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The Maximal Number of Singular Points on Log del Pezzo Surfaces

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Abstract. We prove that a del Pezzo surface with Picard number one has at most four singular points.

1. Intoduction

A *log del Pezzo surface* is a projective algebraic surface X with only quotient singularities and ample anticanonical divisor $-K_X$.

Del Pezzo surfaces naturally appear in the log minimal model program (see, e. g., [7]). The most interesting class of del Pezzo surfaces is the class of surfaces with Picard number 1. It is known that a log del Pezzo surface of Picard number one has at most five singular points (see [8]). In [1] the author proved there is no log del Pezzo surfaces of Picard number one with five singular points. In this paper we give another, simpler proof.

Theorem 1.1. *Let X be a log del Pezzo surface and Picard number is 1. Then X has at most four singular points.*

Recall that a normal complex projective surface is called a *rational homology projective plane* if it has the same Betti numbers as the projective plane $\mathbb{P}^2$. J. Kollár [9] posed the problem to classify rational homology $\mathbb{P}^2$'s with quotient singularities having five singular points. In [4] this problem is solved for the case of numerically effective K_X. Our main theorem solves Kollár's problem in the case where $-K_X$ is ample.

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2. Preliminary Results

We work over complex number field $\mathbb{C}$. We employ the following notation:

- $(-n)$-curve is a smooth rational curve with self intersection number $-n$.
- K_X: the canonical divisor on X.
- $\rho(X)$: the Picard number of X.

Theorem 2.1 (see [8, Corollary 9.2]). *Let X be a rational surface with log terminal singularities and $\rho(X) = 1$. Then*

$$(*) \qquad \sum_{P \in X} \frac{m_P - 1}{m_P} \leq 3,$$

where m_P is the order of the local fundamental group $\pi_1(U_P - \{P\})$ (U_P is a sufficiently small neighborhood of P).

So, every rational surface X with log terminal singularities and Picard number one has at most six singular points. Assume that X has exactly six singular points. Then by $(*)$ all singularities are Du Val. This contradicts the classification of del Pezzo surfaces with Du Val singularities (see, e. g., [3], [10]).

2.2. Thus to prove Theorem 1.1 it is sufficient to show that there is no log del Pezzo surfaces with five singular points and Picard number one. Assume the contrary: there is log del Pezzo surfaces with five singular points and Picard number one. Let $P_1, \ldots, P_5 \in X$ be singular points and $U_{P_i} \ni P_i$ small analytic neighborhood. By Theorem 2.1 the collection of orders of groups $\pi_1(U_{P_1} - P_1), \ldots \pi_1(U_{P_5} - P_5)$ up to permutations is one of the following:

2.2.1. $(2, 2, 3, 3, 3)$, $(2, 2, 2, 4, 4)$, $(2, 2, 2, 3, n')$, $n' = 3$, 4, 5, 6,

2.2.2. $(2, 2, 2, 2, n')$, $\quad n' \geq 2$.

REMARK 2.3. According to the classification of del Pezzo surfaces with Du Val singularities we may assume that there is a non-Du Val singular point. The case 2.2.1 is discussed in [4, Remark 4.2 and Section 6]. Thus it is sufficient to consider case 2.2.2.

2.4. Notation and assumptions. Let X be a del Pezzo surface with log terminal singularities and Picard number $\rho(X) = 1$. We assume that we are in case 2.2.2, i. e. the singular locus of X consists of four points P_1, P_2, P_3, P_4 of type A_1 and one non Du Val singular point P_5 with $|\pi_1(U_{P_5} - P_5)| = n' \geq 3$. Let $\pi\colon \bar{X} \to X$ be the minimal resolution and let $D = \sum_{i=1}^{n} D_i$ be the reduced exceptional divisor, where the D_i are irreducible components. Then there exists a uniquely defined effective $\mathbb{Q}$-divisor $D^\sharp = \sum_{i=1}^{n} \alpha_i D_i$ such that $\pi^*(K_X) \equiv D^\sharp + K_{\bar{X}}$.

LEMMA 2.5 (see, e. g., [13, Lemma 1.5]). *Under the condition of 2.4, let $\Phi\colon \bar{X} \to \mathbb{P}^1$ be a generically $\mathbb{P}^1$-fibration. Let m be the number of irreducible components of D not contained in any fiber of Φ and let d_f be the number of (-1)-curves contained in a fiber f. Then*

(1) $m = 1 + \sum_f (d_f - 1)$, *where f run only over the fibres with $d_f \geq 1$.*

(2) *If $d_f = 1$ and E is the only (-1)-curve in f, then its coefficient in f is at least two.*

The following lemma is a consequence of the Cone Theorem.

LEMMA 2.6 (see, e. g., [13, Lemma 1.3]). *Under the condition of 2.4, every curve on $\bar{X}$ with negative selfintersection number is either (-1)-curve or a component of D.*

DEFINITION 2.7. Let (Y, D) be a projective log surface. (Y, D) is called the *weak log del Pezzo* surface if the pair (Y, D) is klt and the divisor $-(K_Y + D)$ is nef and big.

For example, in the above notation, $(\bar{X}, D^\sharp)$ is a weak del Pezzo surface. Note that if (Y, D) is a weak log del Pezzo surface with $\rho(Y) = 1$ then divisor $-(K_Y + D) = A$ is ample and Y has only log terminal singularities. Hence, Y is a log del Pezzo surface.

Lemma 2.8 (see, e. g., [1, Lemma 2.9]). *Suppose* (Y, D) *is a weak log del Pezzo surface. Let* $f\colon Y \to Y'$ *be a birational contraction. Then* $(Y', D' = f_*D)$ *is also a weak log del Pezzo surface.*

3. Proof of the Main Theorem: The Case where X has Cyclic Quotient Singularities

In this section we assume that X has only cyclic quotient singularities.

The following lemma is very similar to that in [5]. For the convenience of the reader we give a complete proof.

Lemma 3.1. *Under the condition of 2.4, suppose that* P_5 *is a cyclic quotient singularity. Then there exists a generically* $\mathbb{P}^1$*-fibration* $\Phi : \bar{X} \to \mathbb{P}^1$ *such that* $f \cdot D \leq 2$*, where* f *is a fiber of* Φ*.*

Proof. Let $\nu : \hat{X} \to X$ be the minimal resolution of the non Du Val singularity and let $E = \sum E_i$ be the exceptional divisor. By [12, Corollary 1.3] or [8, Lemma 10.4] we have $|-K_X| \neq \varnothing$. Take $B \in |-K_X|$. Then we can write

$$K_{\hat{X}} + \hat{B} = \nu^*(K_X + B) \sim 0,$$

where $\hat{B}$ is an effective integral divisor. We obviously have $\hat{B} \geq E$.

Run the MMP on $\hat{X}$. We obtain a birational morphism $\phi : \hat{X} \to \tilde{X}$ such that $\tilde{X}$ has only Du Val singularities and either $\rho(\tilde{X}) = 2$ and there is a generically $\mathbb{P}^1$-fibration $\psi : \tilde{X} \to \mathbb{P}^1$ or $\rho(\tilde{X}) = 1$. Moreover, ϕ is a composition

$$\hat{X} = X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_n} X_{n+1} = \tilde{X},$$

where ϕ_i is a weighted blowup of a smooth point of X_{i+1} with weights $(1, n_i)$ (see [11]).

Assume that $\rho(\tilde{X}) = 1$, then every singular point on $\tilde{X}$ is of type A_1. By the classification of del Pezzo surfaces with Du Val singularities and Picard number one (see, e. g., [3], [10]) we have $\tilde{X} = \mathbb{P}^2$ or $\tilde{X} = \mathbb{P}(1, 1, 2)$.

Assume that $\rho(\tilde{X}) = 1$ and $\tilde{X} = \mathbb{P}(1, 1, 2)$. Note that ϕ contracts $\rho(\hat{X}) - 1 = \#E$ curves, where $\#E$ number of irreducible component of E. Since $\phi_*(\hat{B})$ has at most two components and $\hat{B} \geq E$, we see that ϕ contracts at most two curves K_1 and K_2 that are not components of E.

Since X has four singular points of type A_1, we see that $\tilde{X}$ has at least two singular points, a contradiction.

Assume that $\rho(\tilde{X}) = 1$ and $\tilde{X} = \mathbb{P}^2$. Since $\phi_*(\hat{B})$ has at most three components, as above, we see that ϕ contracts at most three curves K_1, K_2 and K_3 that are not components of E. Since X has four singular points of type A_1, we see that $\tilde{X}$ has at least one singular point, a contradiction.

Therefore, $\rho(\tilde{X}) = 2$ and there is a generically $\mathbb{P}^1$-fibration $\psi : \tilde{X} \to \mathbb{P}^1$. Let $g : \bar{X} \to \hat{X}$ be the minimal resolution of $\hat{X}$. Let $\Phi' = \psi \circ \phi$ and let f' be a fiber of Φ'. Then $f' \cdot E \leq f' \cdot \hat{B} = -K_{\hat{X}} \cdot f' = 2$. Set $\Phi = \Phi' \circ g$. $\square$

3.2. Let f be a fiber of Φ. By Lemma 3.1 we have the following cases:

3.2.1. f meets exactly one irreducible component D_0 of D and $f \cdot D_0 = 1$.

Let L be a singular fiber of Φ. By Lemma 2.5 (1) the fiber L contains exactly one (-1)-curve F. By Lemma 2.5 (2) F does not meet D_0. Then F meets at most two components of D. Blowup one of the intersection points of F and D. We obtain a surface Y. Let $h : Y \to Y'$ be a contraction of all curves with selfintersection number at most -2. Note that Y' has only log terminal singularities but not of type 2.2.2, a contradiction.

3.2.2. f meets exactly two irreducible components D_1, D_2 of D and $D_1 \cdot f = D_2 \cdot f = 1$.

By Lemma 2.5 (1) there exists a unique singular fiber L such that L has two (-1)-curves F_1 and F_2. Note that one of these curves, say F_1, meets D at one or two points. Blowup one the intersection points of F_1 and D. We obtain a surface Y. Let $h : Y \to Y'$ be a contraction of all curves with selfintersection number at most -2. Note that Y' has only log terminal singularities but not of type 2.2.2, a contradiction.

3.2.3. f meets exactly one irreducible component D_0 of D and $f \cdot D_0 = 2$. Let A be a connected component of D containing D_0.

By Lemma 2.5 (1) every singular fiber of Φ contains exactly one (-1)-curve. Note that every singular fiber of Φ either contains two connected components of $A - D_0$ or the coefficient of a unique (-1)-curve in this fiber is equal to two. If a singular fiber L contains exactly one (-1)-curve with coefficient two, then the dual graph of L is the following:

(**) $$\overset{-2}{\circ} \text{——} \overset{-1}{\circ} \text{——} \overset{-2}{\circ}$$

Since X has five singular points with orders of local fundamental groups $(2,2,2,2,n)$, we see that Φ has two singular fibers L_1, L_2 of type $(**)$ and possibly one more singular fiber L_3. Note that L_3 contains both connected component of $A-D_0$. Let $\mu : \bar{X} \to \mathbb{F}_n$ be the contraction of all (-1)-curves in fibers of Φ, where $\mathbb{F}_n$ is the Hirzebruch surface of degree n (rational ruled surface) and $n=0,1$. Denote $\tilde{D_0} := \mu_* D_0$. Note that $\tilde{D_0} \sim 2M + kf$, where $M^2 = -n$ and $M \cdot f = 1$. Since we contract at most five curves that meet D_0, and $D_0^2 \le -2$, we see that $0 < \tilde{D_0}^2 \le 3$. Hence, $0 < -4n + 4k \le 3$. This is impossible, a contradiction.

4. Proof of the Main Theorem: The Case where X has a Non-Cyclic Quotient Singularity

Under the condition of 2.4, assume X has a non-cyclic singular point, say P. Then there is a unique component D_0 of D such that $D_0 \cdot (D - D_0) = 3$ (see [2]).

Lemma 4.1. *There is a generically $\mathbb{P}^1$-fibration $\Phi : \bar{X} \to \mathbb{P}^1$ such that Φ has a unique section D_0 in D and $D_0 \cdot f \le 3$, where f is a fiber of Φ.*

Proof. Recall that P is not Du Val. Let $h : \bar{X} \to \hat{X}$ be a contraction of all curves in D except D_0. Let $\hat{D}_0 = h_*(D_0)$ then $\hat{X}$ has seven singular points, $\rho(\hat{X}) = 2$ and there is $\nu : \hat{X} \to X$ such that $K_{\hat{X}} + a\hat{D}_0 = \nu^* K_X$. Note that $(\hat{X}, aD_0)$ is a weak log del Pezzo. Let R be the extremal rational curve different from $\hat{D}$. Let $\phi : \hat{X} \to \tilde{X}$ be the contraction of R.

4.2. There are two cases:

4.2.1. $\rho(\tilde{X}) = 1$. Then, by Lemma 2.8, $\tilde{X}$ is a del Pezzo surface. If the number of singular points of $\hat{X}$ on R is at most two, $\tilde{X}$ has at least five singular points and all points are cyclic quotients. Thus assume that there is at least three singular points of $\hat{X}$ on R, say P_1, P_2, P_3. Let $R_1 = \sum_i R_{1i}$, $R_2 = \sum_i R_{2i}$ and $R_3 = \sum_i R_{3i}$ be the exceptional divisors on $\bar{X}$ over P_1, P_2 and P_3, respectively. Let $\bar{R}$ is the proper transformation of R on $\bar{X}$. Since $\bar{R}$ is not component of D, we see that $\bar{R}^2 \ge -1$. Indeed, this follows from Lemma 2.6. Note that matrix of intersection of component $\bar{R} + R_1 + R_2 + R_3$ is not negative definite. Hence, $\bar{R} + E_1 + E_2 + E_3$ can not be contracted, a contradiction.

4.2.2. $\tilde{X} = \mathbb{P}^1$. Let $g : \bar{X} \to \hat{X}$ be the resolution of singularities. Then $\Phi = \phi \circ g : \bar{X} \to \mathbb{P}^1$. Note that there is a unique horizontal curve D_0 in D. Let f be a fiber of Φ. Denote coefficient of D_0 in $D^\sharp$ by α. Then

$$0 > (K_{\bar{X}} + D^\sharp) \cdot f = -2 + \alpha(D_0 \cdot f).$$

Hence, $D_0 \cdot f < \frac{2}{\alpha}$. Since P is not Du Val, we see that $\alpha \geq \frac{1}{2}$. Hence, $D_0 \cdot f \leq 3$. □

By Lemma 2.5 (1) every singular fiber of Φ contains exactly one (-1)-curve. Let B be the exceptional divisor corresponding to the non-cyclic singular point. Note that B contains D_0.

4.3. Consider the following three cases.

4.3.1. $D_0 \cdot f = 1$. Then every singular fiber of Φ contains exactly one connected component of $B - D_0$. On the other hand, $B - D_0$ contains three connected components. Hence X has at most four singular points, a contradiction.

4.3.2. $D_0 \cdot f = 2$. Let F_1, F_2, F_3 be a connected components of $B - D_0$. We may assume F_1 is (-2)-curve (see [2]). Let L_1 be a singular fiber of Φ. Assume that L_1 contains F_1. Then L_1 is of type $(**)$ and L_1 contain F_2. Hence, F_2 is a (-2)-curve. Let L_2 be a singular fiber of Φ. Assume that L_2 contains F_3 and let E be a unique (-1)-curve in L_2. By blowing up the intersection point of E and F_3, we obtain a surface Y. Let $h : Y \to Y'$ be a contraction of all curves with selfintersection number at most -2. Note that Y' has only log terminal singularities but not of type 2.2.2, a contradiction.

4.3.3. $D_0 \cdot f = 3$. Since every component of $D - B$ is a (-2)-curve, we see that every singular fiber of Φ contains a connected component of $B - D_0$. Note that $B - D_0$ contains three connected components. Hence X has at most four singular points, a contradiction.

This completes the proof of Theorem 1.1.

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