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On Charts with Two Crossings I: There Exist No NS-Tangles in a Minimal Chart

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Abstract. In this paper, we establish methods to count the number of crossings and terminal edges of charts. These methods are useful to show that a chart Γ with at most two crossings is a ribbon chart provided that the closure of the surface braid represented by Γ is a disjoint union of spheres.

1. Introduction

S. Kamada introduced *charts* which correspond to surface braids [2],[3]. Charts are oriented labeled graphs in a disk with three kinds of vertices called black vertices, crossings, and white vertices. Kamada also introduced *C-moves* which are local modifications of charts in a disk. A C-move between two charts induces an ambient isotopy between the closures of the corresponding two surface braids. Two charts are said to be *C-move equivalent* if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

For a set X in a space, let $Cl(X)$ be the closure of the set X.

Let Γ be a chart. Let e_1 and e_2 be edges of Γ which connect two white vertices w_1 and w_2 where possibly $w_1 = w_2$. Suppose that the union $e_1 \cup e_2$ bounds an open disk E. Then $Cl(E)$ is called a *bigon* provided that any edge containing w_1 or w_2 does not intersect the open disk E (see Fig. 1). Since e_1 and e_2 are edges of Γ, they do not contain any crossings.

Let Γ be a chart. Let $w(\Gamma)$, $f(\Gamma)$ and $b(\Gamma)$ be the number of white vertices, the number of free edges and the number of bigons in Γ respectively. Let $C(\Gamma) = (w(\Gamma), -f(\Gamma), -b(\Gamma))$. The triplet $C(\Gamma)$ is called an *extended complexity* of the chart Γ (see [2] for complexities of charts).

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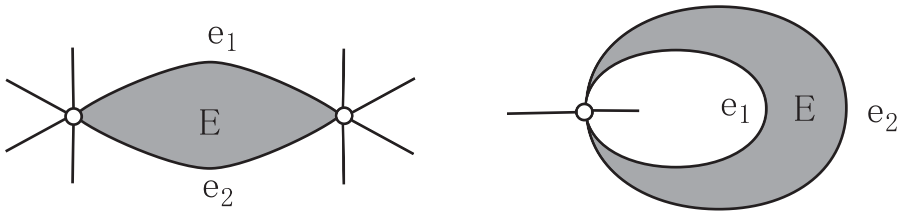

Fig. 1. The edges e_1 and e_2 do not contain crossings.

For each non-negative integer k, let $c(\Gamma)$ be the number of crossings in a chart Γ and $C_k = \{\Gamma \mid c(\Gamma) \leq k\}$. A chart Γ in C_k is said to be *k-minimal* if its extended complexity is minimal among the charts in C_k which are C-move equivalent to the chart Γ with respect to the lexicographical order of the triad of the integers [9].

Let Γ be a chart. For each label m, we denote by Γ_m the subgraph of Γ consisting of edges of label m and their vertices. In this paper,

> *crossings are vertices of Γ but we do not consider crossings as vertices of the subgraph Γ_m.*

A chart Γ with a white vertex is called a *generalized n-chart* if there exist two non-negative integers $p < q$ with $n = q - p$ such that

(i) Γ_i does not have a white vertex except for $p < i < q$, and

(ii) the both Γ_{p+1} and Γ_{q-1} have white vertices.

We will prove the following two theorems in [10].

THEOREM 1.1 ([10, Theorem 1.1]). *Let Γ be a 2-minimal generalized n-chart. If $n \geq 5$, then Γ contains at least $4n - 10$ black vertices.*

THEOREM 1.2 ([10, Theorem 1.2]). *Let Γ be a chart with at most two crossings. If the closure of the surface braid represented by Γ is a disjoint union of spheres, then Γ is a ribbon chart. Hence the closure is a ribbon surface.*

In this paper, to prove the above two theorems we establish methods to count black vertices in graphs often appear in charts. Namely we show the following three key theorems: Theorem 3.5, Theorem 4.8, and Theorem 5.4.

For a graph X in a chart Γ, let

$$w(X) = \text{the number of white vertices in } X.$$

Let Γ be a chart and D a disk. The pair $(D \cap \Gamma, D)$ is called a *tangle* if it satisfies the following two conditions:

(i) ∂D does not contain any white vertices, black vertices nor crossings of the chart Γ, and

(ii) ∂D transversely intersects edges of Γ.

Let Γ be a chart. A tangle $(D \cap \Gamma, D)$ is called an *NS-tangle of label* m (new significant tangle) if it satisfies the following two conditions:

(i) If $i \neq m$, then $\partial D \cap \Gamma_i$ is at most one point, and

(ii) $w(D \cap \Gamma) \geq 1$, and D contains at most one crossing.

The following is the first theorem of this paper.

THEOREM 3.5. *There does not exist any NS-tangle in a k-minimal chart* Γ.

To make the argument simple, we assume that the charts lie on the 2-sphere instead of the disk. In this paper,

all charts are contained in the 2*-sphere* S^2.

We have the special point in the 2-sphere S^2, called *the point at infinity*, denoted by ∞. In this paper, all charts are contained in a disk which does not contain the point at infinity ∞.

For each graph G in S^2, let (see Fig. 2)

$$\begin{aligned}
M(G) &= \text{the maximal subgraph of } G \text{ without vertices of degree 1},\\
Out(G) &= \text{the complementary domain of } M(G) \text{ containing the}\\
&\quad\ \text{point at infinity } \infty,\\
In(G) &= (Cl(Out(G)))^c, \text{ and}\\
Brd(G) &= M(G) \cap Cl(Out(G)).
\end{aligned}$$

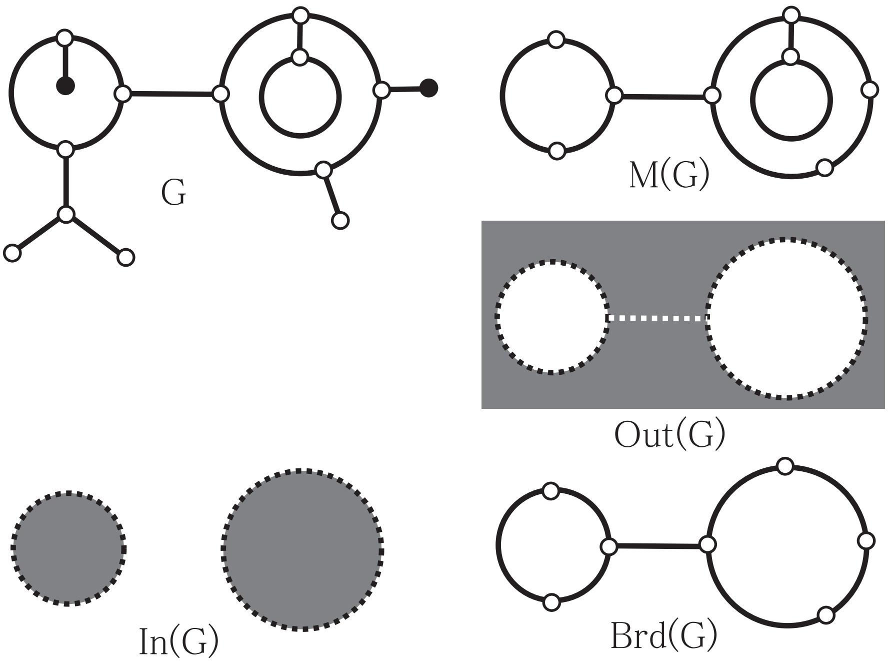

Fig. 2. $Out(G)$ and $In(G)$ are shaded areas.

An edge in a chart is called a *terminal edge* if it contains a white vertex and a black vertex.

A connected component G' of a graph G is a *small component* of G if it satisfies

$$(In(G') - G') \cap G = \emptyset.$$

In Fig. 3, X is a small component of $X \cup Y$, but Y is not a small component of $X \cup Y$.

The following is the second theorem of this paper.

THEOREM 4.8. *Let Γ be a k-minimal chart. Let G be a small component of Γ_n such that $G \cup In(G)$ does not contain any crossing. Then G contains at least two terminal edges of label n.*

Let Γ be a chart, $(D \cap \Gamma, D)$ a tangle and $G_i = D \cap \Gamma_i$ $(i = 1, 2, \cdots)$.

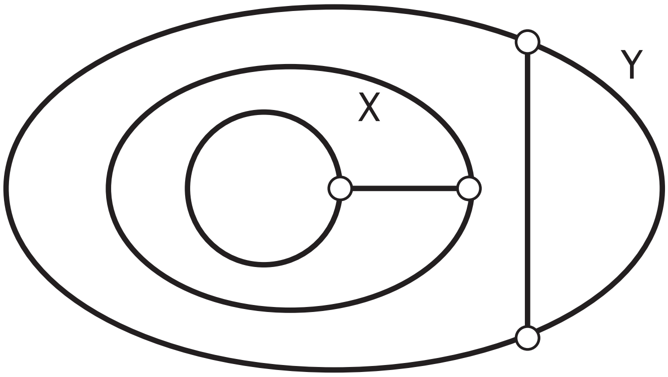

Fig. 3.

The tangle $(D \cap \Gamma, D)$ is called a T*-tangle* of label n (a tangle with at most three labels) if it satisfies the following two conditions:

(i) $G_i = \emptyset$ except for $n - 1 \leq i \leq n + 1$.

(ii) $w(D \cap \Gamma) \geq 1$, but D does not contain any crossing.

If $In(G_n) = \emptyset$ then we say that the T-tangle is *linear*. If $Cl(In(G_n))$ is a disk then we say that the T-tangle is *cellular*.

Let $(D \cap \Gamma, D)$ be a T-tangle of label n. If an edge e of Γ_n intersects ∂D, then $e \cap D$ is called an *exceptional arc* of the T-tangle.

Let $(D \cap \Gamma, D)$ be a T-tangle of a chart Γ. If s is the number of labels in $\{ i \mid \partial D \cap G_i \neq \emptyset \}$, then the T-tangle is called a T_s*-tangle*. Thus a T-tangle means a T_0-tangle, a T_1-tangle, a T_2-tangle or a T_3-tangle.

Let Γ be a chart, and $(D \cap \Gamma, D)$ a cellular T-tangle of label n. The tangle $(D \cap \Gamma, D)$ is *tiny* provided that the closure of each component of $(D - Cl(In(D \cap \Gamma_n))) \cap \Gamma$ is

(i) an arc connecting a point in ∂D and a point in $Brd(D \cap \Gamma_n)$, or

(ii) a terminal edge of label n.

Note. For any cellular T-tangle of label n, let X be the union of all the terminal edges of label n in D each of which intersects $Cl(In(D \cap \Gamma_n))$, and

N a regular neighborhood of $Cl(In(D \cap \Gamma_n)) \cup X$ in D. Then $(N \cap \Gamma, N)$ is a tiny cellular T-tangle of label n.

The following is the third theorem of this paper.

THEOREM 5.4. *Let $(D \cap \Gamma, D)$ be a tiny cellular T_2-tangle of label n in a k-minimal chart Γ which possesses exceptional arcs.*

(1) *The tangle possesses at least two exceptional arcs.*

(2) *If the tangle possesses exactly two exceptional arcs, then D contains at least two terminal edges of label n.*

(3) *If the tangle possesses exactly three exceptional arcs, then D contains at least one terminal edge of label n.*

A surface in $\mathbb{R}^4$ is called a *ribbon surface* if it is the boundary of an immersed handlebody with singularities which are mutually disjoint disks such that the preimage of each disk is a union of a proper disk of the domain and a disk in the interior of the domain, a handlebody. In the words of charts, a ribbon surface is the closure of a surface braid which corresponds to a *ribbon chart* where a ribbon chart is a chart which is C-move equivalent to a chart without white vertices [2].

Kamada showed that any 3-chart is a ribbon chart [2]. Nagase and Hirota extended Kamada's result: Any 4-chart with at most one crossing is a ribbon chart [5]. We showed that any n-chart with at most one crossing is a ribbon chart [9].

We showed that if a 2-minimal 4-chart contains exactly two crossings, then it contains at least eight black vertices [6]. It is well known that if the closure of the surface braid represented by a 4-chart is one sphere, then the chart contains exactly six black vertices. Using this fact we showed that any 4-chart with at most two crossings is a ribbon chart if the chart corresponds to a surface braid whose closure is one sphere [6]. We give another proof of this theorem [11] by using the results developed in this paper and [10].

2. Preliminaries

Let n be a positive integer. An *n-chart* is an oriented labeled graph in a disk, which may be empty or have closed edges without vertices, called *hoops*, satisfying the following four conditions:

(i) Every vertex has degree 1,4, or 6.

(ii) The labels of edges are in $\{1, 2, \ldots, n-1\}$.

(iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled i and $i+1$ alternately for some i, where the orientation and label of each arc are inherited from the edge containing the arc.

(iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i-j| > 1$.

A vertex of degree 1, 4, and 6 is called a *black vertex,* a *crossing*, and a *white vertex* respectively (see Fig. 4). Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward or outward is called a *middle arc* at the white vertex (see Fig. 4c). There are two middle arcs in a small neighborhood of each white vertex.

C-moves are local modifications of charts in a disk as shown in Fig. 5 (see [1], [4] for the precise definition). Kamada originally defined CI-moves as follows: A chart Γ is obtained from a chart Γ' by a *CI-move*, if there exists a disk D such that

(i) the two charts Γ and Γ' intersect the boundary of D transversely or do not intersect the boundary of D,

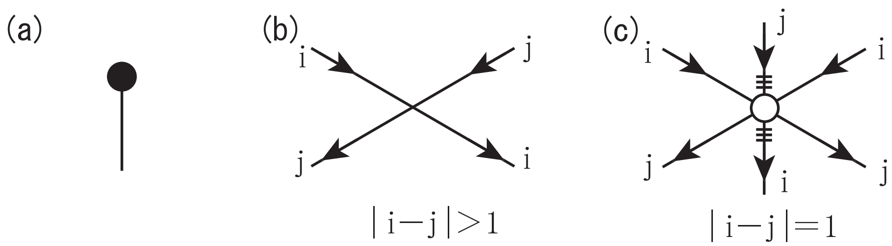

Fig. 4. (a) a black vertex, (b) a crossing, (c) a white vertex. Each arc with three transversal short arcs is a middle arc.

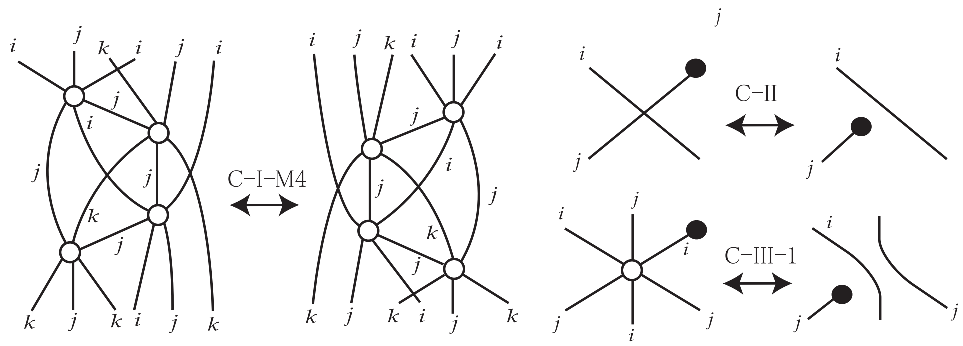

Fig. 5. In the C-III-1 move, the terminal edge does not contain a middle arc at the white vertex in the left figure.

(ii) $\Gamma \cap D^c = \Gamma' \cap D^c$, and

(iii) neither of $\Gamma \cap D$ nor $\Gamma' \cap D$ contains a black vertex,

where $(\cdots)^c$ is the complement of $(\cdots)$.

Let Γ be a chart. An *edge* of Γ is the closure of a connected component of the set obtained by taking out all white vertices and crossings from Γ. On the other hand, an *edge* of Γ_m is the closure of a connected component of the set obtained by taking out all white vertices from Γ_m. An edge of Γ or Γ_m is called a *free edge* if it has two black vertices. An edge of Γ or Γ_m is called a *terminal edge* if it has a white vertex and a black vertex. Note that free edges and terminal edges may contain crossings of Γ.

A hoop is said to be *simple* if one of the complementary domain of the hoop does not contain any white vertices.

A *ring* is a simple closed curve consisting of edges of the same label which contains a crossing but does not contain any white vertices.

We can assume that any k-minimal charts Γ satisfy the following five assumptions (See [9] and [7]):

ASSUMPTION 1. *Any terminal edge of* Γ_m *does not contain a crossing. Hence any terminal edge of* Γ_m *is a terminal edge of* Γ *and any terminal edge of* Γ_m *contains a middle arc.*

Assumption 2. *Any free edge of* Γ_m *does not contain a crossing. Hence any free edge of* Γ_m *is a free edge of* Γ.

Assumption 3. *All free edges and simple hoops in* Γ *are moved into a small neighborhood* U_∞ *of the point at infinity* ∞.

Assumption 4. *Each complementary domain of any ring must contain at least one white vertex.*

Assumption 5. *Hence we can assume that the subgraph obtained from* Γ *by omitting free edges and simple hoops does not meet the set* U_∞. *And also we can assume that* Γ *does not contain free edges nor simple hoops, otherwise mentioned. Therefore we can assume that if an edge of* Γ_m *contains a black vertex, then it is a terminal edge and that each complementary domain of any hoops and rings of* Γ *contains a white vertex, otherwise mentioned.*

Furthermore as shown in [7], we can also assume the following assumption:

Assumption 6. *The point at infinity* ∞ *is moved in any complementary domain of* Γ.

For a set X in a space, let $Int(X), \partial(X)$ be the interior, the boundary of the set X respectively.

3. NS-Tangles

Let Γ be a chart. A tangle $(D \cap \Gamma, D)$ is called an *NR-tangle* (*a new reducible tangle*) *of label* m if it satisfies the following two conditions:

(i) $\partial D \cap (\Gamma - \Gamma_m)$ is at most one point, and

(ii) $w(D \cap \Gamma) \geq 1$ but D does not contain any crossing.

Note that an NR-tangle is a special NS-tangle.

The following two lemmata are proved in [9].

Lemma 3.1 ([9, Theorem 1]). *There does not exist any NR-tangle in any* k*-minimal chart.*

LEMMA 3.2 ([9, Lemma 5.1]). *Let G be a connected graph in S^2. Let D be a disk containing G. Then the following hold:*

(1) *$Out(G)$ is an open disk.*

(2) *Each connected component of $In(G)$ is an open disk whose closure is a disk.*

(3) *A regular neighbourhood of $In(G) \cup G$ in S^2 is a disk, and so is a regular neighbourhood of $In(G) \cup G$ in D.*

Let Γ be a chart, and D a disk. Let m be a label with $D \cap \Gamma_m \neq \emptyset$. A connected component G of $D \cap \Gamma_m$ is a *two-color component* of label m in D provided that

(i) $G \cap \partial D$ consists of at most one point,

(ii) there exists an integer $\delta \in \{+1, -1\}$ such that all the white vertices in G are contained in $\Gamma_{m+\delta}$, and

(iii) G is not an arc contained in a terminal edge.

Note that a two-color component may contain a crossing.

LEMMA 3.3. *Let Γ be a k-minimal chart and D a disk. Then for any two-color component G in D, $G \cup In(G)$ contains at least one crossing.*

PROOF. Suppose that there exists a two-color component G of label m in D such that $G \cup In(G)$ contains no crossing.

Suppose that $w(G) = 0$. Since $G \cup In(G)$ does not contain a crossing, G is not a ring. Thus G must be a hoop. Let U be the open disk bounded by the hoop. Since the hoop is not simple by Assumption 3, U contains a white vertex. Let N be a disk in U such that $U - N$ is a very thin open annulus. Then N contains a white vertex and $\partial N \cap \Gamma = \emptyset$. Since $G \cup In(G)$ does not contain any crossing, $(N \cap \Gamma, N)$ is an NR-tangle. This contradicts Lemma 3.1.

Suppose that $w(G) > 0$. Let N be a regular neighbourhood of $G \cup In(G)$ in D. Then N is a disk by Lemma 3.2(3).

Since $G \cap \partial D$ consists of at most one point, so does $\partial N \cap \Gamma_m$. Since $G \cup In(G)$ does not contain any crossing, neither does N. Now $G \subset N$

implies that $w(N \cap \Gamma) > 0$. Since G is a two-color component of label m, all the white vertices in $Brd(G)$ are contained in $\Gamma_m \cap \Gamma_{m+\delta}$ for an integer $\delta \in \{+1, -1\}$. Thus $\partial N \cap (\Gamma - \Gamma_{m+\delta}) = \partial N \cap \Gamma_m$. Since $\partial N \cap \Gamma_m$ consists of at most one point, $(N \cap \Gamma, N)$ is an NR-tangle of label $m + \delta$. This contradicts Lemma 3.1. □

LEMMA 3.4. *Let Γ be a k-minimal chart and D a disk. If D contains at most one crossing, then any two-color component in D does not contain the crossing.*

PROOF. The proof will follow by contradiction. Suppose that there exists a two-color component G of label m in D such that G contains the crossing. There exists an integer $\delta \in \{+1, -1\}$ such that all the white vertex in G is contained in $\Gamma_{m+\delta}$. Let e be the edge in G containing the crossing and e' the other edge containing the crossing. Let t be the label of e'. Since no terminal edge contains a crossing by Assumption 1, neither e nor e' is a terminal edge. Since e' contains the crossing, e' is not a hoop.

Suppose that $G - e'$ is connected. Then e' is not a ring, and there exists a connected component U in $In(G) - G$ with $U \cap e' \neq \emptyset$. Since G is connected, U is an open disk. Since e' is neither a ring nor a hoop and since $|m - t| \geq 2$, the edge e' contains a white vertex w in U. Let N be a disk in U such that $U - N$ is a very thin open annulus. Then we can assume that $w \in N$, $\partial N \cap \Gamma_t$ is one point, and $\partial N \cap (\Gamma - \Gamma_t) \subset \Gamma_{m+\delta}$. Since D contains only one crossing, N does not contain a crossing any more. Hence $(N \cap \Gamma, N)$ is an NR-tangle of label $m + \delta$. This contradicts Lemma 3.1.

Now $G - e'$ must be disconnected. Let N be a regular neighbourhood of $G \cup In(G)$ and E a regular neighbourhood of e'. Then $N - E$ is disconnected. Let N' be the closure of a connected component of $N - E$. Then N' is a disk. Since e is not a terminal edge, N' contains a white vertex. Now $\partial N' \cap \Gamma_m$ is one point, and $\partial N' \cap (\Gamma - \Gamma_m) \subset \Gamma_{m+\delta}$. Since D contains only one crossing, N does not contain a crossing any more. Hence $(N' \cap \Gamma, N')$ is an NR-tangle of label $m + \delta$. This contradicts Lemma 3.1. □

For a graph X of a chart Γ, let

$$\alpha(X) = min\{ \ i \mid \Gamma_i \cap X \neq \emptyset \ \},$$

$$\beta(X) = max\{ \ i \mid \Gamma_i \cap X \neq \emptyset \ \}.$$

For an NS-tangle $(D \cap \Gamma, D)$ in a k-minimal chart Γ, let

$$n(D) = \beta(D \cap \Gamma) - \alpha(D \cap \Gamma).$$

An NS-tangle $(D \cap \Gamma, D)$ is *minimal* provided that

$$n(D) = \min\{\ n(D') \mid (D' \cap \Gamma, D') \text{ is an NS-tangle in } \Gamma\}.$$

Theorem 3.5. *There does not exist any NS-tangle in a k-minimal chart Γ.*

Proof. Suppose that there exists an NS-tangle. Then there exists a minimal NS-tangle $(D \cap \Gamma, D)$ of label m.

Let $\alpha = \alpha(D \cap \Gamma)$ and $\beta = \beta(D \cap \Gamma)$. Since $w(D \cap \Gamma) \geq 1$, we have $\alpha < \beta$. Hence $\alpha \neq m$ or $\beta \neq m$.

Suppose that $\alpha \neq m$. Then $\alpha < m$. By Condition (i) of an NS-tangle of label m, $\partial D \cap \Gamma_\alpha$ is at most one point.

If $D \cap \Gamma_\alpha$ is an arc contained in a terminal edge, then we take a regular neighborhood N of the arc in D. Then we have $(Cl(D-N) \cap \Gamma, Cl(D-N))$ is an NS-tangle with $Cl(D-N) \cap \Gamma_\alpha = \emptyset$. Thus $\alpha(D)+1 \leq \alpha(Cl(D-N))$. Hence $\beta(Cl(D-N)) - \alpha(Cl(D-N)) \leq \beta(D) - (\alpha(D)+1) < \beta(D) - \alpha(D)$. This contradicts that $(D \cap \Gamma, D)$ is a minimal NS-tangle. Hence there exists a connected component of $D \cap \Gamma_\alpha$ which is not contained in a terminal edge.

Let G be a small component of $D \cap \Gamma_\alpha$ which is not contained in a terminal edge. Then $G \cap \partial D$ is at most one point. Thus G is a two-color component in D. Hence $G \cup In(G)$ contains at least one crossing by Lemma 3.3. Since D contains at most one crossing, $G \cup In(G)$ contains exactly one crossing. Now G does not contain the crossing by Lemma 3.4. Thus there exists a connected component U of $In(G) - G$ which contains the crossing. Since G is connected, U is an open disk.

Let s, t $(s < t)$ be the labels such that $\Gamma_s \cap \Gamma_t$ contains the crossing. Since $\alpha \leq s < s+2 \leq t$, we have $\alpha + 2 \leq t$. Since G does not contain the crossing, we have $G \cap \Gamma_t = \emptyset$.

We show that U contains a white vertex. Suppose that $U \cap \Gamma_t$ contains a ring or a hoop ℓ. Then the open disk bounded by ℓ contains a white vertex by Assumption 3 and 4, and so does U. Suppose that $U \cap \Gamma_t$ does not contain any ring nor a hoop. Since there is no free edge in U by Assumption 3,

$G \cap \Gamma_t = \emptyset$ implies that U contains a white vertex in Γ_t. Either case, the open disk U contains a white vertex, say w.

Let N be a disk in U such that $U - N$ is a very thin open annulus. Then we can assume that $w \in N$ and $\partial N \cap \Gamma \subset \Gamma_{\alpha+1}$. Hence $(N \cap \Gamma, N)$ is an NS-tangle. Since G is a small component of $D \cap \Gamma_\alpha$, we have $N \cap \Gamma_\alpha = \emptyset$. Thus $\alpha(D)+1 \leq \alpha(N)$. Hence $\beta(N)-\alpha(N) \leq \beta(D)-(\alpha(D)+1) < \beta(D)-\alpha(D)$. This contradicts that $(D \cap \Gamma, D)$ is a minimal NS-tangle.

Similarly we have a contradiction for the case $\beta \neq m$. $\square$

LEMMA 3.6. *Let* Γ *be a* k*-minimal chart and* D *a disk. Then for any two-color component* G *in* D, $G \cup In(G)$ *contains at least two crossings.*

PROOF. The proof will follow by contradiction. Suppose that there exists a two-color component G of label m in D such that $G \cup In(G)$ contains at most one crossing. Let $\delta \in \{+1, -1\}$ be the integer such that all the white vertices in G are contained in $\Gamma_{m+\delta}$. By Lemma 3.3 $G \cup In(G)$ contains at least one crossing. Thus $G \cup In(G)$ contains exactly one crossing.

By Lemma 3.4 G does not contain the crossing. Hence the crossing is contained in $In(G) - G$. Let U be a connected component of $In(G) - G$ which contains the crossing. We can show that U contains a white vertex by the same way as the one in Theorem 3.5. Thus $In(G)$ contains a white vertex.

Let N be a regular neighbourhood of $G \cup In(G)$ in D. By Condition (i) for two-color components, we have that $G \cap \partial N$ is at most one point. Thus $(N \cap \Gamma, N)$ is an NS-tangle of label $m + \delta$. This contradicts Theorem 3.5. $\square$

4. The Number of Terminal Edges in Cellular T-Tangles

LEMMA 4.1 [**Boundary Condition Lemma**]. *Let* $(D \cap \Gamma, D)$ *be a tangle in a* k*-minimal chart* Γ *such that* D *does not contain any crossing. Let* $a = \alpha(\partial D \cap \Gamma)$ *and* $b = \beta(\partial D \cap \Gamma)$. *Then* $D \cap \Gamma_i = \emptyset$ *except for* $a \leq i \leq b$.

PROOF. Let $\alpha = \alpha(D \cap \Gamma)$ and $\beta = \beta(D \cap \Gamma)$. Since $\partial D \cap \Gamma \subset D \cap \Gamma$, $\alpha \leq a$ and $b \leq \beta$.

The proof will follow by contradiction. Suppose that $\alpha < a$. Then $\partial D \cap \Gamma_\alpha = \emptyset$. Let G be a small component of $D \cap \Gamma_\alpha$. Then we have that

$G \cap \partial D = \emptyset$ and $D \cap \Gamma_\alpha \neq \emptyset$. Then any vertex of G is contained in $\Gamma_{\alpha+1}$. The condition $G \cap \partial D = \emptyset$ implies that G is a two-color component of label α in D. Now D does not contain any crossing, and neither does $G \cup In(G)$. This contradicts Lemma 3.3.

Similarly we have a contradiction for the case $b < \beta$. □

LEMMA 4.2. *Any linear T-tangle in a k-minimal chart Γ possesses at least two exceptional arcs.*

PROOF. Suppose that there exists a linear T-tangle $(D \cap \Gamma, D)$ of label n with at most one exceptional arc. Since $In(D \cap \Gamma_n) = \emptyset$, $D \cap \Gamma_n$ is a union of trees. Since any white vertex of $D \cap \Gamma_n$ is of degree 3, and since $\partial D \cap \Gamma_n$ consists of at most one point, there exists two terminal edges of label n in $D \cap \Gamma_n$ which contain the same white vertex (see Fig. 6). Since there exists only one middle arc of label n at the white vertex, one of the two terminal edges does not contain a middle arc at the white vertex. Hence by a C-III-1 move we can eliminate the white vertex. This contradict that Γ is k-minimal. □

LEMMA 4.3. *For any k-minimal chart, any open disk bounded by a hoop contains a crossing.*

PROOF. Suppose that an open disk U bounded by a hoop does not

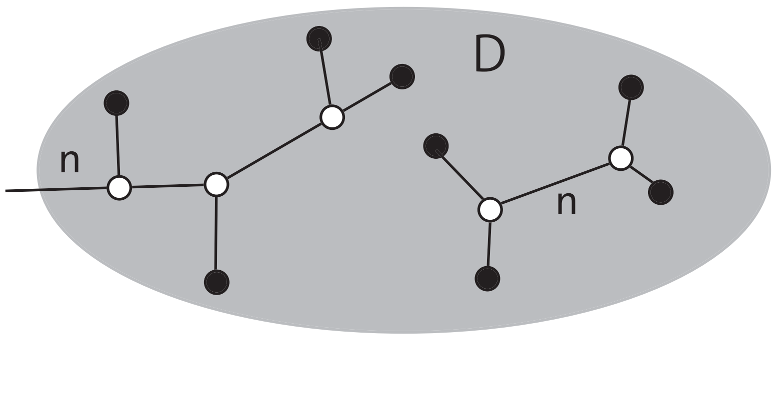

Fig. 6.

contain a crossing. Then $U \cap \Gamma = \emptyset$ by Boundary Condition Lemma (Lemma 4.1). Thus the hoop is simple. This contradicts Assumption 3. □

LEMMA 4.4 ([9, Lemma 4.1]). *Let Γ be a chart and U a complementary domain of Γ_m (possibly U may not be an open disk). If $Cl(U)$ contains no terminal edges of label m, then $Cl(U)$ contains even number of middle arcs of label $m \pm 1$ which intersect ∂U.*

Let Γ be a chart. Let m be a label and Γ_m^* the graph obtained by omitting all the free edges, hoops and rings from Γ_m. A complementary domain U of Γ_m^* is a *reducible complementary domain* of label m provided that

(i) U is an open disk,

(ii) U does not contain a crossing,

(iii) $U \cap (\Gamma_{m-2} \cup \Gamma_{m+2}) = \emptyset$, and

(iv) U does not intersect any middle arc of label $m \pm 1$.

LEMMA 4.5 ([9, Lemma 4.2]). *Let Γ be a k-minimal chart. Then for any label m there does not exist any reducible complementary domain of label m.*

LEMMA 4.6 ([9, Corollary 2.2]). *Let Γ be a k-minimal chart and U a complementary domain of Γ_m. If U contains at most one crossing and if $U \cap (\Gamma_{m-2} \cup \Gamma_{m+2}) = \emptyset$, then $Cl(U)$ does not contain any terminal edge of label m.*

Let $(D \cap \Gamma, D)$ be a tiny cellular T-tangle of label n in a k-minimal chart Γ. An exceptional arc is *essential* if it contains a middle arc of the white vertex on $Brd(D \cap \Gamma_n)$. Let $A = D - In(D \cap \Gamma_n)$ and

$m(D \cap \Gamma, D) =$ the number of middle arcs of label $n \pm 1$ in A each of which does not intersect any exceptional arcs,

$t(D \cap \Gamma, D) =$ the number of terminal edges of label n in A,

$\varepsilon(D \cap \Gamma, D) =$ the number of essential exceptional arcs of the tangle.

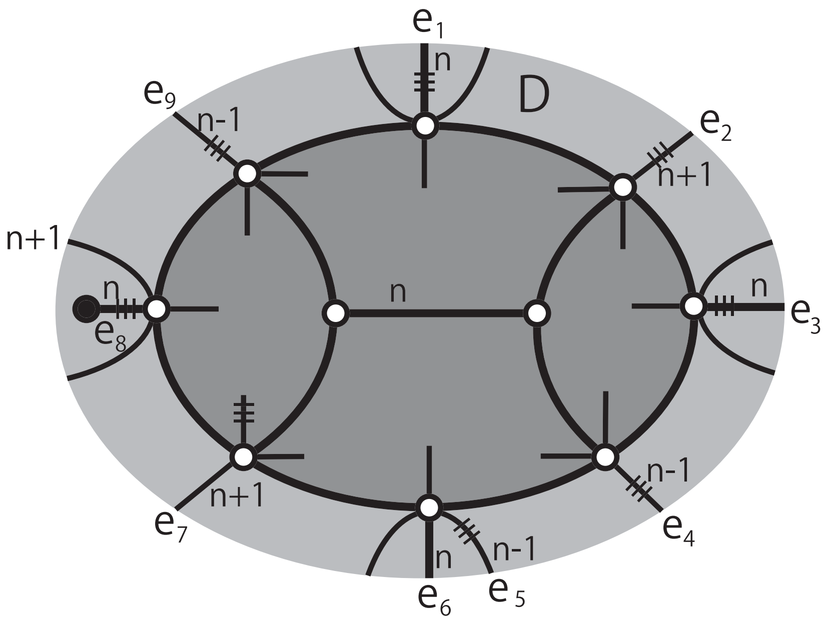

Fig. 7. Each arc with three transversal short arcs is a middle arc.

In Fig. 7, four arcs e_2, e_4, e_5, e_9 contain middle arcs, but e_5 intersects the exceptional arc e_6. Thus we have $m(D \cap \Gamma, D) = 3$. Since e_8 is the only one terminal edge of label n, we have $t(D \cap \Gamma, D) = 1$. Since e_1 and e_3 contain middle arcs among three exceptional arcs e_1, e_3, e_6, we have $\varepsilon(D \cap \Gamma, D) = 2$.

The following lemma is a generalization of Lemma 7.2 in [9]. The proof is almost parallel to the one of Lemma 7.2 in [9].

LEMMA 4.7. *Let $(D \cap \Gamma, D)$ be a tiny cellular T-tangle of label n in a k-minimal chart Γ. If $D \cap \Gamma_n$ is connected, then we have*

$$m(D \cap \Gamma, D) + 2 \leq t(D \cap \Gamma, D) + \varepsilon(D \cap \Gamma, D).$$

PROOF. We prove the lemma by contradiction. Suppose that

$$m(D \cap \Gamma, D) + 2 > t(D \cap \Gamma, D) + \varepsilon(D \cap \Gamma, D).$$

Let

$$\begin{aligned} \varepsilon_0 &= \varepsilon(D \cap \Gamma, D), \\ m_0 &= m(D \cap \Gamma, D), \ \textit{and} \\ t_0 &= t(D \cap \Gamma, D). \end{aligned}$$

Then we have

(1) $m_0 + 2 > t_0 + \varepsilon_0$.

Let $D' = Cl(In(D \cap \Gamma_n))$ and $A = Cl(D - D')$. Let

$V =$ the number of the white vertex in D',

$E =$ the number of the edges of label n in D', and

$F =$ the number of connected components of $D' - \Gamma_n$.

Since $D \cap \Gamma_n$ is connected, each connected component of $D' - \Gamma_n$ is an open disk. Since the T-tangle is cellular, D' is a disk. Thus by Euler formula we have

(2) $V - E + F = 1$.

Let

$p =$ the number of the exceptional arcs of the T-tangle.

On $Brd(In(D \cap \Gamma_n))$, $p + t_0$ is the number of white vertices contained in an exceptional arc or in a terminal edge in the annulus A. Each of the white vertices is contained in exactly two edges of label n in D' locally. Since there is no terminal edge of label n in D' by Lemma 4.6, $V - (p + t_0)$ is the number of the white vertices in D' each of which is contained in the three edges of label n in D' locally. Since each edge in D' possesses two white vertices locally, we have that

(3) $2(p + t_0) + 3(V - (p + t_0)) = 2E$.

Hence by using the equation (3) and the equation obtained by doubling each side of the equation (2), we have

(4) $2V - (3V - p - t_0) + 2F = 2$.

Thus

(5) $2F = 2 + V - p - t_0$.

On the other hand, for each white vertex there exists only one middle arc of label $n \pm 1$. Thus the number of middle arcs of label $n \pm 1$ in D' is

(6) $V - (m_0 + (p - \varepsilon_0))$.

Hence by using the equation (5) and (6), we have

$$(7)\quad \begin{aligned} 2F - (V - m_0 - p + \varepsilon_0) &= 2 + V - p - t_0 - (V - m_0 - p + \varepsilon_0) \\ &= 2 + m_0 - t_0 - \varepsilon_0. \end{aligned}$$

By using the inequality (1), we have

(8) $2F - (V - m_0 - p + \varepsilon_0) > 0$.

There are even number of middle arcs of label $n \pm 1$ in the closure of each connected component of $D' - \Gamma_n$ by Lemma 4.4. If the closure of each connected component of $D' - \Gamma_n$ contains a middle arc of label $n \pm 1$, then the number of middle arcs of label $n \pm 1$ in D' is greater than or equal to $2F$. Thus the last inequality (8) implies that there exists a connected component of $D' - \Gamma_n$ whose closure does not include any middle arc of label $n \pm 1$. Since $D \cap \Gamma_n$ is connected, the connected component is an open disk. Hence the connected component is a reducible complementary domain of label n. This contradicts Lemma 4.5. □

THEOREM 4.8. *Let Γ be a k-minimal chart. Let G be a small component of Γ_n such that $G \cup In(G)$ does not contain any crossing. Then G contains at least two terminal edges of label n.*

PROOF. Let D be a regular neighborhood of $G \cup In(G)$. Then D is a disk by Lemma 3.2(3). Since $G \cup In(G)$ does not contain any crossing, neither does D.

Since G does not contain any crossing, G is not a ring. Further G is not a hoop by Lemma 4.3. Furthermore G is not a free edge by Assumption 3. Thus $w(G) > 0$.

Since D is a regular neighbourhood of $G \cup In(G)$, $G \subset \Gamma_n$ implies that we have $\partial D \cap \Gamma \subset \Gamma_{n-1} \cup \Gamma_{n+1}$. By Boundary Condition Lemma (Lemma 4.1), we have $D \cap \Gamma_i = \emptyset$ except for $i \in \{n-1, n, n+1\}$, namely $D \cap \Gamma \subset \Gamma_{n-1} \cup \Gamma_n \cup \Gamma_{n+1}$. Thus $w(G) > 0$ implies that $(D \cap \Gamma, D)$ is a T-tangle of label n without any exceptional arc. Hence the tangle is not linear by Lemma 4.2.

Since G is a small component of Γ_n, $D \cap \Gamma_n = G$ is connected.

Let $N_1, N_2, \cdots, N_u$ be the connected components of $Cl(In(G))$. See Fig. 8. Since each white vertex of G is of degree 3, $N_1, N_2, \cdots, N_u$ are mutually disjoint disks. Since the tangle is not linear, we have $u \geq 1$. For each $i = 1, 2 \cdots, u$, let X_i be the union of the terminal edges of G intersecting N_i, and D_i a regular neighborhood of $N_i \cup X_i$. Then each $(D_i \cap \Gamma, D_i)$ is a tiny cellular T-tangle of label n.

Let $w_{u+1}, w_{u+2}, \cdots, w_s$ be the white vertices in $G - (\bigcup_{i=1}^{u} D_i)$. The set $G \cup (\bigcup_{i=1}^{u} D_i)$ is deformed to a tree T if we contract each of the disks $N_1, N_2, \cdots, N_u$ to a point which we also call a vertex.

Suppose that the tree T contains only one vertex. Then $u = 1$, and the tangle $(D \cap \Gamma, D)$ is a tiny cellular T-tangle. Since $(D \cap \Gamma, D)$ does not possess any exceptional arc, $\varepsilon(D \cap \Gamma, D) = 0$. Thus by Lemma 4.7 we have

$$m(D \cap \Gamma, D) + 2 \leq t(D \cap \Gamma, D).$$

Hence $t(D \cap \Gamma, D) \geq 2$. Thus there exist at least two terminal edges of label

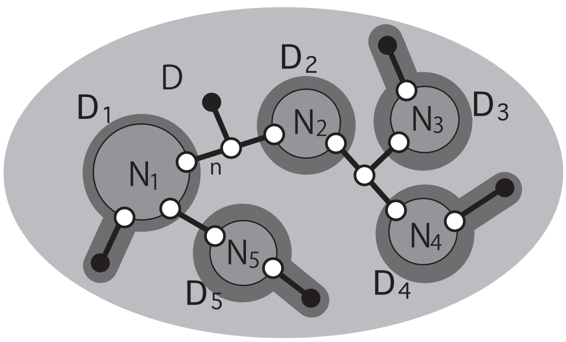

Fig. 8. $(D_3 \cap \Gamma, D_3)$, $(D_4 \cap \Gamma, D_4)$ and $(D_5 \cap \Gamma, D_5)$ are tiny cellular T-tangles with exactly one exceptional arc.

n in D. Since G is a small component of Γ_n, the terminal edges of label n in D are contained in G.

Suppose that the tree T contains at least two vertices. Then there exists at least two vertices v_1 and v_2 of degree 1 in T. Each vertex v_j $(j = 1, 2)$ corresponds to either a disk D_{i_j} or a black vertex contained in a terminal edge. If v_j corresponds to a disk D_{i_j}, then $(D_{i_j} \cap \Gamma, D_{i_j})$ is a tiny cellular T-tangle with exactly one exceptional arc (see Fig. 8). Thus $\varepsilon(D_{i_j} \cap \Gamma, D_{i_j}) \leq 1$. By Lemma 4.7 we have

$$m(D_{i_j} \cap \Gamma, D_{i_j}) + 2 \leq t(D_{i_j} \cap \Gamma, D_{i_j}) + 1.$$

Hence $t(D_{i_j} \cap \Gamma, D_{i_j}) \geq 1$. For the both cases, there exists a terminal edge of label n. Therefore G possesses at least two terminal edges of label n. □

5. The Number of Terminal Edges in Cellular T_2-Tangles

Let h and n be labels of a chart Γ with $|h - n| = 1$. Let $e_1, e_2, \cdots, e_p$ be edges of label n and $w_2, \cdots, w_p$ white vertices with $e_{i-1} \cap e_i = w_i$ $(1 < i \leq p)$. Suppose that there exists a disk D such that (see Fig. 9a)

(1) $(D \cap \Gamma) \subset (\Gamma_h \cup \Gamma_n)$,

(2) if $e_1^* = e_1 \cap D$ and $e_p^* = e_p \cap D$, then each of e_1^* and e_p^* is a non-empty arc,

(3) $D \cap \Gamma_n = \partial D \cap \Gamma_n = e_1^* \cup e_2 \cup e_3 \cup \cdots \cup e_{p-1} \cup e_p^*$,

(4) for each $i = 2, \cdots, p$ there exists an arc e_i' of label h connecting the white vertex w_i and a point on ∂D, and

(5) $D \cap \Gamma_h = e_2' \cup e_3' \cup \cdots \cup e_p'$.

The p-tuple $(e_1^*, e_2, \cdots, e_{p-1}, e_p^*)$ is called a *path of label n between two arcs e_1^* and e_p^**. We often say that

each arc e_i' is an arc *situated between* e_1^* and e_p^*.

The path $(e_1^*, e_2, \cdots, e_{p-1}, e_p^*)$ of label n is an *m&m path* provided that (see Fig. 9b)

(6) e_1^* contains a middle arc at w_2 and e_p^* contains a middle arc at w_p.

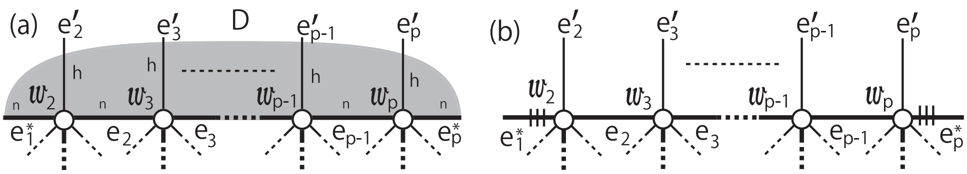

Fig. 9. Each arc with three transversal short arcs is a middle arc.

LEMMA 5.1 ([9, Lemma 3.1(2)]). *Let* Γ *be a* k*-minimal chart. Then for any m&m path* $(e_1^*, e_2, \cdots, e_{p-1}, e_p^*)$ *there exists a middle arc situated between the two arcs* e_1^* *and* e_p^*.

LEMMA 5.2. *Let* $(D \cap \Gamma, D)$ *be a tiny cellular* T_2*-tangle of label* n *in a* k*-minimal chart* Γ. *If the* T_2*-tangle possesses exceptional arcs, then* $D \cap \Gamma_n$ *is connected.*

PROOF. Since the T_2-tangle possesses an exceptional arc, there exists an integer $\delta \in \{+1, -1\}$ with $\partial D \cap \Gamma \subset \Gamma_n \cup \Gamma_{n+\delta}$. Thus we have $D \cap \Gamma \subset \Gamma_n \cup \Gamma_{n+\delta}$ by Boundary Condition Lemma (Lemma 4.1).

Let X be the connected component of $D \cap \Gamma_n$ which contains $Brd(D \cap \Gamma_n)$. Since the tangle is tiny and cellular, X is the only one connected component of $D \cap \Gamma_n$ which intersects ∂D.

Suppose that $(In(D \cap \Gamma_n) - X) \cap \Gamma_n \neq \emptyset$. Let G be a connected component of $In(X) \cap \Gamma_n$. Then $G \cap \partial D = \emptyset$. Since G does not contain any crossing, G is not a ring. Further G is not a hoop by Lemma 4.3. Furthermore G is not a free edge by Assumption 3. Thus $w(G) > 0$. Let N be a regular neighbourhood of $G \cup In(G)$ in D. Then $(N \cap \Gamma, N)$ is an NS-tangle of label $n + \delta$. This contradicts Theorem 3.5.

Thus $(In(D \cap \Gamma_n) - X) \cap \Gamma_n = \emptyset$. Since $(D \cap \Gamma, D)$ is tiny, D is a regular neighbourhood of $Cl(In(D \cap \Gamma_n))$. Hence we have $D \cap \Gamma_n = X$. Thus $D \cap \Gamma_n$ is connected. □

LEMMA 5.3. *Let* $(D \cap \Gamma, D)$ *be a tiny cellular* T_2*-tangle of label* n *in a* k*-minimal chart* Γ. *If the* T_2*-tangle possesses exceptional arcs, then it possesses at least two non-essential exceptional arcs.*

PROOF. Since the T_2-tangle possesses an exceptional arc, there exists an integer $\delta \in \{+1,-1\}$ with $\partial D \cap \Gamma \subset \Gamma_n \cup \Gamma_{n+\delta}$. Thus we have $D \cap \Gamma \subset \Gamma_n \cup \Gamma_{n+\delta}$ by Boundary Condition Lemma (Lemma 4.1).

Now $D \cap \Gamma_n$ is connected by Lemma 5.2. Let p be the number of exceptional arcs of the T_2-tangle. If $p \leq 1$, then $(D \cap \Gamma, D)$ is an NS-tangle. This contradicts Theorem 3.5. Thus we have $p \geq 2$.

We must show that $p - \varepsilon(D \cap \Gamma, D) \geq 2$.

Since the tangle is cellular, $Brd(D \cap \Gamma_n)$ is a simple closed curve. Let $e_1, e_2, \cdots, e_s$ be the edges of $Brd(D \cap \Gamma_n)$ and $w_1, w_2, \cdots, w_s$ white vertices such that

(1) $\partial e_i = \{w_i, w_{i+1}\}$ $(i = 1, 2, \cdots, s)$, where we assume $w_{s+1} = w_1$.

Let

$$\begin{aligned} \varepsilon_0 &= \varepsilon(D \cap \Gamma, D), \\ m_0 &= m(D \cap \Gamma, D), and \\ t_0 &= t(D \cap \Gamma, D). \end{aligned}$$

Let $e_1^*, e_2^*, \cdots, e_t^*$ be the terminal edges of label n or the exceptional arcs in the annulus $Cl(D - D')$. Then $t = t_0 + p$. For each $i = 1, 2, \cdots, t$ let $w_{b_i} = e_i^* \cap D'$. Since $(D \cap \Gamma, D)$ is tiny, we can assume that (see Fig. 10)

(2) $1 = b_1 < b_2 < \cdots < b_t \leq s$, and

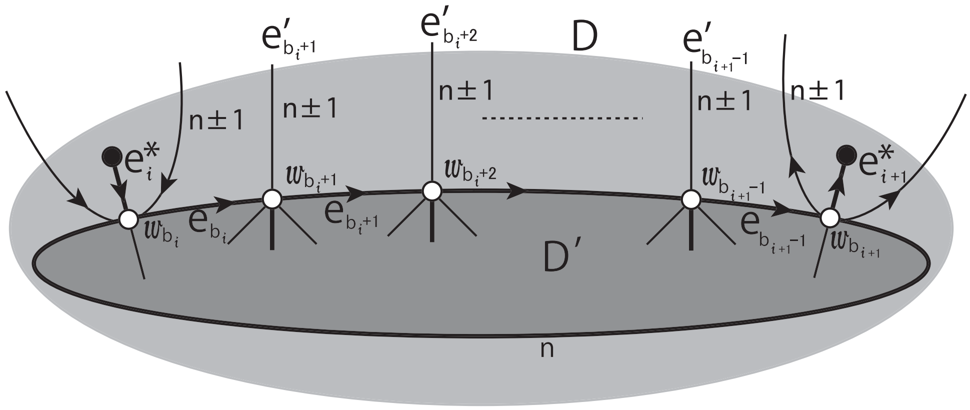

Fig. 10.

(3) for each $i = 1, 2, \cdots, t$ and for each $j = b_i + 1, b_i + 2, \cdots, b_{i+1} - 1$, there exists an arc e'_j of label $n \pm 1$ connecting w_j and a point in ∂D, here we assume the cyclic order $b_{t+1} = b_1$ and $w_{s+i} = w_i$.

Suppose that $p - \varepsilon(D \cap \Gamma, D) = 0$. The exceptional arcs are essential. Thus $D \cap \Gamma \subset \Gamma_n \cup \Gamma_{n+\delta}$ implies that $(e_i^*, e_{b_i}, e_{b_i+1}, \cdots, e_{b_{i+1}-1}, e_{i+1}^*)$ is an $m\&m$ path for each $i = 1, 2, \cdots, t$. Since for each $i = 1, 2, \cdots, t$ there exists a middle arc of label $n \pm 1$ situated between the two arcs e_i^* and e_{i+1}^* by Lemma 5.1. Since each e_i^* and e_{i+1}^* contains a middle arc, the middle of label $n \pm 1$ arc intersects neither e_i^* nor e_{i+1}^*. Thus there exist at least t middle arcs of label $n \pm 1$ in $D - In(D \cap \Gamma_n)$ each of which does not intersect any exceptional arc. Thus we have $m_0 \geq t$. Since $\varepsilon_0 = p$, we have

$$m_0 + 2 \geq t + 2 = t_0 + p + 2 = t_0 + \varepsilon_0 + 2 > t_0 + \varepsilon_0.$$

Since $D \cap \Gamma_n$ is connected, the above inequality contradicts Lemma 4.7.

Suppose that $p - \varepsilon(D \cap \Gamma, D) = 1$. Then all the exceptional arcs are essential except one. Without loss of generality we can assume that e_1^* is the non-essential exceptional arc. Then e_i^* contains a middle arc at w_{b_i} for each $i = 2, 3, \cdots, t$. Since for each $i = 2, 3, \cdots, t - 1$, there exists a middle arcs of label $n \pm 1$ situated between e_i^* and e_{i+1}^* by Lemma 5.1. Since each of e_i^* and e_{i+1}^* contains a middle arc, the middle arc of label $n \pm 1$ intersects neither e_i^* nor e_{i+1}^*. Thus there exist $t - 2$ middle arcs of label $n \pm 1$ in $D - In(D \cap \Gamma_n)$ each of which does not intersect any exceptional arc. Thus we have $m_0 \geq t - 2$. Since $\varepsilon_0 = p - 1$, we have

$$m_0 + 2 \geq t - 2 + 2 = t = t_0 + p = t_0 + \varepsilon_0 + 1 > t_0 + \varepsilon_0.$$

Since $D \cap \Gamma_n$ is connected, the above inequality contradicts Lemma 4.7.

Thus $p - \varepsilon(D \cap \Gamma, D) \geq 2$. Hence the T_2-tangle possesses at least two non-essential exceptional arcs. □

THEOREM 5.4. *Let $(D \cap \Gamma, D)$ be a tiny cellular T_2-tangle of label n in a k-minimal chart Γ which possesses exceptional arcs.*

(1) *The tangle possesses at least two exceptional arcs.*

(2) *If the tangle possesses exactly two exceptional arcs, then D contains at least two terminal edges of label n.*

(3) *If the tangle possesses exactly three exceptional arcs, then D contains at least one terminal edge of label n.*

PROOF. Lemma 5.3 implies Statement (1).
Now $D \cap \Gamma_n$ is connected by Lemma 5.2. By Lemma 4.7 we have

$$m(D \cap \Gamma, D) + 2 \leq t(D \cap \Gamma, D) + \varepsilon(D \cap \Gamma, D).$$

Suppose that the T_2-tangle possesses exactly two exceptional arcs. Then the exceptional arcs are non-essential by Lemma 5.3. Hence $\varepsilon(D \cap \Gamma, D) = 0$. Thus the above inequality implies that

$$m(D \cap \Gamma, D) + 2 \leq t(D \cap \Gamma, D).$$

Since $m(D \cap \Gamma, D) \geq 0$, we have $2 \leq t(D \cap \Gamma, D)$.

Suppose that the T_2-tangle possesses exactly three exceptional arcs. Since there exist at least two non-essential exceptional arcs by Lemma 5.3, there exists at most one essential exceptional arc. Namely $\varepsilon(D \cap \Gamma, D) \leq 1$. Thus we have

$$m(D \cap \Gamma, D) + 2 \leq t(D \cap \Gamma, D) + 1.$$

Thus we have $1 \leq t(D \cap \Gamma, D)$. □

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