J. Math. Sci. Univ. Tokyo
18 (2011), 465–489.

Quasineutral Limit of the Schrödinger-Poisson System in Coulomb Gauge

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Abstract. The zero Debye length asymptotic of the Schrödinger-Poisson system in Coulomb gauge for ill-prepared initial data is studied. We prove that when the scaled Debye length $\lambda \to 0$, the current density defined by the solution of the Schrödinger-Poisson system in the Coulomb gauge converges to the solution of the rotating incompressible Euler equation plus a fast singular oscillating gradient vector field.

1. Introduction

The dimensionless form of the Schrödinger-Poisson system in Coulomb gauge is given by

$$\begin{cases} i\varepsilon\partial_t\psi^\lambda + \dfrac{\varepsilon^2}{2}\Delta_A\psi^\lambda - V'(|\psi^\lambda|^2)\psi^\lambda = \Phi^\lambda\psi^\lambda\,, \\ -\lambda^2\Delta\Phi^\lambda = |\psi^\lambda|^2 - \mathcal{C}^\lambda\,, \end{cases} \tag{1.1}$$

for $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ $(n = 2, 3)$. The superscript λ in the wave function ψ^λ and in the electric potential Φ^λ indicates the λ-dependence, and $\mathcal{C}^\lambda > 0$ is the given impurity (or doping) profile which is given by the difference between the densities of positive charged donor ions and negative charged acceptor ions ([18]). For the following, we shall exclude mobile impurities and assume that $\mathcal{C}^\lambda$ is a function of the position variable x only, i.e. $\mathcal{C}^\lambda = \mathcal{C}^\lambda(x)$. The nonlinear function V' is the first derivative of a twice differentiable nonlinear real-valued function over $\mathbb{R}^+$. Thus, V' is the potential energy and V is the potential energy density of the fields. The gauge is defined by

$$\nabla \cdot A = 0$$

2010 *Mathematics Subject Classification.* 35Q40, 35Q55, 76Y05.
Key words: Schrödinger-Poisson system; Coulomb gauge; rotating incompressible Euler equations; quasi-neutral limit.

where A is the magnetic vector potential, i.e., A is a divergence free vector field. The notation $\varepsilon\nabla_A$ denotes the covariant gradient defined by $\varepsilon\nabla_A \equiv \varepsilon\nabla - iA$, and the Laplacian $\varepsilon^2\Delta_A$ is given by

$$\varepsilon^2\Delta_A \equiv \varepsilon\nabla_A \cdot \varepsilon\nabla_A = \varepsilon^2\Delta - 2\varepsilon i A \cdot \nabla - |A|^2 .$$

The Poisson equation $(1.1)_2$ for the scaled electrostatic potential function Φ^λ serves as a constraint of the charge density. Here, $(1.1)_2$ denotes the second equation in (1.1), and similar notation will be employed in this paper. The dimensionless parameters ε and λ are the scaled Planck's constant and the scaled Debye length respectively, and they are given by

$$\varepsilon^2 = \frac{\hbar^2}{2m\kappa_B T_0 L^2} \quad \text{and} \quad \lambda^2 = \frac{\lambda_0 \kappa_B T_0}{N q^2 L^2}$$

where κ_B is the Boltzmann's constant, T_0 the electron temperature, q the electron charge, m the electron mass, $\lambda_0 > 0$ the permittivity of the system, N the number density of electrons and L the characteristic length. The case for $0 < \varepsilon \ll 1$, i.e., the characteristic length L is much larger compared to the de Broglie length $\hbar/\sqrt{2m\kappa_B T_0}$, is the famous semiclassical limit. Here, we study the zero Debye length asymptotic, $0 < \lambda \ll 1$, in which the characteristic device length L is much larger than the Debye length $\sqrt{\lambda_0\kappa_B T_0/Nq^2}$ and the limit $\lambda \to 0$, and this is referred to the quasi-neutral limit. Since electrons are mobile, plasmas are excellent conductors and any charges that develop are readily neutralized. In many cases, plasmas can be treated as being electrically neutral. This makes the research of the quasi-neutral limit interesting and challenging, especially for topics in plasma physics, fluid dynamics and kinetic theory (see [5, 18] for the physical background).

The connection between the Schrödinger equation and the classical fluid mechanics was noted by Madelung in 1927. This can also be extended to other Schrödinger type equations and the Klein-Gordon equation (see [15] and references therein). Following this idea, we introduce the geometric optic ansatz

$$\psi^\lambda = R^\lambda \exp\left(\frac{i}{\varepsilon} S^\lambda\right) = \sqrt{\rho^\lambda} \exp\left(\frac{i}{\varepsilon} S^\lambda\right) \tag{1.2}$$

the so-called Madelung transformation and define the hydrodynamical variables: density ρ^λ, velocity u^λ and momentum (or current) J^λ respectively by

$$\rho^\lambda = (R^\lambda)^2, \qquad u^\lambda = \nabla S^\lambda, \qquad J^\lambda = \rho^\lambda u^\lambda. \tag{1.3}$$

To study the effect of the magnetic field A, we define the relative velocity u_A^λ and relative momentum (current) J_A^λ with respect to A by

$$u_A^\lambda = u^\lambda - A\,, \qquad J_A^\lambda = J^\lambda - \rho^\lambda A = \rho^\lambda u_A^\lambda\,. \tag{1.4}$$

Thus, we have the following quantum hydrodynamical formulation of the Schrödinger-Poisson system in Coulomb gauge[13]

$$\partial_t \rho^\lambda + \nabla \cdot J_A^\lambda = 0\,, \tag{1.5}$$

$$\begin{aligned} \partial_t J_A^\lambda + \nabla \cdot \left(\frac{J_A^\lambda \otimes J_A^\lambda}{\rho^\lambda} \right) + \nabla P(\rho^\lambda) + \rho^\lambda \nabla \Phi^\lambda + \mathbb{G}_A(J_A^\lambda) \\ + \rho^\lambda \partial_t A = \frac{\varepsilon^2}{4} \nabla \cdot \left(\rho^\lambda \nabla^2 \log \rho^\lambda \right), \end{aligned} \tag{1.6}$$

$$-\lambda^2 \Delta \Phi^\lambda = \rho^\lambda - \mathcal{C}^\lambda(x)\,, \tag{1.7}$$

where $P(\rho^\lambda) = \rho^\lambda V'(\rho^\lambda) - V(\rho^\lambda)$ is the pressure and $\mathbb{G}_A(J_A^\lambda) = (\operatorname{curl} A) \times J_A^\lambda$ plays the role of rotation. The symbol $\otimes$ denotes the tensor product of two vectors. Note that $-\rho^\lambda A$ can be regarded as the background momentum (current) created by the divergence free magnetic field A; and by combining with J^λ, this constitutes the real momentum (current). We observe that the magnetic field A affects both the equation of continuity and the momentum equation, and the dispersive term on the right side (1.6) can be rewritten as follows:

$$\begin{aligned} \frac{\varepsilon^2}{4} \nabla \cdot \left(\rho^\lambda \nabla^2 \log \rho^\lambda \right) &= \frac{\varepsilon^2}{2} \rho^\lambda \nabla \left(\frac{\Delta \sqrt{\rho^\lambda}}{\sqrt{\rho^\lambda}} \right) \\ &= \frac{\varepsilon^2}{4} \Delta \nabla \rho^\lambda - \varepsilon^2 \nabla \cdot \left(\nabla \sqrt{\rho^\lambda} \otimes \nabla \sqrt{\rho^\lambda} \right) \end{aligned}$$

where $\frac{\varepsilon^2}{2} \frac{\Delta \sqrt{\rho^\lambda}}{\sqrt{\rho^\lambda}}$ is the Bohm quantum potential which can be considered as a quantum correction to the pressure. Equations (1.5)–(1.7) comprise a closed system governing ρ^λ, J^λ and Φ^λ that have the perturbation form of the compressible rotating Euler-Poisson equations with the pressure given by $P(\rho^\lambda)$. As noted in [9], if the Euler part of these equations is hyperbolic, then the pressure P must be a strictly increasing function of ρ^λ; in that case, $P'(\rho^\lambda) = \rho^\lambda V''(\rho^\lambda) > 0$. This implies that V must be a strictly convex function of ρ^λ and corresponds to a *defocusing* nonlinear Schrödinger-Poisson system. When $A = 0$, the global existence of weak solutions to a class of quantum hydrodynamics systems (1.5)–(1.7) with arbitrarily large initial data is proved by Antonelli and Marcati in [2].

The asymptotic limit of the 2D Schrödinger-Poisson system under the influence of a large magnetic field has been studied by Puel in [22]. For the "well-prepared" initial data, the solution converges to a dissipative solution of the incompressible Euler equations as both the permittivity and Planck's constant go to zero. The proof uses the notion of dissipative solution to the Euler equations introduced by Lions [17], as well as an adaptation of the modulated energy method introduced by Brenier in the classical context [6]. In fact, the modulated energy method allows one to reduce the proof to elegant convexity arguments, so that neither a WKB analysis nor a Wigner function approach is needed. Let us remark that the "well-prepared assumption" implies that the initial data is monokinetic, namely each position x corresponds to a unique "initial velocity". The initial velocity in the modulated energy eventually gives the initial datum for the limiting Euler equations.

The main purpose of this paper is to study the quasi-neutral limit of the Schrödinger-Poisson system in the Coulomb gauge (1.1) for general ill-prepared initial data. As pointed out in [14], the general doping function determines the limiting behavior of the particle density. By formally letting $\lambda \to 0$ in the Poisson equation; we obtain $\mathcal{C}^\lambda \to \mathcal{C}$ and $\rho^\lambda = \mathcal{C}(x)$. This will affect the limiting behavior for the momentum $\rho^\lambda u^\lambda$; for instance, replace the ρ^λ by $\mathcal{C}$ in (1.7) and let $\lambda \to 0$. Similar to [14], we consider the simplest case

$$\mathcal{C}^\lambda(x) = 1 + \lambda g(x), \tag{1.8}$$

so that the fast oscillating singular term will be produced by the nondivergence free part of initial current (momentum), and this has to be treated carefully in order to pass into the quasineutral limit.

The Schrödinger-Poisson system is a nonlinear Schrödinger equation of the mean field type, in that the potential is given by the Poisson equation with the square of the unknown wave function as the source term. The system has been used to model electron-electron Coulomb interactions, typically in semi-conductor devices [4]. It also serves as a standard model in quantum mechanics describing the electrons moving on a positive charged background. Moreover, the time-dependent Schrödinger-Poisson system can be derived as the weak coupling limit of the N-body linear Schrödinger equation with Coulomb potential [3]. Employing the modulated energy method introduced by Brenier [6], Puel [21] considers the Schrödinger-Poisson system in the quasi-neutral limit and the asymptotic regime where the Planck's constant and the permittivity of the system go to zero simultaneously. It is then shown that the divergence-free components of the current (which is given in terms of the wave function) converge to a dissipative solution (in the sense of Lions) of the Euler equation. The combined semi-classical and quasi-neutral limit of the bipolar defocusing nonlinear Schrödinger-Poisson system in the whole space was discussed by Jüngel and Wang in [11]. The limit system is the compressible Euler equations with a nonlinear pressure depending on the plasma density only. The proof relies upon a modulated energy method and the Wigner transform of the equation. The modulated energy method has been successfully applied to the coupled rotating Schrödinger equations and the coupled Schrödinger equations [12, 16] for well-prepared initial data. Indeed, Brenier's modulated energy method has been extended by Masmoudi [19] (see also [20]) to treat the general initial data allowing the presence of high oscillations in time and this idea is also applied to the quantum hydrodynamic model of semiconductors [14], Navier-Stokes-Poisson system [10] and magnet-hydrodynamic equation [8]. We also refer to [7] for the intuitive discussion of the modulated energy.

The rest of the paper is organized as follows. In Section 2, we derive the equations of charge, momentum and energy associated to the Schrödinger-Poisson system in Coulomb gauge which play the key roles for the estimates of the modulated energy. The main theorem (Theorem 2.2) is presented, and the limit rotating incompressible Euler system and the coupled linear rotating, oscillating gradient vector field are also studied. Section 3 devotes to the proof of the main theorem. It is based on the spectral analysis of the associated highly oscillating wave operator which is similar to the acoustic wave in the low Mach number limit of the compressible fluid equations.

2. Main Results

We now consider the n dimensional ($n = 2, 3$) defocusing nonlinear Schrödinger-Poisson system in Coulomb gauge (1.1) with nonlinear potential $V'(|\psi^\lambda|^2) = |\psi^\lambda|^{2(\gamma-1)} - 1$, $\gamma \geq 2$. The system can be rewritten in the symmetric form by letting $\phi^\lambda =: \lambda\Phi^\lambda$;

$$\begin{cases} i\varepsilon\partial_t\psi^\lambda + \dfrac{\varepsilon^2}{2}\Delta_A\psi^\lambda - \big(|\psi^\lambda|^{2(\gamma-1)} - 1\big)\psi^\lambda = \dfrac{1}{\lambda}\phi^\lambda\psi^\lambda\,, \\ -\lambda\Delta\phi^\lambda = |\psi^\lambda|^2 - 1 - \lambda g(x)\,, \\ \psi^\lambda(x,0) = \psi_0^\lambda(x)\,, \qquad \phi^\lambda(x,0) = \phi_0^\lambda(x)\,, \end{cases} \tag{2.1}$$

where the given function $g(x)$ and the initial data $\psi_0^\lambda(x)$ and $\phi_0^\lambda(x)$ satisfy the compatibility condition

$$-\lambda\Delta\phi_0^\lambda = |\psi_0^\lambda|^2 - 1 - \lambda g(x)\,. \tag{2.2}$$

To avoid the complications at the boundary, we concentrate below on the case where $x \in \mathbb{T}^n$, the n-dimensional torus. We assume $\psi_0^\lambda(x) \in H^s(\mathbb{T}^n)$, $s > \frac{n}{2} + 2$, $g(x) \in C_0^\infty(\mathbb{T}^n)$ and $A \in C^\infty([0,\infty) \times \mathbb{T}^n)$ with $\nabla \cdot A = 0$. Moreover, the Poisson equation (2.2) on the torus $\mathbb{T}^n$ does not necessary have a solution, unless we impose the zero mean condition on the nonhomogeneous term, hence

$$\int_{\mathbb{T}^n} |\psi_0^\lambda|^2 dx = \int_{\mathbb{T}^n} \big(1 + \lambda g(x)\big) dx\,. \tag{2.3}$$

We define the density(charge) ρ^λ and the momentum(current) J^λ in terms of the wave function ψ^λ respectively by

$$\rho^\lambda = |\psi^\lambda|^2 = \psi^\lambda(\psi^\lambda)^*, \qquad J^\lambda = \frac{i\varepsilon}{2}\Big(\psi^\lambda\nabla(\psi^\lambda)^* - (\psi^\lambda)^*\nabla\psi^\lambda\Big)\,. \tag{2.4}$$

Here, $*$ denotes the complex conjugate of a complex number. Hence the velocity u^λ can be defined as the ratio of J^λ and ρ^λ by

$$u^\lambda = \frac{J^\lambda}{\rho^\lambda} = \frac{i\varepsilon}{2}\frac{1}{|\psi^\lambda|^2}\Big(\psi^\lambda\nabla(\psi^\lambda)^* - (\psi^\lambda)^*\nabla\psi^\lambda\Big)\,. \tag{2.5}$$

The energy density is defined by

$$
\begin{aligned}
e^\lambda &= \frac{\varepsilon^2}{2}|\nabla_A\psi^\lambda|^2 + \frac{1}{2}|\nabla\phi^\lambda|^2 + U(|\psi^\lambda|^2) \\
&= \frac{1}{2}\rho^\lambda|u^\lambda - A|^2 + \frac{\varepsilon^2}{2}|\nabla\sqrt{\rho^\lambda}|^2 + \frac{1}{2}|\nabla\phi^\lambda|^2 + U(\rho^\lambda)\,,
\end{aligned} \tag{2.6}
$$

where

$$
U(|\psi^\lambda|^2) = U(\rho^\lambda) = \frac{1}{\gamma}\Big((\rho^\lambda)^\gamma + (\gamma-1) - \gamma\rho^\lambda\Big) \tag{2.7}
$$

is a strictly convex function of ρ^λ, and the minimum occurs at $\rho^\lambda = 1$ and satisfies $U(\rho^\lambda) \geq 0$. The initial charge, velocity, and momentum (current) are given respectively by

$$
\rho^\lambda(x,0) = \rho_0^\lambda(x), \quad u^\lambda(x,0) = u_0^\lambda(x), \quad J^\lambda(x,0) = J_0^\lambda(x) = \rho_0^\lambda(x)u_0^\lambda(x)\,.
$$

Similar to (1.5)–(1.7), we have the following equations of charge (density), momentum (current) and energy associated with the Schrödinger-Poisson system in Coulomb gauge (2.1):

(A) Charge (density) equation

$$
\partial_t\rho^\lambda + \nabla\cdot J_A^\lambda = 0\,. \tag{2.8}
$$

(B) Momentum (current) equation

$$
\begin{aligned}
&\partial_t J_A^\lambda + \nabla\cdot\left(\frac{J_A^\lambda\otimes J_A^\lambda}{\rho^\lambda}\right) + \frac{1}{\lambda}\rho^\lambda\nabla\phi^\lambda + \frac{\gamma-1}{\gamma}\nabla(\rho^\lambda)^\gamma + \mathbb{G}_A(J_A^\lambda) \\
&\qquad + \rho^\lambda\partial_t A = \frac{1}{4}\varepsilon^2\nabla\cdot\Big(\rho^\lambda\nabla^2\log\rho^\lambda\Big)\,.
\end{aligned} \tag{2.9}
$$

(C) Energy equation

$$
\begin{aligned}
&\partial_t\Big[e^\lambda + \nabla\cdot(\phi^\lambda\nabla\phi^\lambda)\Big] - \frac{\varepsilon^2}{2}\nabla\cdot\Big(\nabla_A\psi^\lambda\partial_t(\psi^\lambda)^* + (\nabla_A\psi^\lambda)^*\partial_t\psi^\lambda\Big) \\
&\qquad = -\partial_t A\cdot J_A^\lambda\,.
\end{aligned} \tag{2.10}
$$

Note that the momentum and the energy equations (2.9)–(2.10) are not necessary conservative because of the existence of the magnetic field A. We deduce from the energy equation (2.10) that

$$\frac{d}{dt}\int_{\mathbb{T}^n} e^\lambda dx \le \|\partial_t A\|_{L^\infty(\mathbb{T}^n)}\int_{\mathbb{T}^n}\big|\sqrt{\rho^\lambda}u_A^\lambda\big|\big|\sqrt{\rho^\lambda}\big|dx$$

$$\le C_1\Big(\int_{\mathbb{T}^n} e^\lambda dx + \|\rho^\lambda\|_{L^1(\mathbb{T}^n)}\Big)\,.$$

Using the charge equation (2.8) and applying the Gronwall's inequality, we obtain

$$\int_{\mathbb{T}^n} e^\lambda(\cdot,t)dx \le e^{C_1 t}\Big[\int_{\mathbb{T}^n} e^\lambda(\cdot,0)dx + C_1 t\|\rho_0^\lambda\|_{L^1(\mathbb{T}^n)}\Big]\,. \tag{2.11}$$

This proves the uniform boundness of the total energy. Employing (2.11) and the duality argument, for $s \ge \frac{n}{2}+2$ and for all $f \in C_0^\infty(\mathbb{T}^n)$, we have

$$\begin{aligned}\Big|\int_{\mathbb{T}^n} J_A^\lambda(x,t)f(x)dx\Big| &\le \int_{\mathbb{T}^n}\big|\sqrt{\rho^\lambda}u_A^\lambda\big|\big|\sqrt{\rho^\lambda}f\big|dx\\ \le C\Big(\int_{\mathbb{T}^n}\rho^\lambda f^2dx\Big)^{1/2} &\le C\|\rho^\lambda\|_{L^1(\mathbb{T}^n)}^{1/2}\|f\|_{L^\infty(\mathbb{T}^n)} \le C\|f\|_{H^s(\mathbb{T}^n)}\,.\end{aligned} \tag{2.12}$$

Thus we have shown that $J_A^\lambda \in L^\infty([0,T];H^{-s}(\mathbb{T}^n))$. We now briefly describe the limit system which is composed of the divergence free and the oscillating parts [14, 19]. The divergence free part satisfies the incompressible rotating Euler equation

$$\begin{cases}\partial_t v + \mathcal{P}\nabla\cdot\big(v_A\otimes v_A\big) + \mathcal{P}\mathbb{G}_A(v_A) = 0\,,\\[2ex] \quad\nabla\cdot v = 0\,,\\[2ex] v(x,0) = v_0(x) =: \mathcal{P}(J_0)(x)\,,\end{cases} \tag{2.13}$$

where the divergence free initial condition $v_0(x) \in H^s(\mathbb{T}^n)$, $s > \frac{n}{2}+1$, and $v_A = v - A$ is the relative velocity with respect to the magnetic field A. Let $v \in C([0,T];H^s(\mathbb{T}^n))$ be a divergence-free vector field, then the oscillating terms describing by $(\nabla q, \nabla\phi)$ satisfy the coupled linear systems ([14, 19])

$$\left\{\begin{aligned} &\partial_t \nabla q + \frac{1}{2}\mathcal{Q}\nabla\cdot\big(v_A\otimes\nabla q + \nabla q\otimes v_A\big)\\ &\qquad + \frac{1}{2}\mathcal{Q}\big(g(x)\nabla\phi\big) + \frac{1}{2}\mathcal{Q}\mathbb{G}_A(\nabla q) = 0\,,\\ &\nabla q(x,0) = \nabla q_0(x) =: \mathcal{Q}(J_0)(x)\,, \end{aligned}\right. \tag{2.14}$$

$$\left\{\begin{aligned} &\partial_t \nabla \phi + \frac{1}{2}\mathcal{Q}\nabla\cdot\big(v_A\otimes\nabla \phi + \nabla \phi\otimes v_A\big)\\ &\qquad - \frac{1}{2}\mathcal{Q}\big(g(x)\nabla q\big) + \frac{1}{2}\mathcal{Q}\mathbb{G}_A(\nabla \phi) = 0\,,\\ &\nabla \phi(x,0) = \nabla \phi_0(x)\,, \end{aligned}\right. \tag{2.15}$$

with initial condition $(\nabla q_0, \nabla\phi_0)\in H^s(\mathbb{T}^n)\times H^s(\mathbb{T}^n)$. Here, $\mathcal{P}$ is the Leray projection operator onto the divergence-free vector field and $\mathcal{Q}$ is its orthogonal complement and they are defined for $u\in L^2(\mathbb{T}^n)$ as

$$\mathcal{Q}u = \nabla\Delta^{-1}\nabla\cdot u, \qquad \mathcal{P} = I - \mathcal{Q}, \qquad \nabla\cdot\mathcal{P}u = 0. \tag{2.16}$$

Let R_j denote the Riesz transform which is defined by $R_j = \frac{\partial_j}{\sqrt{-\Delta}}$, i.e. for $f\in L^2$ by $\mathcal{F}(R_j f) = \frac{i\xi_j}{|\xi|}\hat{f}(\xi)$. Then $\mathcal{P}$ is easily defined on L^2 as $\mathcal{P} = I - \mathcal{R}\otimes\mathcal{R}$ where $\mathcal{R}$ is the vector of the Riesz transform: $(\mathcal{P}f)_j = f_j + \sum_k R_j R_k f_k$. Since R_jR_k is a Calderón-Zygmund operator, $\mathcal{P}$ may be defined on many Banach spaces.

For the initial value problem (2.13)–(2.15), we have the following local existence result.

Proposition 2.1. *Let $v_0\in H^s(\mathbb{T}^n)$, $s>\frac{n}{2}+1$, then there exists a function $v\in C([0,T];H^s(\mathbb{T}^n))$ solving the rotating incompressible Euler equation* (2.13)*. For a given divergence free vector field $v\in C([0,T];H^s(\mathbb{T}^n))$, let $(\nabla q_0,\nabla\phi_0)\in(H^s(\mathbb{T}^n))^2$, then there exists $(\nabla q,\nabla\phi)\in\big(C([0,T];H^s(\mathbb{T}^n))\big)^2$ solve the coupled system* (2.14) – (2.15)*. Moreover, we have the energy relation*

$$\partial_t\int_{\mathbb{T}^n} e(\cdot,t)dx = -\int_{\mathbb{T}^n}\partial_t A\cdot v_A dx\,, \tag{2.17}$$

where the energy density $e(x,t)$ *is given by*

$$e(x,t) = \frac{1}{2}\big(|v_A|^2 + |\nabla q|^2 + |\nabla\phi|^2\big)\,. \tag{2.18}$$

The proof of this proposition proceeds the lines of the proof for the standard incompressible Euler equation [17] with modifications because the rotating term $\mathbb{G}_A(v_A)$ contributes nothing to the energy estimate and it is a linear function of v_A only.

Let $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary parts of the complex number z. We introduce the modulated energy H_A^λ defined by

$$\begin{aligned} H_A^\lambda(t) &= \frac{1}{2}\int_{\mathbb{T}^n}\Big|\Big(\varepsilon\nabla_A - i\big(v_A + \mathrm{Re}\big(\mathcal{G}(t)e^{it/\lambda}\big)\big)\Big)\psi^\lambda\Big|^2 dx \\ &\quad + \frac{1}{2}\int_{\mathbb{T}^n}|\nabla\phi^\lambda - \mathrm{Im}(\mathcal{G}(t)e^{it/\lambda})|^2 dx + \int_{\mathbb{T}^n} U(|\psi^\lambda|^2)\,dx\,, \end{aligned} \tag{2.19}$$

where $\mathcal{G}(t) = \nabla q + i\nabla\phi$. Assuming $H_A^\lambda(0) \to 0$ as $\lambda \to 0$, the initial modulated energy $H_A^\lambda(0)$ is given by

$$\begin{aligned} H_A^\lambda(0) &= \frac{1}{2}\int_{\mathbb{T}^n}\big|\big(\varepsilon\nabla_A - i(v_{A0} + \nabla q_0)\big)\psi_0^\lambda\big|^2 dx \\ &\quad + \frac{1}{2}\int_{\mathbb{T}^n}|\nabla\phi_0^\lambda - \nabla\phi_0|^2 dx + \int_{\mathbb{T}^n} U(|\psi_0^\lambda|^2)\,dx\,. \end{aligned} \tag{2.20}$$

Here $v_{A0}(x) = v_A(x,0) = v_0(x) - A$ is the initial value of the relative velocity $v_A(x,t)$. Employing a combination of the equations of charge, momentum and energy of the rotating Schrödinger-Poisson system (2.8)–(2.10) and the limit equations (2.13)–(2.15), we can prove

$$H_A^\lambda(t) \le H_A^\lambda(0) + C\lambda + \int_0^t H_A^\lambda(s)ds\,,$$

and then deduce by the Gronwall's inequality that $H_A^\lambda(t)$ tends to zero as $\lambda \to 0$. The main result of this paper is stated as follows.

THEOREM 2.2. *Let* $(\psi^\lambda, \phi^\lambda)$ *be the solution of the Schrödinger-Poisson system in Coulomb gauge* (2.1) *with initial condition* $(\psi_0^\lambda, \phi_0^\lambda)$*, for which*

$\psi_0^\lambda \in H^s(\mathbb{T}^n)$, $s > \frac{n}{2} + 2$, *and satisfying* (2.2), (2.3) *and* (2.20). *Let* $(v, \nabla q, \nabla\phi)$ *be the solution of the system* (2.13)–(2.15) *and the initial condition* $(v_0, \nabla q_0, \nabla\phi_0) \in \left(H^k(\mathbb{T}^n)\right)^3$, $k > \frac{n}{2} + 1$, *then there exists* $T > 0$ *such that* $H_A^\lambda(t) \to 0$ *as* $\lambda \to 0$ *uniform in* $t \in [0, T]$. *Moreover, we have*

$$\rho^\lambda \to 1 \quad \textit{strongly in} \quad L^\infty\big([0,T]; L^\gamma(\mathbb{T}^n)\big)\,, \tag{2.21}$$

$$J^\lambda \rightharpoonup v \quad \textit{weakly} * \textit{in} \quad L^\infty\big([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)\big)\,, \tag{2.22}$$

$$\nabla\phi^\lambda \rightharpoonup 0 \quad \textit{weakly} * \textit{in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)\,. \tag{2.23}$$

Let us remark that in Proposition 2.1, the initial condition $v_0 \in H^s(\mathbb{T}^n)$, $s > \frac{n}{2} + 1$, but in Theorem 2.2 we require $\psi_0^\lambda \in H^s(\mathbb{T}^n), s > \frac{n}{2} + 2$, because it satisfies the Poisson equation.

3. Proof of the Theorem 2.2

This section presents the proof of the main theorem. For convenience, we will divide the proof into the following four steps.

Step 1. Spectral analysis and cancelation of the oscillations.

Consider the eigenvalue problem of the isometry operator L defined on the space $\mathbb{H} =: L^2(\mathbb{T}^n) \times \{\nabla\varphi : \varphi \in H^1(\mathbb{T}^n)\}$, by

$$LU = \mu U, \qquad U \in \mathbb{H}\,,$$

such that

$$L\begin{pmatrix} w \\ 0 \end{pmatrix} = 0 \quad \text{if} \quad \nabla\cdot w = 0 \quad \text{and} \quad L\begin{pmatrix} \nabla\chi \\ \nabla\varphi \end{pmatrix} = \begin{pmatrix} -\nabla\varphi \\ \nabla\chi \end{pmatrix}. \tag{3.1}$$

Obviously, the operator L has three eigenvalues $\mu = \pm i, 0$ with the corresponding eigenspaces

$$E_{\pm i} = \left\{ \begin{pmatrix} \nabla\varphi \\ \mp i\nabla\varphi \end{pmatrix} : \varphi \in H^1(\mathbb{T}^n) \right\},$$

$$E_0 = \left\{ \begin{pmatrix} \omega \\ 0 \end{pmatrix} : \nabla\cdot\omega = 0, \omega \in H^1(\mathbb{T}^n) \right\}.$$

Let $\mathcal{L}$ be the evolution group associated with the operator L, $\mathcal{L}(\tau) \equiv e^{\tau L}$, $\tau \in \mathbb{R}$. As L is skew-symmetric, the operator $\mathcal{L}(\tau)$ is unitary for all times τ, in all Sobolev space $H^s(\mathbb{T}^n) \times H^s(\mathbb{T}^n)$, $s \geq 1$ (see [14] for the proof). It holds for $W = (\omega, 0)^t + (\nabla\chi, \nabla\varphi)^t$ that

$$\begin{aligned} \mathcal{L}(\tau)W = \begin{pmatrix} \omega \\ 0 \end{pmatrix} + \frac{1}{2}e^{i\tau} \begin{pmatrix} \nabla\chi + i\nabla\varphi \\ -i\nabla\chi + \nabla\varphi \end{pmatrix} \\ + \frac{1}{2}e^{-i\tau} \begin{pmatrix} \nabla\chi - i\nabla\varphi \\ i\nabla\chi + \nabla\varphi \end{pmatrix}. \end{aligned} \tag{3.2}$$

We now rewrite the Schrödinger-Poisson system (2.1) as

$$\begin{cases} \partial_t \rho^\lambda + \nabla \cdot J_A^\lambda = 0\,, \\ \partial_t J_A^\lambda + \dfrac{1}{\lambda}\nabla\phi^\lambda + F^\lambda + G^\lambda = 0\,, \\ \partial_t \nabla\phi^\lambda - \dfrac{1}{\lambda}\mathcal{Q}(J_A^\lambda) = 0\,, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} F^\lambda = \nabla \cdot \left(\rho^\lambda u_A^\lambda \otimes u_A^\lambda\right) - \nabla \cdot (\nabla\phi^\lambda \otimes \nabla\phi^\lambda) - \frac{1}{2}\nabla(|\nabla\phi^\lambda|^2) \\ + g\nabla\phi^\lambda + \frac{\gamma - 1}{\gamma}\nabla(\rho^\lambda)^\gamma + \rho^\lambda \partial_t A + \mathbb{G}_A(J_A^\lambda)\,, \end{aligned} \tag{3.4}$$

and

$$G^\lambda = -\frac{1}{4}\varepsilon^2 \Delta\nabla\rho^\lambda + \varepsilon^2 \nabla \cdot \left(\nabla\sqrt{\rho^\lambda} \otimes \nabla\sqrt{\rho^\lambda}\right). \tag{3.5}$$

It is obvious from $(3.3)_{2,3}$ that $\partial_t J_A^\lambda$ and $\partial_t \nabla\phi^\lambda$ are of order $O(1/\lambda)$ and are highly oscillatory as $\lambda \to 0$. This is the main reason why we have to introduce the wave group in order to filter out the fast oscillating wave. Comparing with Eq.$(3.3)_3$, it is natural to project the momentum equation $(3.3)_2$ on the gradient vector fields

$$\partial_t \mathcal{Q}(J_A^\lambda) + \frac{1}{\lambda}\nabla\phi^\lambda + \mathcal{Q}(F^\lambda + G^\lambda) = 0\,. \tag{3.6}$$

Combing (3.6) with the Poisson equation $(3.3)_3$, we obtain

$$\partial_t \mathcal{Q}(J_A^\lambda) + \frac{1}{\lambda}\nabla\phi^\lambda + \mathcal{Q}(F^\lambda + G^\lambda) = 0\,, \qquad \partial_t \nabla\phi^\lambda - \frac{1}{\lambda}\mathcal{Q}(J_A^\lambda) = 0\,. \tag{3.7}$$

Thus, we will study the evolution of the vector fields $\widetilde{U}^\lambda = \begin{pmatrix} \mathcal{Q}(J_A^\lambda) \\ \nabla\phi^\lambda \end{pmatrix}$. For convenience, let

$$U^\lambda = \begin{pmatrix} \mathcal{P}(J_A^\lambda) \\ 0 \end{pmatrix} + \widetilde{U}^\lambda$$

and rewrite equation (3.7) as

$$\partial_t \widetilde{U}^\lambda = \frac{1}{\lambda} L \widetilde{U}^\lambda - \begin{pmatrix} \mathcal{Q}(F^\lambda + G^\lambda) \\ 0 \end{pmatrix},$$

which can be converted into

$$\partial_t \widetilde{V}^\lambda = -\mathcal{L}\Big(\frac{-t}{\lambda}\Big) \begin{pmatrix} \mathcal{Q}(F^\lambda + G^\lambda) \\ 0 \end{pmatrix}, \qquad \widetilde{V}^\lambda = \mathcal{L}\Big(\frac{-t}{\lambda}\Big)\widetilde{U}^\lambda$$

by applying the operator $\mathcal{L}(\frac{-t}{\lambda})$. We also define

$$V^\lambda = \mathcal{L}\Big(\frac{-t}{\lambda}\Big) U^\lambda = \mathcal{L}\Big(\frac{-t}{\lambda}\Big) \begin{pmatrix} \mathcal{P}(J_A^\lambda) \\ 0 \end{pmatrix} + \widetilde{V}^\lambda\,,$$

then it follows from (3.3) that V^λ satisfies

$$\partial_t V^\lambda = -\mathcal{L}\Big(\frac{-t}{\lambda}\Big) \begin{pmatrix} F^\lambda + G^\lambda \\ 0 \end{pmatrix}. \tag{3.8}$$

To passing the limit, we need more compactness of the solutions sequence in space variables x. By (2.12), one can show that V^λ is uniformly bounded in $L^\infty\big([0,T], H^{-s}(\mathbb{T}^n)\big)$ for $s \geq \frac{n}{2} + 2$. Similar argument as (2.12) shows that $F^\lambda + G^\lambda$ is uniformly bounded in $L^1\big([0,T]; H^{-m}(\mathbb{T}^n)\big)$ for $m \geq \frac{n}{2} + 3$, and by the isometry property of $\mathcal{L}$, we can show that $\partial_t V^\lambda$ is uniformly bounded in $L^1\big([0,T]; H^{-m}(\mathbb{T}^n)\big)$. Therefore, we deduce from the Lions-Aubin's lemma that there exists a subsequence of $\{V^\lambda\}_\lambda$ which we still denote by $\{V^\lambda\}_\lambda$ and $\overline{V} \in L^1\big([0,T]; H^{-m}(\mathbb{T}^n)\big)$, such that

$$V^\lambda \to \overline{V} \quad \text{strongly in} \quad L^1\big([0,T]; H^{-m}(\mathbb{T}^n)\big)\,. \tag{3.9}$$

We also have the similar compactness for $\{\widetilde{V}^\lambda\}_\lambda$;

$$\widetilde{V}^\lambda \to \widetilde{\overline{V}} \quad \text{strongly in} \quad L^1\big([0,T];H^{-m}(\mathbb{T}^n)\big)\,. \tag{3.10}$$

Indeed, the limits $\overline{V}$ and $\widetilde{\overline{V}}$ can be specified as

$$\overline{V} = \begin{pmatrix} \overline{v}-A \\ 0 \end{pmatrix} + \widetilde{\overline{V}}, \qquad \widetilde{\overline{V}} = \begin{pmatrix} \nabla\overline{q} \\ \nabla\overline{\phi} \end{pmatrix}.$$

Similarly, we have

$$V = \begin{pmatrix} v_A \\ 0 \end{pmatrix} + \widetilde{V}, \qquad \widetilde{V} = \begin{pmatrix} \nabla q \\ \nabla\phi \end{pmatrix}.$$

Step 2. Uniform estimate of the modulated energy $H_A^\lambda(t)$.

We observe that

$$\mathrm{Re}(\mathcal{G}(t)e^{it/\lambda}) = \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V}\,, \qquad \mathrm{Im}(\mathcal{G}(t)e^{it/\lambda}) = \mathcal{L}_2(\tfrac{t}{\lambda})\widetilde{V},$$

and the modulated energy (2.19) can be rewritten as

$$\begin{aligned} H_A^\lambda(t) = \frac{1}{2}\int_{\mathbb{T}^n}\Big|\big(\varepsilon\nabla_A - i(v_A + \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V})\big)\psi^\lambda\Big|^2 dx \\ +\frac{1}{2}\int_{\mathbb{T}^n}|\nabla\phi^\lambda - \mathcal{L}_2(\tfrac{t}{\lambda})\widetilde{V}|^2 dx + \int_{\mathbb{T}^n} U(|\psi^\lambda|^2)\,dx\,. \end{aligned}$$

Using the hydrodynamical variables, it can be expressed as

$$\begin{aligned} H_A^\lambda(t) = \frac{1}{2}\int_{\mathbb{T}^n}\rho^\lambda\Big|u_A^\lambda - \Big(v_A + \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V}\Big)\Big|^2 dx + \frac{\varepsilon^2}{2}\int_{\mathbb{T}^n}|\nabla\sqrt{\rho^\lambda}|^2 dx \\ +\frac{1}{2}\int_{\mathbb{T}^n}|\nabla\phi^\lambda - \mathcal{L}_2(\tfrac{t}{\lambda})\widetilde{V}|^2 dx + \int_{\mathbb{T}^n} U(\rho^\lambda)\,dx\,. \end{aligned} \tag{3.11}$$

Moreover, one have

$$\begin{aligned} H_A^\lambda(t) = \int_{\mathbb{T}^n} e^\lambda(\cdot,t)dx + \frac{1}{2}\int_{\mathbb{T}^n}(\rho^\lambda - 1)\big|v_A + \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V}\big|^2 dx \\ + \int_{\mathbb{T}^n} e(\cdot,t)dx - \int_{\mathbb{T}^n} J_A^\lambda\cdot\big(v_A + \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V}\big)dx - \int_{\mathbb{T}^n}\nabla\phi^\lambda\cdot\mathcal{L}_2(\tfrac{t}{\lambda})\widetilde{V}dx. \end{aligned}$$

Applying the energy estimate (2.10) and (2.17)

$$H_A^\lambda(t) = \int_{\mathbb{T}^n} e^\lambda(\cdot,0)\,dx + \int_{\mathbb{T}^n} e(\cdot,0)dx + \frac{1}{2}\int_{\mathbb{T}^n}(\rho^\lambda - 1)\big|v_A + \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V}\big|^2 dx$$

$$-\int_0^t\int_{\mathbb{T}^n}\partial_t A\cdot J_A^\lambda dxds - \int_0^t\int_{\mathbb{T}^n}\partial_t A\cdot v_A dxds$$

$$-\int_{\mathbb{T}^n} J_A^\lambda\cdot\big(v_A + \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V}\big)dx - \int_{\mathbb{T}^n}\nabla\phi^\lambda\cdot\mathcal{L}_2(\tfrac{t}{\lambda})\widetilde{V}dx$$

By simply computations, we rewrite the modulated energy $H_A^\lambda(t)$ as

$$H_A^\lambda(t) = H_A^\lambda(0) + \frac{1}{2}\int_{\mathbb{T}^n}|\rho_0^\lambda - 1||J_0 - A|^2 dx$$

$$+\frac{1}{2}\int_{\mathbb{T}^n}(\rho^\lambda - 1)\big|v_A + \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V}\big|^2 dx$$

$$-\int_0^t\int_{\mathbb{T}^n}\partial_s\Big[J_A^\lambda\cdot\big(v_A + \mathcal{L}_1(\tfrac{s}{\lambda})\widetilde{V}\big)\Big]dxds$$

$$-\int_0^t\int_{\mathbb{T}^n}\partial_s\Big[\nabla\phi^\lambda\cdot\mathcal{L}_2(\tfrac{s}{\lambda})\widetilde{V}\Big]dxds$$

$$-\int_0^t\int_{\mathbb{T}^n}\partial_t A\cdot J_A^\lambda dxds - \int_0^t\int_{\mathbb{T}^n}\partial_t A\cdot v_A dxds\,,$$

then using the compatibility conditions (2.2) and the equation (3.3) together, it yields

$$H_A^\lambda(t) \le H_A^\lambda(0) + C\lambda - \int_0^t\int_{\mathbb{T}^n}\partial_t A\cdot J_A^\lambda dxds - \int_0^t\int_{\mathbb{T}^n}\partial_t A\cdot v_A dxds$$

$$+\int_0^t\int_{\mathbb{T}^n}\big(F^\lambda + G^\lambda + \tfrac{1}{\lambda}\nabla\phi^\lambda\big)\cdot\big(v_A + \mathcal{L}_1(\tfrac{s}{\lambda})\widetilde{V}\big)dxds$$

$$-\int_0^t\int_{\mathbb{T}^n}J_A^\lambda\cdot\partial_s v_A dxds - \int_0^t\int_{\mathbb{T}^n}J_A^\lambda\cdot\partial_s\big(\mathcal{L}_1(\tfrac{s}{\lambda})\widetilde{V}\big)dxds$$

$$
-\int_0^t\int_{\mathbb{T}^n}\tfrac{1}{\lambda}Q(J_A^\lambda)\cdot\big(\mathcal{L}_2(\tfrac{s}{\lambda})\widetilde{V}\big)dxds
$$

$$
-\int_0^t\int_{\mathbb{T}^n}\nabla\phi^\lambda\cdot\partial_s\big(\mathcal{L}_2\big(\tfrac{s}{\lambda}\big)\widetilde{V}\big)dxds\,.
$$

Decomposing J_A^λ into the divergence and curl free parts, and using the fact that v_A and $\mathcal{P}(J_A^\lambda)$ are divergence free, we have

$$
\int_0^t\int_{\mathbb{T}^n}\nabla\phi^\lambda\cdot\big(v_A+\mathcal{L}_1(\tfrac{s}{\lambda})\widetilde{V}\big)dxds+\int_0^t\int_{\mathbb{T}^n}J_A^\lambda\cdot\mathcal{L}_2(\tfrac{s}{\lambda})\widetilde{V}dxds
$$

$$
-\int_0^t\int_{\mathbb{T}^n}\nabla\phi^\lambda\cdot\mathcal{L}_1(\tfrac{s}{\lambda})\widetilde{V}dxds-\int_0^t\int_{\mathbb{T}^n}\mathcal{Q}(J_A^\lambda)\cdot\mathcal{L}_2(\tfrac{s}{\lambda})\widetilde{V}dxds=0\,.
$$

Thus we have the following inequality

$$
H_A^\lambda(t)\le H_A^\lambda(0)+C\lambda-\int_0^t\int_{\mathbb{T}^n}\partial_tA\cdot J_A^\lambda dxds-\int_0^t\int_{\mathbb{T}^n}\partial_tA\cdot v_Adxds
$$

$$
+\int_0^t\int_{\mathbb{T}^n}\big(F^\lambda+G^\lambda\big)\cdot\big(v_A+\mathcal{L}_1(\tfrac{s}{\lambda})\widetilde{V}\big)dxds
$$

$$
-\int_0^t\int_{\mathbb{T}^n}J_A^\lambda\cdot\partial_sv_Adxds
$$

$$
-\int_0^t\int_{\mathbb{T}^n}J_A^\lambda\cdot\mathcal{L}_1(\tfrac{s}{\lambda})\partial_s\widetilde{V}dxds-\int_0^t\int_{\mathbb{T}^n}\nabla\phi^\lambda\cdot\mathcal{L}_2(\tfrac{s}{\lambda})\partial_s\widetilde{V}dxds\,.
$$

To treat the rotating part, we observe the antisymmetric property

$$
\begin{aligned}
\mathbb{G}_A(J_A^\lambda)\cdot\mathcal{L}_1(\tfrac{s}{\lambda})\widetilde{V}&=\big(\operatorname{curl}A\times J_A^\lambda\big)\cdot\mathcal{L}_1(\tfrac{s}{\lambda})\widetilde{V}\\
&=-\big(\operatorname{curl}A\times\mathcal{L}_1(\tfrac{s}{\lambda})\widetilde{V}\big)\cdot J_A^\lambda=-\mathbb{G}_A(\mathcal{L}_1\big(\tfrac{s}{\lambda})\widetilde{V}\big)\cdot J_A^\lambda
\end{aligned}\tag{3.12}
$$

then by straightforward computation, it gives

$$
H_A^\lambda(t)\le H_A^\lambda(0)+C\lambda+R_1^\lambda(t)+\int_0^t\int_{\mathbb{T}^n}R_2^\lambda dxds-\int_0^t\int_{\mathbb{T}^n}U^\lambda\cdot\mathcal{L}(\tfrac{s}{\lambda})\partial_sVdxds
$$

$$
-\int_0^t\int_{\mathbb{T}^n}B_3(\partial_tA)\cdot U^\lambda dxds-\int_0^t\int_{\mathbb{T}^n}\partial_tA\cdot v_Adxds
$$

$$
\begin{aligned}
&-\int_0^t\int_{\mathbb{T}^n} B_3\Big(\mathbb{G}_A\big(\mathcal{L}_1(\tfrac{t}{\lambda})V\big)\Big)\cdot U^\lambda dxds\\
&+\int_0^t\int_{\mathbb{T}^n} B_3\big(\partial_t A\big)\cdot\mathcal{L}(\tfrac{s}{\lambda})V + B_3\big(g\mathcal{L}_2(\tfrac{s}{\lambda})V^\lambda\big)\cdot\mathcal{L}(\tfrac{s}{\lambda})V dxds\\
&+\int_0^t\int_{\mathbb{T}^n} 2\widetilde{B}\big(U^\lambda,\mathcal{L}(\tfrac{s}{\lambda})V\big)\cdot\mathcal{L}(\tfrac{s}{\lambda})V\\
&-\widetilde{B}\big(\mathcal{L}(\tfrac{s}{\lambda})V,\mathcal{L}(\tfrac{s}{\lambda})V\big)\cdot\mathcal{L}(\tfrac{s}{\lambda})V dxds
\end{aligned}
$$

where

$$
\begin{aligned}
R_1^\lambda(t) = &\int_0^t\int_{\mathbb{T}^n}(\rho^\lambda-1)\partial_t A\cdot\mathcal{L}_1(\tfrac{t}{\lambda})V dxds\\
&-\int_0^t\int_{\mathbb{T}^n}\nabla\cdot\big[(\rho^\lambda-1)\mathcal{L}_1(\tfrac{s}{\lambda})V\otimes\mathcal{L}_1(\tfrac{s}{\lambda})V\big]\cdot\mathcal{L}_1(\tfrac{s}{\lambda})V dxds\\
&-\frac{1}{4}\varepsilon^2\int_0^t\int_{\mathbb{T}^n}\Delta\nabla\rho^\lambda\cdot\mathcal{L}_1(\tfrac{s}{\lambda})V dxds + (\gamma-1)\int_0^t\int_{\mathbb{T}^n}\nabla\rho^\lambda\cdot\mathcal{L}_1(\tfrac{s}{\lambda})V dxds
\end{aligned}
$$

and

$$
\begin{aligned}
R_2^\lambda = &\nabla\cdot\Big[\sqrt{\rho^\lambda}\big(u_A^\lambda-\mathcal{L}_1(\tfrac{s}{\lambda})V\big)\otimes\sqrt{\rho^\lambda}\big(u_A^\lambda-\mathcal{L}_1(\tfrac{s}{\lambda})V\big)\Big]\cdot\mathcal{L}_1(\tfrac{s}{\lambda})V\\
&+\varepsilon^2\nabla\cdot\big(\nabla\sqrt{\rho^\lambda}\otimes\nabla\sqrt{\rho^\lambda}\big)\cdot\mathcal{L}_1(\tfrac{s}{\lambda})V\\
&-\nabla\cdot\Big[\big(\nabla\phi^\lambda-\mathcal{L}_2(\tfrac{s}{\lambda})V\big)\otimes\big(\nabla\phi^\lambda-\mathcal{L}_2(\tfrac{s}{\lambda})V\big)\Big]\cdot\mathcal{L}_1(\tfrac{s}{\lambda})V\\
&+\frac{1}{2}\nabla\Big(\big|\nabla\phi^\lambda-\mathcal{L}_2(\tfrac{s}{\lambda})V\big|^2\Big)\cdot\mathcal{L}_1(\tfrac{s}{\lambda})V\\
&+\frac{\gamma-1}{\gamma}\nabla\Big[(\rho^\lambda)^\gamma+(\gamma-1)-\gamma\rho^\lambda\Big]\cdot\mathcal{L}_1(\tfrac{s}{\lambda})V\,.
\end{aligned}
$$

Let ω be a vector in $\mathbb{R}^n$, $W_1=(W_1^1,W_1^2)^t=(\omega_1+\nabla\chi_1,\nabla\varphi_1)^t$ and $W_2=$

$(W_2^1, W_2^2)^t = (\omega_2 + \nabla\chi_2, \nabla\varphi_2)^t$. We define the two vectors B_3 and $\widetilde{B}$ by

$$B_3(\omega) = \begin{pmatrix} \omega \\ 0 \end{pmatrix}, \quad \widetilde{B}(W_1, W_2) = \begin{pmatrix} B_1(W_1^1, W_2^1) + B_2(W_1^2, W_2^2) \\ 0 \end{pmatrix}$$

where B_1 and B_2 are the bilinear forms given respectively by

$$B_1(W_1^1, W_2^1) = \frac{1}{2}\nabla \cdot \left(W_1^1 \otimes W_2^1 + W_2^1 \otimes W_1^1\right),$$

$$B_2(W_1^2, W_2^2) = -\frac{1}{2}\nabla \cdot \left(W_1^2 \otimes W_2^2 + W_2^2 \otimes W_1^2\right) + \frac{1}{2}\nabla\left(W_1^2 \cdot W_2^2\right).$$

Step 3. Rate of Convergence.

It is easy to see that

$$|R_1^\lambda(t)| < C\lambda \qquad \text{and} \qquad \left|\int_0^t \int_{\mathbb{T}^n} R_2^\lambda dxds\right| < C\int_0^t H_A^\lambda(s)ds\,.$$

Using the isometric property of $\mathcal{L}$, we have

$$\begin{aligned} H_A^\lambda(t) \le\; & H_A^\lambda(0) + C\lambda + C\int_0^t H_A^\lambda(s)ds - \int_0^t \int_{\mathbb{T}^n} V^\lambda \cdot \partial_s V dxds \\ & -\int_0^t \int_{\mathbb{T}^n} \mathcal{L}(\tfrac{-s}{\lambda}) B_3(\partial_t A) \cdot V^\lambda dxds \\ & -\int_0^t \int_{\mathbb{T}^n} \mathcal{L}(\tfrac{-s}{\lambda}) \Big[B_3\Big(\mathbb{G}_A\big(\mathcal{L}_1(\tfrac{t}{\lambda})V\big)\Big)\Big] \cdot V^\lambda dxds \\ & +\int_0^t \int_{\mathbb{T}^n} \mathcal{L}(\tfrac{-s}{\lambda}) B_3\Big(g\mathcal{L}_2(\tfrac{s}{\lambda})V^\lambda\Big) \cdot V dxds \\ & +\int_0^t \int_{\mathbb{T}^n} \mathcal{L}(\tfrac{-s}{\lambda}) \Big[2\widetilde{B}\Big(U^\lambda, \mathcal{L}(\tfrac{s}{\lambda})V\Big) - \widetilde{B}\Big(\mathcal{L}(\tfrac{s}{\lambda})V, \mathcal{L}(\tfrac{s}{\lambda})V\Big)\Big] \cdot V dxds\,. \end{aligned}$$

To proceed, we introduce the following notations. Let $W_1 = (W_1^1, W_1^2)^t = (\omega_1 + \nabla\chi_1, \nabla\varphi_1)^t$ and $W_2 = (W_2^1, W_2^2)^t = (\omega_2 + \nabla\chi_2, \nabla\varphi_2)^t$, one can define

$$\overline{\mathcal{B}}(W_1, W_2) = \begin{pmatrix} \mathcal{P}\nabla\cdot(\omega_1\otimes\omega_2) \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \mathcal{Q}\nabla\cdot(\omega_1\otimes\nabla\chi_2 + \nabla\chi_2\otimes\omega_1) \\ \\ \mathcal{Q}\nabla\cdot(\omega_1\otimes\nabla\varphi_2 + \nabla\varphi_2\otimes\omega_1) \end{pmatrix},$$

and

$$\overline{\mathcal{A}}(W_1) = \frac{1}{2}\begin{pmatrix} \mathcal{Q}\big(g\nabla\varphi_1\big) \\ \\ -\mathcal{Q}\big(g\nabla\chi_1\big) \end{pmatrix}.$$

Similarly, we decompose the rotating part $\overline{\mathbb{G}}_A(W_1)$ into the divergence free and the oscillating parts as

$$\overline{\mathbb{G}}_A(W_1) = \begin{pmatrix} \mathcal{P}\mathbb{G}_A(\omega_1) \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \mathcal{Q}\mathbb{G}_A(\nabla\chi_1) \\ \\ \mathcal{Q}\mathbb{G}_A(\nabla\varphi_1) \end{pmatrix}$$

After some computations, we have

$$\begin{aligned} \mathcal{L}\Big(\frac{-t}{\lambda}\Big)\widetilde{B}\left(\mathcal{L}\Big(\frac{t}{\lambda}\Big)V, \mathcal{L}\Big(\frac{t}{\lambda}\Big)V\right) &= \overline{\mathcal{B}}(V,V) + \frac{1}{2}\sum_{k=1}^{3} e^{ik\frac{t}{\lambda}}T_k(V,V) \\ &\quad + \frac{1}{2}\sum_{k=1}^{3} e^{-ik\frac{t}{\lambda}}T_k^*(V,V), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} 2\mathcal{L}\Big(\frac{-t}{\lambda}\Big)\widetilde{B}\left(\mathcal{L}\Big(\frac{t}{\lambda}\Big)V^\lambda, \mathcal{L}\Big(\frac{t}{\lambda}\Big)V\right) &= \overline{\mathcal{B}}(V^\lambda,V) + \overline{\mathcal{B}}(V,V^\lambda) \\ &\quad + \sum_{k=1}^{3} e^{ik\frac{t}{\lambda}}T_k(V^\lambda,V) + \sum_{k=1}^{3} e^{-ik\frac{t}{\lambda}}T_k^*(V,V^\lambda), \end{aligned} \tag{3.14}$$

where $T_k(V,V)$, $k=1,2,3$, is a bilinear form and T_k^* denotes the conjugate of T_k. Now, we study the convergence result of (3.13) and (3.14) (see also [14, 19]). For (3.13), it is easy to see that $\mathcal{L}(\frac{-t}{\lambda})\widetilde{B}\left(\mathcal{L}(\frac{t}{\lambda})V, \mathcal{L}(\frac{t}{\lambda})V\right)$ is bounded in

$L^\infty([0,T];W^{-1,1}(\mathbb{T}^n))$. For any test function $f(t,x)\in L^1([0,T];W^{1,\infty}(\mathbb{T}^n))$, by Riemann-Lebesgue lemma, we have

$$\lim_{\lambda\to 0}\int_0^{t_1}\int_{\mathbb{T}^n} e^{ik\frac{t}{\lambda}}T_k(V,V)f(t,x)dxdt=0\,,$$

this shows

$$\mathcal{L}\Big(\frac{-t}{\lambda}\Big)\widetilde{B}\left(\mathcal{L}\Big(\frac{t}{\lambda}\Big)V,\mathcal{L}\Big(\frac{t}{\lambda}\Big)V\right)\to\overline{\mathcal{B}}(V,V) \tag{3.15}$$

weakly in $L^\infty([0,T];W^{-1,1}(\mathbb{T}^n))$. For (3.14), using the technique of Friedrich's mollifier we get

$$\begin{aligned}\|V^\lambda-\overline{V}\|_{L^1([0,T];H^{-m})}\le\|V^\lambda-V_\delta^\lambda\|_{L^1([0,T];H^{-m})}&\overset{\delta\to 0}{\to}0\\ +\|V_\delta^\lambda-\overline{V}_\delta\|_{L^1([0,T];H^{-m})}&\overset{\lambda\to 0}{\to}0\\ +\|\overline{V}_\delta-\overline{V}\|_{L^1([0,T];H^{-m})}&\overset{\delta\to 0}{\to}0,\end{aligned} \tag{3.16}$$

where $m\ge\frac{n}{2}+2$, $f_\delta=M_\delta * f$ with $M_\delta *$ the Friedrich's mollifier. Using (3.16) and repeating the same argument as in the proof of (3.15), we have

$$2\mathcal{L}\Big(\frac{-t}{\lambda}\Big)\widetilde{B}\left(\mathcal{L}\Big(\frac{t}{\lambda}\Big)V^\lambda,\mathcal{L}\Big(\frac{t}{\lambda}\Big)V\right)\to\overline{\mathcal{B}}(\overline{V},V)+\overline{\mathcal{B}}(V,\overline{V}) \tag{3.17}$$

in the sense of distribution. Moreover, we have the spectral decomposition of the rotating part;

$$\begin{aligned}\mathcal{L}\Big(\frac{-t}{\lambda}\Big)\left[B_3\Big(\mathbb{G}_A\Big(\mathcal{L}_1\Big(\frac{t}{\lambda}\Big)V\Big)\Big)\right]&=\overline{\mathbb{G}}_A(V)\\ &+\sum_{k=1}^{2}e^{ik\frac{t}{\lambda}}G_k\big(\mathbb{G}_A(V)\big)+\sum_{k=1}^{2}e^{-ik\frac{t}{\lambda}}G_k^*\big(\mathbb{G}_A(V)\big)\end{aligned}$$

where G_1 and G_2 are linear functionals of $\mathbb{G}_A(V)$, G_1^* and G_2^* are their complex conjugates respectively. Hence, we deduce from the Riemann-Lebesgue lemma that

$$\mathcal{L}\Big(\frac{-t}{\lambda}\Big)\left[B_3\Big(\mathbb{G}_A\Big(\mathcal{L}_1\Big(\frac{t}{\lambda}\Big)V\Big)\Big)\right]\to\overline{\mathbb{G}}_A(V)$$

weakly in $L^\infty([0,T];W^{-1,2}(\mathbb{T}^n))$. Similarly, we also have

$$\mathcal{L}\Big(\frac{-t}{\lambda}\Big)B_3\left(g\mathcal{L}_2\Big(\frac{t}{\lambda}\Big)V^\lambda\right) = \overline{\mathcal{A}}(\overline{V^\lambda}) + \sum_{k=1}^{2} e^{ik\frac{t}{\lambda}}F_k(gV^\lambda) + \sum_{k=1}^{2} e^{-ik\frac{t}{\lambda}}F_k^*(gV^\lambda)$$

and using the same technique as (3.17), the sequence

$$\mathcal{L}\Big(\frac{-t}{\lambda}\Big)B_3\left(g\mathcal{L}_2\Big(\frac{t}{\lambda}\Big)V^\lambda\right) \to \overline{\mathcal{A}}(\overline{V})$$

in the sense of distribution, where F_1 and F_2 are linear vector functions of V^λ. Let

$$\eta(t) = \limsup_{\lambda\to 0} H^\lambda(t)\,.$$

Using the following equalities (direct calculation)

$$\int_{\mathbb{T}^n} \overline{\mathcal{B}}(\overline{V},V)\cdot V dx = \int_{\mathbb{T}^n} \overline{\mathcal{B}}(V,V)\cdot V dx = 0\,,$$

$$\int_{\mathbb{T}^n} \overline{\mathcal{B}}(V,\overline{V})\cdot V dx = -\int_{\mathbb{T}^n} \overline{\mathcal{B}}(V,V)\cdot \overline{V} dx\,,$$

$$\int_{\mathbb{T}^n} \overline{\mathcal{A}}(\overline{V})\cdot V dx = -\int_{\mathbb{T}^n} \overline{\mathcal{A}}(V)\cdot \overline{V} dx\,,$$

we can show that

$$\int_0^t \int_{\mathbb{T}^n} -\overline{V}\cdot\partial_s V + \overline{\mathcal{B}}(V,\overline{V})\cdot V + \overline{\mathcal{A}}(\overline{V})\cdot V$$

$$- \overline{\mathbb{G}}_A(V)\cdot\overline{V} - B_3(\partial_t A)\cdot\overline{V} dxds = 0\,.$$

Thus, we derive the Gronwall's type inequality

$$\eta(t) \le \eta(0) + C\int_0^t \eta(s)ds$$

which implies $\eta(t) = 0$ for all t because $\eta(0) = 0$. It follows then that if H^λ is small at $t = 0$, then it remains small on the interval of time $[0,T]$.

Step 4. Conclusion of the modulated energy.

Note that $H_A^\lambda(t) \to 0$ as $\lambda \to 0$ uniform in $t \in [0,T]$, and each integral of (3.11) is positive definite, thus we show that each term of (3.11) will converge to zero. First, (2.21) follows from the elementary convexity inequality

$$\left|\rho^\lambda - 1\right|^\gamma \le (\rho^\lambda)^\gamma + (\gamma - 1) - \gamma\rho^\lambda .$$

For (2.23), we use the fact

$$\|\nabla\phi^\lambda - \mathcal{L}_2(\tfrac{t}{\lambda})\widetilde{V}\|_{L^\infty([0,T];L^2(\mathbb{T}^n))} \to 0 , \tag{3.18}$$

then for any $\varphi(x,t) \in L^1([0,T];L^2(\mathbb{T}^n))$

$$\int_0^t \int_{\mathbb{T}^n} \nabla\phi^\lambda \varphi dxds = \int_0^t \int_{\mathbb{T}^n} \big(\nabla\phi^\lambda - \mathcal{L}_2(\tfrac{t}{\lambda})\widetilde{V}\big)\varphi dxds$$

$$+ \int_0^t \int_{\mathbb{T}^n} \big(\mathcal{L}_2(\tfrac{t}{\lambda})\widetilde{V}\big)\varphi dxds ,$$

the first integral converges to zero by (3.18), and the second integral converges to zero by the Riemann-Lebesgue lemma, this proves (2.23). Finally, we deduce from the Hölder inequality that

$$\begin{aligned}
&\big\|J^\lambda - \big(v + \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V})\big)\big\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \\
&\le \big\|J^\lambda - \rho^\lambda\big(v + \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V})\big)\big\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} + \big\|(\rho^\lambda - 1)\big(v + \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V})\big)\big\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^n)} \\
&\le \|\sqrt{\rho^\lambda}\|_{L^{2\gamma}(\mathbb{T}^n)} \Big\|\frac{1}{\sqrt{\rho^\lambda}}\Big(J^\lambda - \rho^\lambda\big(v + \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V}\big)\Big)\Big\|_{L^2(\mathbb{T}^n)} \\
&\quad + \big\|\rho^\lambda - 1\big\|_{L^\gamma(\mathbb{T}^n)} \big\|v + \mathcal{L}_1(\tfrac{t}{\lambda})\widetilde{V}\big\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^n)}
\end{aligned}$$

which converges to 0 as $\lambda \to 0$. Similarly, for any $\varphi(x,t) \in L^1([0,T]; L^{\frac{2\gamma}{\gamma-1}}(\mathbb{T}^n))$, we have

$$\lim_{\lambda\to 0} \int_0^t \int_{\mathbb{T}^n} \big(J^\lambda(x,s) - v(x,s)\big)\varphi(x,s) dxds = 0 ,$$

this proves (2.22).

Acknowledgments. We would like to thank Professor Tai-Ping Liu for stimulating discussion and valuable suggestion in our weekly analysis seminar at Academia Sinica, Taipei, which helps to improve the manuscript. This work is supported by the National Science Council of Taiwan under the grant NSC98-2115-M-009-004-MY3, and the Natural Sciences and Engineering Research Council of Canada. K.C. Wu is also supported by the Tsz-Tza Foundation in Institute of Mathematics, Academia Sinica, Taipei, Taiwan.

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(Received April 6, 2011)
(Revised September 5, 2011)

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