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A Note on Hyperbolic Operators with Log–Zygmund Coefficients

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1. Introduction

Consider the operator

$$L = \partial_t^2 - \sum_{j,k=1}^{n} \partial_{x_j}(a_{jk}(t)\partial_{x_k}). \tag{1.1}$$

Suppose that L is *strictly hyperbolic* with bounded coefficients, i.e. there exist λ_0, $\Lambda_0 > 0$ such that

$$\lambda_0|\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(t)\xi_k\xi_j \leq \Lambda_0|\xi|^2 \tag{1.2}$$

for all $t \in [0,T]$ and for all $\xi \in \mathbb{R}^n$.

It is well–known that if the coefficients a_{jk} are *Lipschitz–continuous* then an *energy estimate* holds for L: for all $s \in \mathbb{R}$ there exists $C_s > 0$ such that

$$\begin{aligned} &\sup_{0\leq t\leq T} \{\|u(t,\cdot)\|_{\mathcal{H}^{s+1}} + \|\partial_t u(t,\cdot)\|_{\mathcal{H}^s}\} \\ &\qquad \leq C_s(\|u(0,\cdot)\|_{\mathcal{H}^{s+1}} + \|\partial_t u(0,\cdot)\|_{\mathcal{H}^s} + \int_0^T \|Lu(t,\cdot)\|_{\mathcal{H}^s}\,dt), \end{aligned} \tag{1.3}$$

for every function $u \in \mathcal{C}^0([0,T],\mathcal{H}^{s+1}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,T],\mathcal{H}^s(\mathbb{R}^n))$ with $Lu \in \mathcal{L}^1([0,T],\mathcal{H}^s(\mathbb{R}^n))$, in particular for all $u \in \mathcal{C}^2([0,T],\mathcal{H}^\infty(\mathbb{R}^n))$ (see e.g. [11, Ch. IX]). The estimate (1.3) implies that the *Cauchy problem* for (1.1) is *$\mathcal{H}^\infty$–well–posed (without loss of derivatives)* if, for instance, the forcing term is null.

If the coefficients a_{jk} are not Lipschitz–continuous, then the estimate (1.3) is no more true in general; nevertheless the $\mathcal{H}^\infty$–well–posedness may

2000 *Mathematics Subject Classification.* 35L15 be recovered from an energy estimate *with loss of derivatives* (see e.g. the estimate (1.5) below), under regularity assumption on the a_{jk}'s weaker than Lipschitz-continuity.

A first result of this type was obtained in the well–known paper of Colombini, De Giorgi and Spagnolo [4]. The regularity condition was the following: there exists $C > 0$ such that

$$\int_0^{T-\varepsilon} |(a_{jk}(t+\varepsilon) - a_{jk}(t))|\, dt \le C\varepsilon \log\left(\frac{1}{\varepsilon}+1\right) \tag{1.4}$$

for all $\varepsilon \in (0, T]$. The energy estimate, deduced from the Fourier transform with respect to x of the equation together with an approximation of the coefficients which is different in different zones of the phase space (the so called *approximate energy technique*, see [5]), is then: there exist C_s, $K > 0$ (K independent of s) such that

$$\begin{aligned} &\sup_{0\le t\le T} \{\|u(t,\cdot)\|_{\mathcal{H}^{s+1-K}} + \|\partial_t u(t,\cdot)\|_{\mathcal{H}^{s-K}}\} \\ &\qquad \le C_s(\|u(0,\cdot)\|_{\mathcal{H}^{s+1}} + \|\partial_t u(0,\cdot)\|_{\mathcal{H}^{s}} + \int_0^T \|Lu(t,\cdot)\|_{\mathcal{H}^{s}}\, dt), \end{aligned} \tag{1.5}$$

for all $u \in \mathcal{C}^2([0,T], \mathcal{H}^\infty(\mathbb{R}^n))$ (on the necessity of some kind of loss of derivatives when the coefficients are not Lipschitz–continuous, see [2]).

Recently, in [12] (see also [13]), Tarama has proved the $\mathcal{H}^\infty$–well–posedness to the Cauchy problem for (1.1) under the condition: there exists $C > 0$ such that

$$\int_\varepsilon^{T-\varepsilon} |(a_{jk}(t+\varepsilon) + a_{jk}(t-\varepsilon) - 2a_{jk}(t))|\, dt \le C\varepsilon \log\left(\frac{1}{\varepsilon}+1\right) \tag{1.6}$$

for all $\varepsilon \in (0, T/2]$. The improvement with respect to [4] is obtained introducing a new type of approximate energy which involves the second derivatives of the approximating coefficients (see par. 3.3 below).

The case of the operator L with coefficients depending on the time variable t and also on the space variables x was considered by Colombini and Lerner in [6]. In this paper the regularity condition was: there exists $C > 0$ such that

$$\sup_{\substack{y,y'\in[0,T]\times\mathbb{R}^n \\ |y'|=\varepsilon}} |(a_{jk}(y+y') - a_{jk}(y))|\, dt \le C\varepsilon \log\left(\frac{1}{\varepsilon}+1\right) \tag{1.7}$$

for all $\varepsilon \in (0, T]$. Here the use of the Littlewood-Paley dyadic decomposition (in place of the Fourier transform with respect to x) together with the approximate energy technique was the crucial point to obtain an energy estimate of the following type: for all $\theta \in (0, 1/4]$ there exist β, $C > 0$ and $T^* \in (0, T]$ such that

$$\begin{aligned}
&\sup_{0 \le t \le T^*} \{\|u(t,\cdot)\|_{\mathcal{H}^{-\theta+1-\beta t}} + \|\partial_t u(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta t}}\} \\
&\qquad \le C\Big(\|u(0,\cdot)\|_{\mathcal{H}^{-\theta+1}} + \|\partial_t u(0,\cdot)\|_{\mathcal{H}^{-\theta}} \\
&\qquad\qquad + \int_0^{T^*} \|Lu(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta t}}\, dt\Big)
\end{aligned} \tag{1.8}$$

for all $u \in \mathcal{C}^2([0, T^*], \mathcal{H}^\infty(\mathbb{R}))$. Results concerning local existence and uniqueness of the solutions to the Cauchy problem for similar hyperbolic problems can be found in [7].

In the present note we will consider the case of one space variable (from now on $n = 1$) and will study the case of the coefficient a depending on t and x, under a regularity condition inspired by (1.6) and (1.7). In particular a will be log–Zygmund–continuous with respect to t, uniformly with respect to x, and log–Lipschitz–continuous with respect to x, uniformly with respect to t (see par. 2 for the precise definitions). The dyadic decomposition technique will be applied as in [6] (see also [3], [9] and [8]) together with Tarama's approximate energy. An energy estimate similar to (1.8) will be obtained.

Before ending this introduction, let us remark that the choice of considering only one space variable is due to the fact that the case of several space variables needs some different and new ideas in the definition of the microlocal energy $e_{\nu,\varepsilon}(t)$ (see par. 3.3 below). This point still remain as an open problem.

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2. Main Result

Let $a : \mathbb{R}^2 \to \mathbb{R}$. We suppose that there exist λ_0, $\Lambda_0 > 0$ such that, for all $(t,x) \in \mathbb{R}^2$,

$$\lambda_0 \le a(t,x) \le \Lambda_0. \tag{2.1}$$

We suppose moreover that there exists $C_0 > 0$ such that, for all τ, $\xi > 0$,

$$\sup_{(t,x)\in\mathbb{R}^2} |a(t+\tau,x) + a(t-\tau,x) - 2a(t,x)| \le C_0\, \tau \log\Big(\frac{1}{\tau}+1\Big), \tag{2.2}$$

$$\sup_{(t,x)\in\mathbb{R}^2} |a(t,x+\xi) - a(t,x)| \le C_0\, \xi \log\Big(\frac{1}{\xi}+1\Big). \tag{2.3}$$

THEOREM 1. *Let $\theta \in (0, 1/2)$. Consider the operator*

$$L = \partial_t^2 - \partial_x(a(t,x)\partial_x). \tag{2.4}$$

Then there exist T, β^*, $C > 0$ such that, for all $u \in \mathcal{C}^2([0,T], \mathcal{H}^\infty(\mathbb{R}))$, the following a–priori estimate holds:

$$\begin{aligned} &\sup_{0\le t\le T} \{\|u(t,\cdot)\|_{\mathcal{H}^{-\theta+1-\beta^* t}} + \|\partial_t u(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta^* t}}\} \\ &\qquad \le C\Big(\|u(0,\cdot)\|_{\mathcal{H}^{-\theta+1}} + \|\partial_t u(0,\cdot)\|_{\mathcal{H}^{-\theta}} + \int_0^T \|Lu(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta^* t}}\, dt\Big). \end{aligned} \tag{2.5}$$

COROLLARY 1. *The Cauchy problem for (2.4) is (locally in time) well–posed in $\mathcal{H}^\infty$.*

3. Proof

3.1. Approximation of the coefficient a

We set

$$a_\varepsilon(t,x) := \iint \rho_\varepsilon(t-s)\rho_\varepsilon(x-y)a(s,y)\, ds\, dy,$$

where $\rho_\varepsilon(s) = \frac{1}{\varepsilon}\rho(\frac{s}{\varepsilon})$ with $\rho \in C_0^\infty(\mathbb{R})$, ρ even, $0 \le \rho \le 1$, $\operatorname{supp}\rho \subseteq [-1,1]$ and $\int \rho(s)\,ds = 1$. We obtain that, for all $\varepsilon \in (0,1]$,

$$\sup_{(t,x)\in\mathbb{R}^2} |a_\varepsilon(t,x) - a(t,x)| \le \frac{C_0}{2}\,\varepsilon \log\Big(\frac{1}{\varepsilon}+1\Big); \tag{3.1}$$

for all $\sigma \in (0,1)$ there exists $c_\sigma > 0$ such that, for all $\varepsilon \in (0,1]$,

$$\sup_{(t,x)\in\mathbb{R}^2} |\partial_t a_\varepsilon(t,x)| \le c_\sigma(\Lambda_0 + C_0)\varepsilon^{\sigma-1}; \tag{3.2}$$

for all $\varepsilon \in (0,1]$,

$$\sup_{(t,x)\in\mathbb{R}^2} |\partial_x a_\varepsilon(t,x)| \le C_0\|\rho'\|_{\mathcal{L}^1} \log\Big(\frac{1}{\varepsilon}+1\Big), \tag{3.3}$$

$$\sup_{(t,x)\in\mathbb{R}^2} |\partial_t^2 a_\varepsilon(t,x)| \le \frac{C_0}{2}\|\rho''\|_{\mathcal{L}^1}\frac{1}{\varepsilon} \log\Big(\frac{1}{\varepsilon}+1\Big), \tag{3.4}$$

$$\sup_{(t,x)\in\mathbb{R}^2} |\partial_t\partial_x a_\varepsilon(t,x)| \le C_0\|\rho'\|^2_{\mathcal{L}^1}\frac{1}{\varepsilon} \log\Big(\frac{1}{\varepsilon}+1\Big). \tag{3.5}$$

In particular, (3.1) is obtained from (2.2) remarking that

$$\begin{aligned} &a_\varepsilon(t,x) - a(t,x)\\ &\quad = \frac{1}{2}\int_{|s|\le\varepsilon} \rho_\varepsilon(s) \int \rho_\varepsilon(x-y)(a(t+s,y) + a(t-s,y) - 2a(t,y))dy\,ds, \end{aligned}$$

where we have used the fact that ρ is an even function. Next

$$\partial_t^2 a_\varepsilon(t,x) = \frac{1}{2}\int_{|s|\le\varepsilon} \rho''_\varepsilon(s) \int \rho_\varepsilon(x-y)(a(t+s,y) + a(t-s,y) - 2a(t,y))dy\,ds,$$

and (3.4) follows. The inequalities (3.3) and (3.5) are deduced from (2.3) in a similar way and, finally, (3.2) is a consequence of the fact that (2.1) and (2.2) imply that for all $\sigma \in (0,1)$ there exists $c'_\sigma > 0$ such that, for all $\tau > 0$,

$$\sup_{(t,x)\in\mathbb{R}^2} |a(t+\tau,x) - a(t,x)| \le c'_\sigma(\Lambda_0 + C_0)\,\tau^\sigma. \tag{3.6}$$

Let us note that a way to obtain (3.6) is to use the characterization of Hölder spaces given by the dyadic decomposition remarking that in such a case it is equivalent to use first or second order difference.

3.2. Dyadic decomposition

We collect here some well–known facts on the Littlewood-Paley decomposition, referring to [1] and [6] for the details. Let $\varphi_0 \in C_0^\infty(\mathbb{R}_\xi)$, $0 \le \varphi_0(\xi) \le 1$, $\varphi_0(\xi) = 1$ if $|\xi| \le 1$, $\varphi_0(\xi) = 0$ if $|\xi| \ge 2$, φ_0 even and φ_0 decreasing on $[0, +\infty)$. We set $\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)$ and, if ν is an integer greater than or equal to 1, $\varphi_\nu(\xi) = \varphi(2^{-\nu}\xi)$. Let w be a tempered distribution in $\mathcal{H}^{-\infty}(\mathbb{R})$; we define

$$\begin{aligned} w_\nu(x) := \varphi_\nu(D_x)w(x) &= \frac{1}{2\pi}\int e^{ix\xi}\varphi_\nu(\xi)\hat{w}(\xi)\,d\xi \\ &= \frac{1}{2\pi}\int \hat{\varphi_\nu}(y)w(x-y)\,dy. \end{aligned}$$

For all ν, w_ν is an entire analytic function belonging to $\mathcal{L}^2$ and for all $m \in \mathbb{R}$ there exists $K_m > 0$ such that

$$\frac{1}{K_m}\sum_{\nu=0}^{\infty}\|w_\nu\|_{\mathcal{L}^2}^2 2^{2m\nu} \le \|w\|_{\mathcal{H}^m}^2 \le K_m\sum_{\nu=0}^{\infty}\|w_\nu\|_{\mathcal{L}^2}^2 2^{2m\nu}. \tag{3.7}$$

Moreover, we have

$$2^{\nu-1}\|w_\nu\|_{\mathcal{L}^2} \le \|\partial_x w_\nu\|_{\mathcal{L}^2} \le 2^{\nu+1}\|w_\nu\|_{\mathcal{L}^2}, \tag{3.8}$$

where the inequality on the right–hand side holds for all $\nu \ge 0$, while the other one holds only for all $\nu \ge 1$.

We end this subsection quoting a result which will be useful in the following (for the proof see [6, Prop. 3.6.]). There exist $C > 0$ and $\nu_0 \in \mathbb{N}$ such that if $a \in \mathcal{L}^\infty(\mathbb{R})$ with $\sup_{x\in\mathbb{R}}|a(x+y) - a(x)| \le C_0\, y\log(\frac{1}{y}+1)$, $y > 0$, then, for all $\nu \ge \nu_0$,

$$\|[\varphi_\nu(D_x), a(x)]\|_{L(\mathcal{L}^2)} \le C(\|a\|_{\mathcal{L}^\infty} + C_0)2^{-\nu}\nu, \tag{3.9}$$

where $[\varphi_\nu(D_x), a(x)]$ is the commutator between $\varphi_\nu(D_x)$ and a, and $\|\cdot\|_{L(\mathcal{L}^2)}$ is the operator norm from $\mathcal{L}^2$ to $\mathcal{L}^2$.

3.3. Approximate energy of the ν-component

Let $T_0 > 0$. Let $u(t,x)$ be a function in $C^2([0,T_0], \mathcal{H}^\infty(\mathbb{R}^n))$. We set $u_\nu(t,x) = \varphi_\nu(D)u(t,x)$. We obtain

$$\partial_t^2 u_\nu = \partial_x(a(t,x)\partial_x u_\nu) + \partial_x([\varphi_\nu, a]\partial_x u) + (Lu)_\nu. \tag{3.10}$$

We introduce the approximate energy of u_ν (see [12]), setting

$$e_{\nu,\varepsilon}(t) := \int_{\mathbb{R}} \frac{1}{\sqrt{a_\varepsilon}} |\partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu|^2 + \sqrt{a_\varepsilon} |\partial_x u_\nu|^2 + |u_\nu|^2 \, dx.$$

We have

$$\begin{aligned}
\frac{d}{dt} e_{\nu,\varepsilon}(t) = \int \frac{2}{\sqrt{a_\varepsilon}} &\mathrm{Re}\left(\partial_t^2 u_\nu \cdot \overline{(\partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu)}\right) dx \\
&+ \int \frac{2}{\sqrt{a_\varepsilon}} \Big(\partial_t (\frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}) - (\frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}})^2\Big) \\
&\quad \times \mathrm{Re}\left(u_\nu \cdot \overline{(\partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu)}\right) dx \\
&\quad + \int \partial_t \sqrt{a_\varepsilon} |\partial_x u_\nu|^2 \, dx + \int 2\sqrt{a_\varepsilon} \mathrm{Re}\,(\partial_x u_\nu \cdot \overline{\partial_x \partial_t u_\nu}) \, dx \\
&\qquad + \int 2\mathrm{Re}\,(u_\nu \cdot \overline{\partial_t u_\nu}) \, dx.
\end{aligned}$$

We replace $\partial_t^2 u_\nu$ by the quantity given by (3.10) and we obtain

$$\begin{aligned}
&\int \frac{2}{\sqrt{a_\varepsilon}} \mathrm{Re}\left(\partial_t^2 u_\nu \cdot \overline{(\partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu)}\right) dx \\
&\qquad = \int \frac{2}{\sqrt{a_\varepsilon}} \mathrm{Re}\left(\partial_x (a(t,x) \partial_x u_\nu) \cdot \overline{(\partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu)}\right) dx \\
&\qquad\quad + \int \frac{2}{\sqrt{a_\varepsilon}} \mathrm{Re}\left((\partial_x([\varphi_\nu, a] \partial_x u) + (Lu)_\nu) \cdot \overline{(\partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu)}\right) dx.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int \frac{2}{\sqrt{a_\varepsilon}} \mathrm{Re}\left(\partial_x (a(t,x) \partial_x u_\nu) \cdot \overline{(\partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu)}\right) dx \\
&\quad = \int 2\, \frac{\partial_x \sqrt{a_\varepsilon}}{a_\varepsilon}\, a\, \mathrm{Re}\left(\partial_x u_\nu \cdot \overline{(\partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu)}\right) dx \\
&\qquad - \int \frac{\partial_t \sqrt{a_\varepsilon}}{a_\varepsilon}\, a\, |\partial_x u_\nu|^2 \, dx - \int 2 \frac{a}{\sqrt{a_\varepsilon}}\, \mathrm{Re}\,(\partial_x u_\nu \cdot \overline{\partial_x \partial_t u_\nu}) \, dx \\
&\qquad - \int \frac{a}{\sqrt{a_\varepsilon}}\, \partial_x (\frac{\partial_t \sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}})\, \mathrm{Re}\,(\partial_x u_\nu \cdot \overline{u_\nu}) \, dx.
\end{aligned}$$

Consequently, we obtain

$$
\begin{aligned}
\frac{d}{dt}e_{\nu,\varepsilon}(t) = \int \frac{2}{\sqrt{a_\varepsilon}}\mathrm{Re}\Big((\partial_x([\varphi_\nu, a]\partial_x u) + (Lu)_\nu)\cdot \overline{(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu)}\Big)\, dx \\
+ \int \frac{2}{\sqrt{a_\varepsilon}}\Big(\partial_t(\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}) - (\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}})^2\Big) \\
\times \mathrm{Re}\Big(u_\nu \cdot \overline{(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu)}\Big)\, dx \\
+ \int \partial_t\sqrt{a_\varepsilon}(1-\frac{a}{a_\varepsilon})|\partial_x u_\nu|^2\, dx \\
+ \int 2(\sqrt{a_\varepsilon} - \frac{a}{\sqrt{a_\varepsilon}})\,\mathrm{Re}\,(\partial_x u_\nu \cdot \overline{\partial_x\partial_t u_\nu})\, dx \\
+ \int 2\,\frac{\partial_x\sqrt{a_\varepsilon}}{a_\varepsilon}\, a\,\mathrm{Re}\Big(\partial_x u_\nu \cdot \overline{(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu)}\Big)\, dx \\
- \int \frac{a}{\sqrt{a_\varepsilon}}\,\partial_x(\frac{\partial_t\sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}})\,\mathrm{Re}\,(\partial_x u_\nu \cdot \overline{u_\nu})\, dx \\
+ \int 2\mathrm{Re}\,(u_\nu \cdot \overline{\partial_t u_\nu})\, dx.
\end{aligned}
$$

From (2.1), (3.2) with e. g. $\sigma = 1/2$, (3.4) we deduce that there exists $C_1 > 0$ depending only on λ_0, Λ_0 and C_0 such that, for all $\nu \in \mathbb{N}$,

$$
\begin{aligned}
\Big|\int \frac{2}{\sqrt{a_\varepsilon}}\Big(\partial_t(\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}) - (\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}})^2\Big)\mathrm{Re}\Big(u_\nu \cdot \overline{(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu)}\Big)\, dx\Big| \\
\le C_1\frac{1}{\varepsilon}\log(\frac{1}{\varepsilon}+1)\, 2^{-\nu}\, e_{\nu,\varepsilon}(t),
\end{aligned}
$$

where, for $\nu \ge 1$, we have used the left-hand side part of (3.8). Similarly from (2.1), (3.1) and (3.2) we deduce that

$$
\Big|\int \partial_t\sqrt{a_\varepsilon}(1-\frac{a}{a_\varepsilon})|\partial_x u_\nu|^2\, dx\Big| \le C_2 \log\Big(\frac{1}{\varepsilon}+1\Big)\, e_{\nu,\varepsilon}(t),
$$

where again C_2 depends only on λ_0, Λ_0 and C_0. From (2.1) and (3.1) we have that

$$\begin{aligned}\int 2(\sqrt{a_\varepsilon}-\frac{a}{\sqrt{a_\varepsilon}})\,\mathrm{Re}\,(\partial_x u_\nu\cdot\overline{\partial_x\partial_t u_\nu})\,dx&\\ \le C_3\varepsilon\log\Big(\frac{1}{\varepsilon}+1\Big)\|\partial_x u_\nu\|_{\mathcal{L}^2}\|\partial_x\partial_t u_\nu\|_{\mathcal{L}^2}&\\ \le C_3\varepsilon\log\Big(\frac{1}{\varepsilon}+1\Big)2^{\nu+1}\|\partial_x u_\nu\|_{\mathcal{L}^2}\|\partial_t u_\nu\|_{\mathcal{L}^2},&\end{aligned}$$

where we have used the right-hand side part of (3.8). Remarking that

$$\|\partial_t u_\nu\|_{\mathcal{L}^2}\le\Big\|\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\Big\|_{\mathcal{L}^2}+\Big\|\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\Big\|_{\mathcal{L}^2},$$

and

$$\Big\|\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_0\Big\|_{\mathcal{L}^2}\le C_3'\varepsilon^{-1/2}\|u_0\|_{\mathcal{L}^2},$$

while, for all $\nu\ge 1$,

$$\Big\|\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\Big\|_{\mathcal{L}^2}\le C_3'\varepsilon^{-1/2}2^{-\nu}\|\partial_x u_\nu\|_{\mathcal{L}^2},$$

we deduce that

$$\Big|\int 2(\sqrt{a_\varepsilon}-\frac{a}{\sqrt{a_\varepsilon}})\,\mathrm{Re}\,(\partial_x u_\nu\cdot\overline{\partial_x\partial_t u_\nu})\,dx\Big|\le C_3''\Big((\varepsilon 2^\nu+1)\log\Big(\frac{1}{\varepsilon}+1\Big)\,e_{\nu,\varepsilon}(t)\Big).$$

Similarly, from (3.3),

$$\Big|\int 2\,\frac{\partial_x\sqrt{a_\varepsilon}}{a_\varepsilon}\,a\,\mathrm{Re}\Big(\partial_x u_\nu\cdot\overline{(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu)}\Big)\,dx\Big|\le C_4\log\Big(\frac{1}{\varepsilon}+1\Big)\,e_{\nu,\varepsilon}(t),$$

and, from (3.2), from (3.3) and from (3.5),

$$\Big|\int\frac{a}{\sqrt{a_\varepsilon}}\,\partial_x(\frac{\partial_t\sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}})\,\mathrm{Re}\,(\partial_x u_\nu\cdot\overline{u_\nu})\,dx\Big|\le C_5\frac{1}{\varepsilon}\log\Big(\frac{1}{\varepsilon}+1\Big)2^{-\nu}e_{\nu,\varepsilon}(t).$$

Finally

$$\Big|\int 2\mathrm{Re}\,(u_\nu\cdot\overline{\partial_t u_\nu})\,dx\Big|\le C_6\varepsilon^{-1/2}2^{-\nu}e_{\nu,\varepsilon}(t).$$

We remark that the constants C_3, C_3', C_3'', C_4, C_5, C_6 depend only on λ_0, Λ_0 and C_0. We choose now $\varepsilon = 2^{-\nu}$. We obtain that there exists $\tilde{C} > 0$ such that, for all $\nu\in\mathbb{N}$,

$$\begin{aligned}\frac{d}{dt}e_{\nu,2^{-\nu}}(t) &\le \tilde{C}(\nu+1)e_{\nu,2^{-\nu}}(t)\\ &+\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\mathrm{Re}\Big((\partial_x([\varphi_\nu,a]\partial_x u)+(Lu)_\nu\Big)\\ &\cdot\overline{\Big(\partial_t u_\nu+\frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}}u_\nu)\Big)}\,dx,\end{aligned}\tag{3.11}$$

where $\tilde{C}$ depends only on λ_0, Λ_0 and C_0.

3.4. Total energy

Let $\theta\in(0,1/2)$. We define the total energy for the function u setting

$$E(t) := \sum_{\nu=0}^{\infty} e^{-2\beta(\nu+1)t}2^{-2\nu\theta}e_{\nu,2^{-\nu}}(t),\tag{3.12}$$

where $\beta>0$ will be fixed later on. Using (3.7), (3.8) and the fact that there exists a constant $c>0$ not depending on ν such that

$$c\,e_{\nu,2^{-\nu}}(t)\le\int_{\mathbb{R}}|\partial_t u_\nu|^2+|\partial_x u_\nu|^2+|u_\nu|^2\,dx\le\frac{1}{c}\,e_{\nu,2^{-\nu}}(t),$$

it is possible to prove that there exist c_θ, $c_\theta'>0$ such that

$$E(0)\le c_\theta(\|\partial_t u(0,\cdot)\|^2_{\mathcal{H}^{-\theta}}+\|u(0,\cdot)\|^2_{\mathcal{H}^{-\theta+1}})\tag{3.13}$$

and

$$E(t)\ge c_\theta'(\|\partial_t u(t,\cdot)\|^2_{\mathcal{H}^{-\theta-\beta^* t}}+\|u(t,\cdot)\|^2_{\mathcal{H}^{-\theta+1-\beta^* t}}),\tag{3.14}$$

where $\beta^* = \beta(\log 2)^{-1}$. From (3.11) we deduce

$$\begin{aligned}
&\frac{d}{dt}E(t) \le (\tilde{C} - 2\beta)\sum_{\nu=0}^{\infty}(\nu+1)e^{-2\beta(\nu+1)t}2^{-2\nu\theta}e_{\nu,2^{-\nu}}(t)\\
&+\sum_{\nu=0}^{\infty}e^{-2\beta(\nu+1)t}2^{-2\nu\theta}\\
&\qquad\times\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\mathrm{Re}\Big(\partial_x([\varphi_\nu, a]\partial_x u)\cdot\overline{\Big(\partial_t u_\nu + \frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}}u_\nu\Big)}\Big)dx\\
&\qquad+\sum_{\nu=0}^{\infty}e^{-2\beta(\nu+1)t}2^{-2\nu\theta}\\
&\qquad\quad\times\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\mathrm{Re}\Big((Lu)_\nu\cdot\overline{\Big(\partial_t u_\nu + \frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}}u_\nu\Big)}\Big)dx.
\end{aligned} \tag{3.15}$$

It is not difficult to show that there exists $\tilde{C}_\theta > 0$ such that

$$\begin{aligned}
\sum_{\nu=0}^{\infty}e^{-2\beta(\nu+1)t}2^{-2\nu\theta}\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\mathrm{Re}\Big((Lu)_\nu\cdot\overline{\Big(\partial_t u_\nu + \frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}}u_\nu\Big)}\Big)dx\\
\le \tilde{C}_\theta E(t)^{1/2}\|Lu(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta^* t}}.
\end{aligned} \tag{3.16}$$

3.5. Estimate of the commutator term

The estimate of the second term in the right–hand side part of (3.15) is essentially the same as that one in [6, Lemma 4.4.]. For the reader's convenience we give here most part of the details. First of all we remark that

$$\Big\|\partial_x\Big(\frac{1}{\sqrt{a_{2^{-\nu}}}}\Big(\partial_t u_\nu + \frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}}u_\nu\Big)\Big)\Big\|_{\mathcal{L}^2} \le C' 2^\nu (e_{\nu,2^{-\nu}}(t))^{1/2},$$

where C' depends only on λ_0, Λ_0 and C_0. We set $\varphi_{-1} := 0$ and we define, for $\mu \ge 0$, $\psi_\mu := \varphi_{\mu-1} + \varphi_\mu + \varphi_{\mu+1}$. Then

$$\psi_\mu(D_x)(\varphi_\mu(D_x)\partial_x u) = \varphi_\mu(D_x)\partial_x u = \partial_x u_\mu,$$

and, consequently,

$$[\varphi_\nu, a]\partial_x u = [\varphi_\nu, a]\Big(\sum_\mu \partial_x u_\mu\Big) = \sum_\mu([\varphi_\nu, a]\psi_\mu)\partial_x u_\mu.$$

Hence

$$
\begin{aligned}
&\Big|\int \frac{2}{\sqrt{a_{2^{-\nu}}}}\mathrm{Re}\,\Big(\partial_x([\varphi_\nu,a]\partial_x u)\cdot\overline{\Big(\partial_t u_\nu+\frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}}u_\nu\Big)}\Big)\,dx\Big|\\
&\quad=\Big|\int\sum_\mu 2\mathrm{Re}\,\Big(([\varphi_\nu,a]\psi_\mu)\partial_x u_\mu\cdot\overline{\partial_x\Big(\frac{1}{\sqrt{a_{2^{-\nu}}}}\Big(\partial_t u_\nu+\frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}}u_\nu\Big)\Big)}\Big)\,dx\Big|\\
&\qquad\le C'\sum_\mu\|([\varphi_\nu,a]\psi_\mu)\partial_x u_\mu\|_{\mathcal{L}^2}2^\nu(e_{\nu,2^{-\nu}}(t))^{1/2}\\
&\qquad\quad\le C''\sum_\mu\|([\varphi_\nu,a]\psi_\mu)\|_{L(\mathcal{L}^2)}(e_{\mu,2^{-\mu}}(t))^{1/2}2^\nu(e_{\nu,2^{-\nu}}(t))^{1/2},
\end{aligned}
$$

where C'' depends only on λ_0, Λ_0 and C_0. This implies that

$$
\begin{aligned}
&\Big|\sum_{\nu=0}^{\infty}e^{-2\beta(\nu+1)t}2^{-2\nu\theta}\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\mathrm{Re}\,\Big(\partial_x([\varphi_\nu,a]\partial_x u)\cdot\overline{\Big(\partial_t u_\nu+\frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}}u_\nu\Big)}\Big)\,dx\Big|\\
&\quad\le C''\sum_{\nu,\mu}k_{\nu,\mu}(\nu+1)^{1/2}e^{-\beta(\nu+1)t}2^{-\nu\theta}(e_{\nu,2^{-\nu}}(t))^{1/2}\\
&\qquad\qquad\qquad\qquad\cdot(\mu+1)^{1/2}e^{-\beta(\mu+1)t}2^{-\mu\theta}(e_{\mu,2^{-\mu}}(t))^{1/2},
\end{aligned}
$$

where

$$k_{\nu,\mu}=e^{-(\nu-\mu)\beta t}\,2^{-(\nu-\mu)\theta}\,2^\nu(\nu+1)^{-1/2}(\mu+1)^{-1/2}\|([\varphi_\nu,a]\psi_\mu)\|_{L(\mathcal{L}^2)}.$$

We remark that if $|\nu-\mu|\ge 3$, then $\varphi_\nu\psi_\mu\equiv 0$ and, consequently, $[\varphi_\nu,a]\psi_\mu=\varphi_\nu([a,\psi_\mu])$, so that from (3.9) we deduce that

$$
\|([\varphi_\nu,a]\psi_\mu)\|_{L(\mathcal{L}^2)}\le\begin{cases} C'''2^{-\nu}(\nu+1) & \text{if } |\nu-\mu|\le 2,\\[2mm] C'''2^{-\max\{\nu,\mu\}}\max\{\nu+1,\mu+1\} & \text{if } |\nu-\mu|\ge 3,\end{cases}
$$

where C''' depends only on Λ_0 and C_0.

We need the following elementary lemma.

LEMMA 1. *There exist two continuous decreasing functions θ_1, θ_2 : $(0,1]\to(0,+\infty)$, with $\lim_{c\to0^+}\theta_j(c)=+\infty$ for $j=1,\ 2$, such that, for all $c\in(0,1]$ and for all $m\ge1$,*

$$\sum_{j=1}^{m}e^{cj}j^{-1/2}\le\theta_1(c)e^{cm}m^{-1/2},\qquad \sum_{j=m}^{+\infty}e^{-cj}j^{1/2}\le\theta_2(c)e^{-cm}m^{1/2}. \tag{3.17}$$

Our aim is to use Schur's Lemma, so we have to estimate

$$\sup_{\mu}\sum_{\nu}|k_{\nu,\mu}|+\sup_{\nu}\sum_{\mu}|k_{\nu,\mu}|.$$

We choose now $\beta>0$ and $T\in(0,T_0]$ in such a way that $\beta T=\frac{\theta}{2}\log 2$ (remark that for the moment only the product βT is fixed). Then for $t\in(0,T]$ we have that

$$\beta t+\theta\log 2\geq\theta\log 2>0, \tag{3.18}$$

and

$$(-\theta+1)\log 2-\beta t\geq(1-\frac{3}{2}\theta)\log 2>0. \tag{3.19}$$

Let $\mu\leq 2$. Then, using the second estimate in (3.17) and (3.18), we have

$$\begin{aligned}\sum_{\nu=0}^{+\infty}|k_{\nu,\mu}|&\leq C'''e^{(\mu+1)\beta t}2^{(\mu+1)\theta}(\mu+1)^{-1/2}\sum_{\nu=0}^{+\infty}e^{-(\nu+1)\beta t}2^{-(\nu+1)\theta}(\nu+1)^{1/2}\\&\leq C'''e^{(\mu+1)\beta t}2^{(\mu+1)\theta}(\mu+1)^{-1/2}\sum_{\nu=0}^{+\infty}e^{-(\beta t+\theta\log 2)(\nu+1)}(\nu+1)^{1/2}\\&\leq C'''e^{2\beta t}2^{2\theta}\theta_2(\beta t+\theta\log 2)\\&\leq C'''2^{3\theta}\theta_2(\theta\log 2).\end{aligned}$$

Let $\mu\geq 3$. We have $\sum_{\nu=0}^{+\infty}|k_{\nu,\mu}|=\sum_{\nu=0}^{\mu-3}|k_{\nu,\mu}|+\sum_{\nu=\mu-2}^{+\infty}|k_{\nu,\mu}|$. Then, from the first one in (3.17) and (3.19), we deduce

$$\begin{aligned}&\sum_{\nu=0}^{\mu-3}|k_{\nu,\mu}|\\&\leq C'''e^{(\mu+1)\beta t}2^{(\mu+1)(\theta-1)}(\mu+1)^{1/2}\sum_{\nu=0}^{\mu-3}e^{-(\nu+1)\beta t}2^{(\nu+1)(-\theta+1)}(\nu+1)^{-1/2}\\&\leq C'''e^{(\mu+1)\beta t}2^{(\mu+1)(\theta-1)}(\mu+1)^{1/2}\sum_{\nu=0}^{\mu-3}e^{(-\beta t+(-\theta+1)\log 2)(\nu+1)}(\nu+1)^{-1/2}\\&\leq C'''e^{(\mu+1)\beta t}2^{(\mu+1)(\theta-1)}(\mu+1)^{1/2}\theta_1(-\beta t+(-\theta+1)\log 2)\\&\qquad\qquad\cdot e^{(-\beta t+(-\theta+1)\log 2)(\mu-2)}(\mu-2)^{-1/2}\\&\leq C'''2^{1+\frac{9}{2}\theta}\theta_1\big((1-\frac{3}{2}\theta)\log 2\big),\end{aligned}$$

and, from the second one in (3.17) and (3.18),

$$\begin{aligned}\sum_{\nu=\mu-2}^{+\infty}|k_{\nu,\mu}| &\le C''' e^{(\mu+1)\beta t}2^{(\mu+1)\theta}(\mu+1)^{-1/2}\\ &\qquad\times\sum_{\nu=\mu-2}^{\infty} e^{-(\nu+1)\beta t}2^{-(\nu+1)\theta}(\nu+1)^{1/2}\\ &\le C''' e^{(\mu+1)\beta t}2^{(\mu+1)\theta}(\mu+1)^{-1/2}\theta_2(\beta t+\theta\log 2)\\ &\qquad\qquad\cdot e^{-(\beta t+\theta\log 2)(\mu-1)}(\mu-1)^{1/2}\\ &\le C''' 2^{3\theta}\theta_2(\theta\log 2).\end{aligned}$$

Considering now $\sum_\mu |k_{\nu,\mu}|$ we have

$$\begin{aligned}\sum_{\mu=0}^{\nu+2}|k_{\nu,\mu}| &\le C''' e^{-(\nu+1)\beta t}2^{-(\nu+1)\theta}(\nu+1)^{1/2}\sum_{\mu=0}^{\nu+2} e^{(\mu+1)\beta t}2^{(\mu+1)\theta}(\mu+1)^{-1/2}\\ &\le C''' e^{-(\nu+1)\beta t}2^{-(\nu+1)\theta}(\nu+1)^{1/2}\theta_1(\beta t+\theta\log 2)\\ &\qquad\qquad\cdot e^{(\beta t+\theta\log 2)(\nu+3)}(\nu+3)^{-1/2}\\ &\le C''' 2^{\frac{7}{2}\theta}\theta_1(\theta\log 2),\end{aligned}$$

and

$$\begin{aligned}&\sum_{\mu=\nu+3}^{+\infty}|k_{\nu,\mu}|\\ &\quad\le C''' e^{-(\nu+1)\beta t}2^{(\nu+1)(-\theta+1)}(\nu+1)^{-1/2}\\ &\qquad\times\sum_{\mu=\nu+3}^{\infty} e^{(\mu+1)\beta t}2^{-(\mu+1)(-\theta+1)}(\mu+1)^{1/2}\\ &\quad\le C''' e^{-(\nu+1)\beta t}2^{(\nu+1)(-\theta+1)}(\nu+1)^{-1/2}\theta_2(-\beta t+(-\theta+1)\log 2)\\ &\qquad\qquad\cdot e^{(-\beta t+(-\theta+1)\log 2)(\nu+4)}(\nu+4)^{1/2}\\ &\quad\le C''' 2^{\frac{9}{2}\theta}\theta_2\big(\big(1-\frac{3}{2}\theta\big)\log 2\big).\end{aligned}$$

Hence there exists a positive function Θ, with $\lim_{\theta\to 0^+}\Theta(\theta)=+\infty$, such that

$$\sup_\mu\sum_{\nu=0}^{+\infty}|k_{\nu,\mu}|+\sup_\nu\sum_{\mu=0}^{+\infty}|k_{\nu,\mu}|\le C'''\Theta(\theta).$$

We finally obtain

$$
\begin{aligned}
&\Big|\sum_{\nu=0}^{\infty} e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} \\
&\times \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \mathrm{Re}\Big(\partial_x([\varphi_\nu, a]\partial_x u)\cdot\overline{\Big(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\Big)}\Big)\, dx\Big| \\
&\qquad\leq C'' C''' \Theta(\theta) \sum_{\nu=0}^{\infty} (\nu+1) e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} e_{\nu,2^{-\nu}}(t).
\end{aligned}
\tag{3.20}
$$

3.6. End of the proof

From (3.15), (3.16) and (3.20) we have that

$$
\begin{aligned}
\frac{d}{dt}E(t) \leq (\tilde{C} + C''C'''\Theta(\theta) - 2\beta) &\sum_{\nu=0}^{\infty} (\nu+1) e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} e_{\nu,2^{-\nu}}(t) \\
&+ \tilde{C}_\theta E(t)^{1/2} \|Lu(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta^* t}}.
\end{aligned}
$$

We fix now β in such a way that $\tilde{C} + C''C'''\Theta(\theta) - 2\beta \leq 0$. Remark that since the product βT was already fixed, this force us to choose T sufficiently small. We obtain

$$
\frac{d}{dt}E(t) \leq \tilde{C}_\theta E(t)^{1/2} \|Lu(t,\cdot)\|_{\mathcal{H}^{-\theta-\beta^* t}},
$$

and the conclusion of the theorem easily follows from (3.13) and (3.14).

Appendix

We give here in some details an example due to S. Tarama concerning a bounded function which is log–Zygmund–continuous but not log–Lipschitz–continuous. The function is the following

$$
\omega(t) = \sum_{n=1}^{\infty} 2^{-n} n \sin(2^n t).
$$

Considering the sequence $t_k = 2^{-k-1}\pi$, $k \geq 1$, it is easy to see that

$$
\omega(t_k) = \sum_{n=1}^{k} 2^{-n} n \sin(2^{n-k-1}\pi) \geq 2^{-k-1} k(k-1),
$$

so that

$$\frac{|\omega(t_k) - \omega(0)|}{|t_k \log t_k|} \geq C_0 k$$

and, consequently, ω is not log–Lipschitz–continuous. To prove that ω is log-Zygmund–continuous we argue as in [12]. Setting $\varepsilon \in (0, 1/2)$ and $\omega(t) = \omega_{1,\varepsilon}(t) + \omega_{2,\varepsilon}(t)$, where

$$\omega_{1,\varepsilon}(t) = \sum_{1 \leq n \leq \frac{|\log \varepsilon|}{\log 2}} 2^{-n} n \sin(2^n t) \quad \text{and} \quad \omega_{2,\varepsilon}(t) = \sum_{n > \frac{|\log \varepsilon|}{\log 2}} 2^{-n} n \sin(2^n t),$$

we easily deduce that $|\omega''_{1,\varepsilon}(t)| \leq C\varepsilon^{-1}|\log \varepsilon|$ while $\omega_{2,\varepsilon}(t)| \leq C\varepsilon|\log \varepsilon|$. Then $|\omega(t+\varepsilon) + \omega(t-\varepsilon) - 2\omega(t)| \leq C'\varepsilon|\log \varepsilon|$ and the conclusion follows. To end let us remark that the function ω is nowhere differentiable (see [10]).

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