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Semistability Criterion for Parabolic Vector Bundles on Curves

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Abstract. We give a cohomological criterion for a parabolic vector bundle on a curve to be semistable. It says that a parabolic vector bundle $\mathcal{E}_*$ with rational parabolic weights is semistable if and only if there is another parabolic vector bundle $\mathcal{F}_*$ with rational parabolic weights such that the cohomologies of the vector bundle underlying the parabolic tensor product $\mathcal{E}_* \otimes \mathcal{F}_*$ vanish. This criterion generalizes the known semistability criterion of Faltings for vector bundles on curves and significantly improves the result in [Bis07].

1. Introduction

We will work over an algebraically closed ground field of characteristic zero.

Let X be an irreducible smooth projective curve. A theorem due to Faltings says that a vector bundle E over X is semistable if and only if there is a vector bundle F over X such that $\mathrm{H}^0(X,\, E\otimes F) = 0 = \mathrm{H}^1(X,\, E\otimes F)$ (see [Fal93, p. 514, Theorem 1.2] and [Fal93, p. 516, Remark]). Let D be a reduced effective divisor on X. For a parabolic vector bundle W_* on X with parabolic divisor D, the underlying vector bundle will be denoted by W_0; see [MS80], [MY92] for parabolic vector bundles. Let r be a positive integer. Denote by $\mathrm{Vect}(X, D, r)$ the category of parabolic vector bundles on X with parabolic structure along D and parabolic weights being integral multiples of $1/r$. In [Bis07] the following theorem was proved:

Theorem 1.1. *There is a parabolic vector bundle $\mathcal{V}_* \in \mathrm{Vect}(X, D, r)$ with the following property: A parabolic vector bundle $\mathcal{E}_*$ is semistable if and only if there is a parabolic vector bundle $\mathcal{F}_* \in \mathrm{Vect}(X, D, r)$ with $\mathrm{H}^0(X,\, (\mathcal{E}_*\otimes \mathcal{V}_* \otimes \mathcal{F}_*)_0) = 0 = \mathrm{H}^1(X,\, (\mathcal{E}_* \otimes \mathcal{V}_* \otimes \mathcal{F}_*)_0)$, where $(\mathcal{E}_* \otimes \mathcal{V}_* \otimes \mathcal{F}_*)_*$ is the parabolic tensor product.*

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Theorem 1.1 was also proved in [Par10]. It should be mentioned that the vector bundle $\mathcal{V}_*$ in Theorem 1.1 is not canonical; it depends upon the choice of a suitable ramified Galois covering $Y \longrightarrow X$ that transforms parabolic bundles in $\mathrm{Vect}(X, D, r)$ into G-linearized vector bundles on Y, where G is the Galois group for the covering. However, many different covers do this.

We prove that $\mathcal{V}_*$ in Theorem 1.1 can be chosen to be the trivial line bundle $\mathcal{O}_X$ equipped with the trivial parabolic structure. More precisely, we prove the following theorem (see Theorem 6.1):

THEOREM 1.2. *A parabolic vector bundle $\mathcal{E}_* \in \mathrm{Vect}(X, D, r)$ is semistable if and only if there is a parabolic vector bundle $\mathcal{F}_* \in \mathrm{Vect}(X, D, r)$ such that*

$$\mathrm{H}^0(X, (\mathcal{E}_* \otimes \mathcal{F}_*)_0) = 0 = \mathrm{H}^1(X, (\mathcal{E}_* \otimes \mathcal{F}_*)_0).$$

Theorem 1.2 is proved by systematically working with stacks. Compare this method with the earlier attempts (cf. [Bis07], [Par10]) that landed in the weaker version given in Theorem 1.1. Note that from Theorem 1.1 it follows immediately that a semistable parabolic vector bundle satisfies the criterion in Theorem 1.2. The nontrivial part is that if a parabolic vector bundle satisfies the criterion in Theorem 1.2, then it is semistable.

2. Parabolic Bundles and Root Stacks

Recall that to give a morphism $X \longrightarrow [\mathbb{A}^1/\mathbb{G}_m]$ is the same as giving a line bundle $\mathcal{L}$ with section s on X (see [Cad07]). Given a positive integer r, there is a natural morphism

$$\theta_r : [\mathbb{A}^1/\mathbb{G}_m] \longrightarrow [\mathbb{A}^1/\mathbb{G}_m]$$

defined by $t \longmapsto t^r$, with $t \in \mathbb{A}^1$. We define the root stack $X_{(\mathcal{L},s,r)}$ to be the fibered product

$$X \times_{[\mathbb{A}^1/\mathbb{G}_m],\theta_r} [\mathbb{A}^1/\mathbb{G}_m].$$

When the section is non-zero, this root stack is an orbifold curve; see [Cad07, Example 2.4.6].

The data $(\mathcal{L}, s)$ corresponds to an effective divisor D on X. We will henceforth assume that this divisor is reduced. Sometime we write $X_{D,r}$ instead of $X_{\mathcal{L},s,r}$.

We think of the ordered set $\frac{1}{r}\mathbb{Z}$ of rational numbers with denominator r as a category. Let j be an integer multiple of $1/r$. Given a functor from the opposite category

$$\mathcal{F}_* \, : \, (\frac{1}{r}\mathbb{Z})^{\mathrm{op}} \longrightarrow \mathrm{Vect}(X)\,,$$

we denote by $\mathcal{F}_*[j]$ its shift by j, so

$$\mathcal{F}_i[j] \, = \, \mathcal{F}_{i+j}\,.$$

There is a natural transformation $\mathcal{F}_*[j] \longrightarrow \mathcal{F}_*$ when $j \, \geq \, 0$.

A *vector bundle with parabolic structure* over D such that the parabolic weights are integral multiples of $1/r$ is a functor

$$\mathcal{F}_* \, : \, (\frac{1}{r}\mathbb{Z})^{\mathrm{op}} \longrightarrow \mathrm{Vect}(X)$$

together with a natural isomorphism

$$j \, : \, \mathcal{F}_* \otimes \mathcal{O}_X(-D) \xrightarrow{\sim} \mathcal{F}[1]$$

such that the following diagram commutes

$$\begin{tikzcd} \mathcal{F}_* \otimes \mathcal{O}_X(-D) \arrow[r] \arrow[dr] & \mathcal{F}[1] \arrow[d] \\ & \mathcal{F}_* \end{tikzcd}$$

(see [MY92], [MS80]). The *underlying vector bundle* of a parabolic vector bundle is the value of this functor at 0. We have previously denoted this by $\mathcal{F}_0$. For a functor $\mathcal{F}_*$ defining a parabolic vector bundle, the value of $\mathcal{F}_*$ at $t \in \frac{1}{r}\mathbb{Z}$ will be denoted by $\mathcal{F}_t$.

Denote by $\mathrm{Vect}(X, D, r)$ the category of vector bundles on X with parabolic structure along D and parabolic weights integral multiples of $1/r$. It is a tensor category.

THEOREM 2.1. *There is an equivalence of tensor categories*

$$F \, : \, \mathrm{Vect}(X_{(\mathcal{L},s,r)}) \xrightarrow{\sim} \mathrm{Vect}(X, D, r).$$

The equivalence preserves parabolic degree and semistability (see § 4 below).

The functor F has the following explicit description. There is a natural root line bundle $\mathcal{N}$ on $X_{(\mathcal{L},s,r)}$. Given a vector bundle $\mathcal{F}$ on the root stack, the corresponding parabolic bundle is the functor defined by

$$l/r \longmapsto \pi_*(\mathcal{F} \otimes \mathcal{N}^l)\,.$$

Proof of Theorem 2.1. See [Bor07, Section 3] and [Bis97]. □

3. Root Stacks as Quotient Stacks

For the map $z \longmapsto z^n$ defined around $0 \in \mathbb{C}$, the *ramification index* at 0 will be $n-1$.

We will need the following theorem :

Theorem 3.1. *Suppose $k = \mathbb{C}$. There is a finite Galois covering $Y \longrightarrow X$ ramified over D with ramification index $r-1$ at each point in D if and only if either $X \neq \mathbb{P}^1$ or $X = \mathbb{P}^1$ with $|D| \neq 1$.*

Proof. See [Nam87, p. 29, Theorem 1.2.15]. □

Corollary 3.2. *Theorem 3.1 holds over any algebraically closed ground field of characteristic zero.*

Proof. This follows from [SGA1, Expose IX, Theorem 4.10]. See also Proposition 7.2.2 in [Mur67, p. 146]. □

Proposition 3.3. *Suppose that either $X \neq \mathbb{P}^1$ or $|D| \neq 1$. Then $X_{(D,r)}$ is a quotient stack.*

Proof. Fix a covering $Y \longrightarrow X$ as in Corollary 3.2. Let G be the Galois group for this covering. Our goal is to show that $X_{(D,r)} = [Y/G]$.

Let R be the ramification divisor in Y. Then the reduced divisor R_{red} produces a morphism

$$Y \longrightarrow X_{(D,r)} \tag{1}$$

via the universal property of root stacks. As R_{red} is G-invariant so is the morphism in (1). Hence we obtain a morphism

$$[Y/G] \longrightarrow X_{(D,r)} .$$

To show that this morphism is an isomorphism is a local condition for the flat topology and follows from [Cad07, Example 2.4.6]. □

4. Semistability

Recall that the *parabolic degree* of a parabolic vector bundle $\mathcal{E}_*$ over X is defined to be

$$\begin{aligned} \deg_{\text{par}}(\mathcal{E}_*) &:= \operatorname{rk}(\mathcal{E}_0)(\deg D - \chi(\mathcal{O}_X)) + \frac{1}{r}(\sum_{i=1}^{r} \chi(\mathcal{E}_{i/r})) \\ &= \operatorname{rk}(\mathcal{E}_0)\deg D + \frac{1}{r}\sum_{i=1}^{r} \deg(\mathcal{E}_{i/r}) \end{aligned}$$

(see [MS80], [Bis97], [Bor07, § 4]). The *slope* is defined as usual :

$$\mu(\mathcal{E}_*) := \frac{\deg_{\text{par}}(\mathcal{E})}{\operatorname{rk}(\mathcal{E})} .$$

A parabolic vector bundle $\mathcal{E}_*$ is said to be *semistable* if

$$\mu(\mathcal{E}_*) \geq \mu(\mathcal{F}_*)$$

for all parabolic subbundles $\mathcal{F}_*$.

Example 4.1. Let us describe all the parabolic semistable bundles on $\mathbb{P}^1$ with one parabolic point, meaning $D = x$, where x is some point on $\mathbb{P}^1$. Let $\mathcal{E}_*$ be a semistable parabolic vector bundle. Then we may write

$$\mathcal{E}_0 = \bigoplus_{k=1}^{m} \mathcal{O}(n_k)^{s_k}$$

[Gro57]. We may assume that the integers n_i are strictly decreasing. A subbundle $\mathcal{F}_*$ is defined by taking

$$\mathcal{F}_{i/r} = \mathcal{O}(n_1)^{s_1} \cap \mathcal{E}_{i/r}$$

for $0 \leq i < r$. This extends to a parabolic subbundle of $\mathcal{E}_*$. We see immediately that

$$\mu(\mathcal{F}_*) > \mu(\mathcal{E}_*)$$

when $m > 1$. Consequently, a parabolic vector bundle $\mathcal{E}_*$ of rank n over $\mathbb{P}^1$ with one parabolic point is semistable if and only if

$$\mathcal{E}_* = (\mathcal{L}_*)^{\oplus n},$$

where $\mathcal{L}_*$ is a parabolic line bundle.

5. Grothendieck-Riemann-Roch Theorem for Deligne-Mumford Stacks

In this section we recall the pertinent results from [Tö99]. An excellent summary of this paper of Töen can be found in the appendix to [Bor07]. We denote by $\mathfrak{X}$ a smooth Deligne-Mumford stack that is proper over our ground field k. We equip it with the étale topology. The category of vector bundles (respectively, coherent sheaves) on $\mathfrak{X}$ is an exact category so we may form the groups

$$K_i(\mathfrak{X}) \qquad (\text{respectively, } G_i(\mathfrak{X}))\,.$$

Let $\mathcal{K}_i$ denote the sheaf in the étale topology on $\mathfrak{X}$ associated to the presheaf

$$(X \longrightarrow \mathfrak{X}) \longmapsto K_i(X)\,.$$

Set

$$\mathrm{H}^i(\mathfrak{X}, \mathbb{Q}) = \mathrm{H}^i(\mathfrak{X}, \mathcal{K}_i \otimes \mathbb{Q})\,.$$

By [Gil81] we have Chern classes and hence Chern characters and Todd classes

$$\mathrm{c}_i^{\mathrm{et}},\ \mathrm{ch}^{\mathrm{et}},\ \mathrm{td}^{\mathrm{et}} : K_0(\mathfrak{X}) \longrightarrow \mathrm{H}^*(\mathfrak{X})\,.$$

Let $I_{\mathfrak{X}} := \mathfrak{X} \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X}$ be the inertia stack of $\mathfrak{X}$. Let μ_∞ denote the group of roots of unity in $\overline{\mathbb{Q}}$, and set $\Lambda := \mathbb{Q}(\mu_\infty)$. If $\mathcal{G}$ is a locally free sheaf on $I_{\mathfrak{X}}$, the inertial action induces an eigenspace decomposition

$$\mathcal{G} = \bigoplus_{\zeta \in \mu_\infty} \mathcal{G}^{(\zeta)}\,.$$

Let

$$\rho_{\mathfrak{X}} : K_0(I_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \Lambda \longrightarrow K_0(I_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \Lambda$$

be the morphism defined by

$$\mathcal{G} \longmapsto \sum \zeta[\mathcal{G}^{(\zeta)}] .$$

We have a morphism, called the *Frobenius character*,

$$\phi_{\mathfrak{X}} : K_0(\mathfrak{X}) \otimes_{\mathbb{Z}} \Lambda \xrightarrow{\pi^*_{\mathfrak{X}}} K_0(I_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \Lambda \xrightarrow{\rho_{\mathfrak{X}}} K_0(I_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \Lambda \longrightarrow K_{0,\mathrm{et}}(I_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \Lambda .$$

The ring K_0 is a lambda ring and we write $\lambda_{-1}(x) = \sum (-1)^i \lambda_i(x)$. Define

$$\alpha_{\mathfrak{X}} := \rho_{\mathfrak{X}}(\lambda_{-1}([\Omega^1_{I_{\mathfrak{X}}/\mathfrak{X}}])) \in K_{0,\mathrm{et}}(I_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \Lambda .$$

Finally define the characteristic classes

$$\mathrm{ch}^{\mathrm{rep}}(x) := \mathrm{ch}^{\mathrm{et}}(\phi_{\mathfrak{X}}(x))$$

and

$$\mathrm{td}^{\mathrm{rep}}(\mathfrak{X}) := \mathrm{ch}^{\mathrm{et}}(\alpha_{\mathfrak{X}}^{-1})\mathrm{td}^{\mathrm{et}}(\mathcal{T}_{I_{\mathfrak{X}}}) .$$

Theorem 5.1. *Denote by* $\int_{\mathfrak{X}}^{\mathrm{rep}}$ *the push-forward* p_* *for* $p : I_{\mathfrak{X}} \longrightarrow \mathrm{Spec}(k)$*. The following holds:*

$$\chi(\mathfrak{X}, \mathcal{F}) = \int_{\mathfrak{X}}^{\mathrm{rep}} \mathrm{td}^{\mathrm{rep}}(\mathfrak{X})\mathrm{ch}^{\mathrm{rep}}(\mathcal{F}) .$$

Proof. See [Tö99, Corollary 4.13]. □

Corollary 5.2. *Suppose that* $\mathfrak{X}$ *is a proper orbifold curve. Then*

$$\mu(\mathcal{F}) = \chi(\mathcal{F}) - \int_{\mathfrak{X}}^{\mathrm{rep}} \mathrm{td}^{\mathrm{rep}}(\mathfrak{X}) .$$

PROOF. We have that $\pi_{\mathfrak{X}}^*(\mathcal{F})$ is an eigensheaf with eigenvector 1 as the stack $\mathfrak{X}$ is generically a variety. There is a diagram

$$\begin{tikzcd} I_{\mathfrak{X}} \arrow[d, "\pi_{\mathfrak{X}}"] \arrow[dr, "p_I"] & \\ \mathfrak{X} \arrow[r, "p"] & \mathrm{Spec}(k). \end{tikzcd}$$

By the projection formula,

$$p_{I,*}(c_1^{\mathrm{et}}(\pi_{\mathfrak{X}}^*\mathcal{F})) = p_*(c_1^{\mathrm{et}}(\mathcal{F}))\,.$$

In view of Theorem 5.1, the result follows from the fact that $\deg(\mathcal{F})) = p_*(c_1^{\mathrm{et}}(\mathcal{F}))$ ([Bor07, Theorem 4.3]) and the usual expression for the Chern character. □

COROLLARY 5.3. *Suppose that there is a vector bundle $\mathcal{E}$ so that $\mathrm{H}^i(\mathfrak{X}, \mathcal{E}\otimes\mathcal{F}) = 0$ for $i = 0, 1$. Then $\mathcal{F}$ is semistable.*

PROOF. Suppose there is a subsheaf $\mathcal{F}'$ of $\mathcal{F}$ with

$$\mu(\mathcal{F}') > \mu(\mathcal{F})\,.$$

Then it follows from Corollary 5.2 that

$$\frac{\chi(\mathcal{E}\otimes\mathcal{F}')}{\mathrm{rank}(\mathcal{E}\otimes\mathcal{F}')} - \frac{\chi(\mathcal{E}\otimes\mathcal{F})}{\mathrm{rank}(\mathcal{E}\otimes\mathcal{F})} > 0\,.$$

Since $\chi(\mathcal{E}\otimes\mathcal{F}) = 0$, this implies that $\mathrm{H}^0(\mathfrak{X}, \mathcal{E}\otimes\mathcal{F}') \neq 0$. But $\mathcal{E}\otimes\mathcal{F}' \subset \mathcal{E}\otimes\mathcal{F}$. Hence $\mathrm{H}^0(\mathfrak{X}, \mathcal{E}\otimes\mathcal{F}) \neq 0$ which is a contradiction. □

6. Semistability Criterion

THEOREM 6.1. *A vector bundle with parabolic structure $\mathcal{E}_* \in \mathrm{Vect}(X, D, r)$ is semistable if and only if there is a parabolic vector bundle $\mathcal{F}_* \in \mathrm{Vect}(X, D, r)$ with*

$$\mathrm{H}^i(X, (\mathcal{E}_*\otimes\mathcal{F}_*)_0) = 0$$

for all i, where $(\mathcal{E}_*\otimes\mathcal{F}_*)_*$ is the parabolic tensor product.

Proof. We have a morphism $\pi : X_{D,r} \longrightarrow X$, and π_* is exact as $\mathrm{char}(k) = 0$. Hence by the Leray spectral sequence,

$$\mathrm{H}^i(X, \pi_*(\mathcal{F})) = \mathrm{H}^i(X_{D,r}, \mathcal{F})$$

for all i.

Suppose that there is a parabolic vector bundle $\mathcal{F}_* \in \mathrm{Vect}(X, D, r)$ with

$$\mathrm{H}^0(X, (\mathcal{E}_* \otimes \mathcal{F}_*)_0) = 0 = \mathrm{H}^1(X, (\mathcal{E}_* \otimes \mathcal{F}_*)_0)\,.$$

Applying Theorem 2.1, we deduce from Corollary 5.3 that $\mathcal{E}_*$ is semistable.

To prove the converse, assume that $\mathcal{E}_*$ is semistable. We break up into two cases.

The case of $\mathbb{P}^1$ with exactly one parabolic point: Applying Example 4.1, we see that

$$\mathcal{E}_0 = \bigoplus \mathcal{O}(n)^m\,.$$

So tensoring with $\mathcal{O}(-n-1)$ does the job.

All other cases: In view of Proposition 3.3 we may assume that we have a quotient stack, so $X_{D,r} = [Y/G]$. Then given a semistable parabolic bundle on X, we obtain a corresponding semistable G-linearized vector bundle $\mathcal{E}$ on Y. We note that this implies that the vector bundle $\mathcal{E}$ is semistable [Bis97, p. 308, Lemma 2.7]. By [Fal93, p. 514, Theorem 1.2], there is a vector bundle $\mathcal{F}$ on Y such that all the cohomology groups of $\mathcal{F} \otimes \mathcal{E}$ vanish. Consider

$$\widetilde{\mathcal{F}} = \bigoplus_{g \in G} g^* \mathcal{F}\,.$$

The vector bundle $\widetilde{\mathcal{F}}$ has a natural G-action and

$$\mathrm{H}^i(Y, \widetilde{\mathcal{F}} \otimes \mathcal{E}) = 0$$

for all i. The vector bundle $\widetilde{\mathcal{F}}$ produces a vector bundle on $[Y/G]$, which will also be denoted by $\widetilde{\mathcal{F}}$. Finally

$$\mathrm{H}^i([Y/G], \widetilde{\mathcal{F}} \otimes \mathcal{E}) = \mathrm{H}^i(Y, \widetilde{\mathcal{F}} \otimes \mathcal{E})^G = 0\,.$$

The theorem now follows. $\square$

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