mathbf{N}$, $m_0 \geq 2$, such that

$$\sum_{k \in \mathbb{Z}^d} (x + m_0 k) p^0_{x+m_0 k} = 0 \tag{2}$$

for every $x \in (\mathbb{Z}/m_0\mathbb{Z})^d$.

We remark that, in the case of (i) in the assumption (A.3), $\{X_n\}$ should be reversible with respect to the constant weight on $\mathbb{Z}^d$.

In the case (i) of (A.3) is satisfied, let m_0 be the arbitrary natural number greater than or equal to 2, and in the case (ii) of (A.3) is satisfied, let m_0 be the one in (ii). Denote by c the covering map from $\mathbb{Z}$ onto $\mathbb{T} = \mathbb{Z}/m_0\mathbb{Z} = \{0, 1, 2, \cdots, m_0 - 1\}$, i.e.,

$$c(x) = x \pmod{m_0}$$

and set

$$c_0(x) = (c(x_1), \cdots, c(x_d)), \quad x = (x_1, \cdots, x_d) \in \mathbb{Z}^d.$$

Let $\mathcal{M}(\mathbb{T}^d)$ be the set of probability measures on $\mathbb{T}^d$ endowed with the topology induced by the metric

$$\rho(\mu, \lambda) = \sqrt{\sum_{x \in \mathbb{T}^d} |\mu(x) - \lambda(x)|^2}, \quad \mu, \lambda \in \mathcal{M}(\mathbb{T}^d). \tag{3}$$

Evidently $(\mathcal{M}(\mathbb{T}^d), \rho)$ is a compactmetric space. For each positive integer n, we define the empirical measure L_n on $\mathbb{T}^d$ of the random walk $\{c_0(X_k)\}$ by

$$L_n\big(\{x\}\big) = \frac{1}{n} \sum_{k=0}^{n-1} \chi_{\{x\}}(c_0(X_k))$$

for all $x \in \mathbb{T}^d$, where $\chi_{\{x\}}$ is the indicator of $x \in \mathbb{T}^d$. Let $P^{(n,y)}_{(0,x)}$ denotes the probability law of the random walk $\{X_n\}$ pinned as $X_0 = x \in \mathbb{Z}^d$ and $X_n = y \in \mathbb{Z}^d$, and let $Q^{(n,y)}_{(0,x)}$ be the law of L_n under $P^{(n,y)}_{(0,x)}$, namely

$$Q^{(n,y)}_{(0,x)}(B) = P^{(n,y)}_{(0,x)}\big(L_n \in B\big)$$

for every Borel subset B in $\mathcal{M}(\mathbb{T}^d)$. In this paper a large deviation principle for $Q^{(n,y)}_{(0,x)}$ will be investigated. For this purpose we will first define the rate function. Making use of the new probability measure $(p^0_z)_{z \in \mathbb{Z}^d}$ defined in (1), a new transition probability measure $p^0(x, y)$ is defined by

$$p^0(x, y) = p^0_{y-x} = \frac{1}{Z_{\lambda_0}} e^{\lambda_0 \cdot (y-x)} p(x, y), \qquad x, y \in \mathbb{Z}^d. \tag{4}$$

Let us denote by $\mathcal{U}_0$ the set of positive periodic functions of period m_0 on $\mathbb{Z}^d$ and set for $u \in \mathcal{U}_0$,

$$\pi_0 u(x) = \sum_{y \in \mathbb{Z}^d} p^0(x, y) u(y). \tag{5}$$

Note that $\pi_0 u \in \mathcal{U}_0$ for all $u \in \mathcal{U}_0$. The rate function I_0 on $\mathcal{M}(\mathbb{T}^d)$ is defined by

$$I_0(\mu) = -\inf_{u \in \mathcal{U}_0} \sum_{x \in \mathbb{T}^d} \log\Big(\frac{\pi_0 u}{u}\Big)(x)\mu(x).$$

Note that this is the rate function for the large deviation principle for the empirical measures of the Markov chain $\{\tilde{X}_n^0\}$ on $\mathbb{T}^d$ with the transition probability

$$\tilde{p}^0(x,y) = \sum_{k\in\mathbb{Z}^d} p^0(x, y + m_0 k),$$

see [2] and [5].

Now we are able to state our main result:

THEOREM 1. *Under the assumption (A.1), (A.2) and (A.3) on the transition probability* $\{p(x,y)\}_{x,y\in\mathbb{Z}^d}$ *of the random walk* $\{X_n\}$*, for all* $x, y \in \mathbb{Z}^d$*, the large deviation principle for* $Q^{(n,y)}_{(0,x)}$ *holds with the rate function* I_0*, i.e.,*

(i) *For any closed* $F \subseteq \mathcal{M}(\mathbb{T}^d)$*,*

$$\limsup_{n\to\infty} \frac{1}{n} \log Q^{(n,y)}_{(0,x)}(F) \le - \inf_{\mu\in F} I_0(\mu).$$

(ii) *For any open* $G \subseteq \mathcal{M}(\mathbb{T}^d)$*,*

$$\liminf_{n\to\infty} \frac{1}{n} \log Q^{(n,y)}_{(0,x)}(G) \ge - \inf_{\mu\in G} I_0(\mu).$$

A corollary of the above theorem is stated as follows.

COROLLARY 1. *If* Φ *is a real-valued weakly continuous functional on* $\mathcal{M}(\mathbb{T}^d)$*, then*

$$\lim_{n\to\infty} \frac{1}{n} \log E^{Q^{(n,y)}_{(0,x)}}[e^{-n\Phi(\cdot)}] = - \inf_{\mu\in\mathcal{M}(\mathbb{T}^d)} [\Phi(\mu) + I_0(\mu)]$$

for any $x, y \in \mathbb{Z}^d$*.*

The remainder of the paper is organized as follows. In section 3, for a given probability measure μ on $\mathbb{T}^d$ whose support coincides with it, we first construct the transition probabilities of a Markov chain from $\{p^0(x,y)\}_{x,y\in\mathbb{Z}^d}$ via a gauge transform such that the invariant measures of the chain is μ. Then we show that under our assumption (A.1)-(A.3), the spectral radius of each of the Markov chain is 1 for every such μ. In section 4 and section 5, we prove the large deviation principle, the upper bound and lower bound respectively. In section 6 we give the examples of random walks for which our assumption of (A.1)-(A.3) are satisfied.

3. Spectral Radius

Let us denote by $\mathcal{M}_c(\mathbb{T}^d)$ the set of probability measures μ whose support coincide with $\mathbb{T}^d$. The following lemma has already been given and proved in [1] as Lemma 1.

LEMMA 1. *For each $\mu \in \mathcal{M}_c(\mathbb{T}^d)$, there exists a $V_0 \in \mathcal{U}_0$ such that*

$$I_0(\mu) = -\sum_{z \in \mathbb{T}^d} \log\left(\frac{\pi_0 V_0}{V_0}\right)\mu(z). \tag{6}$$

Now, with the help of V_0 in Lemma 1, let a new transition probability p^1 on $\mathbb{Z}^d$ be defined by

$$p^1(x,y) = \frac{p^0(x,y)V_0(y)}{\pi_0 V_0(x)} \tag{7}$$

for $x \in \mathbb{Z}^d$ and $y \in \mathbb{Z}^d$, where $\pi_0 V_0(x)$ is defined by (5), and let us denote by $\{X_n^1\}$ the Markov chain with the transition probabilities $\{p^1(x,y)\}_{x,y\in\mathbb{Z}^d}$. Moreover, a transition probability on $\mathbb{T}^d$ is defined by

$$\tilde{p}^1(x,y) = \sum_{k\in\mathbb{Z}^d} p^1(x, y+m_0k) = \frac{\tilde{p}^0(x,y)V_0(y)}{\pi_0 V_0(x)}, \quad x,y \in \mathbb{T}^d. \tag{8}$$

We denote by $\{\tilde{X}_n^1\}$ the Markov chain on $\mathbb{T}^d$ with the transition probabilities $\{\tilde{p}^1(x,y)\}_{x,y\in\mathbb{T}^d}$. Notice that $p^1(x,y)$ and $\tilde{p}^1(x,y)$ are functionals of μ in Lemma 1.

The next lemma has also been given in [1] as Lemma 2.

LEMMA 2. *$\{\tilde{p}^1(x,y)\}_{x,y\in\mathbb{T}^d}$ constitutes the system of one-step transition probabilities of a Markov chain on $\mathbb{T}^d$ having μ as its unique invariant probability measure.*

For any $\mu \in \mathcal{M}_c(\mathbb{T}^d)$, let $\vartheta(\mu) \in \mathbb{R}^d$ be defined by

$$\vartheta(\mu) = \sum_{x,y\in\mathbb{T}^d} \mu(x) \sum_{k\in\mathbb{Z}^d} k \cdot p^1(x, y+m_0k),$$

where the system of the transition probabilities $\{p^1(x,y)\}_{x,y\in\mathbb{Z}^d}$ is defined by (7) with V_0 so chosen as described in Lemma 1.

PROPOSITION 1. *Assume (A.1), (A.2) and (A.3) on the transition probabilities $\{p(x,y)\}_{x,y\in\mathbb{Z}^d}$ of our original random walk. Then for all $\mu \in \mathcal{M}_c(\mathbb{T}^d)$,*

$$\vartheta(\mu) = 0.$$

Let us now recall the definition of the spectral radius of a Markov chain. For the irreducible Markov chain on $\mathbb{Z}^d$ with the n-step transition probabilities $\{p_n(x,y)\}_{x,y\in\mathbb{Z}^d}$, it is known that if

$$\rho(x,y) = \lim_{n\to\infty} p_n(x,y)^{1/n}$$

exists for some $(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, then it exists for all $x, y \in \mathbb{Z}^d$ and does not depend on the choice of (x,y). This $\rho = \rho(x,y)$ is called the spectral radius of the Markov chain.

We denote by $\rho(\mu)$ the spectral radius of the Markov chain given by the transition probabilities $\{p^1(x,y)\}_{x,y\in\mathbb{Z}^d}$ defined by (7), since $\{p^1(x,y)\}_{x,y\in\mathbb{Z}^d}$ are defined given $\mu \in \mathbb{T}^d$. As a result of Proposition 1, combined with Salvatori's theorem (see [6]), we obtain the following.

THEOREM 2. *Assume (A.1), (A.2) and (A.3) on the transition probabilities $\{p(x,y)\}_{x,y\in\mathbb{Z}^d}$ of our original Markov chain. Then for all $\mu \in \mathcal{M}_c(\mathbb{T}^d)$, we have*

$$\rho(\mu) = 1.$$

PROOF OF THEOREM 2. Under our assumption, by Proposition 1, we have $\vartheta(\mu) = 0$, and thus by Theorem(6.7) and Theorem(8.14) in [6], we have

$$\limsup_{n\to\infty} p^1_n(x,y)^{1/n} = 1.$$

Since the Markov chain with the transition probability $\{p^1(x,y)\}$ is irreducible and aperiodic by our assumption (A.1) and (A.2), by Lemma(1.9) of [6], $\lim_{n\to\infty} p^1_n(x,y)^{1/n}$ exists, and thus the theorem follows. □

PROOF OF PROPOSITION 1. For each $\mu \in \mathcal{M}_c(\mathbb{T}^d)$, let the transition probability $\{p^1(x,y)\}_{x,y\in\mathbb{Z}^d}$ be defined by (7) with V_0 so chosen as described in Lemma 1.

Step 1. In the first step, we show the result under the assumption (A.1), (A.2) and (i) of (A.3). We first remark that under the assumption, $\{p^0(x,y)\}_{x,y\in\mathbb{Z}^d}$ is reversible with respect to the constant measure and thus $p^0(x,y) = p^0(y,x)$ for all x and y. Thus, by (7),

$$V_0(x)\pi_0 V_0(x)p^1(x,y) = V_0(y)\pi_0 V_0(y)p^1(y,x), \qquad x,y \in \mathbb{Z}^d$$

and so $p^1(x,y)$ is reversible with respect to

$$\mu'(x) = \frac{V_0(x)\pi_0 V_0(x)}{\sum_{z\in\mathbb{T}^d} V_0(z)\pi_0 V_0(z).}$$

This implies that $\mu = \mu'$ and $\{p^1(x,y)\}_{x,y\in\mathbb{Z}^d}$ is reversible with respect to μ, namely

$$\mu(x)p^1(x,y) = \mu(y)p^1(y,x) \tag{9}$$

for all $x,y \in \mathbb{Z}^d$. We notice also that p^1 is periodic in the sense that

$$p^1(x+m_0k, y+m_0k) = p^1(x,y) \tag{10}$$

for all $x,y \in \mathbb{Z}^d$ and $k \in \mathbb{Z}^d$, since V_0 and $\pi_0 V_0$ are periodic with period m_0 and p^0 is also periodic in the same sense as above. Hence, by (9) and (10),

$$\begin{aligned}
\vartheta(\mu) &= \sum_{x,y\in\mathbb{T}^d} \mu(x) \sum_{k\in\mathbb{Z}^d} k\cdot p^1(x, y+m_0k) \\
&= \sum_{x,y\in\mathbb{T}^d} \mu(y+m_0k) \sum_{k\in\mathbb{Z}^d} k\cdot p^1(y+m_0k, x) \\
&= \sum_{x,y\in\mathbb{T}^d} \mu(y) \sum_{k\in\mathbb{Z}^d} k\cdot p^1(y+m_0k, x) \\
&= \sum_{x,y\in\mathbb{T}^d} \mu(y) \sum_{k\in\mathbb{Z}^d} k\cdot p^1(y, x-m_0k) \\
&= -\sum_{x,y\in\mathbb{T}^d} \mu(y) \sum_{k\in\mathbb{Z}^d} (-k)\cdot p^1(y, x+m_0(-k)) = -\vartheta(\mu),
\end{aligned}$$

and thus $\vartheta(\mu) = 0$.

Step 2. In the remainder of the proof we show the result under the assumption (A.1), (A.2) and (ii) of (A.3). For $x \in \mathbb{T}^d$, we denote $x = (x_1, x')$ with $x_1 \in \mathbb{T}$ and $x' \in \mathbb{T}^{d-1}$. Let $\tilde{\mu}$ be the probability measure on $\mathbb{T}$ given by

$$\tilde{\mu}(x_1) = \sum_{x' \in \mathbb{T}^{d-1}} \mu(x).$$

Another system of transition probabilities $\tilde{q}(x_1, y_1)$ on $\mathbb{T}$ is given by

$$\tilde{q}^1(x_1, y_1) = \frac{1}{\tilde{\mu}(x_1)} \sum_{x', y' \in \mathbb{T}^{d-1}} \tilde{p}^1(x, y)\mu(x). \tag{11}$$

Here, notice that

$$\sum_{y_1 \in \mathbb{T}} \tilde{q}^1(x_1, y_1) = \frac{1}{\tilde{\mu}(x_1)} \sum_{x' \in \mathbb{T}^{d-1}} \mu(x) \sum_{y \in \mathbb{T}^d} \tilde{p}^1(x, y) = \frac{1}{\tilde{\mu}(x_1)} \sum_{x' \in \mathbb{T}^{d-1}} \mu(x) = 1.$$

Now, since μ is the invariant measure for the transition probability $\tilde{p}^1$,

$$\begin{aligned} \sum_{x_1 \in \mathbb{T}} \tilde{\mu}(x_1)\tilde{q}^1(x_1, y_1) &= \sum_{x_1 \in \mathbb{T}} \sum_{x' \in \mathbb{T}^{d-1}, y' \in \mathbb{T}^{d-1}} \mu(x)\tilde{p}^1(x, y) \\ &= \sum_{y' \in \mathbb{T}^{d-1}} \sum_{x \in T^d} \mu(x)\tilde{p}^1(x, y) = \sum_{y' \in \mathbb{T}^{d-1}} \mu(y) = \tilde{\mu}(y_1), \end{aligned} \tag{12}$$

meaning that $\tilde{\mu}$ is a invariant probability measure for the transition probability $\tilde{q}^1$.

We denote $\mathbb{T} = \{0, 1, \cdots, m_0-1\}$ and $\tilde{\mu} = (\tilde{\mu}(0), \cdots, \tilde{\mu}(m_0-1))$. Notice that (12) is a system of homogeneous equations for $(\tilde{\mu}(0), \cdots, \tilde{\mu}(m_0-1))$. Let $\tilde{u} = (\tilde{u}(0), \cdots, \tilde{u}(m_0-1))$ be the solution of the equation (12) with $\tilde{u}(m_0-1) = 1$. Namely, $(\tilde{u}(0), \cdots, \tilde{u}(m_0-2))$ is a solution of

$$\sum_{x=0, x\neq z}^{m_0-2} \tilde{q}^1(x, z)\tilde{u}(x) + \{-1 + \tilde{q}^1(z, z)\}\tilde{u}(z) = -\tilde{q}^1(m_0-1, z)$$

for $z = 0, 1, \cdots, m_0 - 2$. This system of equations can be written as

$$A\tilde{u}^t = -b$$

with $(m_0-1)\times(m_0-1)$ matrix

$$
\begin{aligned}
A &= [a_0,\cdots a_{m_0-2}]\\
&= \begin{bmatrix}
\tilde{q}^1(0,0)-1 & \tilde{q}^1(1,0) & \cdots & \tilde{q}^1(m_0-2,0)\\
\tilde{q}^1(0,1) & \tilde{q}^1(1,1)-1 & \cdots & \tilde{q}^1(m_0-2,1)\\
\vdots & \vdots & \ddots & \vdots\\
\tilde{q}^1(0,m_0-2) & \tilde{q}^1(1,m_0-2) & \cdots & \tilde{q}^1(m_0-2,m_0-2)-1
\end{bmatrix}
\end{aligned}
$$

and

$$
b = \begin{bmatrix}
\tilde{q}^1(m_0-1,0)\\
\tilde{q}^1(m_0-1,1)\\
\vdots\\
\tilde{q}^1(m_0-1,m_0-2)
\end{bmatrix}.
$$

Thus we have

$$
\begin{aligned}
\tilde{u}(x) &= \frac{\det[a_0,\cdots,a_{x-1},-b,a_{x+1},\cdots,a_{m_0-2}]}{\det(A)}\\
&= \frac{(-1)^{m_0-x-1}\det[a_0,\cdots,a_{x-1},a_{x+1},\cdots,a_{m_0-2},b]}{\det(A)}
\end{aligned}
$$

for every $x\in\{0,1,\cdots,m_0-2\}$. Therefore, since $\tilde{\mu}$ is a solution of (12) with $\sum_{x\in\mathbb{T}}\tilde{\mu}(x)=1$, if we set

$$
C=\det(A)+\sum_{x\in\mathbb{T}}(-1)^{m_0-x-1}\det[a_0,\cdots,a_{x-1},a_{x+1},\cdots,a_{m_0-2},b],
$$

then we have

$$
\tilde{\mu}_x = \begin{cases}
\dfrac{(-1)^{m_0-x-1}}{C}\det[a_0,\cdots,a_{x-1},a_{x+1},\cdots,a_{m_0-2},b], \\
\qquad\qquad x=0,1,\cdots,m_0-2\\[2ex]
\dfrac{\det(A)}{C}, \qquad x=m_0-1.
\end{cases}
\tag{13}
$$

Hence, the first coordinate $\vartheta(\mu)_1$ of $\vartheta(\mu)\in\mathbb{R}^d$ is, with $k=(k_1,\cdots,k_d)$,

$$
\vartheta(\mu)_1 = \sum_{x,y\in\mathbb{T}^d}\mu(x)\sum_{k\in\mathbb{Z}^d}k_1\cdot p^1(x,y+m_0k)
$$

$$
\begin{aligned}
&= \sum_{x_1\in\mathbb{T}} \tilde{\mu}(x_1)\left(\frac{1}{\tilde{\mu}(x_1)}\sum_{x'\in\mathbb{T}^{d-1}}\mu(x)\sum_{y\in\mathbb{T}^d}\sum_{k\in\mathbb{Z}^d} k_1\cdot p^1(x,y+m_0k)\right)\\
&= \sum_{x_1\in\mathbb{T}} \tilde{\mu}(x_1)B_{x_1}, \quad \text{say.}
\end{aligned}
$$

By (13), we have

$$
\begin{aligned}
\vartheta(\mu)_1 &= \frac{1}{C}\Bigg\{\sum_{x=0}^{m_0-2}(-1)^{m_0-x-1}\det[a_0,\cdots,a_{x-1},a_{x+1}\cdots,a_{m_0-2},b]B_x\\
&\qquad +\det(A)B_{m_0-1}\Bigg\}\\
&= \frac{1}{C}\det\begin{bmatrix} a_0 & a_1 & \cdots & a_{m_0-2} & b\\ B_0 & B_1 & \cdots & B_{m_0-2} & B_{m_0-1}\end{bmatrix}
\end{aligned}
$$

(14)

$$
\begin{aligned}
&= \frac{1}{C}\det\begin{bmatrix}
\tilde{q}^1(0,0)-1 & \tilde{q}^1(1,0) & \cdots & \tilde{q}^1(m_0-2,0) & \tilde{q}^1(m_0-1,0)\\
\tilde{q}^1(0,1) & \tilde{q}^1(1,1)-1 & \cdots & \tilde{q}^1(m_0-2,1) & \tilde{q}^1(m_0-1,1)\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\tilde{q}^1(0,m_0-2) & \tilde{q}^1(1,m_0-2) & \cdots & \tilde{q}^1(m_0-2,m_0-2)-1 & \tilde{q}^1(m_0-1,m_0-2)\\
B_0 & B_1 & \cdots & B_{m_0-2} & B_{m_0-1}
\end{bmatrix}\\
&= \frac{\det(M)}{C}, \qquad \text{say.}
\end{aligned}
$$

Since given $y_1\in\mathbb{T}$ of $y=(y_1,y')\in\mathbb{T}^d$,

$$
\sum_{y'\in\mathbb{T}^{d-1}}\tilde{p}^1(x,y)-1=\sum_{y'\in\mathbb{T}^{d-1}}\tilde{p}^1(x,y)-\sum_{y\in\mathbb{T}^d}\tilde{p}^1(x,y)=-\sum_{z\in\mathbb{T}^d,z_1\neq y_1}\tilde{p}^1(x,z),
$$

we have

$$
\begin{aligned}
\tilde{q}^1(x_1,y_1)-1 &= \frac{1}{\tilde{\mu}(x_1)}\left(\sum_{x',y'\in\mathbb{T}^{d-1}}\tilde{p}^1(x,y)\mu(x)-\sum_{x'\in\mathbb{T}^{d-1}}\mu(x)\right)\\
&= \frac{1}{\tilde{\mu}(x_1)}\sum_{x'\in\mathbb{T}^{d-1}}\mu(x)\left(\sum_{y'\in\mathbb{T}^{d-1}}\tilde{p}^1(x,y)-1\right)\\
&= -\frac{1}{\tilde{\mu}(x_1)}\sum_{x'\in\mathbb{T}^{d-1}}\mu(x)\sum_{z\in\mathbb{T}^d,z_1\neq y_1}\tilde{p}^1(x,z).
\end{aligned}
$$

Hence, for $x_1 \in \mathbb{T} = \{0, 1, \cdots, m_0 - 1\}$, the (x_1+1)-th column M_{x_1+1} of M is

$$
\begin{bmatrix}
\tilde{q}^1(x_1, 0) \\
\tilde{q}^1(x_1, 1) \\
\vdots \\
\tilde{q}^1(x_1, x_1 - 1) \\
\tilde{q}^1(x_1, x_1) - 1 \\
\tilde{q}^1(x_1, x_1 + 1) \\
\vdots \\
\tilde{q}^1(x_1, m_0 - 2) \\
B(x_1)
\end{bmatrix}
$$

$$
= \frac{1}{\tilde{\mu}(x_1)} \sum_{x' \in \mathbb{T}^{d-1}} \mu(x) \sum_{y' \in \mathbb{T}^{d-1}}
\begin{bmatrix}
\tilde{p}^1((x_1, x'), (0, y')) \\
\tilde{p}^1((x_1, x'), (1, y')) \\
\vdots \\
\tilde{p}^1((x_1, x'), (x_1 - 1, y')) \\
- \sum_{y_1 \in \mathbb{T}, y_1 \neq x_1} \tilde{p}^1((x_1, x'), (y_1, y')) \\
\tilde{p}^1((x_1, x'), (x_1 + 1, y')) \\
\vdots \\
\tilde{p}^1((x_1, x'), (m_0 - 2, y')) \\
\sum_{y_1 \in \mathbb{T}} \sum_{k \in \mathbb{Z}^d} k_1 p^1(x, y + m_0 k)
\end{bmatrix}
$$

$$
= \frac{1}{\tilde{\mu}(x_1)} \sum_{x' \in \mathbb{T}^{d-1}} \mu(x) \sum_{y' \in \mathbb{T}^{d-1}}
\begin{bmatrix}
N_{0, x_1+1} \\
N_{1, x_1+1} \\
\vdots \\
N_{m_0 - 1, x_1+1}
\end{bmatrix}, \quad \text{say.}
$$

Now, recalling (7) and (8), for every $x_1 \in \mathbb{T}$,

$$
\begin{aligned}
& m_0 \cdot N_{m_0, x_1+1} - \sum_{y_1=0}^{m_0-1} (m_0 - y_1) N_{y_1, x_1+1} \\
= \quad & \sum_{y_1=0}^{m_0-1} \sum_{k \in \mathbb{Z}^d} \{m_0 k_1 - (m_0 - y_1) + (m_0 - x_1)\} p^1((x_1, x'), (y_1, y') + m_0 k)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{y_1=0}^{m_0-1} \sum_{k\in\mathbb{Z}^d} (m_0k_1 + y_1 - x_1)p^1((x_1,x'),(y_1,y') + m_0k) \\
&= \sum_{y_1=0}^{m_0-1} \frac{V(y)}{\pi V_0(x)} \sum_{k\in\mathbb{Z}^d} (m_0k_1 + y_1 - x_1)p^0((x_1,x'),(y_1,y') + m_0k) \\
&= 0,
\end{aligned}
$$

where the last equality follows from our assumption (2). Hence, we get

$$
m_0 M_{m_0,x_1+1} - \sum_{y_1=0}^{m_0-1} (m_0 - y_1) M_{y_1,x_1+1} = 0
$$

for every (x_1+1)-th column of M. Therefore we see that $\det(M) = 0$, and so by (14), we have $\vartheta(\mu)_1 = 0$. Obviously, the same argument as above works for other coordinates of $\vartheta(\mu)$, and thus our assertion follows. □

4. Proof of Upper Bounds

We will first prove (i) of the Theorem 1. We denote by (X_n^0, P_x^0) the Markov chain on $\mathbb{Z}^d$ induced by the system of transition probabilities $\{p^0(x,y)\}_{x,y\in Z^d}$. By (4), the n-step transition probability $p_n^0(x,y)$ of (X_n^0, P_x^0) and the n-step transition probability $p_n(x,y)$ of (X_n, P_x) have the relation

$$
p_n^0(x,y) = \frac{e^{\lambda_0 y}}{Z_{\lambda_0}^n e^{\lambda_0 x}} \cdot p_n(x,y) \tag{15}
$$

for every n. Set, for $u \in \mathcal{U}_0$,

$$
V(x) = \pi_0 u(x) \quad \text{and} \quad W(x) = \log\bigl(\frac{V(x)}{u(x)}\bigr)
$$

where π_0 is given by (5). Note that $V \in \mathcal{U}_0$, i.e., periodic and positive. From the definition of the pinned process, noting (4) and (15), we have

$$
E^{P_{(0,x)}^{(n,y)}}\bigl[V(X_{n-1})\exp\bigl(-\sum_{k=0}^{n-1} W(X_k)\bigr)\bigr]
$$

$$
\begin{aligned}
&= \frac{1}{p_n(x,y)} E^{P_x}\big[V(X_{n-1})\exp\big(-\sum_{k=0}^{n-1} W(X_k)\big)p(X_{n-1},y)\big] \\
&= \frac{1}{p_n(x,y)} \sum_{x_1,\cdots,x_{n-1}\in\mathbb{Z}^d} V(x_{n-1}) \\
&\quad \times \exp\big(-\sum_{k=0}^{n-1} W(x_k)\big)p(x_{n-1},y)p(x,x_1)p(x_1,x_2)\cdots p(x_{n-2},x_{n-1}) \\
&= \frac{e^{\lambda_0 y}}{Z_{\lambda_0}\cdot p_n^0(x,y)} E^{P_x^0}\big[V(X_{n-1}^0)\exp\big(-\sum_{k=0}^{n-1} W(X_k^0)\big)\frac{p(X_{n-1}^0,y)}{e^{\lambda_0 X_{n-1}^0}}\big].
\end{aligned}
$$

Hence, since $x \mapsto p(x,y)/e^{\lambda_0 x}$ is bounded in x for every $y \in \mathbb{Z}^d$ in view of our assumption (A.1), we have

$$
\begin{aligned}
&E^{P_{(0,x)}^{(n,y)}}\big[V(X_{n-1})\exp\big(-\sum_{k=0}^{n-1} W(X_k)\big)\big] \\
&\le C_1 \frac{e^{\lambda_0 y}}{Z_{\lambda_0}\cdot p_n^0(x,y)} E^{P_x^0}\big[V(X_{n-1}^0)\exp\big(-\sum_{k=0}^{n-1} W(X_k^0)\big)\big] \\
&= \frac{C_1 e^{\lambda_0 y}}{Z_{\lambda_0}\cdot p_n^0(x,y)} u(x)
\end{aligned}
$$

for some $C_1 > 0$. Here, the last equality can easily be checked inductively in n. Thus we have obtained

$$
(16) \qquad E^{P_{(0,x)}^{(n,y)}}\big[\exp\big(-\sum_{k=0}^{n-1} W(X_k)\big)\big] \le \frac{C_1 e^{\lambda_0 y} u(x)}{Z_{\lambda_0}\cdot p_n^0(x,y)\cdot \inf_{x\in\mathbb{Z}^d} V(x)}.
$$

Thus, for every closed set F in $\mathcal{M}(\mathbb{T}^d)$,

$$
\begin{aligned}
Q_{(0,x)}^{(n,y)}(F) \quad &\le \quad \exp\big(n \sup_{\mu\in F} \sum_{z\in\mathbb{Z}^d} W(z)\mu(z)\big) E^{Q_{(0,x)}^{(n,y)}} \\
&\qquad \times \big[\exp\big(-n \sum_{z\in\mathbb{Z}^d} W(z)\mu(z)\big),\ \mu \in F\big] \\
&\le \quad \exp\big(n \sup_{\mu\in F} \sum_{z\in\mathbb{Z}^d} W(z)\mu(z)\big) E^{P_{(0,x)}^{(n,y)}}\big[\exp\big(-\sum_{k=0}^{n-1} W(X_k)\big)\big]
\end{aligned}
$$

$$\leq \frac{C_1 e^{\lambda_0 y} u(x)}{Z_{\lambda_0} \cdot p_n^0(x,y) \cdot \inf_{x\in\mathbb{Z}^d} V(x)} \times \exp\big(n \sup_{\mu\in F} \sum_{z\in\mathbb{T}^d} \log\big(\frac{\pi_0 u}{u}\big)(z)\mu(z)\big),$$

and hence, by reminding that

$$\lim_{n\to\infty} \frac{1}{n} \log p_n^0(x,y) = 0 \tag{17}$$

as a result of Theorem 2, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log Q_{(0,x)}^{(n,y)}(F) \leq \sup_{\mu\in F} \sum_{z\in\mathbb{T}^d} \log\big(\frac{\pi_0 u}{u}\big)(z)\mu(z)$$

for all $u \in \mathcal{U}_0$, and hence

$$\limsup_{n\to\infty} \frac{1}{n} \log Q_{(0,x)}^{(n,y)}(F) \leq \inf_{u\in\mathcal{U}_0} \sup_{\mu\in F} \sum_{z\in\mathbb{T}^d} \log\big(\frac{\pi_0 u}{u}\big)(z)\mu(z). \tag{18}$$

Now we are able to follow the standard argument in the theory of large deviations, see [2], to get (18) with the order of infimum and supremum in the RHS altered. Since

$$\begin{aligned}\sup_{\mu\in F} \inf_{u\in\mathcal{U}_0} \sum_{z\in\mathbb{T}^d} \log\big(\frac{\pi_0 u}{u}\big)(z)\mu(z) &= -\inf_{\mu\in F}\Big(-\inf_{u\in\mathcal{U}_0} \sum_{z\in\mathbb{T}^d} \log\big(\frac{\pi_0 u}{u}\big)(z)\mu(z)\Big) \\ &= -\inf_{\mu\in F} I_0(\mu),\end{aligned}$$

we see that (i) of Theorem 1 follows.

5. Proof of Lower Bounds

In this section we will prove (ii) of Theorem 1. Let $S(\mu,\varepsilon) = \{\nu \in \mathcal{M}(\mathbb{T}^d);\ \rho(\nu,\mu) \leq \varepsilon\}$. It suffices to verify

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{(0,x)}^{(n,y)}\big(\omega; L_n(\omega,\cdot) \in S(\mu,\varepsilon)\big) \geq -I_0(\mu) \tag{19}$$

for any $\varepsilon > 0$. From the definition of the pinned process and the relation (15), we have

$$P_{(0,x)}^{(n,y)}(\ L_n \in S(\mu,\varepsilon)) \;=\; \frac{1}{p_n(x,y)} E^{P_x}\big[p(X_{n-1},y);\ L_n \in S(\mu,\varepsilon)\big]$$

$$= \frac{Z_{\lambda_0}^{n-1} \cdot e^{\lambda_0 x}}{p_n(x,y)} E^{P_x^0}\big[\frac{p(X_{n-1}^0, y)}{e^{\lambda_0 X_{n-1}^0}};\ L_n \in S(\mu,\varepsilon)\big]$$
$$= \frac{e^{\lambda_0 y}}{Z_{\lambda_0} \cdot p_n^0(x,y)} E^{P_x^0}\big[\frac{p(X_{n-1}^0, y)}{e^{\lambda_0 X_{n-1}^0}};\ L_n \in S(\mu,\varepsilon)\big].$$

Now for each $\mu \in \mathcal{M}_c(\mathbb{T}^d)$, by Lemma 1 and Lemma 2, there is a $V_0 \in \mathcal{U}_0$ such that $I(\mu)$ is given via V_0 by (6), and $\{c(X_n^1)\}$ of the Markov chain (X_n^1, P_x^1) on $\mathbb{Z}^d$ associated with the transition probability $p^1(x,y)$ given by (7) has μ as its invariant measure. Since

$$p^0(x,y) = \frac{p^1(x,y)\pi_0 V_0(x)}{V_0(y)}$$

by (7), we have

$$E^{P_x^0}\big[\frac{p(X_{n-1}^0, y)}{e^{\lambda_0 X_{n-1}^0}};\ L_n \in S(\mu,\varepsilon)\big]$$
$$= \sum_{x_1,\cdots,x_{n-1}\in\mathbb{Z}^d} \frac{p(x_{n-1}, y)}{e^{\lambda_0 x_{n-1}}} \chi_{\{L_n \in S(\mu,\varepsilon)\}} p^0(x,x_1)p^0(x_1,x_2)\cdots p^0(x_{n-2},x_{n-1})$$
$$= E^{P_x^1}\big[\frac{p(X_{n-1}^1, y)}{e^{\lambda_0 X_{n-1}^1}}\big(\prod_{k=0}^{n-1}\frac{\pi_0 V_0(X_k^1)}{V_0(X_k^1)}\big)\frac{V_0(x)}{\pi_0 V_0(X_{n-1}^1)};\ L_n \in S(\mu,\varepsilon)\big],$$

and thus we see that

$$(20)\qquad P_{(0,x)}^{(n,y)}(L_n \in S(\mu,\varepsilon))$$
$$\geq \frac{e^{\lambda_0 y}}{Z_{\lambda_0} \cdot p_n^0(x,y)} \frac{V_0(x)}{\sup_{x\in\mathbb{Z}^d} \pi_0 V_0(x)} E^{P_x^1}$$
$$\times \big[\frac{p(X_{n-1}^1, y)}{e^{\lambda_0 X_{n-1}^1}} \exp\big(\sum_{k=0}^{n-1} \log \frac{\pi_0 V_0(X_k^1)}{V_0(X_k^1)}\big);\ L_n \in S(\mu,\varepsilon)\big].$$

We introduce, for $\varepsilon' > 0$,

$$S_1(n,\mu,\varepsilon')$$
$$= \{\omega;\ \big|\sum_{x\in\mathbb{T}^d} \log\big(\frac{\pi_0 V_0}{V_0}\big)(x) L_n(\omega,\{x\}) - \sum_{x\in\mathbb{T}^d} \log\big(\frac{\pi_0 V_0}{V_0}\big)(x)\mu(x)\big| < \varepsilon'\},$$

and

$$S_2(n,\mu,\varepsilon') = \{\omega;\ L_n(\omega,\cdot) \in S(\mu,\varepsilon)\} \cap S_1(n,\mu,\varepsilon').$$

Then, by Lemma 1,

$$\begin{aligned}
&E^{P_x^1}\Big[\frac{p(X_{n-1}^1,y)}{e^{\lambda_0 X_{n-1}^1}} \exp\Big(\sum_{k=0}^{n-1} \log \frac{\pi_0 V_0(X_k^1)}{V_0(X_k^1)}\Big);\ L_n \in S(\mu,\varepsilon)\Big] \\
(21)\quad &\geq \exp\Big\{n\Big(\sum_{x\in\mathbb{T}^d} \log\Big(\frac{\pi_0 V_0}{V_0}\Big)(x)\mu(x) - \varepsilon'\Big)\Big\} E^{P_x^1}\Big[\frac{p(X_{n-1}^1,y)}{e^{\lambda_0 X_{n-1}^1}}; S_2(n,\mu,\varepsilon')\Big] \\
&= \exp(-nI_0(\mu) - n\varepsilon') E^{P_x^1}\Big[\frac{p(X_{n-1}^1,y)}{e^{\lambda_0 X_{n-1}^1}}; S_2(n,\mu,\varepsilon')\Big].
\end{aligned}$$

From the assumption (A.1), the set $\{x \in \mathbb{Z}^d; p(x,y) > 0\}$ is finite for every y. Denote this set by $\{x_1, \cdots, x_m\}$ and set $\alpha = \inf_{1\leq j\leq m} \frac{p(x_j,y)}{e^{\lambda_0 x_j}}$. Then,

$$\begin{aligned}
&E^{P_x^1}\Big[\frac{p(X_{n-1}^1,y)}{e^{\lambda_0 X_{n-1}^1}}; S_2(n,\mu,\varepsilon')\Big] \\
&\quad = \sum_{j=1}^{m} E^{P_x^1}\Big[\frac{p(X_{n-1}^1,y)}{e^{\lambda_0 X_{n-1}^1}}; S_2(n,\mu,\varepsilon') \cap \{X_{n-1}^1 = x_j\}\Big] \\
&\quad \geq \alpha \sum_{j=1}^{m} P_x^1(S_2(n,\mu,\varepsilon') \cap \{X_{n-1}^1 = x_j\}) \\
&\quad \geq \alpha \sum_{j=1}^{m} P_x^1(X_{n-1}^1 = x_j) - \alpha m P_x^1(\Omega \setminus S_2(n,\mu,\varepsilon')).
\end{aligned}$$

Now, since the large deviation principle for the empirical measures of the irreducible Markov chain$\{c_0(X_n^1)\}$ on $\mathbb{T}^d$ holds, $P_x^1(\Omega \setminus S_2(n,\mu,\varepsilon'))$ decays exponentially fast in n as n gets large. On the other hand, in view of Theorem 2, for any $0 < \delta < 1$

$$P_x^1(X_{n-1}^1 = x_j) = p_{n-1}^1(x,x_j) \geq (1-\delta)^n$$

for sufficiently large n. Thus we have

$$\text{(22)}\qquad \liminf_{n\to\infty} \frac{1}{n} \log E^{P_x^1}\Big[\frac{p(X_{n-1}^1,y)}{e^{\lambda_0 X_{n-1}^1}}; S_2(n,\mu,\varepsilon')\Big] \geq 0.$$

Combining (17), (20), (21) and (22), it follows that

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{(0,x)}^{(n,y)}(\omega;\ L_n(\omega,\cdot) \in S(\mu,\varepsilon)) \geq -I_0(\mu) - \varepsilon'.$$

Since ε' is arbitrary, this implies (19) for $\mu \in \mathcal{M}_c(\mathbb{T}^d)$. Noting that I_0 is lower semi-continuous on $\mathcal{M}(\mathbb{T}^d)$, this can immediately be extended to hold for all $\mu \in \mathcal{M}(\mathbb{T}^d)$, and this completes the proof of Theorem 1.

6. Examples

Example 1. (Nearest neighbor case) Suppose that $p_x = 0$ if $\|x\| > 1$ and that the random walk corresponding to $(p_x)_{x\in\mathbb{Z}^d}$ is irreducible and aperiodic, then the walk is reversible with respect to a measure $\mu = (\mu_x)_{x\in\mathbb{Z}^d}$ satisfying

$$\mu_x = \mu_0 \prod_{i=1}^{d} \left(\frac{p_{e_i}}{p_{-e_i}} \right)^{x_i},$$

with $e_i = (0,\cdots,0,1,0,\cdots,0)$. Namely (A.3), (i) holds and hence Theorem 1 is valid with any $m_0 \in \mathbb{N}$.

The class of reversible, i.i.d. random walks is a slight generalization of the class of nearest i.i.d. random walks and is very special. Therefore we omit more discussion here.

Most of i.i.d. random walks are non-reversible and the situation related to the invariance of spectral radius is complicated as above. Here we only give some numerical examples.

Example 2. (One dimensional case) Let $d = 1$ and the probability measure $(p_x)_{x\in\mathbb{Z}}$ be defined as

$$p_1 = \frac{16}{27}, \quad p_2 = \frac{8}{27}, \quad p_{-1} = \frac{2}{27}, \quad p_{-2} = \frac{1}{27}, \quad p_x = 0\,(\text{otherwise}).$$

Then the corresponding random walk is irreducible and aperiodic.

Minimizing

$$Z_\lambda = \frac{16}{27}e^{\lambda} + \frac{8}{27}e^{2\lambda} + \frac{2}{27}e^{-\lambda} + \frac{1}{27}e^{-2\lambda}, \quad \lambda \in \mathbb{R},$$

we see that its minimum value (which is the spectral radius of (p_x)) is equal to $\dfrac{2}{3}$ and that it is attained at $\lambda = -\log 2$. Making use of these, the associated unbiased probability measure $(p^0_x)_{x\in\mathbb{Z}}$ is given by

$$p^0_1 = \frac{4}{9}, \quad p^0_2 = \frac{1}{9}, \quad p^0_{-1} = \frac{2}{9}, \quad p^0_{-2} = \frac{2}{9}, \quad p^0_x = 0\,(\text{otherwise}).$$

This probability $(p^0_x)_{x\in\mathbb{Z}}$ satisfies the condition (A.3), (ii) with $m_0 = 3$. Thus Theorem 1 holds with $m_0 = 3$.

Example 3. (Two dimensional case) Let $d = 2$ and the probability measure $(p_{(x,y)})_{(x,y)\in\mathbb{Z}^2}$ be defined as

$$\begin{aligned} &p_{(0,1)} = \frac{8}{49},\ p_{(0,-2)} = \frac{4}{49},\ p_{(2,0)} = \frac{16}{49},\ p_{(-1,0)} = \frac{4}{49},\\ &p_{(1,1)} = \frac{16}{49},\ p_{(-2,-2)} = \frac{1}{49},\ p_{(x,y)} = 0\,(\text{otherwise}). \end{aligned}$$

Then the corresponding random walk is irreducible and aperiodic.

An elementary computation shows that the minimum value of

$$Z_{(\lambda_1,\lambda_2)} = \frac{8}{49}e^{\lambda_2} + \frac{4}{49}e^{-2\lambda_2} + \frac{16}{49}e^{2\lambda_1} + \frac{4}{49}e^{-\lambda_1} + \frac{16}{49}e^{\lambda_1+\lambda_2} + \frac{1}{49}e^{-2\lambda_1-2\lambda_2},$$

is equal to $\dfrac{36}{49}$ and is attained at $\lambda_1 = -\log 2$, $\lambda_2 = 0$. Then the unbiased probability $(p^0_{(x,y)})$ associated with $(p_{(x,y)})$ is

$$\begin{aligned} &p^0_{(0,1)} = \frac{2}{9},\ p^0_{(0,-2)} = \frac{1}{9},\ p^0_{(2,0)} = \frac{1}{9},\ p^0_{(-1,0)} = \frac{2}{9},\\ &p^0_{(1,1)} = \frac{2}{9},\ p^0_{(-2,-2)} = \frac{1}{9},\ p^0_{(x,y)} = 0\,(\text{otherwise}). \end{aligned}$$

This probability $(p^0_{(x,y)})$ satisfies (A.3),(ii) with m_0=3. Thus Theorem 1 holds with m_0=3.

Example 4. To conclude the section we present a general result that the direct product of the 1-dimensional probability measures both of which satisfy the assumption (A.1), (A.2) and (A.3)(ii) satisfies the same assumption on the multidimensional setting.

Proposition 2. *Assume that* $p^{0,i} \in \mathcal{M}(\mathbb{Z})$, $i = 1, 2, \cdots, d$ *satisfies (A.3)(ii), namely;*

$$\sum_{k \in \mathbb{Z}} (x_i + m_0 k) p^{0,i}_{x_i + m_0 k} = 0$$

for all $x_i \in \mathbb{T}$, *then* $p^0 \in \mathcal{M}(\mathbb{Z}^d)$ *defined by*

$$p^0_x = p^{0,1}_{x_1} \cdots p^{0,d}_{x_d}, \quad x = (x_1, \cdots, x_d)$$

satisfy (A.3)(ii), or

$$\sum_{k \in \mathbb{Z}^d} (x + m_0 k) p^0_{x + m_0 k} = 0$$

for all $x \in \mathbb{T}^d$.

Proof. For $x \in \mathbb{Z}^d$, let us denote $x = (x_1, x')$. Since

$$\sum_{k' \in \mathbb{Z}^{d-1}} p^0_{x + m_0 k} = p^{0,1}_{x + m_0 k} \sum_{k' \in \mathbb{Z}^{d-1}} \prod_{i=2}^{d} p^{0,i}_{x_i + m_0 k},$$

we have

$$\begin{aligned} &\sum_{k \in \mathbb{Z}^d} (x_1 + m_0 k_1) p^0_{x + m_0 k} \\ &\quad = \sum_{k_1 \in \mathbb{Z}} (x_1 + m_0 k_1) \sum_{k' \in \mathbb{Z}^{d-1}} p^0_{x + m_0 k} \\ &\quad = \left(\sum_{k_1 \in \mathbb{Z}} (x_1 + m_0 k_1) p^{0,1}_{x + m_0 k} \right) \cdot \sum_{k' \in \mathbb{Z}^{d-1}} \prod_{i=2}^{d} p^{0,i}_{x_i + m_0 k} = 0. \end{aligned}$$

The same argument is valid to show that

$$\sum_{k \in \mathbb{Z}^d} (x_i + m_0 k_i) p^0_{x + m_0 k} = 0$$

for every $i = 2, \cdots, d$, and thus the proposition follows. □

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