

Tropical Jacobians in \mathbb{R}^2

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Abstract. By tropical Abel-Jacobi theorem, the Jacobian of a tropical curve is isomorphic to the Picard group. A tropical curve in \mathbb{R}^2 corresponds to an immersion from a tropical curve to \mathbb{R}^2 . In this paper, we show that any principal divisor on a tropical curve is the restriction of a principal divisor on the ambient plane \mathbb{R}^2 .

1. Introduction

The main result of this paper is comparison of the Picard group of a tropical curve X embedded to \mathbb{R}^2 and a potentially larger group obtained by quotienting the divisor group by the equivalence generated by rational functions that extend to the ambient plane. We show that these two groups are equal. In other words, any principal divisor on X is induced from tropical curves in \mathbb{R}^2 .

Let C be a tropical curve in \mathbb{R}^2 . Let $X \rightarrow \mathbb{R}^2$ be the corresponding immersion from a tropical curve X to the affine space \mathbb{R}^2 with no crossing points. (So, if C is reduced, the immersion $X \rightarrow C$ is bijective.) The Jacobian $\text{Jac}(X)$ is defined in [4].

THEOREM 1.1 (Tropical Abel-Jacobi, [4]). *The Abel's map $\mu: \text{Div}^0(X) \rightarrow \text{Jac}(X)$ factors through the Picard group $\text{Pic}^0(X)$:*

$$\begin{array}{ccc} \text{Div}^0(X) & \longrightarrow & \text{Pic}^0(X) \\ & \searrow \mu & \downarrow \phi \\ & & \text{Jac}(X) \end{array}$$

and ϕ is a bijection.

DEFINITION. A tropical curve C in \mathbb{R}^2 is *reduced* if every edge is of weight 1. C is *smooth* if every vertex is 3-valent and of multiplicity 1. (Hence any smooth tropical curve is reduced.)

DEFINITION. The *genus* of a reduced tropical curve C in \mathbb{R}^2 is the first Betti number $b_1(C)$.

For example, a tropical elliptic curve C is a smooth tropical curve of genus 1 (Figure 1).

DEFINITION. A divisor D of C is *principal* if there are tropical curves L, L' in \mathbb{R}^2 such that

$$\Delta(L) = \Delta(L'),$$

$$D = C \cdot L - C \cdot L',$$

where Δ denotes the Newton polygon and $C \cdot L$ denotes the stable intersection. Divisors D, D' of C are *linearly equivalent* ($D \sim D'$) if $D - D'$ is principal.

THEOREM 1.2. Let C be a reduced tropical curve in \mathbb{R}^2 . Let $X \rightarrow \mathbb{R}^2$ be the corresponding immersion of a tropical curve. Let D be a divisor on C . Let \tilde{D} be the corresponding divisor on X . Then D is principal if and only if \tilde{D} is principal.

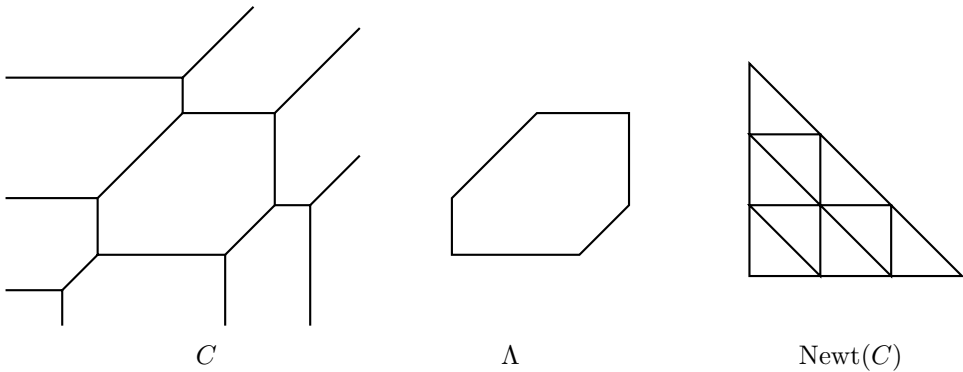


Fig. 1. Tropical elliptic curve

Let $T(C)$ be the quotient group

$$T(C) = \text{Div}^0(C) / \sim .$$

Theorem 1.2 is equivalent to the following statement.

THEOREM 1.3. *Let C be a reduced tropical curve of genus g in \mathbb{R}^2 . Then $T(C)$ is the g -dimensional real torus, and $T(C) \cong \text{Pic}^0(X)$.*

Also we show the following statement in the case of genus 1.

THEOREM 1.4. *Let C be a reduced tropical curve of genus 1 in \mathbb{R}^2 . Let Λ be the cycle in C . Let \mathcal{O} be any point of Λ . Then the map*

$$\begin{aligned} \varphi: \Lambda &\longrightarrow T(C) \\ P &\longmapsto P - \mathcal{O} \end{aligned}$$

is bijective.

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2. Preliminaries

The affine space \mathbb{R}^2 is equipped with interior product

$$(u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2$$

and exterior product

$$(u_1, u_2) \times (v_1, v_2) = u_1 v_2 - v_1 u_2.$$

A *primitive vector* in \mathbb{R}^2 is an integral vector $u = (u_1, u_2)$ such that u_1, u_2 are relatively prime. Let $v \in \mathbb{Z}^2$ be an integral vector. Then there are a primitive vector u and a number $m \in \mathbb{Z}_{>0}$ such that $v = mu$. The number m is called the *lattice length* of v . We write $m = |v|$.

Let C be a 1-dimensional weighted simplicial complex of rational slopes in \mathbb{R}^2 (that is a subset of \mathbb{R}^2 written as a union of line segments equipped with weights of edges). Each finite edge $E \subset C$ has two vertices. Let $V \in E$

be one of the vertices. Let $\text{wt}(E) \in \mathbb{Z}_{>0}$ be the weight of E . The weighted primitive vector of E starting at V is defined to be the integral vector u_E for the direction from V to E such that $|u_E| = \text{wt}(E)$.

A ray E of C is an infinite edge. Any ray E has only one vertex.

DEFINITION. A vertex V of C satisfies the balancing condition if the sum of all weighted primitive vectors starting at V equals 0:

$$\sum_{E \ni V} u_E = 0.$$

A tropical curve in \mathbb{R}^2 is a 1-dimensional weighted simplicial complex of rational slopes such that each vertex satisfies the balancing condition.

The union $C_1 \cup C_2$ of two tropical curves is a tropical curve. Indeed if $V \in C_1 \cap C_2$ is an intersection point of two edges, V is considered as a 4-valent vertex of $C_1 \cup C_2$.

Let C be a tropical curve in \mathbb{R}^2 . Let U_1, \dots, U_r be all connected components of $\mathbb{R}^2 \setminus C$. Let N be a 1-dimensional simplicial complex with vertex set $\text{Ver}(N) = \{w_1, \dots, w_r\}$, $w_i \in \mathbb{Z}^2$.

DEFINITION. N is a Newton complex of C if it satisfies the following conditions for any $i \neq j$.

- i) If $\overline{U_i} \cap \overline{U_j} = \emptyset$, then $[w_i, w_j] \notin \text{Edge}(N)$.
- ii) If $\overline{U_i} \cap \overline{U_j} = E$ for some $E \in \text{Edge}(C)$, then $[w_i, w_j] \in \text{Edge}(N)$, and $w_j - w_i$ has lattice length $\text{wt}(E)$, direction orthogonal to E from U_i to U_j .

PROPOSITION 2.1. Let C, C_1, C_2 be tropical curves in \mathbb{R}^2 .

- 1) The Newton complex $\text{Newt}(C)$ of C exists uniquely up to parallel translation. The convex hull

$$\Delta(C) = \text{Conv}(\text{Newt}(C))$$

is called the Newton polygon of C .

- 2) $\Delta(C_1 \cup C_2)$ equals the Minkowski sum $\Delta(C_1) + \Delta(C_2)$.

PROOF. 1) Let $w_1 = (0, 0)$, and suppose w_1, \dots, w_{k-1} are constructed. Rearranging U_k, \dots, U_r , we may assume $U_i \cap U_k = E_k$ for some $i < k$ and

some $E_k \in \text{Edge}(C)$. Let u_k be the primitive vector of direction orthogonal to E_k from U_i to U_k . Let w_k be the vector defined as follows.

$$(1) \quad w_k - w_i = \text{wt}(E_k)u_k.$$

If U_1, \dots, U_k are adjacent at a common vertex V , we have the balancing condition

$$\sum_{i=1}^k \text{wt}(E_i)u_i = 0,$$

where E_1 is the boundary of U_k and U_1 , and u_1 is the primitive vector of direction orthogonal to E_1 from U_k to U_1 . So we have

$$w_1 - w_k = \text{wt}(E_1)u_1.$$

So this construction does not depend on the choice of U_k .

2) The set of the rays of $C_1 \cup C_2$ corresponds to the set of the rays of C_1 and C_2 . So the set of the edges of $\text{Newt}(C_1 \cup C_2)$ in the boundary of $\Delta(C_1 \cup C_2)$ corresponds to the set of the edges of $\text{Newt}(C_1)$ and $\text{Newt}(C_2)$ in the boundary of $\Delta(C_1)$ and $\Delta(C_2)$. \square

$\text{Newt}(C)$ can be considered as a dual object of C , with correspondence from U_i to w_i (Figure 2). A vertex $V \in C$ corresponds to a polygon $T_V \subset$

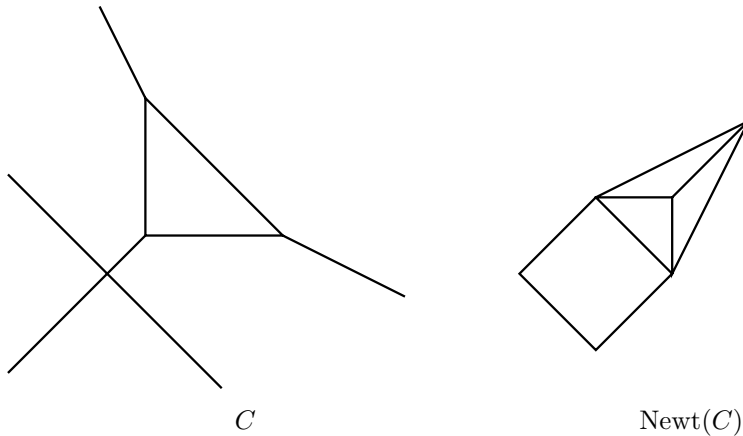


Fig. 2. Newton complex

$\Delta(C)$ as follows.

- i) U_{i_1}, \dots, U_{i_k} are adjacent at V ,
- ii) $T_V = \text{Conv}\{w_{i_1}, \dots, w_{i_k}\}$.

PROPOSITION 2.2 (Global balancing condition). *Let C be a tropical curve in \mathbb{R}^2 . Let Λ be a simple closed curve in \mathbb{R}^2 intersecting edges of C , say E_1, \dots, E_N , transversely. Then*

$$\sum_{i=1}^N u_{E_i} = 0,$$

where u_{E_i} is the weighted primitive vector of E_i starting at the vertex inside Λ .

PROOF. For each vertex $V_j \in C$ inside Λ , the balancing condition

$$\sum_k u_{jk} = 0$$

holds. Thus

$$\sum_{j,k} u_{jk} = 0.$$

On the left side, two weighted primitive vectors of the same edge inside Λ are canceled. Thus we have the required equation. \square

A *tangent vector* (v, P) in \mathbb{R}^2 is a vector $v \in \mathbb{R}^2$ with a starting point $P \in \mathbb{R}^2$. We fix a point $P_0 \in \mathbb{R}^2$. The *moment* of (v, P) is the exterior product

$$\text{moment}(v, P) = \overrightarrow{P_0P} \times v.$$

PROPOSITION 2.3 (Moment condition). *Under the assumption of Proposition 2.2,*

$$\sum_{i=1}^N \text{moment}(u_{E_i}, V_{E_i}) = 0,$$

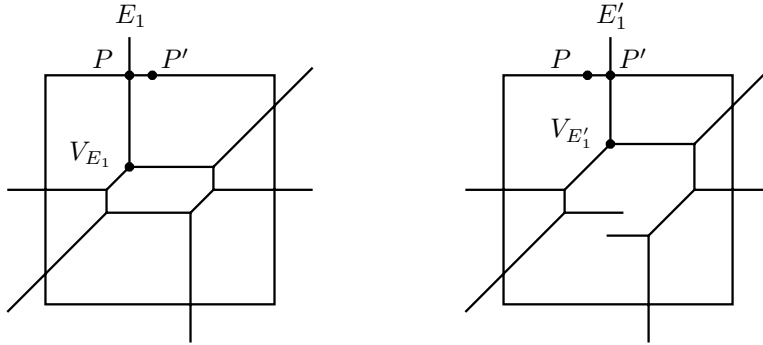


Fig. 3. Moment condition inside Λ . Note that $\text{moment}(u_{E_1}, V_{E_1}) \neq \text{moment}(u_{E'_1}, V_{E'_1})$.

where (u_{E_i}, V_{E_i}) is the weighted primitive tangent vector of E_i starting at the vertex inside Λ . (See Figure 3.)

PROOF. For each vertex $V_j \in C$ inside Λ , the balancing condition

$$\sum_k u_{jk} = 0$$

holds. Thus

$$\sum_{j,k} \overrightarrow{P_0 V_j} \times u_{jk} = 0.$$

On the left side, two weighted primitive vectors of the same edge inside Λ are canceled as follows.

$$\begin{aligned} \overrightarrow{P_0 V_j} \times u_{jk} + \overrightarrow{P_0 V_{j'}} \times u_{j'k'} &= \overrightarrow{P_0 V_j} \times u_{jk} + \overrightarrow{P_0 V_{j'}} \times (-u_{jk}) \\ &= \overrightarrow{V_{j'} V_j} \times u_{jk} \\ &= 0. \end{aligned}$$

Thus we have the required equation. \square

DEFINITION. The *multiplicity* of a vertex $V \in C$ is

$$\text{Mult}(V; C) = 2 \cdot \text{area}(T_V).$$

The *intersection multiplicity* of $V \in C_1 \cap C_2$ is

$$\mu_V = \frac{1}{2} (\text{Mult}(V; C_1 \cup C_2) - \text{Mult}(V; C_1) - \text{Mult}(V; C_2)).$$

(If V is not a vertex of C , we put $\text{Mult}(V; C) = 0$.)

We have a divisor

$$C_1 \cdot C_2 = \sum_{V \in C_1 \cap C_2} \mu_V V.$$

This is called the *stable intersection* of C_1 and C_2 .

By [6], the stable intersection is characterized as the limit of the transversal intersection. If $V \in C_1 \cap C_2$ is a transversal intersection point, we have

$$\mu_V = |u_E \times u_F|,$$

where $E \subset C_1$ and $F \subset C_2$ are edges passing through V .

A tropical Bezout's theorem is proved in [6]. Now we show another proof.

THEOREM 2.4 (Tropical Bezout). *Let C_1, C_2 be tropical curves in \mathbb{R}^2 . Then the following formula holds.*

$$\deg(C_1 \cdot C_2) = \text{area}(\Delta(C_1) + \Delta(C_2)) - \text{area}(\Delta(C_1)) - \text{area}(\Delta(C_2)),$$

where $\Delta(C_1) + \Delta(C_2)$ is the *Minkowski sum*.

PROOF. The statement follows from Proposition 2.1. \square

For example, let $c, d \geq 1$, and suppose $\Delta(C_1) = \text{Conv}\{(0, 0), (c, 0), (0, c)\}$, $\Delta(C_2) = \text{Conv}\{(0, 0), (d, 0), (0, d)\}$ (C_2 is said to be *projective* of degree d). Then $\Delta(C_1) + \Delta(C_2) = \text{Conv}\{(0, 0), (c + d, 0), (0, c + d)\}$, and $\deg(C_1 \cdot C_2) = \frac{1}{2}(c + d)^2 - \frac{1}{2}c^2 - \frac{1}{2}d^2 = cd$.

3. Reduced Tropical Curves

Let C be a reduced tropical curve.

LEMMA 3.1. *Let $u \in \mathbb{Z}^2$ be a primitive vector. Then for given $\varepsilon > 0$, there is a primitive vector $v \in \mathbb{Z}^2$ such that*

$$u \times v = 1, \quad |\theta(u) - \theta(v)| < \varepsilon,$$

where $\theta(u)$ denotes the angle of u .

This lemma is easy.

LEMMA 3.2. *Let E be an edge of C , and let P, P', Q, Q' be points of E such that $\overrightarrow{PP'} = \overrightarrow{QQ'}$. Then $P' - P \sim Q' - Q$.*

PROOF. (Figure 4) We may assume that P', Q lie on the interval $[P, Q']$, and that Q' lies in the interior $\text{Int}(E)$. Let v_1, v_2, v_3 be primitive vectors such that

$$(2) \quad |u_E \times v_i| = 1 \quad (i = 1, 2, 3),$$

$$\theta(u_E) - \varepsilon < \theta(v_1) < \theta(u_E) < \theta(v_2) < \theta(u_E) + \varepsilon < \theta(v_3).$$

Then we have a triangle, with vertices P and $R_1, R_2 \in \mathbb{R}^2$, such that

$$\begin{aligned} \overrightarrow{PR_i} &\text{ has direction } v_i \quad (i = 1, 2), \\ \overrightarrow{R_1R_2} &\text{ has direction } v_3, \\ Q' &\in [R_1, R_2]. \end{aligned}$$

Take $\varepsilon > 0$ small enough so that this triangle is disjoint from $C \setminus E$.

Now we take two tropical curves L, M as follows. L consists of one vertex P and three rays $L_0, L_1, L_2 \subset \mathbb{R}^2$, with $R_1 \in L_1, R_2 \in L_2$. L_0 is parallel to E . $\text{wt}(L_1) = \text{wt}(L_2) = 1$. (The balancing condition at P follows from equation (2).) M consists of one finite edge $M_0 \subset \mathbb{R}^2$, four rays $M_1, M_2, M_3, M_4 \subset \mathbb{R}^2$, and two vertices $V_1, V_2 \in \mathbb{R}^2$. M_0 is parallel to $[R_1, R_2]$, and passes through Q . $V_1 \in [P, R_1], V_2 \in [P, R_2]$. For $i = 1, 2$, M_i has vertex V_i and passes through R_i . M_3, M_4 are parallel to E . $\text{wt}(M_0) = \text{wt}(M_1) = \text{wt}(M_2) = 1$.

Move L by $\overrightarrow{PP'}$, and denote it by L' . Move M by $\overrightarrow{QQ'}$, and denote it by M' . Note that the intersection of L' and C at P' has multiplicity 1. Now we have

$$(C \cdot L - P) - (C \cdot L' - P') = (C \cdot M - Q) - (C \cdot M' - Q').$$

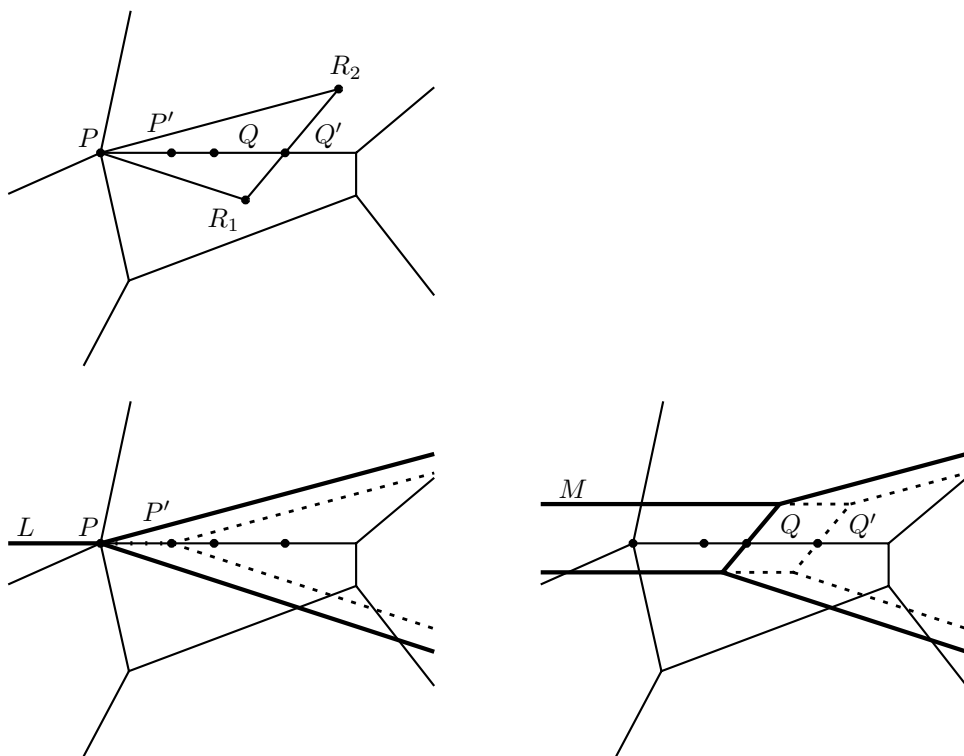


Fig. 4.

Thus $P' - P$ is linearly equivalent to $Q' - Q$. \square

COROLLARY 3.3. *Let E be any edge, and suppose that all interior points of E are linearly equivalent. Then all points of E are linearly equivalent.*

LEMMA 3.4. *Let E be a ray of C . Then all points of E are linearly equivalent.*

PROOF. (Figure 5, left) Let $P, Q \in \text{Int}(E)$. Take a primitive vector v so that

$$|u_E \times v| = 1, \quad |\theta(u_E) - \theta(v)| < \varepsilon.$$

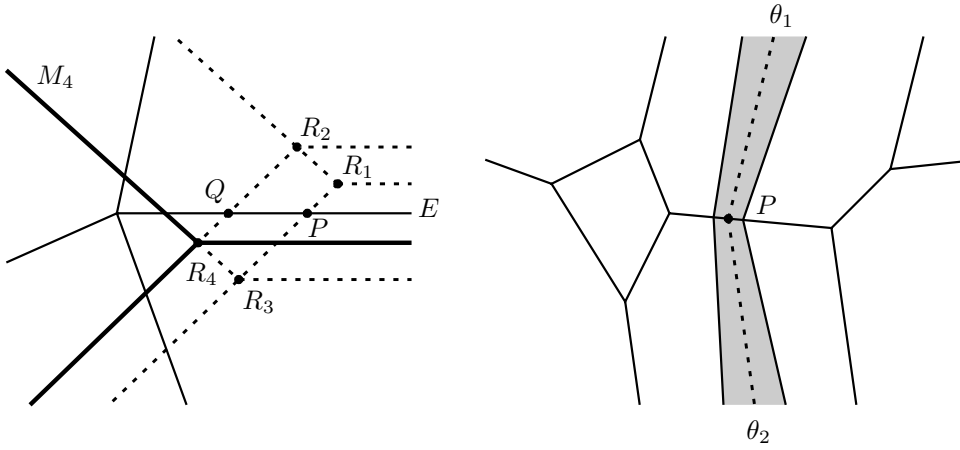


Fig. 5.

There is a parallelogram $R_1R_2R_3R_4$ such that

$$\begin{aligned} \overrightarrow{R_3R_1}, \overrightarrow{R_4R_2} &\text{ have direction } v, \\ \overrightarrow{R_2R_1}, \overrightarrow{R_4R_3} &\text{ have direction } w := u_E - v, \\ P &\in [R_1, R_3], Q \in [R_2, R_4]. \end{aligned}$$

Take $\varepsilon > 0$ small enough so that this parallelogram is disjoint from $C \setminus E$.

Let M_1 be a tropical curve, consisting of one vertex R_1 and three rays $L_0, L_1, L_2 \subset \mathbb{R}^2$, such that L_0 has direction u_E , $R_3 \in L_1$, $R_2 \in L_2$, $\text{wt}(L_0) = \text{wt}(L_1) = 1$. Move M_1 by $\overrightarrow{R_1R_i}$, and denote it by $M_i (i = 2, 3, 4)$. Then we have

$$C \cdot M_1 + C \cdot M_4 - Q = C \cdot M_2 + C \cdot M_3 - P.$$

Thus $P \sim Q$. \square

DEFINITION. An edge $E \subset C$ is *trivial* if $C \setminus \text{Int}(E)$ is disconnected.

LEMMA 3.5. *Let E be a trivial edge of C . Then all points of E are linearly equivalent.*

PROOF. (Figure 5, right) Let $P \in \text{Int}(E)$. Since $C \setminus \text{Int}(E)$ is disconnected, E is a boundary of two unbounded convex open sets U_1, U_2 of

$\mathbb{R}^2 \setminus C$. For $i = 1, 2$, let θ_i be the angle of any unbounded direction of U_i . For $\delta, \varepsilon > 0$, let

$$W_i = \left\{ P + a + w \mid \begin{array}{l} a, w \in \mathbb{R}^2, |a| < \delta, P + a \in E, \\ |\theta(w) - \theta_i| < \varepsilon \end{array} \right\},$$

$$W = W_1 \cup W_2.$$

Take $\delta, \varepsilon > 0$ small enough so that W intersects $C \setminus E$ only at points of rays of C .

Let v be a primitive vector such that $u_E \times v = 1$. Take $w, w' \in \mathbb{Z}^2$ so that

$$w - w' = v,$$

$$|\theta(w) - \theta_1| < \varepsilon,$$

$$|\theta(w') - \theta_1| < \varepsilon.$$

Let L be a tropical curve consisting of three rays $L_0, L_1, L_2 \subset \mathbb{R}^2$ such that

$$L_1 \cup L_2 \subset W,$$

$$u_{L_1} = w,$$

$$|\theta(L_2) - \theta_2| < \varepsilon,$$

$$L_1 \cap E = P.$$

Moving L on the direction of L_0 , the intersection point P changes to other point $Q \in E$, but all other intersection points of L and C are stable except for points of rays of C . We have

$$(u_E \times w)P \sim (u_E \times w)Q.$$

Similarly,

$$(u_E \times w')P \sim (u_E \times w')Q.$$

Thus $P \sim Q$. \square

4. Parameter Space of Tropical Plane Curves

Let L be a tropical curve with Newton complex \mathcal{N} . We fix a vertex $V_0 = (b_1, b_2)$ of L . Let E_1, \dots, E_l be all finite edges of L . Let a_i be the lattice length of E_i . Then all tropical curves with Newton complex \mathcal{N} are parametrized by $a_1, \dots, a_l > 0$ and $b_1, b_2 \in \mathbb{R}$. Let $\mathcal{P}(\mathcal{N}, \mathbb{R}^2) \subset \mathbb{R}^{l+2}$ be the parameter space.

PROPOSITION 4.1. $\mathcal{P}(\mathcal{N}, \mathbb{R}^2)$ is connected.

PROOF. Let $\Gamma_1, \dots, \Gamma_g$ be all convex cycles of L . Let $E_{i(j,1)}, \dots, E_{i(j,s_j)}$ be all edges of Γ_j . Let $u_{j,k}$ be the primitive vector of $E_{i(j,k)}$ of positive direction. Then the equation

$$(3) \quad a_{i(j,1)}u_{j,1} + \dots + a_{i(j,s_j)}u_{j,s_j} = 0$$

is satisfied for any $L \in \mathcal{P}(\mathcal{N}, \mathbb{R}^2)$. Let $H_j \subset \mathbb{R}^{l+2}$ be the linear subspace defined by equation (3). Then

$$\mathcal{P}(\mathcal{N}, \mathbb{R}^2) = \{(a_1, \dots, a_l, b_1, b_2) \mid a_1, \dots, a_l > 0\} \cap (H_1 \cap \dots \cap H_g).$$

Thus $\mathcal{P}(\mathcal{N}, \mathbb{R}^2)$ is a relatively open convex cone in \mathbb{R}^{l+2} , which is connected. \square

For tropical curves L, L' , the notation $\text{Newt}(L') \subset \text{Newt}(L)$ means that any edge of $\text{Newt}(L')$ is an edge of $\text{Newt}(L)$.

DEFINITION. A tropical curve L' is a *degeneration* of L if $\Delta(L') = \Delta(L)$ and $\text{Newt}(L') \subset \text{Newt}(L)$.

The set of all degenerations of L is parametrized by $\overline{\mathcal{P}(\mathcal{N}, \mathbb{R}^2)}$. If a Newton polygon Δ is fixed, all tropical curves have a common degeneration, which is a tropical curve consisting of one vertex. The set of the Newton polygons is countably infinite. Therefore, the space $\mathcal{P}(\mathbb{R}^2)$ of all tropical curves is a disjoint union of countably many closed cones in affine spaces.

COROLLARY 4.2. Tropical curves $L, L' \in \mathcal{P}(\mathbb{R}^2)$ lie in the same connected component if and only if $\Delta(L) = \Delta(L')$.

5. Proof of Theorem 1.4

Let Λ be the cycle in C . (C is a graph with first Betti number $b_1(C) = 1$. So there is a unique cycle.) Λ is equipped with the positive direction defined as follows. For any edge $E \subset \Lambda$ with vertices V_1, V_2 , the direction from V_1 to V_2 is positive if and only if

$$\overrightarrow{V_1 V_2} \times u_{E'} < 0,$$

where E' is any edge of any tropical curve L that intersects E transversely, and $u_{E'}$ is the weighted primitive vector of E' starting at the vertex inside Λ .

Let \mathcal{O} be any point of Λ . We consider the map $\pi: \mathbb{R} \rightarrow \Lambda$ with the following properties.

- i) $\pi(0) = \mathcal{O}$.
- ii) π is increasing with respect to the positive direction of Λ .
- iii) π is compatible with the lattice length. i.e. If $\pi[a, b]$ ($a, b \in \mathbb{R}$) is contained in an edge of primitive vector u , then

$$\text{length}(\pi[a, b]) = |u|(b - a).$$

From Lemma 3.4 and Lemma 3.5, $T(C)$ is generated by $\{P - \mathcal{O} | P \in \Lambda\}$. Λ has the group structure induced by π . The following lemma 5.1 says that the map

$$\varphi: \Lambda \longrightarrow T(C)$$

is a surjective homomorphism.

PROOF OF THAT φ IS A HOMOMORPHISM. Let $a, b \in \mathbb{R}$. We have $\pi(a + b) - \pi(b) = \pi(a) - \mathcal{O}$ from lemma 5.1. $\pi(a + b) - \mathcal{O} = (\pi(a) - \mathcal{O}) + (\pi(b) - \mathcal{O})$. Thus $\varphi(a + b) = \varphi(a) + \varphi(b)$. \square

LEMMA 5.1. *If $\pi(a) = P$, $\pi(a') = P'$, $\pi(b) = Q$, $\pi(b') = Q'$, $a' - a = b' - b$, then $P' - P \sim Q' - Q$.*

PROOF. (Figure 6) We may assume that $a' - a > 0$ is small enough so that P, P' lie on the same edge E , and Q, Q' on F . We may also assume that E, F are adjacent at a common vertex R , and $E \neq F$. From Lemma

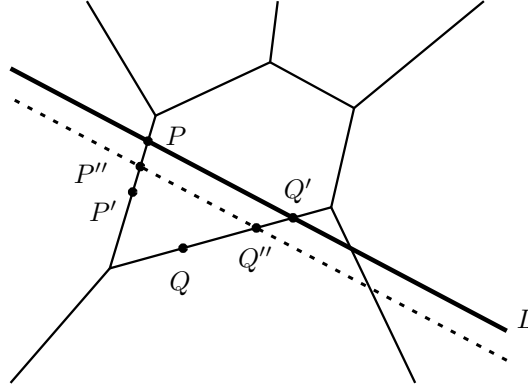


Fig. 6.

3.2, we may assume that P, P', Q, Q' are interior points of edges, and $[P, Q']$ has rational slope.

Let L be the line passing through P, Q' . Then

$$E \cdot L = |u_E \times u_L|P,$$

$$F \cdot L = |u_F \times u_L|Q.$$

Let P'', Q'' be points of E, F such that

$$\overrightarrow{PP''} = \frac{1}{|u_E \times u_L|} \overrightarrow{PP'},$$

$$\overrightarrow{Q'Q''} = \frac{1}{|u_F \times u_L|} \overrightarrow{Q'Q}.$$

Then

$$\begin{aligned} |\overrightarrow{PP''} \times u_L| &= \frac{|\overrightarrow{PP''}|}{|u_E|} |u_E \times u_L| \\ &= \frac{|\overrightarrow{PP'}|}{|u_E|} \end{aligned}$$

$$\begin{aligned}
&= \frac{|\overrightarrow{Q'Q}|}{|u_F|} \quad (\text{because } a' - a = b' - b) \\
&= |\overrightarrow{Q'Q''} \times u_L|,
\end{aligned}$$

which means that $[P'', Q'']$ is parallel to L . Thus

$$|u_E \times u_L|(P'' - P) \sim |u_F \times u_L|(Q' - Q'').$$

The interval $[P, P']$ is divided into $|u_E \times u_L|$ -number of intervals with the same length. From lemma 3.2, we have $P' - P \sim |u_E \times u_L|(P'' - P)$. Thus $P' - P \sim Q' - Q$. \square

Let E_1, \dots, E_N be all edges of Λ ordered by the positive direction. Let $\lambda: C \rightarrow \Lambda$ be the canonical surjection. For a tropical curve $L \in \mathcal{P}(\mathbb{R}^2)$, we define $\sigma(L) \in \Lambda$ as follows.

$$C \cdot L = P_1 + \dots + P_r,$$

$$\sigma(L) = \lambda(P_1) + \dots + \lambda(P_r).$$

LEMMA 5.2. $\sigma: \mathcal{P}(\mathbb{R}^2) \rightarrow \Lambda$ is locally constant.

PROOF. σ is continuous by definition of the stable intersection. Let $\{L_t | 0 \leq t \leq 1\}$ be a continuous family of tropical curves with Newton complex \mathcal{N} such that C intersects L_t transversely for any t . Let $P_{ijt} \in E_i$ be the points such that $E_i \cdot L_t = \sum_j P_{ijt}$. Then there are edges $L_{ijt} \subset L_t$, and vectors $u_{ij} \in \mathbb{R}^2$ such that

i) $E_i \cap L_{ijt} = P_{ijt}$,

ii) u_{ij} is the weighted primitive vector of L_{ijt} starting at the vertex inside Λ .

Let μ_{ij} be the multiplicity $\mu_{P_{ijt}}$ (which is constant for t). For L_0 and L_1 , we have the moment condition inside Λ :

$$\sum_{i,j} \text{moment}(u_{ij}/\mu_{ij}, P_{ij0}) = 0,$$

$$\sum_{i,j} \text{moment}(u_{ij}/\mu_{ij}, P_{ij1}) = 0.$$

From these,

$$\sum_{i,j} \left(\overrightarrow{P_{ij0}P_{ij1}} \times u_{ij}/\mu_{ij} \right) = 0.$$

Since $u_{E_i} \times u_{ij} = -\mu_{ij}$, this means

$$\sum_{i,j} (\lambda(P_{ij0}) - \lambda(P_{ij1})) = 0.$$

Thus $\sigma(L_0) = \sigma(L_1)$. \square

PROOF OF THE INJECTIVITY OF φ . Suppose

$$P - Q = C \cdot L - C \cdot L',$$

$$P, Q \in \Lambda,$$

$$\Delta(L) = \Delta(L').$$

From Corollary 4.2 and Lemma 5.2, we have $\sigma(L) = \sigma(L')$. Thus $P = Q$. \square

6. Proof of Theorem 1.3

Let H be the subgroup of $\text{Div}^0(C)$ generated by all $(P' - P) - (Q' - Q)$ that satisfies the assumption of Lemma 3.2. We have a homomorphism

$$\text{Div}^0(C)/H \longrightarrow \mathbb{T}(C).$$

Let E_1, \dots, E_N be all finite edges of C . Let V_i, V'_i be the vertices of E_i . We have a homomorphism

$$\begin{aligned} \pi_i: \mathbb{R} &\longrightarrow \text{Div}^0(C)/H \\ c &\longmapsto m(P - V_i), \end{aligned}$$

where $m > 0$ is any integer such that

$$\frac{c}{m} \leq \text{length}([V_i, V'_i]),$$

and $P \in E_i$ is the point such that

$$\frac{c}{m} = \text{length}([V_i, P]).$$

This is well-defined by definition of H . So we have a homomorphism

$$\pi: \mathbb{R}^N \longrightarrow \text{Div}^0(C)/H.$$

Let $F = \text{Ker } \pi$. The composite map

$$\mathbb{R}^N/F \longrightarrow \text{Div}^0(C)/H \longrightarrow \text{T}(C)$$

is surjective because of Lemma 3.4.

We fix a vertex $\mathcal{O} \in C$. Let $\lambda: C \rightarrow \mathbb{R}^N/F$ be the map such that

- i) $\pi(\lambda(P)) = P - \mathcal{O}$ for $P \in E_i$,
- ii) every ray of C is contracted by λ .

Let $\tilde{\lambda}_i: E_i \rightarrow \mathbb{R}^N$ be a lift of $\lambda|_{E_i}: E_i \rightarrow \mathbb{R}^N/F$ with the following condition.

$$\tilde{\lambda}_i(P) = \tilde{\lambda}_i(V_i) + \delta e_i,$$

where

$$\delta = \text{length}([V_i, P]).$$

LEMMA 6.1. *F is a free abelian group of rank g .*

PROOF. Let $\Lambda_1, \dots, \Lambda_g$ be cycles generating $H_1(C, \mathbb{Z})$. Let V_{j1}, \dots, V_{js_j} be all vertices of Λ_j , and take $V_{j0} = V_{js_j}$. Let $a_{jk} \in \mathbb{R}^N$ be the vector such that $\pi(a_{jk}) = V_{jk} - V_{j,k-1}$. Let F' be the abelian group generated by a_1, \dots, a_g , where $a_j = \sum_k a_{jk}$. Then we have $F' \subset F$. The canonical map

$$\lambda': C \longrightarrow \mathbb{R}^N/F'$$

induces a homomorphism

$$\lambda'': \text{Div}^0(C)/H \rightarrow \mathbb{R}^N/F'.$$

So we have $F' = F$. \square

For a tropical curve $L \in \mathcal{P}(\mathbb{R}^2)$, we define $\sigma(L) \in \mathbb{R}^N/F$ as follows.

$$C \cdot L = P_1 + \cdots + P_r,$$

$$\sigma(L) = \lambda(P_1) + \cdots + \lambda(P_r).$$

The map

$$\sigma: \mathcal{P}(\mathbb{R}^2) \longrightarrow \mathbb{R}^N/F$$

is continuous by definition of the stable intersection.

Given a continuous path $\gamma: [0, 1] \rightarrow \mathcal{P}(\mathbb{R}^2)$, let

$$\tilde{\sigma}_\gamma: [0, 1] \rightarrow \mathbb{R}^N$$

be a lift of $\sigma|_\gamma: [0, 1] \rightarrow \mathbb{R}^N/F$, and let

$$v(\gamma) = \tilde{\sigma}_\gamma(1) - \tilde{\sigma}_\gamma(0).$$

Let $W \subset \mathbb{R}^N$ be the set of all $v(\gamma)$.

LEMMA 6.2. *W is a linear subspace in \mathbb{R}^N .*

PROOF. W is a subgroup of \mathbb{R}^N . Indeed, $v(\gamma) + v(\gamma') \in W$, because $\sigma(L \cup L') = \sigma(L) + \sigma(L')$.

Recall that each component of $\mathcal{P}(\mathbb{R}^2)$ is a closed cone in an affine space. Given a Newton complex \mathcal{N} , let

$$\mathcal{P}(\mathcal{N}, C) = \{L \in \mathcal{P}(\mathbb{R}^2) \mid \text{Newt}(C \cup L) = \mathcal{N}\}.$$

Every element of W is written as $v(\gamma_1) + \cdots + v(\gamma_s)$ for linear paths

$$\gamma_j: [0, 1] \rightarrow \overline{\mathcal{P}(\mathcal{N}_j, C)}$$

and Newton complexes $\mathcal{N}_1, \dots, \mathcal{N}_s$. For $0 \leq t \leq 1$, we have

$$t(v(\gamma_1) + \cdots + v(\gamma_s)) = v(\gamma_{1,t}) + \cdots + v(\gamma_{s,t}),$$

where $\gamma_{j,t}: [0, t] \rightarrow \overline{\mathcal{P}(\mathcal{N}_j, C)}$ is a restriction of γ_j . So W is a linear subspace in \mathbb{R}^N . \square

By Corollary 4.2, W is the kernel of the surjection $\mathbb{R}^N/F \rightarrow \mathsf{T}(C)$. So we have an isomorphism

$$\mathbb{R}^N/(F + W) \cong \mathsf{T}(C).$$

Note that \mathbb{R}^N/W is a vector space generated by e_1, \dots, e_N . Theorem 1.3 follows from the following lemma.

LEMMA 6.3. *Suppose that $\mathcal{X} := \overline{C \setminus (E_1 \cup \dots \cup E_g)}$ is a maximal tree in C . Then $\langle e_1, \dots, e_g \rangle$ is a basis of \mathbb{R}^N/W over \mathbb{R} .*

PROOF OF THAT $\langle e_1, \dots, e_g \rangle$ IS A GENERATOR. For $g + 1 \leq j \leq N$, there is a piecewise linear curve $M \subset \mathbb{R}^2$ containing two rays such that $\mathcal{X} \cap M$ is a single point of E_j . The following lemma says that $e_j \in \mathbb{Q}\langle e_1, \dots, e_g \rangle$ in \mathbb{R}^N/W . \square

In the proof of the following lemma, the notation $[P, Q, \infty)$ means the ray with vertex P that contains Q .

LEMMA 6.4. *Let S, T be rays in \mathbb{R}^2 such that*

$$S \cap C = \emptyset, \quad T \cap C = \emptyset.$$

Let $P_0, \dots, P_r, P'_0, \dots, P'_r \in \mathbb{R}^2$ be rational points such that

$$P_0, P'_0 \in S, \quad P_r, P'_r \in T,$$

$$[P_i, P'_i] \cap C = \emptyset,$$

$\overrightarrow{P_{i-1}P_i}$ and $\overrightarrow{P'_{i-1}P'_i}$ have the same direction.

Then there are integers $m_1, \dots, m_r > 0$ such that

$$\sum_{i=1}^r m_i (C \cdot [P_{i-1}, P_i]) \sim \sum_{i=1}^r m_i (C \cdot [P'_{i-1}, P'_i]).$$

PROOF. Let L be the tropical curve with vertex P_i , consisting of three rays L_0, L_1, L_2 such that

$$P_{i-1} \in L_1, \quad P_{i+1} \in L_2,$$

$$[P_i, P'_i] \text{ is parallel to } L_0.$$

Let $w_i = \text{wt}(L_1)$, $v_i = \text{wt}(L_2)$. Move L by $\overrightarrow{P_i P'_i}$, and denote it by L' . Comparing $C \cdot L$ and $C \cdot L'$, we have

$$\begin{aligned} C \cdot (w_i[P_i, P_{i-1}, \infty) \cup v_i[P_i, P_{i+1}, \infty)) \\ \sim C \cdot (w_i[P'_i, P'_{i-1}, \infty) \cup v_i[P'_i, P'_{i+1}, \infty)) \end{aligned}$$

for $1 \leq i \leq r-1$. Similarly,

$$C \cdot (w_0[P_0, S] \cup v_0[P_0, P_1, \infty)) \sim C \cdot (w_0[P'_0, S] \cup v_0[P'_0, P'_1, \infty)),$$

$$C \cdot (w_r[P_r, P_{r-1}, \infty) \cup v_r[P_r, T]) \sim C \cdot (w_r[P'_r, P'_{r-1}, \infty) \cup v_r[P'_r, T]).$$

Taking a suitable linear combination and canceling lines, we have the statement. \square

PROOF OF THAT $\langle e_1, \dots, e_g \rangle$ ARE LINEARLY INDEPENDENT. Suppose

$$a_1 e_1 + \dots + a_g e_g = v(\gamma),$$

$$\gamma: [0, 1] \longrightarrow \mathcal{P}(\mathbb{R}^2).$$

For $1 \leq j \leq g$, there is a piecewise linear closed curve Λ in $\mathcal{X} \cup E_j$. Let $E_{i(0)}, \dots, E_{i(s)}$ be all edges of Λ , and take $E_{i(0)} = E_j$. Let $\varepsilon_k \in \{1, -1\}$ be the sign of $E_{i(k)}$ with respect to the positive direction of Λ : The sign is positive if the coordinate of \mathbb{R}^N is increasing. Let $h: \mathbb{R}^N \rightarrow \mathbb{R}$ be the linear map defined by

$$h = \varepsilon_0 \check{e}_{i(0)} + \dots + \varepsilon_s \check{e}_{i(s)},$$

where \check{e}_i denotes the dual basis. Then we have

$$h(\tilde{\sigma}_\gamma(1)) - h(\tilde{\sigma}_\gamma(0)) = h(v(\gamma)) = \varepsilon_0 a_j.$$

To complete the proof, we show that the map $h \circ \tilde{\sigma}_\gamma : [0, 1] \rightarrow \mathbb{R}$ is constant for any continuous path γ .

We may assume that C intersects $L_t := \gamma(t)$ transversely for any t . There are points $P_{kmt} \in E_{i(k)}$, edges $L_{kmt} \subset L_t$, and vectors $u_{km} \in \mathbb{R}^2$ such that

$$\text{i) } E_{i(k)} \cdot L_t = \sum_m P_{kmt},$$

$$\text{ii) } E_{i(k)} \cap L_{kmt} = P_{kmt},$$

iii) u_{km} is the weighted primitive vector of L_{kmt} starting at the vertex inside Λ .

Let μ_{km} be the multiplicity $\mu_{P_{kmt}}$ (which is constant for t). For L_0 and L_1 , we have the moment condition inside Λ :

$$\sum_{k,m} \text{moment}(u_{km}/\mu_{km}, P_{km0}) = 0,$$

$$\sum_{k,m} \text{moment}(u_{km}/\mu_{km}, P_{km1}) = 0.$$

From these,

$$\sum_{k,m} \left(\overrightarrow{P_{km0}P_{km1}} \times u_{km}/\mu_{km} \right) = 0.$$

Let u_k be the primitive vector of $E_{i(k)}$ starting at $V_{i(k)}$. Since $u_k \times u_{km} = -\varepsilon_k \mu_{km}$, we have

$$\sum_{k,m} h \left(\tilde{\lambda}_{i(k)}(P_{km0}) - \tilde{\lambda}_{i(k)}(P_{km1}) \right) = 0.$$

Thus $h(\tilde{\sigma}_\gamma(0)) = h(\tilde{\sigma}_\gamma(1))$. \square

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