

## *Smooth Right Quasigroup Structures on 1-Manifolds*

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**Abstract.** Smooth loop structures on one-manifolds for which the groups topologically generated by right translations are locally compact, are known. In this article we study smooth right loop structures on one-manifolds.

### 1. Introduction

The smooth loop (quasigroup) structures on one dimensional manifolds are completely classified. One dimensional manifolds are (i)  $\mathbb{R}$ , (ii)  $\mathbb{S}^1$  and (iii) Alexandroff half line [1, 2], [[3], page 235]. But on the Alexandroff half line, no smooth quasigroup structure can be defined. However on  $\mathbb{R}$  and  $\mathbb{S}^1$  we have smooth quasigroup structures [2, 3] which are obtained by looking at copies of  $\mathbb{S}^1$  appearing as transversals to 2-dimensional subgroups of  $PSL(2, \mathbb{R})$  and that on  $\mathbb{R}$  appear as transversals to special type of subgroups in the universal cover of  $PSL(2, \mathbb{R})$ . This paper is devoted mainly to study smooth right quasigroup structures on  $\mathbb{R}$  and  $\mathbb{S}^1$  obtained by deformation of their group structures.

### 2. Preliminaries

Let  $S$  be a nonempty set and  $\circ$  be a binary operation on  $S$ . Then the groupoid  $(S, \circ)$  is called a right quasigroup if for all  $x, y \in S$ , the equation  $X \circ x = y$ , where  $X$  is unknown in the equation, has a unique solution in  $S$ . If there exists  $e \in S$  such that  $e \circ x = x = x \circ e$  for all  $x \in S$ , then  $(S, \circ)$  is called a right quasigroup with identity also called a right loop. A right quasigroup (right loop)  $(S, \circ)$  is called a quasigroup (loop) if the equation  $x \circ X = y$ , where  $X$  is unknown in the equation and  $x, y \in S$ , has a unique solution in  $S$ . Throughout the paper a right quasigroup will always be assumed to contain the identity.

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Let  $S$  be a fixed right transversal to a subgroup  $H$  in a group  $G$ . Then every right transversal to  $H$  in  $G$  determines and is determined uniquely by a map  $g : S \rightarrow H$  such that  $g(e) = e$ , the identity of  $G$  (see, [5]). The right transversal  $S_g$  determined by a map  $g : S \rightarrow H$  is given by  $S_g = \{g(x)x | x \in S\}$ .  $S$  and  $S_g$  are right quasigroups with identities with respect to the operation  $o$  on  $S$  and  $o'$  on  $S_g$  given by

$$\{xoy\} = Hxy \cap S$$

and

$$\{g(x)x o' g(y)y\} = S_g \cap Hg(x)xg(y)y$$

respectively. Further,  $H$  acts on  $S$  from right through an action  $\theta$  given by  $\{x\theta h\} = Hxh \cap S, \forall x \in S, h \in H$

PROPOSITION 2.1. [5] *The right quasigroup  $(S_g, o')$  is isomorphic to the right quasigroup  $(S, o_g)$  where  $o_g$  is the operation given by  $x o_g y = x\theta g(y)oy$ .*

The operation  $o_g$  will be termed as the deformation of  $o$  through the map  $g : S \rightarrow H$ .

DEFINITION 2.2. Let  $H$  be a subgroup of a group  $G$ . Then a map  $s : G/H \rightarrow G$  is called a *section* if  $s(H) = e$ , the identity of group  $G$  and  $\nu s = \mathbf{1}$  where  $\nu$  denotes the quotient map given by  $\nu(x) = Hx$ .

If  $s$  is a section then the image  $s(G/H)$  is a right transversal to  $H$  in  $G$ . Conversely every right transversal  $S$  determines a section  $s$  given by  $\{s(Hx)\} = S \cap Hx$ .

PROPOSITION 2.3. [5] *Let  $H$  be a closed subgroup of a topological group  $G$  and  $S$  a right transversal to  $H$  in  $G$ . Suppose that the section  $s$  from the quotient space  $G/rH$  (the set of right cosets of  $H$  in  $G$ ) to  $G$  given by  $\{s(Hg)\} = S \cap Hg$  is continuous. Then the binary operation  $o$  on  $S$  given by  $\{xoy\} = S \cap Hxy$  and the map  $\chi : S \times S \rightarrow S$  given by  $\chi(x, y) ox = y$  are continuous (Here  $S$  is given the subspace topology).*

PROPOSITION 2.4. [5] *Let  $S$  be a right transversal to a closed subgroup  $H$  of a topological group  $G$  which is the image of a continuous section  $s : G/H \rightarrow G$ . Then the right transversal  $S_g = \{g(x)x \mid x \in S\}$  corresponding to the map  $g : S \rightarrow H$  with  $g(e) = e$  is the image of a continuous section if and only if  $g$  is continuous from  $S$  (with subspace topology) to  $H$ .*

### 3. Main Results

The main results of the paper are:

THEOREM 3.1. *The circle group  $\mathbb{S}^1 = \left\{ \overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}} \mid 0 \leq t < \pi \right\}$  is a right transversal to the subgroup  $H = \left\{ \overline{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}} \mid a > 0, b \in \mathbb{R} \right\}$  in the projective special linear group  $PSL(2, \mathbb{R})$ . Let  $g$  be a map from  $\mathbb{S}^1$  to  $H$  given by*

$$(1) \quad g \left( \overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}} \right) = \overline{\begin{pmatrix} u(t) & v(t) \\ 0 & (u(t))^{-1} \end{pmatrix}}$$

where  $u : [0, \pi) \rightarrow \mathbb{R} \setminus \{0\}$  and  $v : [0, \pi) \rightarrow \mathbb{R}$  are smooth maps with  $u(0) = 1$ ,  $v(0) = 0$ . Then  $(\mathbb{S}^1, o_g)$  is a smooth right quasigroup with identity where  $o_g$  is given by

$$(2) \quad \begin{aligned} & \overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}} o_g \overline{\begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}} \\ & = \overline{\begin{pmatrix} \cos(t_1 + s) & \sin(t_1 + s) \\ -\sin(t_1 + s) & \cos(t_1 + s) \end{pmatrix}} \end{aligned}$$

and

$$(3) \quad t_1 = \tan^{-1} \left( \frac{u(s) \sin t}{u(s)^{-1} \cos t - v(s) \sin t} \right).$$

PROOF. A two dimensional subgroup of  $PSL(2, \mathbb{R})$  is a conjugate of the subgroup

$$H = \left\{ \overline{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}} \mid a > 0, b \in \mathbb{R} \right\}.$$

The circle group  $\mathbb{S}^1$  is given by  $\mathbb{S}^1 = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mid 0 \leq t < 2\pi \right\}$ .

Clearly  $\mathbb{S}^1 \subseteq SL(2, \mathbb{R})$ . The subgroup  $\mathbb{S}^1/\{I, -I\}$  of the group  $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{I, -I\}$  is again a circle group which is given by

$$\mathbb{S}^1 = \left\{ \overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}} \mid 0 \leq t < \pi \right\}.$$

We claim that  $\mathbb{S}^1$  is a right transversal to the subgroup  $H$  in the group  $PSL(2, \mathbb{R})$ . To prove this it is sufficient to prove that  $PSL(2, \mathbb{R}) = H\mathbb{S}^1$  and  $H \cap \mathbb{S}^1 = \{\bar{I}\}$ .

Let  $\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \in PSL(2, \mathbb{R})$ . Then  $ad - bc = 1$ . Thus  $c$  and  $d$  will not be zero simultaneously. If  $c \neq 0$ . Without loss we may assume that  $c > 0$ . Then

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \overline{\begin{pmatrix} u(t) & v(t) \\ 0 & (u(t))^{-1} \end{pmatrix}} \overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}}$$

where  $u(t) = a \cos t + b \sin t$ ,  $v(t) = -a \sin t + b \cos t$  and  $t = \tan^{-1}(\frac{-c}{d})$ .

If  $c = 0$  then  $d = a^{-1}$ . In this case,

$$\overline{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}} = \begin{cases} \overline{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}} \overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} & \text{if } a > 0 \\ \overline{\begin{pmatrix} -a & -b \\ 0 & -a^{-1} \end{pmatrix}} \overline{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}} & \text{if } a < 0 \end{cases}$$

Thus  $PSL(2, \mathbb{R}) = H\mathbb{S}^1$ . Clearly  $H \cap \mathbb{S}^1 = \{\bar{I}\}$ . This shows that  $\mathbb{S}^1$

is a right transversal to  $H$  in  $PSL(2, \mathbb{R})$ . The induced right quasigroup structure  $o$  on  $\mathbb{S}^1$  is given by

$$(4) \quad \overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}} o \overline{\begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}} \\ = \overline{\begin{pmatrix} \cos(t+s) & \sin(t+s) \\ -\sin(t+s) & \cos(t+s) \end{pmatrix}}$$

and the action  $\theta$  of  $H$  on  $\mathbb{S}^1$  is given by

$$(5) \quad \overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}} \theta \overline{\begin{pmatrix} u(s) & v(s) \\ 0 & (u(s))^{-1} \end{pmatrix}} = \overline{\begin{pmatrix} \cos t_1 & \sin t_1 \\ -\sin t_1 & \cos t_1 \end{pmatrix}}$$

where  $t_1$  is given by Eq.(3).

Let  $g : \mathbb{S}^1 \rightarrow H$  be a map given by Eq.(1). Since  $u, v$  are smooth, therefore  $g$  is smooth. By Eq.(4), Eq.(5) and Proposition 2.1, the right quasigroup structure  $o_g$  on  $\mathbb{S}^1$  is given by

$$\overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}} o_g \overline{\begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}} \\ = \overline{\begin{pmatrix} \cos(t_1+s) & \sin(t_1+s) \\ -\sin(t_1+s) & \cos(t_1+s) \end{pmatrix}}$$

where  $t_1$  is given by Eq.(3). By Proposition 2.4, the operation  $o_g$  is smooth.  $\square$

REMARK 3.2. In general  $(\mathbb{S}^1, o_g)$  need not be a loop and it may not have right inverse property. For example, if we take  $u$  and  $v$  given by  $u(t) = \sin \frac{t}{2} + \cos \frac{t}{2}$  and  $v(t) = \sin t \cos t$  then  $\overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \in \mathbb{S}^1$  has no right

inverse. However for certain  $g$ , for example

$$g \left( \overline{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}} \right) = \overline{\begin{pmatrix} 1 & \frac{1}{4}t \sin t \\ 0 & 1 \end{pmatrix}},$$

$(\mathbb{S}^1, o_g)$  is indeed a loop.

Consider the additive group of real numbers. Since a differentiable function can be approximated by a polynomial function. In this light, we establish the following

**PROPOSITION 3.3.** *Let  $n \geq 3$ . Then a polynomial  $P_n(x, y) = \sum_{i=0}^n a_i y^i + x \sum_{j=0}^{n-1} b_j y^j$  of degree  $n$  in two variables  $x$  and  $y$  over the field  $\mathbb{R}$  determines a smooth right quasigroup structure  $o$  with identity  $0$  given by  $xoy = P_n(x, y)$  if and only if  $P_n(x, y) = x + y + x\phi(y)$  where  $\phi(y)$  is a polynomial in  $y$  of degree  $n - 1$  such that  $\phi(a) \neq -1$  for every  $a \in \mathbb{R}$  and  $\phi(0) = 0$ . In particular  $P_{2n+1}(x, y) = x + y + xy^{2n}$  where  $n \in \mathbb{N}$  determines a smooth right quasigroup structure on  $\mathbb{R}$ .*

**PROOF.** Suppose that  $n \geq 3$  and  $P_n(x, y) = \sum_{i=0}^n a_i y^i + x \sum_{j=0}^{n-1} b_j y^j$ . Suppose that the structure  $o$  given by  $xoy = P_n(x, y)$  is a right quasigroup structure with identity  $0$ . Then  $0o0 = 0$  and  $xo0 = x = 0ox$  implies that

$$(6) \quad a_0 = 0, \quad a_1 = b_0 = 1, \quad a_i = 0, \quad \forall i \geq 2$$

Thus  $P_n(x, y) = x + y + x\phi(y)$ , where  $\phi(y) = \sum_{j=1}^{n-1} b_j y^j$  is a polynomial of degree  $n - 1$  such that  $\phi(0) = 0$ . Consider the equation  $Xoa = 0$  where  $a \in \mathbb{R}$  and  $X$  is unknown in the equation. Then the existence of solution to the equation implies that  $\phi(a) \neq -1$ .

Conversely, suppose that  $P_n(x, y) = x + y + x\phi(y)$  and  $xoy = P_n(x, y)$ , where  $\phi(y)$  is a polynomial of degree  $n - 1$  such that  $\phi(a) \neq -1$  for every  $a \in \mathbb{R}$  and  $\phi(0) = 0$ . Let  $a, b \in \mathbb{R}$ . Then  $U = \frac{b - a}{1 + \phi(a)}$  is the solution to the equation  $Xoa = b$ , where  $X$  is unknown in the equation. Also  $xo0 = x = 0ox$ . The smoothness of polynomial function implies the smoothness of  $o$ . Thus  $(\mathbb{R}, o)$  is a smooth right quasigroup with identity. In particular

if  $\phi(y) = y^{2n}, n \in \mathbb{N}$  then  $\phi(a) = a^{2n} \geq 0$  for every  $a \in \mathbb{R}$ . Thus the result follows.  $\square$

More generally we have the following proposition:

**PROPOSITION 3.4.** *Let  $\phi(y)$  be a differentiable function on  $\mathbb{R}$  such that  $\phi(y) \neq -1$ , for all  $y \in \mathbb{R}$  and  $\phi(0) = 0$ . Then  $(\mathbb{R}, o)$  is a smooth right quasigroup with identity 0, where  $xoy = x + y + x\phi(y)$ .*

**REMARK 3.5.** The right quasigroup  $(\mathbb{R}, o)$  where  $xoy = x + y + xy^{2n}$  does not have right inverse property for 1 has no right inverse.

Smooth right quasigroup structures on  $\mathbb{R}$  are obtained also in the following manner :

**THEOREM 3.6.** *Let  $\phi : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$  be a differentiable map such that  $\phi(0) = 1$ . Then  $(\mathbb{R}, \odot)$  is a smooth right quasigroup with identity 0, where  $\odot$  is given by  $x \odot y = y + x[\phi(y)]^{-2}$ .*

**PROOF.** Consider the Borel subgroup  $\mathcal{B} = \left\{ \overline{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}} \mid \alpha, \beta \in \mathbb{R}, \alpha > 0 \right\}$  of the projective special linear group  $PSL(2, \mathbb{R})$ . Then  $\mathcal{B} = HS, H \cap S = \{\overline{I_2}\}$  where

$$H = \left\{ \overline{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}} \mid \alpha \in \mathbb{R}, \alpha > 0 \right\} \quad \text{and} \quad S = \left\{ \overline{\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}} \mid \beta \in \mathbb{R} \right\}.$$

Let  $A_\alpha = \overline{\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}}, \forall \alpha \in \mathbb{R}$ . Then  $(S, o)$  is a right quasigroup with identity  $\overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$  where  $o$  is given by  $A_\alpha o A_\beta = A_{\alpha+\beta}$ . Let  $g : S \rightarrow H$  be a deformation map given by  $g(A_\alpha) = \overline{\begin{pmatrix} \phi(\alpha) & 0 \\ 0 & \phi(\alpha)^{-1} \end{pmatrix}}$  where  $\phi : \mathbb{R} \rightarrow$

$\mathbb{R} \setminus \{0\}$  is a differentiable map such that  $\phi(0) = 1$ . The induced right quasigroup structure  $o_g$  on  $S$  is given by

$$A_\alpha o_g A_\beta = S \cap H[g(A_\alpha)A_\alpha g(A_\beta)A_\beta].$$

Since,

$$g(A_\alpha)A_\alpha g(A_\beta)A_\beta = g(A_\alpha)g(A_\beta) \begin{pmatrix} 1 & \beta + \alpha\phi(\beta)^{-2} \\ 0 & 1 \end{pmatrix}.$$

Therefore  $A_\alpha o_g A_\beta = A_{\beta + \alpha\phi(\beta)^{-2}}$ . Clearly the map  $\psi : \mathbb{R} \rightarrow S$  defined by  $\psi(\alpha) = A_\alpha$  is bijective. This in turn induces the operation  $\odot$  on  $\mathbb{R}$  given by  $\alpha \odot \beta = \beta + \alpha\phi(\beta)^{-2}$  so that  $\psi$  is an isomorphism. Clearly the operation  $\odot$  on  $\mathbb{R}$  is smooth.  $\square$

**REMARK 3.7.** The right quasigroup structure  $o$  on  $\mathbb{R}$  given by  $xoy = x + y + x\phi(y)$ , is determined by the structure  $\odot$  on  $\mathbb{R}$  given by  $x \odot y = y + x[\psi(y)]^{-2}$ , where  $\psi(y) = \frac{1}{+\sqrt{1+\phi(y)}}$  and  $\phi$  is a differentiable map from  $\mathbb{R}$  into  $(-1, \infty)$  such that  $\phi(0) = 0$ . Further, none of the right loops on  $\mathbb{R}$  are loops except the trivial one which is the additive group of real numbers.

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