

The Product Formula for Local Constants in Torsion Rings

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Abstract. Let p be a rational prime and K a local field of residue characteristic p . In this paper, we prove the product formula for local ε_0 -constants defined in [Y1].

1. Introduction

This paper is a continuation of the author's article [Y1]. Let K be a complete discrete valuation field whose residue field k is finite of characteristic p (such a field will be called a *p-local field*). Let q denote the cardinality of k . Let W_K be the Weil group of K . In [Y1], we defined the local constants $\varepsilon_{0,R}(V, \psi) \in R^\times$, generalizing Deligne's $\varepsilon_0(V, \psi, dx)$, for triples $(R, (\rho, V), \psi)$ where R is a *strict p'-coefficient ring* (cf. loc. cit.), (ρ, V) is an object in $\text{Rep}(W_K, R)$, and $\psi : K \rightarrow R^\times$ is a non-trivial continuous additive character.

In this paper we will concentrate on the case where $\text{char } K = p$. When R_0 is the ring of integers of a finite extension of \mathbb{Q}_ℓ for a prime $\ell \neq p$, the product formula of Deligne-Laumon describes the determinant of Frobenius on the étale cohomologies of a smooth R_0 -sheaf on a curve over k as a product of local ε_0 -constants. In this paper, we generalize the product formula to the case where R_0 is a pro-finite p' -coefficient ring, giving evidence that our construction provides a good theory of local ε_0 -constants.

Let us briefly review the contents of this paper. After recalling in § 3 some basic facts necessary in this paper, we give, in § 4, the statement of the product formula which is the main result of this paper. The next four sections are devoted to the proof of the product formula. In § 9, we give an application of our product formula to Saito's theorem in [Sa1].

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2. Notation

Let \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} denote the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Let $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{\geq 0}$) be the ordered set of positive (resp. non-negative) integers. We also define $\mathbb{Q}_{\geq 0}$, $\mathbb{Q}_{>0}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ in a similar way. For a prime number ℓ , let \mathbb{F}_ℓ denote the finite field of ℓ elements. For a ring R , let R^\times denote the group of units in R . For a finite extension L/K of fields, let $[L : K]$ denote the degree of L over K . For a subgroup H of a group G of finite index, its index is denoted by $[G : H]$.

Throughout this paper, we fix once for all a prime number p . We consider a complete discrete valuation field K whose residue field k is finite of characteristic p . We say such a field K is a *p-local field*.

For a p -local field K , let \mathcal{O}_K denote its ring of integers, \mathfrak{m}_K the maximal ideal of \mathcal{O}_K , and $v_K : K^\times \rightarrow \mathbb{Z}$ the normalized valuation. We also denote by W_K the Weil group of K , by $\text{rec} = \text{rec}_K : K^\times \xrightarrow{\cong} W_K^{\text{ab}}$ the reciprocity map in the local class field theory, which sends a prime element of K to a lift of geometric Frobenius of k .

If L/K is a finite separable extension of p -local fields, let $e_{L/K} \in \mathbb{Z}$, $f_{L/K} \in \mathbb{Z}$, $D_{L/K} \in \mathcal{O}_L/\mathcal{O}_L^\times$, $d_{L/K} \in \mathcal{O}_K/\mathcal{O}_K^{\times 2}$ denotes the ramification index of L/K , the residual degree of L/K , the different of L/K , the discriminant of L/K respectively.

For a topological group (or more generally for a topological monoid) G and a commutative topological ring R , let $\text{Rep}(G, R)$ denote the category whose object is a pair (ρ, V) of a finitely generated free R -module V and a continuous group homomorphism $\rho : G \rightarrow GL_R(V)$ (we endow $GL_R(V)$ with the topology induced from the direct product topology of $\text{End}_R(V)$), and whose morphisms are R -linear maps compatible with actions of G .

A sequence

$$0 \rightarrow (\rho', V') \rightarrow (\rho, V) \rightarrow (\rho'', V'') \rightarrow 0$$

of morphisms in $\text{Rep}(G, R)$ is called a *short exact sequence* in $\text{Rep}(G, R)$ if $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ is the short exact sequence of R -modules.

In this paper, a noetherian local ring with residue field of characteristic $\neq p$ is called a p' -coefficient ring. Any p' -coefficient ring (R, \mathfrak{m}_R) is considered as a topological ring with the \mathfrak{m}_R -preadic topology. A *strict p' -coefficient ring* is a p' -coefficient ring R with an algebraically closed residue field such that $(R^\times)^p = R^\times$.

3. Review of Basic Facts

3.1. Ramification subgroups

Let K be a p -local field. We denote its residue field by k . We fix a separable closure \overline{K} of K , and denote by \overline{k} the residue field of the valuation field \overline{K} . Let $G = W_K$ denote the Weil group of K . Let $G^v = G \cap \text{Gal}(\overline{K}/K)^v$ and $G^{v+} = G \cap \text{Gal}(\overline{K}/K)^{v+}$ be the *upper numbering ramification subgroups* of G . These subgroups have the following properties:

- G^v and G^{v+} are closed normal subgroups of G .
- $G^v \supset G^{v+} \supset G^w$ for every $v, w \in \mathbb{Q}_{\geq 0}$ with $w > v$.
- G^{v+} is equal to the closure of $\bigcup_{w>v} G^w$.
- $G^0 = I_K$, the inertia subgroup of W_K . $G^{0+} = P_K$, the wild inertia subgroup of W_K . In particular, G^w for $w > 0$ and G^{w+} for $w \geq 0$ are pro p -groups.
- For $w \in \mathbb{Q}$, $w > 0$, G^w/G^{w+} is an abelian group which is killed by p .

3.2. Character sheaves

Let S be a scheme of characteristic p , (R, \mathfrak{m}_R) a complete p' -coefficient ring, and G a commutative group scheme over S . An *invertible character R -sheaf* on G is a smooth invertible R -sheaf (that is, a pro-system of smooth invertible R/\mathfrak{m}_R^n -sheaves) \mathcal{L} on G such that $\mathcal{L} \boxtimes \mathcal{L} \cong \mu^* \mathcal{L}$, where $\mu : G \times_S G \rightarrow G$ is the group law. We have $i^* \mathcal{L} \cong \mathcal{L}$, where $i : G \rightarrow G$ is the inverse morphism. If $\mathcal{L}_1, \mathcal{L}_2$ are two invertible character R -sheaf on G , then so is $\mathcal{L}_1 \otimes_R \mathcal{L}_2$.

LEMMA 3.1 (Orthogonality relation). *Suppose that S is quasi-compact and quasi-separated, and that the structure morphism $\pi : G \rightarrow S$ is compactifiable. Let \mathcal{L} be an invertible character R -sheaf on G such that $\mathcal{L} \otimes_R R/\mathfrak{m}_R$ is non-trivial. Then we have $R\pi_! \mathcal{L} = 0$.*

PROOF. We may assume that R is a field. Since $R\mathrm{pr}_1(\mathcal{L} \boxtimes \mathcal{L}) \cong (\pi^*R\pi_1\mathcal{L}) \otimes \mathcal{L}$ and $R\mathrm{pr}_1(\mu^*\mathcal{L}) \cong \pi^*R\pi_1\mathcal{L}$, we have $(\pi^*R^i\pi_1\mathcal{L}) \otimes \mathcal{L} \cong \pi^*R^i\pi_1\mathcal{L}$ for all i . Hence $R^i\pi_1\mathcal{L} = 0$ for all i . \square

The following lemma will be used in the subsequent paper [Y2]:

LEMMA 3.2. *Suppose further that S and G are noetherian and connected, and that R is a finite ring. Let \mathcal{L} be a smooth invertible R -sheaf on G . Then \mathcal{L} is an invertible character R -sheaf if and only if there is a finite etale homomorphism $G' \rightarrow G$ of commutative S -group schemes with a constant kernel H_S and a homomorphism $\chi : H \rightarrow R^\times$ of groups such that \mathcal{L} is the sheaf defined by G' and χ .*

PROOF. The if part is easy. We prove the only if part. Let \mathcal{L} be an invertible character R -sheaf on G , $\rho : \pi_1(G) \rightarrow R^\times$ a representation of the etale fundamental group $\pi_1(G)$ of G corresponding to \mathcal{L} , and $f : X \rightarrow G$ the finite etale Galois covering of G corresponding to $\mathrm{Ker} \rho$. Let us define a group law on X . Let $e : S \rightarrow G$ be the unit section. Since $(\mathcal{L}|_{e(S)})^{\otimes 2} \cong \mathcal{L}|_{e(S)}$, there exists a section $e' : S \rightarrow X$ satisfying $e = f \circ e'$. Since $\mathcal{L} \boxtimes \mathcal{L} \cong \mu^*\mathcal{L}$, there exists a finite etale morphism $X \times_S X \rightarrow X \times_{G,\mu}(G \times_S G)$ over $G \times_S G$. Then there exists a unique morphism $\mu' : X \times_S X \rightarrow X$ over S which is the composition of the above morphism, the projection $X \times_{G,\mu}(G \times_S G) \rightarrow X$, and an automorphism $X \rightarrow X$ over G such that $\mu' \circ (e' \times e') = e'$. Since $i^*\mathcal{L} \cong \mathcal{L}$, there exists a unique isomorphism $i' : X \times_{G,i} G \xrightarrow{\cong} X$ over G . It is a routine to check that (e', μ', i') defines a structure of a commutative S -group scheme on X . Hence the assertion follows. \square

4. Product Formula (Statement)

Let k be a finite field of characteristic p with q elements, X_0 a proper smooth connected curve over k , $U_0 \subset X_0$ a non-empty open subscheme of X_0 , $j_0 : U_0 \hookrightarrow X_0$ the inclusion, $X = X_0 \otimes_k \bar{k}$, $U = U_0 \otimes_k \bar{k}$, $j = j_0 \times id : U \hookrightarrow X$, R a strict p' -coefficient ring, $R_0 \subset R$ a finite subring, and \mathcal{F} a smooth R_0 -flat R_0 -sheaf on U_0 .

Define the global ε -constant $\varepsilon_{R_0}(U_0, \mathcal{F})$ as

$$\varepsilon_{R_0}(U_0, \mathcal{F}) = \det(-\mathrm{Fr}_q; R\Gamma_c(U, \mathcal{F}))^{-1} = \det(-\mathrm{Fr}_q; R\Gamma_c(X, j_{0,!}\mathcal{F}))^{-1}.$$

Let $\omega \in \Gamma(U_0, \Omega_{U_0/k}^1)$ be a non-zero differential on U . Fix a non-trivial additive character $\psi : k \rightarrow R_0^\times$. For a closed point $x \in X_0$, we denote by $\kappa(x)$ the residue field at x , by $q_x = \#\kappa(x)$ the cardinality of $\kappa(x)$, by K_x the completion of the function field of X_0 at x , by \mathcal{F}_x the isomorphism class in $\text{Rep}(W_{K_x}, R)$ corresponding to the pull-back of \mathcal{F} by the canonical morphism $\text{Spec}(K_x) \rightarrow U$, and by $\psi_{\omega,x} : K_x \rightarrow R^\times$ the additive character given by

$$\psi_{\omega,x}(a) = \psi(\text{Tr}_{\kappa(x)/k}(\text{Res}_x(a\omega)))$$

for $a \in K_x$. Here Res_x is the residue homomorphism at x .

THEOREM 4.1 (Product formula for $(U_0, \mathcal{F}, \omega)$). *In the above notation, we have*

$$\varepsilon_{R_0}(U_0, \mathcal{F}) = q^{\frac{1}{2}\chi(X)\text{rank}(\mathcal{F})} \prod_{x \in X_0 - U_0} \varepsilon_{0,R}(\mathcal{F}_x, \psi_{\omega,x}),$$

where $\chi(X)$ is the Euler number of X .

LEMMA 4.2.

- (1) *Let $V_0 \subset U_0$ be a open dense subscheme. Then the product formula for $(U_0, \mathcal{F}, \omega)$ is equivalent to that for $(V_0, \mathcal{F}|_{V_0}, \omega|_{V_0})$.*
- (2) *Let $\omega' \in \Gamma(U_0, \Omega_{U_0/k}^1)$ be another non-zero differential on U_0 . Then the product formula for $(U_0, \mathcal{F}, \omega)$ is equivalent to that for $(U_0, \mathcal{F}, \omega')$.*
- (3) *Let Y_0 be another proper smooth connected curve over k , $f : Y_0 \rightarrow X_0$ a finite morphism such that the restriction $V_0 = U_0 \times_{X_0} Y_0 \rightarrow U_0$ of f to V_0 is etale, \mathcal{G} a smooth R_0 -flat R_0 -sheaf on V_0 . Then the product formula for $(V_0, \mathcal{G}, f^*\omega)$ is equivalent to that for $(U_0, (f|_{V_0})_*\mathcal{G}, \omega)$.*
- (4) *Let R'_0 be another finite subring of R containing R_0 . Then the product formula for $(U_0, \mathcal{F}, \omega)$ is equivalent to that for $(U_0, \mathcal{F} \otimes_{R_0} R'_0, \omega)$.*

PROOF. (1) Let $x \in U_0 - V_0$. Since \mathcal{F} is unramified at x and $\psi_{\omega,x}$ has conductor 0,

$$\varepsilon_{0,R}(\mathcal{F}_x, \psi_{\omega,x}) = (-1)^{\text{rank } \mathcal{F}_x} \det(\mathcal{F}_x)(\text{Fr}_x).$$

Let $j' : V_0 \hookrightarrow U_0$ be the canonical inclusion. For $x \in U_0$, let $i_x : x \hookrightarrow U_0$ be the canonical morphism. By the short exact sequence

$$0 \rightarrow j'_! j'^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow \bigoplus_{x \in U_0 - V_0} i_{x,*} i_x^* \mathcal{F} \rightarrow 0,$$

we have

$$\varepsilon_{R_0}(U_0, \mathcal{F}) = \varepsilon_{R_0}(V_0, \mathcal{F}|_{V_0}) \cdot \prod_{x \in U_0 - V_0} \det(-\mathrm{Fr}_x; \mathcal{F}_x).$$

Hence the lemma follows.

(2) By shrinking U_0 if necessary, we may assume that there exists an invertible element $f \in \Gamma(U_0, \mathcal{O}_{X_0}^\times)$ satisfying $\omega' = f\omega$. For $x \in X_0$, we have

$$\varepsilon_{0,R}(\mathcal{F}_x, \psi_{\omega',x}) = \det(\mathcal{F}_x)(\mathrm{rec}_{K_x}(f)) q_x^{\mathrm{ord}_x(f)} \varepsilon_{0,R}(\mathcal{F}_x, \psi_{\omega,x}).$$

Since $\det(\mathcal{F}_x) = (\det \mathcal{F})_x$ is unramified at $x \in U$, we have, by global class field theory,

$$\prod_{x \in X_0 - U_0} \det(\mathcal{F}_x)(\mathrm{rec}_{K_x}(f)) = \prod_{x \in X_0} \det(\mathcal{F}_x)(\mathrm{rec}_{K_x}(f)) = 1.$$

Hence the lemma follows from

$$\prod_{x \in X_0 - U_0} q_x^{\mathrm{ord}_x(f)} = \prod_{x \in X_0} q_x^{\mathrm{ord}_x(f)} = 1.$$

(3) Since $(f|_{V_0})_* = (f|_{V_0})_!$, we have

$$R\Gamma_c(V_0 \otimes_k \bar{k}, \mathcal{G}) = R\Gamma_c(U, (f|_{V_0})_* \mathcal{G}).$$

Hence $\varepsilon_{R_0}(U_0, (f|_{V_0})_* \mathcal{G}) = \varepsilon_{R_0}(V_0, \mathcal{G})$.

On the other hand, for $x \in X_0$, we have

$$((f|_{V_0})_* \mathcal{G})_x \cong \bigoplus_{f(y)=x} \mathrm{Ind}_{W_{K_y}}^{W_{K_x}} \mathcal{G}_y.$$

By [Y1, Thm. 5.6] we have

$$\begin{aligned} \prod_{x \in X_0 - U_0} \varepsilon_{0,R}(((f|_{V_0})_* \mathcal{G})_x, \psi_{\omega,x}) &= \prod_{y \in Y_0 - V_0} \varepsilon_{0,R}(\mathrm{Ind}_{W_{K_y}}^{W_{K_{f(y)}}} \mathcal{G}_y, \psi_{\omega,f(y)}) \\ &= \prod_{y \in Y_0 - V_0} \varepsilon_{0,R}(\mathcal{G}_y, \psi_{\omega,f(y)} \circ \mathrm{Tr}_{K_y/K_{f(y)}}) \cdot \lambda_R(K_y/K_{f(y)}, \psi_{\omega,f(y)})^{\mathrm{rank} \mathcal{F}_y}. \end{aligned}$$

Since $\psi_{\omega, f(y)} \circ \text{Tr}_{K_y/K_{f(y)}} = \psi_{f^*\omega, y}$, we have

$$\begin{aligned} & \prod_{x \in X_0 - U_0} \varepsilon_{0,R}((f|_{V_0})_* \mathcal{G})_x, \psi_{\omega, x} \\ &= \prod_{y \in Y_0 - V_0} \varepsilon_{0,R}(\mathcal{G}_y, \psi_{f^*\omega, y}) \cdot \lambda_R(K_y/K_{f(y)}, \psi_{\omega, f(y)})^{\text{rank } \mathcal{F}_y}. \end{aligned}$$

To prove the lemma, it suffices to prove

$$q^{\frac{1}{2}\chi(X)} \prod_{y \in Y_0 - V_0} \lambda_R(K_y/K_{f(y)}, \psi_{\omega, f(y)}) = q^{\frac{1}{2}\chi(Y)}.$$

which follows from similar computation for trivial $\overline{\mathbb{Q}}_\ell$ -sheaf on V_0 , where ℓ is the residue characteristic of R .

(4) is obvious. \square

Therefore, to prove Theorem 4.1, we may assume that $X_0 = \mathbb{P}_k^1$, $U_0 \subset \mathbb{A}_k^1$, and that \mathcal{F} is unramified at ∞ .

5. Key Proposition

Let K be a p -local field of characteristic p whose residue field is k . We fix a prime element π_K in K . We identify $X = \text{Spec}(\bigoplus_{n \in \mathbb{Z}} \mathfrak{m}_K^n / \mathfrak{m}_K^{n+1})$ with $\mathbb{G}_{m,k}$ via the canonical isomorphism $\bigoplus_{n \in \mathbb{Z}} \mathfrak{m}_K^n / \mathfrak{m}_K^{n+1} \cong k[\pi_K^{\pm 1}]$. Let $i_0 : \text{Spec}(K) \rightarrow X$ be the morphism whose associated homomorphism of coordinate rings is given by the canonical inclusion

$$\bigoplus_{n \in \mathbb{Z}} \mathfrak{m}_K^n / \mathfrak{m}_K^{n+1} \cong k[\pi_K^{\pm 1}] \hookrightarrow k((\pi_K)) \cong K.$$

Let R_0 be a finite p' -coefficient ring which contains a primitive p -th root of unity. For an object V in $\text{Rep}(W_K, R_0)$, let \mathcal{F}_V be the canonical extension of V (cf. [Kz1, p. 76, Thm. 1.4.1]). We exchange the roles of 0 and ∞ in [Kz1]; the sheaf \mathcal{F}_V is a smooth R_0 -flat R_0 -sheaf on $X = \mathbb{G}_{m,k}$ which is tame at ∞ such that $i_0^* \mathcal{F}_V$ is the sheaf on $\text{Spec}(K)$ corresponding to V .

Fix a non-trivial additive character $\phi_0 : \mathfrak{m}_K^{-1} / \mathcal{O}_K \rightarrow R_0^\times$ and let $\mathcal{L} = \mathcal{L}_{\phi_0}$ be the Artin-Schreier sheaf on $\text{Spec}(\bigoplus_{n \geq 0} \mathfrak{m}_K^n / \mathfrak{m}_K^{n+1})$, and \mathcal{L}' its restriction to X . For an object V in $\text{Rep}(W_K, R_0)$, set $\mathcal{F}'_V = \mathcal{F}_V \otimes_{R_0} \mathcal{L}'$.

PROPOSITION 5.1. *Suppose that V is totally wildly ramified. Let $r = \text{rank } V + \text{sw}(V)$. For $r' \in \mathbb{Z}_{\geq 0}$, let $\tilde{s}_{r'} : \text{Sym}^{r'} \mathbb{G}_{m,k} \rightarrow \mathbb{G}_{m,k}$ be the morphism induced by the product map $s_{r'} : \mathbb{G}_{m,k}^{r'} \rightarrow \mathbb{G}_{m,k}$. Let $\mathfrak{S}_{r'}$ be the r' -th symmetric group and let $\Gamma_{\text{ext}}^{r'}(\mathcal{F}'_V)$ denote the $\mathfrak{S}_{r'}$ -invariant part of the direct image of $\mathcal{F}'_V^{\boxtimes r'}$ under the quotient morphism $\mathbb{G}_{m,k}^{r'} \rightarrow \text{Sym}^{r'} \mathbb{G}_{m,k}$. Then*

- (1) *For $r' > r$, we have $R\tilde{s}_{r',!} \Gamma_{\text{ext}}^{r'} \mathcal{F}'_V = 0$.*
- (2) *The complex $R\tilde{s}_{r,!} \Gamma_{\text{ext}}^r \mathcal{F}'_V$ is supported on a closed point of $\mathbb{G}_{m,k}$.*

LEMMA 5.2. *Let (R, \mathfrak{m}) be a local ring, K^\bullet a complex of finitely generated free R -modules which is bounded above. Suppose that $K \otimes_R R/\mathfrak{m}$ is acyclic. Then K is also acyclic.*

PROOF. Assume that K^\bullet is not acyclic. Let i be the maximal integer such that $K^i \neq 0$. If $d^{i-1} : K^{i-1} \rightarrow K^i$ is surjective, then $\text{Ker } d^{i-1}$ is a finitely generated free R -module and the complex

$$\dots \rightarrow K^{i-2} \rightarrow \text{Ker } d^{i-1} \rightarrow 0 \rightarrow \dots$$

is quasi-isomorphic to K^\bullet . Hence we may assume that d^{i-1} is not surjective. Since

$$0 = \text{Coker} (K^{i-1} \otimes_R R/\mathfrak{m} \rightarrow K^i \otimes_R R/\mathfrak{m}) \cong (\text{Coker } d^{i-1}) \otimes_R R/\mathfrak{m},$$

we have $\text{Coker } d^{i-1} = 0$. This is a contradiction. \square

Since the geometric stalks of $R\tilde{s}_{r',!} \Gamma_{\text{ext}}^{r'} \mathcal{F}'_V$ are bounded above and have constructible cohomologies, we may assume, by the above lemma, that R_0 is a finite field in proving Proposition 5.1.

PROOF OF PROPOSITION 5.1 (1). We may assume that R_0 is a field. Assume that $R\tilde{s}_{r',!} \Gamma_{\text{ext}}^{r'} \mathcal{F}'_V \neq 0$. Take the minimum i satisfying $R^i \tilde{s}_{r',!} \Gamma_{\text{ext}}^{r'} \mathcal{F}'_V \neq 0$ and put $\mathcal{H} = R^i \tilde{s}_{r',!} \Gamma_{\text{ext}}^{r'} \mathcal{F}'_V$. Since $R\Gamma_c(\mathbb{G}_{m,\bar{k}}, R\tilde{s}_{r',!} \Gamma_{\text{ext}}^{r'} \mathcal{F}'_V) = \bigwedge^{r'} H_c^1(\mathbb{G}_{m,\bar{k}}, \mathcal{F}'_V) = 0$, we have $H_c^0(\mathbb{G}_{m,\bar{k}}, \mathcal{H}) = H_c^1(\mathbb{G}_{m,\bar{k}}, \mathcal{H}) = 0$. We see that \mathcal{H} is a smooth R_0 -sheaf on $\mathbb{G}_{m,k}$ which is tame both at 0 and ∞ . In fact, there exists a non-empty open subscheme $U \subset \mathbb{G}_{m,k}$ such that $\mathcal{H}|_U$ is smooth. Take the maximum such U and let

R be the rank of $\mathcal{H}|_U$. By the Grothendieck-Ogg-Shafarevich formula, we have

$$\dim_{R_0} H_c^2(\mathbb{G}_{m,\bar{k}}, \mathcal{H}) = - \sum_{x \in \mathbb{G}_{m,k} - U} (R - \dim_{R_0} \mathcal{H}_{\bar{x}}) - \sum_{x \in \mathbb{P}_k^1 - U} \text{sw}_x(\mathcal{H}|_U).$$

Hence $U = \mathbb{G}_{m,k}$ and $\text{sw}_0(\mathcal{H}) = \text{sw}_\infty(\mathcal{H}) = 0$. By replacing k by its finite extension, we may assume that the representation of $\pi_1(\mathbb{G}_{m,k})$ corresponding to \mathcal{H} is abelian. By replacing R_0 by its finite extension, there exists an invertible smooth R_0 -subsheaf \mathcal{W} of \mathcal{H} . Let $\mathcal{M} = \mathcal{W}^{-1}$. Since $s_{r'}^* \mathcal{M} \cong \mathcal{M}^{\boxtimes r}$, we have $\tilde{s}_{r'}^* \mathcal{M} \cong \Gamma_{\text{ext}}^r \mathcal{M}$. Hence, by replacing \mathcal{F}'_V by $\mathcal{F}'_V \otimes \mathcal{M}$, we have $R\Gamma_c(\mathbb{G}_{m,\bar{k}}, R\tilde{s}_{r'}^* \Gamma_{\text{ext}}^{r'}(\mathcal{F}'_V \otimes \mathcal{M})) \neq 0$. This is a contradiction. \square

The following lemma is proved in the same way as in the proof of [DH, p. 101, Prop. 2.2].

LEMMA 5.3. *Let K be a p -local field, R_0 a finite field of characteristic $\neq p$, V a totally wild object in $\text{Rep}(W_K, R_0)$. Then there exist a finite extension R'_0 of R_0 , finitely many finite separable extensions L_1, \dots, L_m of K , integers $n_1, \dots, n_m \in \mathbb{Z}$, and wild object χ_i in $\text{Rep}(W_{L_i}, R'_0)$ of rank one such that*

$$V = \sum_i n_i \text{Ind}_{W_{L_i}}^{W_K} \chi_i$$

in the Grothendieck group of objects in $\text{Rep}(W_K, R'_0)$.

LEMMA 5.4. *Let $k = \bar{k}$ be an algebraically closed field of characteristic p , Y a smooth irreducible affine curve over k , $Y \hookrightarrow \bar{Y}$ the smooth completion of Y . Suppose that $\bar{Y} - Y$ consists of more than or equal to two points. We set $\bar{Y} - Y = \{q_1, \dots, q_m\}$. Let R_0 be a finite p' -coefficient ring, and \mathcal{F} a smooth invertible R_0 -sheaf on Y which is wild at all points q_1, \dots, q_m . Let $g(\bar{Y})$ denote the genus of \bar{Y} . Set $s_i := \text{sw}_{q_i}(\mathcal{F}) > 0$ for each i and put*

$$r := 2g(\bar{Y}) - 2 + \sum_i (1 + s_i).$$

Define an effective divisor \mathfrak{m}_0 on \bar{Y} as

$$\mathfrak{m}_0 := \sum_i [q_i].$$

Let $J_{\mathfrak{m}_0}$ be the generalized Jacobian of \overline{Y} with modulus \mathfrak{m}_0 . Take a closed point P on Y and let $h : \mathrm{Sym}^r Y \rightarrow J_{\mathfrak{m}_0}$ be the morphism given by $D \mapsto D - r[P]$. Then $Rh_! \Gamma_{\mathrm{ext}}^r \mathcal{F}$ is supported on a closed point on $J_{\mathfrak{m}_0}$.

PROOF. We set

$$\mathfrak{m} := \sum_i (1 + s_i)[q_i].$$

Let $J_{\mathfrak{m}}$ (resp. $J_{\mathfrak{m}-\mathfrak{m}_0}$) be the generalized Jacobian of \overline{Y} with modulus \mathfrak{m} (resp. $\mathfrak{m}-\mathfrak{m}_0$). We write the morphism $h : \mathrm{Sym}^r Y \rightarrow J_{\mathfrak{m}_0}$ as the composition

$$\mathrm{Sym}^r Y \xrightarrow{g} J_{\mathfrak{m}} \xrightarrow{h'} J_{\mathfrak{m}-\mathfrak{m}_0} \xrightarrow{h''} J_{\mathfrak{m}_0}.$$

By the geometric class field theory, there exists a character sheaf \mathcal{G} on $J_{\mathfrak{m}}$ such that $\mathcal{F} = h^* \mathcal{G}$. Furthermore, the restriction of \mathcal{G} on the kernel of $J_{\mathfrak{m}} \rightarrow J_{\mathfrak{m}-[q_i]}$ is non-trivial for every i .

Define the reduced closed subschemes $K, K' \subset J_{\mathfrak{m}}$ with $K' \subset K$ as

$$K := \{\mathrm{div}(\omega) + \mathfrak{m} - r[P] ; \mathrm{Supp}(\mathrm{div}(\omega) + \mathfrak{m}) \subset Y\},$$

and by

$$K' := \{\mathrm{div}(\omega) + \mathfrak{m} - r[P] \in K ; \sum_i \mathrm{Res}_{q_i} \omega = 0\}.$$

By [Se3, V, §1, 2] and Riemann-Roch theorem [Se3, IV, §2, 6. Thm. 1], $g^{-1}(J_{\mathfrak{m}} - K) \rightarrow J_{\mathfrak{m}} - K$ is a vector bundle of rank $g(\overline{Y}) - 1$, $g^{-1}(K - K')$ is empty and $g^{-1}(K') \rightarrow K'$ is a vector bundle of rank $g(\overline{Y})$. We have $R(h'' \circ h')_! \mathcal{G} = 0$ by the orthogonality relation. Hence to prove the lemma, it suffices to show that

$$R(h'' \circ h'|_K)_! (\mathcal{G}|_K) = 0$$

and that

$$R(h'' \circ h'|_{K'})_! (\mathcal{G}|_{K'})$$

is supported on a closed point of $J_{\mathfrak{m}_0}$.

Since K is a translation of a sub k -group of $J_{\mathfrak{m}}$ which contains the kernel of $h'' \circ h'$, the first assertion follows from the orthogonality relation of character sheaves (§ 3.2). We prove the second assertion.

Take prime elements $\pi_{q_1}, \dots, \pi_{q_m}$ of $\mathcal{O}_{\bar{Y}, q_1}, \dots, \mathcal{O}_{\bar{Y}, q_m}$. Take a meromorphic differential ω_0 on \bar{Y} such that

$$\frac{\omega_0}{d\pi_{q_i}} \in \frac{1}{\pi_{q_i}^{1+s_i}} + \mathcal{O}_{\bar{Y}, q_i}$$

for all i .

Then K and K' are the translations of the sub k -group scheme

$$\begin{aligned} C_{\mathfrak{m}} &= \left(\prod_i (\mathcal{O}_{\bar{Y}, q_i} / \mathfrak{m}_{\bar{Y}, q_i}^{1+s_i})^\times \right) / \mathbb{G}_{m, k} \\ &\cong \{ \text{div}(f) \in J_{\mathfrak{m}} ; \text{Supp}(\text{div}(f)) \subset Y \} \end{aligned}$$

of $J_{\mathfrak{m}}$ and the sub k -scheme

$$\left\{ \left(\sum_{j=0}^{s_i} b_{i,j} \pi_{q_i}^j \right)_i \in C_{\mathfrak{m}} ; \sum_i b_{i, s_i} = 0 \right\}$$

of $C_{\mathfrak{m}}$ by the class of $\text{div}(\omega_0) + \mathfrak{m} - r[P]$.

The image $h'(K)$ is the translation of the sub k -group scheme

$$\begin{aligned} C_{\mathfrak{m}-\mathfrak{m}_0} &= \left(\prod_i (\mathcal{O}_{\bar{Y}, q_i} / \mathfrak{m}_{\bar{Y}, q_i}^{s_i})^\times \right) / \mathbb{G}_{m, k} \\ &\cong \{ \text{div}(f) \in J_{\mathfrak{m}-\mathfrak{m}_0} ; \text{Supp}(\text{div}(f)) \subset Y \} \end{aligned}$$

of $J_{\mathfrak{m}-\mathfrak{m}_0}$ by the class of $\text{div}(\omega_0) + \mathfrak{m} - r[P]$.

Let

$$Q = \text{div}(\omega_0) + \mathfrak{m} - r[P] + \left(\sum_{j=0}^{s_i-1} b_{i,j} \pi_{q_i}^j \right)_i$$

be a closed point in $h'(K)$, $\tilde{Q} \in K$ be the lift of Q defined as

$$\tilde{Q} = \text{div}(\omega_0) + \mathfrak{m} - r[P] + \sum_{j=0}^{s_i-1} b_{i,j} \pi_{q_i}^j + 0)_i.$$

Then the fiber $K' \cap (h')^{-1}(Q)$ of $h'|_{K'}$ at Q is equal to the translation of

$$\left\{ (1 + b_{i, s_i} \pi_{q_i}^{s_i})_i \in C_{\mathfrak{m}} ; \sum_i b_{i, 0} b_{i, s_i} = 0 \right\}$$

by \tilde{Q} .

It is easy to see that there is a unique closed point $(b'_{i,0})_i \in (\mathbb{G}_{m,k}^{m+1})/\mathbb{G}_{m,k}$ such that $R(h'|_{K'})!(\mathcal{G}|_{K'})_Q = 0$ for $Q \in h'(K)$ as above with $(b_{i,0})_i \neq (b'_{i,0})_i$. This completes the proof. \square

In view of the proof of the above lemma, we have:

COROLLARY 5.5. *In the notation of the above lemma, suppose that there exists a finite subfield k_0 of k such that Y is the base change of a smooth curve Y_0 over k_0 , that q_1 comes from a k_0 -rational point of the completion \bar{Y}_0 of Y_0 , and that \mathcal{F} is defined on Y_0 . Let $\bar{Y}_0 - Y_0 = \{q'_1, \dots, q'_{m'}\}$. Fix a non-trivial additive character $\psi : k_0 \rightarrow R_0^\times$. Take a non-zero meromorphic differential ω_0 on \bar{Y}_0 . For each i , let $K_{q'_i}$ be the completion of the function field of Y_0 at q'_i . Take a non-zero meromorphic function f on \bar{Y}_0 such that for every i , the class of f in $K_{q'_i}/1 + \mathfrak{m}_{K_{q'_i}}$ is equal to the refined swan conductor $\text{rsw}_{\psi, \omega_0, q'_1}(\mathcal{F}_{q'_i})$.*

Then $Rh_! \Gamma_{\text{ext}}^r \mathcal{F}$ is supported on the closed point on $J_{\mathfrak{m}_0}$ corresponding to the divisor $\text{div}(\omega_0) - \text{div}(f) + \mathfrak{m} - r[P]$.

COROLLARY 5.6. *Proposition 5.1 (2) holds if V is of the form $V = \text{Ind}_{W_L}^{W_K} \chi$ for a finite separable totally ramified extension L of K and a rank one wild object χ in $\text{Rep}(W_L, R_0)$.*

PROOF. Let $f : Y \rightarrow \mathbb{G}_{m,k}$ be the finite etale extension corresponding to L/K by the theory of canonical extension. Let $Y \hookrightarrow \bar{Y}$ be its smooth compactification, let $\bar{f} : \bar{Y} \rightarrow \mathbb{P}_k^1$ be the morphism induced by f . Since L/K is totally ramified, Y is geometrically irreducible and $\bar{f}^{-1}(0)$ consists of one k -rational point q_1 . Set $(\bar{Y} - Y)(\bar{k}) = \{q_1, q_2, \dots, q_m\}$. There exists a smooth invertible R_0 -sheaf \mathcal{F}_1 on Y which is wildly ramified at q_1 and tame at q_2, \dots, q_m such that $\mathcal{F}_V \cong f_* \mathcal{F}_1$. Then $Y, \bar{Y}, q_1, \dots, q_m$, and $\mathcal{F} = \mathcal{F}_1 \otimes f^* \mathcal{L}$ satisfy the conditions of the above lemma.

Let $J_{\mathfrak{m}_0}$ be the generalized Jacobian as in the above lemma. Take a point $P \in Y(\bar{k})$ such that $f(P) = 1 \in \mathbb{G}_{m,k}$. By using P , there is a canonical

morphism $\mathrm{Sym}^r Y \rightarrow J_{\mathfrak{m}_0}$. We have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Sym}^r Y & \xrightarrow{\mathrm{Sym}^r f} & \mathrm{Sym}^r \mathbb{G}_{m,k} \\ h \downarrow & & \downarrow \bar{s}_r \\ J_{\mathfrak{m}_0} & \longrightarrow & \mathbb{G}_{m,k}. \end{array}$$

Hence the assertion follows from the above lemma and $\Gamma_{\mathrm{ext}}^r \mathcal{F}'_V \cong (\mathrm{Sym}^r f)_* \Gamma_{\mathrm{ext}}^r \mathcal{F}$. \square

Therefore, to prove Proposition 5.1 (2), it suffices to prove the following lemma:

LEMMA 5.7.

(1) *Let*

$$0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$$

be an exact sequence of totally wild objects in $\mathrm{Rep}(W_K, R_0)$. If Proposition 5.1 (2) holds for two of V , V' and V'' , then it holds for all of V , V' and V'' .

(2) *Let K'/K be a finite unramified extension, V a totally wild object in $\mathrm{Rep}(W'_K, R_0)$. If Proposition 5.1 (2) holds for V , then it holds for all of $\mathrm{Ind}_{W'_K}^{W_K} V$.*

PROOF. We prove only the assertion 1. The assertion 2 is obtained by a similar method.

We set $r := \mathrm{sw}(V) + \mathrm{rank} V$, $r' := \mathrm{sw}(V') + \mathrm{rank} V'$, and $r'' := \mathrm{sw}(V'') + \mathrm{rank} V''$. The sheaf $\Gamma_{\mathrm{ext}}^r \mathcal{F}'_V$ has a natural filtration, whose graded pieces are equal to $\vee_{r_1, r_2, *} (\Gamma_{\mathrm{ext}}^{r_1} \mathcal{F}'_{V'} \boxtimes \Gamma_{\mathrm{ext}}^{r_2} \mathcal{F}'_{V''})$ for some $r_1, r_2 \in \mathbb{Z}_{\geq 0}$ with $r_1 + r_2 = r$, where $\vee_{r_1, r_2} : \mathrm{Sym}^{r_1} \mathbb{G}_{m,k} \times \mathrm{Sym}^{r_2} \mathbb{G}_{m,k} \rightarrow \mathrm{Sym}^r \mathbb{G}_{m,k}$ is the canonical map.

We have the following commutative diagram :

$$\begin{array}{ccc} \mathrm{Sym}^{r_1} \mathbb{G}_{m,k} \times \mathrm{Sym}^{r_2} \mathbb{G}_{m,k} & \longrightarrow & \mathrm{Sym}^r \mathbb{G}_{m,k} \\ \bar{s}_{r_1} \times \bar{s}_{r_2} \downarrow & & \downarrow \bar{s}_r \\ \mathbb{G}_{m,k} \times \mathbb{G}_{m,k} & \xrightarrow{\mu} & \mathbb{G}_{m,k}, \end{array}$$

where $\mu : \mathbb{G}_{m,k} \times \mathbb{G}_{m,k} \rightarrow \mathbb{G}_{m,k}$ is the multiplication map. By Proposition 5.1 1, we have

$$R\tilde{s}_{r,!}\Gamma_{\text{ext}}^r \mathcal{F}'_V \cong R\mu_!(R\tilde{s}_{r'}\Gamma_{\text{ext}}^{r'} \mathcal{F}'_{V'} \boxtimes R\tilde{s}_{r''}\Gamma_{\text{ext}}^{r''} \mathcal{F}'_{V''}).$$

Hence the assertion follows from simple computation. \square

This completes the proof of Proposition 5.1. \square

REMARK 5.8. In view of Corollary 5.5 and the proof of Proposition 5.1, we have the following refinement of Proposition 5.1 (2): in the notation of Proposition 5.1, fix a non-trivial additive character $\psi : k \rightarrow R_0^\times$. Take a non-zero meromorphic differential ω on $\mathbb{G}_{m,k}$. Let K_0 and K_∞ , be the completions of the function field of $\mathbb{G}_{m,k}$ at 0 and ∞ , respectively. We identify K_0 with K by the morphism i_0 . Take a non-zero meromorphic function $f \in k(\pi_K)^\times$ on $\mathbb{G}_{m,k}$ such that the class of f in $K_0^\times/1 + \mathfrak{m}_{K_0}$ and $K_\infty^\times/1 + \mathfrak{m}_{K_\infty}$ are equal to the refined Swan conductors $\text{rsw}_{\psi,\omega,0}(V)$ and $(\text{rsw}_{\psi,\omega,\infty}(\mathcal{L}))^{\text{rank } V}$, respectively. Let $g(\pi_K) \in k(\pi_K)^\times$ be a rational function such that

$$\text{div}(g) = \text{div}(\omega_0) - \text{div}(f) + \mathfrak{m} - r[1]$$

Then $R\tilde{s}_{r,!}\Gamma_{\text{ext}}^r \mathcal{F}'_V$ is supported on the k -rational point $\frac{g(0)}{g(\infty)} \in k^\times = \mathbb{G}_{m,k}(k)$.

6. Determinant of \mathcal{F}'_V

Let the notation be the same as in the beginning of § 5. Fix a non-trivial additive character $\psi_0 : k \rightarrow R_0^\times$. Let $\psi : K \rightarrow R_0^\times$ be the continuous additive character given by $\psi(x) = \psi_0(\text{Res}(x \frac{d\pi_K}{\pi_K}))$. Let $\beta \in k^\times$ be the unique element satisfying $\phi_0(\frac{x}{\pi_K}) = \psi_0(-\beta x)$ for all $x \in k$.

PROPOSITION 6.1. *Let V and (ρ, W) be two objects in $\text{Rep}(W_K, R_0)$. Suppose that V is totally wild and that W is tamely ramified. Then we have*

$$\begin{aligned} & \det(-\text{Fr}; R\Gamma_c(\mathcal{F}'_{V \otimes W}))^{-1} \\ &= \det(-\text{Fr}; R\Gamma_c(\mathcal{F}'_V))^{-\text{rank } W} \cdot \det W(\text{rec}(\text{rsw}_\psi(V) \cdot (\beta\pi_K)^{\text{rank } V})). \end{aligned}$$

REMARK 6.2. Let $K_\infty = k(\frac{1}{\pi_K})$ and let $\psi_\infty : K_\infty \rightarrow R_0^\times$ be the continuous additive character given by $\psi_\infty(x) = \psi_0(\text{Res}(x \frac{d\pi_K}{\pi_K}))$. Then we have $\text{rsw}_{\psi_\infty}(\mathcal{L}_\infty) = \beta^{-1} \pi_K^{-1}$. Hence $\text{rsw}_{\psi_\infty}((\mathcal{F}'_V)_\infty) = \beta^{-\text{rank } V} \pi_K^{-\text{rank } V}$. Therefore, we have

$$\begin{aligned} \det(\mathcal{F}_W)_\infty(\text{rec}_{K_\infty}(\text{rsw}_{\psi_\infty}((\mathcal{F}'_V)_\infty))) &= \det(\mathcal{F}_W)_\infty(\text{rec}_{K_\infty}(\beta \cdot \pi_K))^{-\text{rank } V} \\ &= \det W(\text{rec}(\beta \cdot \pi_K))^{\text{rank } V}. \end{aligned}$$

PROOF. We set $G = \pi_1^{tm}(\mathbb{G}_{m,k})$, $I = \pi_1^{tm}(\mathbb{G}_{m,\bar{k}})$, and $I_m = I/I^m$. For every integer m which is prime to p , let $\pi_m : \mathbb{G}_{m,k} \rightarrow \mathbb{G}_{m,k}$ be the m -th power map, $W_m = H_c^1(\mathbb{G}_{m,\bar{k}}, (\pi_{m,*} R_0) \otimes_{R_0} \mathcal{F}'_V)$. W_m is a free $R[I_m]$ -module of rank r with a semi-linear action of G , where $r = \text{rank } V + \text{sw}(V)$. If n, m are two positive integers which are prime to p with $m|n$, the canonical map $W_n \rightarrow W_m$ induces an isomorphism

$$W_n \otimes_{R_0[I_n]} R_0[I_m] \cong W_m.$$

Let $\widehat{W} = \varprojlim_m W_m$ be the projective limit of W_m . \widehat{W} is a free $R_0[[I]]$ -module of rank r with a semi-linear action of G .

Consider the maximal exterior power $\det_{R_0[[I]]} \widehat{W} = \bigwedge_{R_0[[I]]}^r \widehat{W}$ of \widehat{W} . It is a free $R_0[[I]]$ -module of rank one with a semi-linear action of G . We note that the action of $I \subset G$ on $\det_{R_0[[I]]} \widehat{W}$ does not necessarily coincide with the action of $I \subset R_0[[I]]^\times$.

Take a lift $\widetilde{\text{Fr}}_q \in G$ of the geometric Frobenius Fr_q . An argument similar to that in [Y1, Remark 11.3] shows that the eigenvalue of $\widetilde{\text{Fr}}_q$ on $\det_{R_0[[I]]} \widehat{W}$ gives a well-defined element $\widehat{u} \in (R_0[[I]]^\times)_G$ in the G -coinvariant. As in [Y1, § 11.2], we have

LEMMA 6.3. *The determinant of $\det(\text{Fr}_k; H_{c,\text{et}}^1(\mathbb{G}_{m,\bar{k}}, \mathcal{F}'_{V \otimes_{R_0} W}))$ is equal to*

$$\det(\widetilde{\text{Fr}}_q; W)^r \cdot \det \left(\int_I \rho(g)^{-1} d\widehat{u}(g) \right).$$

PROOF. Take a sufficiently divisible m which is prime to p such that the restriction $\pi_m^* \mathcal{F}_W$ is geometrically constant. By the same argument as

in [Y1, § 11.1] we have $H_c^1(\mathbb{G}_{m,\bar{k}}, \mathcal{F}'_{V \otimes_{R_0} W}) \cong (W_m \otimes_{R_0} W)_{I_m}$. The lemma follows from an argument similar to that in [Y1, § 11.1]. \square

Let us go back to the proof of Proposition 6.1. We consider W_K/W_K^{0+} as a subgroup of G in a canonical way. Then $\widetilde{\text{Fr}}_q^r$ lies in W_K/W_K^{0+} . The element $\text{rs}w_\psi(V) \cdot (\beta\pi_K)^{\text{rank } V}$ in $K^\times/1 + \mathfrak{m}_K$ and $\text{rec}^{-1}(\widetilde{\text{Fr}}_q^r) \in K^\times/1 + \mathfrak{m}_K$ only differ by an element in k^\times which we denote by $a_{\widetilde{\text{Fr}}_q, \psi} \in k^\times$, that is

$$\text{rs}w_\psi(V) \cdot (\beta\pi_K)^{\text{rank } V} = a_{\widetilde{\text{Fr}}_q, \psi} \cdot \text{rec}^{-1}(\widetilde{\text{Fr}}_q^r).$$

We consider $a_{\widetilde{\text{Fr}}_q, \psi}$ as an element in $I/I^{q-1} \subset (R_0[[I]])_G^\times$ by the canonical isomorphism $I/I^{q-1} \cong k^\times$. Then it suffices to prove that

$$\widehat{u} = a_{\widetilde{\text{Fr}}_q, \psi} \cdot \det(\text{Fr}_k; H_c^1(\mathbb{G}_{m,\bar{k}}, \mathcal{F}'_V)) \text{ in } (R_0[[I]])_G^\times.$$

By the same way as in [Y1, § 11.3], we have

$$\det_{R_0[[I]]} \widehat{W} \cong \varprojlim_m R\Gamma_c(\mathbb{G}_{m,\bar{k}}, (\pi_{m,*} R_0) \otimes_{R_0} R\widetilde{s}_{r,!} \Gamma_{\text{ext}}^r \mathcal{F}'_V)[r].$$

By Proposition 5.1, $R\widetilde{s}_{r,!} \Gamma_{\text{ext}}^r$ is supported on a closed point P in $\mathbb{G}_{m,k}$. By computation of rank, P is a k -rational point, $R^i \widetilde{s}_{r,!} \Gamma_{\text{ext}}^r \mathcal{F}'_V = 0$ for $i \neq r$, and the geometric stalk $R^i \widetilde{s}_{r,!} \Gamma_{\text{ext}}^r \mathcal{F}'_V$ at P is a free R_0 -module of rank one. Hence there exists a prime element π'_K in K such that for every positive integer m with $(m, p) = 1$, $\det_{R_0[[I_m]]} W_m$ is isomorphic to the free R_0 -module

$$\left(\bigoplus_{x^m = \pi'_K} R_0 \right) \otimes_{R_0} \det_{R_0} H_c^1(\mathbb{G}_{m,\bar{k}}, \mathcal{F}'_V),$$

endowed with the canonical action of G . Hence there exists an element $a' \in I/I^{q-1}$ such that $\widehat{u} = a' \cdot \det(\text{Fr}_k; H_c^1(\mathbb{G}_{m,\bar{k}}, \mathcal{F}'_V))$ in $(R_0[[I]])_G^\times$. It suffices to prove that $a' = a_{\widetilde{\text{Fr}}_q, \psi}$. This assertion follows from Remark 5.8. \square

COROLLARY 6.4. *Let R be a strict p' -coefficient ring which contains R_0 , V a totally wild object in $\text{Rep}(W_K, R_0)$. Then the product formula holds for $\mathcal{F}'_V \otimes_{R_0} R$.*

PROOF. By Lemma 3 and [Y1, Prop. 9.14], we may assume that V is of the form $V = V_1 \otimes_{R_0} V_2$, where V_1 is a tamely ramified object in

$\text{Rep}(W_K, R_0)$ and V_2 is the base change of an object in $\text{Rep}(W_K, R'_0)$ by a local ring homomorphisms $R'_0 \rightarrow R_0$, where R'_0 is a p' -coefficient ring which is a complete discrete valuation ring with a finite residue field whose field of fractions is of characteristic zero. By [Lau2, p. 187, Thm. 3.2.1.1], the product formula holds for V_2 . Since V_2 is totally wild, the product formula also holds for V by [Y1, Prop. 8.3] and Proposition 6.1. \square

7. Fourier Transforms

7.1. Deligne-Laumon's global Fourier transform $F^{(0, \infty')}$

Let k be a finite field of characteristic p , R_0 a finite p' -coefficient ring, U a non-empty open subscheme of the affine line \mathbb{A}_k^1 .

Take a non-trivial additive character $\phi : k \rightarrow R_0^\times$. Let \mathcal{L}_ϕ be the Artin-Schreier sheaf on \mathbb{A}_k^1 associated to ϕ . Let $\langle , \rangle : \mathbb{A}_k^1 \times_k \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ be the product map. According to [Lau2, p. 148, 1.4.1], let us denote the sheaf $\langle , \rangle^* \mathcal{L}_\phi$ on $\mathbb{A}_k^1 \times_k \mathbb{A}_k^1$ by $\mathcal{L}_\phi(xx')$. We also use the notation $\mathcal{L}_\psi(s.x')$, $\mathcal{L}_\psi(x.s')$ and $\mathcal{L}_\psi((x-s).x')$ in [Lau2, p. 148–149, 1.4.2, and p. 195]. Let $\overline{\mathcal{L}}_\phi(xx')$ be the extension by zero of $\mathcal{L}_\phi(xx')$ by the canonical inclusion $\mathbb{A}_k^1 \times_k \mathbb{A}_k^1 \hookrightarrow \mathbb{P}_k^1 \times_k \mathbb{P}_k^1$.

For a smooth R_0 -flat R_0 -sheaf \mathcal{F} on U which is at most tamely ramified at $\infty \in \mathbb{P}_k^1 - \mathbb{A}_k^1$, define the *global Fourier transform* $\text{FT}_{\phi,U}(\mathcal{F})$ of \mathcal{F} as

$$\text{FT}_{\phi,U}(\mathcal{F}) := R^1 \text{pr}_{2,!}(\text{pr}_1^* \mathcal{F} \otimes (\mathcal{L}_\phi(xx')|_{U \times \mathbb{G}_{m,k}})),$$

where $\text{pr}_1 : U \times \mathbb{G}_{m,k} \rightarrow U$ and $\text{pr}_2 : U \times \mathbb{G}_{m,k} \rightarrow \mathbb{G}_{m,k}$ are projections.

LEMMA 7.1. $\text{FT}_{\phi,U}(\mathcal{F})$ is a smooth R_0 -flat R_0 -sheaf on $\mathbb{G}_{m,k}$.

PROOF. It is easy to see that $\text{FT}_{\phi,U}(\mathcal{F})$ is a R_0 -flat R_0 -sheaf on $\mathbb{G}_{m,k}$ whose all geometric fibers are of a same rank. To prove the smoothness of $\text{FT}_{\phi,U}(\mathcal{F})$, we may assume that R_0 is a field. Let \mathcal{G} be the extension by zero of the sheaf $\text{pr}_1^* \mathcal{F} \otimes (\mathcal{L}_\phi(xx')|_{U \times \mathbb{G}_{m,k}})$ on $U \times \mathbb{G}_{m,k}$ to $\mathbb{P}_k^1 \times \mathbb{G}_{m,k}$. Then $(\mathbb{P}_k^1 \times \mathbb{G}_{m,k}, \mathcal{G}, \text{pr}_2)$ is universally locally acyclic by [Lau1, p. 186, Thm. 2.1.1]. Hence $\text{FT}_{\phi,U}(\mathcal{F})$ is smooth. \square

Let $\alpha : U \hookrightarrow \mathbb{P}_k^1$ be the inclusion map, and $D_{0'}$ the henselizations of \mathbb{P}_k^1 at 0. Let $\overline{\text{pr}}_1 : \mathbb{P}^1 \times D_{0'} \rightarrow \mathbb{P}^1$, $\overline{\text{pr}}_2 : \mathbb{P}^1 \times D_{0'} \rightarrow D_{0'}$ be projections. Let $\eta_{0'}$ be the generic point of $D_{0'}$. Let us consider the vanishing cycle

$$R\Phi_{\eta_{\infty'}}(\overline{\text{pr}}_1^*(\alpha! \mathcal{F}) \otimes (\overline{\mathcal{L}}_\phi(xx')|_{\mathbb{P}^1 \times D_{0'}})) \in D(\mathbb{P}^1 \times \eta_{0'}, R_0)$$

for the projection $\overline{\mathrm{pr}}_2$.

The following lemma is proved in a manner similar to that in [Lau2, p. 160, Prop. 2.3.2.1]:

LEMMA 7.2.

- (1) *The restriction of $R\Phi_{\eta_{0'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes \mathcal{L})$ to $U \times \eta_{0'}$ is zero.*
- (2) *For any closed point $s \in \mathbb{P}_k^1$, let \bar{s} denote a geometric point of \mathbb{P}_k^1 over s . Then for $i \neq -1$,*

$$R^i\Phi_{\eta_{0'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes \mathcal{L})_{(\infty, \bar{\sigma}')}$$

is zero.

- (3) *We have a canonical distinguished triangle*

$$R\Gamma_c(U \otimes_k \bar{k}, \mathcal{F}) \rightarrow \mathrm{FT}_{\phi, U}(\mathcal{F})_{\eta_{0'}} \rightarrow R^{-1}\Phi_{\eta_{0'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes \mathcal{L})_{(\infty, \bar{\sigma}')} \xrightarrow{+1} .$$

Let $D_{\infty'}$ be the henselizations of \mathbb{P}_k^1 at ∞ . Let $\overline{\mathrm{pr}}_1 : \mathbb{P}^1 \times D_{\infty'} \rightarrow \mathbb{P}^1$, $\overline{\mathrm{pr}}_2 : \mathbb{P}^1 \times D_{\infty'} \rightarrow D_{\infty'}$ be projections. Let $\eta_{\infty'}$ be the generic point of $D_{\infty'}$. Let us consider the vanishing cycle

$$R\Phi_{\eta_{\infty'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes (\overline{\mathcal{L}}_{\phi}(xx')|_{\mathbb{P}^1 \times D_{\infty'}})) \in D(\mathbb{P}^1 \times \eta_{\infty'}, R_0)$$

for the projection $\overline{\mathrm{pr}}_2$.

LEMMA 7.3.

- (1) *The restriction of $R\Phi_{\eta_{\infty'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes \mathcal{L})$ to $U \times \eta_{\infty'}$ is zero.*
- (2) *For any closed point $s \in \mathbb{P}_k^1$, let \bar{s} denote a geometric point of \mathbb{P}_k^1 over s . Then for $s \in \mathbb{P}_k^1 - U$ and for $i \neq -1$,*

$$R^i\Phi_{\eta_{\infty'}}(\overline{\mathrm{pr}}_1^*(\alpha_!\mathcal{F}) \otimes \mathcal{L})_{(\bar{s}, \overline{\infty}')}$$

is zero.

- (3) For any closed point s on a curve C over k , let G_s denote the absolute Galois group of the fraction field of the henselization of C at s . Then we have

$$\mathrm{FT}_{\phi,U}(\mathcal{F})_{\bar{\eta}_{\infty'}} = \bigoplus_{s \in \mathbb{P}_k^1 - U} \mathrm{Ind}_{G_{s \times_k \infty'}}^{G_{\infty'}} R^{-1} \Phi_{\bar{\eta}_{\infty'}}(\overline{\mathrm{pr}}_1^*(\alpha! \mathcal{F}) \otimes \mathcal{L})_{(\bar{s}, \bar{\infty}')}.$$

REMARK 7.4. We will see later that

$$R^{-1} \Phi_{\bar{\eta}_{\infty'}}(\overline{\mathrm{pr}}_1^*(\alpha! \mathcal{F}) \otimes \mathcal{L})_{(\bar{\infty}, \bar{\infty}')}.$$

is zero.

PROOF. The assertion (1) follows from [KL, 2.4]. The assertions (2) and (3) are proved in a manner similar to that in [Lau2, p. 161, Prop. 2.3.3.1]. \square

7.2. Laumon's local Fourier transform $F^{(0, \infty')}$

Let K be a p -local field of characteristic p with residue field k , π_K a prime element in K . Since the subring $\cap_n (K)^{p^n}$ of K is canonically isomorphic to k , the field K has a canonical structure of k -algebra. Let \mathcal{O}_K^h be the henselization of $k[\pi_K]_{(\pi_K)}$, $K^h = \mathrm{Frac}(\mathcal{O}_K^h)$. Let R_0 be a finite local ring on which p is invertible. For an object V in $\mathrm{Rep}(W_K, R_0)$, let \tilde{V} denote the étale R_0 -sheaf on $\mathrm{Spec}(K^h)$ corresponding to V . Let $j_K : \mathrm{Spec}(K^h) \rightarrow \mathrm{Spec}(\mathcal{O}_K^h)$ be the canonical inclusion. We consider the R_0 -sheaf $j_{K,!} \tilde{V}$ on $\mathrm{Spec}(\mathcal{O}_K^h)$.

Suppose further that there exists a non-trivial additive character $\phi : \mathbb{F}_p \rightarrow R_0^\times$. Let $A = \mathbb{A}_k^1 = \mathrm{Spec}(k[t])$ be the affine line over k , \mathcal{L}_ϕ be the smooth étale R_0 -sheaf on A defined by the Artin-Schreier equation $s^p - s = t$ and ϕ . Let $\mathcal{L}_\phi^{(0, \infty')}$ be the pull-back of \mathcal{L}_ϕ by the morphism from $\mathrm{Spec}(\mathcal{O}_K^h) \times_k \mathrm{Spec}(K^h)$ to A whose associated homomorphism of coordinate rings is given by $t \mapsto \frac{\pi_K \otimes 1}{1 \otimes \pi_K}$, and set $\overline{\mathcal{L}}_\phi^{(0, \infty')} = (id \times j_K)_! \mathcal{L}_\phi^{(0, \infty')}$. Let us consider the vanishing cycle

$$R^1 \Phi_{\bar{\eta}'}(\mathrm{pr}_1^*(j_{K,!} \tilde{V}) \otimes_{R_0} \overline{\mathcal{L}}_\phi^{(0, \infty')})$$

relative to $\mathrm{pr}_2 : \mathrm{Spec}(\mathcal{O}_K^h) \times_k \mathrm{Spec}(K^h) \rightarrow \mathrm{Spec}(\mathcal{O}_K^h)$.

DEFINITION 7.5. For an object V in $\text{Rep}(W_K, R_0)$, let

$$F^{(0, \infty')}(V) := R^1 \Phi_{\bar{\eta}'}(\text{pr}_1^*(j_{K,!} \tilde{V}) \otimes_{R_0} \overline{\mathcal{L}}_\phi^{(0, \infty')})_{\bar{t}, \bar{t}'},$$

where $\bar{t} = \bar{t}'$ is the spectrum of an algebraic closure of the residue field at the closed point of $\text{Spec}(\mathcal{O}_K^h)$. Then $F^{(0, \infty')}(V)$ is also an object in $\text{Rep}(W_K, R_0)$.

We also define objects $F^{(\infty, 0')}(V)$ and $F^{(\infty, \infty')}(V)$ in $\text{Rep}(W_K, R_0)$ in a manner similar to that in [Lau2, p. 163, Defn. 2.4.2.3]. The objects $F^{(0, \infty')}(V)$, $F^{(\infty, 0')}(V)$ and $F^{(\infty, \infty')}(V)$ in $\text{Rep}(W_K, R_0)$ are called the *local Fourier transforms* of V .

The following lemma is easily checked:

LEMMA 7.6 (cf. [Lau2, 2.4.2.1]). *Let $\pi : \text{Spec}(\mathcal{O}_K^h) \rightarrow A$ and $\pi' : \text{Spec}(K^h) \rightarrow A$ denote the morphisms whose associated homomorphisms of coordinate rings are given by $t \mapsto \pi_K$, and $t \mapsto \frac{1}{\pi_K}$, respectively. Then for any smooth R_0 -sheaf \mathcal{F} on a nonempty open subscheme $U \subset A$ such that $\pi^* \alpha_! \mathcal{F} \cong j_{K,!} \tilde{V}$, where $\alpha : U \hookrightarrow \mathbb{P}_k^1$ is the canonical inclusion, we have*

$$F^{(0, \infty')}(V) = (\pi \times \pi')^* R\Phi_{\eta'_0}(\overline{\text{pr}}_1^*(\alpha)_! \mathcal{F} \otimes_{R_0} \overline{\mathcal{L}}_\phi)_{(\bar{t}, \bar{t}')}.$$

Similar statements hold for $F^{(\infty, 0')}$ and $F^{(\infty, \infty')}$.

As a corollary, we have the exactness of the three functors $F^{(0, \infty')}$, $F^{(\infty, 0')}$, and $F^{(\infty, \infty')}$.

PROPOSITION 7.7 (cf. [Lau2, p. 165, 2.5.3]). *Let $V \in \text{Rep}(W_K, R_0)$ be a tamely ramified object. Then we have*

- (1) $F^{(0, \infty')}(V)$ is a tamely ramified object with the same rank as V .
- (2) $F^{(\infty, 0')}(V)$ is a tamely ramified object with the same rank as V . If V is unramified, then $F^{(\infty, 0')}(V) \cong V(-1)$.
- (3) $F^{(\infty, \infty')}(V) = 0$.

PROOF. There exists a smooth R_0 -flat R_0 -sheaf \mathcal{F} on $\mathbb{G}_{m,k}$ which is tamely ramified both at 0 and at ∞ such that the geometric stalk of \mathcal{F} at η_0

can be identified with V . Then global Fourier transform $\mathrm{FT}_{\phi, \mathbb{G}_{m,k}}(\mathcal{F})$ of \mathcal{F} is a smooth R_0 -sheaf on $\mathbb{G}_{m,k}$ whose geometric stalks are free R_0 -modules of rank $\mathrm{rank} V$. Take any element $a \in k^\times$ and let $t_a : \mathbb{G}_{m,\bar{k}} \rightarrow \mathbb{G}_{m,\bar{k}}$ be the translation by a . It is easy to check that $t_a^*(\mathrm{FT}_{\phi, \mathbb{G}_{m,k}}(\mathcal{F})|_{\mathbb{G}_{m,\bar{k}}})$ is isomorphic to $\mathrm{FT}_{\phi, \mathbb{G}_{m,k}}(\mathcal{F})|_{\mathbb{G}_{m,\bar{k}}}$ for any $a \in \bar{k}$. By [V, p. 336, Prop. 1.1], $\mathrm{FT}_{\phi, \mathbb{G}_{m,k}}(\mathcal{F})$ is also tamely ramified both at 0 and at ∞ . Hence (2) follows from Lemma 7.2 (3).

By Lemma 7.3 (3), $F^{(0,\infty')}(V) \oplus F^{(\infty,\infty')}(\mathcal{F}_{\bar{\eta}_\infty})$ is a tamely ramified object with the same rank as V . Hence to complete the proof of the proposition, it suffices to prove that $F^{(\infty,\infty')}(V) = 0$.

Let $1 \in \mathrm{Rep}(W_K, R_0)$ be the trivial representation. By considering the constant sheaf R_0 on $\mathbb{G}_{m,k} - \{1\}$, we have $2 \mathrm{rank} F^{(0,\infty')}(1) + \mathrm{rank} F^{(\infty,\infty')}(1) = 2$. Hence $F^{(\infty,\infty')}(1) = 0$. For general V , there exists a smooth R_0 -sheaf \mathcal{F}' on $\mathbb{G}_{m,k} - \{1\}$ such that $\mathcal{F}'_{\bar{\eta}_\infty}$ trivial and that $\mathcal{F}'_{\bar{\eta}_0}$ is isomorphic to V . Using this \mathcal{F}' , we have $\mathrm{rank} F^{(0,\infty')}(V) = \mathrm{rank} V$. Hence the assertion follows. \square

COROLLARY 7.8. *Let V be an object in $\mathrm{Rep}(W_K, R_0)$. Then $\mathrm{rank} F^{(0,\infty')}(V) = \mathrm{rank} V + \mathrm{sw}(V)$ and $\mathrm{sw}(F^{(0,\infty')}(V)) = \mathrm{sw}(V)$.*

PROOF. We have $F^{(0,\infty')} \cong (\mathrm{FT}_{\phi, \mathbb{G}_{m,k}}(\mathcal{F}_V))_{\bar{\eta}_{\infty'}}$. Applying the Grothendieck-Ogg-Shafarevich formula to \mathcal{F}_V , we have $\mathrm{rank} F^{(0,\infty')}(V) = \mathrm{rank} V + \mathrm{sw}(V)$. By Proposition 7.7, $\mathrm{FT}_{\phi, \mathbb{G}_{m,k}}(\mathcal{F}_V)$ is tamely ramified at $0'$. Since

$$\begin{aligned} R\Gamma_c(\mathbb{G}_{m,\bar{k}}, \mathrm{FT}_{\phi, \mathbb{G}_{m,k}}(\mathcal{F}_V)) \\ \cong R\Gamma_c(\mathbb{G}_{m,\bar{k}} \times \mathbb{G}_{m,\bar{k}}, \mathcal{F}_V \otimes (\mathcal{L}_\phi(xx')|_{\mathbb{G}_{m,k} \times \mathbb{G}_{m,k}})[1]) \\ \cong R\Gamma_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F}_V), \end{aligned}$$

the second assertion follows from the Grothendieck-Ogg-Shafarevich formula. \square

Let U be a non-empty open subscheme of A , \mathcal{F} a smooth R_0 -flat R_0 -sheaf on U which is unramified at ∞ , $\alpha : U \hookrightarrow \mathbb{P}_k^1$ the canonical inclusion. Let $k\{t\}$ be the henselization of $k[t]_{(t)}$. For any closed point $s \in A$, let $P_s(t) \in k[t]$ be the unique monic irreducible polynomial which vanishes at s . Let $A_{(s)}$ be the henselization of A at s , η_s the generic point of $A_{(s)}$. Let

$\pi_s : A_{(s)} \rightarrow \text{Spec}(k\{t\})$ be the morphism whose associated homomorphism of coordinate rings is given by $t \mapsto P_s(t)$.

PROPOSITION 7.9 (cf. [Lau2, p. 194, Thm. 3.4.2]). *Let $S = A - U$ be the complement of U . Then*

(1) *For all $s \in S$,*

$$\det(F^{(0, \infty')})(\pi_{s,*}(\mathcal{F}|_{\eta_s}))$$

is tamely ramified.

(2)

$$\bigotimes_{s \in S} \det(F^{(0, \infty')})(\pi_{s,*}(\mathcal{F}|_{\eta_s}))$$

is unramified and is isomorphic to

$$\det(R\Gamma_c(U \otimes_k \bar{k}, \mathcal{F}))^{-1} \otimes \det(\mathcal{F}_\infty(-1)).$$

PROOF. (1) Consider the global Fourier transform

$$\mathcal{F}' = \text{FT}_{\phi, U}(\mathcal{F})$$

of \mathcal{F} . The sheaf \mathcal{F}' is a smooth R_0 -flat R_0 -sheaf on $\mathbb{G}_{m, k}$. There exists a distinguished triangle

$$R\Gamma_c(U \otimes_k \bar{k}, \mathcal{F})[1] \rightarrow \mathcal{F}'_{\bar{\eta}'_0} \rightarrow \mathcal{F}_\infty(-1) \xrightarrow{+1}.$$

Hence $\det(\mathcal{F}')$ is unramified at 0 and

$$(j'_* \det(\mathcal{F}'))_{\bar{\eta}'_0} \cong \det(R\Gamma_c(U \otimes_k \bar{k}, \mathcal{F}))^{-1} \otimes \det(\mathcal{F}_\infty(-1)),$$

where $j' : \mathbb{G}_{m, k} \hookrightarrow A$.

(2) Since

$$\mathcal{F}'_{\bar{\eta}'_\infty} \cong \bigoplus_{s \in S} \text{Ind}_{G_s \times_k \infty'}^{G_s} (R^{-1} \Phi_{\bar{\eta}'_\infty}(\mathcal{M}(\mathcal{F})))_{\bar{s}, \bar{\infty}'},$$

where

$$\begin{aligned} & (R^{-1} \Phi_{\bar{\eta}'_\infty}(\mathcal{M}(\mathcal{F})))_{\bar{s}, \bar{\infty}'} \\ & \cong \bar{\mathcal{L}}_\phi(s.x')_{\bar{\eta}'_\infty} \otimes_{R_0} (R^{-1} \Phi_{\bar{\eta}'_\infty}(\bar{\text{p}}\Gamma^*(\alpha_1 \mathcal{F}) \otimes \bar{\mathcal{L}}_\phi((x-s).x'))_{\bar{s}, \bar{\infty}'}), \end{aligned}$$

we have

$$\det(\mathcal{F})_{\overline{\eta}_{\infty'}} \cong \mathcal{L}_{\phi}(\delta_{\mathcal{F}}.x')_{\overline{\eta}_{\infty'}} \otimes_{R_0} \bigotimes_{s \in S} \det(F^{0, \infty'}(\pi_{s, *}(F|_{\eta_s}))),$$

where

$$\delta_{\mathcal{F}} = \sum_{s \in S} (\text{rank } \mathcal{F} + \text{sw}_s(\mathcal{F})) \cdot \text{Tr}(s) \in k.$$

Hence, $\mathcal{L}(-\delta_{\mathcal{F}}.x') \otimes j'_* \det(\mathcal{F}')$ is a smooth invertible R_0 -sheaf on A which is tamely ramified at ∞ . By the global class field theory, $\mathcal{L}(-\delta_{\mathcal{F}}.x') \otimes j'_* \det(\mathcal{F}')$ must be geometrically constant. Therefore

$$\bigotimes_{s \in S} \det(F^{(0, \infty')}(\pi_s(F|_{\eta_s})))$$

is an unramified object which is isomorphic to

$$\det(R\Gamma_c(U \otimes_k \overline{k}, \mathcal{F}))^{-1} \otimes \det(\mathcal{F}_{\infty}(-1)).$$

Hence the proposition follows. \square

8. End of Proof

DEFINITION 8.1. Let $\psi : K \rightarrow R_0^{\times}$ be a non-trivial continuous additive character. Let $a \in K^{\times}$ be the unique element satisfying $\psi(x) = \phi(\text{Tr}_{k/\mathbb{F}_p}(\text{Res}(axd\pi_K)))$ for every $x \in K$. Define the ε'_{0, R_0} -constant $\varepsilon'_{0, R_0}(V, \psi, \phi, \pi_K) \in R_0^{\times}$ as

$$\begin{aligned} \varepsilon'_{0, R_0}(V, \psi, \phi, \pi_K) &:= (-1)^{\text{rank } V + \text{sw}V} \\ &\cdot \det V(\text{rec}(a))q^{v_K(a)} \det(F^{(0, \infty')}(V))(\text{rec}(\pi_K)), \end{aligned}$$

PROPOSITION 8.2. Let $A = \text{Spec } k[t]$, U and \mathcal{F} be as above. In the notation of Theorem 4.1, suppose that $X_0 = \mathbb{P}_k^1$, $U_0 = U$, $\omega = -dt$ and $\mathcal{F} = \mathcal{F}$. For $x \in A - U_0$, let $\pi_x = P_x(t)$ and let $\pi_{\infty} = \frac{1}{t}$. Then we have

$$\varepsilon_{R_0}(U_0, \mathcal{F}) = q^{\frac{1}{2}\chi(X)\text{rank}(\mathcal{F})} \prod_{x \in X_0 - U_0} \varepsilon'_{0, R_0}(\mathcal{F}_x, \psi_{\omega, x}, \phi, \pi_x).$$

PROOF. It follows from Proposition 7.9 (2). \square

Hence to prove Theorem 4.1, it suffice to prove the following proposition:

PROPOSITION 8.3. *For any object V in $\text{Rep}(W_K, R_0)$ and for any non-trivial continuous additive character $\psi : K \rightarrow R_0^\times$, we have*

$$\varepsilon_{0,R}(V \otimes_{R_0} R, \psi) = \varepsilon'_{0,R_0}(V, \psi, \phi, \pi_K).$$

LEMMA 8.4. *Proposition 8.3 holds for (V, ψ) if V is a tamely ramified object.*

PROOF. We may assume that the character ψ is of the form

$$\psi(x) = \phi(\text{Tr}_{k/\mathbb{F}_p}(\text{Res}(x \frac{d\pi_K}{\pi_K}))).$$

Let V be a tamely ramified object in $\text{Rep}(W_K, R_0)$, and \mathcal{F}_V the canonical extension sheaf on $\mathbb{G}_{m,k}$ corresponding to V . Consider the global Fourier transform $\text{FT}_{\phi, \mathbb{G}_{m,k}}(\mathcal{F}_V)$.

Then for every closed point x in $\mathbb{G}_{m,k}$, we have, by [Y1, Thm. 10.5] and [Y1, Thm. 5.6]

$$\det(\text{Fr}_x ; \text{FT}_{\phi, \mathbb{G}_{m,k}}(\mathcal{F}_V)) = \det(\text{Fr}_x ; \mathcal{F}_V) \cdot (q \cdot \varepsilon_{0,R}(V \otimes_{R_0} R, \psi))^{\deg(x)}.$$

Hence $\det(\text{FT}_{\phi, \mathbb{G}_{m,k}}(\mathcal{F}_V)) \otimes_{R_0} \det(\mathcal{F}_V)^{\otimes -1}$ is geometrically constant. The lemma follows by comparing the traces of the geometric Frobenius at 1 and at ∞ on $\det(\text{FT}_{\phi, \mathbb{G}_{m,k}}(\mathcal{F}_V)) \otimes_{R_0} \det(\mathcal{F}_V)^{\otimes -1}$. \square

COROLLARY 8.5. *Let $\phi : k \rightarrow R_0^\times$ be a non-trivial character and $\chi : W_K^{ab} \rightarrow R_0^\times$ be the character given by*

$$\chi(\text{rec}(x)) = \phi(\text{Res}(\frac{dx}{\pi_K x})).$$

Then Proposition 8.3 holds for (V, ψ) if V is of the form $V = W \otimes \chi$ for a tamely ramified object W in $\text{Rep}(W_K, R_0)$.

PROOF. This follows from the definition of $\varepsilon_{0,R}(W, \psi)$. \square

PROOF OF PROPOSITION 8.3. By Lemma 8.4, we may assume that V is totally wild. Let us consider the sheaf \mathcal{F}'_V . By Corollary 6.4, the product formula holds for $\mathcal{F}'_V \otimes_{R_0} R$. Hence the assertion follows from Proposition 8.2 and Corollary 8.5. \square

This completes the proof of Theorem 4.1. \square

9. Application to Saito's Formula

Let k be a finite field of q elements with characteristic p , X/k a smooth projective variety of pure dimension n , and $U \subset X$ an open subscheme such that $D = X - U$ is a divisor with simple normal crossing.

Let $c_{X,U}$ be the element in $\pi_1^{tm}(U)^{ab}$ which is introduced and is called the *relative canonical cycle* in [Sa1, p. 402]; the prime-to- p part of $c_{X,U}$ is equal to

$$\begin{aligned} &(-1)^n c_n(\Omega_{X/k}^1(\log D), \text{Res}) \in H^{2n}(X \pmod D, \widehat{\mathbb{Z}}'(n)) \\ &:= \prod_{l \neq p} H^{2n}(U, j_l \mathbb{Z}_l(n)_U), \end{aligned}$$

where $j : U \hookrightarrow X$ is the canonical inclusion and $c_n(\Omega_{X/k}^1(\log D), \text{Res})$ is the relative top chern class for the partially trivialized locally free sheaf $(\Omega_{X/k}^1(\log D), \text{Res})$ which is introduced in [Sa1, p. 391], and the p -part of $c_{X,U}$ is the image of usual chern class $(-1)^n c_n(\Omega_{X/k}^1(\log D))$ by the reciprocity map $CH_0(X) \rightarrow \pi_1(X)^{ab}$.

Let R_0 be a finite p' -coefficient ring, $\psi_0 : k \rightarrow R_0^\times$ a non-trivial additive character. Let $\widehat{\mathbb{Z}}'(1)$ be the group

$$\widehat{\mathbb{Z}}'(1) := \varprojlim_{k'} (k')^\times,$$

where k' runs over all finite extensions of k in a fixed algebraic closure of k , and the projective limit is taken with respect to the norm maps. Let \mathcal{C}_k be the exact category of object V in $\text{Rep}(\widehat{\mathbb{Z}}'(1), R_0)$ which satisfies $q^*V \cong V$, where q^*V is the pull-back of V by the q -power map $\widehat{\mathbb{Z}}'(1) \xrightarrow{\cong} \widehat{\mathbb{Z}}'(1)$. For each object V in \mathcal{C}_k , we define an element $\tau(V, \psi_0) \in R_0^\times$ in the following way: take a triple (R, \widetilde{V}, ψ) , where R is a strict p' -coefficient ring containing R_0 , \widetilde{V} is a tamely ramified object in $\text{Rep}(W_{k((t))}, R)$ such that

$\text{Res}_{W_{k((t))}^0}^{W_{k((t))}}$ is identified with $V \otimes_{R_0} R$ via the canonical isomorphism $\widehat{\mathbb{Z}}'(1) \cong W_{k((t))}^0/W_{k((t))}^{0+}$, and $\psi : k((t)) \rightarrow R^\times$ is a continuous additive character of conductor -1 whose restriction to $k[[t]]$ induces ψ_0 . We put $\tau(V, \psi_0) := (-q)^{\text{rank } V} \varepsilon_{0,R}(\widetilde{V}, \psi)$. Then $\tau(V, \psi_0)$ belongs to R_0^\times and is independent of the choice of (R, \widetilde{V}, ψ) .

LEMMA 9.1 (cf. [Sa1, p. 400–401, Lem. 1 (1)]). *Let k' be a finite extension of k and $r \in \mathbb{Z}_{>0}$ be a positive integer. Let $r = ms$ be the decomposition of r into the prime-to- p part m and the p -part s . For an object V in $\mathcal{C}_{k'}$, let $\text{Ind } V$ be the object in \mathcal{C}_k defined as*

$$\text{Ind } V = \bigoplus_{i=0}^{[k':k]-1} (q^i)^*(\text{Ind}_r V),$$

where $\text{Ind}_r V$ is the induced representation of V by the multiplication-by- r map $\widetilde{\mathbb{Z}}'(1) \rightarrow \widetilde{\mathbb{Z}}'(1)$. Then we have

$$\tau(\text{Ind } V, \psi_0) = (\det V)(m) \tau(V, \psi_0^{(s)} \circ \text{Tr}_{k'/k}) \cdot (\tau_{k,m,\psi_0})^{[k':k] \cdot \text{rank } V}.$$

Here $\det V$ is considered as a character of $\widehat{\mathbb{Z}}'(1)/(q-1)\widehat{\mathbb{Z}}'(1) = k^\times$ and $\psi_0^{(s)}$ is the additive character of k given by $\psi_0^{(s)}(x) = \psi_0(x^s)$, and

$$\tau_{k,m,\psi_0} = \begin{cases} \left(\frac{m}{k}\right) \left(\left(\frac{2}{k}\right) \tau_k\left(\left(\frac{1}{k}\right), \psi_0\right)\right)^{m-1} & \text{if } p \neq 2, \\ \left(\frac{m}{k}\right)' q^{\frac{m-1}{2}} & \text{if } p = 2, \end{cases}$$

Here $\left(\frac{m}{k}\right)' = (-1)^{\frac{m^2-1}{8} \cdot [k:\mathbb{F}_2]}$.

PROOF. We may assume that $s = 1$. Let $L = k'((t))$ and $K = k((t^m)) \subset L$. Take a triple (R, \widetilde{V}, ψ) , where R is a strict p' -coefficient ring containing R_0 , \widetilde{V} is a tamely ramified object in $\text{Rep}(W_L, R)$ such that $\text{Res}_{W_L^0}^{W_L}$ is identified with $V \otimes_{R_0} R$, and $\psi : K \rightarrow R^\times$ is a continuous additive character of conductor -1 whose restriction to \mathcal{O}_K induces ψ_0 . Then we have $\tau(\text{Ind } V, \psi_0) = (-q)^{\text{rank } \text{Ind } V} \varepsilon_{0,R}(\text{Ind}_{W_L}^{W_K} \widetilde{V}, \psi)$ and $\tau(V, \psi_0 \circ \text{Tr}_{k'/k}) = (-q)^{[k':k] \text{rank } V} (\det V)(m)^{-1} \varepsilon_{0,R}(V, \psi \circ \text{Tr}_{L/K})$. By [Y1, Thm. 5.6], it suffices to prove that

$$\tau_{k,m,\psi_0}^{[k':k]} = \frac{(-q)^{[k':k]m}}{(-q)^{[k':k]}} \cdot \lambda_R(L/K, \psi).$$

Let $L_0 = k'((t^m))$ be the maximal unramified subextension of L/K . By [Y1, Prop. 6.5 (5)] we have

$$\begin{aligned} \lambda_R(L/K, \psi) &= \lambda_R(L_0/K, \psi)^m \cdot \lambda_R(L/L_0, \psi \circ \text{Tr}_{L_0/K}) \\ &= (-1)^{([k':k]-1)m} \cdot \lambda_R(L/L_0, \psi \circ \text{Tr}_{L_0/K}). \end{aligned}$$

By [Y1, Lem. 6.7], we have

$$\begin{aligned} &\lambda_R(L/L_0, \psi \circ \text{Tr}_{L_0/K}) \\ &= \begin{cases} q^{-\frac{m-1}{2}[k':k]} \left(\frac{(-1)^{\frac{m-1}{2}m}}{k'} \right) & \text{if } m \text{ is odd and } p \neq 2, \\ q^{-\frac{m-1}{2}[k':k]} (-1)^{\frac{m^2-1}{8}[k':\mathbb{F}_2]} & \text{if } m \text{ is odd and } p = 2, \\ q^{-\frac{m}{2}[k':k]} \tau_R(L/L_0, \psi \circ \text{Tr}_{L_0/K}) \left(\frac{(-1)^{\frac{m}{2}-1} \frac{m}{2}}{k'} \right) & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

Hence the assertion follows. \square

Let V be an object in $\text{Rep}(\pi_1^{tm}(U), R_0)$. We define $\tau_{D/k}(V, \psi_0) \in R_0^\times$ in the following way (cf. [Sa1, p. 403]). Let $(D_i)_{i \in I}$ denote the family of irreducible components of D . For $i \in I$, let k_i be the constant field of D_i , $c_i \in \mathbb{Z}$ the Euler number of $(D_i - \bigcup_{i' \neq i} D_{i'}) \otimes_{k_i} \bar{k}_i$, and $V_i \in \mathcal{C}_{k_i}$ the restriction of V by the morphism $\widehat{\mathbb{Z}}'(1)_{\bar{k}_i} \rightarrow \pi_1(U)^{tm}$ which is canonically defined up to conjugacy. Then we put

$$\tau_{D/k}(V, \psi_0) := \prod_{i \in I} \tau_{k_i}(V_i, \psi_0 \circ \text{Tr}_{k_i/k})^{c_i}.$$

PROPOSITION 9.2 (cf. [Sa1, p. 403, Thm. 1]). *Assume that $p \neq 2$. Let \mathcal{F} be the smooth R_0 -sheaf on U corresponding to an object V in $\text{Rep}(\pi_1^{tm}(U), R_0)$. Then the global ε_0 -constant*

$$\varepsilon_{0,R_0}(U/k, \mathcal{F}) = \det(-\text{Fr}_k; R\Gamma_c(U \otimes_k \bar{k}, \mathcal{F}))^{-1} \in R_0^\times$$

satisfies

$$\varepsilon_{0,R_0}(U/k, \mathcal{F}) = \det V(-c_{X,U}) \cdot \tau_{D/k}(V, \psi_0) \cdot \varepsilon_{0,R_0}(U/k, R_0)^{\text{rank } V}.$$

Let K be a p -local field with the residue field k , X a proper flat generically smooth \mathcal{O}_K -scheme which is regular of dimension n as a scheme, and $U \subset X$ an open subscheme contained in $X \otimes_{\mathcal{O}_K} K$. Assume that the complement $D = X - U$ is a divisor with simple normal crossing of X . Let $(D_i)_{i \in I}$ be the family of irreducible components of D and let $I_0 \subset I$ be the subset of $i \in I$ such that $D_i \subset X \otimes_{\mathcal{O}_K} k$. Let r_i be the multiplicity of $X \otimes_{\mathcal{O}_K} k$ at D_i and m_i be the prime-to- p part of r_i . For a subset $J \subset I$, let $D_J = \bigcap_{i \in J} D_i$. We assume that $D_{J,K} = \bigcap_{i \in J} D_i \otimes_{\mathcal{O}_K} K$ is smooth over K for all $J \subset I$ and that (X, U) is tame over \mathcal{O}_K in the sense of [Sa1, p. 404], that is, the \mathcal{O}_X -module $\Omega_{X/\mathcal{O}_K}^1(\log D/\log k)$ of differentials with logarithmic poles is locally free. We define (the tame part of) the relative canonical cycle

$$c_{X,U/\mathcal{O}_K} \in H_{X_k}^{2n}(X \bmod D, \widehat{\mathbb{Z}}^I(n))$$

in the following way (cf. [Sa1, p. 404]). For $i \in I$, let $D_i^* = D_i - \bigcup_{i' \neq i} D_{i'}$. For $i \in I_0$, we also denote, by abuse of notation, by c_{D_i, D_i^*} the prime-to- p part in $H^{2n-2}(D_i \bmod D, \widehat{\mathbb{Z}}^I(n-1))$ of the relative canonical cycle $c_{D_i, D_i^*} \in \pi_1^{tm}(U)^{ab}$. Then we put

$$c_{X,U/\mathcal{O}_K} = i_* \left(- \sum_{i \in I_0} m_i \cup c_{D_i, D_i^*} \right) \in H_{X_k}^{2n}(X \bmod D, \widehat{\mathbb{Z}}^I(n)),$$

where $i_* : \bigoplus_{i \in I_0} H^{2n-1}(D_i \bmod D, \widehat{\mathbb{Z}}^I(n)) \rightarrow H_{X_k}^{2n}(X \bmod D, \widehat{\mathbb{Z}}^I(n))$ is as in [Sa1, p403], and $m_i \in \mathbb{Z}$ is considered as an element in $H^1(D_i \bmod D, \widehat{\mathbb{Z}}(1))$.

Let $h : R_0 \rightarrow R$ be a local ring homomorphism from R_0 to a strict p' -coefficient ring R , $\psi : K \rightarrow R^\times$ a non-trivial continuous additive character with $\text{ord } \psi = -1$, and $\psi_0 : k \rightarrow R^\times$ the character given by the restriction $\psi|_{\mathcal{O}_K}$. For an object (ρ, V) in $\text{Rep}(\pi_1^{tm}(U), R_0)$, we define $\tau_{D/k}(\rho, \psi_0) \in R_0^\times$ in a manner similar to that in [Sa1, p. 404], that is,

$$\tau_{D/k}(\rho, \psi_0) = \prod_{i \in I_0^t} \tau_{k_i}(\rho_i, \psi_0 \circ \text{Tr}_{k_i/k})^{c_i},$$

where $I_0^t = \{i \in I_0; p \nmid r_i\}$.

In view of the proof of [Sa1, p. 403, Thm. 1] given in [Sa1, p. 409–415], to prove Proposition 9.2, it suffices to prove the following proposition:

PROPOSITION 9.3 (cf. [Sa1, p. 405, Thm. 2]). *Assume that $p \neq 2$. Let \mathcal{F} be the the smooth R_0 -sheaf on U corresponding to (ρ, V) . Then the local ε_0 -constant*

$$\varepsilon_{0,R}(U/K, \mathcal{F}, \psi) = \varepsilon_{0,R}(R\Gamma_c(U \otimes_K \overline{K}, \mathcal{F}) \otimes_{R_0} R, \psi) \in R^\times$$

is equal to the image by h of

$$\det \rho(-c_{X,U/\mathcal{O}_K})\tau_{D/k}(\rho, \psi_0) \times \begin{cases} \left((-1)^{\chi_c(U_{\overline{K}})} \left(\frac{M}{k}\right) \left(\left(\frac{2}{k}\right) \tau_k \left(\left(\frac{1}{k}\right), \psi_0\right)\right)^{\chi_c(U_{\overline{K}}) - \chi_c^s(U_{\overline{K}})} \right)^{\text{rank } V} & \text{if } p \neq 2, \\ \left((-1)^{\chi_c(U_{\overline{K}})} \left(\frac{M}{k}\right)' q^{\frac{1}{2}(\chi_c(U_{\overline{K}}) - \chi_c^s(U_{\overline{K}}))} \right)^{\text{rank } V} & \text{if } p = 2, \end{cases}$$

where $\chi_c(U_{\overline{K}})$ and $\chi_c^s(U_{\overline{K}})$ are as in [Sa1, p. 405, Thm. 2].

PROOF. We fix a subset $J \subset I$ and a connected component E of D_J and study the restriction $R\Psi\mathcal{F}|_{E^*}$ of the complex of the nearby cycles $R\Psi\mathcal{F}$ on $E^* = E - \bigcup_{i \in J} D_i$. For $i \in J$, let I_i and $P_i \dots \alpha_i : I_i/P_i \rightarrow \pi_1^{tm}(U)$. We may assume that the image of α_i is commutative to each other. Let $J_0 = J \cap I_0$ and M be the complex concentrated on degree 0 and -1 :

$$\mathbb{Z}^{J_0} \rightarrow \mathbb{Z} ; e_i \mapsto r_i.$$

Then the canonical morphism

$$\widehat{\mu} : \prod_{i \in J_0} I_i/P_i \rightarrow I_K/P_K$$

is canonically identified with $\mu \otimes id_{\widehat{\mathbb{Z}}'(1)}$.

In view of the proof of [Sa1, p. 405, Thm. 2] given in [Sa1, p. 405–408], in order to prove the proposition, it suffices to prove the following analogue of [Sa1, p. 407, Cor. 1]. \square

LEMMA 9.4 (cf. [Sa1, p. 407, Cor. 1]). *On each stratum E^* , $R\Psi\mathcal{F}|_{E^*}$ is a successive extension of bounded complexes of smooth R_0 -flat R_0 -sheaves which is tamely ramified along the boundary $E - E^*$. If $J = \{i\}$ for some $i \in I_0$, then $R^q\Psi\mathcal{F}|_{E^*} = 0$ for $q \neq 0$ and $R^0\Psi\mathcal{F}|_{E^*} = \text{Ind}_{I_i/P_i}^{I_K/P_K} \rho_i$ as a representation of I_K/P_K . If J otherwise, we have*

$$[R\Psi\mathcal{F}|_{E^*}] = 0$$

in the Grothendieck group of the smooth R_0 -flat R_0 -sheaves on E^* which is tamely ramified along the boundary $E - E^*$.

PROOF. We use the notation in the proof of [Sa1, p. 405–406, Prop. 6]. Let $R_0[[G]] = \varprojlim_d R_0[G_d]$. Consider the $R_0[[G]]$ -sheaf $(\varphi_{d,!}\varphi_d^*\mathcal{F})_d$. Since

$$\mathcal{F} \cong (\varphi_{d,!}\varphi_d^*\mathcal{F})_d \otimes_{R_0[[G]]}^{\mathbb{L}} R_0,$$

we have

$$R\Psi\mathcal{F}|_{E^*} \cong (\mathcal{F}_d|_{E^*} \otimes_{R_0} \mathrm{Ind}_{I_{K_d}}^{I_K} R\Psi_d R_{0,U_d})_d \otimes_{R_0[[G]]}^{\mathbb{L}} R_0.$$

Hence there exists a filtration on $R\Psi\mathcal{F}|_{E^*}$ such that each graded piece is isomorphic to

$$(\mathcal{F}_d|_{E^*} \otimes_{R_0} \mathrm{Ind}_{I_{K_d}}^{I_K} R_0)_d \otimes_{R_0[[G]]}^{\mathbb{L}} R_0 \otimes_{R_0} \bigwedge^q \mathrm{Hom}(H_1(M)(1), R_0).$$

Hence the lemma follows. \square

This completes the proof of Proposition 9.2. \square

References

- [De1] Deligne, P., *Les constantes des équations fonctionnelles des fonctions L*, in Modular functions in one variable II, 501–597, Lecture Notes in Math. 349, Springer, Berlin (1973).
- [De2] Deligne, P., *Cohomologie étale. Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 $\frac{1}{2}$. Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier*, Lecture Notes in Math. 569, Springer, Berlin (1977).
- [De3] Deligne, P., *Les constantes des équations fonctionnelles des fonctions L*, Seminaire a l’IHES (1980).
- [DH] Deligne, and G. Henniart, *Sur la variation, par torsion, des constantes locales d’équations fonctionnelles de fonctions L*, Invent. Math. **64** (1981), no. 1, 89–118.
- [Dr] Drinfeld, V. G., *Two-dimensional ℓ -adic representations of the fundamental group of a curve over a finite residue field and automorphic forms on $GL(2)$* , 85–114.
- [He] Henniart, G., *Galois ε -factors modulo roots of unity*, Invent. Math. **78** (1984), 117–126.

- [K1] Kato, K., *Swan conductors with differential values*, in Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986), 315–342, Adv. Stud. Pure Math. **12**, North-Holland, Amsterdam-New York (1987).
- [Kz1] Katz, N. M., Local-to-global extensions of representations fundamental groups, Ann. Inst. Fourier (Grenoble) **36** (1986), no. 4, 69–106.
- [KL] Katz, N. M. and G. Laumon, Transformation de Fourier et majoration de sommes exponentielles, Inst. Hautes Études Sci. Publ. Math. **62** (1985), 361–418.
- [Lau1] Laumon, G., *Semi-continuité du conducteur de Swan (d’après P. Deligne)*, in Caractéristique d’Euler-Poincaré, Astérisque, 82-83, Soc. Math. France, Paris.
- [Lau2] Laumon, G., Transformation de Fourier, constantes d’équations fonctionnelles et conjecture de Weil, Inst. Hautes Études Sci. Publ. Math. **65** (1987), 131–210.
- [R] Raynaud, M., *Caractéristique d’Euler-Poincaré d’un faisceau et cohomologie des variétés abéliennes*, Séminaire Bourbaki, Exp. 286 (1965).
- [Sa1] Saito, T., ε -factor of a tamely ramified sheaf on a variety, Invent. Math. **113** (1993), 389–417.
- [Sa2] Saito, T., *Ramification groups and local constants*, UTMS preprint 96-19, University of Tokyo (1996).
- [Se1] Serre, J.-P., Groupes proalgébriques, Inst. Hautes Études Sci. Publ. Math. **7** (1960), 1–67.
- [Se2] Serre, J.-P., *Corps locaux*, Hermann, Paris (1968).
- [Se3] Serre, J.-P., *Groupes algébriques et corps de classes*, Hermann, Paris (1975).
- [SGA4] Artin, M., Grothendieck, A. and J. L. Verdier (ed., with the collaboration of P. Deligne and B. Saint-Donat) *Théorie des topos et cohomologie étale des schémas. Tome 3*, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Lecture Notes in Math. 305, Springer, Berlin-New York (1973).
- [SGA5] Illusie, L. (ed.), *Cohomologie l -adique et fonctions L* , Séminaire de Géométrie Algébrique du Bois-Marie 1965-1966 (SGA 5), Lecture Notes in Math. 589, Springer, Berlin-New York (1977).
- [T1] Tate, J., *Fourier analysis in number fields, and Hecke’s zeta-functions*, in Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), 305–347.
- [V] Verdier, J.-L., *Spécialisation de faisceaux et monodromie modérée*, in Analysis and topology on singular spaces, II, III (Luminy, 1981), 332–364, Astérisque, 101-102, Soc. Math. France, Paris (1983).
- [Y1] Yasuda, S., *Local constants in torsion rings*, preprint (2001).

- [Y2] Yasuda, S., Local ε_0 -characters in torsion rings, *J. Théor. Nombres Bordx.* **19** (2007), 763–797.

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