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A Generalized Cartan Decomposition for the Double Coset Space $SU(2n+1)\backslash SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C})$

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Abstract. This paper gives a generalization of the Cartan decomposition for the non-symmetric space $SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C})$. Our method uses the herringbone stitch introduced by T. Kobayashi [6], and as a corollary, we prove that $SU(2n+1)$ acts on the spherical variety $SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C})$ in a strongly visible fashion with slice of real dimension $2n$.

1. Introduction and Statement of Main Results

Let G be a non-compact reductive Lie group, H a closed subgroup of G, and K a maximal compact subgroup of G. Our concern here is with the K-action on the homogeneous space G/H, or the double coset space decomposition $K\backslash G/H$.

For symmetric spaces G/H, it is known that there is an analogue of the Cartan decomposition $G = KAH$ where A is a non-compact abelian subgroup of dimension $\mathrm{rank}_{\mathbb{R}} G/H$ (see [1, Theorem 4.1]), which generalizes the classical fact that any positive definite real matrix is diagonalizable by an orthogonal matrix. For general non-symmetric spaces G/H, as was pointed out in [3], there does not always exist an abelian subgroup A such that the multiplication map $K \times A \times H \to G$ is surjective. However, if K acts on G/H in a visible fashion in the sense of [4, 5], we expect that there is a nice decomposition $G = KAH$ for some nice subgroup (or subset) A even for non-symmetric spaces G/H.

This new line of investigation was initiated by T. Kobayashi [4, 6] for (generalized) flag varieties, and then also studied for $\mathbb{C}^\times$-bundles over a

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complexification of Hermitian symmetric spaces of non-tube type [12]. This paper treats a new setting where

$$G_{\mathbb{C}}/H_{\mathbb{C}} = SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C}).$$

This space is a non-symmetric complex Stein manifold, and has a $G_{\mathbb{C}}$-equivariant holomorphic fiber bundle structure

$$K_{\mathbb{C}}/H_{\mathbb{C}} \to G_{\mathbb{C}}/H_{\mathbb{C}} \to G_{\mathbb{C}}/K_{\mathbb{C}}$$

over the complex symmetric space $G_{\mathbb{C}}/K_{\mathbb{C}} = SL(2n+1,\mathbb{C})/GL(2n,\mathbb{C})$. The space which we studied in [12] also has the same fiber bundle structure and its fiber is one-dimensional. In contrast to the case [12], the fiber $K_{\mathbb{C}}/H_{\mathbb{C}} = GL(2n,\mathbb{C})/Sp(n,\mathbb{C})$ is of higher dimension in our setting.

The main result of this paper is an explicit orbit decomposition on $G_{\mathbb{C}}/H_{\mathbb{C}}$ by the action of a maximal compact subgroup $G_u = SU(2n+1)$:

THEOREM 1.1. *Let $G_{\mathbb{C}} = SL(2n+1,\mathbb{C})$ and $H_{\mathbb{C}} = Sp(n,\mathbb{C})$. We take a maximal compact subgroup $G_u = SU(2n+1)$ of $G_{\mathbb{C}}$. Then, there exists a $2n$-dimensional 'slice' A in $G_{\mathbb{C}}$ such that we have a generalized Cartan decomposition $G_{\mathbb{C}} = G_u A H_{\mathbb{C}}$. In particular, A is of the form*

$$A \simeq \mathbb{R}^2 \cdot \underbrace{\mathbb{T}\cdot\cdots\cdot\mathbb{T}}_{n-1} \cdot \mathbb{R}^{n-1}.$$

The complex homogeneous space $G_{\mathbb{C}}/H_{\mathbb{C}}$ is not a symmetric space, but is still a spherical variety in the sense that a Borel subgroup of $G_{\mathbb{C}}$ acts on $G_{\mathbb{C}}/H_{\mathbb{C}}$ with an open orbit (see [10]). Further, Theorem 1.1 brings a new example of (strongly) visible actions on complex manifolds.

The notion of (strongly) visible actions has been introduced by Kobayashi [4]. Let us recall it briefly. Suppose that a Lie group L acts holomorphically on a connected complex manifold D. We say that this action is *strongly visible* if there exist a subset S of D and an anti-holomorphic diffeomorphism σ of D such that the following conditions are satisfied ([8, Definition 4.1]):

(V.1) $L \cdot S$ is open in D,

(S.1) $$\sigma|_S = \mathrm{id}_S,$$

(S.2) σ preserves each L-orbit in D.

In this paper, we do not discuss the singularity of our slice S (cf. [5, Definition 3.3.1]), but we show $K \cdot S = G/H$ instead of (V.1) (here $L = K$, $S = AH/H$, $D = G/H$).

The significance of strongly visible actions is an application to representation theory, namely, the multiplicity-free property propagates from fibers to the space of holomorphic sections for L-equivariant holomorphic vector bundle, if L acts on the base space in a strongly visible fashion (see [5, 8]). Recently, there have been found a number of concrete examples of strongly visible actions in connection with multiplicity-free representations (see [6, 7, 11, 12, 13]).

By Theorem 1.1, we give a new example of strongly visible actions:

Theorem 1.2. *The action of $G_u = SU(2n+1)$ on $G_{\mathbb{C}}/H_{\mathbb{C}} = SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C})$ is strongly visible.*

This paper is organized as follows: In Section 2, we consider a decomposition for a certain double coset space of compact non-symmetric type (see Proposition 2.2). This result serves as a preparation for the proof of Theorem 1.1. In Section 3, we show Theorem 1.1, particularly, we give a concrete description of a subset A. The method 'herringbone stitch' which was introduced by Kobayashi [6] plays an important role to our proof. In Section 4, we prove Theorem 1.2. In Section 5, we give an alternative proof of Theorem 1.2 by using a symmetric space which is biholomorphic to the non-symmetric space $G_{\mathbb{C}}/H_{\mathbb{C}}$.

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2. $SU(2)^n$-Orbits on S^{4n-1}

We begin with a concrete description of the double coset space $K_1\backslash K/K_2$, where

$$(K, K_1, K_2) := (SU(2n), 1 \times SU(2n-1), SU(2)^n).$$

Here, $SU(2)^n$ stands for the direct product group of n-copies of $SU(2)$.

Neither (K, K_1) nor (K, K_2) is a symmetric pair. However, by the observation that the homogeneous space K/K_1 is diffeomorphic to the unit sphere S^{4n-1} in $\mathbb{C}^{2n}$, the double coset space $K_1\backslash K/K_2$ is equivalent to the K_2-orbit decomposition of S^{4n-1}. Then, the latter object is manageable by an elementary method as follows.

Let $\{\vec{e}_1, \dots, \vec{e}_{2n}\}$ be the standard orthonormal basis of $\mathbb{C}^{2n}$. We let $SU(2)^n$ act on $\mathbb{C}^{2n}$ by

$$(g_1, \dots, g_n) \cdot {}^t(v_1, \dots, v_{2n}) = \begin{pmatrix} g_1\, {}^t(v_1, v_2) \\ g_2\, {}^t(v_3, v_4) \\ \vdots \\ g_n\, {}^t(v_{2n-1}, v_{2n}) \end{pmatrix}. \tag{2.1}$$

Clearly, $SU(2)^n$ preserves the unit sphere $S^{4n-1} = \{v \in \mathbb{C}^{2n} : \|v\| = 1\}$.

Now, we define an $(n-1)$-dimensional submanifold in S^{4n-1} by

$$T_0 = \{\sum_{j=1}^{n} r_j \vec{e}_{2j-1} \in S^{4n-1} : r_1, \dots, r_n \in \mathbb{R}\} \simeq S^{n-1}.$$

We claim:

LEMMA 2.1. $S^{4n-1} = SU(2)^n \cdot T_0$.

PROOF. We take an element ${}^t(v_1, \dots, v_{2n})$ of S^{4n-1}. Let us consider each component in the right-hand side of (2.1).

Since $SU(2)$ acts transitively on the unit sphere S^3 in $\mathbb{C}^2$, there exists $g_j^{(0)} \in SU(2)$ such that $g_j^{(0)}\, {}^t(v_{2j-1}, v_{2j}) = {}^t(\sqrt{|v_{2j-1}|^2 + |v_{2j}|^2}, 0)$. Thus, we obtain

$$(g_1^{(0)}, \dots, g_n^{(0)}) \cdot {}^t(v_1, \dots, v_{2n}) = \sum_{j=1}^{n} \sqrt{|v_{2j-1}|^2 + |v_{2j}|^2}\, \vec{e}_{2j-1}.$$

This means that any $SU(2)^n$-orbit in S^{4n-1} meets T_0. Hence, Lemma 2.1 has been proved. $\square$

Next, for $j = 1, 2, \ldots, n-1$, we define one-dimensional subgroups of $SU(2n)$ by

$$B_j := \exp \mathbb{R}(E_{2j+1,2j-1} - E_{2j-1,2j+1}),$$

and set

$$B := B_{n-1}B_{n-2} \cdots B_2 B_1.$$

We note that B_j's do not commute with each other, and that B is no longer a subgroup of $SU(2n)$.

We take a submanifold T in the homogeneous space K/K_1 as

$$T := B \cdot o_{K_1},$$

where o_{K_1} is the base point of K/K_1. Then, T is diffeomorphic to T_0 via $K/K_1 \simeq S^{4n-1}$. Hence, Lemma 2.1 implies that $K/K_1 = K_2 \cdot T$, and consequently $K = K_2 B K_1$.

We set $C := B^{-1} = \{g^{-1} : g \in B\}$. By the definition of B, we have

$$C = B_1 B_2 \cdots B_{n-2} B_{n-1}. \tag{2.2}$$

Therefore, we have proved:

Proposition 2.2. *$K = K_1 C K_2$, where* $C \simeq \underbrace{\mathbb{T} \cdot \cdots \cdot \mathbb{T}}_{n-1}$.

3. Proof of Theorem 1.1

In Section 3, we give a proof of Theorem 1.1.

Retain the settings as in Section 1, namely, $G_{\mathbb{C}} = SL(2n+1,\mathbb{C})$, $H_{\mathbb{C}} = Sp(n,\mathbb{C})$ and $G_u = SU(2n+1)$. We realize $Sp(n,\mathbb{C})$ and $SU(2n+1)$ as the following standard matrix groups:

$$Sp(n,\mathbb{C}) = \{g \in SL(2n,\mathbb{C}) : {}^t g J_n g = J_n\},$$

$$SU(2n+1) = \{g \in SL(2n+1,\mathbb{C}) : {}^t \overline{g} g = I_{2n+1}\},$$

where I_{2n+1} stands for the unit matrix, J_n for the $2n$ by $2n$ matrix as

$$J_n = \sum_{j=1}^{n} (E_{2j,2j-1} - E_{2j-1,2j}) \in SL(2n, \mathbb{R}),$$

and ${}^t g$ denotes the transposed matrix of g. We then realize $Sp(n, \mathbb{C})$ in $SL(2n+1, \mathbb{C})$ by

$$Sp(n, \mathbb{C}) \hookrightarrow SL(2n+1, \mathbb{C}),\ h \mapsto \begin{pmatrix} 1 & \\ & h \end{pmatrix}. \tag{3.1}$$

The key ingredient of the proof for Theorem 1.1 is 'herringbone stitch' method which was introduced in [6] for the study of visible actions on generalized flag varieties.

3.1. Herringbone stitch method

We will explain how to apply herringbone stitch method to our double coset space $G_u \backslash G_{\mathbb{C}} / H_{\mathbb{C}}$.

First, we take a subgroup $L_{\mathbb{C}} := 1 \times SL(2n, \mathbb{C})$ containing $H_{\mathbb{C}}$ and consider the homogeneous space $G_{\mathbb{C}}/L_{\mathbb{C}}$. Although $G_{\mathbb{C}}/L_{\mathbb{C}}$ is not a symmetric space, there exists an abelian subgroup A_1 such that $G_{\mathbb{C}} = G_u A_1 L_{\mathbb{C}}$ (see Section 3.2).

Second, we consider the complex symmetric space $L_{\mathbb{C}}/H_{\mathbb{C}}$. For a maximal compact subgroup L_u of $L_{\mathbb{C}}$, we have $L_{\mathbb{C}} = L_u A_2 H_{\mathbb{C}}$ by taking an abelian subgroup A_2 (see Section 3.3).

The third step begins with the observation that L_u does not commute with A_1. Then, we take U_1 to be the centralizer of A_1 in L_u, and U_2 to be the centralizer of A_2 in $H_u := L_u \cap H_{\mathbb{C}}$. We then find a subset A_3 such that the multiplication map $U_1 \times A_3 \times U_2 \to L_u$ is surjective (see Section 3.4).

Finally, we use herringbone stitch method consisting of $G_u \backslash G_{\mathbb{C}} / L_{\mathbb{C}}$, $L_u \backslash L_{\mathbb{C}} / H_{\mathbb{C}}$, and $U_1 \backslash L_u / U_2$ (see Section 3.5).

3.2. Cartan decomposition for $G_{\mathbb{C}}/L_{\mathbb{C}}$

This subsection deals with a Cartan decomposition for $G_{\mathbb{C}}/L_{\mathbb{C}} \simeq SL(2n+1, \mathbb{C})/SL(2n, \mathbb{C})$. We notice that this space $G_{\mathbb{C}}/L_{\mathbb{C}}$ is non-symmetric, but has a nice structure, namely, it is a complexification of a homogeneous space $G_{\mathbb{R}}/L_{\mathbb{R}}$ which is an S^1-bundle over an irreducible Hermitian symmetric space $G_{\mathbb{R}}/K_{\mathbb{R}}$.

$$
\begin{array}{ccccccccc}
 & & H_{\mathbb{C}} & & \supset & & H_u & & \\
 & & & \subset & & & & \supset & \\
 & & & & L_{\mathbb{C}} & & & & U_2 \\
 & & & \supset & & \supset & & \supset & \\
 & & G_{\mathbb{C}} & & & & L_u & & \\
 & \subset & & & & \subset & & & \\
G_u & & \supset & & U_1 & & & &
\end{array}
$$

Table 3.1. Herringbone stitch for the double coset space $G_u\backslash G_{\mathbb{C}}/H_{\mathbb{C}}$.

In this setting, we write $K_{\mathbb{C}}$ for the complexification of $K_{\mathbb{R}}$, and set $L_{\mathbb{C}} := [K_{\mathbb{C}}, K_{\mathbb{C}}]$ which is of complex codimension one in $K_{\mathbb{C}}$. Then, $G_{\mathbb{R}}/K_{\mathbb{R}}$ is of non-tube type if and only if the action of a maximal compact subgroup of $G_{\mathbb{C}}$ on the non-symmetric homogeneous space $G_{\mathbb{C}}/L_{\mathbb{C}}$ is strongly visible ([12, Theorem 1.1]). For its proof, we gave a generalized Cartan decomposition for $G_{\mathbb{C}}/L_{\mathbb{C}}$ by taking a non-compact abelian subgroup of dimension $(\operatorname{rank} G_{\mathbb{R}}/K_{\mathbb{R}}) + 1$ ([12, Section 4.1]).

We apply this result to $G_{\mathbb{R}}/K_{\mathbb{R}} = SU(1,2n)/S(U(1)\times U(2n))$. Then, $G_{\mathbb{R}}/K_{\mathbb{R}}$ is a non-tube type Hermitian symmetric space of rank one, and $G_{\mathbb{C}}/L_{\mathbb{C}}$ is isomorphic to $SL(2n+1,\mathbb{C})/SL(2n,\mathbb{C})$. Hence, we have:

Lemma 3.1. *$G_{\mathbb{C}} = G_u A_1 L_{\mathbb{C}}$, where $A_1 \simeq \mathbb{R}^2$.*

Explicitly, A_1 is taken as $A_1 = (\exp \mathfrak{a}_0)(\exp \mathfrak{b}_0)$, where

$$
\begin{aligned}
\mathfrak{a}_0 &= \mathbb{R}(E_{1,2} + E_{2,1}), \\
\mathfrak{b}_0 &= \mathbb{R}\{(2n-1)(E_{1,1} + E_{2,2}) - 2(E_{3,3} + \cdots + E_{2n+1,2n+1})\}.
\end{aligned}
$$

3.3. Cartan decomposition for $L_{\mathbb{C}}/H_{\mathbb{C}}$

Next, we consider a Cartan decomposition for $L_{\mathbb{C}}/H_{\mathbb{C}} \simeq SL(2n,\mathbb{C})/Sp(n,\mathbb{C})$.

Since $L_{\mathbb{C}}/H_{\mathbb{C}}$ is symmetric, we can apply Flensted–Jensen [1, Theorem 4.1] to $L_{\mathbb{C}}/H_{\mathbb{C}}$. We take a maximal compact subgroup L_u of $L_{\mathbb{C}}$. Then, we have:

LEMMA 3.2. *$L_{\mathbb{C}} = L_u A_2 H_{\mathbb{C}}$, where $A_2 \simeq \mathbb{R}^{n-1}$.*

An abelian A_2 here is taken as $A_2 = \exp \mathfrak{a}_2$, where

$$\mathfrak{a}_2 = \left\{ \sum_{j=1}^{n} t_j (E_{2j,2j} + E_{2j+1,2j+1}) : t_1, \dots, t_n \in \mathbb{R}, \sum_{j=1}^{n} t_j = 0 \right\}. \tag{3.2}$$

3.4. Double coset decomposition for $U_1 \backslash L_u / U_2$

Let us consider a double coset decomposition for $U_1 \backslash L_u / U_2$.

The centralizer U_1 of A_1 in L_u is realized as the block diagonal matrices $I_2 \times SU(2n-1)$, and the centralizer U_2 of A_2 in $L_u \cap H_{\mathbb{C}}$ is realized as $1 \times SU(2)^n$. Then, we have the following natural bijection

$$U_1 \backslash L_u / U_2 \simeq (1 \times SU(2n-1)) \backslash SU(2n) / SU(2)^n = K_1 \backslash K / K_2. \tag{3.3}$$

We recall that Proposition 2.2 gives the decomposition $K = K_1 C K_2$ by taking non-abelian C (see (2.2) for definition). Then, the bijection (3.3) gives rise to the decomposition formula as follows:

LEMMA 3.3. *We set $A_3 := 1 \times C$. Then, we have $L_u = U_1 A_3 U_2$.*

We note that the set A_3 is not a subgroup of L_u.

3.5. Proof of Theorem 1.1

Finally, we complete the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. We set

$$A := A_1 A_3 A_2. \tag{3.4}$$

The proof uses herringbone stitch method by using Lemmas 3.1–3.3 (see Table 3.1) together with the commuting properties $A_jU_j = U_jA_j$ for $j = 1, 2$ as follows:

$$\begin{aligned}
G_{\mathbb{C}} &= G_uA_1L_{\mathbb{C}} && \text{by Lemma 3.1} \\
&= G_uA_1(L_uA_2H_{\mathbb{C}}) && \text{by Lemma 3.2} \\
&= G_uA_1(U_1A_3U_2)A_2H_{\mathbb{C}} && \text{by Lemma 3.3} \\
&= G_uU_1(A_1A_3A_2)U_2H_{\mathbb{C}} && \text{by the commuting properties} \\
&= G_uAH_{\mathbb{C}} && \text{by } U_1 \subset G_u, U_2 \subset H_{\mathbb{C}}.
\end{aligned}$$

Therefore, Theorem 1.1 has been proved. □

4. Proof of Theorem 1.2

Let D be the complex non-symmetric space $G_{\mathbb{C}}/H_{\mathbb{C}} = SL(2n+1,\mathbb{C})/Sp(n,\mathbb{C})$. In Section 4, we give a proof of Theorem 1.2, namely, the G_u-action on D is strongly visible.

For this, we will define a subset S and an anti-holomorphic diffeomorphism σ satisfying the conditions (V.1)–(S.2) (see Section 1) as follows.

Retain the notations of Theorem 1.1. Here, we set

$$S := AH_{\mathbb{C}}/H_{\mathbb{C}}. \tag{4.1}$$

Let σ be the standard complex conjugation on $G_{\mathbb{C}}$, namely,

$$\sigma(g) = \overline{g} \quad (g \in G_{\mathbb{C}}).$$

It is clear that $H_{\mathbb{C}}$ is σ-stable. Then, σ induces an anti-holomorphic diffeomorphism on D, which we use the same letter to denote.

Proof of Theorem 1.2. We need to verify the conditions (V.1)–(S.2) for the above choice of S and σ.

Theorem 1.1 gives rise to the G_u-orbit decomposition of D as follows:

$$D = G_u \cdot S, \tag{4.2}$$

which implies the condition (V.1).

By definition, we have

$$\sigma(g \cdot o_{H_{\mathbb{C}}}) = \sigma(g) \cdot o_{H_{\mathbb{C}}} \quad (g \in G_{\mathbb{C}}), \tag{4.3}$$

where $o_{H_{\mathbb{C}}}$ is the base point of D. Obviously, $\sigma|_A = \mathrm{id}_A$, and then, $\sigma|_S = \mathrm{id}_S$. Hence, the condition (S.1) has been verified.

We see that σ preserves each G_u-orbit in D. Let x be an element of D. According to the G_u-orbit decomposition (4.2), we write $x = g \cdot s$ for some $g \in G_u$ and some $s \in S$. As $\sigma|_S = \mathrm{id}_S$, we compute that

$$\sigma(x) = \sigma(g) \cdot \sigma(s) = \sigma(g) \cdot s = (\sigma(g)g^{-1}) \cdot x. \tag{4.4}$$

Since G_u is σ-stable, we obtain $\sigma(g)g^{-1} \in G_u$. The equality (4.4) means that $\sigma(x) \in G_u \cdot x$ for any $x \in D$. Hence, the condition (S.2) has been verified.

Therefore, Theorem 1.2 has been proved. □

REMARK 4.1. For the strongly visible G_u-action on $G_{\mathbb{C}}/H_{\mathbb{C}}$, the dimension of our slice S coincides with the rank of the spherical variety $G_{\mathbb{C}}/H_{\mathbb{C}}$ (see [10]).

5. Alternative Proof of Theorem 1.2

We have already given a proof of our main results, namely, Theorems 1.1 and 1.2. We end this paper by providing an alternative proof of Theorem 1.2 by replacing the role of Theorem 1.1 with another decomposition result (see Theorem 5.1).

The key idea here is the fact that the non-symmetric space $G_{\mathbb{C}}/H_{\mathbb{C}}$ is biholomorphic to the complex symmetric space $G'_{\mathbb{C}}/H'_{\mathbb{C}}$ for some $G'_{\mathbb{C}} \supset G_{\mathbb{C}}$. This is discussed in Section 5.1. Then, by using known results for symmetric $G'_{\mathbb{C}}/H'_{\mathbb{C}}$ (Section 5.2), we show that the G_u-action on $G'_{\mathbb{C}}/H'_{\mathbb{C}}$ is strongly visible (Section 5.3), giving another proof of Theorem 1.2.

5.1. Complex symmetric space $SL(2n+2,\mathbb{C})/Sp(n+1,\mathbb{C})$

Let $G'_{\mathbb{C}} := SL(2n+2,\mathbb{C})$ and $H'_{\mathbb{C}} := Sp(n+1,\mathbb{C})$. Then, $G'_{\mathbb{C}}/H'_{\mathbb{C}}$ is a complex symmetric space. We realize $G_{\mathbb{C}} = SL(2n+1,\mathbb{C})$ as a subgroup of $G'_{\mathbb{C}}$ by

$$\iota : G_{\mathbb{C}} \hookrightarrow G'_{\mathbb{C}},\ g \mapsto \begin{pmatrix} 1 & \\ & g \end{pmatrix},$$

and $H_{\mathbb{C}} = Sp(n,\mathbb{C})$ as a subgroup of $H'_{\mathbb{C}}$ by

$$H_{\mathbb{C}} \hookrightarrow H'_{\mathbb{C}},\ h \mapsto \begin{pmatrix} I_2 & \\ & h \end{pmatrix}.$$

Then, the map ι naturally induces an open embedding $\widetilde{\iota}$ from the non-symmetric homogeneous space $G_{\mathbb{C}}/H_{\mathbb{C}}$ to the symmetric space $G'_{\mathbb{C}}/H'_{\mathbb{C}}$

$$\widetilde{\iota} : G_{\mathbb{C}}/H_{\mathbb{C}} \hookrightarrow G'_{\mathbb{C}}/H'_{\mathbb{C}},\ g \cdot o_{H_{\mathbb{C}}} \to \iota(g) \cdot o_{H'_{\mathbb{C}}}, \tag{5.1}$$

where $o_{H'_{\mathbb{C}}}$ is the base point of $G'_{\mathbb{C}}/H'_{\mathbb{C}}$.

Here, the complex dimension of $G_{\mathbb{C}}/H_{\mathbb{C}}$ equals that of $G'_{\mathbb{C}}/H'_{\mathbb{C}}$. Applying [2, Lemma 5.1], we see that the open embedding $\widetilde{\iota}$ becomes a surjective diffeomorphism because all the groups here are reductive algebraic groups. In particular, we obtain the following bijection induced from $\iota : G_{\mathbb{C}} \hookrightarrow G'_{\mathbb{C}}$

$$G_u\backslash G_{\mathbb{C}}/H_{\mathbb{C}} \simeq G_u\backslash G'_{\mathbb{C}}/H'_{\mathbb{C}}.$$

5.2. Double coset decomposition for $G_u\backslash G'_{\mathbb{C}}/H'_{\mathbb{C}}$

In this subsection, we give a concrete description of the double coset space

$$G_u\backslash G'_{\mathbb{C}}/H'_{\mathbb{C}} = SU(2n+1)\backslash SL(2n+2,\mathbb{C})/Sp(n+1,\mathbb{C}).$$

The points here are that $G'_{\mathbb{C}}/H'_{\mathbb{C}}$ is a complex reductive symmetric space and that $G_u\backslash G'_{\mathbb{C}}$ is not a Riemannian symmetric space. In fact, $G_u \subsetneq SU(2n+2) \subsetneq G'_{\mathbb{C}}$.

THEOREM 5.1. *Let* $G'_{\mathbb{C}} = SL(2n+2,\mathbb{C})$, $H'_{\mathbb{C}} = Sp(n+1,\mathbb{C})$, *and* $G_u = SU(2n+1)$. *Then, there exists a* $2n$*-dimensional slice* A' *in* $G'_{\mathbb{C}}$ *such that we have* $G'_{\mathbb{C}} = G_u A' H'_{\mathbb{C}}$. *In particular,* A' *is of the form*

$$A' \simeq \underbrace{\mathbb{T}\cdot\dots\cdot\mathbb{T}}_{n}\cdot\mathbb{R}^n.$$

The proof for Theorem 5.1 is analogous to Theorem 1.1. First, for a maximal compact subgroup G'_u of $G'_{\mathbb{C}}$, we have $G'_{\mathbb{C}} = G'_u A'_1 H'_{\mathbb{C}}$ by taking an abelian A'_1. Next, we take K' to be the centralizer of A'_1 in $H'_u := G'_u \cap H'_{\mathbb{C}}$. Then, we find a subset A'_2 such that $G'_u = G_u A'_2 K'$. Combining the decomposition formula for $G'_u\backslash G'_{\mathbb{C}}/H'_{\mathbb{C}}$ and $G_u\backslash G'_u/K'$ we again apply herringbone stitch method to $G_u\backslash G'_{\mathbb{C}}/H'_{\mathbb{C}}$ (see Table 5.1).

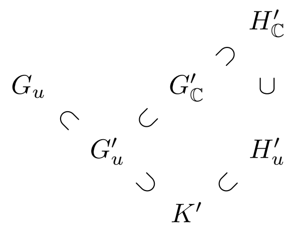

Table 5.1. Herringbone stitch for the double coset space $G_u\backslash G'_{\mathbb{C}}/H'_{\mathbb{C}}$.

PROOF OF THEOREM 5.1. First, applying the Cartan decomposition [1, Theorem 4.1] to the complex symmetric space $G'_{\mathbb{C}}/H'_{\mathbb{C}}$, we have

$$G'_{\mathbb{C}} = G'_u A'_1 H'_{\mathbb{C}} \tag{5.2}$$

by taking a non-compact abelian subgroup $A'_1 = \exp \mathfrak{a}'_1 \simeq \mathbb{R}^n$, where

$$\mathfrak{a}'_1 = \left\{ \sum_{j=1}^{n+1} t_j (E_{2j-1,2j-1} + E_{2j,2j}) : t_1, \dots, t_{n+1} \in \mathbb{R},\ \sum_{j=1}^{n+1} t_j = 0 \right\}. \tag{5.3}$$

Next, the centralizer K' of A'_1 in $G'_u \cap H'_{\mathbb{C}}$ is isomorphic to the direct product group of $(n+1)$-copies of $SU(2)$, namely, $K' = SU(2)^{n+1}$. Then, we can apply Proposition 2.2 to the double coset space $G_u\backslash G'_u/K' = (1 \times SU(2n+1))\backslash SU(2n+2)/SU(2)^{n+1}$. For $j = 1, 2, \ldots, n$, we define one-dimensional subgroups B'_j of G'_u by

$$B'_j := \exp \mathbb{R}(E_{2j-1,2j+1} - E_{2j+1,2j-1})$$

and set

$$A'_2 = B'_1 B'_2 \cdots B'_n \simeq \underbrace{\mathbb{T} \cdot \cdots \cdot \mathbb{T}}_{n}. \tag{5.4}$$

It follows from Proposition 2.2 that we get

$$G'_u = G_u A'_2 K'. \tag{5.5}$$

Finally, we take a subset A' in $G'_{\mathbb{C}}$ as

$$A' := A'_2 A'_1. \tag{5.6}$$

Combining (5.2) and (5.5), we conclude

$$\begin{aligned} G'_{\mathbb{C}} &= G'_u A'_1 H'_{\mathbb{C}} && \text{by (5.2)} \\ &= (G_u A'_2 K') A'_1 H'_{\mathbb{C}} && \text{by (5.5)} \\ &= G_u (A'_2 A'_1) K' H'_{\mathbb{C}} && \text{by } A'_1 K' = K' A'_1 \\ &= G_u A' H'_{\mathbb{C}} && \text{by } K' \subset H_{\mathbb{C}}. \end{aligned}$$

Therefore, Theorem 5.1 has been proved. □

5.3. Alternative proof of Theorem 1.2

Finally, we shall see that Theorem 5.1 leads us to another proof of Theorem 1.2.

We set

$$S' := A' H'_{\mathbb{C}} / H'_{\mathbb{C}}, \tag{5.7}$$

It follows from Theorem 5.1 that we have $G'_{\mathbb{C}}/H'_{\mathbb{C}} = G_u \cdot S'$.

We define an anti-holomorphic involution σ' of $G'_{\mathbb{C}}$ by

$$\sigma'(g) = \overline{g} \quad (g \in G'_{\mathbb{C}}).$$

This induces an anti-holomorphic diffeomorphism of $G'_{\mathbb{C}}/H'_{\mathbb{C}}$. By using the same argument as in the proof of Theorem 1.2 (see Section 4), we verify the conditions (S.1) and (S.2).

Consequently, we have proved that the G_u-action on $G'_{\mathbb{C}}/H'_{\mathbb{C}}$ is strongly visible. In view of $G_{\mathbb{C}}/H_{\mathbb{C}} \simeq G'_{\mathbb{C}}/H'_{\mathbb{C}}$, the G_u-action on $G_{\mathbb{C}}/H_{\mathbb{C}}$ is strongly visible.

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