

A Remark on the Geometric Jacquet Functor

By Noriyuki ABE and Yoichi MIEDA

Abstract. We give an action of N on the geometric Jacquet functor defined by Emerton-Nadler-Vilonen.

Let $G_{\mathbb{R}}$ be a reductive linear algebraic group over \mathbb{R} , $G_{\mathbb{R}} = K_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ an Iwasawa decomposition, and $M_{\mathbb{R}}$ the centralizer of $A_{\mathbb{R}}$ in $K_{\mathbb{R}}$. Then $P_{\mathbb{R}} = M_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$ is a Langlands decomposition of a minimal parabolic subgroup. We use lower-case fraktur letters to denote the corresponding Lie algebras and omit the subscript “ \mathbb{R} ” to denote complexifications. Fix a Cartan involution θ such that $K = \{g \in G \mid \theta(g) = g\}$. For a (\mathfrak{g}, K) -module V , the Jacquet module $J(V)$ of V is defined by the space of \mathfrak{n} -finite vectors in $\varprojlim_{k \rightarrow \infty} V/\theta(\mathfrak{n})^k V$ [Cas80].

For simplicity, assume that V has the same infinitesimal character as the trivial representation. Denote the category of Harish-Chandra modules with the same infinitesimal characters as the trivial representation by HC_{ρ} . Let X be the flag variety of G , $\mathrm{Perv}_K(X)$ the category of K -equivariant perverse sheaves on X . By the Beilinson-Bernstein correspondence and the Riemann-Hilbert correspondence, we have the localization functor $\Delta: \mathrm{HC}_{\rho} \rightarrow \mathrm{Perv}_K(X)$. Emerton-Nadler-Vilonen gave a geometric description of $J(V)$ by the following way [ENV04]. Fix a cocharacter $\nu: \mathbb{G}_m \rightarrow A$ which is positive on the roots in \mathfrak{n} . Define $a: \mathbb{G}_m \times X \rightarrow X$ by $a(t, x) = \nu(t)x$. Consider the following diagram,

$$X \simeq \{0\} \times X \rightarrow \mathbb{A}^1 \times X \leftarrow \mathbb{G}_m \times X \xrightarrow{a} X.$$

Let $R\psi$ be the nearby cycle functor with respect to $\mathbb{A}^1 \times X \rightarrow \mathbb{A}^1$. Then the geometric Jacquet functor Ψ is defined by

$$\Psi(\mathcal{F}) = R\psi(a^*(\mathcal{F})).$$

2010 *Mathematics Subject Classification.* 22E46, 14F05.

THEOREM 1 (Emerton-Nadler-Vilonen [ENV04, Theorem 1.1]). *We have $\Delta \circ J \simeq \Psi \circ \Delta: \mathrm{HC}_\rho \rightarrow \mathrm{Perv}_K(X)$.*

It is easy to see that $J(V)$ is a (\mathfrak{g}, N) -module for a (\mathfrak{g}, K) -module V . Hence $\Psi(\mathcal{F})$ is N -equivariant for $\mathcal{F} \in \mathrm{Perv}_K(X)$. (See also [ENV04, Remark 1.3].)

In this paper, we give the action of N on $\Psi(\mathcal{F})$ in a geometric way. Roughly speaking, this action is given by the “limit” of the action of K .

We use the following lemma.

LEMMA 2. *Let \mathcal{X} be a scheme of finite type over \mathbb{A}^1 , \mathcal{X}^0 (resp. \mathcal{X}_0) the inverse image of \mathbb{G}_m (resp. $\{0\}$), and \mathcal{G} a smooth group scheme over \mathbb{A}^1 . Assume that the action of \mathcal{G} on \mathcal{X} over \mathbb{A}^1 is given. If \mathcal{F}^0 is a \mathcal{G}^0 -equivariant perverse sheaf on \mathcal{X}^0 , then $R\psi(\mathcal{F}^0)$ is \mathcal{G}_0 -equivariant.*

PROOF. Define $m: \mathcal{G} \times_{\mathbb{A}^1} \mathcal{X} \rightarrow \mathcal{X}$ by $m(g, x) = gx$. Then m is a smooth morphism. Let $m^0: \mathcal{G}^0 \times_{\mathbb{G}_m} \mathcal{X}^0 \rightarrow \mathcal{X}^0$ and $m_0: \mathcal{G}_0 \times \mathcal{X}_0 \rightarrow \mathcal{X}_0$ be its restrictions. Since \mathcal{F}^0 is \mathcal{G}^0 -equivariant, we are given an isomorphism $(m^0)^*(\mathcal{F}^0) \simeq \mathrm{pr}_2^*(\mathcal{F}^0)$. Hence we have an isomorphism $R\psi(m^0)^*(\mathcal{F}^0) \simeq R\psi\mathrm{pr}_2^*(\mathcal{F}^0)$. By the smooth base change theorem, we get an isomorphism $m_0^*R\psi(\mathcal{F}^0) \simeq \mathrm{pr}_2^*R\psi(\mathcal{F}^0)$ [SGA7 II, Exposé XIII]. It is immediate that this isomorphism gives a \mathcal{G}_0 -equivariant structure on $R\psi(\mathcal{F}^0)$. \square

Set

$$\mathcal{K}^0 = \{(t, k) \in \mathbb{G}_m \times G \mid k \in \mathrm{Ad}(\nu(t)^{-1})(K)\} \subset \mathbb{A}^1 \times G.$$

Let \mathcal{K} be the closure of \mathcal{K}^0 in $\mathbb{A}^1 \times G$. It is a closed sub-group scheme of $\mathbb{A}^1 \times G$. Then \mathcal{K} is flat over \mathbb{A}^1 . By [Oor66], a group scheme of finite type over a field of characteristic zero is reduced. Hence each fiber of $\mathcal{K} \rightarrow \mathbb{A}^1$ is reduced. Therefore, $\mathcal{K} \rightarrow \mathbb{A}^1$ is smooth. Hence \mathcal{K} is smooth over \mathbb{A}^1 .

Let Σ be the restricted root system of $(\mathfrak{g}, \mathfrak{a})$, Σ^+ the positive system corresponding to \mathfrak{n} , and \mathfrak{g}_α the restricted root space for $\alpha \in \Sigma$. Then \mathfrak{k} is spanned by \mathfrak{m} and $\{X + \theta(X) \mid X \in \mathfrak{g}_\alpha, \alpha \in \Sigma^+\}$. Since

$$\mathrm{Ad}(\nu(t)^{-1})(X + \theta(X)) = t^{-\langle \nu, \alpha \rangle} (X + t^{2\langle \nu, \alpha \rangle} \theta(X))$$

for $X \in \mathfrak{g}_\alpha$, the Lie algebra of $\mathrm{Ad}(\nu(t)^{-1})(K)$ is spanned by

$$\mathfrak{m} \text{ and } \{X + t^{2\langle \nu, \alpha \rangle} \theta(X) \mid X \in \mathfrak{g}_\alpha, \alpha \in \Sigma^+\}.$$

Hence the neutral component of \mathcal{K}_0 is $M^\circ N$ where M° is the neutral component of M . Since $MK^\circ = K$ and $\text{Ad}(\nu(t)^{-1})(M) = M$, we have $\mathcal{K}_0 = MM^\circ N = MN$.

Define $\tilde{a}: \mathcal{K}^0 \times_{\mathbb{G}_m} (\mathbb{G}_m \times X) \rightarrow K \times X$ (resp. $\tilde{m}: \mathcal{K}^0 \times_{\mathbb{G}_m} (\mathbb{G}_m \times X) \rightarrow \mathbb{G}_m \times X$, $m: K \times X \rightarrow X$) by $\tilde{a}((t, k), (t, x)) = (\text{Ad}(\nu(t))k, \nu(t)x)$ (resp. $\tilde{m}((t, k), (t, x)) = (t, kx)$, $m(k, x) = kx$). Then we have the following commutative diagrams

$$\begin{array}{ccc}
 \mathcal{K}^0 \times_{\mathbb{G}_m} (\mathbb{G}_m \times X) & \xrightarrow{\tilde{m}} & \mathbb{G}_m \times X \\
 \tilde{a} \downarrow & & \downarrow a \\
 K \times X & \xrightarrow{m} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{K}^0 \times_{\mathbb{G}_m} (\mathbb{G}_m \times X) & \xrightarrow{\text{pr}_2} & \mathbb{G}_m \times X \\
 \tilde{a} \downarrow & & \downarrow a \\
 K \times X & \xrightarrow{\text{pr}_2} & X.
 \end{array}$$

Let $\mathcal{F} \in \text{Perv}_K(X)$. Then $m^*\mathcal{F} \simeq \text{pr}_2^*\mathcal{F}$. Hence we get $\tilde{a}^*m^*\mathcal{F} \simeq \tilde{a}^*\text{pr}_2^*\mathcal{F}$. By the above diagrams, $\tilde{m}^*a^*\mathcal{F} \simeq \text{pr}_2^*a^*\mathcal{F}$. Therefore $a^*(\mathcal{F})$ is \mathcal{K}^0 -equivariant. By Lemma 2, $\Psi(\mathcal{F}) = R\psi(a^*(\mathcal{F}))$ is $\mathcal{K}_0 = MN$ -equivariant.

REMARK 3. Let Γ be the quasi-inverse functor of Δ . Then N acts on $\Gamma(\Psi(\mathcal{F}))$. Moreover, this becomes a (\mathfrak{g}, N) -module [Kas89, 9.1.1], namely, the infinitesimal action of N coincides with the action of $\mathfrak{n} \subset \mathfrak{g}$. We also have that $J(\Gamma(\mathcal{F}))$ is a (\mathfrak{g}, N) -module. Hence both N -actions have the same infinitesimal actions. Since the action of N is determined by its infinitesimal action, the N -action we defined above coincides with the N -action on $J(\Gamma(\mathcal{F}))$.

References

- [Cas80] Casselman, W., *Jacquet modules for real reductive groups*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978) (Helsinki), Acad. Sci. Fennica, 1980, pp. 557–563.
- [ENV04] Emerton, M., Nadler, D. and K. Vilonen, A geometric Jacquet functor, *Duke Math. J.* **125** (2004), no. 2, 267–278.
- [Kas89] Kashiwara, M., *Representation theory and D-modules on flag varieties*, *Astérisque* (1989), no. 173–174, 9, 55–109, Orbits unipotentes et représentations, III.
- [Oor66] Oort, F., Algebraic group schemes in characteristic zero are reduced, *Invent. Math.* **2** (1966), 79–80.
- [SGA7 II] *Groupes de monodromie en géométrie algébrique. II*, Lecture Notes in Mathematics, Vol. 340, Springer-Verlag, Berlin, 1973, Séminaire de

Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par
P. Deligne et N. Katz.

(Received April 30, 2010)

(Revised June 18, 2010)

Noriyuki Abe
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro, Tokyo
153-8914 Japan
E-mail: abenori@ms.u-tokyo.ac.jp

Yoichi Mieda
Graduate School of Mathematics
Kyushu University
744 Motoooka, Nishi-ku, Fukuoka
819-0395 Japan
E-mail: mieda@math.kyushu-u.ac.jp