J. Math. Sci. Univ. Tokyo
18 (2011), 35–49.

Periodic Solutions for a Kind of Third-Order Delay Differential Equations with a Deviating Argument

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Abstract. In this paper, by using the continuation theorem of coincidence degree theory and analysis techniques, we establish a new result on the existence and uniqueness of a T-periodic solution for the third-order delay differential equation with a deviating argument of the following form

$$\dddot{x}(t) + f(t, x(t))\ddot{x}(t) + g(x(t))\dot{x}(t) + h(t, x(t - r(t))) = p(t).$$

1. Introduction

One of the most attractive areas of the qualitative theory of differential equations is the existence of periodic solutions.

Existence and uniqueness of periodic solutions of delay differential equations are of great interest in mathematics and its applications to the modeling of various practical problems.

In recent years, by using Mawhin's continuation theorem of coincidence degree theory, the existence and uniqueness of periodic solutions for some types of first and second-order delay differential equations with deviating arguments were studied, for example, [7 − 10, 12, 13], etc. Besides it is worth-mentioning that there are a few results on the same topic for third-order delay differential equations, for example, [3, 11, 15]. Gui [3] established criteria for existence of positive periodic solutions to the following third-order neutral delay differential equation with deviating arguments

$$\dddot{x}(t) + a\ddot{x}(t) + g(\dot{x}(t - \tau(t))) + f(x(t - \tau(t))) = p(t),$$

where a is a positive constant; g, f and p are real continuous functions and are defined on $\mathbb{R}$; $\tau(t), p(t)$ are periodic with period ω.

2010 *Mathematics Subject Classification.* 34C25.
Key words: Continuation theorem, Coincidence degree, Existence and uniqueness, Third-order delay differential equations, Deviating argument.

The main purpose of this paper is to investigate sufficient conditions ensuring the existence and uniqueness of a T-periodic solution to third-order delay differential equation with a deviating argument of the form

$$\dddot{x}(t) + f(t,x(t))\,\ddot{x}(t) + g(x(t))\,\dot{x}(t) + h(t,x(t-r(t))) = p(t), \tag{1.1}$$

where $g, r, p : \mathbb{R} \to \mathbb{R}$ and $f, h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, r and p are T-periodic, f and h are T-periodic in their first argument and $T > 0$.

2. Preliminary Results

In this section, we give some technical yet elementary results, that will serve us well in the section that follows.

For ease of exposition throughout this paper we will adopt the following notation:

$$|x|_k = (\int_0^T |x(t)|^k dt)^{\frac{1}{k}}, k \geq 1, \quad |x|_\infty = \max_{t\in[0,T]} |x(t)|$$

and

$$|p|_\infty = \max_{t\in[0,T]} |p(t)|.$$

Let

$$X = \{x | x \in C^2(\mathbb{R}, \mathbb{R}), x(t+T) = x(t), \text{ for all } t \in \mathbb{R}\}$$

and

$$Y = \{y | y \in C(\mathbb{R}, \mathbb{R}), y(t+T) = y(t), \text{ for all } t \in \mathbb{R}\},$$

be two Banach spaces with the norms

$$\|x\|_X = \max\{|x|_\infty, |\dot{x}|_\infty, |\ddot{x}|_\infty\} \text{ and } \|y\|_Y = |y|_\infty.$$

Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \{x | x \in X, \dddot{x}(t) \in C(\mathbb{R}, \mathbb{R})\},$$

and for $x \in D(L)$,

$$Lx = \dddot{x}(t). \tag{2.1}$$

We also define a nonlinear operator $N : X \to Y$ by setting

$$Nx = -f(t, x(t))\ddot{x}(t) - g(x(t))\dot{x}(t) - h(t, x(t - r(t))) + p(t). \tag{2.2}$$

Then we notice that

$$KerL = \mathbb{R} \text{ and } ImL = \{y | y \in Y, \int_0^T y(s)ds = 0\}.$$

Thus the operator L is a Fredholm operator with index zero.

Define the continuous projector $P : X \to KerL$ and the averaging projector $Q : Y \to Y$ by setting

$$Px(t) = \frac{1}{T}\int_0^T x(s)ds \text{ and } Qy(t) = \frac{1}{T}\int_0^T y(s)ds.$$

Hence $ImP = KerL$ and $KerQ = ImL$. Denoting by $L_P^{-1} : ImL \to D(L) \cap KerP$ the inverse of $L|_{D(L)\cap KerP}$, we have

$$L_P^{-1}y(t) = -\frac{t}{T}\int_0^T (t-s)y(s)ds + \int_0^T (t-s)y(s)ds.$$

Therefore we can see from (2.2) and the above equation, that N is L-compact on $\bar{\Omega}$, where Ω is an open bounded subset in X.
In view of (2.1) and (2.2) the operator equation

$$Lx = \lambda Nx,$$

is equivalent to the following equation

$$\dddot{x}(t) + \lambda\{f(t, x(t))\ddot{x}(t) + g(x(t))\dot{x}(t) + h(t, x(t - r(t)))\} = \lambda p(t), \quad \lambda \in (0, 1). \tag{2.3}$$

To prove the main result, we introduce the continuation theorem of coincidence degree theory formulated in [2].

Lemma 2.1. *let X and Y be two Banach spaces. Suppose that $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero and $N : X \to Y$ is L-compact on $\bar{\Omega}$, where Ω is an open bounded subset in X. Moreover assume that all the following conditions are satisfied:*

(a) $Lx \neq \lambda Nx$, *for all* $x \in \partial\Omega \cap D(L)$ *and* $\lambda \in (0,1)$;

(b) $QNx \neq 0$, *for all* $x \in \partial\Omega \cap KerL$;

(c) *The Brower degree*

$$deg\{QN, \Omega \cap KerL, 0\} \neq 0.$$

Then equation $Lx = Nx$ *has at least one solution on* $\bar{\Omega}$.

LEMMA 2.2. *It is convenient to introduce the following assumptions:*

(i) *Assume that there exist non-negative constants* c_1, c_2 *and* c_3 *such that*

$$|f(t,x)| \leq c_1, \text{ for all } t, x \in \mathbb{R}$$

and

$$|g(x)| \leq c_2, \ \ |g(x_1) - g(x_2)| \leq c_3|x_1 - x_2|, \text{ for all } x, x_1, x_2 \in \mathbb{R}.$$

(ii) *Suppose that there exists a constant* $d > 0$ *such that*

$$x\{h(t,x) - p(t)\} < 0, \text{ for all } t \in \mathbb{R} \text{ and } |x| \geq d.$$

If $x(t)$ *is a T-periodic solution of (2.3), then*

$$|x|_\infty \leq d + \frac{1}{2}\sqrt{T}|\dot{x}|_2. \tag{2.4}$$

(iii) *Suppose that* (i) *and* (ii) *hold and there exists a non-negative constant* b *such that*

$$|h(t,x_1) - h(t,x_2)| \leq b|x_1 - x_2|, \text{ for all } t, x_1, x_2 \in \mathbb{R} \tag{2.5}$$

and

$$c_1\frac{T}{2} + c_2\frac{T^2}{4} + b\frac{T^3}{8} < 1.$$

If $x(t)$ *is a T-periodic solution of* (1.1), *then*

$$|\dot{x}|_\infty \leq \frac{1}{4}\frac{[bd + \max\{|h(t,0)| : 0 \leq t \leq T\} + |p|_\infty]T^2}{1 - (c_1\frac{T}{2} + c_2\frac{T^2}{4} + b\frac{T^3}{8})} := k. \tag{2.6}$$

(*iv*) *Suppose that* (*i*), (*ii*), (2.5) *and* (2.6) *hold,* $f(t,x) \equiv f(t)$ *for all* $t, x \in \mathbb{R}$ *and* $h(t,x)$ *is a strictly monotone decreasing function in* x *such that*

$$c_1\frac{T}{2} + c_2\frac{T^2}{4} + (c_3k + b)\frac{T^3}{8} < 1.$$

Then (1.1) *has at most one T-periodic solution.*

Proof (*ii*). Let $x(t) \in X$ be a T-periodic solution of (2.3) for a certain $\lambda \in (0,1)$. Then by integrating (2.3) over $[0,T]$ together with condition (*i*) implies that

$$\int_0^T \{h(t, x(t - r(t))) - p(t)\}dt = 0.$$

This implies that there exists $\xi \in [0,T]$ such that

$$h(\xi, x(\xi - r(\xi))) - p(\xi) = 0.$$

Taking this together with (*ii*) as appropriate we have

$$|x(\xi - r(\xi))| < d.$$

let $\xi - r(\xi) = mT + t_0$, where $t_0 \in [0,T]$ and m is an integer then we obtain

$$\begin{aligned} |x(t)| &= |x(t_0) + \int_{t_0}^t \dot{x}(s)ds| \\ &\le |x(\xi - r(\xi))| + |\int_{t_0}^t \dot{x}(s)ds| \\ &< d + \int_{t_0}^t |\dot{x}(s)|ds, \quad t \in [t_0, t_0 + T]. \end{aligned}$$

Since $x(t)$ is the T-periodic solution, for $t \in [t_0, t_0 + T]$ we get

$$\begin{aligned} |x(t)| &= |x(t_0 + T) + \int_{t_0+T}^t \dot{x}(s)ds| \\ &\le |x(t_0 + T)| + |\int_t^{t_0+T} \dot{x}(s)ds| \\ &\le d + \int_t^{t_0+T} |\dot{x}(s)|ds, \quad t \in [t_0, t_0 + T]. \end{aligned}$$

Combining the above two inequalities we find

$$|x(t)| \leq d + \frac{1}{2}\int_0^T |\dot{x}(s)|ds.$$

Using the Schwarz inequality yields

$$|x(t)| \leq d + \frac{1}{2}\sqrt{T}(\int_0^T |\dot{x}(s)|^2 ds)^{\frac{1}{2}} = d + \frac{1}{2}\sqrt{T}|\dot{x}|_2.$$

Therefore we have

$$|x|_\infty = \max_{t\in[0,T]} |x(t)| \leq d + \frac{1}{2}\sqrt{T}|\dot{x}|_2.$$

This completes the proof of condition (ii) in Lemma 2.2. □

Proof (iii). Let $x(t)$ be a T-periodic solution of (1.1) for a certain $\lambda \in (0,1)$.
Multiplying (1.1) by $\dddot{x}(t)$ and then integrating it over $[0,T]$ implies

$$\int_0^T |\dddot{x}(t)|^2 dt = -\int_0^T f(t,x(t))\ddot{x}(t)\dddot{x}(t)dt - \int_0^T g(x(t))\dot{x}(t)\dddot{x}(t)dt$$
$$- \int_0^T h(t,x(t-r(t)))\dddot{x}(t)dt + \int_0^T p(t)\dddot{x}(t)dt.$$

By using condition (i) we find

$$|\dddot{x}(t)|_2^2 \leq c_1\int_0^T |\ddot{x}(t)||\dddot{x}(t)|dt + c_2\int_0^T |\dot{x}(t)||\dddot{x}(t)|dt + \int_0^T |p(t)||\dddot{x}(t)|dt$$
$$+ \int_0^T \{|h(t,x(t-r(t))) - h(t,0)| + |h(t,0)|\}|\dddot{x}(t)|dt.$$

From (2.5) we get

$$|\dddot{x}(t)|_2^2 \leq c_1\int_0^T |\ddot{x}(t)||\dddot{x}(t)|dt + c_2\int_0^T |\dot{x}(t)||\dddot{x}(t)|dt$$
$$+ b\int_0^T |x(t-r(t))||\dddot{x}(t)|dt$$
$$+ \int_0^T |h(t,0)||\dddot{x}(t)|dt + \int_0^T |p(t)||\dddot{x}(t)|dt$$

$$\leq c_1 \int_0^T |\ddot{x}(t)||\dddot{x}(t)|dt + c_2 \int_0^T |\dot{x}(t)||\dddot{x}(t)|dt + b|x|_\infty \int_0^T |\dddot{x}(t)|dt$$
$$+ \max\{|h(t,0)| : 0 \leq t \leq T\} \int_0^T |\dddot{x}(t)|dt + |p|_\infty \int_0^T |\dddot{x}(t)|dt.$$

Thus from (2.4) we obtain

$$\begin{aligned}|\dddot{x}(t)|_2^2 \leq\ & c_1 \int_0^T |\ddot{x}(t)||\dddot{x}(t)|dt + c_2 \int_0^T |\dot{x}(t)||\dddot{x}(t)|dt \\ & + \frac{1}{2} b\sqrt{T}|\dot{x}|_2 \int_0^T |\dddot{x}(t)|dt \\ & + [bd + \max\{|h(t,0)| : 0 \leq t \leq T\} + |p|_\infty] \int_0^T |\dddot{x}(t)|dt.\end{aligned}$$

By using the Cauchy-Schwarz inequality we find

$$\begin{aligned}|\dddot{x}(t)|_2^2 \leq\ & c_1|\ddot{x}|_2|\dddot{x}|_2 + c_2|\dot{x}|_2|\dddot{x}|_2 + \frac{1}{2}bT|\dot{x}|_2|\dddot{x}|_2 \\ & + [bd + \max\{|h(t,0)| : 0 \leq t \leq T\} + |p|_\infty]\sqrt{T}|\dddot{x}|_2.\end{aligned} \tag{2.7}$$

Since $x(0) = x(T)$ there exists a constant $\xi \in [0,T]$ such that $\dot{x}(\xi) = 0$ and

$$\begin{aligned}|\dot{x}(t)| &= |\dot{x}(\xi) + \int_\xi^t \ddot{x}(s)ds| \\ &\leq \int_\xi^t |\ddot{x}(s)|ds, \quad t \in [\xi, T+\xi].\end{aligned} \tag{2.8}$$

Again

$$\begin{aligned}|\dot{x}(t)| &= |\dot{x}(\xi+T) + \int_{\xi+T}^t \ddot{x}(s)ds| \\ &\leq |\dot{x}(\xi+T)| + \int_t^{\xi+T} |\ddot{x}(s)|ds = \int_t^{\xi+T} |\ddot{x}(s)|ds, \quad t \in [0,T].\end{aligned} \tag{2.9}$$

The inequalities (2.8) and (2.9) imply that

$$2|\dot{x}(t)| \leq \int_\xi^t |\ddot{x}(s)|ds + \int_t^{\xi+T} |\ddot{x}(s)|ds$$

$$= \int_0^T |\ddot{x}(s)|ds, \quad t \in [0,T].$$

Therefore by using Schwarz inequality we have

$$|\dot{x}(t)| \leq \frac{1}{2}\sqrt{T}\Big(\int_0^T |\ddot{x}(s)|^2 ds\Big)^{\frac{1}{2}}, \quad \text{for all} \ \ t \in [0,T], \tag{2.10}$$

so

$$|\dot{x}|_\infty \leq \frac{1}{2}\sqrt{T}|\ddot{x}|_2, \tag{2.11}$$

$$|\dot{x}|_2 \leq \sqrt{T} \max_{t\in[0,T]} |\dot{x}(s)| \leq \frac{1}{2}T\Big(\int_0^T |\ddot{x}(s)|^2 ds\Big)^{\frac{1}{2}} = \frac{1}{2}T|\ddot{x}|_2. \tag{2.12}$$

Since $x(t)$ is periodic function for $t \in [0,T]$ and by using the above similar technique we find

$$|\ddot{x}(t)| \leq \frac{1}{2}\int_0^T |\dddot{x}(t)|dt.$$

Which together with Cauchy-Schwarz inequality implies

$$|\ddot{x}|_\infty \leq \frac{1}{2}\sqrt{T}\Big(\int_0^T |\dddot{x}(s)|^2 ds\Big)^{\frac{1}{2}} = \frac{1}{2}\sqrt{T}|\dddot{x}|_2, \tag{2.13}$$

$$|\ddot{x}|_2 \leq \sqrt{T} \max_{t\in[0,T]} |\ddot{x}(s)| \leq \frac{1}{2}\sqrt{T}\int_0^T |\dddot{x}(s)|ds \leq \frac{1}{2}T|\dddot{x}|_2. \tag{2.14}$$

By substituting from (2.14) in (2.12) we get

$$|\dot{x}|_2 \leq \frac{1}{4}T^2|\dddot{x}|_2. \tag{2.15}$$

By substituting from (2.14) in (2.11) we have

$$|\dot{x}|_\infty \leq \frac{1}{4}T^{\frac{3}{2}}|\dddot{x}|_2. \tag{2.16}$$

From (2.4) and (2.15) we obtain

$$|x|_\infty \leq d + \frac{1}{8}T^{\frac{5}{2}}|\dddot{x}|_2. \tag{2.17}$$

Then by substituting from (2.14) and (2.15) in (2.7) we find

$$|\dddot{x}|_2^2 \le c_1\frac{T}{2}|\dddot{x}|_2^2 + c_2\frac{T^2}{4}|\dddot{x}|_2^2 + b\frac{T^3}{8}|\dddot{x}|_2^2 + [bd + \max\{|h(t,0)| : 0 \le t \le T\} + |p|_\infty] \quad . \tag{2.18}$$

Thus we get

$$(1 - c_1\frac{T}{2} - c_2\frac{T^2}{4} - b\frac{T^3}{8})|\dddot{x}|_2^2 \le [bd + \max\{|h(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}|\dddot{x}|_2.$$

Therefore we obtain

$$|\dddot{x}|_2 \le \frac{[bd + \max\{|h(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}}{1 - c_1\frac{T}{2} - c_2\frac{T^2}{4} - b\frac{T^3}{8}}. \tag{2.19}$$

By substituting from (2.19) in (2.16) we find

$$|\dot{x}|_\infty \le \frac{[bd + \max\{|h(t,0)| : 0 \le t \le T\} + |p|_\infty]T^2}{4(1 - c_1\frac{T}{2} - c_2\frac{T^2}{4} - b\frac{T^3}{8})} := k.$$

This completes the proof of condition (iii) in Lemma 2.2. □

PROOF (iv). Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.1), then we have

$$\dddot{x}_1(t) - \dddot{x}_2(t) + f(t)\ddot{x}_1(t) - f(t)\ddot{x}_2(t) + g(x_1(t))\dot{x}_1(t) - g(x_2(t))\dot{x}_2(t) + h(t, x_1(t - r(t))) - h(t, x_2(t - r(t))) = 0.$$

Set

$$z(t) = x_1(t) - x_2(t).$$

Then we find

$$\dddot{z}(t) + f(t)\ddot{x}_1(t) - f(t)\ddot{x}_2(t) + g(x_1(t))\dot{x}_1(t) - g(x_2(t))\dot{x}_2(t) + h(t, x_1(t - r(t))) - h(t, x_2(t - r(t))) = 0. \tag{2.20}$$

Since $x_1(t)$ and $x_2(t)$ are two T-periodic solutions, by integrating (2.20) over $[0, T]$ together condition (i), we obtain

$$\int_0^T \{h(t, x_1(t - r(t))) - h(t, x_2(t - r(t)))\}dt = 0.$$

Using the integral mean-value theorem, it follows that there exists a constant $\gamma \in [0, T]$ such that

$$h(\gamma, x_1(\gamma - r(\gamma))) - h(\gamma, x_2(\gamma - r(\gamma))) = 0. \tag{2.21}$$

Let $\gamma - r(\gamma) = nT + \bar{\gamma}$, where $\bar{\gamma} \in [0, T]$ and n is an integer. Then (2.21) together with condition (iii) implies that there exists a constant $\bar{\gamma} \in [0, T]$ such that

$$z(\bar{\gamma}) = x_1(\bar{\gamma}) - x_2(\bar{\gamma}) = x_1(\gamma - r(\gamma)) - x_2(\gamma - r(\gamma)) = 0. \tag{2.22}$$

Thus

$$|z(t)| = |z(\bar{\gamma}) + \int_{\bar{\gamma}}^{t} \dot{z}(s)ds| \le \int_{\bar{\gamma}}^{t} |\dot{z}(s)|ds.$$

Again

$$|z(t)| = |z(\bar{\gamma} + T) + \int_{\bar{\gamma}+T}^{t} \dot{z}(s)ds| \le \int_{t}^{\bar{\gamma}+T} |\dot{z}(s)|ds.$$

Hence by using Schwarz inequality we have

$$\begin{aligned} 2|z(t)| &\le \int_{\bar{\gamma}}^{\bar{\gamma}+T} |\dot{z}(s)|ds = \int_{0}^{T} |\dot{z}(s)|ds \\ &\le \sqrt{T}(\int_{0}^{T} |\dot{z}(s)|^2 ds)^{\frac{1}{2}} = \sqrt{T}|\dot{z}|_2. \end{aligned}$$

Therefore

$$|z|_\infty \le \frac{1}{2}\sqrt{T}|\dot{z}|_2. \tag{2.23}$$

Multiplying (2.20) by $\dddot{z}(t)$ and then integrating it over $[0, T]$ it follows

$$\begin{aligned} |\dddot{z}(t)|_2^2 = &- \int_{0}^{T} f(t)\{\ddot{x}_1(t) - \ddot{x}_2(t)\}\dddot{z}(t)dt \\ &- \int_{0}^{T} \{g(x_1(t))\dot{x}_1(t) - g(x_2(t))\dot{x}_2(t)\}\dddot{z}(t)dt \\ &- \int_{0}^{T} \{h(t, x_1(t - r(t))) - h(t, x_2(t - r(t)))\}\dddot{z}(t)dt. \end{aligned}$$

From (i) and (2.5) we get

$$
\begin{aligned}
|\dddot{z}(t)|_2^2 \leq & \int_0^T |f(t)||\ddot{x}_1(t)-\ddot{x}_2(t))||\dddot{z}(t)|dt \\
& + \int_0^T |g(x_1(t))||\dot{x}_1(t)-\dot{x}_2(t)||\dddot{z}(t)|dt \\
& + \int_0^T |g(x_1(t))-g(x_2(t))||\dot{x}_2(t))||\dddot{z}(t)|dt \\
& + b\int_0^T |x_1(t-r(t))-x_2(t-r(t))||\dddot{z}(t)|dt \\
& \leq c_1\int_0^T |\ddot{z}(t)||\dddot{z}(t)|dt + c_2\int_0^T |\dot{z}(t)||\dddot{z}(t)|dt \\
& + c_3\int_0^T |z(t)||\dot{x}_2(t))||\dddot{z}(t)|dt \\
& + b\int_0^T |z(t-r(t))||\dddot{z}(t)|dt.
\end{aligned}
$$

By using Schwarz inequality we have

$$|\dddot{z}|_2^2 \leq c_1|\ddot{z}|_2|\dddot{z}|_2 + c_2|\dot{z}|_2|\dddot{z}|_2 + c_3|z|_\infty|\dot{x}_2|_\infty\sqrt{T}|\dddot{z}|_2 + b|z|_\infty\sqrt{T}|\dddot{z}|_2.$$

From (2.14), (2.15), (2.23) and (2.6) we obtain

$$|\dddot{z}|_2^2 \leq \frac{1}{2}c_1T|\dddot{z}|_2^2 + \frac{1}{4}c_2T^2|\dddot{z}|_2^2 + \frac{1}{8}c_3kT^3|\dddot{z}|_2^2 + \frac{1}{8}bT^3|\dddot{z}|_2^2.$$

Thus we find

$$\{1-(c_1\frac{T}{2}+c_2\frac{T^2}{4}+c_3k\frac{T^3}{8}+b\frac{T^3}{8})\}|\dddot{z}|_2^2 \leq 0. \tag{2.24}$$

Since $z(t), \dot{z}(t), \ddot{z}(t)$ and $\dddot{z}(t)$ are T-periodic and continuous functions, in view of (iv), (2.12), (2.22) and (2.24) we have

$$z(t) \equiv \dot{z}(t) \equiv \ddot{z}(t) \equiv \dddot{z}(t) = 0, \text{for all } t \in \mathbb{R}.$$

Thus $x_1(t) \equiv x_2(t)$, for all $t \in \mathbb{R}$. Therefore (1.1) has at most one T-periodic solution.

This completes the proof of condition (iv) in Lemma 2.2.

So the proof of Lemma 2.2 is now completed. □

3. Main Result

Theorem 3.1. *Suppose that* (i) *and* (iv) *hold, then* (1.1) *has a unique T-periodic solution.*

Proof. Condition (iv) of Lemma 2.2 states that (1.1) has at most one T-periodic solution. Thus to prove Theorem 3.1 it suffices to show that (1.1) has at least one T-periodic solution. To do this, we shall apply Lemma 2.1. First we will claim that the set of all possible T-periodic solutions of (2.3) is bounded.
Let $x(t)$ be a T-periodic solution of (2.3). Multiplying (2.3) by $\dddot{x}(t)$ and then integrating it over $[0,T]$ we obtain

$$\int_0^T |\dddot{x}(t)|^2 dt = -\lambda \int_0^T f(t,x(t))\ddot{x}(t)\dddot{x}(t)dt - \lambda \int_0^T g(x(t))\dot{x}(t)\dddot{x}(t)dt \\ - \lambda \int_0^T h(t,x(t-r(t)))\dddot{x}(t)dt + \lambda \int_0^T p(t)\dddot{x}(t)dt.$$

In view of $(i),(iv)$, (2.4), (2.7) and the inequality of Schwarz we have

$$|\dddot{x}|_2^2 \le (c_1\frac{T}{2} + c_2\frac{T^2}{4} + b\frac{T^3}{8})|\dddot{x}|_2^2 \\ + [bd + \max\{|h(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}|\dddot{x}|_2,$$

which together with (2.13), (2.16), (2.17) and (2.19) implies that there exist positive constants D_1, D_2 and D_3 such that

$$|\ddot{x}|_\infty \le \frac{1}{2}\sqrt{T}|\dddot{x}|_2 := D_1,$$

$$|\dot{x}|_\infty \le \frac{T}{2}D_1 := D_2,$$

$$|x|_\infty \le d + \frac{T^2}{4}D_1 := D_3.$$

Let $D_0 = \max\{D_1, D_2, D_3\}$ and take $\Omega = \{x | x \in X, \ \ \|x\| < D_0\}$.
If $x \in \partial\Omega \cap KerL = \partial\Omega \cap \mathbb{R}$, then x is a constant with $x(t) = D_0$ or $x(t) = -D_0$. Then

$$QNx = \frac{1}{T}\int_0^T \{-f(t,x(t))\ddot{x}(t) - g(x(t))\dot{x}(t) - h(t,x(t-r(t))) + p(t)\}dt$$

$$= \frac{1}{T}\int_0^T \{-h(t, \pm D_0) + p(t)\}dt \neq 0.$$

So the conditions (a) and (b) in Lemma 2.1 hold.
Furthermore define a continuous function $H(x, \mu)$ by setting

$$H(x, \mu) = (1-\mu)x - \mu.\frac{1}{T}\int_0^T \{h(t, x) - p(t)\}dt, \quad \mu \in [0, 1].$$

It follows from (ii) that

$$xH(x, \mu) \neq 0, \quad \text{for all } x \in \partial\Omega \cap KerL.$$

Thus $H(x, \mu)$ is a homotopy.
Hence by using the homotopy invariance theorem we have

$$\begin{aligned} deg\{QN, \Omega \cap KerL, 0\} &= deg\{-\frac{1}{T}\int_0^T [h(t, x) - p(t)]dt, \Omega \cap KerL, 0\} \\ &= deg\{x, \Omega \cap KerL, 0\} \neq 0. \end{aligned}$$

So condition (c) of Lemma 2.1 is satisfied.
In view of all the discussions above, we conclude from Lemma 2.1 that Theorem 3.1 is proved. □

4. Example

Example 4.1. Let $h(t, x) = -\frac{x}{6\pi}$, for all $t, x \in \mathbb{R}$. Then the equation

$$\dddot{x}(t) + \frac{3}{8}(\sin 4t)\ddot{x}(t) + \frac{3}{8}(\sin x)\dot{x}(t) + \frac{x(t - \sin^2 t)}{6\pi} = \frac{1}{6\pi}e^{-\cos^2 t}, \tag{4.1}$$

has a unique $\frac{\pi}{4}$-periodic solution.

Proof. By (4.1) we have

$$d = 1, \ b = \frac{1}{6\pi}, \ c_1 = c_2 = c_3 = \frac{3}{8}, \ r(t) = \sin^2 t, \ T = \frac{\pi}{4} \text{ and}$$
$$p(t) = \frac{1}{6\pi}e^{-\cos^2 t}.$$

Then

$$\begin{aligned}&\frac{1}{4}\frac{[bd+\max\{|h(t,0)|:0\le t\le T\}+|p|_\infty]T^2}{1-(c_1\frac{T}{2}+c_2\frac{T^2}{4}+b\frac{T^3}{8})}\\&=\frac{1}{4}\frac{(\frac{1}{6\pi}+\frac{1}{6\pi})\frac{\pi^2}{16}}{(1-\frac{3}{8}\frac{\pi}{8}-\frac{3}{8}\frac{\pi^2}{64}-\frac{1}{6\pi}\frac{\pi^3}{512})}:=k=0.02,\end{aligned}$$

$$\begin{aligned}&c_1\frac{T}{2}+c_2\frac{T^2}{4}+(c_3k+b)\frac{T^3}{8}\\&=\frac{3}{8}(\frac{\pi}{8}+\frac{\pi^2}{64})+(\frac{3}{8}\frac{2}{100}+\frac{1}{6\pi})\frac{\pi^3}{512}=0.20835<1.\end{aligned}$$

It is obvious that the assumptions (i) and (iv) hold.
Hence by Theorem 3.1, equation (4.1) has a unique $\frac{\pi}{4}$-periodic solution. □

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(Received November 22, 2010)

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