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Matrix Coefficients with Minimal K-Types of the Spherical and Non-Spherical Principal Series Representations of $SL(3, \mathbf{R})$

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Abstract. We compute the holonomic system of rank 6 for the radial part of the matrix coefficients of spherical and non-spherical principal series representations of $SL(3, \mathbf{R})$. We obtain six power series solutions corresponding to the set of six characteristic roots, and express the matrix coefficients by linear combinations of these power series. Among others, the c-functions of non-spherical principal series are obtained.

1. Introduction

It is a classical result to have the matrix coefficient of the spherical (or class one) principal series of a semisimple real Lie group as a linear combination of asymptotic power series solutions [1]. Among others there appear c-functions as the coefficients of this linear combination, which are obtained as certain explicit products of the gamma factors. The functions of this kind were firstly computed explicitly by G. Schiffmann [7] only for the case of class one representations (see also chap.9 of Warner [8]). It is generally believed that we have a similar formula for non-spherical case. But in spite of this folklore, we find few references even for small Lie groups. One of the reasons is that the inductive argument used in [7] does not work for non-spherical case.

In this paper, we investigate the matrix coefficients of the spherical and non-spherical principal series representations of $G = SL(3, \mathbf{R})$. Let us introduce the contents of this paper. We take $K = SO(3, \mathbf{R})$ as a maximal compact subgroup of G. Given a principal series representation (π, H_π) of G, the matrix coefficients of π are regarded as the elements of the space of

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the spherical functions defined by

$$\begin{aligned} &C^{\infty}_{\eta,\tau}(K\backslash G/K) \\ &\quad := \{\phi : G \to V_\eta \otimes V_\tau \,|\, \phi(k_L g k_R^{-1}) \\ &\qquad\qquad = (\eta(k_L) \otimes \tau(k_R))\, \phi(g), k_L, k_R \in K, g \in G\} \end{aligned}$$

with some finite dimensional representations (η, V_η), (τ, V_τ) of K. We consider the following two cases. If π is the spherical principal series representation, then we take $\eta = \tau = \mathbf{1}$, the trivial representation of K, and if π is the non-spherical principal series representation, we take $\eta = \tau = \tau_2$, the three dimensional tautological representation of K. In other words, we restrict the vectors of the matrix coefficients to the elements of the minimal K-type of π.

The basic method is to construct and investigate the differential equations satisfied by the spherical functions we mentioned above. Let $\mathfrak{g}$ be the Lie algebra of G and $U(\mathfrak{g})$ its universal enveloping algebra. The spherical function attached to the class one principal series is completely determined by the actions of the Capelli elements Cp_2, Cp_3, which are the generators of the center of $U(\mathfrak{g})$ together with the regularity at the identity of G. This part is more or less a classical fact. Meanwhile, to investigate the spherical functions attached to the non-spherical principal series representations, we construct the following two kinds of differential equations: (a) the differential equations characterized by the action of the Casimir element of degree two (this is also one of the generators of the center of $U(\mathfrak{g})$), (b) the differential equations characterized by the action of the gradient operator (or the Dirac-Schmid operator). We have three different non-spherical principal series with the same infinitesimal characters $Z(\mathfrak{g}) \to \mathbf{C}$. We cannot distinguish them only by the elements of $Z(\mathfrak{g})$. This is the reason we need the gradient operator which has distinct eigenvalues for different non-spherical principal series. This part seems to be new. In both cases, we obtain six power series solutions corresponding to the six characteristic roots. Since the explicit forms of the coefficients of the power series solutions are quite complicated, they are introduced in the "Appendix", section 7 of this paper.

The main theorems in this paper are Theorem 5.6 and Theorem 6.6, which give the exact power series expansions of the matrix coefficients of the spherical and non-spherical principal series representations. We express the matrix coefficients by the linear combinations of the power series solutions. The method we use is classical (for example, introduced in [3]). We investigate a part of the monodromy data of our holonomic system to have the unique solutions invariant under the fundamental group of the regular part of the split Cartan subgroup in $SL(3,\mathbf{R})$.

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2. Preliminaries

2.1. Notation

Let $G = SL(3,\mathbf{R})$ and fix $K = SO(3,\mathbf{R})$ as a maximal compact subgroup of G, and set $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{sl}(3,\mathbf{R}), \mathfrak{k} = \mathrm{Lie}(K) = \mathfrak{so}(3)$. Put

$$A := \left\{ \mathrm{diag}(a_1, a_2, a_3) \in G \,\middle|\, \prod_{i=1}^{3} a_i = 1, a_i \in \mathbf{R}_{>0} \right\}$$

and set $\mathfrak{a} = \mathrm{Lie}(A)$. The Cartan involution $\theta : G \to G$ is defined by $g \mapsto ({}^t g)^{-1}$ $(g \in G)$, and its Lie algebra version is $\theta : \mathfrak{g} \to \mathfrak{g}, X \mapsto -{}^t X$. Then

$$K = G^{\theta} = \{g \in G |\, \theta(g) = g\}$$

and

$$\mathfrak{k} = \mathfrak{g}^{\theta} = \{X \in \mathfrak{g} |\, \theta(X) = X\}.$$

Put

$$\mathfrak{p} = \mathfrak{g}^{-\theta} = \{X \in \mathfrak{g} |\, \theta(X) = -X\}.$$

Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, called the Cartan decomposition. Let $E_{i,j}$ $(1 \leq i, j \leq 3)$ be the matrix unit with 1 at the (i,j)-th entry and 0 at other entries. Put

$$H_{i,j} := E_{i,i} - E_{j,j} \in \mathfrak{a}\, (i \neq j).$$

Put $X_{i,j} = E_{i,j} + E_{j,i}\, (i \neq j) \in \mathfrak{p}$ and $K_{i,j} = E_{i,j} - E_{j,i}\, (i \neq j) \in \mathfrak{k}$.

2.2. The principal series representations

Let P_0 be a minimal parabolic subgroup of G given by the upper triangular matrices in G, and $P_0 = MAN$ be the Langlands decomposition of P_0 with

$$M = K \cap \{\text{diagonals in}\, G\}, \text{ and}$$
$$N = \left\{ \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \in G \,\middle|\, x_i \in \mathbf{R}, i = 1,2,3 \right\}.$$

To define a principal series representation with respect to the minimal parabolic subgroup P_0 of G, we firstly fix a character σ of M and a linear form $\nu \in \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C} = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$. We write $\nu(\mathrm{diag}(t_1, t_2, t_3)) = \nu_1 t_1 + \nu_2 t_2$. Then we can define a representation $\sigma \otimes a^{\nu}$ of MA, and extend this to P_0 by the identification $P_0/N \simeq MA$. Then we set

$$\pi_{\sigma,\nu} = C^{\infty}\mathrm{Ind}_{P_0}^{G}(\sigma \otimes a^{\nu+\rho} \otimes 1_N).$$

Here ρ is the half sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$ given by $a^{\rho} = a_1^2 a_2$, for $a = \mathrm{diag}(a_1, a_2, a_3) \in A$. The representation space is

$$C^{\infty}_{(M,\sigma)}(K) = \{f \in C^{\infty}(K) | f(mk) = \sigma(m) f(k), m \in M, k \in K\}$$

and the action of G is defined by

$$(\pi(x)f)(k) = a(kx)^{\nu+\rho} f(\kappa(kx))\, (x \in G, k \in K).$$

Here, for $g \in G$, $g = n(g)a(g)\kappa(g)$ $(n(g) \in N, a(g) \in A, \kappa(g) \in K)$ is the Iwasawa decomposition. Next, we define characters σ_j $(j = 0,1,2,3)$ of M as follows. The group M consisting of four elements is a finite abelian group of (2,2)-type, and its elements except for the unity are given by

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since M is commutative, all irreducible unitary representations of M are 1-dimensional. For any $\sigma \in \widehat{M}$, we have $\sigma^2 = 1$. Therefore, the set $\widehat{M}$ consisting of 4 characters $\{\sigma_j | j = 0,1,2,3\}$, where each σ_j, except for the trivial character σ_0, is specified by the following table of values at the elements m_i $(i = 1,2,3)$.

	m_1	m_2	m_3
σ_1	1	-1	-1
σ_2	-1	1	-1
σ_3	-1	-1	1

The correspondence of a character of M and the minimal K-type of $\pi_{\sigma,\nu}$ is as follows ([6]).

PROPOSITION 2.1. 1) *If σ is the trivial character of M, the representation $\pi_{\sigma,\nu}$ is spherical or class one. That is, it has a unique K-invariant vecter in $H_{\sigma,\nu}$.*
2) *If σ is not trivial, the minimal K-type of the restriction $\pi_{\sigma,\nu}|_K$ to K is a 3-dimensional representation of K, which is isomorphic to the unique standard one (τ_2, V_2). In this case, we call $\pi_{\sigma,\nu}$ the non-spherical principal series representation. The multiplicity of this minimal K-type is one:*

$$\dim_{\mathbf{C}}\mathrm{Hom}_K(\tau_2, H_{\sigma,\nu}) = 1.$$

2.3. The definition of spherical functions

Let (π, H_π) be the principal series representation of $G = SL(3,\mathbf{R})$. We want to study the matrix coefficient

$$\Phi_{w,v} : G \to \mathbf{C}, \quad g \mapsto \langle w, \pi(g)v\rangle \quad (w \in H_\pi^*, v \in H_\pi).$$

Let (τ_L, V_L) be a K-type of H_π^* and (τ_R, V_R) be a K-type of H_π. And let $\iota : \tau_L \boxtimes \tau_R \to \pi^* \boxtimes \pi$ be a $K\times K$ embedding. The bilinear form $(w,v) \mapsto \Phi_{w,v}$ is the element of $\mathrm{Hom}_{G\times G}(H_\pi^* \otimes H_\pi, C^\infty(G))$. We define a homomorphism $\mathrm{Hom}_{G\times G}(H_\pi^* \otimes H_\pi, C^\infty(G)) \to \mathrm{Hom}_{K\times K}(V_L \otimes V_R, C^\infty(G))$ by $\Phi \mapsto \Phi \circ \iota$. The space $\mathrm{Hom}_{K\times K}(V_L \otimes V_R, C^\infty(G))$ is identified with a space

$$\begin{aligned} &C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K) \\ &\qquad := \{F : G \to V_L^* \otimes V_R^* | F(k_1 g k_2) \\ &\qquad\qquad = (\tau_L^* \boxtimes \tau_R^*)(k_1, k_2^{-1})F(g),\ k_1, k_2 \in K, g \in G\} \end{aligned}$$

by the correspondence

$$\langle F_\phi(g), v_1 \otimes v_2\rangle = \phi(v_1 \otimes v_2)(g) \quad (\forall (v_1, v_2) \in V_L \times V_R)$$

for $\phi \in \mathrm{Hom}_{K\times K}(V_L \otimes V_R, C^\infty(G))$, $F_\phi \in C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K)$. Hence the matrix coefficient

$$\phi(v_1 \otimes v_2)(g) = \langle v_1, \pi(g)v_2\rangle$$

with $v_1 \in V_L$, $v_2 \in V_R$ is determined by $F_\phi \in C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K)$. Elements of $C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K)$ are called spherical functions. In particular, the element of $C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K)$ which is realized as the image of the bilinear form $(w,v) \mapsto \Phi_{w,v}$ ($w \in H_\pi^*, v \in H_\pi$) is called the spherical function attached to the principal series representation π. Because of the Cartan double coset decomposition $G = KAK$, spherical functions are determined by their restriction to A.

3. The Double Coset Cartan Decomposition

Because G has the double coset decomposition $G = KAK$, we consider the decomposition of the standard elements in $\mathfrak{p}$ with respect to the double coset decomposition:

$$\mathfrak{g} = \mathrm{Ad}(a^{-1})\mathfrak{k} + \mathfrak{a} + \mathfrak{k}.$$

Here $a \in A$ is a regular element in A. For $x \in \mathbf{R}_{>0}$, put $sh(x) = \frac{1}{2}(x - \frac{1}{x})$, $ch(x) = \frac{1}{2}(x + \frac{1}{x})$. We have the following decomposition:

Lemma 3.1. *We have*

$$X_{i,j} = -\frac{1}{sh(\frac{a_i}{a_j})}\mathrm{Ad}(a^{-1})K_{i,j} + 0 + \frac{ch(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_j})}K_{i,j} \quad ;$$

$$H_{i,j} = 0 + H_{i,j} + 0$$

with respect to the decomposition $\mathfrak{g} = \mathrm{Ad}(a^{-1})\mathfrak{k} + \mathfrak{a} + \mathfrak{k}$.

4. The $(\mathfrak{g}, K)$-Modules of Principal Series Representations

4.1. The Capelli elements

The center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ has two independent generators, and they are obtained as Capelli elements because

$\mathfrak{g} = \mathfrak{sl}_3$ is of type A_2 (see [2]). For $i = 1,2,3$, we put

$$E'_{i,i} = E_{i,i} - \frac{1}{3}\left(\sum_{k=1}^{3} E_{k,k}\right).$$

The following proposition ([6], Proposition 3.1) gives the explicit description of the independent generators of $Z(\mathfrak{g})$.

PROPOSITION 4.1. *The independent generators $\{Cp_2, Cp_3\}$ of $Z(\mathfrak{g})$ are given as follows:*

$$\begin{aligned} Cp_2 =& (E'_{1,1} - 1)E'_{2,2} + E'_{2,2}(E'_{3,3} + 1) + (E'_{1,1} - 1)(E'_{3,3} + 1) \\ & - E_{2,3}E_{3,2} - E_{1,3}E_{3,1} - E_{1,2}E_{2,1} \\ Cp_3 =& (E'_{1,1} - 1)E'_{2,2}(E'_{3,3} + 1) + E_{1,2}E_{2,3}E_{3,1} + E_{1,3}E_{2,1}E_{3,2} \\ & - (E'_{1,1} - 1)E_{2,3}E_{3,2} - E_{1,3}E'_{2,2}E_{3,1} - E_{1,2}E_{2,1}(E'_{3,3} + 1). \end{aligned}$$

4.2. Reduction of Capelli elements

To compute the actions of Capelli elements on spherical functions attached to the spherical principal series, we may regard the above two elements as elements in $Z(\mathfrak{g})\,(\mathrm{mod} U(\mathfrak{g})\,\mathfrak{k})$, because these functions are annihilated by the right action of $\mathfrak{k}$. Cp_2, Cp_3 $(\mathrm{mod} U(\mathfrak{g})\mathfrak{k})$ are given in [6]. They are as follows:

LEMMA 4.2. *The Capelli elements Cp_2,Cp_3 satisfy the following congruences:*

$$\begin{aligned} Cp_2 \equiv& (E'_{1,1} - 1)E'_{2,2} + E'_{2,2}(E'_{3,3} + 1) + (E'_{1,1} - 1)(E'_{3,3} + 1) \\ & - E_{2,3}^2 - E_{1,3}^2 - E_{1,2}^2 \quad (\mathrm{mod}\, U(\mathfrak{g})\mathfrak{k}), \\ Cp_3 \equiv& (E'_{1,1} - 1)E'_{2,2}(E'_{3,3} + 1) + E_{1,2}E_{2,3}E_{1,3} + E_{1,3}E_{1,2}E_{2,3} - E_{1,3}^2 \\ & - E_{2,3}^2(E'_{1,1} - 1) - E_{1,3}^2E'_{2,2} - E_{1,2}^2(E'_{3,3} + 1) \\ & (\mathrm{mod}\, U(\mathfrak{g})\mathfrak{k}). \end{aligned}$$

4.3. Eigenvalues of Cp_2, Cp_3

In order to construct the partial differential equations satisfied by spherical functions attached to the spherical principal series, we have to compute the eigenvalues of the actions of the Capelli elements Cp_2, Cp_3. For the spherical principal series, $\sigma = \sigma_0$ is the trivial character of M. Let f_0 be the generator of the minimal K-type in $H_{\sigma_0,\nu}$ normalized such that $f_0|K \equiv 1$. The actions of Cp_2, Cp_3 on f_0 are computed in [6], and the result is as follows:

PROPOSITION 4.3. *The Capelli elements Cp_2, Cp_3 act on f_0 by scalar multiples, and the eigenvalues are given as follows:*

$$Cp_2 f_0 = S_2\left(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(-\nu_1 + 2\nu_2), -\frac{1}{3}(\nu_1 + \nu_2)\right) f_0,$$

$$Cp_3 f_0 = S_3\left(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(-\nu_1 + 2\nu_2), -\frac{1}{3}(\nu_1 + \nu_2)\right) f_0.$$

Here, $S_2(a,b,c) = ab + bc + ca$, $S_3(a,b,c) = abc$.

5. The Matrix Coefficient of the Spherical Principal Series Representation

5.1. Construction of the differential equations

We put

$$y_1 = y_1(a) := a_1/a_2,\ y_2 = y_2(a) := a_2/a_3$$

for $a = \mathrm{diag}(a_1, a_2, a_3) \in A$. By definition of the action of Lie algebra, we have the following formula.

LEMMA 5.1. *For $f(y_1, y_2) = f(a) \in C^\infty(A)$, we have*

$$H_{1,2} f = \left(2y_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial y_2}\right) f,\ H_{2,3} f = \left(-y_1 \frac{\partial}{\partial y_1} + 2y_2 \frac{\partial}{\partial y_2}\right) f.$$

Now we want to construct the partial differential equations satisfied by spherical functions attached to the spherical principal series representation $\pi_{\sigma_0,\nu}$. We define differential operators ∂_1, ∂_2 by

$$\partial_i := y_i \frac{\partial}{\partial y_i} \quad (i = 1, 2).$$

By direct computations, we have the following two lemmas.

LEMMA 5.2. *For $1 \le i, j \le 3$ such that $i \neq j$, we have*

$$[K_{i,j}, \mathrm{Ad}(a^{-1})K_{i,j}] = -2sh\left(\frac{a_i}{a_j}\right) H_{i,j}.$$

LEMMA 5.3. *For $i, j, k \in \{1,2,3\}$ such that $i \neq j, j \neq k, k \neq i$, we have*

$$[K_{i,j}, \mathrm{Ad}(a^{-1})K_{j,k}] = \frac{sh(\frac{a_j}{a_k})}{sh(\frac{a_i}{a_k})}\mathrm{Ad}(a^{-1})K_{i,k} + \frac{sh(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_k})}K_{i,k},$$

$$[K_{i,j}, \mathrm{Ad}(a^{-1})K_{k,i}] = \frac{sh(\frac{a_i}{a_k})}{sh(\frac{a_j}{a_k})}\mathrm{Ad}(a^{-1})K_{j,k} - \frac{sh(\frac{a_i}{a_j})}{sh(\frac{a_j}{a_k})}K_{j,k}.$$

By combining Lemma 5.1, Lemma 5.2, and Lemma 5.3, the actions of Cp_2, Cp_3 in Lemma 4.2 on f_0 are obtained by direct computations. The eigenvalues are obtained in Proposition 4.3. Therefore, we can construct two differential equations characterized by the actions of Capelli elements Cp_2, Cp_3.

Let us compute the action of Cp_2 on the bi- K-invariant spherical function $F(y_1, y_2) \in C^\infty(K\backslash G/K)|_A$. Firstly, since we have

$$E'_{1,1} = \frac{2}{3}H_{1,2} + \frac{1}{3}H_{2,3},$$

$$E'_{2,2} = -\frac{1}{3}H_{1,2} + \frac{1}{3}H_{2,3},$$

$$E'_{3,3} = -\frac{1}{3}H_{1,2} - \frac{2}{3}H_{2,3},$$

the actions of $E'_{1,1}$, $E'_{2,2}$, $E'_{3,3}$ on F are given by ∂_1, $-\partial_1 + \partial_2$, $-\partial_2$ respectively. Therefore, the action of $(E'_{1,1} - 1)E'_{2,2} + E'_{2,2}(E'_{3,3} + 1) + (E'_{1,1} - 1)(E'_{3,3} + 1)$ on F is given by

$$\begin{aligned}&(\partial_1 - 1)(-\partial_1 + \partial_2) + (-\partial_1 + \partial_2)(-\partial_2 + 1) + (\partial_1 - 1)(-\partial_2 + 1)\\ &= -\partial_1^2 + \partial_1\partial_2 - \partial_2^2 + \partial_1 + \partial_2 - 1.\end{aligned}$$

Next, since

$$E_{2,3} = \frac{1}{2}(K_{2,3} + X_{2,3})$$
$$= \frac{1}{2}\left\{-\frac{1}{sh(\frac{a_2}{a_3})}\mathrm{Ad}(a^{-1})K_{2,3} + \left(\frac{ch(\frac{a_2}{a_3})}{sh(\frac{a_2}{a_3})} + 1\right)K_{2,3}\right\}$$

and F is annihilated by the action of $\mathrm{Ad}(a^{-1})\mathfrak{k}U(\mathfrak{g})$ and $U(\mathfrak{g})\mathfrak{k}$, the action of $E_{2,3}^2$ on F is equivalent to that of

$$-\frac{1}{4sh(\frac{a_2}{a_3})}\left(\frac{ch(\frac{a_2}{a_3})}{sh(\frac{a_2}{a_3})} + 1\right)K_{2,3}\mathrm{Ad}(a^{-1})K_{2,3}.$$

By using Lemma 5.2, this equals

$$-\frac{1}{4sh(\frac{a_2}{a_3})}\left(\frac{ch(\frac{a_2}{a_3})}{sh(\frac{a_2}{a_3})} + 1\right)\cdot\left\{-2sh\left(\frac{a_2}{a_3}\right)H_{2,3}\right\} = \frac{1}{2}\left(\frac{ch(\frac{a_2}{a_3})}{sh(\frac{a_2}{a_3})} + 1\right)H_{2,3}.$$

Since $\frac{ch(\frac{a_2}{a_3})}{sh(\frac{a_2}{a_3})} = \frac{y_2^2+1}{y_2^2-1}$ and the action of $H_{2,3}$ is given by $-\partial_1+2\partial_2$, we conclude that

$$(E_{2,3}^2F)(y_1,y_2) = \frac{1}{2}\left(\frac{y_2^2+1}{y_2^2-1} + 1\right)(-\partial_1 + 2\partial_2)F(y_1,y_2).$$

Similarly, we have

$$(E_{1,3}^2F)(y_1,y_2) = \frac{1}{2}\left(\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} + 1\right)(\partial_1 + \partial_2)F(y_1,y_2),$$

$$(E_{1,2}^2F)(y_1,y_2) = \frac{1}{2}\left(\frac{y_1^2+1}{y_1^2-1} + 1\right)(2\partial_1 - \partial_2)F(y_1,y_2).$$

From Proposition 4.3, the eigenvalue χ_{Cp_2} is given by

$$\chi_{Cp_2} = -\frac{1}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2).$$

By combining these results, we have following differential equation:

$$
\begin{aligned}
&(-\partial_1^2+\partial_1\partial_2-\partial_2^2+\partial_1+\partial_2-1)F(y_1,y_2)\\
&-\frac{1}{2}\left(-\frac{y_2^2+1}{y_2^2-1}+\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}+2\frac{y_1^2+1}{y_1^2-1}+2\right)\partial_1F(y_1,y_2)\\
&-\frac{1}{2}\left(2\frac{y_2^2+1}{y_2^2-1}+\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}-\frac{y_1^2+1}{y_1^2-1}+2\right)\partial_2F(y_1,y_2)\\
&=-\frac{1}{3}(\nu_1^2-\nu_1\nu_2+\nu_2^2)F(y_1,y_2).
\end{aligned}
$$

By multiplying both sides by -2, we have

$$
\begin{aligned}
&2(\partial_1^2-\partial_1\partial_2+\partial_2^2)F(y_1,y_2)\\
&+\left(-\frac{y_2^2+1}{y_2^2-1}+\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}+2\frac{y_1^2+1}{y_1^2-1}\right)\partial_1F(y_1,y_2)\\
&+\left(2\frac{y_2^2+1}{y_2^2-1}+\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}-\frac{y_1^2+1}{y_1^2-1}\right)\partial_2F(y_1,y_2)\\
&+\left\{-\frac{2}{3}(\nu_1^2-\nu_1\nu_2+\nu_2^2)+2\right\}F(y_1,y_2)=0.
\end{aligned}
$$

The construction of the differential equation characterized by the action of Cp_3 is more complicated, but the way is similar. We omit this computation. After all, we have the following two differential equations.

THEOREM 5.4. *Let* $F\in C^\infty_{\mathbf{1},\mathbf{1}}(K\backslash G/K)$ *be a spherical function attached to the spherical principal series representation* $\pi_{\sigma_0,\nu}$. *Then its restriction to* A*:* $F|_A=F(y_1,y_2)$ *satisfies two partial differential equations:*

$$
\begin{aligned}
&2(\partial_1^2-\partial_1\partial_2+\partial_2^2)F\\
&+\left(-\frac{y_2^2+1}{y_2^2-1}+\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}+2\frac{y_1^2+1}{y_1^2-1}\right)\partial_1F\\
&+\left(2\frac{y_2^2+1}{y_2^2-1}+\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}-\frac{y_1^2+1}{y_1^2-1}\right)\partial_2F\\
&+\left\{-\frac{2}{3}(\nu_1^2-\nu_1\nu_2+\nu_2^2)+2\right\}F=0,
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&\partial_1^2\partial_2 F - \partial_1\partial_2^2 F + \left(-1 + \frac{y_2^2}{y_2^2-1} + \frac{y_1^2y_2^2}{y_1^2y_2^2-1}\right)\partial_1^2 F \\
&+ \left(-\frac{2y_2^2}{y_2^2-1} + \frac{2y_1^2}{y_1^2-1}\right)\partial_1\partial_2 F + \left(1 - \frac{y_1^2}{y_1^2-1} - \frac{y_1^2y_2^2}{y_1^2y_2^2-1}\right)\partial_2^2 F \\
&+ \left(\frac{1}{2} - \frac{y_1^2y_2^2}{(y_2^2-1)(y_1^2y_2^2-1)} + \frac{3y_1^2y_2^2}{(y_1^2-1)(y_2^2-1)}\right. \\
&\left.+\frac{y_1^2y_2^2}{(y_1^2-1)(y_1^2y_2^2-1)} - \frac{y_1^2y_2^2+1}{2(y_1^2y_2^2-1)} - \frac{y_2^2}{y_2^2-1} - \frac{2y_1^2}{y_1^2-1}\right)\partial_1 F \\
&+ \left(-\frac{3}{2} - \frac{y_1^2y_2^2}{(y_2^2-1)(y_1^2y_2^2-1)} - \frac{3y_1^2y_2^2}{(y_1^2-1)(y_2^2-1)}\right. \\
&\left.+\frac{y_1^2y_2^2}{(y_1^2-1)(y_1^2y_2^2-1)} - \frac{y_1^2y_2^2+1}{2(y_1^2y_2^2-1)} + \frac{2y_2^2}{y_2^2-1} + \frac{y_1^2}{y_1^2-1}\right)\partial_2 F \\
&+ \frac{1}{27}(2\nu_1-\nu_2)(2\nu_2-\nu_1)(\nu_1+\nu_2)F = 0.
\end{aligned} \tag{5.2}
$$

5.2. The expansion of the matrix coefficients in terms of the power series around $y_1 = y_2 = 0$

For the spherical function $F \in C^\infty_{\mathbf{1},\mathbf{1}}(K\backslash G/K)$ above, we want to find its series expansion at the origin $y_1 = 0$, $y_2 = 0$ by solving (5.1) and (5.2). Firstly, we put

$$
F(y_1, y_2) = \sum_{n,m=0}^{\infty} a_{n,m} y_1^{n+\mu_1} y_2^{m+\mu_2} \quad (a_{0,0} \neq 0). \tag{5.3}
$$

The first task is to compute the characteristic roots (μ_1, μ_2). By substituting (5.3) for F into the equation (5.1) , and picking up the coefficient of $y_1^{n+\mu_1} y_2^{m+\mu_2}$, we have the following equation satisfied by $\{a_{n,m}\}$:

$$
\begin{aligned}
&\{2(n'-4)^2-2(n'-4)(m'-4)+2(m'-4)^2\\
&+2(n'-4)+2(m'-4)+\lambda\}a_{n-4,m-4}\\
&+\{-2(n'-4)^2+2(n'-4)(m'-2)-2(m'-2)^2\\
&-4(n'-4)+2(m'-2)-\lambda\}a_{n-4,m-2}\\
&+\{-2(n'-2)^2+2(n'-2)(m'-4)-2(m'-4)^2\\
&+2(n'-2)-4(m'-4)-\lambda\}a_{n-2,m-4}\\
&+\{2(n'-2)^2-2(n'-2)m'+2m'^2+2(n'-2)-4m'+\lambda\}a_{n-2,m}\\
&+\{2n'^2-2n'(m'-2)+2(m'-2)^2-4n'+2(m'-2)+\lambda\}a_{n,m-2}\\
&+(-2n'^2+2n'm'-2m'^2+2n'+2m'-\lambda)a_{n,m}=0.
\end{aligned}
\tag{5.4}
$$

Here, $\lambda := -\frac{2}{3}(\nu_1^2-\nu_1\nu_2+\nu_2^2)+2$, $n' := n+\mu_1$, $m' = m+\mu_2$, and $a_{i,j}=0$ if $i<0$ or $j<0$.

PROPOSITION 5.5. *The characteristic roots (μ_1,μ_2) take following six values:*

$$
\begin{aligned}
(\mu_1,\mu_2) = &\left(\frac{1}{3}(2\nu_1-\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1\right),\\
&\left(\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1\right),\\
&\left(\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1\right),\\
&\left(-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1\right),\\
&\left(\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1\right),\\
&\left(-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1\right).
\end{aligned}
\tag{5.5}
$$

PROOF. Because $a_{i,j}=0$ (if $i<0$ or $j<0$), and $a_{0,0}\neq 0$, by putting $n=m=0$ in (5.4), we have

$$-2(\mu_1^2-\mu_1\mu_2+\mu_2^2-\mu_1-\mu_2)-\lambda=0.$$

This equation is equivalent to

$$(\mu_1-1)^2-(\mu_1-1)(\mu_2-1)+(\mu_2-1)^2=\frac{1}{3}(\nu_1^2-\nu_1\nu_2+\nu_2^2). \tag{5.6}$$

Next, by computing the recurrence equation given by equation (5.2), and substituting $n=m=0$ in the coefficient of $a_{n,m}$, we have

$$\mu_1^2\mu_2-\mu_1\mu_2^2-\mu_1^2+\mu_2^2+\mu_1-\mu_2=-\frac{1}{27}(2\nu_1-\nu_2)(2\nu_2-\nu_1)(\nu_1+\nu_2).$$

This equation is equivalent to

$$\begin{aligned}&\{(\mu_1-1)-(\mu_2-1)\}(\mu_1-1)(\mu_2-1)\\&=-\frac{1}{27}(2\nu_1-\nu_2)(2\nu_2-\nu_1)(\nu_1+\nu_2).\end{aligned} \tag{5.7}$$

By combining (5.6) and (5.7), we have the result. □

Since the explicit forms of these power series are quite complicated, and we do not use them at all in the computation of c-functions, we leave them to the "Appendix", section 7 of this paper.

Now we have known that the equations (5.1) and (5.2) have six power series solutions corresponding to the six characteristic roots in Proposition 5.5. And the coefficients of each power series satisfy the recurrence relation (5.4). For a characteristic root (α,β), we express the power seires solution corresponding to (α,β) by $\psi_{\alpha,\beta}$. We assume that the constant term of $\psi_{\alpha,\beta}$ is 1. By Proposition 5.5, β takes following three values:

$$\beta_1=\frac{1}{3}(\nu_1+\nu_2)+1,\beta_2=-\frac{1}{3}(2\nu_1-\nu_2)+1,\beta_3=-\frac{1}{3}(2\nu_2-\nu_1)+1.$$

And for each β_i, we have two power series solutions. Therefore, we can write matrix coefficient F by

$$F(y_1,y_2)=\sum_{i=1}^{3}c_ia_i(y_1,y_2)y_2^{\beta_i}.$$

Here, c_i $(i=1,2,3)$ are some constants and $a_i(y_1,y_2)$ $(i=1,2,3)$ are some analytic functions around $y_2=0$. By substituting $a_i(y_1,y_2)y_2^{\beta_i}$ into the equation (5.1), we have

$$\begin{aligned}
&y_2^{\beta_i}\Big\{2(\partial_1^2 a_i(y_1,y_2) - \partial_1\partial_2 a_i(y_1,y_2) - \beta_i\partial_1 a_i(y_1,y_2) \\
&+ \partial_2^2 a_i(y_1,y_2) + 2\beta_i\partial_2 a_i(y_1,y_2) + \beta_i^2 a_i(y_1,y_2)) \\
&+ \left(-\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} + 2\frac{y_1^2+1}{y_1^2-1}\right)\partial_1 a_i(y_1,y_2) \\
&+ \left(2\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} - \frac{y_1^2+1}{y_1^2-1}\right)(\partial_2 a_i(y_1,y_2) + \beta_i a_i(y_1,y_2)) \\
&+ \left(-\frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2\right) a_i(y_1,y_2)\Big\} = 0.
\end{aligned}$$

Dividing both sides by $y_2^{\beta_i}$ and taking the limit $y_2 \to 0$, then we obtain

$$\begin{aligned}
&2\partial_1^2 a_i(y_1,0) + \left(2\frac{y_1^2+1}{y_1^2-1} - 2\beta_i\right)\partial_1 a_i(y_1,0) \\
&+ \left(2\beta_i^2 - \frac{y_1^2+1}{y_1^2-1}\beta_i - 3\beta_i - \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2\right) a_i(y_1,0) = 0.
\end{aligned} \tag{5.8}$$

We put $y_1^2 = u$, $f_i(u) = a_i(y_1,0)$. Then the equation (5.8) becomes

$$\begin{aligned}
&8u^2\frac{d^2 f_i}{du^2} + \left(4\frac{u+1}{u-1} - 4\beta_i + 8\right)u\frac{df_i}{du} \\
&+ \left(2\beta_i^2 - \frac{u+1}{u-1}\beta_i - 3\beta_i - \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2\right) f_i = 0.
\end{aligned} \tag{5.9}$$

Next, we put $f_i(u) = u^x g_i(u)$ ($x \in \mathbf{C}$) and substitute this into (5.9). Then we have

$$\begin{aligned}
&8u^2\frac{d^2 g_i}{du^2} + \left(4\frac{u+1}{u-1} - 4\beta_i + 8 + 16x\right)u\frac{dg_i}{du} \\
&+ \Big(8x^2 + (4-4\beta_i)x + 2\beta_i^2 - 4\beta_i - \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) \\
&+2 + \frac{8x - 2\beta_i}{u-1}\Big) g_i = 0.
\end{aligned} \tag{5.10}$$

Now, we choose x_i satisfying

$$8x_i^2 + (4-4\beta_i)x_i + 2\beta_i^2 - 4\beta_i - \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2 = 0$$

and substitute $x = x_i$ into (5.10). Then we have

$$8u^2\frac{d^2g_i}{du^2} + \left(4\frac{u+1}{u-1} - 4\beta_i + 8 + 16x_i\right)u\frac{dg_i}{du} + \frac{8x_i - 2\beta_i}{u-1}g_i = 0. \tag{5.11}$$

Finally, we put $u = \frac{1}{\zeta}$ and substitute this into (5.11). Then we have

$$\begin{aligned}&\zeta(\zeta-1)\frac{d^2g_i}{d\zeta^2} + \left(-\frac{1}{2}(\beta_i - 4x_i + 1) + \frac{1}{2}(\beta_i - 4x_i + 3)\zeta\right)\frac{dg_i}{d\zeta}\\ &+ (-x_i + \frac{\beta_i}{4})g_i = 0.\end{aligned} \tag{5.12}$$

(5.12) is a Gaussian hypergeometric differential equation, and if we define p_i, q_i as the complex numbers satisfying

$$1 + p_i + q_i = \frac{1}{2}(\beta_i - 4x_i + 3),$$

$$p_iq_i = -x_i + \frac{\beta_i}{4}$$

and define r_i by

$$r_i = \frac{1}{2}(\beta_i - 4x_i + 1),$$

then the solution is expressed by

$$\begin{aligned}&P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & p_i & ;\zeta \\ 1-r_i & r_i - p_i - q_i & q_i & \end{matrix}\right\}\\ &= P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & p_i & ;1-\zeta \\ r_i - p_i - q_i & 1-r_i & q_i & \end{matrix}\right\}.\end{aligned}$$

Here, $P\{\quad\}$ denotes Riemann's P-function. The regular solution is,

$$\begin{aligned}g_i(y_1) &= {}_2F_1\left(p_i, q_i; 1 - r_i + p_i + q_i; 1 - \zeta\right)\\ &= {}_2F_1\left(p_i, q_i; 1 - r_i + p_i + q_i; 1 - \frac{1}{y_1^2}\right).\end{aligned}$$

(See [9]). Since ${}_2F_1$ satisfies a formula ([5])

$$\begin{aligned}
&{}_2F_1(a,b;c;z) \\
&= (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}{}_2F_1\left(a, c-b; 1+a-b; \frac{1}{1-z}\right) \\
&\quad + (1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}{}_2F_1\left(b, c-a; 1+b-a; \frac{1}{1-z}\right),
\end{aligned} \tag{5.13}$$

we have

$$\begin{aligned}
g_i(y_1) =& y_1^{2p_i}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(q_i-p_i)}{\Gamma(1-r_i+q_i)\Gamma(q_i)}{}_2F_1(p_i, 1-r_i+p_i; 1+p_i-q_i; y_1^2) \\
&+ y_1^{2q_i}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(p_i-q_i)}{\Gamma(1-r_i+p_i)\Gamma(p_i)} \\
&\times {}_2F_1(q_i, 1-r_i+q_i; 1+q_i-p_i; y_1^2).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
a_i(y_1, 0) &= u^{x_i}g_i(y_1) = y_1^{2x_i}g_i(y_1) \\
&= y_1^{2(p_i+x_i)}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(q_i-p_i)}{\Gamma(1-r_i+q_i)\Gamma(q_i)} \\
&\quad \times {}_2F_1(p_i, 1-r_i+p_i; 1+p_i-q_i; y_1^2) \\
&\quad + y_1^{2(q_i+x_i)}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(p_i-q_i)}{\Gamma(1-r_i+p_i)\Gamma(p_i)} \\
&\quad \times {}_2F_1(q_i, 1-r_i+q_i; 1+q_i-p_i; y_1^2).
\end{aligned} \tag{5.14}$$

Next, for $i = 1,2,3$, we compute (x_i, p_i, q_i, r_i). Although the equation for x_i has two different solutions in general, the result doesn't depend on the choice of x_i.

A) In case of $\beta_i = \beta_1 = \frac{1}{3}(\nu_1+\nu_2)+1$, $x_1 = \frac{1}{3}\nu_1 - \frac{1}{6}\nu_2$, $p_1 = -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2}$, $q_1 = \frac{1}{2}$, $r_1 = -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1$.

B) In case of $\beta_i = \beta_2 = -\frac{1}{3}(2\nu_1 - \nu_2)+1$, $x_2 = -\frac{1}{6}\nu_1 + \frac{1}{3}\nu_2$, $p_2 = -\frac{1}{2}\nu_2 + \frac{1}{2}$, $q_2 = \frac{1}{2}$, $r_2 = -\frac{1}{2}\nu_2 + 1$.

C) In case of $\beta_i = \beta_3 = -\frac{1}{3}(2\nu_2 - \nu_1) + 1$, $x_3 = \frac{1}{3}\nu_1 - \frac{1}{6}\nu_2$, $p_3 = -\frac{1}{2}\nu_1 + \frac{1}{2}$, $q_3 = \frac{1}{2}$, $r_3 = -\frac{1}{2}\nu_1 + 1$.

By substituting these results into equation (5.14), we have

$$\begin{aligned}
&a_1(y_1, 0) \\
&= y_1^{\frac{1}{3}(2\nu_2-\nu_1)+1} \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})} \\
&\times {}_2F_1\left(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1; y_1^2\right) \\
&+ y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})} \\
&\times {}_2F_1\left(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1; y_1^2\right),
\end{aligned}$$

$$\begin{aligned}
&a_2(y_1, 0) \\
&= y_1^{-\frac{1}{3}(\nu_1+\nu_2)+1} \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})} {}_2F_1\left(-\frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_2 + 1; y_1^2\right) \\
&+ y_1^{\frac{1}{3}(2\nu_2-\nu_1)+1} \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})} {}_2F_1\left(\frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\nu_2 + 1; y_1^2\right),
\end{aligned}$$

$$\begin{aligned}
&a_3(y_1, 0) \\
&= y_1^{-\frac{1}{3}(\nu_1+\nu_2)+1} \frac{\Gamma(\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})} {}_2F_1\left(-\frac{1}{2}\nu_1 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_1 + 1; y_1^2\right) \\
&+ y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} \frac{\Gamma(-\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})} {}_2F_1\left(\frac{1}{2}\nu_1 + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\nu_1 + 1; y_1^2\right).
\end{aligned}$$

Therefore, by compairing the leading terms, we have

$$\begin{aligned}
&\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1}(y_1, y_2) \\
&= y_1^{\frac{1}{3}(2\nu_2-\nu_1)+1} y_2^{\frac{1}{3}(\nu_1+\nu_2)+1} {}_2F_1\left(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1; y_1^2\right) \\
&+ (\text{higher order terms with respect to } y_2),
\end{aligned}$$

$$\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}(y_1,y_2)$$
$$= y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} y_2^{\frac{1}{3}(\nu_1+\nu_2)+1} {}_2F_1\left(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1;y_1^2\right)$$
$$+ (\text{higher order terms with respect to } y_2),$$

$$\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}(y_1,y_2)$$
$$= y_1^{-\frac{1}{3}(\nu_1+\nu_2)+1} y_2^{-\frac{1}{3}(2\nu_1-\nu_2)+1} {}_2F_1\left(-\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};-\frac{1}{2}\nu_2+1;y_1^2\right)$$
$$+ (\text{higher order terms with respect to } y_2),$$

$$\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}(y_1,y_2)$$
$$= y_1^{\frac{1}{3}(2\nu_2-\nu_1)+1} y_2^{-\frac{1}{3}(2\nu_1-\nu_2)+1} {}_2F_1\left(\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};\frac{1}{2}\nu_2+1;y_1^2\right)$$
$$+ (\text{higher order terms with respect to } y_2),$$

$$\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}(y_1,y_2)$$
$$= y_1^{-\frac{1}{3}(\nu_1+\nu_2)+1} y_2^{-\frac{1}{3}(2\nu_2-\nu_1)+1} {}_2F_1\left(-\frac{1}{2}\nu_1+\frac{1}{2},\frac{1}{2};-\frac{1}{2}\nu_1+1;y_1^2\right)$$
$$+ (\text{higher order terms with respect to } y_2),$$

$$\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}(y_1,y_2)$$
$$= y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} y_2^{-\frac{1}{3}(2\nu_2-\nu_1)+1} {}_2F_1\left(\frac{1}{2}\nu_1+\frac{1}{2},\frac{1}{2};\frac{1}{2}\nu_1+1;y_1^2\right)$$
$$+ (\text{higher order terms with respect to } y_2),$$

and

$$
\begin{aligned}
F = c_1 & \left\{ \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right. \\
& \left. + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\} \\
& + c_2 \left\{ \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\
& \left. + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right\} \\
& + c_3 \left\{ \frac{\Gamma(\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right. \\
& \left. + \frac{\Gamma(-\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right\}.
\end{aligned} \tag{5.15}
$$

The next work is to determine the values of c_1, c_2, c_3. To do this, we apply the same method to y_2-part. That is, for $\alpha_1 = -\frac{1}{3}(\nu_1 + \nu_2) + 1, \alpha_2 = \frac{1}{3}(2\nu_1 - \nu_2) + 1, \alpha_3 = \frac{1}{3}(2\nu_2 - \nu_1) + 1$, we can write

$$F(y_1, y_2) = \sum_{i=1}^{3} d_i b_i(y_1, y_2) y_1^{\alpha_i}$$

and by investigating the differential equations satisfied by $b_i(0, y_2)$ $(i = 1, 2, 3)$ and compairing the leading terms with respect to y_2, we have

$$
\begin{aligned}
F = d_1 &\left\{ \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\
&\left. + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1} \right\} \\
+ d_2 &\left\{ \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1} \right. \\
&\left. + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1} \right\} \\
+ d_3 &\left\{ \frac{\Gamma(\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\
&\left. + \frac{\Gamma(-\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1} \right\}.
\end{aligned}
\tag{5.16}
$$

By comparing the coefficients of $\psi_{\alpha,\beta}$ in the equation (5.15) and (5.16), we can determine c_i, d_i $(i = 1, 2, 3)$ up to constant multiples. In particular,

$$
c_1 = \frac{\Gamma(-\frac{1}{2}\nu_1)\Gamma(-\frac{1}{2}\nu_2)}{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})}, \; c_2 = \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1)}{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})},
$$

$$
c_3 = \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_2)}{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})}.
$$

For the power series $\sum_{n,m=0}^{\infty} a_{n,m} y_1^{n+\mu_1} y_2^{m+\mu_2}$ $(a_{0,0} \neq 0)$, we call $a_{0,0}$ the first term of this power series. We completely determined the six coefficients appearing in the linear combination of power series. Summing up, we have the following theorem:

Theorem 5.6. *Let F be a spherical function attached to the spherical principal series representation $\pi_{\sigma_0,\nu}$, and $\psi_{\alpha,\beta}$ be the power series solution around $y_1 = y_2 = 0$ corresponding to the characteristic root (α, β) whose first term is equal to 1. Then we have*

$$
F = \frac{\Gamma(-\frac{1}{2}\nu_1)\Gamma(-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}
$$

$$+\frac{\Gamma(-\frac{1}{2}\nu_1)\Gamma(-\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}$$
$$\times\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}$$
$$+\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1)\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}$$
$$+\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1)\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}$$
$$+\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}$$
$$+\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})}$$
$$\times\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}.$$

6. The Matrix Coefficients of the Non-Spherical Principal Seires Representations

In this section, we investigate the matrix coefficients of the non-spherical principal series representations whose minimal K-type is the 3-dimensional tautological representation of K. Let $\tau_2 : K = SO(3) \hookrightarrow GL(3, \mathbf{R})$ be the tautological representation. Then we say that

$$\{s_1 = {}^t(1,0,0), s_2 = {}^t(0,1,0), s_3 = {}^t(0,0,1)\}$$

is the natural basis of this representation τ_2. We consider a spherical function $\Psi \in C^\infty_{\tau_2,\tau_2}(K\backslash G/K)$ attached to the non-spherical principal series representation $\pi_{\sigma_i,\nu}$ $(i = 1,2,3)$. Ψ can be written in terms of the basis $\{s_i | i = 1,2,3\}$:

$$\Psi(g) = \sum_{i=1}^{3}\sum_{j=1}^{3} d_{ij}(g) s_i^L \otimes s_j^R.$$

Note that Ψ satisfies

$$\Psi(k_1 g k_2^{-1}) = \sum_{i=1}^{3}\sum_{j=1}^{3} d_{ij}(g)(\tau_2(k_1)s_i^L) \otimes (\tau_2(k_2)s_j^R)$$

for $k_1, k_2 \in K$, $g \in G$.

LEMMA 6.1. *For $a \in A$, we have $d_{ij}(a) = 0$ if $i \neq j$.*

PROOF. A subgroup M of G is defined by

$$\begin{aligned} M = Z_K(A) &= \{k \in K | ak = ka \quad (\forall a \in A)\} \\ &= \{\mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3) | \epsilon_i \in \{\pm 1\}, \epsilon_1\epsilon_2\epsilon_3 = 1\}. \end{aligned}$$

Then for $m \in M$, $a \in A$, we have

$$\tau_L(m)\Psi(a) = \Psi(ma) = \Psi(am) = \tau_R(m^{-1})\Psi(a).$$

Therefore, for example, for $m_3 = \mathrm{diag}(-1, -1, 1) \in M$, we have

$$\tau_2(m_3) \otimes \Psi(a) = 1 \otimes \tau_2(m_3^{-1})\Psi(a).$$

Hence

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} (d_{ij}(a)) = (d_{ij}(a)) \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

From this, we have $d_{13}(a) = d_{31}(a) = d_{23}(a) = d_{32}(a) = 0$. Similarly, the actions of the other elements of M show that $d_{ij}(a) = 0$ if $i \neq j$. □

6.1. The action of Casimir operator

We use the same coordinate $y_1 = \frac{a_1}{a_2}, y_2 = \frac{a_2}{a_3}$ $(a = \mathrm{diag}(a_1, a_2, a_3) \in A)$ as in section 5. We compute the action of the Casimir element, which is an element of $Z(\mathfrak{g})$. The Casimir operator C of $SL(3,\mathbf{R})$ is decomposed into two parts with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$:

$$C = C(\mathfrak{p}) + C(\mathfrak{k}).$$

Here,

$$C(\mathfrak{p}) = \frac{2}{3}(H_{1,2}^2 + H_{1,2}H_{2,3} + H_{2,3}^2) + \frac{1}{2}\sum_{i<j} X_{i,j}^2,$$

$$C(\mathfrak{k}) = -\frac{1}{2}\sum_{i<j} K_{i,j}^2.$$

Firstly, we consider the action of $C(\mathfrak{p})$.

$$\begin{aligned}
&\left(\text{The action of } \quad \frac{2}{3}(H_{1,2}^2 + H_{1,2}H_{2,3} + H_{2,3}^2)\right)\\
&= \frac{2}{3}\{(2\partial_1 - \partial_2)^2 + (2\partial_1 - \partial_2)(-\partial_1 + 2\partial_2) + (-\partial_1 + 2\partial_2)^2\}\\
&= 2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2).
\end{aligned}$$

Next,

$$\begin{aligned}
X_{i,j}^2 &= \left\{-\frac{1}{sh(\frac{a_i}{a_j})}\mathrm{Ad}(a^{-1})K_{i,j} + \frac{ch(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_j})}K_{i,j}\right\}^2\\
&= \frac{1}{sh(\frac{a_i}{a_j})^2}(\mathrm{Ad}(a^{-1})K_{i,j})^2 - 2\frac{ch(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_j})^2}\mathrm{Ad}(a^{-1})K_{i,j}\cdot K_{i,j}\\
&+ \frac{ch(\frac{a_i}{a_j})^2}{sh(\frac{a_i}{a_j})^2}K_{i,j}^2 - \frac{ch(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_j})^2}[K_{i,j}, \mathrm{Ad}(a^{-1})K_{i,j}].
\end{aligned}$$

The bracket product above is given by Lemma 5.2. Therefore, we have

$$\begin{aligned}
\frac{1}{2}\sum_{i<j} X_{i,j}^2 &= \frac{1}{2}\frac{1}{sh(y_1)^2}(\mathrm{Ad}(a^{-1})K_{1,2})^2 - \frac{ch(y_1)}{sh(y_1)^2}\mathrm{Ad}(a^{-1})K_{1,2}\cdot K_{1,2}\\
&+ \frac{1}{2}\frac{ch(y_1)^2}{sh(y_1)^2}K_{1,2}^2 + \frac{ch(y_1)}{sh(y_1)}H_{1,2}\\
&+ \frac{1}{2}\frac{1}{sh(y_1y_2)^2}(\mathrm{Ad}(a^{-1})K_{1,3})^2 - \frac{ch(y_1y_2)}{sh(y_1y_2)^2}\mathrm{Ad}(a^{-1})K_{1,3}\cdot K_{1,3}\\
&+ \frac{1}{2}\frac{ch(y_1y_2)^2}{sh(y_1y_2)^2}K_{1,3}^2 + \frac{ch(y_1y_2)}{sh(y_1y_2)}H_{1,3}\\
&+ \frac{1}{2}\frac{1}{sh(y_2)^2}(\mathrm{Ad}(a^{-1})K_{2,3})^2 - \frac{ch(y_2)}{sh(y_2)^2}\mathrm{Ad}(a^{-1})K_{2,3}\cdot K_{2,3}\\
&+ \frac{1}{2}\frac{ch(y_2)^2}{sh(y_2)^2}K_{2,3}^2 + \frac{ch(y_2)}{sh(y_2)}H_{2,3}.
\end{aligned}$$

The actions of $(\mathrm{Ad}(a^{-1})K_{i,j})^2$, $(\mathrm{Ad}(a^{-1})K_{i,j})K_{i,j}$, $K_{i,j}^2$ on $\Psi(g) = \sum_i\sum_j d_{i,j}(g)s_i^L \otimes s_j^R$ are given by

$$(\mathrm{Ad}(a^{-1})K_{i,j})^2\Psi(a) = -d_{ii}(a)s_{ii}^{LR} - d_{jj}(a)s_{jj}^{LR},$$

$$(\mathrm{Ad}(a^{-1})K_{i,j})K_{i,j}\Psi(a) = d_{jj}(a)s_{ii}^{LR} + d_{ii}(a)s_{jj}^{LR},$$
$$K_{i,j}^2\Psi(a) = -d_{ii}(a)s_{ii}^{LR} - d_{jj}(a)s_{jj}^{LR}$$

on A. Here, we put $s_{ij}^{LR} := s_i^L \otimes s_j^R$. Therefore, we have

$$\begin{aligned}
&C(\mathfrak{p})\Psi(a)\\
&= 2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)\Psi(a)\\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1\Psi(a)\\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2\Psi(a)\\
&- \frac{1}{2}\frac{1}{sh(y_1)^2}\{d_{11}(a)s_{11}^{LR} + d_{22}(a)s_{22}^{LR}\}\\
&- \frac{1}{2}\frac{1}{sh(y_1y_2)^2}\{d_{11}(a)s_{11}^{LR} + d_{33}(a)s_{33}^{LR}\}\\
&- \frac{1}{2}\frac{1}{sh(y_2)^2}\{d_{22}(a)s_{22}^{LR} + d_{33}(a)s_{33}^{LR}\}\\
&+ \frac{ch(y_1)}{sh(y_1)^2}\{d_{22}(a)s_{11}^{LR} + d_{11}(a)s_{22}^{LR}\} + \frac{ch(y_1y_2)}{sh(y_1y_2)^2}\{d_{33}(a)s_{11}^{LR} + d_{11}(a)s_{33}^{LR}\}\\
&+ \frac{ch(y_2)}{sh(y_2)^2}\{d_{33}(a)s_{22}^{LR} + d_{22}(a)s_{33}^{LR}\}\\
&- \frac{1}{2}\frac{ch(y_1)^2}{sh(y_1)^2}\{d_{11}(a)s_{11}^{LR} + d_{22}(a)s_{22}^{LR}\}\\
&- \frac{1}{2}\frac{ch(y_1y_2)^2}{sh(y_1y_2)^2}\{d_{11}(a)s_{11}^{LR} + d_{33}(a)s_{33}^{LR}\}\\
&- \frac{1}{2}\frac{ch(y_2)^2}{sh(y_2)^2}\{d_{22}(a)s_{22}^{LR} + d_{33}(a)s_{33}^{LR}\}.
\end{aligned}$$

Next, the action of $C(\mathfrak{k}) = -\frac{1}{2}\sum_{i<j} K_{i,j}^2$ is given as follows:

$$\begin{aligned}
&C(\mathfrak{k})\Psi(a)\\
&= \frac{1}{2}\{d_{11}(a)s_{11}^{LR} + d_{22}(a)s_{22}^{LR}\} + \frac{1}{2}\{d_{22}(a)s_{22}^{LR} + d_{33}(a)s_{33}^{LR}\}\\
&+ \frac{1}{2}\{d_{11}(a)s_{11}^{LR} + d_{33}(a)s_{33}^{LR}\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(C\Psi)(a)\\
&= 2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)\Psi(a)\\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1\Psi(a)\\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2\Psi(a)\\
&- \frac{1}{sh(y_1)^2}\{d_{11}(a)s_{11}^{LR} + d_{22}(a)s_{22}^{LR}\} - \frac{1}{sh(y_1y_2)^2}\{d_{11}(a)s_{11}^{LR} + d_{33}(a)s_{33}^{LR}\}\\
&- \frac{1}{sh(y_2)^2}\{d_{22}(a)s_{22}^{LR} + d_{33}(a)s_{33}^{LR}\}\\
&+ \frac{ch(y_1)}{sh(y_1)^2}\{d_{22}(a)s_{11}^{LR} + d_{11}(a)s_{22}^{LR}\} + \frac{ch(y_1y_2)}{sh(y_1y_2)^2}\{d_{33}(a)s_{11}^{LR} + d_{11}(a)s_{33}^{LR}\}\\
&+ \frac{ch(y_2)}{sh(y_2)^2}\{d_{33}(a)s_{22}^{LR} + d_{22}(a)s_{33}^{LR}\}.
\end{aligned}$$

The next step is to compute the eigenvalue λ of the Casimir operator C. We compute the action on $f \in H_{\pi_{\sigma_i},\nu}$ such that $f(e) = 1$. Firstly, $\frac{2}{3}(H_{1,2}^2 + H_{1,2}H_{2,3} + H_{2,3}^2)$ acts on f by scalar multiplication. Its scalar is given by

$$\begin{aligned}
&\frac{2}{3}\{(\nu_1 - \nu_2 + 1)^2 + (\nu_1 - \nu_2 + 1)(\nu_2 + 1) + (\nu_2 + 1)^2\}\\
&= \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2 + 3\nu_1 + 3).
\end{aligned}$$

Next,

$$\begin{aligned}
\frac{1}{2}\sum_{i<j} X_{i,j}^2 - \frac{1}{2}\sum_{i<j} K_{i,j}^2 &= \frac{1}{2}\sum_{i<j}(E_{i,j} + E_{j,i})^2 - \frac{1}{2}\sum_{i<j}(E_{i,j} - E_{j,i})^2\\
&= \sum_{i<j}(E_{i,j}E_{j,i} + E_{j,i}E_{i,j}).
\end{aligned}$$

Since $Xf(e) = 0$ for $X \in \mathfrak{n}$, the action of $E_{i,j}E_{j,i}$ is 0. On the other hand, since

$$[E_{i,j}, E_{j,i}] = H_{i,j},$$

we have

$$E_{j,i}E_{i,j} = E_{i,j}E_{j,i} - H_{i,j}.$$

Thus

$$\begin{aligned}&\left(\frac{1}{2}\sum_{i<j} X_{i,j}^2 - \frac{1}{2}\sum_{i<j} K_{i,j}^2\right) f\\ &= -\left(\sum_{i<j} H_{i,j}\right) f\\ &= -2H_{1,3}f = -2(\nu_1+2)f.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\lambda &= \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2 + 3\nu_1 + 3) - 2(\nu_1+2)\\ &= \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) - 2.\end{aligned}$$

For $\Psi(g) = \sum_{i=1}^3 \sum_{j=1}^3 d_{ij}(g) s_i^L \otimes s_j^R \in C_{\tau_2,\tau_2}^\infty(K\backslash G/K)$, we put

$$d_{11}(a) = F(a) = F(y_1, y_2),$$

$$d_{22}(a) = G(a) = G(y_1, y_2),$$

$$d_{33}(a) = H(a) = H(y_1, y_2).$$

Then, by compairing the coefficients of s_{ii}^{LR} in both sides of the equation $C\Psi = \lambda\Psi$, we have the following theorem:

THEOREM 6.2. *Let $\Psi = {}^t(F,G,H)$ be a spherical function attached to the non-spherical principal series representation $\pi_{\sigma_i,\nu}$ $(i=1,2,3)$ restricted* *to A. Then, F, G, H satisfy the following differential equations:*

$$
\begin{aligned}
&2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)F(y_1, y_2) \\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1 F(y_1, y_2) \\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2 F(y_1, y_2) \\
&- \left(\frac{1}{sh(y_1)^2} + \frac{1}{sh(y_1y_2)^2}\right)F(y_1, y_2) \\
&+ \frac{ch(y_1)}{sh(y_1)^2}G(y_1, y_2) + \frac{ch(y_1y_2)}{sh(y_1y_2)^2}H(y_1, y_2) \\
&= \lambda F(y_1, y_2),
\end{aligned}
\tag{6.1}
$$

$$
\begin{aligned}
&2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)G(y_1, y_2) \\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1 G(y_1, y_2) \\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2 G(y_1, y_2) \\
&- \left(\frac{1}{sh(y_1)^2} + \frac{1}{sh(y_2)^2}\right)G(y_1, y_2) \\
&+ \frac{ch(y_1)}{sh(y_1)^2}F(y_1, y_2) + \frac{ch(y_2)}{sh(y_2)^2}H(y_1, y_2) \\
&= \lambda G(y_1, y_2),
\end{aligned}
\tag{6.2}
$$

$$
\begin{aligned}
&2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)H(y_1, y_2) \\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1 H(y_1, y_2) \\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2 H(y_1, y_2) \\
&- \left(\frac{1}{sh(y_2)^2} + \frac{1}{sh(y_1y_2)^2}\right)H(y_1, y_2) \\
&+ \frac{ch(y_1y_2)}{sh(y_1y_2)^2}F(y_1, y_2) + \frac{ch(y_2)}{sh(y_2)^2}G(y_1, y_2) \\
&= \lambda H(y_1, y_2).
\end{aligned}
\tag{6.3}
$$

Here, $\lambda = \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) - 2$.

6.2. The gradient operator

For a spherical function $\Psi(g) \in C^\infty_{\tau_2,\tau_2}(K\backslash G/K)$ attached to the non-spherical principal series representation $\pi_{\sigma_i,\nu}$ $(i = 1,2,3)$, we define the right gradient operator ∇^R as follows:

Definition 6.3. For the orthonormal basis $\{X_i\}_{i=1}^5$ of $\mathfrak{p}$, the right gradient operator ∇^R is defined by

$$\nabla^R\Psi := \sum_{i=1}^{5} R_{X_i}\Psi \otimes X_i^*.$$

Here, X_i^* is the dual basis of X_i with respect to the inner product $(X,Y) \in \mathfrak{p}\times\mathfrak{p} \to \mathrm{Tr}(XY) \in \mathbf{C}$.

If we take $\{H_{1,2}, H_{2,3}, X_{1,2}, X_{2,3}, X_{1,3}\}$ as a basis of $\mathfrak{p}$, its dual basis is $\{\frac{1}{3}(2H_{1,2}+H_{2,3}), \frac{1}{3}(H_{1,2}+2H_{2,3}), \frac{1}{2}X_{1,2}, \frac{1}{2}X_{2,3}, \frac{1}{2}X_{1,3}\}$. Therefore,

$$\begin{aligned}\nabla^R\Psi =& \frac{1}{3}R_{H_{1,2}}\Psi \otimes (2H_{1,2}+H_{2,3}) + \frac{1}{3}R_{H_{2,3}}\Psi\otimes(H_{1,2}+2H_{2,3}) \\ &+ \frac{1}{2}\sum_{i<j} R_{X_{i,j}}\Psi \otimes X_{i,j}.\end{aligned}$$

Claim 1. We define $\{w_i | 0 \le i \le 4\} \subset \mathfrak{p}_\mathbf{C} = \mathfrak{p}\otimes_\mathbf{R}\mathbf{C}$ by

$$\begin{aligned} w_0 &:= -2(H_{2,3} - \sqrt{-1}X_{2,3}) \\ w_4 &:= -2(H_{2,3} + \sqrt{-1}X_{2,3}) \\ w_2 &:= \frac{2}{3}(2H_{1,2}+H_{2,3}) \\ w_1 &:= X_{1,3} + \sqrt{-1}X_{1,2} \\ w_3 &:= -X_{1,3} + \sqrt{-1}X_{1,2}. \end{aligned}$$

Then $\{w_i | 0 \le i \le 4\}$ becomes a basis of $\mathfrak{p}_\mathbf{C}$.

With this basis, the gradient operator ∇^R is written as

$$\begin{aligned}\nabla^R\Psi =&\frac{1}{16}R_{w_4}\Psi\otimes w_0+\frac{1}{16}R_{w_0}\Psi\otimes w_4-\frac{1}{4}R_{w_3}\Psi\otimes w_1\\&-\frac{1}{4}R_{w_1}\Psi\otimes w_3+\frac{3}{8}R_{w_2}\Psi\otimes w_2\\&=\frac{1}{4}\left(\frac{1}{4}R_{w_4}\Psi\otimes w_0+\frac{1}{4}R_{w_0}\Psi\otimes w_4-R_{w_3}\Psi\otimes w_1\right.\\&\left.-R_{w_1}\Psi\otimes w_3+\frac{3}{2}R_{w_2}\Psi\otimes w_2\right).\end{aligned}$$

K acts on $\mathfrak{p}_{\mathbf{C}}$ by adjoint action. We denote this representation by (τ_4, W_4). By the Clebsh-Gordan theorem, $\tau_2\otimes\tau_4$ has the irreducible decomposition

$$\tau_2\otimes\tau_4\cong\tau_2\oplus\tau_4\oplus\tau_6.$$

Here, each τ_n is a $(n+1)$-dimensional irreducible representation of K. In this decomposition, the projector of K-modules

$$pr_2:\tau_2\otimes\tau_4\to\tau_2$$

is described as in the following table:

Table 1. Table of $pr_2(s_j\otimes w_k)$.

	w_0	w_1	w_2	w_3	w_4
s_1	0	$-\frac{1}{4}(s_3+\sqrt{-1}s_2)$	$-\frac{1}{3}s_1$	$\frac{1}{4}(s_3-\sqrt{-1}s_2)$	0
s_2	$\frac{1}{2}(s_2-\sqrt{-1}s_3)$	$-\frac{\sqrt{-1}}{4}s_1$	$\frac{1}{6}s_2$	$-\frac{\sqrt{-1}}{4}s_1$	$\frac{1}{2}(s_2+\sqrt{-1}s_3)$
s_3	$-\frac{1}{2}(s_3+\sqrt{-1}s_2)$	$-\frac{1}{4}s_1$	$\frac{1}{6}s_3$	$\frac{1}{4}s_1$	$\frac{1}{2}(-s_3+\sqrt{-1}s_2)$

$\nabla^R\Psi$ is a $(\tau_2\otimes\tau_2)\otimes\mathfrak{p}_{\mathbf{C}}$-valued function. Then, by mapping $s_i^L\otimes s_j^R\otimes w_k$ to $s_i^L\otimes s_j^R w_k$ (here, $s_j^R w_k := pr_2(s_j^R\otimes w_k)$) , we have a K-homomorphism

$$\tilde{pr}_2\circ\nabla^R : C^\infty_{\tau_2,\tau_2}(K\backslash G/K)\to C^\infty_{\tau_2,\tau_2}(K\backslash G/K).$$

Since the minimal K-type τ_2 is of multiplicity one, $\tilde{pr}_2\circ\nabla^R$ is a map of scalar multiplication.

We compute $4\tilde{pr}_2(\nabla^R\Psi)(a)$ for $\Psi(g)=\sum_i\sum_j d_{ij}(g)s_i^L\otimes s_j^R$, $a\in A$.

1)

$$\begin{aligned}\frac{1}{4}p\tilde{r}_2(R_{w_4}\Psi\otimes w_0)(a) &= \frac{1}{4}p\tilde{r}_2(R_{-2(H_{2,3}+\sqrt{-1}X_{2,3})}\Psi\otimes w_0)(a)\\ &= -\frac{1}{2}p\tilde{r}_2(R_{H_{2,3}}\Psi\otimes w_0)(a)\\ &\quad -\frac{\sqrt{-1}}{2}p\tilde{r}_2(R_{X_{2,3}}\Psi\otimes w_0)(a)\end{aligned}$$

Firstly,

$$\begin{aligned}&-\frac{1}{2}p\tilde{r}_2(R_{H_{2,3}}\Psi\otimes w_0)(a)\\ &= -\frac{1}{2}(-\partial_1+2\partial_2)\sum_{i=1}^{3}d_{ii}(a)s_i^L\otimes s_i^R w_0\\ &= -\frac{1}{4}(-\partial_1+2\partial_2)d_{22}(a)s_{22}^{LR}+\frac{\sqrt{-1}}{4}(-\partial_1+2\partial_2)d_{22}(a)s_{23}^{LR}\\ &+\frac{\sqrt{-1}}{4}(-\partial_1+2\partial_2)d_{33}(a)s_{32}^{LR}+\frac{1}{4}(-\partial_1+2\partial_2)d_{33}(a)s_{33}^{LR}.\end{aligned}$$

Next, since

$$X_{2,3} = -\frac{1}{sh(y_2)}\mathrm{Ad}(a^{-1})K_{2,3}+\frac{ch(y_2)}{sh(y_2)}K_{2,3},$$

we have

$$\begin{aligned}&-\frac{\sqrt{-1}}{2}p\tilde{r}_2(R_{X_{2,3}}\Psi\otimes w_0)(a)\\ &= -\frac{\sqrt{-1}}{2}p\tilde{r}_2\left(-\frac{1}{sh(y_2)}R_{\mathrm{Ad}(a^{-1})K_{2,3}}\Psi\otimes w_0+\frac{ch(y_2)}{sh(y_2)}R_{K_{2,3}}\Psi\otimes w_0\right)(a)\\ &= \left(\frac{1}{4sh(y_2)}d_{33}(a)-\frac{ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{22}^{LR}\\ &+\left(-\frac{\sqrt{-1}}{4sh(y_2)}d_{33}(a)+\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{23}^{LR}\\ &+\left(-\frac{\sqrt{-1}}{4sh(y_2)}d_{22}(a)+\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{32}^{LR}\\ &+\left(-\frac{1}{4sh(y_2)}d_{22}(a)+\frac{ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{33}^{LR}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{4}p\tilde{r}_2(R_{w_4}\Psi\otimes w_0)(a)\\
&= -\frac{1}{4}(-\partial_1+2\partial_2)d_{22}(a)s_{22}^{LR}+\frac{\sqrt{-1}}{4}(-\partial_1+2\partial_2)d_{22}(a)s_{23}^{LR}\\
&+\frac{\sqrt{-1}}{4}(-\partial_1+2\partial_2)d_{33}(a)s_{32}^{LR}+\frac{1}{4}(-\partial_1+2\partial_2)d_{33}(a)s_{33}^{LR}\\
&+\left(\frac{1}{4sh(y_2)}d_{33}(a)-\frac{ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{22}^{LR}\\
&+\left(-\frac{\sqrt{-1}}{4sh(y_2)}d_{33}(a)+\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{23}^{LR}\\
&+\left(-\frac{\sqrt{-1}}{4sh(y_2)}d_{22}(a)+\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{32}^{LR}\\
&+\left(-\frac{1}{4sh(y_2)}d_{22}(a)+\frac{ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{33}^{LR}.
\end{aligned}$$

2) Similarly, we have

$$\begin{aligned}
&\frac{1}{4}p\tilde{r}_2(R_{w_0}\Psi\otimes w_4)(a)\\
&= -\frac{1}{4}(-\partial_1+2\partial_2)d_{22}(a)s_{22}^{LR}-\frac{\sqrt{-1}}{4}(-\partial_1+2\partial_2)d_{22}(a)s_{23}^{LR}\\
&-\frac{\sqrt{-1}}{4}(-\partial_1+2\partial_2)d_{33}(a)s_{32}^{LR}+\frac{1}{4}(-\partial_1+2\partial_2)d_{33}(a)s_{33}^{LR}\\
&+\left(\frac{1}{4sh(y_2)}d_{33}(a)-\frac{ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{22}^{LR}\\
&+\left(\frac{\sqrt{-1}}{4sh(y_2)}d_{33}(a)-\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{23}^{LR}\\
&+\left(\frac{\sqrt{-1}}{4sh(y_2)}d_{22}(a)-\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{32}^{LR}\\
&+\left(-\frac{1}{4sh(y_2)}d_{22}(a)+\frac{ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{33}^{LR}.
\end{aligned}$$

3)

$$\begin{aligned}
&-p\tilde{r}_2(R_{w_3}\Psi\otimes w_1)(a)\\
&=p\tilde{r}_2(R_{X_{1,3}}\Psi\otimes w_1)(a)-\sqrt{-1}p\tilde{r}_2(R_{X_{1,2}}\otimes w_1)(a).
\end{aligned}$$

Firstly,

$$
\begin{aligned}
&\tilde{pr}_2(R_{X_{1,3}}\Psi\otimes w_1)(a)\\
&= -\frac{1}{sh(y_1y_2)}R_{\mathrm{Ad}(a^{-1})K_{1,3}}\left(\sum_{i=1}^{3} d_{ii}(a)s_i^L\otimes s_i^R\right)w_1\\
&+\frac{ch(y_1y_2)}{sh(y_1y_2)}R_{K_{1,3}}\left(\sum_{i=1}^{3} d_{ii}(a)s_i^L\otimes s_i^R\right)w_1\\
&= -\frac{1}{sh(y_1y_2)}(-d_{11}(a)s_3^L\otimes s_1^Rw_1+d_{33}(a)s_1^L\otimes s_3^Rw_1)\\
&-\frac{ch(y_1y_2)}{sh(y_1y_2)}(-d_{11}(a)s_1^L\otimes s_3^Rw_1+d_{33}(a)s_3^L\otimes s_1^Rw_1)\\
&=\left(\frac{1}{4sh(y_1y_2)}d_{33}(a)-\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{11}(a)\right)s_{11}^{LR}\\
&+\left(-\frac{\sqrt{-1}}{4sh(y_1y_2)}d_{11}(a)+\frac{\sqrt{-1}ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s_{32}^{LR}\\
&+\left(-\frac{1}{4sh(y_1y_2)}d_{11}(a)+\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s_{33}^{LR}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&-\sqrt{-1}\tilde{pr}_2(R_{X_{1,2}}\Psi\otimes w_1)(a)\\
&=\frac{\sqrt{-1}}{sh(y_1)}R_{\mathrm{Ad}(a^{-1})K_{1,2}}\left(\sum_{i=1}^{3} d_{ii}(a)s_i^L\otimes s_i^R\right)w_1\\
&-\frac{\sqrt{-1}ch(y_1)}{sh(y_1)}R_{K_{1,2}}\left(\sum_{i=1}^{3} d_{ii}(a)s_i^L\otimes s_i^R\right)w_1\\
&=\frac{\sqrt{-1}}{sh(y_1)}(-d_{11}(a)s_2^L\otimes s_1^Rw_1+d_{22}(a)s_1^L\otimes s_2^Rw_1)\\
&+\frac{\sqrt{-1}ch(y_1)}{sh(y_1)}(-d_{11}(a)s_1^L\otimes s_2^Rw_1+d_{22}(a)s_2^L\otimes s_1^Rw_1)\\
&=\left(\frac{1}{4sh(y_1)}d_{22}(a)-\frac{ch(y_1)}{4sh(y_1)}d_{11}(a)\right)s_{11}^{LR}\\
&+\left(\frac{\sqrt{-1}}{4sh(y_1)}d_{11}(a)-\frac{\sqrt{-1}ch(y_1)}{4sh(y_1)}d_{22}(a)\right)s_{23}^{LR}
\end{aligned}
$$

$$+\left(-\frac{1}{4sh(y_1)}d_{11}(a)+\frac{ch(y_1)}{4sh(y_1)}d_{22}(a)\right)s_{22}^{LR}.$$

Therefore, we have

$$\begin{aligned}
&-\tilde{pr}_2(R_{w_3}\Psi\otimes w_1)(a)\\
&=\left(\frac{1}{4sh(y_1)}d_{22}(a)+\frac{1}{4sh(y_1y_2)}d_{33}(a)\right.\\
&\left.-\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{11}(a)-\frac{ch(y_1)}{4sh(y_1)}d_{11}(a)\right)s_{11}^{LR}\\
&+\left(\frac{\sqrt{-1}}{4sh(y_1)}d_{11}(a)-\frac{\sqrt{-1}ch(y_1)}{4sh(y_1)}d_{22}(a)\right)s_{23}^{LR}\\
&+\left(-\frac{\sqrt{-1}}{4sh(y_1y_2)}d_{11}(a)+\frac{\sqrt{-1}ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s_{32}^{LR}\\
&+\left(-\frac{1}{4sh(y_1)}d_{11}(a)+\frac{ch(y_1)}{4sh(y_1)}d_{22}(a)\right)s_{22}^{LR}\\
&+\left(-\frac{1}{4sh(y_1y_2)}d_{11}(a)+\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s_{33}^{LR}.
\end{aligned}$$

4) Similarly, we have

$$\begin{aligned}
&-\tilde{pr}_2(R_{w_1}\Psi\otimes w_3)(a)\\
&=\left(\frac{1}{4sh(y_1y_2)}d_{33}(a)-\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{11}(a)\right.\\
&\left.+\frac{1}{4sh(y_1)}d_{22}(a)-\frac{ch(y_1)}{4sh(y_1)}d_{11}(a)\right)s_{11}^{LR}\\
&+\left(-\frac{\sqrt{-1}}{4sh(y_1)}d_{11}(a)+\frac{\sqrt{-1}ch(y_1)}{4sh(y_1)}d_{22}(a)\right)s_{23}^{LR}\\
&+\left(\frac{\sqrt{-1}}{4sh(y_1y_2)}d_{11}(a)-\frac{\sqrt{-1}ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s_{32}^{LR}\\
&+\left(-\frac{1}{4sh(y_1)}d_{11}(a)+\frac{ch(y_1)}{4sh(y_1)}d_{22}(a)\right)s_{22}^{LR}\\
&+\left(-\frac{1}{4sh(y_1y_2)}d_{11}(a)+\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s_{33}^{LR}.
\end{aligned}$$

5) Finally,

$$
\begin{aligned}
&\frac{3}{2}\tilde{pr}_2(R_{w_2}\Psi \otimes w_2)(a) \\
&= \frac{3}{2}\tilde{pr}_2(R_{\frac{2}{3}(2H_{1,2}+H_{2,3})}\Psi \otimes w_2)(a) \\
&= R_{2H_{1,2}+H_{2,3}}\left(\sum_{i=1}^{3} d_{ii}(a)s_i^L \otimes s_i^R\right) w_2 \\
&= -\partial_1 d_{11}(a)s_{11}^{LR} + \frac{1}{2}\partial_1 d_{22}(a)s_{22}^{LR} + \frac{1}{2}\partial_1 d_{33}(a)s_{33}^{LR}.
\end{aligned}
$$

By combining these results, we have

$$
\begin{aligned}
&4\tilde{pr}_2(\nabla_R\Psi)(a) \\
&= \left\{-\left(\partial_1 + \frac{ch(y_1)}{2sh(y_1)} + \frac{ch(y_1y_2)}{2sh(y_1y_2)}\right) d_{11}(a)\right. \\
&\left.+\frac{1}{2sh(y_1)}d_{22}(a) + \frac{1}{2sh(y_1y_2)}d_{33}(a)\right\} s_{11}^{LR} \\
&+\left\{-\frac{1}{2sh(y_1)}d_{11}(a) + \left(\partial_1 - \partial_2 + \frac{ch(y_1)}{2sh(y_1)} - \frac{ch(y_2)}{2sh(y_2)}\right) d_{22}(a)\right. \\
&\left.+\frac{1}{2sh(y_2)}d_{33}(a)\right\} s_{22}^{LR} \\
&+\left\{-\frac{1}{2sh(y_1y_2)}d_{11}(a) - \frac{1}{2sh(y_2)}d_{22}(a)\right. \\
&\left.+\left(\partial_2 + \frac{ch(y_2)}{2sh(y_2)} + \frac{ch(y_1y_2)}{2sh(y_1y_2)}\right) d_{33}(a)\right\} s_{33}^{LR}.
\end{aligned}
$$

This equals $\lambda_i \sum_{j=1}^{3} d_{jj}(a)s_{jj}^{LR}$, where λ_i $(i = 1,2,3)$ are some constants depending on the choice of $\sigma = \sigma_i$ of the principal series representation. The eigenvalues λ_i $(i = 1,2,3)$ are computed in [6], and they are $\lambda_1 = -\frac{1}{3}(2\nu_1 - \nu_2)$, $\lambda_2 = \frac{1}{3}(\nu_1 - 2\nu_2)$, $\lambda_3 = \frac{1}{3}(\nu_1 + \nu_2)$. Summing up, we have the following result:

THEOREM 6.4. *Let* $\Psi(g) = \sum_{i=1}^{3}\sum_{j=1}^{3} d_{ij}(g)s_i^L \otimes s_j^R \in C^{\infty}_{\tau_2,\tau_2}(K\backslash G/K)$ *be a spherical function attached to the non-spherical principal series representation* $\pi_{\sigma_i,\nu}$ $(i = 1,2,3)$*. Put*

$$d_{11}(a) = F(y_1,y_2), d_{22}(a) = G(y_1,y_2), d_{33}(a) = H(y_1,y_2).$$

Then F, G, H satisfy the following differential equations:

$$\begin{aligned} &-\left(\partial_1 + \frac{ch(y_1)}{2sh(y_1)} + \frac{ch(y_1y_2)}{2sh(y_1y_2)}\right) F(y_1,y_2) + \frac{1}{2sh(y_1)} G(y_1,y_2) \\ &+ \frac{1}{2sh(y_1y_2)} H(y_1,y_2) = \lambda_i F(y_1,y_2) \end{aligned} \tag{6.4}$$

$$\begin{aligned} &- \frac{1}{2sh(y_1)} F(y_1,y_2) + \left(\partial_1 - \partial_2 + \frac{ch(y_1)}{2sh(y_1)} - \frac{ch(y_2)}{2sh(y_2)}\right) G(y_1,y_2) \\ &+ \frac{1}{2sh(y_2)} H(y_1,y_2) = \lambda_i G(y_1,y_2) \end{aligned} \tag{6.5}$$

$$\begin{aligned} &- \frac{1}{2sh(y_1y_2)} F(y_1,y_2) - \frac{1}{2sh(y_2)} G(y_1,y_2) \\ &+ \left(\partial_2 + \frac{ch(y_2)}{2sh(y_2)} + \frac{ch(y_1y_2)}{2sh(y_1y_2)}\right) H(y_1,y_2) = \lambda_i H(y_1,y_2). \end{aligned} \tag{6.6}$$

Here, $\lambda_1 = -\frac{1}{3}(2\nu_1 - \nu_2)$, $\lambda_2 = \frac{1}{3}(\nu_1 - 2\nu_2)$, $\lambda_3 = \frac{1}{3}(\nu_1 + \nu_2)$.

6.3. The expansion of the matrix coefficients in terms of the power series around $y_1 = y_2 = 0$

We put

$$F(y_1,y_2) = \sum_{n,m=0}^{\infty} a_{n,m} y_1^{n+\mu_1} y_2^{m+\mu_2},$$

$$G(y_1,y_2) = \sum_{n,m=0}^{\infty} b_{n,m} y_1^{n+\mu_1} y_2^{m+\mu_2},$$

$$H(y_1,y_2) = \sum_{n,m=0}^{\infty} c_{n,m} y_1^{n+\mu_1} y_2^{m+\mu_2}.$$

($(a_{0,0}, b_{0,0}, c_{0,0}) \neq (0,0,0)$). We want to find the recurrence relations between $(a_{n,m})$, $(b_{n,m})$, $(c_{n,m})$ and the values of characteristic roots (μ_1, μ_2). Hereafter, we assume that $1, \nu_1, \nu_2$ are linearly independent over $\mathbf{Q}$. By inserting these power series into the Casimir equation (6.1), (6.2), (6.3) and the gradient equation (6.4), (6.5), (6.6) and picking up the coefficient of $y_1^{n+\mu_1} y_2^{m+\mu_2}$, we have the following recurrence relations:

$$\begin{aligned}
&\{2(n'^2 - n'm' + m'^2 - n' - m') - \lambda\}a_{n,m} \\
&+ 2\sum_{k=1}^{\infty}(-2n' + m' + 2k)a_{n-2k,m} + 2\sum_{k=1}^{\infty}(n' - 2m' + 4k)a_{n,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(-n' - m' + 2k)a_{n-2k,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(2k-1)b_{n-2k+1,m} + 2\sum_{k=1}^{\infty}(2k-1)c_{n-2k+1,m-2k+1} = 0,
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
&\{2(n'^2 - n'm' + m'^2 - n' - m') - \lambda\}b_{n,m} \\
&+ 2\sum_{k=1}^{\infty}(-2n' + m' + 2k)b_{n-2k,m} + 2\sum_{k=1}^{\infty}(n' - 2m' + 2k)b_{n,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(-n' - m' + 4k)b_{n-2k,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(2k-1)a_{n-2k+1,m} + 2\sum_{k=1}^{\infty}(2k-1)c_{n,m-2k+1} = 0,
\end{aligned} \tag{6.8}$$

$$\begin{aligned}
&\{2(n'^2 - n'm' + m'^2 - n' - m') - \lambda\}c_{n,m} \\
&+ 2\sum_{k=1}^{\infty}(-2n' + m' + 4k)c_{n-2k,m} + 2\sum_{k=1}^{\infty}(n' - 2m' + 2k)c_{n,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(-n' - m' + 2k)c_{n-2k,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(2k-1)a_{n-2k+1,m-2k+1} + 2\sum_{k=1}^{\infty}(2k-1)b_{n,m-2k+1} = 0,
\end{aligned} \tag{6.9}$$

$$\begin{aligned}
&(-n' + 1 - \lambda_i)a_{n,m} + \sum_{k=1}^{\infty} a_{n-2k,m} + \sum_{k=1}^{\infty} a_{n-2k,m-2k} \\
&- \sum_{k=1}^{\infty} b_{n-2k+1,m} - \sum_{k=1}^{\infty} c_{n-2k+1,m-2k+1} = 0,
\end{aligned} \tag{6.10}$$

$$(n' - m' - \lambda_i)b_{n,m} - \sum_{k=1}^{\infty} b_{n-2k,m} + \sum_{k=1}^{\infty} b_{n,m-2k} + \sum_{k=1}^{\infty} a_{n-2k+1,m} - \sum_{k=1}^{\infty} c_{n,m-2k+1} = 0, \tag{6.11}$$

$$(m' - 1 - \lambda_i)c_{n,m} - \sum_{k=1}^{\infty} c_{n,m-2k} - \sum_{k=1}^{\infty} c_{n-2k,m-2k} + \sum_{k=1}^{\infty} a_{n-2k+1,m-2k+1} + \sum_{k=1}^{\infty} b_{n,m-2k+1} = 0. \tag{6.12}$$

Here, $n' = n + \mu_1$, $m' = m + \mu_2$, λ is the eigenvalue of the Casimir operator given in Theorem 6.2, and λ_i $(i = 1, 2, 3)$ are the eigenvalues of the gradient operator given in Theorem 6.4. In these identities, we assume that $a_{i,j}, b_{i,j}, c_{i,j} = 0$ if $i < 0$ or $j < 0$. Note that in this computation, we used the power series expansions

$$\frac{ch(y)}{sh(y)} = -1 - 2\sum_{k=1}^{\infty} y^{2k}, \frac{1}{sh(y)} = -2\sum_{k=1}^{\infty} y^{2k-1},$$

$$\frac{1}{sh(y)^2} = 4\sum_{k=1}^{\infty} ky^{2k}, \frac{ch(y)}{sh(y)^2} = 2\sum_{k=1}^{\infty}(2k-1)y^{2k-1}$$

$(|y| < 1)$. By inserting $n = m = 0$ into (6.7), (6.8), (6.9), we have

$$2(\mu_1^2 - \mu_1\mu_2 + \mu_2^2 - \mu_1 - \mu_2) - \lambda = 0. \tag{6.13}$$

Moreover, by inserting $n = m = 0$ into (6.10), (6.11), (6.12), we have

$$(-\mu_1 + 1 - \lambda_i)a_{0,0} = 0$$

$$(\mu_1 - \mu_2 - \lambda_i)b_{0,0} = 0$$

$$(\mu_2 - 1 - \lambda_i)c_{0,0} = 0.$$

Since $(a_{0,0}, b_{0,0}, c_{0,0}) \neq (0, 0, 0)$, at least one of $-\mu_1+1-\lambda_i$, $\mu_1-\mu_2-\lambda_i$, $\mu_2-1-\lambda_i$ is 0. By combining this with the equation (6.13), we can compute the values of (μ_1, μ_2). (Because of the assumption of the linearly independence of $1, \nu_1, \nu_2$, we know that just one of $-\mu_1 + 1 - \lambda_i$, $\mu_1 - \mu_2 - \lambda_i$, $\mu_2 - 1 - \lambda_i$ is 0, and the other two are not 0.)

PROPOSITION 6.5. 1) *In case of* $\sigma = \sigma_1$, $\lambda_i = \lambda_1 = -\frac{1}{3}(2\nu_1 - \nu_2)$.
a) If $-\mu_1 + 1 - \lambda_1 = 0$,

$$(\mu_1, \mu_2) = \left(\frac{2\nu_1 - \nu_2 + 3}{3}, \frac{\nu_1 - 2\nu_2 + 3}{3}\right), \left(\frac{2\nu_1 - \nu_2 + 3}{3}, \frac{\nu_1 + \nu_2 + 3}{3}\right)$$

and $a_{0,0} \neq 0, b_{0,0} = 0, c_{0,0} = 0$.
b) If $\mu_1 - \mu_2 - \lambda_1 = 0$,

$$(\mu_1, \mu_2) = \left(\frac{2\nu_2 - \nu_1 + 3}{3}, \frac{\nu_1 + \nu_2 + 3}{3}\right), \left(\frac{-\nu_1 - \nu_2 + 3}{3}, \frac{\nu_1 - 2\nu_2 + 3}{3}\right)$$

and $a_{0,0} = 0, b_{0,0} \neq 0, c_{0,0} = 0$.
c) If $\mu_2 - 1 - \lambda_1 = 0$,

$$(\mu_1, \mu_2) = \left(\frac{2\nu_2 - \nu_1 + 3}{3}, \frac{\nu_2 - 2\nu_1 + 3}{3}\right), \left(\frac{-\nu_1 - \nu_2 + 3}{3}, \frac{\nu_2 - 2\nu_1 + 3}{3}\right)$$

and $a_{0,0} = 0, b_{0,0} = 0, c_{0,0} \neq 0$.
2) *The change of notation* $\lambda_1 \mapsto \lambda_2$, $\nu_1 \mapsto \nu_2$, $\nu_2 \mapsto \nu_1$ *in* 1) *gives the values of* (μ_1, μ_2) *in case of* $\sigma = \sigma_2$, *and the change of notation* $\lambda_1 \mapsto \lambda_3$, $\nu_1 \mapsto -\nu_1$, $\nu_2 \mapsto -\nu_1 + \nu_2$ *in* 1) *gives the values of* (μ_1, μ_2) *in case of* $\sigma = \sigma_3$.

Because of the same reasons we mentioned in section 5, we leave the explicit formulas for the power series to the "Appendix", section 7 of this paper. Now, we have known that there exsists six power series corresponding to the six characteristic roots given in Proposition 6.5, and the coefficients of power series satisfy the recurrence relations from (6.7) to (6.12). Firstly, we take $\sigma = \sigma_1$ for the character of M. Let $\Psi = {}^t(F, G, H)$ be the matrix coefficient with K-type of three dimensional tautological representation, and $\psi_{\alpha,\beta} = {}^t(f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta})$ be the power series solution around $y_1 = y_2 = 0$ corresponding to the characteristic root (α, β) whose first term is ${}^t(1, 0, 0)$ or ${}^t(0, 1, 0)$ or ${}^t(0, 0, 1)$.

By Proposition 6.5, α takes following three values:

$$\alpha_1 = -\frac{1}{3}(\nu_1 + \nu_2) + 1, \alpha_2 = \frac{1}{3}(2\nu_1 - \nu_2) + 1, \alpha_3 = \frac{1}{3}(2\nu_2 - \nu_1) + 1.$$

Therefore, we can write

$$F(y_1, y_2) = \sum_{i=1}^{3} d_i b_i^{(1)}(y_1, y_2) y_1^{\alpha_i}$$
$$G(y_1, y_2) = \sum_{i=1}^{3} d_i b_i^{(2)}(y_1, y_2) y_1^{\alpha_i}$$
$$H(y_1, y_2) = \sum_{i=1}^{3} d_i b_i^{(3)}(y_1, y_2) y_1^{\alpha_i}.$$

Here, $d_i (i = 1, 2, 3)$ are some constants and $b_i^{(j)}(y_1, y_2)$ are analytic functions for $0 < y_2 < 1$ and $0 < y_1 \ll 1$. By inserting $F = b_2^{(1)}(y_1, y_2) y_1^{\alpha_2}, G = b_2^{(2)}(y_1, y_2) y_1^{\alpha_2}, H = b_2^{(3)}(y_1, y_2) y_1^{\alpha_2}$ into the equation (6.1), we have

$$\begin{aligned}
&y_1^{\alpha_2} \Big\{ 2(\partial_1^2 b_2^{(1)}(y_1, y_2) + 2\alpha_2 \partial_1 b_2^{(1)}(y_1, y_2) + \alpha_2^2 b_2^{(1)}(y_1, y_2) \\
&- \partial_1 \partial_2 b_2^{(1)}(y_1, y_2) - \alpha_2 \partial_2 b_2^{(1)}(y_1, y_2) + \partial_2^2 b_2^{(1)}(y_1, y_2)) \\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1 y_2)}{sh(y_1 y_2)} - \frac{ch(y_2)}{sh(y_2)} \right) (\alpha_2 b_2^{(1)}(y_1, y_2) + \partial_1 b_2^{(1)}(y_1, y_2)) \\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1 y_2)}{sh(y_1 y_2)} + 2\frac{ch(y_2)}{sh(y_2)} \right) \partial_2 b_2^{(1)}(y_1, y_2) \\
&- \left(\frac{1}{sh(y_1)^2} + \frac{1}{sh(y_1 y_2)^2} \right) b_2^{(1)}(y_1, y_2) \\
&+ \frac{ch(y_1)}{sh(y_1)^2} b_2^{(2)}(y_1, y_2) + \frac{ch(y_1 y_2)}{sh(y_1 y_2)^2} b_2^{(3)}(y_1, y_2) \\
&- \lambda b_2^{(1)}(y_1, y_2) \Big\} = 0.
\end{aligned}$$

By dividing both sides by $y_1^{\alpha_2}$ and taking the limit $y_1 \to 0$, we have

$$\begin{aligned}
&2\partial_2^2 b_2^{(1)}(0, y_2) + \left(2\frac{y_2^2 + 1}{y_2^2 - 1} - 2\alpha_2 \right) \partial_2 b_2^{(1)}(0, y_2) \\
&+ \left(2\alpha_2^2 - \frac{y_2^2 + 1}{y_2^2 - 1}\alpha_2 - 3\alpha_2 - \lambda \right) b_2^{(1)}(0, y_2) = 0.
\end{aligned}$$

Note that because of Proposition 6.5, since $a_{0,0} \neq 0$ if $\alpha = \alpha_2$, $b_2^{(1)}(0, y_2)$ is not identically 0. Hereafter the same statement holds for all the functions we compute. This equation is the same type of equation as (5.8). By solving this, we have

$$\begin{aligned} b_2^{(1)}(0,y_2) =& y_2^{-\frac{1}{3}(2\nu_2-\nu_1)+1}\frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}{}_2F_1\left(-\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};-\frac{1}{2}\nu_2+1;y_2^2\right) \\ &+ y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}\frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}{}_2F_1\left(\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};\frac{1}{2}\nu_2+1;y_2^2\right) \end{aligned}$$

and by compairing the leading terms, we have

$$\begin{aligned} & f_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}(y_1,y_2) \\ &= y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}y_2^{-\frac{1}{3}(2\nu_2-\nu_1)+1}{}_2F_1\left(-\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};-\frac{1}{2}\nu_2+1;y_2^2\right) \\ &+ (\text{higher order terms with respect to } y_1), \end{aligned}$$

$$\begin{aligned} & f_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}(y_1,y_2) \\ &= y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}{}_2F_1\left(\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};\frac{1}{2}\nu_2+1;y_2^2\right) \\ &+ (\text{higher order terms with respect to } y_1) \end{aligned}$$

and

$$\begin{aligned} \Psi = d_2 \Bigg\{ & \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1} \\ & + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\Bigg\} \\ & + (\text{linear combination of the other four solutions}). \end{aligned} \tag{6.14}$$

Next, by Proposition 6.5, β takes following three values:

$$\beta_1 = \frac{1}{3}(\nu_1+\nu_2)+1, \beta_2 = -\frac{1}{3}(2\nu_1-\nu_2)+1, \beta_3 = -\frac{1}{3}(2\nu_2-\nu_1)+1.$$

Therefore, we can write

$$F(y_1,y_2)=\sum_{i=1}^{3}c_ia_i^{(1)}(y_1,y_2)y_2^{\beta_i},$$
$$G(y_1,y_2)=\sum_{i=1}^{3}c_ia_i^{(2)}(y_1,y_2)y_2^{\beta_i},$$
$$H(y_1,y_2)=\sum_{i=1}^{3}c_ia_i^{(3)}(y_1,y_2)y_2^{\beta_i}.$$

By inserting $F=a_2^{(1)}(y_1,y_2)y_2^{\beta_2}$, $G=a_2^{(2)}(y_1,y_2)y_2^{\beta_2}$, $H=a_2^{(3)}(y_1,y_2)y_2^{\beta_2}$ into the equation (6.3) and applying the same method, we have

$$\begin{aligned}\Psi=c_2\Bigg\{&\frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\\ &+\frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\Bigg\}\\ &+(\text{linear combination of the other four solutions}).\end{aligned}\tag{6.15}$$

Next, we take $i=1$ or 3. By inserting $F=a_i^{(1)}(y_1,y_2)y_2^{\beta_i}, G=a_i^{(2)}(y_1,y_2)y_2^{\beta_i}, H=a_i^{(3)}(y_1,y_2)y_2^{\beta_i}$ into the equation (6.1), we have

$$\begin{aligned}&y_2^{\beta_i}\Big\{2(\partial_1^2a_i^{(1)}(y_1,y_2)-\partial_1\partial_2a_i^{(1)}(y_1,y_2)-\beta_i\partial_1a_i^{(1)}(y_1,y_2)\\ &+\partial_2^2a_i^{(1)}(y_1,y_2)+2\beta_ia_i^{(1)}(y_1,y_2)+\beta_i^2a_i^{(1)}(y_1,y_2))\\ &+\left(2\frac{ch(y_1)}{sh(y_1)}+\frac{ch(y_1y_2)}{sh(y_1y_2)}-\frac{ch(y_2)}{sh(y_2)}\right)\partial_1a_i^{(1)}(y_1,y_2)\\ &+\left(-\frac{ch(y_1)}{sh(y_1)}+\frac{ch(y_1y_2)}{sh(y_1y_2)}+2\frac{ch(y_2)}{sh(y_2)}\right)(\beta_ia_i^{(1)}(y_1,y_2)+\partial_2a_i^{(1)}(y_1,y_2))\\ &-\left(\frac{1}{sh(y_1)^2}+\frac{1}{sh(y_1y_2)^2}\right)a_i^{(1)}(y_1,y_2)\\ &+\frac{ch(y_1)}{sh(y_1)^2}a_i^{(2)}(y_1,y_2)+\frac{ch(y_1y_2)}{sh(y_1y_2)^2}a_i^{(3)}(y_1,y_2)\\ &-\lambda a_i^{(1)}(y_1,y_2)\Big\}=0.\end{aligned}$$

By dividing both sides by $y_2^{\beta_i}$ and taking the limit $y_2 \to 0$, we have

$$\begin{aligned}
&2\partial_1^2 a_i^{(1)}(y_1,0) + \left(2\frac{y_1^2+1}{y_1^2-1} - 2\beta_i\right)\partial_1 a_i^{(1)}(y_1,0)\\
&+ \left(2\beta_i^2 - \frac{y_1^2+1}{y_1^2-1}\beta_i - 3\beta_i - \frac{4y_1^2}{(y_1^2-1)^2} - \lambda\right) a_i^{(1)}(y_1,0)\\
&+ \frac{2(y_1^3+y_1)}{(y_1^2-1)^2} a_i^{(2)}(y_1,0) = 0.
\end{aligned} \tag{6.16}$$

Next, by inserting $F = a_i^{(1)}(y_1,y_2)y_2^{\beta_i}, G = a_i^{(2)}(y_1,y_2)y_2^{\beta_i}, H = a_i^{(3)}(y_1,y_2)y_2^{\beta_i}$ into the equation (6.4), we have

$$\begin{aligned}
&y_2^{\beta_i}\Big\{ - \partial_1 a_i^{(1)}(y_1,y_2) - \frac{ch(y_1)}{2sh(y_1)} a_i^{(1)}(y_1,y_2)\\
&- \frac{ch(y_1y_2)}{2sh(y_1y_2)} a_i^{(1)}(y_1,y_2) + \frac{1}{2sh(y_1)} a_i^{(2)}(y_1,y_2)\\
&+ \frac{1}{2sh(y_1y_2)} a_i^{(3)}(y_1,y_2) - \lambda_1 a_i^{(1)}(y_1,y_2)\Big\} = 0.
\end{aligned}$$

By dividing both sides by $y_2^{\beta_i}$ and taking the limit $y_2 \to 0$, we have

$$\begin{aligned}
\frac{y_1}{y_1^2-1} a_i^{(2)}(y_1,0) &= \partial_1 a_i^{(1)}(y_1,0)\\
&\quad + \left(\frac{y_1^2+1}{2(y_1^2-1)} + \lambda_1 - \frac{1}{2}\right) a_i^{(1)}(y_1,0).
\end{aligned} \tag{6.17}$$

By combining equations (6.16) and (6.17) to eliminate $a_i^{(2)}(y_1,0)$, we have

$$\begin{aligned}
&2\partial_1^2 a_i^{(1)}(y_1,0) + \left(4\frac{y_1^2+1}{y_1^2-1} - 2\beta_i\right)\partial_1 a_i^{(1)}(y_1,0)\\
&+ \left(\frac{y_1^2+1}{y_1^2-1}(2\lambda_1 - \beta_i - 1) + 2\beta_i^2 - 3\beta_i - \lambda + 1\right) a_i^{(1)}(y_1,0) = 0.
\end{aligned}$$

We put $y_1^2 = u$, $f_i(u) := a_i^{(1)}(y_1,0)$ and define a differential operator $\tilde{\partial}_1$ by $\tilde{\partial}_1 := u\frac{d}{du}$. Then the equation becomes

$$\begin{aligned}
&8\tilde{\partial}_1{}^2 f_i(u) + \left(8 - 4\beta_i + \frac{16}{u-1}\right)\tilde{\partial}_1 f_i(u)\\
&+ \left(2\beta_i^2 - 4\beta_i + 2\lambda_1 - \lambda + \frac{4\lambda_1 - 2\beta_i - 2}{u-1}\right) f_i(u) = 0.
\end{aligned}$$

Next, we put $f_i(u) = u^x g_i(u)$ $(x \in \mathbf{C})$. Then the equation becomes

$$8\tilde{\partial_1}^2 g_i(u) + \left(8 - 4\beta_i + \frac{16}{u-1} + 16x\right)\tilde{\partial_1} g_i(u)$$
$$+\left(8x^2 + (8-4\beta_i)x + 2\beta_i^2 - 4\beta_i + 2\lambda_1 - \lambda + \frac{16x + 4\lambda_1 - 2\beta_i - 2}{u-1}\right) g_i(u)$$
$$= 0.$$

We take $x = x_i$ as the number satisfying

$$8x_i^2 + (8 - 4\beta_i)x_i + 2\beta_i^2 - 4\beta_i + 2\lambda_1 - \lambda = 0.$$

Then we have

$$8u^2 \frac{d^2 g_i}{du^2} + \left(16 - 4\beta_i + 16x_i + \frac{16}{u-1}\right) u \frac{dg_i}{du}$$
$$+ \frac{16x_i + 4\lambda_1 - 2\beta_i - 2}{u-1} g_i(u) = 0.$$

Finally, we put $u = \frac{1}{\zeta}$. Then the equation becomes

$$\zeta(\zeta - 1)\frac{d^2 g_i}{d\zeta^2}$$
$$+\left(\left(\frac{1}{2}\beta_i - 2x_i + 2\right)\zeta - \frac{1}{2}\beta_i + 2x_i\right)\frac{dg_i}{d\zeta} \tag{6.18}$$
$$+\left(-2x_i - \frac{1}{2}\lambda_1 + \frac{1}{4}\beta_i + \frac{1}{4}\right) g_i = 0.$$

(6.18) is a Gaussian hypergeometric differential equation, and if we define p_i,q_i by complex numbers satisfying

$$1 + p_i + q_i = \frac{1}{2}\beta_i - 2x_i + 2$$

$$p_i q_i = -2x_i - \frac{1}{2}\lambda_1 + \frac{1}{4}\beta_i + \frac{1}{4}$$

and r_i by

$$r_i = \frac{1}{2}\beta_i - 2x_i,$$

then the general solution is expressed by

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & p_i & ;\zeta \\ 1-r_i & r_i-p_i-q_i & q_i & \end{matrix}\right\}$$
$$= P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & p_i & ;1-\zeta \\ r_i-p_i-q_i & 1-r_i & q_i & \end{matrix}\right\}.$$

The regular solution is given by

$$\begin{aligned} g_i(y_1) &= {}_2F_1\left(p_i, q_i; 1-r_i+p_i+q_i; 1-\zeta\right) \\ &= {}_2F_1\left(p_i, q_i; 1-r_i+p_i+q_i; 1-\frac{1}{y_1^2}\right). \end{aligned}$$

Since ${}_2F_1$ satisfies a formula (5.13), we have

$$\begin{aligned} g_i(y_1) =& y_1^{2p_i}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(q_i-p_i)}{\Gamma(1-r_i+q_i)\Gamma(q_i)}{}_2F_1(p_i, 1-r_i+p_i; 1+p_i-q_i; y_1^2) \\ &+ y_1^{2q_i}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(p_i-q_i)}{\Gamma(1-r_i+p_i)\Gamma(p_i)} \\ &\times {}_2F_1(q_i, 1-r_i+q_i; 1+q_i-p_i; y_1^2). \end{aligned}$$

Therefore, we have

$$\begin{aligned} a_i^{(1)}(y_1,0) &= u^{x_i}g_i(y_1) = y_1^{2x_i}g_i(y_1) \\ &= y_1^{2(p_i+x_i)}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(q_i-p_i)}{\Gamma(1-r_i+q_i)\Gamma(q_i)} \\ &\times {}_2F_1(p_i, 1-r_i+p_i; 1+p_i-q_i; y_1^2) \\ &+ y_1^{2(q_i+x_i)}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(p_i-q_i)}{\Gamma(1-r_i+p_i)\Gamma(p_i)} \\ &\times {}_2F_1(q_i, 1-r_i+q_i; 1+q_i-p_i; y_1^2) \end{aligned} \tag{6.19}$$

for $i=1,3$. The values (x_i, p_i, q_i, r_i) $(i=1,3)$ are given as follows:
A) When $i=1$, we have

$$\begin{aligned} x_1 &= \frac{1}{3}\nu_1 - \frac{1}{6}\nu_2, \quad p_1 = -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1, \\ q_1 &= \frac{1}{2}, \quad r_1 = -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2}. \end{aligned}$$

B) When $i=3$, we have

$$x_3=\frac{1}{3}\nu_1-\frac{1}{6}\nu_2,\quad p_3=-\frac{1}{2}\nu_1+1,$$
$$q_3=\frac{1}{2},\quad r_3=-\frac{1}{2}\nu_1+\frac{1}{2}.$$

By inserting these results into (6.19), we have

$$\begin{aligned}
&a_1^{(1)}(y_1,0)\\
&=y_1^{\frac{1}{3}(2\nu_2-\nu_1)+2}\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2-\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\\
&\times{}_2F_1\left(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1,\frac{3}{2};-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{3}{2};y_1^2\right)\\
&+y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}\frac{2\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}\\
&\times{}_2F_1\left(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1,\frac{1}{2};\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2};y_1^2\right),\\
&a_3^{(1)}(y_1,0)\\
&=y_1^{-\frac{1}{3}(\nu_1+\nu_2)+2}\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1+1)}{}_2F_1\left(-\frac{1}{2}\nu_1+1,\frac{3}{2};-\frac{1}{2}\nu_1+\frac{3}{2};y_1^2\right)\\
&+y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}\frac{2\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)}{}_2F_1\left(\frac{1}{2}\nu_1+1,\frac{1}{2};\frac{1}{2}\nu_1+\frac{1}{2};y_1^2\right).
\end{aligned}$$

Here, we used $\Gamma(\frac{1}{2})=\sqrt{\pi}$, $\Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{2}$. From the equation of $a_1^{(1)}(y_1,0)$, we have

$$\begin{aligned}
&f_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}(y_1,y_2)\\
&=y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}{}_2F_1\left(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1,\frac{1}{2};\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2};y_1^2\right)\\
&\quad+(\text{higher order terms with respect to } y_2),
\end{aligned}$$

$$\begin{aligned}
&f_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}(y_1,y_2)\\
&=y_1^{\frac{1}{3}(2\nu_2-\nu_1)+2}y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}{}_2F_1\left(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1,\frac{3}{2};-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{3}{2};y_1^2\right)\\
&+(\text{higher order terms with respect to } y_2).
\end{aligned}$$

And since the coefficient of $y_1^{\frac{1}{3}(2\nu_2-\nu_1)+2}y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}$ of $f_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}$ in $\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}$ is $\frac{1}{\nu_1-\nu_2-1}$, if the coefficient of $\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}$ is $\frac{2\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}$, the coefficient of $\Psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}$ is given by

$$\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2-\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\times(\nu_1-\nu_2-1)=\frac{2\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}.$$

Therefore, we have

$$\begin{aligned}\Psi =&c_1\left\{\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\right.\\ &\left.+\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\right\}\\ &+(\text{linear combination of the other four solutions}).\end{aligned}\tag{6.20}$$

Similarly, from the equation of $a_3^{(1)}(y_1,0)$, we have

$$\begin{aligned}\Psi =&c_3\left\{\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\right.\\ &\left.+\frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\right\}\\ &+(\text{linear combination of the other four solutions}).\end{aligned}\tag{6.21}$$

Next, we insert $F=b_i^{(1)}(y_1,y_2)y_1^{\alpha_i}$, $G=b_i^{(2)}(y_1,y_2)y_1^{\alpha_i}$, $H=b_i^{(3)}(y_1,y_2)y_1^{\alpha_i}$ ($i=1,3$) into the equation (6.3), (6.6). By applying the same method as we used above to $b_i^{(j)}(y_1,y_2)$ ($i=1,3,j=1,2,3$) (eliminate $b_i^{(2)}(0,y_2)$ and construct the differential equation with respect to $b_i^{(3)}(0,y_2)$), we have

$$\begin{aligned}\Psi =&d_1\left\{\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\right.\\ &\left.+\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\right\}\\ &+(\text{linear combination of the other four solutions})\end{aligned}\tag{6.22}$$

and

$$
\begin{aligned}
\Psi =& d_3 \left\{ \frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\
&\left. +\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1} \right\} \\
&+(\text{linear combination of the other four solutions}).
\end{aligned}
\tag{6.23}
$$

Now we have six equations with respect to the matrix coefficient Ψ (i.e. (6.20), (6.15), (6.21), (6.22), (6.14), (6.23)). By combining these equations, we obtain two different expressions of Ψ. That is,

$$
\begin{aligned}
\Psi =& c_1 \left\{ \frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1} \right. \\
&\left. +\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1} \right\} \\
&+c_2 \left\{ \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\
&\left. +\frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1} \right\} \\
&+c_3 \left\{ \frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1} \right. \\
&\left. +\frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1} \right\} \\
=& d_1 \left\{ \frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\
&\left. +\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1} \right\}
\end{aligned}
$$

$$+ d_2 \left\{ \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\}$$
$$+ d_3 \left\{ \frac{\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + 1)} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + 1)} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\}.$$

By comparing the coefficients of $\psi_{\alpha,\beta}$, we have

$$c_1 = \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2)}{\Gamma(-\frac{1}{2}\nu_1 + 1)\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})}, \ c_2 = \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})}{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)\Gamma(\frac{1}{2}\nu_1 + 1)},$$

$$c_3 = \frac{\Gamma(\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)}$$

up to the same constant multiples. Thus we obtained the expression of Ψ in case of $\sigma = \sigma_1$. Note that since the transform $\nu_1 \mapsto \nu_2, \nu_2 \mapsto \nu_1$ does not change the eigenvalue of Casimir operator λ and change the eigenvalue of gradient operator λ_1 to λ_2, this transform gives the expression of Ψ in case of $\sigma = \sigma_2$. Similarly, the transform $\nu_1 \mapsto -\nu_1, \nu_2 \mapsto -\nu_1 + \nu_2$ gives the expression in case of $\sigma = \sigma_3$. Trerefore, we obtained the following theorem.

THEOREM 6.6. *Let* $\Psi = {}^t(F, G, H) \in C^\infty_{\tau_2,\tau_2}(K\backslash G/K)|_A$ *be a spherical function attached to the non-spherical principal series* $\pi_{\sigma_1,\nu}$ *whose minimal* K*-type is three dimensional tautological representation* τ_2*, and* $\psi_{\alpha,\beta} = {}^t(f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta})$ *be the power series solution around* $y_1 = y_2 = 0$ *corresponding to the characteristic root* (α, β) *whose first term is* ${}^t(1, 0, 0)$ *or*

${}^t(0,1,0)$ *or* ${}^t(0,0,1)$. *Then we have*

(6.24)

$$\Psi = \frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}$$
$$\times \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}$$
$$+\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}$$
$$+\frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1+1)\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}$$
$$+\frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1+1)\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}$$
$$+\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}$$
$$\times \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}$$
$$+\frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1+1)\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}$$
$$\times \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}.$$

The transform $\nu_1 \mapsto \nu_2, \nu_2 \mapsto \nu_1$ *in* (6.24) *gives the expression of* Ψ *in case of* $\sigma = \sigma_2$ *and the transform* $\nu_1 \mapsto -\nu_1, \nu_2 \mapsto -\nu_1+\nu_2$ *in* (6.24) *gives the expression of* Ψ *in case of* $\sigma = \sigma_3$.

7. Appendix

We give the explicit formulas of the coefficients of power series solutions of both spherical and non-spherical case under certain assumptions without proof. The spherical functions corresponding to the matrix coefficients with minimal K-types are expressed by the linear combinations of these power series (Theorem 5.6, Theorem 6.6). There is a similar result in [6], in the case of Whittaker functions.

7.1. In case of spherical principal series representation

Firstly, we consider the matrix coefficient attached to the spherical principal series. Let $F(y_1, y_2)$ be the matrix coefficient of the spherical principal series representation restricted to A. Next, we put

$$F(y_1,y_2) = sh(y_1)^{-\frac{1}{2}} sh(y_2)^{-\frac{1}{2}} sh(y_1y_2)^{-\frac{1}{2}} G(y_1,y_2) \quad (0 < y_1, y_2 < 1)$$

and compute the power series of G at the origin $y_1 = y_2 = 0$. We put

$$G(y_1,y_2) = \sum_{n,m=0}^{\infty} \tilde{a}_{n,m} y_1^{n+\tilde{\mu}_1} y_2^{m+\tilde{\mu}_2} \quad (\tilde{a}_{0,0} \neq 0). \tag{7.1}$$

The characteristic roots take six values:

$$(\tilde{\mu}_1, \tilde{\mu}_2) = (\mu_1 - 1, \mu_2 - 1)$$

for the six values (μ_1,μ_2) given in Proposition 5.5. We put

$$p(n,m) = n^2 - nm + m^2 + (2\tilde{\mu}_1 - \tilde{\mu}_2)n + (2\tilde{\mu}_2 - \tilde{\mu}_1)m.$$

Assume that $p(n,m) \neq 0$ if $(n,m) \neq (0,0)$. Let $\mathbf{P}_{n,m}$ be the family of all sets $\{p(2n_k, 2m_k), \cdots, p(2n_0, 2m_0)\}$ such that

$$n_k = n, m_k = m, n_0 = m_0 = 0$$

and

$$(n_{i+1}, m_{i+1}) = (n_i + l_i, m_i) \, or \, (n_i, m_i + l_i) \, or \, (n_i + l_i, m_i + l_i)$$

$$(\exists l_i \in \mathbf{Z}_{>0}), \quad (i = 0, \cdots, k-1).$$

Here, k depends on each set. For $\{p(2n_k, 2m_k), \cdots, p(2n_0, 2m_0)\} \in \mathbf{P}_{n,m}$ and $0 \leq i \leq k-1$, we define $d_i \in \mathbf{Z}$ by $d_i = -l_i$. And we put

$$C_{(n_1,\cdots n_k; m_1,\cdots m_k)} = \prod_{i=0}^{k-1} d_i. \tag{7.2}$$

Then, $\tilde{a}_{n,m} = 0$ if n or m is odd, and

$$\tilde{a}_{2n,2m} = \sum_{\{p(2n_k,2m_k),\cdots,p(2n_0,2m_0)\} \in \mathbf{P}_{n,m}} \frac{C_{(n_1,\cdots n_k; m_1,\cdots m_k)}}{p(2n_k,2m_k)\cdots p(2n_1,2m_1)} \tilde{a}_{0,0}. \tag{7.3}$$

for $(n,m) \neq (0,0)$.

7.2. In case of non-spherical principal series representations

Next, we give the explicit formulas for the power series of the matrix coefficients of the non-spherical principal series. We give the power series solution of the equations obtained in Theorem 6.3 and Theorem 6.5. Firstly, we modify F, G, H by

$$\tilde{F}(y_1, y_2) = sh(y_1)^{\frac{1}{2}} sh(y_2)^{\frac{1}{2}} sh(y_1 y_2)^{\frac{1}{2}} F(y_1, y_2)$$

$$\tilde{G}(y_1, y_2) = sh(y_1)^{\frac{1}{2}} sh(y_2)^{\frac{1}{2}} sh(y_1 y_2)^{\frac{1}{2}} G(y_1, y_2)$$

$$\tilde{H}(y_1, y_2) = sh(y_1)^{\frac{1}{2}} sh(y_2)^{\frac{1}{2}} sh(y_1 y_2)^{\frac{1}{2}} H(y_1, y_2).$$

We put

$$\tilde{F}(y_1, y_2) = \sum_{n,m=0}^{\infty} \tilde{a}_{n,m} y_1^{n+\tilde{\mu}_1} y_2^{m+\tilde{\mu}_2} \tag{7.4}$$

$$\tilde{G}(y_1, y_2) = \sum_{n,m=0}^{\infty} \tilde{b}_{n,m} y_1^{n+\tilde{\mu}_1} y_2^{m+\tilde{\mu}_2} \tag{7.5}$$

$$\tilde{H}(y_1, y_2) = \sum_{n,m=0}^{\infty} \tilde{c}_{n,m} y_1^{n+\tilde{\mu}_1} y_2^{m+\tilde{\mu}_2} \tag{7.6}$$

$$(\tilde{a}_{0,0}, \tilde{b}_{0,0}, \tilde{c}_{0,0}) \neq (0, 0, 0).$$

Now, we put

$$\begin{aligned} p(n, m) &= q(n, m) = r(n, m) \\ &= n^2 - nm + m^2 + (2\tilde{\mu}_1 - \tilde{\mu}_2)n + (2\tilde{\mu}_2 - \tilde{\mu}_1)m. \end{aligned}$$

Though $p(n, m)$, $q(n, m)$, $r(n, m)$ are the same polynomials, we use the different symbols. By doing so, the expressions of the coefficients $\tilde{a}_{n,m}$, $\tilde{b}_{n,m}$, $\tilde{c}_{n,m}$ become a little easier. The characteristic roots take six values:

$$(\tilde{\mu}_1, \tilde{\mu}_2) = (\mu_1 - 1, \mu_2 - 1)$$

for the six values (μ_1, μ_2) given in Proposition 6.5.

The following formula gives the explicit expressions of the coefficients $(\tilde{a}_{n,m})$, $(\tilde{b}_{n,m})$, $(\tilde{c}_{n,m})$. Let $\mathbf{P}_{n,m}$ be the family of all sets $\{\alpha_k(n_k,m_k),\cdots,\alpha_0(n_0,m_0)\}$ satisfying the following rules:

A) $\alpha_i = p$ or q or r $(i=0,\cdots,k)$, $(n_k,m_k)=(n,m)$, $(n_0,m_0)=(0,0)$,

B) The relations of $\alpha_i(n_i,m_i)$ and $\alpha_{i-1}(n_{i-1},m_{i-1})$ are as follows. And for each correspondence, we associate one number (the number after the symbol ;).

In case of $\alpha_i(n_i,m_i)=p(n_i,m_i)$,

$$\alpha_{i-1}(n_{i-1},m_{i-1}) = \begin{cases} p(n_i-2l_i,m_i)\,;l_i \quad \text{or} \\ p(n_i,m_i-2l_i)\,;-l_i \quad \text{or} \\ p(n_i-2l_i,m_i-2l_i)\,;l_i \quad \text{or} \\ q(n_i-2l_i+1,m_i)\,;-(2l_i-1) \quad \text{or} \\ r(n_i-2l_i+1,m_i-2l_i+1)\,;-(2l_i-1). \end{cases} \tag{7.7}$$

In case of $\alpha_i(n_i,m_i)=q(n_i,m_i)$,

$$\alpha_{i-1}(n_{i-1},m_{i-1}) = \begin{cases} q(n_i-2l_i,m_i)\,;l_i \quad \text{or} \\ q(n_i,m_i-2l_i);\ l_i \quad \text{or} \\ q(n_i-2l_i,m_i-2l_i)\,;-l_i \quad \text{or} \\ p(n_i-2l_i+1,m_i)\,;-(2l_i-1) \quad \text{or} \\ r(n_i,m_i-2l_i+1)\,;-(2l_i-1). \end{cases} \tag{7.8}$$

In case of $\alpha_i(n_i,m_i)=r(n_i,m_i)$,

$$\alpha_{i-1}(n_{i-1},m_{i-1}) = \begin{cases} r(n_i-2l_i,m_i)\,;-l_i \quad \text{or} \\ r(n_i,m_i-2l_i);\ l_i \quad \text{or} \\ r(n_i-2l_i,m_i-2l_i)\,;l_i \quad \text{or} \\ p(n_i-2l_i+1,m_i-2l_i+1)\,;-(2l_i-1) \quad \text{or} \\ q(n_i,m_i-2l_i+1)\,;-(2l_i-1). \end{cases} \tag{7.9}$$

For each i, we denote the number after the symbol ; of each correspondence by d_i.

We put

$$(\delta^a_{n,m}, \delta^b_{n,m}, \delta^c_{n,m}) = \begin{cases} (\tilde{a}_{0,0}, \tilde{b}_{0,0}, \tilde{c}_{0,0}) & (n; \text{even}, m; \text{even}) \\ (\tilde{a}_{0,0}, \tilde{c}_{0,0}, \tilde{b}_{0,0}) & (n; \text{even}, m; \text{odd}) \\ (\tilde{b}_{0,0}, \tilde{a}_{0,0}, \tilde{c}_{0,0}) & (n; \text{odd}, m; \text{even}) \\ (\tilde{c}_{0,0}, \tilde{b}_{0,0}, \tilde{a}_{0,0}) & (n; \text{odd}, m; \text{odd}) \end{cases}$$

And we put

$$\mathbf{P}^p_{n,m} := \mathbf{P}_{n,m} \cap \{\alpha_k = p\}, \ \mathbf{P}^q_{n,m} := \mathbf{P}_{n,m} \cap \{\alpha_k = q\},$$
$$\mathbf{P}^r_{n,m} := \mathbf{P}_{n,m} \cap \{\alpha_k = r\}.$$

Then we have

$$\tilde{a}_{n,m} = \sum_{\{\alpha_k(n_k,m_k),\cdots,\alpha_0(n_0,m_0)\}\in\mathbf{P}^p_{n,m}} \frac{\left(\prod_{i=1}^k d_i\right) \delta^a_{n,m}}{\alpha_k(n_k,m_k)\cdots\alpha_1(n_1,m_1)}, \tag{7.10}$$

$$\tilde{b}_{n,m} = \sum_{\{\alpha_k(n_k,m_k),\cdots,\alpha_0(n_0,m_0)\}\in\mathbf{P}^q_{n,m}} \frac{\left(\prod_{i=1}^k d_i\right) \delta^b_{n,m}}{\alpha_k(n_k,m_k)\cdots\alpha_1(n_1,m_1)}, \tag{7.11}$$

$$\tilde{c}_{n,m} = \sum_{\{\alpha_k(n_k,m_k),\cdots,\alpha_0(n_0,m_0)\}\in\mathbf{P}^r_{n,m}} \frac{\left(\prod_{i=1}^k d_i\right) \delta^c_{n,m}}{\alpha_k(n_k,m_k)\cdots\alpha_1(n_1,m_1)}. \tag{7.12}$$

for $(n,m) \neq (0,0)$. Here, $\tilde{a}_{n,m} = 0$ (resp. $\tilde{b}_{n,m} = 0, \tilde{c}_{n,m} = 0$) if $\mathbf{P}^p_{n,m} = \emptyset$ (resp. $\mathbf{P}^q_{n,m} = \emptyset$, $\mathbf{P}^r_{n,m} = \emptyset$).

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