J. Math. Sci. Univ. Tokyo
15 (2008), 449–459.

Gap Conjecture for 3-Dimensional Canonical Thresholds

By Yuri Prokhorov

Abstract. We prove that the interval (5/6, 1) contains no 3-dimensional canonical thresholds.

1. Introduction

We work over the complex number field $\mathbb{C}$.

Let $(X \ni P)$ be a three-dimensional canonical singularity and let $S \subset X$ be a $\mathbb{Q}$-Cartier divisor. The *canonical threshold* of the pair (X, S) is

$$\operatorname{ct}(X, S) := \sup\{c \mid \text{the pair } (X, cS) \text{ is canonical}\}.$$

It is easy to see that $\operatorname{ct}(X, S)$ is rational and non-negative. Moreover, if S is effective and integral, then $\operatorname{ct}(X, S) \in [0, 1]$. Define the subset $\mathcal{T}_n^{\mathrm{can}} \subset [0, 1]$ as follows

$$\mathcal{T}_n^{\mathrm{can}} := \{\operatorname{ct}(X, S) \mid \dim X = n,\ S \text{ is integral and effective}\}.$$

The following conjecture is an analog of corresponding conjectures for log canonical thresholds and minimal discrepancies, see [Sho88], [Kol92], [Kol97], [MP04], [Kol08].

Conjecture 1.1. *The set $\mathcal{T}_n^{\mathrm{can}}$ satisfies the ascending chain condition.*

The conjecture is interesting for applications to birational geometry, see, e.g., [Cor95]. It was shown in [BS06] that much more general form of 1.1 follows from ACC for minimal log discrepancies and weak Borisov-Alexeev conjecture. The important particular case of 1.1 is the following

---

1991 *Mathematics Subject Classification.* 14B05, 14J17, 14E30, 32S25.

The author was partially supported by the Russian Foundation for Basic Research (grants No 06-01-72017-MNTI_a, 08-01-00395-a) and Leading Scientific Schools (grants No NSh-1983.2008.1, NSh-1987.2008.1).

CONJECTURE 1.2 (cf. [Kol08]). $\epsilon_n^{\mathrm{can}} := 1 - \sup(\mathcal{T}_n^{\mathrm{can}} \setminus \{1\}) > 0.$

The aim of this note is to prove Conjecture 1.2 for $n = 3$ in a precise form:

THEOREM 1.3. $\epsilon_3^{\mathrm{can}} = 1/6.$

An analog of this theorem for log canonical thresholds was proved by J. Kollár [Kol94]: $\epsilon_3^{\mathrm{lc}} = 1/42$.

Note that replacing $(X \ni P)$ with its terminal $\mathbb{Q}$-factorial modification we may assume that $(X \ni P)$ is terminal. Thus the following is a stronger form of Theorem 1.3:

THEOREM 1.4. *Let $(X \ni P)$ be a three-dimensional terminal singularity and let $S \subset X$ be an (integral) effective Weil $\mathbb{Q}$-Cartier divisor such that the pair (X, S) is not canonical. Then $\mathrm{ct}(X, S) \le 5/6$ and this bound is sharp. Moreover, if $(X \ni P)$ is singular, then $\mathrm{ct}(X, S) \le 4/5$.*

In Section 3 we give examples where the values 5/6 and 4/5 in the above theorem are achieved (see Examples 3.10 and 3.11).

The proof is rather standard. We use the classification of terminal singularities and weighted blowups techniques, cf. [Kaw92], [Kol94], [Mar96].

Acknowledgments. The author would like to thank the referee for careful reading the manuscript and helpful suggestions.

2. Preliminaries

2.1. Notation. For a polynomial ϕ, $\mathrm{ord}_0\,\phi$ denotes the order of vanishing of ϕ at 0 and ϕ_d is the homogeneous component of degree d.

Throughout this paper we let $(X \ni P)$ be the germ of a three-dimensional terminal singularity and let $S \subset X$ be an effective Weil $\mathbb{Q}$-Cartier divisor such that the pair (X, S) is not canonical. Put $c := \mathrm{ct}(X, S) > 0$. Since (X, S) is not canonical, $c < 1$. We assume that $c > 1/2$.

LEMMA 2.2. *In the above notation the singularity $(S \ni P)$ is not Du Val.*

Proof. Assume that $(S \ni P)$ is Du Val. Since $X \ni P$ is an isolated singularity, by the inversion of adjunction [Sho93, §3] we see that the pair (X, S) is PLT. Further, since K_S is Cartier lifting its nonwhere vanishing section to X we can show that $K_X + S$ is also Cartier. Hence, the pair (X, S) is canonical. □

Lemma 2.3. *In the above notation S is reduced, irreducible and normal.*

Proof. Indeed, otherwise by blowing up a curve in the singular locus of S we get $c \leq 1/2$. □

2.4. We use the techniques of weighted blowups. For definitions and basic properties we refer, for example, to [Mar96], [Rei87]. By fixing coordinates $x_1, \dots, x_n$ we regard the affine space $\mathbb{C}^n$ as a toric variety. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a weight (a primitive lattice vector in the positive octant) and let $\sigma_\alpha \colon \mathbb{C}^n_{\boldsymbol{\alpha}} \to \mathbb{C}^n$ be the weighted blowup with weight $\boldsymbol{\alpha}$ ($\boldsymbol{\alpha}$-blowup). The exceptional divisor $E_{\boldsymbol{\alpha}}$ is irreducible and determines a discrete valuation $v_{\boldsymbol{\alpha}}$ of the function field $\mathbb{C}(\mathbb{C}^n)$ such that $v_{\boldsymbol{\alpha}}(x_i) = \alpha_i$.

2.5. Now let $X \subset \mathbb{C}^n$ be a hypersurface given by the equation $\phi = 0$ and let $X_{\boldsymbol{\alpha}} \subset \mathbb{C}^n_{\boldsymbol{\alpha}}$ be its proper transform. Fix an irreducible component G of $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ such that $X_{\boldsymbol{\alpha}}$ is smooth at the generic point of G. Let v_G be the corresponding discrete valuation of $\mathbb{C}(X)$. Write

$$E_{\boldsymbol{\alpha}} \mid_{X_{\boldsymbol{\alpha}}} = m_G G + (\text{other components}).$$

Assume that $m_G = 1$ and G is not a toric subvariety in $\mathbb{C}^n_{\boldsymbol{\alpha}}$. Then the discrepancy of G with respect to K_X is computed by the formula

$$a(G, K_X) = |\boldsymbol{\alpha}| - 1 - v_{\boldsymbol{\alpha}}(\phi), \quad |\boldsymbol{\alpha}| = \sum \alpha_i,$$

see [Mar96]. Let $S \subset X$ be a Cartier divisor and let ψ be a local defining equation of S in $\mathcal{O}_{0,X}$. Then $v_G(\psi) = v_{\boldsymbol{\alpha}}(\psi)$ and the discrepancy of G with respect to $K_X + cS$ is computed by the formula

$$a(G, K_X + cS) = a(G, K_X) - cv_G(\psi) = |\boldsymbol{\alpha}| - 1 - v_{\boldsymbol{\alpha}}(\phi) - cv_{\boldsymbol{\alpha}}(\psi).$$

Therefore,

$$c \leq a(G, K_X)/v_{\boldsymbol{\alpha}}(\psi) = (|\boldsymbol{\alpha}| - 1 - v_{\boldsymbol{\alpha}}(\phi))/v_{\boldsymbol{\alpha}}(\psi).$$

DEFINITION 2.6 (cf. [Mar96]). A weight $\boldsymbol{\alpha}$ is said to be *admissible* if $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ contains at least one reduced non-toric component.

3. Gorenstein Case

In this section we consider the case where $(X \ni P)$ is either smooth or an index one singularity.

LEMMA 3.1. *If $(X \ni P)$ is smooth, then $c \leq 5/6$.*

PROOF. Let $c > 5/6$. We may assume that $X = \mathbb{C}^3$. Let $\psi(x, y, z) = 0$ be an equation of S. Consider a weighted blowup $\sigma_{\boldsymbol{\alpha}} \colon \mathbb{C}^3_{\boldsymbol{\alpha}} \to \mathbb{C}^3$ with a suitable weight $\boldsymbol{\alpha}$. Let $E_{\boldsymbol{\alpha}}$ be the exceptional divisor. Recall that $(S \ni P)$ is not Du Val. Since S is normal, up to analytic coordinate change there are the following cases (cf. [KM98, 4.25]):

3.2. Case $\operatorname{ord}_0 \psi \geq 3$**.** Take $\boldsymbol{\alpha} = (1, 1, 1)$ (usual blowup of 0). Then $a(E_{\boldsymbol{\alpha}}, K_X) = 2$, $v_{\boldsymbol{\alpha}}(\psi) = \operatorname{ord}_0 \psi \geq 3$. Hence $c \leq a(E_{\boldsymbol{\alpha}}, K_X)/v_{\boldsymbol{\alpha}}(\psi) \leq 2/3$, a contradiction.

3.3. Case $\psi = x^2 + \eta(y, z)$**, where** $\operatorname{ord}_0 \eta \geq 4$**.** Take $\boldsymbol{\alpha} = (2, 1, 1)$. Then $a(E_{\boldsymbol{\alpha}}, K_X) = 3$, $v_{\boldsymbol{\alpha}}(\psi) = 4$. Hence $c \leq a(E_{\boldsymbol{\alpha}}, K_X)/v_{\boldsymbol{\alpha}}(\psi) \leq 3/4$, a contradiction.

3.4. Case $\psi = x^2 + y^3 + \eta(y, z)$**, where** $\operatorname{ord}_0 \eta \geq 4$**.** We may assume that $\eta(y, z) = u_a y z^a + u_b z^b$ (see, e.g., [KM98, 4.25]). Since he singularity $(S \ni P)$ is not Du Val, we have $a \geq 4$, $b \geq 6$ and u_a, u_b are either units or zero. Take $\boldsymbol{\alpha} = (3, 2, 1)$. Then $a(E_{\boldsymbol{\alpha}}, K_X) = 5$, $v_{\boldsymbol{\alpha}}(\psi) = 6$. Hence $c \leq a(E_{\boldsymbol{\alpha}}, K_X)/v_{\boldsymbol{\alpha}}(\psi) = 5/6$, a contradiction. $\square$

LEMMA 3.5. *Assume that $(X \ni P)$ is a Gorenstein terminal singularity and $(X \ni P)$ is not smooth. Then $c \leq 4/5$.*

PROOF. Let $c > 4/5$. We may assume that X is a hypersurface in $\mathbb{C}^4$ (it is an isolated cDV-singularity [Rei80]). Let $\phi(x, y, z, t) = 0$ be the equation of X. Since $(X \ni P)$ is a cDV-singularity, $\operatorname{ord}_0 \phi = 2$. According to [Mar96], in a suitable coordinate system (x, y, z, t), there is an admissible weighted blowup $\sigma_{\boldsymbol{\alpha}} \colon \mathbb{C}^4_{\boldsymbol{\alpha}} \to \mathbb{C}^4$ such that at least for one component G of

$E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ we have $a(G, K_X) = 1$. Then $c \le 1/v_{\boldsymbol{\alpha}}(\psi)$, so $v_{\boldsymbol{\alpha}}(\psi) = 1$. This means, in particular, that $\operatorname{ord}_0 \psi = 1$. Up to coordinate change we may assume that $\psi = t$. Write

$$\phi = \eta(x, y, z) + t\zeta(x, y, z, t).$$

Then S is a hypersurface in $\mathbb{C}^3_{x,y,z}$ given by $\eta(x, y, z) = 0$. As in the proof of Lemma 3.1, using Morse Lemma we get the following cases:

3.6. Case $\operatorname{ord}_0 \eta \ge 3$**.** Since $\operatorname{ord}_0 \phi = 2$, ζ contains a linear term. Take $\boldsymbol{\alpha} = (1, 1, 1, 2)$. By the terminality condition [Rei87, Th. 4.6], we have $4 = v_{\boldsymbol{\alpha}}(xyzt) - 1 > v_{\boldsymbol{\alpha}}(\phi)$.

First we assume that ζ contains at least one of the terms x, y, or z. By symmetry we may assume that ζ contains x. After the analytic coordinate change $x \hookleftarrow \zeta(x, y, z, t)$ we obtain

$$\phi = \eta(x, y, z) + tx.$$

In the affine chart $U_x := \{x \ne 0\}$ the map $\sigma_{\boldsymbol{\alpha}}^{-1}$ is given by

$$x \mapsto x', \quad y \mapsto y'x', \quad z \mapsto z'x', \quad t \mapsto t'x'^2. \tag{3.7}$$

$E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ is given in $\sigma_{\boldsymbol{\alpha}}^{-1}(U_x) \simeq \mathbb{C}^4$ by

$$x' = \eta_3(1, y', z') + t' = 0.$$

Hence $\boldsymbol{\alpha}$ is admissible, i.e., $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ has a reduced non-toric component G. Then $a(G, K_X) = 1$, $v_G(\psi) = 2$ and $c \le a(G, K_X)/v_G(\psi) = 1/2$, a contradiction.

Now we assume that ζ does not contain any of the terms x, y, z. Then ζ contains t. So,

$$\phi = \eta(x, y, z) + t^2 + t\xi(x, y, z, t), \quad \operatorname{ord}_0 \xi \ge 2.$$

Further, $v_{\boldsymbol{\alpha}}(\eta) \le 3$ and $\eta_3 \ne 0$. We claim that $\boldsymbol{\alpha}$ is admissible. Using (3.7) we see that $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ is given in $\sigma_{\boldsymbol{\alpha}}^{-1} \simeq \mathbb{C}^4$ by the equations $x' = \eta_3(1, y', z') = 0$. If η_3 is not a cube of a linear form, then $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ has a reduced non-toric component G. Then, as above, $c \le 1/2$, a contradiction.

Finally assume that ζ does not contain any of the terms x, y, z and η_3 is a cube of a linear form. Then, as above, $\eta_3 \neq 0$ and up to linear coordinate change we have $\eta_3(x,y,z) = y^3$. So,

$$\phi = y^3 + \eta^\bullet(x,y,z) + t^2 + t\xi(x,y,z,t), \quad \operatorname{ord}_0 \xi \geq 2, \quad \operatorname{ord}_0 \eta^\bullet \geq 4.$$

Put $\boldsymbol{\alpha}' = (2,2,2,3)$. Again, in the affine chart $U_x := \{x \neq 0\}$ the map $\sigma_{\boldsymbol{\alpha}'}^{-1}$ is given by $x \mapsto x'^2$, $y \mapsto y'x'^2$, $z \mapsto z'x'^2$, $t \mapsto t'x'^3$, where $\sigma_{\boldsymbol{\alpha}'}^{-1}(U_x) \simeq \mathbb{C}^4/\boldsymbol{\mu}_2(1,0,0,1)$ and

$$E_{\boldsymbol{\alpha}'} \cap X_{\boldsymbol{\alpha}'} \cap \sigma_{\boldsymbol{\alpha}'}^{-1}(U_x) = \{x' = 0,\ y'^3 + t'^2 = 0\}.$$

Thus $\boldsymbol{\alpha}'$ is admissible and for some component G' of $X_{\boldsymbol{\alpha}'} \cap E_{\boldsymbol{\alpha}'}$ we have $a(G', K_X) = 2$, $v_{G'}(\psi) = 3$, $c \leq 2/3$, a contradiction.

3.8. Case $\eta = x^2 + \xi(y,z)$, where $\operatorname{ord}_0 \xi \geq 4$. By Morse Lemma we may assume that ζ does not depend on x. Write the linear part of ζ in the form $\zeta_1 = \delta_1 y + \delta_2 z + \delta_3 t$, $\delta_i \in \mathbb{C}$. Take $\boldsymbol{\alpha} = (2,1,1,3)$. In the affine chart $U_y := \{y \neq 0\}$ the map $\sigma_{\boldsymbol{\alpha}}^{-1}$ is given by $x \mapsto x'y'^2$, $y \mapsto y'$, $z \mapsto z'y'$, $t \mapsto t'y'^3$ and

$$E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}} \cap \sigma_{\boldsymbol{\alpha}}^{-1}(U_y) = \{y' = 0,\ x'^2 + \xi_4(1,z') + \delta_1 t' + \delta_2 t'z' = 0\}.$$

If either $\delta_1 \neq 0$ or $\delta_2 \neq 0$ or $\xi_4 \neq 0$, then $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ is reduced (at least over U_y). Hence, $\boldsymbol{\alpha}$ is admissible and for some component G of $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ we have $c \leq a(G, K_X)/v_G(\psi) = 2/3$, a contradiction. Thus $\delta_1 = \delta_2 = 0$ and $\xi_4 = 0$. Then we can write

$$\phi = x^2 + \xi(y,z) + \delta_3 t^2 + t\zeta^\bullet(y,z,t), \quad \operatorname{ord}_0 \xi \geq 5, \quad \operatorname{ord}_0 \zeta^\bullet \geq 2.$$

Take $\boldsymbol{\alpha}' = (2,1,1,2)$. In the affine chart $U_y := \{y \neq 0\}$ the map $\sigma_{\boldsymbol{\alpha}'}^{-1}$ is given by $x \mapsto x'y'^2$, $y \mapsto y'$, $z \mapsto z'y'$, $t \mapsto t'y'^2$ and

$$E_{\boldsymbol{\alpha}'} \cap X_{\boldsymbol{\alpha}'} \cap \sigma_{\boldsymbol{\alpha}'}^{-1}(U_y) = \{y' = 0,\ x'^2 + \delta_3 t'^2 + t'\zeta^\bullet_{(2)}(1,z',0) = 0\},$$

where $\zeta^\bullet_{(2)}(y,z,t) = \zeta^\bullet_{(2)}(y,z,0)$ is the degree 2 weighted homogeneous part of $\zeta^\bullet$. If $\delta_3 \neq 0$ or $\zeta^\bullet_{(2)} \neq 0$, as above, $\boldsymbol{\alpha}'$ is admissible and $c \leq 1/2$, a contradiction. Thus $\delta_3 = 0$, $\zeta^\bullet_{(2)} = 0$, and

$$\phi = x^2 + \xi(y,z) + \delta t^3 + t\zeta^\circ(y,z,t), \quad \delta \in \mathbb{C}, \quad \operatorname{ord}_0 \xi \geq 5, \quad \operatorname{ord}_0 \zeta^\circ \geq 3.$$

Applying the terminality condition [Rei87, Th. 4.6] with weight $(2,1,1,1)$ we get $\delta \neq 0$.

Take $\boldsymbol{\alpha}'' = (3,1,1,2)$. Again by the terminality condition $\xi_5 \neq 0$ or $\zeta^\circ_{(3)} \neq 0$, where $\zeta^\circ_{(3)}$ is the degree 3 weighted homogeneous part of ζ°. As above we get

$$E_{\boldsymbol{\alpha}''} \cap X_{\boldsymbol{\alpha}''} \cap \sigma_{\boldsymbol{\alpha}''}^{-1}(U_x) = \{x' = 0,\ \xi_5(y', z') + t'\zeta^\circ_{(3)}(y', z', t') = 0\}.$$

If either $\zeta^\circ_{(3)} \neq 0$ or ξ_5 has a factor of multiplicity 1, then $\boldsymbol{\alpha}''$ is admissible and $c \leq 1/2$, a contradiction.

Therefore, we may assume that $\zeta^\circ_{(3)} = 0$ and ξ_5 has only multiple factors. Up to linear coordinate change of y and z we can write $\xi_5 = y^5$ or $\xi_5 = y^2z^3$. Take $\boldsymbol{\alpha}''' = (3,2,1,2)$. Then $\boldsymbol{\alpha}'''$ is admissible and $c \leq 1/2$, a contradiction.

3.9. Case $\eta = x^2 + y^3 + \xi(y,z)$**, where** $\operatorname{ord}_0 \xi \geq 4$**.** As in [KM98, 4.25] we may assume that $\xi(y,z) = u_a yz^a + u_b z^b$. Since the singularity $(S \ni P)$ is not Du Val, we have $a \geq 4$, $b \geq 6$ and u_a, u_b are either units or zero. Write the linear part of ζ in the form $\zeta_1 = cz + \ell(x,y,t)$. Let $\xi_{(6)}$ is the degree 6 weighted homogeneous part of ξ with respect to $\operatorname{wt}(y,z) = (2,1)$. Clearly, $\xi_{(6)}$ is a linear combination of z^6 and yz^4. Take $\boldsymbol{\alpha} = (3,2,1,5)$. In the affine chart $U_z := \{z \neq 0\}$ the map $\sigma_{\boldsymbol{\alpha}}^{-1}$ is given by $x \mapsto x'z'^3$, $y \mapsto y'z'^2$, $z \mapsto z'$, $t \mapsto t'z'^5$ and

$$E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}} \cap \sigma_{\boldsymbol{\alpha}}^{-1}(U_z) = \{z' = 0,\ x'^2 + y'^3 + \xi_{(6)}(y', 1) + \delta t' = 0\},$$

where δ is a constant and $\xi_{(6)}(y',1)$ contains no y'^3. Hence $\boldsymbol{\alpha}$ is admissible, i.e., $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ has a reduced non-toric component G. Then $a(G, K_X) = 4$, $v_G(\psi) = 5$, and $c \leq a(G,K_X)/v_G(\psi) \leq 4/5$, a contradiction. $\square$

The following examples show that bounds $\operatorname{ct}(X,S) \leq 5/6$ and $\leq 4/5$ in Theorem 1.4 are sharp.

Example 3.10. Let $X = \mathbb{C}^3$ and let $S = S^d$ is given by $x^2 + y^3 + z^d$, $d \geq 6$. Then $\operatorname{ct}(\mathbb{C}^3, S^d) = 5/6$. We prove this by descending induction on $\lfloor d/6 \rfloor$. Take $\boldsymbol{\alpha} = (3,2,1)$ and consider the $\boldsymbol{\alpha}$-blowup $\sigma_{\boldsymbol{\alpha}} \colon \mathbb{C}^3_{\boldsymbol{\alpha}} \to \mathbb{C}^3$. Let $S_{\boldsymbol{\alpha}} \subset X_{\boldsymbol{\alpha}}$ be the proper transform of S. We have $a(E_{\boldsymbol{\alpha}}, K_X) = 5$ and $v_{\boldsymbol{\alpha}}(\psi) = 6$. Hence, $\operatorname{ct}(\mathbb{C}^3, S^d) \leq 5/6$. Further,

$$K_{\mathbb{C}^3_{\boldsymbol{\alpha}}} + \tfrac{5}{6} S_{\boldsymbol{\alpha}} = \sigma_{\boldsymbol{\alpha}}^*(K_{\mathbb{C}^3} + \tfrac{5}{6} S).$$

Thus it is sufficient to show that $\mathrm{ct}(X_{\boldsymbol{\alpha}}, \frac{5}{6}S_{\boldsymbol{\alpha}})$ is canonical. We have three affine charts:

- $U_x := \{x \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon\ x \mapsto x'^3$, $y \mapsto y'x'^2$, $z \mapsto z'x'$, $S_{\boldsymbol{\alpha}}$ is given in $\sigma_{\boldsymbol{\alpha}}^{-1}(U_x) \simeq \mathbb{C}^3/\boldsymbol{\mu}_3(-1,2,1)$ by the equation $1 + y'^3 + z'^d x'^{d-6} = 0$. Hence, in this chart, $S_{\boldsymbol{\alpha}}$ is smooth and does not pass through a (unique) singular point of $\sigma_{\boldsymbol{\alpha}}^{-1}(U_x)$.
- $U_y := \{y \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon\ x \mapsto x'y'^3$, $y \mapsto y'^2$, $z \mapsto z'y'$, $S_{\boldsymbol{\alpha}}$ is given in $\sigma_{\boldsymbol{\alpha}}^{-1}(U_y) \simeq \mathbb{C}^3/\boldsymbol{\mu}_2(3,-1,1)$ by the equation $x'^2 + 1 + z'^d y'^{d-6} = 0$. Again, in this chart, $S_{\boldsymbol{\alpha}}$ is smooth and does not pass through a (unique) singular point of $\sigma_{\boldsymbol{\alpha}}^{-1}(U_y)$.
- $U_z := \{z \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon\ x \mapsto x'z'^3$, $y \mapsto y'z'^2$, $z \mapsto z'$, $S_{\boldsymbol{\alpha}}$ is given in $\sigma_{\boldsymbol{\alpha}}^{-1}(U_z) \simeq \mathbb{C}^3$ by the equation $x'^2 + y'^3 + z'^{d-6} = 0$. In this chart, $(X_{\boldsymbol{\alpha}}, S_{\boldsymbol{\alpha}}) \simeq (\mathbb{C}^3, S^{d-6})$.

Thus $X_{\boldsymbol{\alpha}}$ has only terminal singularities, $S_{\boldsymbol{\alpha}}$ does not pass through any singular point of $X_{\boldsymbol{\alpha}}$, and the pair $(X_{\boldsymbol{\alpha}}, S_{\boldsymbol{\alpha}})$ is terminal in charts U_x and U_y. In the chart U_z the pair by induction $(X_{\boldsymbol{\alpha}}, \frac{5}{6}S_{\boldsymbol{\alpha}})$ is canonical (moreover, $(X_{\boldsymbol{\alpha}}, S_{\boldsymbol{\alpha}})$ is canonical if $d \leq 11$). Therefore, $\mathrm{ct}(X, S) = 5/6$.

Example 3.11. Let $X \subset \mathbb{C}^4$ is given by $x^2 + y^3 + z^d + tz = 0$, $d \geq 7$ and let S cut out by $t = 0$. Take $\boldsymbol{\alpha} = (3,2,1,5)$ and consider the $\boldsymbol{\alpha}$-blowup $\sigma_{\boldsymbol{\alpha}}\colon X_{\boldsymbol{\alpha}} \to X$. Let $S_{\boldsymbol{\alpha}} \subset X_{\boldsymbol{\alpha}}$ be the proper transform of S. We see below that $\boldsymbol{\alpha}$ is admissible. Moreover, the exceptional divisor $G := E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ is reduced and irreducible. We have four charts:

- $U_x := \{x \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon x \mapsto x^3$, $y \mapsto yx^2$, $z \mapsto zx$, $t \mapsto tx^5$, $X_{\boldsymbol{\alpha}}$ is given in $\sigma_{\boldsymbol{\alpha}}^{-1}(U_x) \simeq \mathbb{C}^4/\boldsymbol{\mu}_3(-1,2,1,5)$ by the equation $1 + y^3 + z^d x^{d-6} + tz = 0$ and $S_{\boldsymbol{\alpha}}$ by two equations $x = 1 + y^3 + tz = 0$. Hence, in this chart, both $X_{\boldsymbol{\alpha}}$ and $S_{\boldsymbol{\alpha}}$ are smooth.
- $U_y := \{y \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon x \mapsto xy^3$, $y \mapsto y^2$, $z \mapsto zy$, $t \mapsto ty^5$, $\sigma_{\boldsymbol{\alpha}}^{-1}(U_y) \simeq \mathbb{C}^4/\boldsymbol{\mu}_2(3,-1,1,5)$, $X_{\boldsymbol{\alpha}} = \{x^2 + 1 + z^d y^{d-6} + tz = 0\}$, and $S_{\boldsymbol{\alpha}} = \{y = x^2 + 1 + tz = 0\}$. As above, both $X_{\boldsymbol{\alpha}}$ and $S_{\boldsymbol{\alpha}}$ are smooth in this chart.
- $U_z := \{z \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1}\colon x \mapsto xz^3$, $y \mapsto yz^2$, $z \mapsto z$, $t \mapsto tz^5$, $\sigma_{\boldsymbol{\alpha}}^{-1}(U_z) \simeq \mathbb{C}^4$, $X_{\boldsymbol{\alpha}} = \{x^2 + y^3 + z^{d-6} + t = 0\}$, and $S_{\boldsymbol{\alpha}} = \{z = x^2 + y^3 + t = 0\}$. As above, both $X_{\boldsymbol{\alpha}}$ and $S_{\boldsymbol{\alpha}}$ are smooth in this chart.

- $U_t := \{t \neq 0\}$. Here $\sigma_{\boldsymbol{\alpha}}^{-1} \colon x \mapsto xt^3$, $y \mapsto yt^2$, $z \mapsto zt$, $t \mapsto t^5$, $\sigma_{\boldsymbol{\alpha}}^{-1}(U_t) \simeq \mathbb{C}^4/\boldsymbol{\mu}_5(3,2,1,-1)$, $X_{\boldsymbol{\alpha}} = \{x^2 + y^3 + z^d t^{d-6} + z = 0\}$, and $S_{\boldsymbol{\alpha}} = \{t = x^2 + y^3 + z = 0\}$. The variety $X_{\boldsymbol{\alpha}}$ has a unique singular point Q at the origin and this point is terminal of type $\frac{1}{5}(3,2,-1)$. In this case, $S_{\boldsymbol{\alpha}} \in |-K_{U_t}|$ and the pair $(U_t, S_{\boldsymbol{\alpha}})$ is canonical.

Thus we have $a(G, K_X) = 4$, $v_{\boldsymbol{\alpha}}(\psi) = 5$, and $a(G, K_X + \frac{4}{5}S) = 0$. Therefore,

$$K_{X_{\boldsymbol{\alpha}}} + \tfrac{4}{5}S_{\boldsymbol{\alpha}} = \sigma_{\boldsymbol{\alpha}}^*(K_X + \tfrac{4}{5}S).$$

Since the pair $K_{X_{\boldsymbol{\alpha}}} + \frac{4}{5}S_{\boldsymbol{\alpha}}$ is canonical, $\mathrm{ct}(X, S) = 4/5$.

4. Non-Gorenstein Case

Now we assume that $(X \ni P)$ is a (terminal) point of index $r > 1$. Let $\pi \colon (X^\sharp \ni P^\sharp) \to (X \ni P)$ be the index-one cover and let $S^\sharp := \pi^{-1}(S)$.

Lemma 4.1. *If $(X \ni P)$ is a cyclic quotient singularity, then $\mathrm{ct}(X, S) \leq 1/2$.*

Proof. By our assumption we have $X \simeq \mathbb{C}^3/\boldsymbol{\mu}_r(a, -a, 1)$ for some $r \geq 2$, $1 \leq a < r$, $\gcd(a, r) = 1$. Assume that $c = \mathrm{ct}(X, S) > 1/2$. Let $\psi = 0$ be a defining equation of $S^\sharp$. Consider the weighted blowup $\sigma_{\boldsymbol{\alpha}} \colon X_{\boldsymbol{\alpha}} \to X$ with weights $\boldsymbol{\alpha} = \frac{1}{r}(a, r-a, 1)$. Then $a(E_{\boldsymbol{\alpha}}, K_X) = 1/r$. Since $a(E_{\boldsymbol{\alpha}}, K_X) - cv_{\boldsymbol{\alpha}}(\psi) \geq 0$, we have $v_{\boldsymbol{\alpha}}(\psi) \leq a(E_{\boldsymbol{\alpha}}, K_X)/c < 2a(E_{\boldsymbol{\alpha}}, K_X) = 2/r$ and so $v_{\boldsymbol{\alpha}}(\psi) = 1/r$. Thus we may assume that ψ contains x_3 (if $a \equiv \pm 1$ we possibly have to permute coordinates). Then $S^\sharp \simeq \mathbb{C}^2$ is smooth and $S \simeq \mathbb{C}^2/\boldsymbol{\mu}_r(a, -a)$, i.e., S is Du Val of type A_{r-1}. □

Lemma 4.2. *If $(X \ni P)$ is a terminal singularity of index $r > 1$ and $\mathrm{ct}(X, S) > 1/2$, then $K_X + S \sim 0$.*

Proof. By Lemma 4.1 $(X \ni P)$ is not a cyclic quotient singularity. There is an analytic $\boldsymbol{\mu}_r$-equivariant embedding $(X^\sharp, P^\sharp) \subset (\mathbb{C}^4, 0)$. Let (x_1, x_2, x_3, x_4) be coordinates in $\mathbb{C}^4$, let $\phi = 0$ be an equation of $X^\sharp$, and let $\psi = 0$ be an equation of $S^\sharp$. We can take (x_1, x_2, x_3, x_4) and ϕ to be semi-invariants such that one of the following holds [Mor85] (see also [Rei87]):

- **Main series.** $\mathrm{wt}(x_1, x_2, x_3, x_4; \phi) \equiv (a, -a, 1, 0; 0) \mod r$, where $\gcd(a, r) = 1$.

- **Case** $cAx/4$**.** $r = 4$, $\operatorname{wt}(x_1, x_2, x_3, x_4; \phi) \equiv (1, 3, 1, 2; 2) \mod 4$.

In both cases $\operatorname{wt}(x_1 x_2 x_3 x_4) - \operatorname{wt}\phi \equiv \operatorname{wt} x_3 \mod r$. According to [Kaw92] there is a weight $\boldsymbol{\alpha}$ such that for the corresponding $\boldsymbol{\alpha}$-blowup $\sigma_{\boldsymbol{\alpha}}\colon X_{\boldsymbol{\alpha}} \subset W \to X \subset \mathbb{C}^4/\boldsymbol{\mu}_r$ the exceptional divisor $E_{\boldsymbol{\alpha}} \cap X_{\boldsymbol{\alpha}}$ has a reduced component G of discrepancy $a(G, K_X) = 1/r$. Moreover, $r\alpha_i \equiv \operatorname{wt} x_i \mod r$, $i = 1, 2, 3, 4$. Since $c > 1/2$, we have $1/r - cv_{\boldsymbol{\alpha}}(\psi) \geq 0$, i.e., $rv_{\boldsymbol{\alpha}}(\psi) < 2$, so $rv_{\boldsymbol{\alpha}}(\psi) = 1$. In particular, $\operatorname{wt}\psi \equiv 1 \mod r$.

Let ω be a section of $\mathcal{O}_X(-K_X)$. Then ω can be written as

$$\omega = \lambda(\partial\phi/\partial x_4)(dx_1 \wedge dx_2 \wedge dx_3)^{-1},$$

where λ is a semi-invariant function with

$$\operatorname{wt}\lambda - \operatorname{wt}(x_1 x_2 x_3 x_4) + \operatorname{wt}\phi \equiv \operatorname{wt}\omega \equiv 0 \mod r.$$

Thus, $\operatorname{wt}\psi \equiv \operatorname{wt}\lambda \mod r$. Hence, $S \sim -K_X$. $\square$

LEMMA 4.3. *If $(X \ni P)$ is a terminal singularity of index $r > 1$, then $c \leq 4/5$.*

PROOF. Since π is étale in codimension one, we have $K_{X^\sharp} + cS^\sharp = \pi^*(K_X + cS)$. Hence the pair $(X^\sharp, cS^\sharp)$ is canonical (see, e.g., [Kol97, 3.16.1]). Assume that $c > 4/5$. By Lemma 4.1 the point $(X^\sharp \ni P^\sharp)$ is singular. Then by Lemma 3.5 the pair $(X^\sharp, S^\sharp)$ is canonical. Therefore, $(S^\sharp \ni P^\sharp)$ is a Du Val singularity. Then the singularity $(S \ni P) = (S^\sharp \ni P^\sharp)/\boldsymbol{\mu}_r$ is log terminal. On the other hand, by Lemma 4.2 the divisor K_S is Cartier. Hence, $(S \ni P)$ is Du Val, a contradiction. $\square$

References

[BS06] Birkar, C. and V. V. Shokurov, Mld's vs thresholds and flips, E-print math.AG/0609539.

[Cor95] Corti, A., Factoring birational maps of threefolds after Sarkisov, J. Algebr. Geom. **4**(2), (1995), 223–254.

[Kaw92] Kawamata, Y., The minimal discrepancy coefficients of terminal singularities in dimension three. Appendix to V.V. Shokurov's paper "3-fold log flips", Russ. Acad. Sci., Izv., Math. **40**(1), (1992), 95–202.

[KM98] Kollár, J. and S. Mori, *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kol92] Kollár, J., editor, *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[Kol94] Kollár, J., Log surfaces of general type; some conjectures, In *Classification of algebraic varieties (L'Aquila, 1992)*, volume 162 of *Contemp. Math.*, pages 261–275. Amer. Math. Soc., Providence, RI, 1994.

[Kol97] Kollár, J., Singularities of pairs, In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997.

[Kol08] Kollár, J., Which powers of holomorphic functions are integrable? Preprint, arXiv.org:0805.0756, 2008.

[Mar96] Markushevich, D., Minimal discrepancy for a terminal cDV singularity is 1, J. Math. Sci. Univ. Tokyo **3**(2), (1996), 445–456.

[MP04] McKernan, J. and Y. Prokhorov, Threefold thresholds, Manuscripta Math. **114**(3), (2004), 281–304.

[Mor85] Mori, S., On 3-dimensional terminal singularities, Nagoya Math. J. **98** (1985), 43–66.

[Rei80] Reid, M., Canonical 3-folds, In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 273–310. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[Rei87] Reid, M., Young person's guide to canonical singularities, In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

[Sho88] Shokurov, V. V., Problems about Fano varieties, In *Birational Geometry of Algebraic Varieties, Open Problems*, pages 30–32, Katata, 1988.

[Sho93] Shokurov, V. V., 3-fold log flips, Russ. Acad. Sci., Izv., Math. **40**(1), (1993), 95–202.

(Received June 26, 2008)
(Revised October 30, 2008)

Department of Higher Algebra
Faculty of Mathematics and Mechanics
Moscow State Lomonosov University
Vorobievy Gory, Moscow
119 899, RUSSIA
E-mail: prokhoro@mech.math.msu.su