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Ultraregular Generalized Functions of Colombeau Type

By Khaled Benmeriem and Chikh Bouzar

Abstract. General algebras of ultradifferentiable generalized functions satisfying some regularity assumptions are introduced. We give a microlocal analysis within these algebras related to the regularity type and the ultradifferentiable property. As a particular case we obtain new algebras of Gevrey generalized functions.

1. Introduction

The current research in the regularity problem in Colombeau algebra $\mathcal{G}(\Omega)$ is based, e.g. see [6], on the Oberguggenberger subalgebra $\mathcal{G}^{\infty}(\Omega)$: the first measure of regularity within the Colombeau algebra. This subalgebra $\mathcal{G}^{\infty}(\Omega)$ plays the same role as $\mathcal{C}^{\infty}(\Omega)$ in $\mathcal{D}'(\Omega)$, and has indicated the importance of the asymptotic behavior of the representative nets of a Colombeau generalized function in studying regularity problems. However, the $\mathcal{G}^{\infty}-$regularity does not exhaust the regularity questions inherent to the Colombeau algebra $\mathcal{G}(\Omega)$, see [12]. Candidates proposed for measuring the regularity within $\mathcal{G}(\Omega)$ make accent on the growth in ε and its asymptotic behavior of the nets of smooth functions representing a Colombeau generalized function.

The aim of this paper is to introduce and to study new general classes of generalized functions measuring regularity both by the asymptotic behavior of the nets of smooth functions representing a Colombeau generalized function and by their ultradifferentiable smoothness. We define $\mathcal{G}^{M,\mathcal{R}}(\Omega)$ subalgebras of $\mathcal{G}(\Omega)$ representing classes of nets $(u_{\varepsilon})_{\varepsilon}$ of smooth functions having simultaneously ultradifferentiable smoothness of Denjoy-Carleman type $M = (M_p)_{p\in\mathbb{Z}_+}$ and $\mathcal{R}$-regular asymptotic behavior in ε. Elements of $\mathcal{G}^{M,\mathcal{R}}(\Omega)$ are called ultraregular generalized functions. The importance of ultradifferentiable functions in the study of partial differential equations is

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well established, see [8], [5] and [13]. The recently introduced $\mathcal{R}$-regular generalized functions [3] promise to give more understanding of Colombeau generalized functions. Ultraregular generalized functions of $(M, \mathcal{R})$ type will without doubt contribute to regularity theory in Colombeau algebra.

Sections two and three recall regular generalized functions and ultradifferentiable functions and give some of their main properties. Section four introduces the algebras $\mathcal{G}^{M,\mathcal{R}}(\Omega)$ of ultraregular generalized functions and show their important properties. Section five is devoted to the $\mathcal{G}^{M,\mathcal{A}}(\Omega)-$ microlocal analysis of Colombeau generalized functions.

2. Regular Generalized Functions

For a deep study of Colombeau generalized functions see [2] and [4]. Let Ω be a non void open subset of $\mathbb{R}^n$, define $\mathcal{X}(\Omega)$ as the space of elements $(u_\varepsilon)_\varepsilon$ of $C^\infty(\Omega)^{]0,1]}$ such that, for every compact set $K \subset \Omega$, $\forall \alpha \in \mathbb{Z}_+^n, \exists m \in \mathbb{Z}_+$, $\exists C > 0, \exists \eta \in]0,1], \forall \varepsilon \in]0,\eta]$,

$$\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq C\varepsilon^{-m}.$$

By $\mathcal{N}(\Omega)$ we denote the elements $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$ such that for every compact set $K \subset \Omega, \forall \alpha \in \mathbb{Z}_+^n, \forall m \in \mathbb{Z}_+$, $\exists C > 0, \exists \eta \in]0,1], \forall \varepsilon \in]0,\eta]$,

$$\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq C\varepsilon^{m}.$$

Definition 1. The Colombeau algebra, denoted by $\mathcal{G}(\Omega)$, is the quotient algebra

$$\mathcal{G}(\Omega) = \frac{\mathcal{X}(\Omega)}{\mathcal{N}(\Omega)}.$$

$\mathcal{G}(\Omega)$ is a commutative and associative differential algebra containing $\mathcal{D}'(\Omega)$ as a subspace and $C^\infty(\Omega)$ as a subalgebra. The subalgebra of generalized functions with compact support, denoted $\mathcal{G}_C(\Omega)$, is the space of elements f of $\mathcal{G}(\Omega)$ satisfying : there exist a representative $(f_\varepsilon)_{\varepsilon \in]0,1]}$ of f and a compact subset K of $\Omega, \forall \varepsilon \in]0,1]$, $\operatorname{supp} f_\varepsilon \subset K$.

One defines the subalgebra of regular elements $\mathcal{G}^{\infty}(\Omega)$, introduced by Oberguggenberger in [11], as the quotient algebra

$$\frac{\mathcal{X}^{\infty}(\Omega)}{\mathcal{N}(\Omega)},$$

where $\mathcal{X}^{\infty}(\Omega)$ is the space of elements $(u_{\varepsilon})_{\varepsilon}$ of $C^{\infty}(\Omega)^{]0,1]}$ such that, for every compact $K \subset \Omega$, $\exists m \in \mathbb{Z}_{+}, \forall \alpha \in \mathbb{Z}_{+}^{n}$, $\exists C > 0, \exists \eta \in]0,1], \forall \varepsilon \in]0,\eta]$,

$$\sup_{x \in K} |\partial^{\alpha} u_{\varepsilon}(x)| \leq C\varepsilon^{-m}.$$

It is proved in [11] the following fundamental result

$$\mathcal{G}^{\infty}(\Omega) \cap \mathcal{D}'(\Omega) = C^{\infty}(\Omega).$$

This means that the subalgebra $\mathcal{G}^{\infty}(\Omega)$ plays in $\mathcal{G}(\Omega)$ the same role as $C^{\infty}(\Omega)$ in $\mathcal{D}'(\Omega)$, consequently one can introduce a local analysis by defining the generalized singular support of $u \in \mathcal{G}(\Omega)$. This was the first notion of regularity in Colombeau algebra. Recently, different measures of regularity in algebras of generalized functions have been proposed, see [1], [3], [10] and [12]. For our needs we recall the essential definitions and results on $\mathcal{R}$ -regular generalized functions, see [3].

Definition 2. Let $(N_m)_{m\in\mathbb{Z}_+}, (N'_m)_{m\in\mathbb{Z}_+}$ be two elements of $\mathbb{R}_{+}^{\mathbb{Z}_+}$, we write $(N_m)_{m\in\mathbb{Z}_+} \leq (N'_m)_{m\in\mathbb{Z}_+}$, if $\forall m \in \mathbb{Z}_+$, $N_m \leq N'_m$. A non void subspace $\mathcal{R}$ of $\mathbb{R}_{+}^{\mathbb{Z}_+}$ is said regular, if the following (R1)-(R3) are all satisfied :

For all $(N_m)_{m\in\mathbb{Z}_+} \in \mathcal{R}$ and $(k,k') \in \mathbb{Z}_{+}^{2}$, there exists $(N'_m)_{m\in\mathbb{Z}_+} \in \mathcal{R}$ such that

(R1) $$N_{m+k} + k' \leq N'_m \ , \forall m \in \mathbb{Z}_+.$$

For all $(N_m)_{m\in\mathbb{Z}_+}$ and $(N'_m)_{m\in\mathbb{Z}_+}$ in $\mathcal{R}$, there exists $(N"_m)_{m\in\mathbb{Z}_+} \in \mathcal{R}$ such that

(R2) $$\max\left(N_m, N'_m\right) \leq N"_m \ , \forall m \in \mathbb{Z}_+.$$

For all $(N_m)_{m\in\mathbb{Z}_+}$ and $(N'_m)_{m\in\mathbb{Z}_+}$ in $\mathcal{R}$, there exists $(N"_m)_{m\in\mathbb{Z}_+} \in \mathcal{R}$ such that

(R3) $$N_{l_1} + N'_{l_2} \leq N"_{l_1+l_2} \ , \forall (l_1,l_2) \in \mathbb{Z}_{+}^{2}.$$

Define the $\mathcal{R}-$regular moderate elements of $\mathcal{X}(\Omega)$, by

$$\mathcal{X}^{\mathcal{R}}(\Omega)=\Big\{(u_\varepsilon)_\varepsilon\in\mathcal{X}(\Omega)\mid\forall K\subset\subset\Omega,\exists N\in\mathcal{R},\forall\alpha\in\mathbb{Z}_+^n,$$
$$\exists C>0,\exists\eta\in]0,1],\forall\varepsilon\in]0,\eta]:\sup_{x\in K}|\partial^\alpha u_\varepsilon(x)|\leq C\varepsilon^{-N_{|\alpha|}}\Big\}$$

and its ideal

$$\mathcal{N}^{\mathcal{R}}(\Omega)=\Big\{(u_\varepsilon)_\varepsilon\in\mathcal{X}(\Omega)\mid\forall K\subset\subset\Omega,\forall N\in\mathcal{R},\forall\alpha\in\mathbb{Z}_+^n,$$
$$\exists C>0,\exists\eta\in]0,1],\forall\varepsilon\in]0,\eta]:\sup_{x\in K}|\partial^\alpha u_\varepsilon(x)|\leq C\varepsilon^{N_{|\alpha|}}\Big\}.$$

Proposition 1. *1. The space $\mathcal{X}^{\mathcal{R}}(\Omega)$ is a subalgebra of $\mathcal{X}(\Omega)$, stable by differentiation.*
2. The set $\mathcal{N}^{\mathcal{R}}(\Omega)$ is an ideal of $\mathcal{X}^{\mathcal{R}}(\Omega)$.

Remark 1. From (R1), one can show that $\mathcal{N}^{\mathcal{R}}(\Omega)=\mathcal{N}(\Omega)$.

Definition 3. The algebra of $\mathcal{R}$-regular generalized functions, denoted by $\mathcal{G}^{\mathcal{R}}(\Omega)$, is the quotient algebra

$$\mathcal{G}^{\mathcal{R}}(\Omega)=\frac{\mathcal{X}^{\mathcal{R}}(\Omega)}{\mathcal{N}(\Omega)}.$$

Example 1. The Colombeau algebra $\mathcal{G}(\Omega)$ is obtained when $\mathcal{R}=\mathbb{R}_+^{\mathbb{Z}_+}$, i.e. $\mathcal{G}^{\mathbb{R}_+^{\mathbb{Z}_+}}(\Omega)=\mathcal{G}(\Omega)$.

Example 2. When

$$\mathcal{A}=\Big\{(N_m)_{m\in\mathbb{Z}_+}\in\mathcal{R}:\exists a\geq 0,\exists b\geq 0,N_m\leq am+b,\forall m\in\mathbb{Z}_+\Big\},$$

we obtain a differential subalgebra denoted by $\mathcal{G}^{\mathcal{A}}(\Omega)$.

Example 3. When $\mathcal{R}=\mathcal{B}$ the set of all bounded sequences, we obtain the Oberguggenberger subalgebra, i.e. $\mathcal{G}^{\mathcal{B}}(\Omega)=\mathcal{G}^\infty(\Omega)$.

Example 4. If we take $\mathcal{R} = \{0\}$, the condition (R1) is not hold, however we can define

$$\mathcal{G}^0(\Omega) = \frac{\mathcal{X}^{\mathcal{R}}(\Omega)}{\mathcal{N}(\Omega)}$$

as the algebra of elements which have all derivatives locally uniformly bounded for small ε, see [12].

We have the following inclusions

$$\mathcal{G}^0(\Omega) \subset \mathcal{G}^{\mathcal{B}}(\Omega) \subset \mathcal{G}^{\mathcal{A}}(\Omega) \subset \mathcal{G}(\Omega).$$

3. Ultradifferentiable Functions

We recall some classical results of ultradifferentiable functions spaces. A sequence of positive numbers $(M_p)_{p\in\mathbb{Z}_+}$ is said to satisfy the following conditions :

$(H1)$ Logarithmic convexity, if

$$M_p^2 \leq M_{p-1}M_{p+1},\ \forall p \geq 1.$$

$(H2)$ Stability under ultradifferential operators, if there are constants $A > 0$ and $H > 0$ such that

$$M_p \leq AH^p M_q M_{p-q}, \forall p \geq q.$$

$(H3)'$ Non-quasi-analyticity, if

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < +\infty.$$

Remark 2. Some results remain valid, see [7], when $(H2)$ is replaced by the following weaker condition :

$(H2)'$ Stability under differential operators, if there are constants $A > 0$ and $H > 0$ such that

$$M_{p+1} \leq AH^p M_p, \forall p \in \mathbb{Z}_+.$$

The associated function of the sequence $(M_p)_{p\in\mathbb{Z}_+}$ is the function defined by

$$\widetilde{M}(t) = \sup_p \ln \frac{t^p}{M_p}, t \in \mathbb{R}_+^*.$$

Some needed results of the associated function are given in the following propositions proved in [7].

Proposition 2. *A positive sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfies condition $(H1)$ if and and only if*

$$M_p = M_0 \sup_{t>0} \frac{t^p}{\exp\left(\widetilde{M}(t)\right)}, \qquad p = 0, 1, ...$$

Proposition 3. *A positive sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfies condition $(H2)$ if and and only if , $\exists A > 0, \exists H > 0, \forall t > 0$,*

$$2\widetilde{M}(t) \leq \widetilde{M}(Ht) + \ln(AM_0). \tag{3.1}$$

Remark 3. We will always suppose that the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfies the condition $(H1)$ and $M_0 = 1$.

A differential operator of infinite order $P(D) = \sum_{\gamma\in\mathbb{Z}_+^n} a_\gamma D^\gamma$ is called an ultradifferential operator of class $M = (M_p)_{p\in\mathbb{Z}_+}$, if for every $h > 0$ there exist a constant $c > 0$ such that $\forall\gamma \in \mathbb{Z}_+^n$,

$$|a_\gamma| \leq c\frac{h^{|\gamma|}}{M_{|\gamma|}}. \tag{3.2}$$

The class of ultradifferentiable functions of class M, denoted by $\mathcal{E}^M(\Omega)$, is the space of all $f \in C^\infty(\Omega)$ satisfying for every compact subset K of Ω , $\exists c > 0, \forall\alpha \in \mathbb{Z}_+^n$,

$$\sup_{x\in K} |\partial^\alpha f(x)| \leq c^{|\alpha|+1} M_{|\alpha|}. \tag{3.3}$$

This space is also called the space of Denjoy-Carleman.

Example 5. If $(M_p)_{p\in\mathbb{Z}_+} = (p!^{\sigma})_{p\in\mathbb{Z}_+}, \sigma > 1$, we obtain $\mathcal{E}^{\sigma}(\Omega)$ the Gevrey space of order σ, and $\mathcal{A}(\Omega) := \mathcal{E}^1(\Omega)$ is the space of real analytic functions defined on the open set Ω.

The basic properties of the space $\mathcal{E}^M(\Omega)$ are summarized in the following proposition, for the proof see [9] and [7].

PROPOSITION 4. *The space $\mathcal{E}^M(\Omega)$ is an algebra. Moreover, if $(M_p)_{p\in\mathbb{Z}_+}$ satisfies $(H2)'$, then $\mathcal{E}^M(\Omega)$ is stable by any differential operator of finite order with coefficients in $\mathcal{E}^M(\Omega)$ and if $(M_p)_{p\in\mathbb{Z}_+}$ satisfies $(H2)$ then any ultradifferential operator of class M operates also as a sheaf homomorphism. The space $\mathcal{D}^M(\Omega) = \mathcal{E}^M(\Omega) \cap \mathcal{D}(\Omega)$ is well defined and is not trivial if and only if the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfies $(H3)'$.*

REMARK 4. The strong dual of $\mathcal{D}^M(\Omega)$, denoted $\mathcal{D}'^M(\Omega)$, is called the space of Roumieu ultraditributions.

4. Ultraregular Generalized Functions

In the same way as $\mathcal{G}^{\infty}(\Omega)$, $\mathcal{G}^{\mathcal{R}}(\Omega)$ forms a sheaf of differential subalgebras of $\mathcal{G}(\Omega)$, consequently one defines the generalized $\mathcal{R}-$singular support of $u \in \mathcal{G}(\Omega)$, denoted by $\operatorname{singsupp}_{\mathcal{R}} u$, as the complement in Ω of the largest set Ω' such that $u_{/\Omega'} \in \mathcal{G}^{\mathcal{R}}(\Omega')$, where $u_{/\Omega'}$ means the restriction of the generalized function u on Ω'. This new notion of regularity is linked with the asymptotic limited growth of generalized functions. Our aim in this section is to introduce a more precise notion of regularity within the Colombeau algebra taking into account both the asymptotic growth and the smoothness property of generalized functions. We introduce general algebras of ultradifferentiable $\mathcal{R}-$regular generalized functions of class M, where the sequence $M = (M_p)_{p\in\mathbb{Z}_+}$ satisfies the conditions $(H1)$ with $M_0 = 1, (H2)$ and $(H3)'$.

DEFINITION 4. The space of ultraregular moderate elements of class M, denoted $\mathcal{X}^{M,\mathcal{R}}(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^{\infty}(\Omega)^{]0,1]}$ satisfying for every compact K of Ω, $\exists N\in\mathcal{R},\exists C>0,\exists\varepsilon_0\in\left]0,1\right],\forall\alpha\in\mathbb{Z}_+^n,\forall\varepsilon\leq\varepsilon_0$,

$$\sup_{x\in K}\left|\partial^{\alpha}f_{\varepsilon}\left(x\right)\right|\leq C^{|\alpha|+1}M_{|\alpha|}\varepsilon^{-N_{|\alpha|}}. \tag{4.1}$$

The space of null elements is defined as $\mathcal{N}^{M,\mathcal{R}}\left(\Omega\right):=\mathcal{N}\left(\Omega\right)\cap\mathcal{X}^{M,\mathcal{R}}\left(\Omega\right).$

The main properties of the spaces $\mathcal{X}^{M,\mathcal{R}}\left(\Omega\right)$ and $\mathcal{N}^{M,\mathcal{R}}\left(\Omega\right)$ are given in the following proposition.

Proposition 5. *1) The space $\mathcal{X}^{M,\mathcal{R}}\left(\Omega\right)$ is a subalgebra of $\mathcal{X}\left(\Omega\right)$ stable by action of differential operators*
2) The space $\mathcal{N}^{M,\mathcal{R}}\left(\Omega\right)$ is an ideal of $\mathcal{X}^{M,\mathcal{R}}\left(\Omega\right).$

Proof. 1) Let $\left(f_{\varepsilon}\right)_{\varepsilon},\left(g_{\varepsilon}\right)_{\varepsilon}\in\mathcal{X}^{M,\mathcal{R}}\left(\Omega\right)$ and K a compact subset of Ω, then

$\exists N\in\mathcal{R},\exists C_1>0,\exists\varepsilon_1\in\left]0,1\right]$, such that $\forall\beta\in\mathbb{Z}_+^n,\forall x\in K,\forall\varepsilon\leq\varepsilon_1$,

$$\left|\partial^{\beta}f_{\varepsilon}\left(x\right)\right|\leq C_1^{|\beta|+1}M_{|\beta|}\varepsilon^{-N_{|\beta|}}, \tag{4.2}$$

$\exists N'\in\mathcal{R},\exists C_2>0,\exists\varepsilon_2\in\left]0,1\right]$, such that $\forall\beta\in\mathbb{Z}_+^n,\forall x\in K,\forall\varepsilon\leq\varepsilon_2$,

$$\left|\partial^{\beta}g_{\varepsilon}\left(x\right)\right|\leq C_2^{|\beta|+1}M_{|\beta|}\varepsilon^{-N'_{|\beta|}}. \tag{4.3}$$

It clear from (R2) that $\left(f_{\varepsilon}+g_{\varepsilon}\right)_{\varepsilon}\in\mathcal{X}^{M,\mathcal{R}}\left(\Omega\right).$ Let $\alpha\in\mathbb{Z}_+^n$, then

$$\left|\partial^{\alpha}\left(f_{\varepsilon}g_{\varepsilon}\right)\left(x\right)\right|\leq\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\left|\partial^{\alpha-\beta}f_{\varepsilon}\left(x\right)\right|\left|\partial^{\beta}g_{\varepsilon}\left(x\right)\right|.$$

From (R3) $\exists N"\in\mathcal{R}$ such that, $\forall\beta\leq\alpha$, $N_{|\alpha-\beta|}+N'_{|\beta|}\leq N"_{|\alpha|}$, and from $(H1)$, we have $M_pM_q\leq M_{p+q}$, then for $\varepsilon\leq\min\left\{\varepsilon_1,\varepsilon_2\right\}$ and $x\in K$, we have

$$\begin{aligned}\frac{\varepsilon^{N"_{|\alpha|}}}{M_{|\alpha|}}\left|\partial^{\alpha}\left(f_{\varepsilon}g_{\varepsilon}\right)\left(x\right)\right| &\leq \sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\frac{\varepsilon^{N_{|\alpha-\beta|}}}{M_{|\alpha-\beta|}}\left|\partial^{\alpha-\beta}f_{\varepsilon}\left(x\right)\right|\times\\ &\quad\times\frac{\varepsilon^{N'_{|\beta|}}}{M_{|\beta|}}\left|\partial^{\beta}g_{\varepsilon}\left(x\right)\right|\\ &\leq \sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}C_1^{|\alpha-\beta|+1}C_2^{|\beta|+1}\\ &\leq C^{|\alpha|+1},\end{aligned}$$

where $C = \max\{C_1C_2, C_1 + C_2\}$, then $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{X}^{M,\mathcal{R}}(\Omega)$.

Let now $\alpha, \beta \in \mathbb{Z}_+^n$, where $|\beta| = 1$, then for $\varepsilon \leq \varepsilon_1$ and $x \in K$, we have

$$\left|\partial^\alpha \left(\partial^\beta f_\varepsilon\right)(x)\right| \leq C_1^{|\alpha|+2} M_{|\alpha|+1} \varepsilon^{-N_{|\alpha|+1}}.$$

From (R1), $\exists N' \in \mathcal{R}$, such that $N_{|\alpha|+1} \leq N'_{|\alpha|}$, and from $(H2)'$, $\exists A > 0, H > 0$, such that $M_{|\alpha|+1} \leq AH^{|\alpha|} M_{|\alpha|}$, we have

$$\begin{aligned}\left|\partial^\alpha \left(\partial^\beta f_\varepsilon\right)(x)\right| &\leq AC_1^2 (C_1 H)^{|\alpha|} M_{|\alpha|} \varepsilon^{-N'_{|\alpha|}} \\ &\leq C^{|\alpha|+1} M_{|\alpha|} \varepsilon^{-N'_{|\alpha|}},\end{aligned}$$

which means $\left(\partial^\beta f_\varepsilon\right)_\varepsilon \in \mathcal{X}^{M,\mathcal{R}}(\Omega)$.

2) The facts that $\mathcal{N}^{M,\mathcal{R}}(\Omega) = \mathcal{N}(\Omega) \cap \mathcal{X}^{M,\mathcal{R}}(\Omega) \subset \mathcal{X}^{M,\mathcal{R}}(\Omega)$ and $\mathcal{N}(\Omega) = \mathcal{N}^{\mathcal{R}}(\Omega)$ is an ideal of $\mathcal{X}^{\mathcal{R}}(\Omega)$ give that $\mathcal{N}^{M,\mathcal{R}}(\Omega)$ is an ideal of $\mathcal{X}^{M,\mathcal{R}}(\Omega)$. □

The following definition introduces the algebra of ultraregular generalized functions.

Definition 5. The algebra of ultraregular generalized functions of class $M = (M_p)_{p\in\mathbb{Z}_+}$, denoted $\mathcal{G}^{M,\mathcal{R}}(\Omega)$, is the quotient algebra

$$\mathcal{G}^{M,\mathcal{R}}(\Omega) = \frac{\mathcal{X}^{M,\mathcal{R}}(\Omega)}{\mathcal{N}^{M,\mathcal{R}}(\Omega)}. \tag{4.4}$$

The basic properties of $\mathcal{G}^{M,\mathcal{R}}(\Omega)$ are given in the following assertion.

Proposition 6. *$\mathcal{G}^{M,\mathcal{R}}(\Omega)$ is a differential subalgebra of $\mathcal{G}(\Omega)$.*

Proof. The algebraic properties hold from proposition 5. □

Example 6. If we take the set $\mathcal{R} = \mathcal{B}$ we obtain as a particular case the algebra $\mathcal{G}^{M,\mathcal{B}}(\Omega)$ of [10] denoted there by $\mathcal{G}^L(\Omega)$.

Example 7. If we take $(M_p)_{p\in\mathbb{Z}_+} = (p!^\sigma)_{p\in\mathbb{Z}_+}$ we obtain a new subalgebra $\mathcal{G}^{\sigma,\mathcal{R}}(\Omega)$ of $\mathcal{G}(\Omega)$ called the algebra of Gevrey regular generalized functions of order σ.

Example 8. If we take both the set $\mathcal{R} = \mathcal{B}$ and $(M_p)_{p\in\mathbb{Z}_+} = (p!^{\sigma})_{p\in\mathbb{Z}_+}$ we obtain a new algebra, denoted $\mathcal{G}^{\sigma,\infty}(\Omega)$, that we will call the Gevrey-Oberguggenberger algebra of order σ.

REMARK 5. In [1] is introduced an algebra of generalized Gevrey ultradistributions containing the classical Gevrey space $\mathcal{E}^{\sigma}(\Omega)$ as a subalgebra and the space of Gevrey ultradistributions $\mathcal{D}'_{3\sigma-1}(\Omega)$ as a subspace.

It is not evident how to obtain, without more conditions, that $\mathcal{X}^{M,\mathcal{R}}(\Omega)$ is stable by action of ultradifferential operators of class M, however we have the following result.

PROPOSITION 7. *Suppose that the regular set $\mathcal{R}$ satisfies as well the following condition : For all $(N_k)_{k\in\mathbb{Z}_+} \in \mathcal{R}$, there exist an $(N_k^*)_{k\in\mathbb{Z}_+} \in \mathcal{R}$, and positive numbers $h > 0, L > 0, \forall m \in \mathbb{Z}_+, \forall \varepsilon \in]0,1]$,*

$$\sum_{k\in\mathbb{Z}_+} h^k \varepsilon^{-N_{k+m}} \leq L\varepsilon^{-N_m^*}. \tag{4.5}$$

Then the algebra $\mathcal{X}^{M,\mathcal{R}}(\Omega)$ is stable by action of ultradifferential operators of class M.

PROOF. Let $(f_\varepsilon)_\varepsilon \in \mathcal{X}^{M,\mathcal{R}}(\Omega)$ and $P(D) = \sum_{\beta\in\mathbb{Z}_+^n} a_\beta D^\beta$ be an ultradifferential operator of class M, then for any compact set K of Ω, $\exists (N_m)_{m\in\mathbb{Z}_+} \in \mathcal{R}, \exists C > 0, \exists \varepsilon_0 \in]0,1]$, such that $\forall \alpha \in \mathbb{Z}_+^n, \forall x \in K, \forall \varepsilon \leq \varepsilon_1$,

$$|\partial^\alpha f_\varepsilon(x)| \leq C^{|\alpha|+1} M_{|\alpha|} \varepsilon^{-N_{|\alpha|}}.$$

For every $h > 0$ there exists a $c > 0$ such that $\forall \beta \in \mathbb{Z}_+^n$,

$$|a_\beta| \leq c\frac{h^{|\beta|}}{M_{|\beta|}}.$$

Let $\alpha \in \mathbb{Z}_+^n$, then

$$\begin{aligned} \frac{h^{|\alpha|}}{M_{|\alpha|}} |\partial^\alpha (P(D) f_\varepsilon)(x)| &\leq \sum_{\beta\in\mathbb{Z}_+^n} |a_\beta| \frac{h^{|\alpha|}}{M_{|\alpha|}} \left|\partial^{\alpha+\beta} f_\varepsilon(x)\right| \\ &\leq c\sum_{\beta\in\mathbb{Z}_+^n} \frac{h^{|\beta|}}{M_{|\beta|}} \frac{h^{|\alpha|}}{M_{|\alpha|}} \left|\partial^{\alpha+\beta} f_\varepsilon(x)\right|. \end{aligned}$$

From $(H2)$ and $(f_\varepsilon)_\varepsilon \in \mathcal{X}^{M,\mathcal{R}}(\Omega)$, then $\exists C > 0, \exists A > 0, \exists H > 0,$

$$\frac{h^{|\alpha|}}{M_{|\alpha|}} \left|\partial^\alpha \left(P\left(D\right) f_\varepsilon\right)(x)\right| \leq A^{|\alpha|} \sum_{\beta \in \mathbb{Z}_+^n} h^{|\beta|} \varepsilon^{-N_{|\alpha+\beta|}},$$

consequently by condition (4.5), there exist $(N_k^*)_{k\in\mathbb{Z}_+} \in \mathcal{R}, h > 0, L > 0, \forall \varepsilon \in \left]0,1\right],$

$$\frac{h^{|\alpha|}}{M_{|\alpha|}} \left|\partial^\alpha \left(P\left(D\right) f_\varepsilon\right)(x)\right| \leq A^{|\alpha|} L \varepsilon^{-N^*_{|\alpha|}}$$

which shows that $(P(D) f_\varepsilon)_\varepsilon \in \mathcal{X}^{M,\mathcal{R}}(\Omega)$. □

Example 9. The sets $\{0\}$ and $\mathcal{B}$ satisfy the condition (4.5).

The space $\mathcal{E}^M(\Omega)$ is embedded into $\mathcal{G}^{M,\mathcal{R}}(\Omega)$ for all $\mathcal{R}$ by the canonical map

$$\begin{array}{ccc} \sigma : \mathcal{E}^M(\Omega) & \rightarrow & \mathcal{G}^{M,\mathcal{R}}(\Omega) \\ u & \rightarrow & [u_\varepsilon] \end{array},$$

where $u_\varepsilon = u$ for all $\varepsilon \in \left]0,1\right]$, which is an injective homomorphism of algebras.

PROPOSITION 8. *The following diagram*

$$\begin{array}{ccccc} \mathcal{E}^M(\Omega) & \rightarrow & C^\infty(\Omega) & \rightarrow & \mathcal{D}'(\Omega) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G}^{M,\mathcal{B}}(\Omega) & \rightarrow & \mathcal{G}^{\mathcal{B}}(\Omega) & \rightarrow & \mathcal{G}(\Omega) \end{array}$$

is commutative.

PROOF. The embeddings in the diagram are canonical except the embedding $\mathcal{D}'(\Omega) \rightarrow \mathcal{G}(\Omega)$, which is now well known in framework of Colombeau generalized functions, see [4] for details. The commutativity of the diagram is then obtained easily from the commutativity of the classical diagram

$$\begin{array}{ccc} C^\infty(\Omega) & \rightarrow & \mathcal{D}'(\Omega) \\ & \searrow & \downarrow \\ & & \mathcal{G}(\Omega) \end{array} \qquad \square$$

A fundamental result of regularity in $\mathcal{G}(\Omega)$ is the following.

THEOREM 1. *We have* $\mathcal{G}^{M,\mathcal{B}}(\Omega) \cap \mathcal{D}'(\Omega) = \mathcal{E}^{M}(\Omega)$.

PROOF. Let $u = cl\,(u_\varepsilon)_\varepsilon \in \mathcal{G}^{M,\mathcal{B}}(\Omega) \cap C^\infty(\Omega)$, i.e. $(u_\varepsilon)_\varepsilon \in \mathcal{X}^{M,\mathcal{B}}(\Omega)$, then we have for every compact set $K \subset \Omega, \exists N \in \mathbb{Z}_+, \exists c > 0, \exists \eta \in]0,1]$,

$$\forall \alpha \in \mathbb{Z}_+^n, \forall \varepsilon \in]0,\eta] : \sup_{x\in K} |\partial^\alpha u(x)| \leq c^{|\alpha|+1} M_{|\alpha|} \varepsilon^{-N}.$$

When choosing $\varepsilon = \eta$, we obtain

$$\forall \alpha \in \mathbb{Z}_+^n, \sup_{x\in K} |\partial^\alpha u(x)| \leq c^{|\alpha|+1} M_{|\alpha|} \eta^{-N} \leq c_1^{|\alpha|+1} M_{|\alpha|},$$

where c_1 depends only on K. Then u is in $\mathcal{E}^M(\Omega)$. This shows that $\mathcal{G}^{M,\mathcal{B}}(\Omega) \cap C^\infty(\Omega) \subset \mathcal{E}^M(\Omega)$. As the reverse inclusion is obvious, then we have proved $\mathcal{G}^{M,\mathcal{B}}(\Omega) \cap C^\infty(\Omega) = \mathcal{E}^M(\Omega)$. Consequently

$$\begin{aligned}
\mathcal{G}^{M,\mathcal{B}}(\Omega) \cap \mathcal{D}'(\Omega) &= \left(\mathcal{G}^{M,\mathcal{B}}(\Omega) \cap \mathcal{G}^{\mathcal{B}}(\Omega)\right) \cap \mathcal{D}'(\Omega) \\
&= \mathcal{G}^{M,\mathcal{B}}(\Omega) \cap \left(\mathcal{G}^{\mathcal{B}}(\Omega) \cap \mathcal{D}'(\Omega)\right) \\
&= \mathcal{G}^{M,\mathcal{B}}(\Omega) \cap C^\infty(\Omega) \\
&= \mathcal{E}^M(\Omega) \ \square
\end{aligned}$$

PROPOSITION 9. *The algebra* $\mathcal{G}^{M,\mathcal{R}}(\Omega)$ *is a sheaf of subalgebras of* $\mathcal{G}(\Omega)$.

PROOF. The sheaf property of $\mathcal{G}^{M,\mathcal{R}}(\Omega)$ is obtained in the same way as the sheaf properties of $\mathcal{G}^{\mathcal{R}}(\Omega)$ and $\mathcal{E}^M(\Omega)$. $\square$

We can now give a new tool of $\mathcal{G}^{M,\mathcal{R}}$-local regularity analysis.

DEFINITION 6. Define the $(M,\mathcal{R})$-singular support of a generalized function $u \in \mathcal{G}(\Omega)$, denoted by $\text{singsupp}_{M,\mathcal{R}}(u)$, as the complement of the largest open set Ω' such that $u \in \mathcal{G}^{M,\mathcal{R}}(\Omega')$.

The basic property of $\text{singsupp}_{M,\mathcal{R}}$ is summarized in the following proposition, which is easy to prove by the facts above.

Proposition 10. *Let* $P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ *be a generalized linear partial differential operator with* $\mathcal{G}^{M,\mathcal{R}}(\Omega)$ *coefficients, then*

$$\operatorname{singsupp}_{M,\mathcal{R}}(P(x,D)u) \subset \operatorname{singsupp}_{M,\mathcal{R}}(u), \forall u \in \mathcal{G}(\Omega) \tag{4.6}$$

We can now introduce a local generalized analysis in the sense of Colombeau algebra. Indeed, a generalized linear partial differential operator with $\mathcal{G}^{M,\mathcal{R}}(\Omega)$ coefficients $P(x,D)$ is said $(M,\mathcal{R})-$hypoelliptic in Ω, if

$$\operatorname{singsupp}_{M,\mathcal{R}}(P(x,D)u) = \operatorname{singsupp}_{M,\mathcal{R}}(u), \forall u \in \mathcal{G}(\Omega) \tag{4.7}$$

Such a problem in this general form is still in the beginning. Of course, a microlocalization of the problem (4.7) will lead to a more precise information about solutions of generalized linear partial differential equations. A first attempt is done in the following section.

5. Affine Ultraregular Generalized Functions

Although we have defined a tool for a local $(M,\mathcal{R})-$analysis in $\mathcal{G}(\Omega)$, it is not clear how to microlocalize this concept in general. We can do it in the general situation of affine ultraregularity. This is the aim of this section.

Definition 7. Define the affine regular sequences

$$\mathcal{A} = \left\{ (N_m)_{m\in\mathbb{Z}_+} \in \mathcal{R} : \exists a \geq 0, \exists b \geq 0, N_m \leq am + b, \forall m \in \mathbb{Z}_+ \right\}.$$

A basic $(M,\mathcal{A})-$microlocal analysis in $\mathcal{G}(\Omega)$ can be developed due to the following result.

Proposition 11. *Let* $f = cl(f_\varepsilon)_\varepsilon \in \mathcal{G}_C(\Omega)$, *then* f *is* $\mathcal{A}-$ *ultraregular of class* $M = (M_p)_{p\in\mathbb{Z}_+}$ *if and only if* $\exists a \geq 0, \exists b \geq 0, \exists C > 0, \exists k > 0$, $\exists \varepsilon_0 \in]0,1], \forall \varepsilon \leq \varepsilon_0$, *such that*

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq C\varepsilon^{-b} \exp\left(-\widetilde{M}(k\varepsilon^a|\xi|)\right), \forall \xi \in \mathbb{R}^n, \tag{5.1}$$

where $\mathcal{F}$ *denotes the Fourier transform.*

Proof. Suppose that $f = cl\,(f_\varepsilon)_\varepsilon \in \mathcal{G}_C\,(\Omega) \cap \mathcal{G}^{M,\mathcal{A}}\,(\Omega)$, then $\exists K$ compact of $\Omega, \exists C > 0, \exists N \in \mathcal{A}, \exists \varepsilon_1 > 0, \forall \alpha \in \mathbb{Z}_+^n, \forall x \in K, \forall \varepsilon \leq \varepsilon_0$, $\mathrm{supp} f_\varepsilon \subset K$, such that

$$|\partial^\alpha f_\varepsilon| \leq C^{|\alpha|+1} M_{|\alpha|} \varepsilon^{-N_{|\alpha|}}, \tag{5.2}$$

so we have, $\forall \alpha \in \mathbb{Z}_+^n$,

$$|\xi^\alpha|\,|\mathcal{F}\,(f_\varepsilon)\,(\xi)| \leq \left| \int \exp\,(-ix\xi)\,\partial^\alpha f_\varepsilon\,(x)\,dx \right|$$

then, $\exists C > 0, \forall \varepsilon \leq \varepsilon_0$,

$$|\xi|^{|\alpha|}\,|\mathcal{F}\,(f_\varepsilon)\,(\xi)| \leq C^{|\alpha|+1} M_{|\alpha|} \varepsilon^{-N_{|\alpha|}}.$$

Therefore

$$\begin{aligned} |\mathcal{F}\,(f_\varepsilon)\,(\xi)| &\leq C \inf_\alpha \left\{ \frac{C^{|\alpha|} M_{|\alpha|}}{|\xi|^{|\alpha|}} \varepsilon^{-N_{|\alpha|}} \right\} \\ &\leq C\varepsilon^{-b} \inf_\alpha \left\{ \left(\frac{\varepsilon^{-a} C}{|\xi|} \right)^{|\alpha|} M_{|\alpha|} \right\} \\ &\leq C\varepsilon^{-b} \exp\left(-\widetilde{M}\left(\frac{\varepsilon^a\,|\xi|}{\sqrt{n}C} \right) \right). \end{aligned}$$

Hence $\exists C > 0, \exists k > 0$,

$$|\mathcal{F}\,(f_\varepsilon)\,(\xi)| \leq C\varepsilon^{-b} \exp\left(-\widetilde{M}\,(k\varepsilon^a\,|\xi|) \right)\ ,$$

i.e. we have (5.1).

Suppose now that (5.1) is valid, then from inequality (3.1) of the Proposition 3, $\exists C, C', C'' > 0, \exists \varepsilon_0 \in \,]0,1]\,, \forall \varepsilon \leq \varepsilon_0$,

$$\begin{aligned} |\partial^\alpha f_\varepsilon\,(x)| &\leq C\varepsilon^{-b} \sup_\xi |\xi|^{|\alpha|} \exp\left(-\widetilde{M}\left(\frac{k}{H} \varepsilon^a\,|\xi| \right) \right) \times \\ &\qquad \times \int \exp\left(-\widetilde{M}\left(\frac{k}{H} \varepsilon^a\,|\xi| \right) \right) d\xi \\ &\leq C'\varepsilon^{-a|\alpha|-b} \sup_\xi \left| \frac{k}{H} \varepsilon^a \xi \right|^{|\alpha|} \exp\left(-\widetilde{M}\left(\frac{k}{H} \varepsilon^a\,|\xi| \right) \right) \\ &\leq C''\varepsilon^{-a|\alpha|-b} \sup_\eta |\eta|^{|\alpha|} \exp\left(-\widetilde{M}\,(|\eta|) \right). \end{aligned}$$

Due to the Proposition 2, then $\exists C>0, \exists N \in \mathcal{A}$, such that

$$\left|\partial^{\alpha} f_{\varepsilon}(x)\right| \leq C^{|\alpha|+1} M_{|\alpha|} \varepsilon^{-N_{|\alpha|}},$$

where $C=\max\left(C, \frac{1}{k}\right)$, and $N_m=am+b$, then $f \in \mathcal{G}^{M,\mathcal{A}}(\Omega)$. □

COROLLARY 1. *Let $f=cl\left(f_{\varepsilon}\right)_{\varepsilon} \in \mathcal{G}_C(\Omega)$, then f is a Gevrey affine ultraregular generalized function of order σ, i.e. $f \in \mathcal{G}^{\sigma,\mathcal{A}}(\Omega)$, if and only if $\exists a \geq 0, \exists b \geq 0, \exists C>0, \exists k>0$, $\exists \varepsilon_0>0, \forall \varepsilon \leq \varepsilon_0$, such that*

$$\left|\mathcal{F}\left(f_{\varepsilon}\right)(\xi)\right| \leq C \varepsilon^{-b} \exp \left(-k \varepsilon^{a}|\xi|^{\frac{1}{\sigma}}\right), \forall \xi \in \mathbb{R}^{n}. \tag{5.3}$$

In particular, f is a Gevrey generalized function of order σ, i.e. $f \in \mathcal{G}^{\sigma,\infty}(\Omega)$, if and only if $\exists b \geq 0, \exists C>0, \exists k>0$, $\exists \varepsilon_0>0, \forall \varepsilon \leq \varepsilon_0$, such that

$$\left|\mathcal{F}\left(f_{\varepsilon}\right)(\xi)\right| \leq C \varepsilon^{-b} \exp \left(-k|\xi|^{\frac{1}{\sigma}}\right), \forall \xi \in \mathbb{R}^{n}. \tag{5.4}$$

Using the above results, we can define the concept of $\mathcal{G}^{M,\mathcal{A}}-$wave front of $u \in \mathcal{G}(\Omega)$ and give the basic elements of a $(M,\mathcal{A})-$generalized microlocal analysis within the Colombeau algebra $\mathcal{G}(\Omega)$.

DEFINITION 8. Define the cone $\sum_{\mathcal{A}}^{M}(f) \subset \mathbb{R}^{n} \backslash \{0\}, f \in \mathcal{G}_C(\Omega)$, as the complement of the set of points having a conic neighborhood Γ such that $\exists a \geq 0, \exists b \geq 0, \exists C>0, \exists k>0$, $\exists \varepsilon_0>0, \forall \varepsilon \leq \varepsilon_0$, such that

$$\left|\mathcal{F}\left(f_{\varepsilon}\right)(\xi)\right| \leq C \varepsilon^{-b} \exp \left(-\widetilde{M}\left(k \varepsilon^{a}|\xi|\right)\right), \forall \xi \in \mathbb{R}^{n}. \tag{5.5}$$

PROPOSITION 12. *For every $f \in \mathcal{G}_C(\Omega)$, we have*
1. The set $\sum_{\mathcal{A}}^{M}(f)$ is a closed subset.
2. $\sum_{\mathcal{A}}^{M}(f)=\emptyset \Longleftrightarrow f \in \mathcal{G}^{M,\mathcal{A}}(\Omega)$.

PROOF. The proof of 1. is clear from the definition, and 2. holds from Proposition 11. □

Proposition 13. *For every $f \in \mathcal{G}_C(\Omega)$, we have*

$$\sum_{\mathcal{A}}^{M}(\psi f) \subset \sum_{\mathcal{A}}^{M}(f), \forall \psi \in \mathcal{E}^M(\Omega).$$

Proof. Let $\xi_0 \notin \sum_{\mathcal{A}}^{M}(f)$, i.e. $\exists \Gamma$ a conic neighborhood of $\xi_0, \exists a \geq 0, \exists b > 0, \exists k_1 > 0, \exists c_1 > 0, \exists \varepsilon_1 \in]0,1]$, such that $\forall \xi \in \Gamma, \forall \varepsilon \leq \varepsilon_1$,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c_1 \varepsilon^{-b} \exp\left(-\widetilde{M}(k_1 \varepsilon^a |\xi|)\right), \tag{5.6}$$

Let $\chi \in \mathcal{D}^M(\Omega), \chi = 1$ on neighborhood of supp$f$, then $\chi\psi \in \mathcal{D}^M(\Omega), \forall \psi \in \mathcal{E}^M(\Omega)$, hence, see [7], $\exists k_2 > 0, \exists c_2 > 0, \forall \xi \in \mathbb{R}^n$,

$$|\mathcal{F}(\chi\psi)(\xi)| \leq c_2 \exp\left(-\widetilde{M}(k_2 |\xi|)\right). \tag{5.7}$$

Let Λ be a conic neighborhood of ξ_0 such that, $\overline{\Lambda} \subset \Gamma$, then we have, for $\xi \in \Lambda$,

$$\begin{aligned} \mathcal{F}(\chi\psi f_\varepsilon)(\xi) &= \int_{\mathbb{R}^n} \mathcal{F}(f_\varepsilon)(\eta)\, \mathcal{F}(\chi\psi)(\xi - \eta)\, d\eta \\ &= \int_A \mathcal{F}(f_\varepsilon)(\eta)\, \mathcal{F}(\chi\psi)(\xi - \eta)\, d\eta + \\ &\quad + \int_B \mathcal{F}(f_\varepsilon)(\eta)\, \mathcal{F}(\chi\psi)(\xi - \eta)\, d\eta, \end{aligned}$$

where $A = \{\eta; |\xi - \eta| \leq \delta(|\xi| + |\eta|)\}$ and $B = \{\eta; |\xi - \eta| > \delta(|\xi| + |\eta|)\}$. Take δ sufficiently small such that $\eta \in \Gamma, \frac{|\xi|}{2} < |\eta| < 2|\xi|, \forall \in \Lambda, \forall \eta \in A$, then $\exists c > 0, \forall \varepsilon \leq \varepsilon_1$,

$$\begin{aligned} \left| \int_A \mathcal{F}(f_\varepsilon)(\eta)\, \mathcal{F}(\chi\psi)(\xi - \eta)\, d\eta \right| &\leq c\varepsilon^{-b} \exp\left(-\widetilde{M}\left(k_1 \varepsilon^a \frac{|\xi|}{2}\right)\right) \times \\ &\quad \times \int_A \exp\left(-\widetilde{M}(k_2 |\xi - \eta|)\right) d\eta, \end{aligned}$$

so $\exists c > 0, \exists k > 0$,

$$\left| \int_A \mathcal{F}(f_\varepsilon)(\eta)\, \mathcal{F}(\chi\psi)(\xi - \eta)\, d\eta \right| \leq c\varepsilon^{-b} \exp\left(-\widetilde{M}(k\varepsilon^a |\xi|)\right). \tag{5.8}$$

As $f \in \mathcal{G}_C(\Omega)$, then $\exists q \in \mathbb{Z}_+, \exists m > 0, \exists c > 0, \exists \varepsilon_2 > 0, \forall \varepsilon \leq \varepsilon_2$,

$$|\mathcal{F}(f_\varepsilon)(\xi)| \leq c\varepsilon^{-q} |\xi|^m .$$

Hence for $\varepsilon \leq \min(\varepsilon_1, \varepsilon_2)$, $\exists c > 0$, such that we have

$$\begin{aligned}\left|\int_B \mathcal{F}(f_\varepsilon)(\eta)\, \mathcal{F}(\chi\psi)(\xi-\eta)\, d\eta\right| \\ &\leq c\varepsilon^{-q} \int_B |\eta|^m \exp\left(-\widetilde{M}(k_2|\xi-\eta|)\right) d\eta \\ &\leq c\varepsilon^{-q} \int_B |\eta|^m \exp\left(-\widetilde{M}(k_2\delta(|\xi|+|\eta|))\right) d\eta.\end{aligned}$$

We have, from Proposition 3, i.e. inequality (3.1), $\exists H > 0, \exists A > 0, \forall t_1 > 0, \forall t_2 > 0$,

$$-\widetilde{M}(t_1+t_2) \leq -\widetilde{M}\left(\frac{t_1}{H}\right) - \widetilde{M}\left(\frac{t_2}{H}\right) + \ln A \ , \tag{5.9}$$

so

$$\begin{aligned}\left|\int_B \mathcal{F}(f_\varepsilon)(\eta)\, \mathcal{F}(\chi\psi)(\xi-\eta)\, d\eta\right| \ &\leq \ cA\varepsilon^{-q} \exp\left(-\widetilde{M}\left(\frac{k_2\delta}{H}|\xi|\right)\right) \times \\ &\quad \times \int_B |\eta|^m \exp\left(-\widetilde{M}\left(\frac{k_2\delta}{H}|\eta|\right)\right) d\eta.\end{aligned}$$

Hence $\exists c > 0, \exists k > 0$, such that

$$\left|\int_B \widehat{f}_\varepsilon(\eta)\, \widehat{\psi}(\xi-\eta)\, d\eta\right| \leq c\varepsilon^{-q} \exp\left(-\widetilde{M}(k\varepsilon^a|\xi|)\right). \tag{5.10}$$

Consequently, (5.8) and (5.10) give $\xi_0 \notin \sum_{\mathcal{A}}^M(\psi f)$. □

We define $\sum_{\mathcal{A}}^M(f)$ of a generalized function f at a point x_0 and the affine wave front set of class M in $\mathcal{G}(\Omega)$.

Definition 9. Let $f \in \mathcal{G}(\Omega)$ and $x_0 \in \Omega$, the cone of affine singular directions of class $M = (M_p)$ of f at x_0, denoted by $\sum_{\mathcal{A},x_0}^M(f)$, is

$$\sum\nolimits_{\mathcal{A},x_0}^M(f) = \bigcap\Big\{\sum\nolimits_{\mathcal{A}}^M(\phi f) : \phi \in \mathcal{D}^M(\Omega), \ \phi = 1 \text{ on a neighborhood of } x_0\Big\}. \tag{5.11}$$

The following lemma gives the relation between the local and microlocal $(M,\mathcal{A})-$analysis in $\mathcal{G}(\Omega)$.

Lemma 1. *Let $f \in \mathcal{G}(\Omega)$, then*

$$\sum\nolimits_{\mathcal{A},x_0}^{M}(f)=\emptyset \Longleftrightarrow x_0 \notin \operatorname{singsupp}_{M,\mathcal{A}}(f).$$

Proof. See the proof of the analogical Lemma in [1]. □

Definition 10. A point $(x_0,\xi_0) \notin WF_{\mathcal{A}}^{M}(f) \subset \Omega\times\mathbb{R}^n\backslash\{0\}$, if there exist $\phi \in \mathcal{D}^M(\Omega), \phi \equiv 1$ on a neighborhood of x_0, a conic neighborhood Γ of ξ_0, and numbers $a \geq 0, b \geq 0, k > 0, c > 0, \varepsilon_0 \in \left]0,1\right]$, such that $\forall\varepsilon \leq \varepsilon_0, \forall\xi \in \Gamma$,

$$\left|\mathcal{F}(\phi f_\varepsilon)(\xi)\right| \leq c\varepsilon^{-b}\exp\left(-\widetilde{M}\left(k\varepsilon^a|\xi|\right)\right).$$

Remark 6. A point $(x_0,\xi_0) \notin WF_{\mathcal{A}}^{M}(f) \subset \Omega\times\mathbb{R}^n\backslash\{0\}$ means $\xi_0 \notin \sum\nolimits_{\mathcal{A},x_0}^{M}(f)$.

The basic properties of $WF_{\mathcal{A}}^{M}$ are given in the following proposition.

Proposition 14. *Let $f \in \mathcal{G}(\Omega)$, then*

1) The projection of $WF_{\mathcal{A}}^{M}(f)$ on Ω is the $\operatorname{singsupp}_{M,\mathcal{A}}(f)$.

2) If $f \in \mathcal{G}_C(\Omega)$, then the projection of $WF_{\mathcal{A}}^{M}(f)$ on $\mathbb{R}^n\backslash\{0\}$ is $\sum\nolimits_{\mathcal{A}}^{M}(f)$.

3) $WF_{\mathcal{A}}^{M}(\partial^\alpha f) \subset WF_{\mathcal{A}}^{M}(f), \forall\alpha \in \mathbb{Z}_+^n$.

4) $WF_{\mathcal{A}}^{M}(gf) \subset WF_{\mathcal{A}}^{M}(f), \forall g \in \mathcal{G}^{M,\mathcal{A}}(\Omega)$.

Proof. 1) and 2) holds from the definition, Proposition 12 and Lemma 1.

3) Let $(x_0,\xi_0) \notin WF_{\mathcal{A}}^{M}(f)$. Then $\exists\phi \in \mathcal{D}^M(\Omega), \phi \equiv 1$ on $\overline{U}$, where U is a neighborhood of x_0, there exist a conic neighborhood Γ of ξ_0, and $\exists a \geq 0, \exists b > 0, \exists k_2 > 0, \exists c_1 > 0, \exists\varepsilon_0 \in \left]0,1\right]$, such that $\forall\xi \in \Gamma, \forall\varepsilon \leq \varepsilon_0$,

$$\left|\mathcal{F}(\phi f_\varepsilon)(\xi)\right| \leq c_1\varepsilon^{-b}\exp\left(-\widetilde{M}\left(k_2\varepsilon^a|\xi|\right)\right). \tag{5.12}$$

We have, for $\psi \in \mathcal{D}^M(U)$ such that $\psi(x_0) = 1$,

$$\begin{aligned}
|\mathcal{F}(\psi \partial f_\varepsilon)(\xi)| &= |\mathcal{F}(\partial(\psi f_\varepsilon))(\xi) - \mathcal{F}((\partial\psi) f_\varepsilon)(\xi)| \\
&\leq |\xi| |\mathcal{F}(\psi\phi f_\varepsilon)(\xi)| + |\mathcal{F}((\partial\psi)\phi f_\varepsilon)(\xi)|.
\end{aligned}$$

As $WF^M_{\mathcal{A}}(\psi f) \subset WF^M_{\mathcal{A}}(f)$, so (5.12) holds for both $|\mathcal{F}(\psi\phi f_\varepsilon)(\xi)|$ and $|\mathcal{F}((\partial\psi)\phi f_\varepsilon)(\xi)|$. Then

$$\begin{aligned}
|\xi| |\mathcal{F}(\psi\phi f_\varepsilon)(\xi)| &\leq c\varepsilon^{-b} |\xi| \exp\left(-\widetilde{M}(k_2\varepsilon^a |\xi|)\right) \\
&\leq c'\varepsilon^{-b-a} \exp\left(-\widetilde{M}(k_3\varepsilon^a |\xi|)\right),
\end{aligned}$$

with some $c' > 0, k_3 > 0, (k_3 < k_2)$, such that

$$\varepsilon^a |\xi| \leq c' \exp\left(\widetilde{M}(k_2\varepsilon^a |\xi|) - \widetilde{M}(k_3\varepsilon^a |\xi|)\right)$$

for ε sufficiently small. Hence (5.12) holds for $|\mathcal{F}(\psi\partial f_\varepsilon)(\xi)|$, which proves $(x_0, \xi_0) \notin WF^M_{\mathcal{A}}(\partial f)$.

4) Let $(x_0, \xi_0) \notin WF^M_{\mathcal{A}}(f)$. Then there exist $\phi \in \mathcal{D}^M(\Omega), \phi(x) = 1$ on a neighborhood U of x_0, a conic neighborhood Γ of ξ_0, and numbers $a_1 \geq 0, b_1 > 0, k_1 > 0, c_1 > 0, \varepsilon_1 \in]0,1]$, such that $\forall\varepsilon \leq \varepsilon_1, \forall\xi \in \Gamma$,

$$|\mathcal{F}(\phi f_\varepsilon)(\xi)| \leq c_1\varepsilon^{-b_1} \exp\left(-\widetilde{M}(k_1\varepsilon^{a_1} |\xi|)\right).$$

Let $\psi \in \mathcal{D}^M(\Omega)$ and $\psi = 1$ on suppϕ, then $\mathcal{F}(\phi g_\varepsilon f_\varepsilon) = \mathcal{F}(\psi g_\varepsilon) * \mathcal{F}(\phi f_\varepsilon)$. We have $\psi g \in \mathcal{G}^{M,\mathcal{A}}(\Omega)$, then $\exists c_2 > 0$, $\exists a_2 \geq 0, \exists b_2 > 0, \exists k_2 > 0, \exists \varepsilon_2 > 0, \forall\xi \in \mathbb{R}^n, \forall\varepsilon \leq \varepsilon_2$,

$$|\mathcal{F}(\psi g_\varepsilon)(\xi)| \leq c_2\varepsilon^{-b_2} \exp\left(-\widetilde{M}(k_2\varepsilon^{a_2} |\xi|)\right). \tag{5.13}$$

We have

$$\begin{aligned}
\mathcal{F}(\phi g_\varepsilon f_\varepsilon)(\xi) &= \int_A \mathcal{F}(\phi f_\varepsilon)(\eta) \mathcal{F}(\psi g_\varepsilon)(\xi - \eta) \, d\eta + \\
&\quad + \int_B \mathcal{F}(\phi f_\varepsilon)(\eta) \mathcal{F}(\psi g_\varepsilon)(\xi - \eta) \, d\eta,
\end{aligned}$$

where A and B are the same as in the proof of Proposition 13. By the same reasoning we obtain the proof. □

A micolocalization of Proposition 10 is expressed in the following result.

Corollary 2. *Let* $P(x,D)=\sum_{|\alpha|\leq m} a_\alpha(x)\,D^\alpha$ *be a generalized linear partial differential operator with* $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ *coefficients, then*

$$WF_{\mathcal{A}}^{M}(P(x,D)f)\subset WF_{\mathcal{A}}^{M}(f),\forall f\in\mathcal{G}(\Omega)$$

The reverse inclusion will give a generalized microlocal ultraregularity of linear partial differential operator with coefficients in $\mathcal{G}^{M,\mathcal{A}}(\Omega)$. The first case of $\mathcal{G}^{\infty}$-microlocal hypoellipticity has been studied in [6]. A general interesting problem of $(M,\mathcal{A})$-generalized microlocal elliptic ultraregularity is to prove the following inclusion

$$WF_{\mathcal{A}}^{M}(f)\subset WF_{\mathcal{A}}^{M}(P(x,D)f)\cup Char(P),\forall f\in\mathcal{G}(\Omega),$$

where $P(x,D)$ is a generalized linear partial differential operator with $\mathcal{G}^{M,\mathcal{A}}(\Omega)$ coefficients and $Char(P)$ is the set of generalized characteristic points of $P(x,D)$.

Remark 7. We can extend the generalized Hörmander's result on the wave front set of the product, the proof has the same steps as the proof of Theorem 26 in [1]. Let $f,g\in\mathcal{G}(\Omega)$, if $\forall x\in\Omega$,

$$(x,0)\notin\left\{(x,\xi+\eta):(x,\xi)\in WF_{\mathcal{A}}^{M}(f),(x,\eta)\in WF_{\mathcal{A}}^{M}(g)\right\}, \tag{5.14}$$

then we have

$$WF_{\mathcal{A}}^{M}(fg)\subseteq\left(WF_{\mathcal{A}}^{M}(f)+WF_{\mathcal{A}}^{M}(g)\right)\cup WF_{\mathcal{A}}^{M}(f)\cup WF_{\mathcal{A}}^{M}(g). \tag{5.15}$$

References

[1] Benmeriem, K. and C. Bouzar, Generalized Gevrey ultradistributions, To appear in New York J. of Mathematics.

[2] Colombeau, J. F., Elementary introduction to new generalized functions, North-Holland, 1985.

[3] Delcroix, A., Regular rapidly decreasing nonlinear generalized functions, Application to microlocal regularity, J. Math. Anal. Appl. **327** (2007), 564–584.

[4] Grosser, M., Kunzinger, M., Oberguggenberger, M. and R. Steinbauer, Geometric theory of generalized functions with applications to general relativity, Kluwer Academic Press, 2001.

[5] Hörmander, L., Distribution theory and Fourier analysis, Springer, 2nd edition 1990.

[6] Hörmann, G., Oberguggenberger, M. and S. Pilipovic, Microlocal hypoellipticity of linear differential operators with generalized functions as coefficients, Trans. Amer. Math. Soc. **358** (2006), 3363–3383.

[7] Komatsu, H., Ultradistributions I, J. Fac. Sci. Univ. Tokyo, Sect. IA **20** (1973), 25–105.

[8] Lions, J. L. and E. Magenes, Non-homogeneous boundary value problems and application, T. 3, Springer, 1973.

[9] Mandelbrojt, S., Séries adhérentes. Ré gularisation des suites, Applications, Gauthier-Villars, 1952.

[10] Marti, J. A., G^L-Microlocal analysis of generalized functions, Integral Transforms Spec. Func. **17** (2006), (2/3), 119–125.

[11] Oberguggenberger, M., Multiplication of distributions and applications to partial differential equations, Longman Scientific & Technical, 1992.

[12] Oberguggenberger, M., Regularity theory in Colombeau algebras, Bull. T. CXXXIII. Acad. Serbe Sci. Arts, Cl. Sci. Math. Nat. Sci. Math., 31, 2006.

[13] Rodino, L., Linear partial differential operators in Gevrey spaces, World Scientific, 1993.

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Khaled BENMERIEM
Centre Universitaire de Mascara
Algeria
E-mail: benmeriemkhaled@yahoo.fr

Chikh BOUZAR
Oran-Essenia University
Oran 31000, Algeria
E-mail: bouzar@yahoo.com
bouzar@univ-oran.dz