J. Math. Sci. Univ. Tokyo
17 (2010), 79–121.

Uniform Estimate for Distributions of the Sum of i.i.d. Random Variables with Fat Tail

By Hirotaka Fushiya and Shigeo Kusuoka*

Abstract. The research on asymptotic behavior of distributions of the sum of i.i.d random variables has a long history and a lot of facts are known. The authors consider the case where the distribution of a random variable has the second moment but has a fat tail, and they show a new limit theorem for large deviations.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and X_n, $n = 1, 2, \ldots$, be independent identically distributed random variables with the same probability law μ.

In the present paper we assume that
(A-1) $E[X_1^2] = 1$ and $E[X_1] = 0$.
Let $F : \mathbf{R} \to [0,1]$ and $\bar{F} : \mathbf{R} \to [0,1]$ be given by

$$F(x) = \mu((-\infty, x]) = P(X_1 \leqq x) \text{ and}$$
$$\bar{F}(x) = \mu((x, \infty)) = P(X_1 > x), \qquad x \in \mathbf{R}.$$

We also assume the following.
(A-2) $\bar{F}(x)$ is a regularly varying function of index $-\alpha$ for some $\alpha > 2$, as $x \to \infty$, i.e., if we let

$$L(x) = x^{\alpha}\bar{F}(x), \qquad x \geqq 1,$$

then $L(x) > 0$ for any $x \geqq 1$, and for any $a > 0$

$$\frac{L(ax)}{L(x)} \to 1, \qquad x \to \infty.$$

*Research supported by the 21st century COE project, Graduate School of Mathematical Sciences, The University of Tokyo.

2000 *Mathematics Subject Classification.* 60F05, 62E20.

(A-3) $|x|^{\alpha+2}F(x) \to 0, \quad x \to -\infty.$

Recently people in finance are interested in computing the quantile of the distribution of $\sum_{k=1}^n X_k$ for the purpose of measuring market risk.

There are many works on this topic. In particular, there are many results on large deviation results (e.g. Borovkov-Borovkov [1], also see books, Borovkov-Borovkov [2] and Petrov [7]). However, there are not so many results on uniform estimates. Nagaev [5] and [6] proved the following theorem (also see Linnik [4]), and this is the best result so far to our best knowledge.

THEOREM 1 (Nagaev). *Assume* (A-1)-(A-3). *Then we have*

$$\sup_{s\in[1,\infty)} \left|\frac{P(\sum_{k=1}^n X_k > sn^{1/2})}{\Phi_0(s) + n\bar{F}(sn^{1/2})} - 1\right| \to 0, \qquad n \to \infty.$$

Here $\Phi_0 : \mathbf{R} \to \mathbf{R}$ is given by

$$\Phi_0(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-\frac{y^2}{2})dy, \qquad x \in \mathbf{R}.$$

In this paper, we show two theorems. Combining them, we can improve Nagaev's result a little bit.

Let us explain our results. We assume the following assumption furthermore.

(A-4) The probability law μ is absolutely continuous and has a density function $\rho : \mathbf{R} \to [0,\infty)$ which is right continuous and has a finite total variation.

To state our theorem (Theorem 2), we need some preparations.

Let K be an integer such that $K - 1 < \alpha \leqq K$. Then $K \geqq 3$. From the assumptions (A-2) and (A-3), we see that the probability law μ has $(K-1)$-th moment. So let η_k, $k = 1, \dots, K-1$, be given by

$$\eta_k = \int_{\mathbf{R}} x^k \mu(dx).$$

Then we see that $\eta_1 = 0$ and $\eta_2 = 1$. Also, let us define $\Phi_k : \mathbf{R} \to \mathbf{R}$, $k = 1, 2, \dots$, by

$$\Phi_1(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) = -\frac{d}{dx}\Phi_0(x),$$

and

$$\Phi_k(x) = (-1)^{k-1} \frac{d^{k-1}}{dx^{k-1}} \Phi_1(x), \qquad k = 2, 3, \dots .$$

THEOREM 2. *Assume* (A-1)-(A-4). *Then there are* $\delta > 0$ *and* $C > 0$ *such that*

$$\sup_{s \in [1,\infty)} |P(\sum_{k=1}^{n} X_k > sn^{1/2}) - G(n,s)| \leqq Cn^{-(\alpha-2)/2-\delta}, \qquad n = 3, 4, \dots .$$

Here

$$G(n,s)$$

$$= \Phi_0(s) + n \int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx - \sum_{k=1}^{K-1} \frac{n^{-(k-2)/2}}{k!} \Phi_k(s) \int_0^{\infty} x^k \mu(dx)$$

$$+ \frac{n^{-(K-2)/2}}{K!} \Phi_K(s) \int_{-\infty}^{0} x^K \mu(dx) + \sum_{k=3}^{K-1} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_k) \Phi_k(s)$$

$$+ \sum_{k=K}^{3(K-1)} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_{K-1}, 0, \dots, 0) \Phi_k(s),$$

and q_k*'s are polynomials defined in the next section.*

For the next theorem we assume the following also.
(A-5) There is an $x_0 > 0$ such that $\bar{F}$ is twice continuously differentiable on (x_0, ∞) and that

$$x^2 \frac{d^2}{dx^2} \log \bar{F}(x) \to \alpha, \qquad x \to \infty.$$

Then we have the following.

THEOREM 3. *Assume the assumptions* (A-1)- (A-5) *and let* $\beta : \mathbf{N} \to (0, \infty)$ *be such that*

$$\frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \qquad n \to \infty.$$

Then we have

$$\sup_{s \geqq n^{1/2}\beta(n)} \frac{s^2}{n} \left| \frac{P(\sum_{k=1}^n X_k > s)}{n\bar{F}(s)} - (1 + \frac{\alpha(\alpha+1)n}{2s^2}) \right| \to 0, \qquad n \to \infty.$$

Let

$$H(n,s)$$
$$= \Phi_0(s) + n\int_{-\infty}^s \bar{F}((s-x)n^{1/2})\Phi_1(x)dx - \sum_{k=1}^2 \frac{n^{-(k-2)/2}}{k!}\Phi_k(s)\int_0^\infty x^k \mu(dx)$$

for $s \geqq 1$, and $n \geqq 1$.

Then we also show the following.

THEOREM 4. *Assume* (A-1)-(A-5). *Then there exist a* $C > 0$, $\delta > 0$ *and* $n_0 \geqq 1$ *such that*

$$\sup_{s \in [1,\infty)} \left| \frac{P(\sum_{k=1}^n X_k > sn^{1/2})}{H(n,s)} - 1 \right| \leqq Cn^{-\delta}, \qquad n \geqq n_0.$$

Note that by Theorem 3, we see that

$$2(\log n)^2 (\frac{P(\sum_{k=1}^n X_k > (\log n)n^{1/2})}{\Phi_0(\log n) + n\bar{F}((\log n)n^{1/2})} - 1) \to \alpha(\alpha+1) \qquad n \to \infty.$$

Therefore we see that $H(n,s)$ is a better approximation for $P(\sum_{k=1}^n X_k > sn^{1/2})$ than $\Phi_0(s) + n\bar{F}(sn^{1/2})$.

The authors thank the referee for useful comments.

2. Algebraic Preparation

In this section, we think of formal power series in z. First, we think of the following formal power series in z.

$$(1) \quad \log(1 + \sum_{k=2}^\infty \frac{a_k}{k!} z^k) = \sum_{\ell=1}^\infty \frac{(-1)^{\ell-1}}{\ell} (\sum_{k=2}^\infty \frac{a_k}{k!} z^k)^\ell = \sum_{\ell=2}^\infty c_\ell(a_2, \dots, a_\ell)\frac{z^\ell}{\ell!}$$

Then we see that $c_\ell(a_2, \dots, a_\ell)$, $\ell \geqq 2$, are polynomials in $a_2, \dots, a_\ell$, and

$$c_\ell(t^2 a_2, \dots, t^\ell a_\ell) = t^\ell c_\ell(a_1, \dots, a_\ell)$$

for any $t, a_1, \dots, a_\ell \in \mathbf{R}$. Moreover, we see that

$$c_2(a_2) = a_2 \quad \text{and} \quad c_\ell(a_2, \dots, a_{\ell-1}, a_\ell) = c_\ell(a_2, \dots, a_{\ell-1}, 0) + a_\ell, \quad \ell \geqq 2.$$

We also think of the following formal power series in z.

$$\exp(y^{-3} \sum_{\ell=3}^{\infty} c_\ell(a_2, \dots, a_\ell) \frac{(yz)^\ell}{\ell!})$$

$$(2) \quad = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} (\sum_{\ell=3}^{\infty} c_\ell(a_2, \dots, a_\ell) \frac{y^{\ell-3} z^\ell}{\ell!})^k = 1 + \sum_{k=3}^{\infty} q_k(y, a_2, \dots, a_k) z^k.$$

Then we see that $q_k(y, a_2, \dots, a_k)$, $k \geqq 3$, are polynomials in y, $a_2, \dots, a_\ell$. Note that

$$q_k(y, t^2 a_2 \dots, t^k a_k) = t^k q_k(y, a_2, \dots, a_k)$$

and that

$$q_k(y, a_2, \dots, a_k) = q_k(y, a_2, \dots, a_{k-1}, 0) + \frac{y^{k-3}}{k!} a_k, \qquad k \geqq 3.$$

Also we have

$$\exp(y^{-6} \sum_{\ell=3}^{\infty} c_\ell(a_2, \dots, a_\ell) \frac{(y^3 z)^\ell}{\ell!})$$

$$= \exp((y^2)^{-3} \sum_{\ell=3}^{\infty} c_\ell(y^2 a_2, \dots, y^\ell a_\ell) \frac{(y^2 z)^\ell}{\ell!})$$

$$(3) \quad = 1 + \sum_{k=3}^{\infty} q_k(y^2, y^2 a_2, \dots, y^k a_k) z^k = 1 + \sum_{k=3}^{\infty} y^k q_k(y^2, a_2, \dots, a_k) z^k$$

as a formal power series in z.

3. Property of the Function L

PROPOSITION 5. *We have*

$$\sup_{1/2 \leqq a \leqq 2} \frac{L(ax)}{L(x)} \to 1, \qquad x \to \infty,$$

and

$$\inf_{1/2 \leqq a \leqq 2} \frac{L(ax)}{L(x)} \to 1, \qquad x \to \infty.$$

PROOF. Since the proof is similar, we prove the first equation only. If not, there are $\varepsilon > 0$, $\{a_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ such that $1/2 \leqq a_n \leqq 2$, $x_n \geqq 1$, $n = 1, 2, \dots$, $x_n \to \infty$, $n \to \infty$, and that

$$\frac{L(a_n x_n)}{L(x_n)} > 1 + \varepsilon, \qquad n = 1, 2, \dots .$$

Then taking a subsequence if necessary, we may assume that there is an $a \in [1/2, 2]$ such that $a_n \to a$, $n \to \infty$. Then we see that for any $m \geqq 3$ there is a $n(m) \geqq 1$ such that

$$\begin{aligned}(a - \frac{1}{m})^{-\alpha} L((a - \frac{1}{m})x_n) &= \bar{F}((a - \frac{1}{m})x_n) \geqq \bar{F}(a_n x_n) \\ &= a_n^{-\alpha} L(a_n x_n), \qquad n \geqq n(m).\end{aligned}$$

So we have

$$(1 - \frac{1}{ma})^{-\alpha} \geqq \lim_{n\to\infty} \frac{L(a_n x_n)}{L((a - 1/m)x_n)} \geqq 1 + \varepsilon, \qquad m \geqq 3.$$

Since m is arbitrary, this implies a contradiction. □

PROPOSITION 6. *For any $\varepsilon \in (0,1)$, there is an $M \geqq 1$ such that*

$$M^{-1} y^{-\varepsilon} \leqq \frac{L(yx)}{L(x)} \leqq M y^{\varepsilon} \qquad x, y \geqq 1.$$

PROOF. For any $\varepsilon \in (0,1)$ there is an $m \geqq 1$ such that

$$\left|\frac{L(ex)}{L(x)} - 1\right| \leqq \varepsilon \qquad x \geqq e^m.$$

Let

$$C = \sup_{x \in [1, e^m]} \left(\frac{L(ex)}{L(x)} + \frac{L(x)}{L(ex)} \right) < \infty.$$

Then we have

$$C^{-m}(1-\varepsilon)^n \leqq \frac{L(e^n x)}{L(x)} \leqq C^m (1+\varepsilon)^n, \qquad x \geqq 1,\ n \geqq 0.$$

For any $y \geqq 1$, there is an $n \geqq 1$ such that $e^{n-1} \leqq y \leqq e^n$. Then we have

$$\bar{F}(e^{n-1}x) \geqq \bar{F}(yx) \geqq \bar{F}(e^n x).$$

So we have for any $x, y \geqq 1$

$$(e^{-1}yx)^{-\alpha} L(e^{n-1}x) \geqq (e^{n-1}x)^{-\alpha} L(e^{n-1}x) \geqq (yx)^{-\alpha} L(yx)$$

$$\geqq (e^n x)^{-\alpha} L(e^n x) \geqq (eyx)^{-\alpha} L(e^n x),$$

which implies

$$C^{-m} e^{-\alpha} (1-\varepsilon)^n \leqq \frac{L(yx)}{L(x)} \leqq C^m e^{\alpha} (1+\varepsilon)^{n-1}.$$

Therefore we have

$$C^{-m} e^{-\alpha} (1-\varepsilon) y^{\log(1-\varepsilon)} \leqq \frac{L(yx)}{L(x)} \leqq C^m e^{\alpha} y^{\log(1+\varepsilon)}, \qquad x \geqq 1,\ y \geqq 1.$$

This implies our assertion. $\square$

The following is known as Karamata's theorem (c.f.[3] Appendix), but we give a proof.

PROPOSITION 7. (1) *For any* $\beta < -1$,

$$\frac{1}{t^{\beta+1} L(t)} \int_t^{\infty} x^{\beta} L(x) dx \to -\frac{1}{\beta+1}, \qquad t \to \infty.$$

(2) *For any* $\beta > -1$,

$$\frac{1}{t^{\beta+1} L(t)} \int_1^{t} x^{\beta} L(x) dx \to \frac{1}{\beta+1}, \qquad t \to \infty.$$

(3) *Let* $f : [1, \infty) \to (0, \infty)$ *be given by*

$$f(t) = \int_1^t x^{-1} L(x) dx \qquad t \geqq 1.$$

Then f *is slowly varying.*

PROOF. Note that for $t > 1$

$$\frac{1}{t^{\beta+1} L(t)} \int_t^\infty x^\beta L(x) dx = \int_1^\infty x^\beta \frac{L(tx)}{L(t)} dx, \text{ if } \beta < -1$$

and

$$\frac{1}{t^{\beta+1} L(t)} \int_1^t x^\beta L(x) dx = \int_{1/t}^1 x^\beta (\frac{L(t)}{L(tx)})^{-1} dx \text{ if } \beta > -1$$

Then the assertions (1) and (2) follow from this equation and Proposition 5.

Let us prove (3). If $\lim_{t\to\infty} f(t) < \infty$, the assertion is obvious. So we assume that $\lim_{t\to\infty} f(t) = \infty$. Then for any $a > 0$ and $t_0 > 1$

$$f(at) = \int_{1/a}^t x^{-1} L(ax) dx = \int_{1/a}^{t_0} x^{-1} L(ax) dx + \int_{t_0}^t x^{-1} L(x) \frac{L(ax)}{L(x)} dx.$$

So we have

$$\inf_{x \geqq t_0} \frac{L(ax)}{L(x)} \leqq \underline{\lim}_{t\to\infty} \frac{f(at)}{f(t)} \leqq \overline{\lim}_{t\to\infty} \frac{f(at)}{f(t)} \leqq \sup_{x \geqq t_0} \frac{L(ax)}{L(x)}.$$

Therefore by Proposition 6 and Lebesgue's convergence theorem, we have our assertion. □

4. Estimate for Moments and Characteristic Functions

Remind that K is an integer such that $K - 1 < \alpha \leqq K$ and

$$\eta_k = \int_{-\infty}^\infty x^k \mu(dx), \qquad k = 1, 2, \dots, K - 1.$$

Then by the assumption (A4) we have $\eta_1 = 0$ and $\eta_2 = 1$. Note that

$$1 - \bar{F}(t) \geqq 1 - \int_2^\infty \frac{x^2}{4} \mu(dx) \geqq \frac{3}{4}$$

for any $t \geqq 2$. Let

$$\eta_k(t) = \int_{(-\infty,t]} x^k \mu(dx), \qquad t > 0,\ k = 1, 2, \dots, K+1,$$

and

$$\bar{\eta}_k(t) = \int_{(t,\infty)} x^k \mu(dx), \qquad t > 0,\ k = 0, 1, 2, \dots, K-1.$$

Then we have

$$\eta_k(t) = \int_{(-\infty,0)} x^k \mu(dx) + k \int_0^t x^{k-1} \bar{F}(x) dx - t^k \bar{F}(t),$$
$$t > 0,\ k = 1, 2, \dots, K+1,$$

and

$$\bar{\eta}_k(t) = k \int_t^\infty x^{k-1} \bar{F}(x) dx + t^k \bar{F}(t) \qquad t > 0,\ k = 0, 1, 2, \dots, K-1.$$

Then by Propositions 6 and 7 we have the following.

PROPOSITION 8. *For any $\varepsilon > 0$, there is a $C(\varepsilon) > 0$ such that*

$$L(t) \leqq C(\varepsilon) t^{\varepsilon},$$

$$|\eta_K(t)| \leqq C(\varepsilon) t^{-\alpha+K+\varepsilon},$$

$$|\bar{\eta}_k(t)| \leqq C(\varepsilon) t^{-\alpha+k+\varepsilon}, \qquad k = 0, 1, 2, \dots K-1,$$

and

$$\int_{(-\infty,t]} |x|^{K+1} \mu(dx) \leqq C(\varepsilon) t^{-\alpha+K+1+\varepsilon}$$

for any $t \geqq 1$.

The following is well known.

PROPOSITION 9. (1) *For any $m \geqq 0$, let $r_{e,m} : \mathbf{R} \to \mathbf{C}$ be given by*

$$r_{e,m}(t) = \exp(it) - (1 + \sum_{k=1}^m \frac{(it)^k}{k!}), \qquad t \in \mathbf{R}.$$

Then we have

$$|r_{e,m}(t)| \leqq \frac{|t|^{m+1}}{(m+1)!} \qquad t \in \mathbf{R}.$$

(2) *For any* $m \geqq 1$, *let* $r_{l,m} : \{z \in \mathbf{C};\ |z| \leqq 1/2\} \to \mathbf{C}$ *be given by*

$$r_{l,m}(z) = \log(1+z) - \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} z^k, \qquad z \in \mathbf{C},\ |z| \leqq 1/2.$$

Then we have

$$|r_{l,m}(z)| \leqq 2|z|^{m+1}, \qquad z \in \mathbf{C},\ |z| \leqq 1/2.$$

Let $\mu(t)$, $t > 0$, be a probability measure on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ given by

$$\mu(t)(A) = (1 - \bar{F}(t))^{-1} \mu(A \cap (-\infty, t]),$$

for any $A \in \mathcal{B}(\mathbf{R})$.

Let $\varphi(\cdot; \mu(t))$, $t > 0$, be the characteristic function of the probability measure $\mu(t)$, i.e.,

$$\varphi(\xi; \mu(t)) = \int_{\mathbf{R}} \exp(ix\xi)\mu(t)(dx), \qquad \xi \in \mathbf{R}.$$

By the assumption (A3), we see that the density function $\rho(x) \to 0$ as $|x| \to \infty$. Also we see that the probability measure $\mu(t)$, $t \geqq 2$, is absolutely continuous and its density function is $(1 - \bar{F}(t))^{-1}\rho(x)1_{(-\infty,t]}(x)$, whose total variation is dominated by twice of that of ρ.

Therefore we have the following.

PROPOSITION 10. (1) *For any* $t \geqq 2$ *and* $\xi \in \mathbf{R}$,

$$\begin{aligned} i\xi\varphi(\xi; \mu(t)) &= (1 - \bar{F}(t))^{-1} \int_{\mathbf{R}} i\xi e^{i\xi x} \rho(x) 1_{(-\infty,t]}(x) dx \\ &= -(1 - \bar{F}(t))^{-1} \int_{\mathbf{R}} e^{i\xi x} d(\rho(x) 1_{(-\infty,t]}(x)). \end{aligned}$$

(2) *There is a* $C > 0$ *such that*

$$|\varphi(\xi, \mu(t))| \leqq C(1 + |\xi|)^{-1} \textit{ for any } t \geqq 2 \textit{ and } \xi \in \mathbf{R}.$$

Then we have the following.

PROPOSITION 11. (1) *There is a* $c_0 > 0$ *such that*

$$|\varphi(\xi, \mu(t))| \leqq (1 + c_0|\xi|^2)^{-1/4} \text{ for any } t \geqq 2 \text{ and } \xi \in \mathbf{R}.$$

(2) *For any* $t \geqq 2$, $\xi \in \mathbf{R}$, *and integers* n, m *with* $n \geqq m$,

$$|\varphi(n^{-1/2}\xi, \mu(t))|^n \leqq (1 + \frac{c_0}{m}|\xi|^2)^{-m/4}.$$

PROOF. Let $g(x) = \rho(x)1_{(-2,2)}(x)$, $x \in \mathbf{R}$. Then we have

$$p = \int_{\mathbf{R}} g(x)dx \geqq 1 - \int_{\mathbf{R}} \frac{x^2}{4}\rho(x)dx \geqq 3/4.$$

Note that

$$\begin{aligned} &|\varphi(\xi, \mu(t))|^2 \\ &= (1 - \bar{F}(t))^{-2} \int_{\mathbf{R}} \int_{\mathbf{R}} \exp(i\xi(x-y))\rho(x)1_{(-\infty,t]}(x)\rho(y)1_{(-\infty,t]}(y)dxdy \\ &\leqq (1 - p^2) + \int_{\mathbf{R}} \int_{\mathbf{R}} \exp(i\xi(x-y))g(x)g(y)dxdy = 1 - f(\xi), \end{aligned}$$

where

$$f(\xi) = \int_{\mathbf{R}} \int_{\mathbf{R}} (1 - \cos(\xi(x-y)))g(x)g(y)dxdy.$$

So we see that

$$\lim_{\xi \to 0} |\xi|^{-2} f(\xi) = \frac{1}{2} \int_{\mathbf{R}} \int_{\mathbf{R}} (x-y)^2 g(x)g(y)dxdy > 0.$$

Also, it is easy to see that $f(\xi) > 0$, for all $\xi \in \mathbf{R} \setminus \{0\}$, and so we see that

$$a(r) = \inf_{|\xi| \leqq r} |\xi|^{-2} f(\xi) > 0 \qquad \text{for all } \ r > 0.$$

Therefore we see that

$$|\varphi(\xi, \mu(t))| \leqq (1 - a(r)|\xi|^2)^{1/2} \leqq (1 + a(r)|\xi|^2)^{-1/4}, \qquad |\xi| \leqq r.$$

Also by Proposition 10(2), we see that there is an $r_0 > 0$ such that

$$|\varphi(\xi, \mu(t))| \leqq (1 + |\xi|^2)^{-1/4}, \qquad |\xi| \geqq r_0$$

So we have the assertion (1).

It is easy to chack that $(1 + x/\beta)^\beta \geqq 1 + x$ for any $\beta \geqq 1$ and $x \geqq 0$. Therefore if $n \geqq m$, we have

$$(1 + c_0|n^{-1/2}\xi|^2)^{n/m} \geqq 1 + \frac{c_0}{m}|\xi|^2.$$

This implies the assertion (2). □

5. Asymptotic Expansion of Characteristic Functions

Let

$$\varphi_1(\xi, t) = -\sum_{k=1}^{K-1} \frac{(i\xi)^k}{k!}\bar{\eta}_k(t) + \frac{(i\xi)^K}{K!}\eta_K(t)$$

and

$$\psi_0(n, \xi) = \sum_{k=3}^{K-1} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_k)(i\xi)^k$$

$$+ \sum_{k=K}^{3(K-1)} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_{K-1}, 0, \dots, 0)(i\xi)^k$$

for $t \geqq 2$, $n \geqq 1$ and $\xi \in \mathbf{R}$. Let $\delta = ((\alpha-2)\wedge 1)/(4(K+2))$, $\delta' = \delta/(4(K+2))$, and $t_n = n^{1/2-\delta}$, $n = 1, 2, 3, \dots$. Then $t_n \geqq 2$ for any $n \geqq 8$.

In this section, we prove the following.

LEMMA 12. *Let*

$$R_{n,0}(\xi) = \exp(\frac{1}{2}\xi^2)\varphi(n^{-1/2}\xi, \mu(t_n))^n - (1 + \psi_0(n, \xi) + n\varphi_1(n^{-1/2}\xi, t_n))$$

$$R_{n,1}(\xi) = \exp(\frac{1}{2}\xi^2)\varphi(n^{-1/2}\xi, \mu(t_n))^n - 1$$

$$R_{n,2}(\xi) = \exp(\frac{1}{2}\xi^2)\varphi(n^{-1/2}\xi, \mu(t_n))^{n-1} - 1$$

Then there is a $C > 0$ *such that*

$$|R_{n,0}(\xi)| \leqq Cn^{-(\alpha-2)/2-\delta/4}|\xi|$$

and

$$|R_{n,1}(\xi)| + |R_{n,2}(\xi)| \leqq Cn^{-2K\delta}|\xi|$$

for any $n \geqq 8$ *and* $\xi \in \mathbf{R}$ *with* $|\xi| \leqq n^{\delta'}$.

We make some preparations to prove this lemma. First we prove the following.

PROPOSITION 13. *Let*

$$\varphi_0(\xi) = \sum_{k=2}^{K-1} \frac{(i\xi)^k}{k!}\eta_k,$$

and

$$R_0(\xi,t) = \varphi(\xi;\mu(t)) - (1 + \varphi_0(\xi) + \varphi_1(\xi,t)).$$

Then we have for any $n \geqq 8$, *and* $\xi \in \mathbf{R}$ *with* $|\xi| \leqq n^{\delta'}$,

$$|\varphi(n^{-1/2}\xi;\mu(t_n)) - 1| \leqq \frac{2\sqrt{3}}{3}n^{-1/2}|\xi|,$$

$$|\varphi_1(n^{-1/2}\xi,t_n)| \leqq KC(\delta)n^{-\alpha/2+(K+1)\delta}|\xi|$$

and

$$|R_0(n^{-1/2}\xi,t_n)| \leqq 3C(\delta)n^{-\alpha/2-\delta/4}|\xi|.$$

Here $C(\delta)$ *is as in Proposition 8.*

PROOF. We can easily see that

$$\varphi(\xi;\mu(t)) = \int_{\mathbf{R}} \exp(ix\xi)\mu(t)(dx)$$

$$= 1 + \sum_{k=1}^{K} \frac{(i\xi)^k}{k!}\eta_k(t) + \int_{(-\infty,t]} r_{e,K}(x\xi)\mu(dx)$$

$$+ \bar{F}(t)(1-\bar{F}(t))^{-1}\int_{(-\infty,t]} r_{e,0}(x\xi)\mu(dx)$$

So we see that

$$R_0(\xi,t) = \bar{F}(t)(1-\bar{F}(t))^{-1}\int_{(-\infty,t]} r_{e,0}(x\xi)\mu(dx) + \int_{(-\infty,t]} r_{e,K}(x\xi)\mu(dx).$$

By Propositions 8 and 9 we have

$$|\varphi_1(\xi,t)| \leqq C(\delta)\sum_{k=1}^{K}\frac{|\xi|^k}{k!}t^{-\alpha+k+\delta}, \qquad \xi\in\mathbf{R},\ t\geqq 2,$$

and

$$|R_0(\xi,t)| \leqq C(\delta)|\xi|t^{-\alpha+\delta}\int_{\mathbf{R}}|x|\mu(t)(dx) + C(\delta)|\xi|^{K+1}t^{-\alpha+K+1+\delta}, \qquad \xi\in\mathbf{R},\ t\geqq 2.$$

Also, we have

$$|\varphi(\xi;\mu(t))-1| \leqq |\xi|\int_{\mathbf{R}}|x|\mu(t)(dx) \leqq (1-\bar{F}(t))^{-1/2}|\xi| \leqq \frac{2\sqrt{3}}{3}|\xi|, \qquad \xi\in\mathbf{R},\ t\geqq 2.$$

Note that

$$(n^{-1/2+\delta'})^k(n^{1/2-\delta})^{-\alpha+k+\delta} = n^{-\alpha/2+(\alpha+1/2)\delta-k(\delta-\delta')-\delta^2}.$$

So we have our assertion. □

PROPOSITION 14. *Let*

$$\psi_1(\xi) = \sum_{k=3}^{K-1}\frac{(i\xi)^k}{k!}c_k(\eta_2,\dots,\eta_k) + \frac{(i\xi)^K}{K!}c_K(\eta_2,\dots,\eta_{K-1},0), \qquad \xi\in\mathbf{R}.$$

Also, for any $n\geqq 8$, and $\xi\in\mathbf{R}$ with $|\xi|\leqq n^{\delta'}$, let

$$R_1(n,\xi) = \log(\varphi(n^{-1/2}\xi,\mu(t_n))) - \{-\frac{1}{2n}\xi^2 + \psi_1(n^{-1/2}\xi) + \varphi_1(n^{-1/2}\xi,t_n)\}.$$

Then there is a constant $C>0$ such that

$$|R_1(n,\xi)| \leqq Cn^{-\alpha/2-\delta/4}|\xi|$$

for any $n \geqq 8$, *and* $\xi \in \mathbf{R}$ *with* $|\xi| \leqq n^{\delta'}$.

PROOF. Let

$$R_{1,1}(\xi) = \sum_{k=1}^{K} \frac{(-1)^{k-1}}{k} (\varphi_0(\xi))^k + \frac{1}{2}\xi^2 - \psi_1(\xi).$$

Note that

$$\log(1 + \sum_{k=2}^{K-1} \eta_k \frac{z^k}{k!})$$

$$= \sum_{k=2}^{K-1} c_k(\eta_2, \dots, \eta_k) \frac{z^k}{k!} + \sum_{k=K}^{\infty} c_k(\eta_2, \dots, \eta_{K-1}, 0, \dots, 0) \frac{z^k}{k!}$$

as a formal power series of z. So we see that there is a constant $C > 0$ such that

$$|R_{1,1}(\xi)| \leqq C|\xi|^{K+1} \tag{4}$$

for any $\xi \in \mathbf{R}$ with $|\xi| \leqq 1$.

We can easily see that

$$R_1(n, \xi)$$

$$= \log(1 + \varphi_0(n^{-1/2}\xi) + \varphi_1(n^{-1/2}\xi, t_n) + R_0(n^{-1/2}\xi, t_n))$$

$$-\{-\frac{1}{2n}\xi^2 + \psi_1(n^{-1/2}\xi) + \varphi_1(n^{-1/2}\xi, t_n)\}$$

$$= R_{1,1}(n^{-1/2}\xi) + r_{l,K}(\varphi(n^{-1/2}\xi, \mu(t_n)) - 1) + R_0(n^{-1/2}\xi, t_n)$$

$$+ \sum_{k=2}^{K} (-1)^{k-1} (\varphi_0(n^{-1/2}\xi))^{k-1} (\varphi_1(n^{-1/2}\xi, t_n) + R_0(n^{-1/2}\xi, t_n))$$

$$+ \sum_{k=1}^{K} \frac{(-1)^{k-1}}{k} \sum_{j=2}^{k} \binom{k}{j} (\varphi_0(n^{-1/2}\xi))^{k-j} (\varphi_1(n^{-1/2}\xi, t_n) + R_0(n^{-1/2}\xi, t_n))^j.$$

Then we have our assertion from Equation (4) and Proposition 13.

PROPOSITION 15. *Let*

$$R_2(n, \xi) = \exp(n\psi_1(n^{-1/2}\xi)) - (1 + \psi_0(n, \xi)).$$

Then there is a constant $C > 0$ such that

$$|R_2(n,\xi)| \leqq Cn^{-(\alpha-2)/2-1/4}|\xi|$$

for any $n \geqq 8$, and $\xi \in \mathbf{R}$ with $|\xi| \leqq n^{\delta'}$.

PROOF. Note that

$$\exp(y^{-6}(\sum_{k=3}^{K-1} \frac{(y^3 z)^k}{k!} c_k(\eta_2, \dots, \eta_k) + \sum_{k=K}^{\infty} \frac{(y^3 z)^k}{k!} c_k(\eta_2, \dots, \eta_{K-1}, 0, \dots, 0)))$$

$$= 1 + \sum_{k=3}^{K-1} y^k q_k(y^2, \eta_2, \dots, \eta_k) z^k + \sum_{k=K}^{\infty} y^k q_k(\eta_2, \dots, \eta_{K-1}, 0, \dots, 0))) z^k$$

as a formal power series in z. This implies our assertion. $\square$

Now let us prove Lemma 12.

Note that for any $n \geqq 8$, and $\xi \in \mathbf{R}$ with $|\xi| \leqq n^{\delta'}$,

$$\begin{aligned} &\exp(\frac{1}{2}\xi^2)\varphi(n^{-1/2}\xi; \mu(t_n))^n \\ &= \exp(n\varphi_1(n^{-1/2}\xi, t_n) + n\psi_1(n^{-1/2}\xi) + nR_1(n,\xi))) \\ &= (1 + n\varphi_1(n^{-1/2}\xi, t_n) + r_{e,1}(n\varphi_1(n^{-1/2}\xi, t_n))) \\ &\times (1 + \psi_0(n,\xi) + R_2(n,\xi))(1 + r_{e,0}(nR_1(n,\xi))). \end{aligned}$$

So we see that

$$\begin{aligned} R_{n,0}(n,\xi) &= r_{e,0}(nR_1(n,\xi)) \exp(n\varphi_1(n^{-1/2}\xi, t_n) + n\psi_1(n^{-1/2}\xi)) \\ &+ R_2(n,\xi) \exp(n\varphi_1(n^{-1/2}\xi, t_n)) \\ &+ r_{e,1}(n\varphi_1(n^{-1/2}\xi, t_n)) + \psi_0(n,\xi)(n\varphi_1(n^{-1/2}\xi, t_n) + r_{e,1}(n\varphi_1(n^{-1/2}\xi, t_n))). \end{aligned}$$

Thus we have the first equation from Propositions 13, 14, 15.

Also, we have

$$R_{n,1}(n,\xi) = \exp(n\varphi_1(n^{-1/2}\xi, t_n) + n\psi_1(n^{-1/2}\xi) + nR_1(n,\xi))) - 1,$$

and

$$\begin{aligned} R_{n,2}(n,\xi) = \exp(&(n-1)\varphi_1(n^{-1/2}\xi, t_n) + (n-1)\psi_1(n^{-1/2}\xi) \\ &+ (n-1)R_1(n,\xi)) - \frac{\xi^2}{n}) - 1. \end{aligned}$$

So, again from Propositions 13, 14, 15 we have the second equation.

6. Proof of Theorem 2

First, we prove the following.

LEMMA 16. *Let ν be a probability measure on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ such that $\int_{\mathbf{R}} x^2 \nu(dx) < \infty$. Also, assume that there is a constant $C > 0$ such that the characteristic function $\varphi(\cdot, \nu) : \mathbf{R} \to \mathbf{C}$ satisfies*

$$|\varphi(\xi;\nu)| \leqq C(1+|\xi|)^{-2}, \qquad \xi \in \mathbf{R}.$$

Then for any $x \in \mathbf{R}$

$$\nu((x,\infty)) = \Phi_0(x) + \frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{-ix\xi}}{i\xi}(\varphi(\xi,\nu) - \exp(-\frac{\xi^2}{2}))d\xi.$$

PROOF. From the assumption, ν has a continuous density function β and

$$\beta(x) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{-ix\xi}\varphi(\xi,\nu)d\xi.$$

So we have

$$\nu((x, x+n])) = \Phi_0(x) - \Phi_0(x+n)$$

$$+\frac{1}{2\pi}\int_{\mathbf{R}}(\int_x^{x+n} e^{-iz\xi}dz)(\varphi(\xi,\nu) - \exp(-\frac{\xi^2}{2}))d\xi.$$

$$= \Phi_0(x) - \Phi_0(x+n) + \frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{-ix\xi} - e^{-i(x+n)\xi}}{i\xi}(\varphi(\xi,\nu) - \exp(-\frac{\xi^2}{2}))d\xi.$$

Since

$$\int_{\mathbf{R}} \frac{1}{|\xi|}|\varphi(\xi,\nu) - \exp(-\frac{\xi^2}{2})|d\xi < \infty,$$

letting $n \to \infty$, we have the assertion. □

Note that

$$P(\sum_{k=1}^n X_k > sn^{1/2}) = \sum_{m=0}^n I_m(n,s),$$

where

$$I_m(n,s) = P(\sum_{k=1}^n X_k > sn^{1/2}, \sum_{k=1}^n 1_{\{X_k > t_n\}} = m), \qquad m = 0, 1, \dots, n.$$

Then we have

$$I_m(n,s) = \binom{n}{m} P(\sum_{k=1}^{n} X_k > sn^{1/2},\ X_i > t_n,\ i=1,\dots,m,$$
$$X_j \leqq t_n,\ j=m+1,\dots,n),$$

for $m = 0, 1, \dots, n$.

PROPOSITION 17. *There is a $C > 0$ such that*

$$\sum_{m=2}^{n} I_m(n,s) \leqq Cn^{-(\alpha-2)/2-\delta}$$

for any $s \geqq 1$ and $n \geqq 8$.

PROOF. We see that by Proposition 8

$$\sum_{m=2}^{n} I_m(n,s) \leqq \sum_{m=2}^{n} \frac{n(n-1)}{m(m-1)} \binom{n-2}{m-2} \bar{F}(t_n)^m (1-\bar{F}(t_n))^{n-m}$$
$$\leqq \frac{n(n-1)}{2} \bar{F}(t_n)^2 \leqq C(\delta)^2 n^{-(\alpha-2)/2-\delta}.$$

This implies our assertion. □

PROPOSITION 18. *There is a $C > 0$ such that*

$$\sup_{s\in[1,\log n]} |I_0(n,s) - \{(1-n\bar{F}(t_n))\Phi_0(s) - \sum_{k=1}^{K-1} \frac{n^{-(k-2)/2}}{k!} \bar{\eta}_k(t_n)\Phi_k(s)$$
$$+ \frac{(n^{1/2})^{K-2}}{K!} \eta_K(t_n)\Phi_K(s) + g(n,s)\}| \leqq Cn^{-(\alpha-2)/2-\delta/4}$$

for any $n \geqq 8$. Here

$$g(n,s) = \sum_{k=3}^{K-1} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_k)\Phi_k(s)$$
$$+ \sum_{k=K}^{3(K-1)} n^{-k/6} q_k(n^{-1/3}, \eta_2, \dots, \eta_{K-1}, 0, \dots, 0)\Phi_k(s).$$

Proof. Note that

$$I_0(n,s) = (1-\bar{F}(t_n))^n \mu(t_n)^{*n}((sn^{1/2},\infty))$$
$$= I_{0,0}(n,s) + I_{0,1}(n,s) + I_{0,2}(n,s),$$

where

$$I_{0,0}(n,s) = \mu(t_n)^{*n}((sn^{1/2},\infty)),$$
$$I_{0,1}(n,s) = -n\bar{F}(t_n)\mu(t_n)^{*n}((sn^{1/2},\infty)),$$
$$I_{0,2}(n,s) = ((1-\bar{F}(t_n))^n - 1 + n\bar{F}(t_n))\mu(t_n)^{*n}((sn^{1/2},\infty)).$$

We remark that

$$\Phi_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\xi)^{k-1}\exp(-i\xi x - \frac{\xi^2}{2})d\xi, \qquad k = 1,2,\dots .$$

By Proposition 11 and Lemma 16, we have

$$I_{0,0}(n,s)$$
$$= \Phi_0(s) + \frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi}(\varphi(n^{-1/2}\xi,\mu(t_n))^n - \exp(-\frac{\xi^2}{2}))d\xi.$$

Let

$$\tilde{R}_{0,0}(n,s) = I_{0,0}(n,s)$$
$$-\{\Phi_0(s) + \frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi}(\psi_0(n,\xi) + n\varphi_1(n^{-1/2}\xi,t_n))e^{-\xi^2/2}d\xi\}$$

Then by Lemma 12 we have

$$|\tilde{R}_{0,0}(n,s)|$$
$$\leqq \int_{|\xi|\leqq n^{\delta'}} \frac{|R_{n,0}(\xi)|}{|\xi|}\exp(-\frac{\xi^2}{2})d\xi$$
$$+\int_{|\xi|>n^{\delta'}} \frac{1}{|\xi|}(|\varphi(n^{-1/2}\xi,\mu(t_n))|^n + \exp(-\frac{\xi^2}{2}))d\xi$$
$$+\int_{|\xi|>n^{\delta'}} (|\psi_0(n,\xi)| + n|\varphi_1(n^{-1/2}\xi,t_n)|)e^{-\xi^2/2}d\xi$$

So by Proposition 11 and Lemma 12, we see that there is a $C_0 > 0$ such that

$$(5) \qquad |\tilde{R}_{0,0}(n,s)| \leqq C_0 n^{-(\alpha-2)/2-\delta/4}, \qquad n \geqq 8,\ s \geqq 1.$$

Also, we see that

$$\frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi} n\varphi_1(n^{-1/2}\xi, t_n)\exp(-\frac{1}{2}\xi^2)d\xi$$

$$= -\sum_{k=1}^{K-1} \frac{(n^{-1/2})^{k-2}}{k!}\bar{\eta}_k(t_n)\Phi_k(s) + \frac{(n^{-1/2})^{K-2}}{K!}\eta_K(t_n)\Phi_K(s),$$

and

$$\frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi}\psi_0(n,\xi)\exp(-\frac{1}{2}\xi^2)d\xi = g(n,s)$$

Similarly by Lemma 12, we see that there is a $C_1 > 0$ such that

$$(6) \qquad \sup_{s\in[1,\log n]} |I_{0,1}(n,s) - n\bar{F}(t_n)\Phi_0(s)| \leqq C_1 n^{-(\alpha-2)/2-\delta}, \qquad n \geqq 8.$$

Note that $|(1-x)^n - (1-nx)| \leqq n^2x^2$ for any $x \in [0,1]$, $n \geqq 1$. So we have

$$|I_{0,2}(n,s)| \leqq n^2\bar{F}(t_n)^2 \leqq C(\delta)^2 n^{-(\alpha-2)/2-\delta}.$$

This and Equations 5, 6 imply our assertion. □

Proposition 19. *There is a $C > 0$ such that*

$$\sup_{s\in[1,\log n]} |I_1(n,s) - \{n\int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx + n\bar{F}(t_n)\Phi_0(s)$$

$$-\sum_{k=1}^{K} \frac{n^{-(k-2)/2}}{k!}\Phi_k(s)\int_0^{t_n} x^k\mu(dx)\}| \leqq Cn^{-(\alpha-2)/2-\delta/4}.$$

Proof. We see that

$$I_1(n,s) = n(1-\bar{F}(t_n))^{n-1}\int_{\mathbf{R}} P(X_1 + x > sn^{1/2},\ X_1 > t_n)\ \mu(t_n)^{*(n-1)}(dx)$$

$$= n(1 - \bar{F}(t_n))^{n-1} \int_{-\infty}^{\infty} \bar{F}((sn^{1/2} - x) \vee t_n)\mu(t_n)^{*(n-1)}(dx)$$
$$= nJ_0(n,s) + nJ_1(n,s) + nJ_2(n,s),$$

where

$$J_0(n,s) = \int_{-\infty}^{\infty} \bar{F}((s-x)n^{1/2} \vee t_n)\Phi_1(x)dx, \tag{7}$$

$$\begin{aligned} &J_1(n,s) \\ &= \int_{-\infty}^{\infty} \bar{F}((sn^{1/2} - x) \vee t_n)(\mu(t_n)^{*(n-1)}(dx) - n^{-1/2}\Phi_1(xn^{1/2})dx), \end{aligned} \tag{8}$$

and

$$J_2(n,s) = -(1 - (1 - \bar{F}(t_n))^{n-1})I_1(n,s). \tag{9}$$

Note that

$$J_0(n,s) = J_{0,0}(n,s) + J_{0,1}(n,s) + J_{0,2}(n,s),$$

where

$$J_{0,0}(n,s) = \int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx,$$
$$J_{0,1}(n,s) = -\int_{s-n^{-\delta}}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx$$
$$= -\int_{0}^{n^{-\delta}} \bar{F}(xn^{1/2})\Phi_1(s-x)dx,$$

and

$$J_{0,2}(n,s) = \bar{F}(t_n)\int_{s-n^{-\delta}}^{\infty} \Phi_1(x)dx = \bar{F}(t_n)\Phi_0(s - n^{-\delta}).$$

We see that

$$J_{0,1}(n,s) = -\sum_{k=1}^{K} \frac{1}{(k-1)!}\Phi_k(s)\int_0^{n^{-\delta}} \bar{F}(xn^{1/2})x^{k-1}dx + R_{J,1}(n,s),$$

where

$$R_{J,1}(n,s) = -\int_0^{n^{-\delta}} \bar{F}(xn^{1/2})(\Phi_1(s-x) - \sum_{k=1}^{K} \frac{x^{k-1}}{(k-1)!}\Phi_k(s))dx.$$

Then

$$|R_{J,1}(n,s)|$$

$$\leqq \sup_{x\in[0,n^{-\delta}]} |\Phi_{K+1}(s-x)|(\int_{n^{-1/2}}^{n^{-\delta}} x^K (xn^{1/2})^{-\alpha} L(xn^{1/2})dx + \int_0^{n^{-1/2}} x^K dx).$$

$$\leqq \sup_{x\in\mathbf{R}} |\Phi_{K+1}(x)|(C(\delta)n^{-\alpha/2+\delta/2}\int_0^{n^{-\delta}} x^{\delta+(K-\alpha)}dx + n^{-(K+1)/2})$$

$$\leqq \sup_{x\in\mathbf{R}} |\Phi_{K+1}(x)|(C(\delta)+1)n^{-\alpha/2-\delta/2}. \tag{10}$$

Also, we see that

$$J_{0,2}(n,s)$$

$$= \bar{F}(t_n)\Phi_0(s) + \sum_{k=1}^K \bar{F}(t_n)\frac{(n^{-\delta})^k}{k!}\Phi_k(s) + R_{J,2}(n,s),$$

where

$$R_{J,2}(n,s) = \bar{F}(t_n)(\Phi_0(s-n^{-\delta}) - \sum_{k=0}^K \frac{(-n^{-\delta})^k}{k!}\frac{d^k\Phi_0}{dx^k}(s)).$$

We see that

$$\begin{aligned}|R_{J,2}(n,s)| &\leqq \bar{F}(t_n)n^{-(K+1)\delta}\sup_{x\in\mathbf{R}}|\Phi_{K+1}(x)| \\ &\leqq C(\delta)\sup_{x\in\mathbf{R}}|\Phi_{K+1}(x)|n^{-\alpha/2-\delta/4}.\end{aligned} \tag{11}$$

It is easy to see that

$$\int_0^{n^{-\delta}} \bar{F}(xn^{1/2})x^{k-1}dx = n^{-k/2}\int_0^{t_n}\bar{F}(x)x^{k-1}dx$$

$$= n^{-k/2}(-\frac{1}{k}\int_0^{t_n} x^k\mu(dx) + \frac{n^{\delta k}}{k}\bar{F}(t_n)), \qquad k=1,\dots,K.$$

So we have

$$J_{0,1}(n,s) + J_{0,2}(n,s) = \bar{F}(t_n)\Phi_0(s)$$

$$(12) \qquad -\sum_{k=1}^{K} \frac{n^{-k/2}}{k!} \Phi_k(s) \int_0^{t_n} x^k \mu(dx) + R_{J,1}(n,s) + R_{J,2}(n,s)$$

Also, we have

$$J_1(n,s) = J_{1,1}(n,s) + J_{1,2}(n,s)$$

where

$$J_{1,1}(n,s) = \bar{F}(t_n)(\mu(t_n)^{*(n-1)}(((s-n^{-\delta})n^{1/2}, \infty)) - \Phi_0(s-n^{-\delta}))$$

and

$$J_{1,2}(n,s) = \int_{-\infty}^{s-n^{-\delta}} dx \bar{F}((s-x)n^{1/2})$$

$$\times \frac{1}{2\pi} \int_{\mathbf{R}} e^{-ix\xi}(\varphi(n^{-1/2}\xi; \mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2}))d\xi$$

By Proposition 11 and Lemma 16, we see that there is a $C_1 > 0$ such that

$$|\mu(t_n)^{*(n-1)}((xn^{1/2}, \infty)) - \Phi_0(x)|$$

$$\leqq |\int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{\xi}(\varphi(n^{-1/2}\xi; \mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2}))d\xi|$$

$$\leqq \int_{|\xi|>n^{\delta'}} \frac{1}{|\xi|}(|\varphi(\xi; \mu(t_n))|^{n-1} + \exp(-\frac{\xi^2}{2}))d\xi$$

$$+ \int_{|\xi|<n^{\delta'}} \frac{1}{|\xi|}|R_{n,2}(\xi)| \exp(-\frac{\xi^2}{2}))d\xi$$

$$\leqq C_1 n^{-2K\delta}, \text{ for any } x \in \mathbf{R} \text{ and } n \geqq 8.$$

Therefore we have

$$|J_{1,1}(n,s)| \leqq C_1 \bar{F}(t_n) n^{-2K\delta} \leqq C(\delta) C_1 n^{-\alpha/2-\delta}.$$

Similarly by Lemma 16, we see that there is a $C_2 > 0$ such that

$$|\int_{\mathbf{R}} e^{-ix\xi}(\varphi(n^{-1/2}\xi; \mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2}))d\xi|$$

$$\leqq C_2 n^{-2K\delta}, \text{ for any } x \in \mathbf{R} \text{ and } n \geqq 8.$$

Then we have

$$|J_{1,2}(n,s)| \leqq C_2 n^{-2K\delta} C(\delta) \int_{n^{-\delta}}^{\infty} (xn^{1/2})^{-\alpha+\delta} dx \leqq C_2 C(\delta) n^{-\alpha/2-\delta}$$

So we see that there is a $C > 0$ such that

$$\sup_{s \in [1, \log n]} |J_1(n,s)| \leqq C n^{-(\alpha-2)/2-\delta} \tag{13}$$

Note that

$$|J_2(n,s)| \leqq n^2 \bar{F}(t_n)^2 \tag{14}$$

So Equations (7) - (14) imply our assertion. $\square$

PROPOSITION 20. *Then there is a $C > 0$ such that*

$$\sup_{s \in [1, \log n)} |P(\sum_{k=1}^{n} X_k > sn^{1/2}) - G(n,s)| \leqq C n^{-(\alpha-2)/2-\delta/4}, \qquad n = 3, 4, \ldots .$$

PROOF. Note that

$$\bar{\eta}_k(t_n) + \int_0^{t_n} x^k \mu(dx) = \int_0^{\infty} x^k \mu(dx), \qquad k = 1, 2, \ldots, K-1,$$

and

$$\eta_K(t_n) - \int_0^{t_n} x^K \mu(dx) = \int_{-\infty}^{0} x^K \mu(dx).$$

So our assetion is an easy consequence of Propositions 17, 18, 19. $\square$

PROPOSITION 21. *There is a $C > 0$ such that*

$$\sup_{s \in [\log n, \infty)} I_0(n,s) \leqq C n^{-(\alpha-2)/2-\delta/4}, \qquad n = 3, 4, \ldots .$$

PROOF. We have

$$I_0(n,s) \leqq \mu(t_n)^{*n}((sn^{1/2}, \infty))$$

$$= \Phi_0(s) + \frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi}(\varphi(n^{-1/2}\xi, \mu(t_n))^n - \exp(-\frac{\xi^2}{2}))d\xi.$$

$$= \Phi_0(s) + \frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi}(\varphi(n^{-1/2}\xi, \mu(t_n))^n - \exp(-\frac{\xi^2}{2}))d\xi.$$

$$\leqq \Phi_0(s) + \frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi}(\psi_0(n,\xi) + n\varphi_1(n^{-1/2}\xi, t_n)\exp(-\frac{\xi^2}{2}))d\xi.$$

$$+\frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{-is\xi}}{i\xi}R_{n,0}(\xi)\exp(-\frac{\xi^2}{2}))d\xi.$$

Since $\sup_{s \geqq \log n} |\Phi_k(s)|$ is of $O(n^{-M})$ for any $M \geqq 1$, we have our assetion similar to the proof of Proposition 18.

Proposition 22. *There is a $C > 0$ such that*

$$\sup_{s\in[\log n,\infty)} |I_1(n,s) - n\int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx| \leqq Cn^{-(\alpha-2)/2-\delta/4}.$$

Proof. Remind that

$$I_1(n,s) = n(1-\bar{F}(t_n))^{n-1}\int_{-\infty}^{\infty} \bar{F}((sn^{1/2}-x)\vee t_n)\mu(t_n)^{*(n-1)}(dx)$$

$$= n(1-\bar{F}(t_n))^{n-1}\{\int_{-\infty}^{\infty} \bar{F}(((s-x)n^{1/2})\vee t_n)\Phi_1(x)dx$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{F}(((s-x)n^{1/2})\vee t_n)$$

$$\times(\int_{\mathbf{R}} \frac{e^{-ix\xi}}{i\xi}(\varphi(n^{-1/2}\xi, \mu(t_n))^{n-1} - \exp(-\frac{\xi^2}{2}))d\xi\}.$$

Then similarly to the proof of Propositions 19 and 21, we have our assertion. □

Now Theorem 2 is a consequence of Propositions 20 and 22.

7. Preliminary for Theorem 3

PROPOSITION 23. *Let Y be a random variable , and assume that*

$$E[|Y|^2] < \infty \quad \textit{and} \qquad E[Y] = 0.$$

Then for any $s \in \mathbf{R} \setminus \{0\}$ and $b > 0$

$$E[\exp(sY1_{\{|Y| \leqq b\}})] \leqq 1 + |s|^2(1 + \frac{1}{|s|b})\exp(|s|b)E[|Y|^2].$$

PROOF. First, note that

$$|\exp(x) - 1| \leqq 1 \vee \exp(x),$$

and

$$|\exp(x) - 1| = |\int_0^x e^y dy| \leqq |x|(1 \vee \exp(x)), \qquad x \in \mathbf{R}.$$

So we have

$$|\exp(x) - (1 + x)| = |\int_0^x (e^y - 1)dy| \leqq (|x| \wedge |x|^2)(1 \vee \exp(x))$$

for any $x \in \mathbf{R}$. Therefore we see that

$$|\exp(x) - (1 + x)| \leqq |x|^2 \exp(|x|), \qquad x \in \mathbf{R}.$$

This implies that

$$|E[\exp(sY1_{\{|Y| \leqq b\}})] - (1 + E[sY1_{\{|Y| \leqq b\}}])|$$
$$\leqq |s|^2 \exp(|s|b)E[|Y|^2].$$

Since

$$|E[sY1_{\{|Y| \leqq b\}}]| = |sE[Y, |Y| > b]| \leqq |s|b^{-1}E[|Y|^2],$$

we have our assertion. □

PROPOSITION 24. *Let X be a random variable and assume that*

$$E[|X|^2] < \infty \textit{ and } E[X] = 0.$$

Then for any $t > 0$ *and* $n \geqq 1$

$$n \log E[\exp(\pm \frac{1}{tn^{1/2}} X 1_{\{|X| \leqq tn^{1/2}\}})] \leqq \frac{6}{t^2} E[|X|^2].$$

PROOF. Let $Y = (1/t)X$, $s = \pm n^{-1/2}$, $b = n^{1/2}$, and apply Proposition 23. Since $\log(1+x) \leqq x$, $x \geqq 0$, we have our assertion. □

Now let X_n, $n = 1, 2, \dots$, be independent identically distributed random variables. Throughout this section we assume that

$$E[|X_1|^2] < \infty \text{ and } E[X_1] = 0.$$

PROPOSITION 25. *For any* $s, t > 0$ *and* $\varepsilon > 0$

$$P(|\sum_{k=1}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}}| \geqq sn^{1/2}) \leqq 2 \exp(\frac{6}{t^2} E[|X_1|^2]) \exp(-\frac{s}{t}).$$

PROOF. We see that

$$P(\pm \sum_{k=1}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}} \geqq sn^{1/2})$$

$$\leqq \exp(-\frac{s}{t}) E[\exp(\frac{\pm 1}{tn^{1/2}} \sum_{k=1}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}})]$$

$$\leqq \exp(-\frac{s}{t}) E[\exp(\frac{\pm 1}{tn^{1/2}} X_1 1_{\{|X_1| \leqq tn^{1/2}\}})]^n.$$

Then by Proposition 24 we have our assertion.

Let $F : \mathbf{R} \to [0, 1]$ and $\bar{F} : \mathbf{R} \to [0, 1]$ be given by

$$F(x) = P(X_1 \leqq x), \qquad x \in \mathbf{R}$$

and

$$\bar{F}(x) = P(X_1 > x), \qquad x \in \mathbf{R}.$$

Then we have the following.

PROPOSITION 26. (1) *For any* $t, s > 0$, *and* $n \geqq 2$,

$$P(|\sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}}| > sn^{1/2}) \leqq 2\exp(\frac{6}{t^2}E[|X_1|^2])\exp(-\frac{s}{t}).$$

(2) *For any* $s, t > 0$, $\varepsilon \in (0,1)$ *with* $t < (1-\varepsilon)s$,

$$|P(\sum_{k=1}^{n} X_k > sn^{1/2}) - nP(X_1 + \sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}} > sn^{1/2},$$

$$|\sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}}| \leqq \varepsilon sn^{1/2})|$$

$$\leqq 2n(n-1)(F(-tn^{1/2}) + \bar{F}(tn^{1/2}))^2 + 2\exp(\frac{6}{t^2}E[|X_1|^2])\exp(-\frac{s}{t})$$

$$+2n(F(-tn^{1/2}) + \bar{F}(tn^{1/2}))\exp(\frac{6}{t^2}E[|X_1|^2])\exp(-\frac{\varepsilon s}{2t})$$

PROOF. Note that

$$P(|\sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}}| > sn^{1/2})$$

$$= P(|\sum_{k=1}^{n-1} X_k 1_{\{|X_k| \leqq \tilde{t}(n-1)^{1/2}\}}| > \tilde{s}(n-1)^{1/2}),$$

where

$$\tilde{t} = t(\frac{n}{n-1})^{1/2}, \qquad \tilde{s} = s(\frac{n}{n-1})^{1/2}.$$

So we have the assertion (1) from Proposition 25.

Let us denote

$$\tilde{F}(x) = P(|X_1| > x) \leqq F(-x) + \bar{F}(x), \qquad x > 0.$$

Note that

$$P(\sum_{k=1}^{n} X_k > sn^{1/2}) = \sum_{m=0}^{n} I_m,$$

where

$$I_m = P(\sum_{k=1}^{n} X_k > sn^{1/2}, \sum_{k=1}^{n} 1_{\{|X_k|>tn^{1/2}\}} = m), \qquad m = 0, 1, \ldots, n.$$

Then we have

$$I_m = \binom{n}{m} P(\sum_{k=1}^{n} X_k > sn^{1/2},\ |X_i| > tn^{1/2},\ i = 1, \ldots, m,$$
$$|X_j| \leqq tn^{1/2},\ j = m+1, \ldots, n),$$

for $m = 0, 1, \ldots, n$. So we see that

$$\sum_{m=2}^{n} I_m \leqq \sum_{m=2}^{n} \frac{n(n-1)}{m(m-1)} \binom{n-2}{m-2} \tilde{F}(tn^{1/2})^m (1 - \tilde{F}(tn^{1/2}))^{n-m}$$
$$\leqq \frac{n(n-1)}{2} \tilde{F}(tn^{1/2})^2. \tag{15}$$

Also, by Proposition 25, we have

$$I_0 \leqq 2 \exp(-\frac{s}{t}) \exp(\frac{6}{t^2} E[|X_1|^2]). \tag{16}$$

Let

$$A_1 = \{|X_1| > tn^{1/2}\}, \qquad A_2 = \{|X_k| \leqq tn^{1/2}, k = 2, 3, \ldots, n\},$$
$$B_1 = \{X_1 + \sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}} > sn^{1/2}\},$$

and

$$B_2 = \{|\sum_{k=2}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}}| \leqq \varepsilon sn^{1/2}\}.$$

Note that $B_1 \cap B_2 \subset A_1$, since $t < (1-\varepsilon)s$. So we see that

$$|P(B_1 \cap A_1 \cap A_2) - P(B_1 \cap B_2)|$$
$$\leqq P(B_1 \cap B_2^c \cap A_1 \cap A_2) + P(B_1 \cap B_2 \cap A_1 \cap A_2^c)$$
$$\leqq P(A_1)P(B_2^c) + P(A_1)P(A_2^c). \tag{17}$$

Note that

$$P(A_2^c) \leqq \sum_{k=2}^{n} P(|X_k| > tn^{1/2}) = (n-1)\tilde{F}(tn^{1/2}).$$

Also, by the assertion (1) we have

$$P(B_2^c) \leqq 2\exp(\frac{6}{t^2}E[|X_1|^2])\exp(-\frac{\varepsilon s}{2t}).$$

Since $I_1 = nP(B_1 \cap A_1 \cap A_2)$, we have the assertion from Equations (15), (16) and (17).

This completes the proof. □

8. Some Estimates

In this section, we assume that (A-1) and (A-5).

Let $g : (x_0, \infty) \to \mathbf{R}$, $H : [-1/2, 1/2] \times (2x_0, \infty) \to (0, \infty)$ and $R : [-1/2, 1/2] \times (2x_0, \infty) \to (0, \infty)$ be given by

$$g(x) = x^2 \frac{d^2}{dx^2}(\log \bar{F})(x) - \alpha, \qquad x > x_0,$$

$$H(y; x) = \frac{\bar{F}(x(1+y))}{\bar{F}(x)}, \qquad y \in [-1/2, 1/2],\ x > 2x_0,$$

and

$$R(y; x) = H(y; x) - \{1 - \alpha y + \frac{\alpha(\alpha+1)y^2}{2}\}, \qquad y \in [-1/2, 1/2],\ x > 2x_0,$$

We prove the following in this section.

Proposition 27. *There are functions $a : (2x_0, \infty) \to \mathbf{R}$, $c : (2x_0, \infty) \to [0, \infty)$ and a constant $C > 0$ such that $a(x) \to 0$ and $c(x) \to 0$, as $x \to \infty$, and that*

$$|R(y; x) - a(x)y| \leqq C(c(x)y^2 + |y|^3), \qquad y \in [-1/2, 1/2],\ x > 2x_0.$$

First we prove the following.

PROPOSITION 28. (1) *For any* $x > x_0$,

$$\frac{d}{dx}\log(x^\alpha \bar{F}(x)) = -\int_x^\infty \frac{g(z)}{z^2}.$$

(2) *For any* $y \in [-1/2, 1/2]$ *and* $x > 2x_0$,

$$H(y; x) = (1+y)^{-\alpha}\exp(-\int_0^y dy' \int_{1+y'}^\infty \frac{g(xz)}{z^2}dz).$$

PROOF. Note that

$$g(x) = x^2 \frac{d^2}{dx^2}(\log(x^\alpha \bar{F}(x)))$$

and $g(x) \to 0$ as $x \to \infty$. Then we see that

$$\frac{d}{dy}(\log(y^\alpha \bar{F}(y))) - \frac{d}{dx}(\log(x^\alpha \bar{F}(x))) = \int_x^y \frac{g(z)}{z^2}dz, \tag{18}$$

and so we see that

$$c_0 = \lim_{y\to\infty} \frac{d}{dy}(\log(y^\alpha \bar{F}(y)))$$

exists. Note that

$$\exp(\int_x^{2x} \frac{d}{dy}(\log(y^\alpha \bar{F}(y)))dy) = \frac{L(2x)}{L(x)} \to 1, \qquad x \to \infty.$$

So we see that $c_0 = 0$. Therefore letting $y \to \infty$ in Equation (18) we have the assertion (1).

By the assertion (1), we have

$$\frac{d}{dy}\log((1+y)^\alpha H(y; x)) = -x\int_{x(1+y)}^\infty \frac{g(z)}{z^2}dz = -\int_{1+y}^\infty \frac{g(xz)}{z^2}dz.$$

Since $H(0; x) = 1$, we have the assertion (2). □

PROPOSITION 29. *Let* $\tilde{a} : (2x_0, \infty) \to \mathbf{R}$ *and* $\tilde{c} : (2x_0, \infty) \to \mathbf{R}$ *be given by*

$$\tilde{a}(x) = \frac{d}{dy}((1+y)^\alpha H(y, x))|_{y=0},$$

and

$$\tilde{c}(x) = \sup_{y \in [-1/2, 1/2]} |\frac{d^2}{dy^2}((1+y)^\alpha H(y,x))|.$$

Then $\tilde{a}(x) \to 0$ *and* $\tilde{c}(x) \to 0,$ *as* $x \to \infty,$ *and that*

$$|H(y;x) - (1+y)^{-\alpha} - \tilde{a}(x) y (1+y)^{-\alpha}| \leqq 2^\alpha \tilde{c}(x) y^2, \qquad y \in [-1/2, 1/2], \ x > 2x_0.$$

Proof. By Proposition 11 We have

$$\frac{d}{dy}((1+y)^\alpha H(y;x)) = -(1+y)^\alpha H(y;x) \int_{1+y}^\infty \frac{g(xz)}{z^2} dz$$

and so

$$\tilde{a}(x) = -\int_1^\infty \frac{g(xz)}{z^2} dz$$

Similarly, we have

$$\frac{d^2}{dy^2}((1+y)^\alpha H(y;x))$$

$$= (1+y)^\alpha H(y;x) \{ (\int_{1+y}^\infty \frac{g(xz)}{z^2} dz)^2 - (1+y)^{-2} g(x(1+y)) \}$$

Therefore we have

$$\tilde{c}(x) \leqq 2^\alpha \{ (\int_{1/2}^\infty \frac{|g(xz)|}{z^2} dz)^2 + 4 \sup\{ |g(x(1+y))|; \ y \in [-1/2, 1/2] \} \}$$

$$\times \exp(\int_{1/2}^\infty \frac{|g(xz)|}{z^2} dz)$$

These imply that $\tilde{a}(x) \to 0$, $\tilde{c}(x) \to 0$, as $x \to \infty$. Also we have

$$|(1+y)^\alpha H(y;x) - (1 + \tilde{a}(x))| \leqq \tilde{c}(x) y^2, \qquad x \geqq 2x_0, \ y \in [-1/2.1/2].$$

This implies our assetion. $\square$

Now Proposition 27 is an easy corollary to Proposition 29.

9. Proof of Thoerem 3.

In this section, we assume that X_n, $n = 1, 2, \dots$, are i.i.d. random variables, $\alpha > 2$ and (A-1) - (A-5) are satisfied. Let $p = (\alpha + 2)/2$ and $\beta = (\alpha + p)/2$. Then we see that $E[|X_1|^p] < \infty$ and there is a $C_0 > 1$ such that

$$F(-x) + \bar{F}(x) \leqq C_0 x^{-\beta}, \qquad x \geqq 1.$$

Proposition 30. *Let $b(x) = E[X_1, |X_1| \leqq x] = -E[X_1, |X_1| > x]$, $x > 0$. Then we have the following.*
(1) $|b(x)| \leqq E[|X_1|^p]^{1/p}(F(-x) + \bar{F}(x))^{1-1/p} \leqq C_0 x^{-\beta(p-1)/p} E[|X_1|^p]^{1/p}$, $x \geqq 1$.
(2) *There is a constant $C_1 > 1$ only dependent on p such that*

$$E[|\sum_{k=1}^{n} X_k 1_{\{|X_k| \leqq x\}}|^p]^{1/p} \leqq C_1 n^{1/2}(E[|X_1|^p]^{1/p} + |b(x)|) + n|b(x)|$$

$$\leqq C_1 E[|X_1|^p]^{1/p}(1 + C_0)(n^{1/2} + n x^{-\beta(p-1)/p})$$

for any $n = 1, 2, \dots$, and $x \geqq 1$.

Proof. The assertion (1) is an easy consequence of Hölder's inequality. So we prove the assertion (2). Since $E[X_k 1_{\{|X_k| \leqq x\}} - b(x)] = 0$, $k = 1, 2, \dots$, we see by Burkholder-Davis-Gundy's theorem that there is a constant $C_1 > 0$ depending on p only such that

$$E[|\sum_{k=1}^{n}(X_k 1_{\{|X_k| \leqq x\}} - b(x))|^p]^{1/p} \leqq C_1 E[|\sum_{k=1}^{n}(X_k 1_{\{|X_k| \leqq x\}} - b(x))^2|^{p/2}]^{1/p}$$

Then by Hölder's inequality, we have

$$E[|\sum_{k=1}^{n}(X_k 1_{\{|X_k| \leqq x\}} - b(x))|^p]^{1/p} \leqq C_1 E[n^{p/2-1} \sum_{k=1}^{n} |X_k 1_{\{|X_k| \leqq x\}} - b(x)|^p]^{1/p}$$

$$= C_1 n^{1/2} E[|X_1 1_{\{|X_1| \leqq x\}} - b(x)|^p]^{1/p} \leqq C_1 n^{1/2}(E[|X_1 1_{\{|X_1| \leqq x\}}|^p]^{1/p} + |b(x)|)$$

This implies our assertion. □

Let $a : (2x_0, \infty) \to \mathbf{R}$ and $c : (2x_0, \infty) \to [0, \infty)$ be as in Proposition 27. Also, let

$$Y_n(t) = \sum_{k=1}^{n} X_k 1_{\{|X_k| \leqq tn^{1/2}\}}, \qquad n \geqq 2,\ t > 0.$$

Then we have the following.

PROPOSITION 31. *Let $r \in ((\alpha+2)/(2\alpha), 1)$. Then for any $\varepsilon \in (0, 1/2)$*

$$\begin{aligned}\overline{\lim}_{n\to\infty} \sup\{s^2 E[|H(-\frac{1}{sn^{1/2}}Y_n(t), sn^{1/2}),\\ |Y_n(t)| \leqq \varepsilon sn^{1/2}] - (1 + \frac{\alpha(\alpha+1)}{2s^2})|;\\ s \geqq (\log n)^{1/2},\ t \geqq (\log n)^{-1} s^{(1+r)/2}\} = 0.\end{aligned}$$

PROOF. Let $s \geqq (\log n)^{1/2}$, $t \geqq (\log n)^{-1} s^{(1+r)/2}$, and $n \geqq 3$. Then $tn^{1/2} \geqq 1$. Note that

$$r\beta(p-1)/p > 1 + \frac{3(\alpha-2)}{8} > 1.$$

We see that

$$\begin{aligned}&E[H(-\frac{1}{sn^{1/2}}Y_n(t), sn^{1/2}),\ |Y_n(t)| \leqq \varepsilon sn^{1/2}] - (1 + \frac{\alpha(\alpha+1)}{2s^2})\\ &= \frac{a(sn^{1/2}) - \alpha}{sn^{1/2}} E[Y_n(t)] + \frac{\alpha(\alpha+1)}{2s^2 n}(E[Y_n(t)^2] - n)\\ &-E[1 + \frac{a(sn^{1/2}) - \alpha}{sn^{1/2}} Y_n(t) + \frac{\alpha(\alpha+1)}{2s^2 n} Y_n(t)^2, |(sn^{1/2})^{-1} Y_n(t)| > \varepsilon]\\ &\quad + E[R(-\frac{1}{sn^{1/2}}Y_n(t), sn^{1/2}), |(sn^{1/2})^{-1} Y_n(t)| \leqq \varepsilon].\end{aligned}$$

Note that

$$\begin{aligned}s|E[Y_n(t)]| &= ns|b(tn^{1/2})| \leqq C_0 E[|X_1|^p]^{1/p} s(tn^{1/2})^{-\beta(p-1)/p}\\ &\leqq C_0 E[|X_1|^p]^{1/p} (n^{1/2}(\log n)^{-1})^{-\beta(p-1)/p} s^{1-(1+r)\beta(p-1)/2p},\end{aligned}$$

$$E[Y_k(t)^2] - n = n(E[(X_1 1_{|X_1| \leqq tn^{1/2}})^2] - b(tn^{1/2})^2) + E[Y_n(t)]^2 - n$$

$$= -nE[X_1^2, |X_1| > tn^{1/2}] + n(n-1)b(tn^{1/2})^2,$$

and

$$n^{-p/2}E[|Y_n(t)|^p] \leqq C_1^p(1+C_0)^p E[|X_1|^p](1+t^{-\beta(p-1)/p)}n^{1/2(1-\beta(p-1)/p)})^p.$$

So we see that

$$\begin{aligned}\frac{1}{n}|E[Y_k(t)^2] - n| &\leqq E[X_1^2, |X_1| > tn^{1/2}] \\ &\quad + C_0(n-1)n^{-\beta(p-1)/p}(\log n)^{2\beta(p-1)/p}E[|X_1|^p]^{2/p}, \\ s^2(sn^{1/2})^{-k}&E[|Y_n(t)|^k, |(sn^{1/2})^{-1}Y_n(t)| > \varepsilon] \\ &\leqq s^{2-p}\varepsilon^{-p+k}n^{-p/2}E[|Y_n(t)|^p], \qquad k = 0,1,2,\end{aligned}$$

and

$$\begin{aligned}s^2 E[|R(-\frac{1}{sn^{1/2}}Y_n(t), sn^{1/2})|, |(sn^{1/2})^{-1}Y_n(t)| \leqq \varepsilon], \\ \leqq C_2(|c(sn^{1/2})|n^{-1}E[Y_n(t)^2] + \varepsilon^{-p}s^{2-p}n^{-p/2}E[|Y_n(t)|^p]).\end{aligned}$$

Combining them, we have our assertion. □

Now we prove Theorem 3. Let $\beta : \mathbf{N} \to (0,\infty)$ be such that

$$\frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \qquad n \to \infty.$$

Assume that Theorem 3 is not valid. Then there is a sequence of positive numbers $\{s'_n\}_{n=1}^\infty$ such that $s'_n \geqq n^{1/2}\beta(n)$, $n = 1,2,\dots$, and

$$\overline{\lim}_{n\to\infty}\frac{(s'_n)^2}{n}|\frac{P(\sum_{k=1}^n X_k > s'_n)}{n\bar{F}(s'_n)} - (1 + \frac{\alpha(\alpha+1)vn}{(s'_n)^2})| > 0.$$

Let $s_n = n^{-1/2}s'_n \geqq \beta(n)$. Let us take an $r \in ((\alpha+2)/(2\alpha), 1)$ and fix it. Let t_n, $n = 1,2,\dots$, be a sequence of positive numbers given by

$$t_n = (\log n)^{-1/2} + (\log n)^{-1}s_n^{(1+r)/2}, \qquad n \geqq 2.$$

Then we have the following.

$$t_n s_n \geqq \frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \quad n \to \infty, \tag{19}$$

$$\frac{2s_n}{t_n} \geqq ((\log n)^{1/2} s_n) \wedge ((\log n) s_n^{(1-r)/2}), \quad n \geqq 2, \tag{20}$$

$$\frac{2s_n}{(\log n)t_n} \geqq (\frac{\beta(n)}{(\log n)^{1/2}}) \wedge (s_n^{(1-r)/2}) \to \infty, \quad n \to \infty, \tag{21}$$

and

$$\frac{(t_n n^{1/2})^2}{(s'_n)^{1+r}} \geqq \frac{(\log n)^{-2} s_n^{1+r} n}{s_n^{1+r} n^{(1+r)/2}} \to \infty, \quad n \to \infty. \tag{22}$$

Therefore by Equation (22), we have

$$\frac{(F(-t_n n^{1/2}) + \bar{F}(t_n n^{1/2}))^2}{\bar{F}(s'_n)^{2r}} \to 0, \quad n \to \infty.$$

Since $2r - 1 > 2/\alpha$, we have

$$(s'_n)^2 \frac{(F(-t_n n^{1/2}) + \bar{F}(t_n n^{1/2}))^2}{\bar{F}(s'_n)} \to 0, \quad n \to \infty. \tag{23}$$

Also, by Equations (19), (20) and (21) we see that for any $m \geqq 1$

$$(ns'_n)^m \exp(\frac{m}{t_n^2} - \frac{1}{m}\frac{s_n}{t_n})$$

$$= \exp(\frac{m}{t_n^2}(1 - \frac{1}{3m^2} t_n s_n)) n^{3m/2} \exp(-(\log n)\frac{1}{3m}\frac{s_n}{(\log n)t_n})$$

$$\times (s_n)^m \exp(-\frac{1}{3m}\frac{s_n}{t_n}) \to 0, \qquad n \to \infty.$$

Note that

$$\frac{(n+1)P(X_{n+1} + Y_n(t_{n+1}) > s_{n+1}(n+1)^{1/2},\ |Y_n(t_{n+1})| \leqq \varepsilon s_{n+1}(n+1)^{1/2})}{(n+1)\bar{F}(s_{n+1}(n+1)^{1/2})}$$

$$= E[H(-\frac{1}{s_{n+1}(n+1)^{1/2}} Y_n(t_{n+1}), s_{n+1}(n+1)^{1/2}),\ |Y_n(t_{n+1})| \leqq \varepsilon s_{n+1}(n+1)^{1/2}].$$

From Proposition 31, we see that

$$s_{n+1}^2 |E[H(-\frac{1}{s_{n+1}(n+1)^{1/2}} Y_n(t_{n+1}), s_{n+1}(n+1)^{1/2}),$$

$$|Y_n(t_{n+1})| \leqq \varepsilon s_{n+1}(n+1)^{1/2}]$$

$$- (1 + \frac{\alpha(\alpha+1)(n+1)}{s_{n+1}^2(n+1)})| \to 0$$

as $n \to \infty$, by letting $s = s_{n+1}(1 + 1/n)^{1/2}$, $t = t_{n+1}$. Then from this, Proposition 26(2), Equations (19), (20), (21), (22) and (23), we have

$$\frac{(s_n')^2}{n}|\frac{P(\sum_{k=1}^n X_k > s_n')}{n\bar{F}(s_n')} - (1 + \frac{\alpha(\alpha+1)n}{(s_n')^2})| \to 0$$

as $n \to \infty$. This is a contradiction.

This proves Theorem 3.

10. Proof of Theorem 4

Let $\hat{F}_n : [1, \infty) \to [0, 1]$, $n \geqq 1$, be given by

$$\hat{F}_n(s) = \int_{-\infty}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx.$$

Then we have the following.

PROPOSITION 32. *Let $\beta : \mathbf{N} \to (0, \infty)$ be such that*

$$\frac{\beta(n)}{(\log n)^{1/2}} \to \infty, \qquad n \to \infty.$$

Then

$$\sup_{s \geqq \beta(n)} s^2|\frac{\hat{F}_n(s)}{\bar{F}(n^{1/2}s)} - (1 + \frac{\alpha(\alpha+1)n}{s^2})| \to 0, \qquad n \to \infty.$$

PROOF. By Proposition 27, we see that

$$|\frac{\hat{F}_n(s)}{\bar{F}(sn^{1/2})} - (1 + \frac{\alpha(\alpha+1)}{s^2})|$$

$$\leqq \int_{[-s/2,s/2]} |R(y/s; sn^{1/2}) - a(sn^{1/2})(y/s)|\Phi_1(y)dy$$

$$+ \int_{[-s/2,s/2]^c} 4(1 + (|a(sn^{1/2})| + \alpha(\alpha+1))(\frac{y^2}{s^2})\Phi_1(y)dy.$$

This and Proposition 27 imply our assetion. $\square$

It is well known (e.g. Williams [8]) that there is a $C_0 > 0$ such that

$$|\Phi_k(x)| \leqq C_0(1+x)^{k-1}\Phi_1(x), \qquad x \geqq 0,\ k = 1, \dots, 3K, \tag{24}$$

and

$$C_0^{-1}\Phi_1(x) \leqq x\Phi_0(x) \leqq C_0\Phi_1(x), \qquad x \geqq 1. \tag{25}$$

Let

$$H_0(n,s) = \Phi_0(s) + n\hat{F}_n(s),$$

$$A_1(n,s) = \sum_{k=1}^{2} \frac{n^{-(k-2)/2}}{k!}\Phi_k(s)\int_0^\infty x^k \mu(dx),$$

and

$$A(n,s) = n\hat{F}_n(s) - A_1(n,s).$$

First we prove the following.

PROPOSITION 33.

$$\sup_{s\in[1,\log n]} \frac{|A(n,s) - n\bar{F}(n^{1/2}s)|}{H_0(n,s)} \to 0, \qquad n \to \infty.$$

PROOF. Let us take a $\gamma \in (0, (\alpha-2)/(4\alpha))$ and fix it. Let $s \geqq 0$ and $n \geqq 3$. Note that

$$\hat{F}_n(s) = \sum_{k=1}^{4} I_k(n,s),$$

where

$$I_1(n,s) = \int_{s-n^{-\gamma}}^{s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx,$$

$$I_2(n,s) = \int_{-s}^{7s/8} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx,$$

$$I_3(n,s) = \int_{7s/8}^{s-n^{-\gamma}} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx,$$

and

$$I_4(n,s) = \int_{-\infty}^{-s} \bar{F}((s-x)n^{1/2})\Phi_1(x)dx.$$

It is easy to see that

$$I_1(n,s) = n^{-1/2}\int_0^{n^{(1/2-\gamma)}} \bar{F}(y)\Phi_1(s-n^{-1/2}y)dy.$$

Let

$$R(n,s,y) = \Phi_1(s-n^{-1/2}y) - (\Phi_1(s) + n^{-1/2}y\Phi_2(s))$$

Then for $y \in [0, sn^{1/2-\gamma}]$

$$|R(n,s,y)| \leqq n^{-1}y^2 \sup_{z\in[s-n^{-\gamma},s]} |\Phi_3(z)|$$

$$\leqq C_0 n^{-1}y^2(1+s)^2\Phi_1(s-n^{-\gamma}) = C_0 n^{-1}y^2(1+s)^2\Phi_1(s)\exp(sn^{-\gamma} - n^{-2\gamma}/2)$$

$$\leqq C_0^2 n^{-1}y^2(1+s)^3\exp(n^{-\gamma}s)\Phi_0(s).$$

So we see that

$$n|I_1(n,s) - \sum_{k=1}^{2}\frac{n^{-k/2}}{k!}\Phi_k(s)\int_0^\infty x^k\mu(dx)|$$

$$\leqq C_0^2(1+s)^3 n^{-1/2}\exp(n^{-\gamma}s)(\int_0^{n^{1/2-\gamma}} y^2\bar{F}(y)dy)\Phi_0(s)$$

$$+C_0^2(1+s)n^{1/2}(\int_{n^{1/2-\gamma}}^\infty \bar{F}(y)dy)\Phi_0(s) + C_0^2(1+s)^2(\int_{n^{1/2-\gamma}}^\infty y\bar{F}(y)dy)\Phi_0(s)$$

This implies that

$$\sup_{s\in[1,\log n]} \Phi_0(s)^{-1}|nI_1(n,s) - \sum_{k=1}^{2}\frac{n^{-(k-2)/2}}{k!}\Phi_k(s)\int_0^\infty x^k\mu(dx)| \to 0, \quad n\to\infty. \tag{26}$$

Note that

$$I_2(n,s) = \bar{F}(sn^{1/2})\int_{-s}^{7s/8}(1-\frac{x}{s})^{-\alpha}\frac{L((s-x)n^{1/2})}{L(sn^{1/2})}\Phi_1(x)dx$$

It is easy to see that

$$\sup_{s\in[(\log n)^{1/4},\log n]}\left|\int_{-s}^{7s/8}(1-\frac{x}{s})^{-\alpha}\frac{L((s-x)n^{1/2})}{L(sn^{1/2})}\Phi_1(x)dx-1\right|\to 0,\qquad n\to\infty$$

Also we see that

$$n|I_2(n,s)|\leqq n\bar{F}(sn^{1/2})8^{\alpha}\int_{-s}^{7s/8}\frac{L((s-x)n^{1/2})}{L(sn^{1/2})}\Phi_1(x)dx$$

Therefore we have

$$\sup_{s\in[1,(\log n)^{1/4}]}\Phi_0(s)^{-1}(n|I_2(n,s)|+n\bar{F}(sn^{1/2}))\to 0,\qquad n\to\infty.$$

Thus we have

$$\sup_{s\in[1,\log n]}H_0(n,s)^{-1}|nI_2(n,s)-n\bar{F}(sn^{1/2})|\to 0,\qquad n\to\infty. \tag{27}$$

Note that $\sqrt{3}/2\leqq 7/8$. Then we have

$$\Phi_1(7s/8)\leqq(\Phi_1(s))^{3/4},$$

and so we have

$$\begin{aligned}nI_3(n,s)&\leqq ns\bar{F}(n^{1/2-\gamma})\Phi_1(7s/8)\\&\leqq(n\bar{F}(n^{1/2}\log n))^{1/2}(s\Phi_1(s))^{3/4}\frac{ns^{1/4}\bar{F}(n^{1/2-\gamma})}{(n\bar{F}(n^{1/2}\log n))^{1/2}}.\end{aligned}$$

Since

$$\sup_{n\geqq 3}\sup_{s\in[1,\log n]}\frac{ns^{1/4}\bar{F}(n^{1/2-\gamma})}{(n\bar{F}(n^{1/2}\log n))^{1/2}}<\infty,$$

we see that there is a constant $C>0$ such that

$$nI_3(n,s)\leqq C(n\bar{F}(sn^{1/2}))^{1/2}\Phi_0(s)^{3/4}\leqq C(n\bar{F}(sn^{1/2}))^{1/4}H_0(n,s),\qquad n\geqq 3,\ s\in[1,\log n].$$

So we have

$$\sup_{s\in[1,\log n]}H_0(n,s)^{-1}|nI_3(n,s)|\to 0,\qquad n\to\infty. \tag{28}$$

Also we have

$$n|I_4(n,s))| \leqq n\bar{F}(2sn^{1/2})\Phi_0(s).$$

So this equation, Equations (26) (27) and (28) imply our assertion. □

PROPOSITION 34. (1) *There is a $C > 0$ such that*

$$\Phi_0(s) + |A_1(n,s)| \leqq Cn^{-2}\bar{F}(n^{1/2}s), \qquad n \geqq 2,\ s \geqq \log n.$$

(2)

$$\sup_{s\in[1,\infty)} \frac{|A(n,s) - n\bar{F}(n^{1/2}s)|}{H_0(n,s)} \to 0, \qquad n \to \infty.$$

PROOF. The assertion (1) is obvious from Equations (24) and (25). To prove the assertion (2), because of Proposition 33, it is sufficient to prove

$$\sup_{s\in[\log n,\infty)} \frac{|A(n,s) - n\bar{F}(n^{1/2}s)|}{H_0(n,s)} \to 0, \qquad n \to \infty.$$

However, by Theorem 3 and Propsosition 32, we see that

$$\sup_{s\in[\log n,\infty)} |\frac{n\hat{F}(n^{1/2}s)}{n\bar{F}(n^{1/2}s)} - 1| \to 0, \qquad n \to \infty.$$

Therefore, combining this with the assertion (1), we have the assertion (2). □

Now let us prove Theorem 4.

By Proposition 34(2), we see that

$$\sup_{s\in[1,\infty)} |\frac{H(n,s)}{H_0(n,s)} - 1| \to 0, \qquad n \to \infty.$$

Therefore there is an n_0 such that

$$H(n,s) \geqq \frac{1}{2}H_0(n,s), \qquad n \geqq n_0,\ s \geqq 1.$$

By Equation (24), we see that there is a $C > 0$ such that

$$\sup_{s\in[1,n^{1/12}]} |G(n,s) - H(n,s)| \leqq Cn^{-1/12}, \qquad n \geqq 1$$

Then combining this with Theorem 2, we see that there are $C > 0$ and $\delta_0 \in (0, 1/12)$ such that

$$\sup_{s \in [1, n^{\delta_0}]} \left| \frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{H(n,s)} - 1 \right| \leqq C n^{-\delta_0}, \qquad n \geqq n_0.$$

On the other hand, by Theorem 3 and Propsosition 32 we see that there is a $C > 0$ such that

$$\sup_{s \in [n^{\delta_0}, \infty)} \left| \frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{n\hat{F}_n(s)} - 1 \right| \leqq C n^{-2\delta_0}, \qquad n \geqq n_0.$$

So we see by Proposition 34 that we see that there is a $C > 0$ such that

$$\sup_{s \in [n^{\delta_0}, \infty)} \left| \frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{H(n,s)} - 1 \right| \leqq C n^{-2\delta_0}, \qquad n \geqq n_0.$$

These imply Theorem 4.

References

[1] Borovkov, A. and K. Borovkov, On the probabilities of large deviations of random walks I. The distributions with regularly varying tails, Theory Probab. Appl. **46** (2002), 193–213.

[2] Borovkov, A. and K. Borovkov, Asyptotic Analysis of Random Walks: Heavy Tailed Distributions, Cambridge University Press 2008, Cambridge.

[3] Embrechts, P., Klüppelberg, C. and T. Milkosch, Modelling Extremal Events, Springer, Berlin Heidelberg, 1997.

[4] Linnik, Yu. V., On the probability of large deviation for sums of independent variables, Proc. 4th Berkley Symp. Math. Statist. Probability 2, 289–306, 1961, Univ. Cal. Press.

[5] Nagaev, S. V., Large deviations of sums of independent random variables, Ann. Probab. **7** (1979), 745–789.

[6] Nagaev, S. V., On large deviations for sums of independent random variables, Doctoral dissertations, Institute of Mathematics, Tashkent, 1970.

[7] Petrov, V., *Limit Theorems of Probability Theory*, Clarendon Press, 1995, Oxford.

[8] Williams, D., *Probability with Martingales*, Cambridge University Press 1991, Cambridge.

(Received June 25, 2009)
(Revised April 22, 2010)

Hirotaka FUSHIYA
School of Social Informatics
Aoyama Gakuin University
Japan

Shigeo KUSUOKA
Graduate School of Mathematical Sciences
The University of Tokyo
Komaba 3-8-1, Meguro-ku
Tokyo 153-8914, Japan