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A Note on Semistable Barsotti-Tate Groups

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Abstract. We show that the Dieudonné crystal associated to a Barsotti-Tate group with potentially semistable reduction over a smooth curve is overconvergent. As a corollary, we obtain the rationality of the L-function associated to this group.

1. Introduction

Let $U/\mathbb{F}_p$ be a smooth curve and G/U a Barsotti-Tate group. Assume G/U has potentially semistable reduction (see 4.2 for a precise definition). We show that the Dieudonné crystal as defined in [1] is overconvergent in the sense of Berthelot. As a corollary we get the rationality of the L-function associated to G/U. In the third section we study the local situation, that is, semitable Barsotti-Tate groups over a complete discrete valuation field of equal characteristic p. Using Extension groups in the category of Dieudonné crystals and their interpretation in terms of syntomic cohomology (as defined in [13]) we prove that the Dieudonné crystal associated to such group extends to a log Dieudonné crystal over the ring of integers. Using the glueing properties of overconvergent F-isocrystals over smooth curves proved in [14], we deduce from section three the overconvergence of the Dieudonné crystal associated to G/U and the rationality of its L-function in the last section. We end both sections three and four by some open questions.

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3. Semistable Barsotti-Tate Groups and Extensions

In this section, we extend the Dieudonné crystal of a semistable Barsotti-Tate group over a complete discrete valuation field of equal characteristic $p > 0$ to a log Dieudonné crystal.

3.1. Let k be a perfect field of characteristic p endowed with its Frobenius σ, $W := W(k)$ the ring of Witt vectors and $K = Frac(W)$. We denote $\eta := \mathrm{Spec}(k((t)))$. Let G_η/η a Barsotti-Tate group. Following [5]:

3.2 Definition. The Barsotti-Tate group G_η/η is called semistable if there exists a filtration:

$$0 \subset G_\eta^\mu \subset G_\eta^f \subset G_\eta$$

by Barsotti-Tate groups such that the following conditions hold:

1. G_η^f and G_η/G_η^μ extend to Barsotti-Tate groups G_1 and G_2 over $k[[t]]$. In this case, the composed map

$$G_\eta^f \hookrightarrow G_\eta \to G_\eta/G_\eta^\mu$$

   extends to a map $G_1 \to G_2$.

2. $G_1^\mu := \mathrm{Ker}\,(G_1 \to G_2)$ and $G_2^{\acute{e}t} := coker(G_1 \to G_2)$ are Barsotti-Tate groups over $k[[t]]$.

3. G_1^μ is of multiplicative type and $G_2^{\acute{e}t}$ is étale.

3.3 Remark. It has been shown in [5], 2.5, that an abelian variety A over η has semistable reduction if and only if its associated Barsotti-Tate group $G_\eta := \varinjlim_n A[p^n]$ is semistable.

3.4. Let S be a fine log-scheme over $\mathrm{Spec}(k)$ endowed with the trivial log-structure. We denote the absolute Frobenius of S by σ_S, lying above σ. We work on the log crystalline site with the étale topology, denoted

$Crys(S/W)$([7]). An object of $Crys(S/W)$ is a pair (S', P), where S' is an étale scheme over S, P is a p.d.-thickening of S' over W with respect to the p.d.-structure of (p), and we are given an isomorphism between the inverse image of the log structure of P on S' and the inverse image of the log structure of S on S'. Morphisms of $Crys(S/W)$ are defined in the evident way. The topology of $Crys(S/W)$ is given by the étale topology of each P. In the applications of this paper, S is mainly one of the followings :

1. S/k is a proper smooth curve with the log structure on S associated to the divisor $S \setminus U$, for some open subset U.
2. $S = U$ with the trivial log structure.
3. $S = \mathrm{Spec}(k[[t]])$ with the log structure associated to the closed point.
4. $S = \eta$ with the trivial log structure.

3.5. A crystal E on $Crys(S/W)$ is called a Dieudonné crystal if it is a finite locally free crystal endowed with linear operators $F : \sigma_S^* E \to E$ and $V : E \to \sigma_S^* E$ called respectively Frobenius and Verschiebung such that $FV = p$ and $VF = p$. If (D, F_D, V_D) is a Dieudonné crystal on $Crys(S/W)$, its $\mathcal{O}_{S/W}$-dual $D^\vee$ is endowed with a structure of Dieudonné crystal such that $F_{D^\vee} = (V_D)^\vee$ and $V_{D^\vee} = (F_D)^\vee$.

3.6. Let G be a Barsotti-Tate group over S. By the crystalline Dieudonné theory (see for example [1], [2], [4]), the Dieudonné crystal $\mathbb{D}(G)$ on $Crys(S/W)$ is defined by forgetting the log structures of objects of $Crys(S/W)$ ($\mathbb{D}$ is a contravariant functor). More precisely, let π denote the canonical morphism from S to S_{triv}, the scheme S endowed with the trivial log-structure. Then $\pi^*\mathbb{D}(G)$ is a Dieudonné crystal on $Crys(S/W)$ that we still denote $\mathbb{D}(G)$. The $\mathcal{O}_{S/W}$-dual of $\mathbb{D}(G)$ will be denoted by $D(G)$, so that $D(.)$ becomes a covariant functor. We will furthermore, denote by $\mathbf{1} := D(\mathbb{Q}_p/\mathbb{Z}_p)$ the Dieudonné crystal $(\mathcal{O}_{S/W}, F = p, V = id)$ and by $\mathbf{1}(1) := D(\mu_{p^\infty})$ the Dieudonné crystal $(\mathcal{O}_{S/W}, F = id, V = p)$. The Dieudonné crystals $\mathbf{1}$ and $\mathbf{1}(1)$ are dual to each other.

3.7. We recall the construction of the syntomic cohomology as defined in [13] in the case $S = \mathrm{Spec}(k[[t]])$ with the log structure associated to the closed point. Let D be a Dieudonné crystal over S/W. The syntomic complex $\mathcal{S}_D$ is the total complex associated to the bicomplex

$$\begin{array}{ccccc} & D^0 & \stackrel{\nabla}{\to} & D\otimes\Omega^1_{\mathcal{Y}} & \\ \mathbf{1}-F_1 & \downarrow & & \downarrow & 1-F_2 \\ & D & \stackrel{\nabla}{\to} & D\otimes\Omega^1_{\mathcal{Y}} & \end{array}$$

We explain the notations: $\mathcal{Y} = Spf(W[[t]])$ is endowed with the log-structure associated to $\mathbb{N} \to W[[t]]$ sending n to t^n. It is a log smooth formal lifting of S and we denote $\sigma_{\mathcal{Y}}$ a lifting of the Frobenius of S sending the variable t to t^p. By abuse of notation, we still denote (D, ∇, F_D, V_D) the realization of the Dieudonné crystal D at the p.d. thickening $(S \subset \mathcal{Y})$ endowed with its connection, Frobenius and Verschiebung. Consider the composed map

$$D \stackrel{\iota}{\to} \sigma^*_{\mathcal{Y}}D \to \sigma^*_{\mathcal{Y}}D/V_D(D)$$

where ι is the map sending $x \to 1\otimes x$. Set $Lie(D)$ to be the image of the above map. Then $Lie(D)$ is a locally free $\mathcal{O}_S$-module (see [13], 5.3) and we denote D^0, the kernel of the surjective map $D \to Lie(D)$. Finally, we explain the Frobenius operators. The map $F_1 : D^0 \to D$ is constructed as follows: the composed map

$$\tilde{F}_1 : D^0 \stackrel{\mathbf{1}}{\to} D \stackrel{\iota}{\to} \sigma^*_{\mathcal{Y}}D \stackrel{F_D}{\to} D,$$

is in $p.D$ (see [13], 5.8.1) and we set $F_1 := p^{-1}\tilde{F}_1$. On the other side, remark that $\sigma_{\mathcal{Y}}(\Omega^1_{\mathcal{Y}}) \subset p.\Omega^1_{\mathcal{Y}}$ so that we can define a map

$$F_2 := F_D \circ \iota \otimes p^{-1}\sigma_{\mathcal{Y}}.$$

3.8 PROPOSITION. *Assume k is algebraically closed and let $S =$ $\mathrm{Spec}(k[[t]])$ endowed with the log structure associated to the closed point. Then, we have:*

$$H^i(S, \mathcal{S}_{\mathbf{1}(1)}) = H^i(\eta, \mathcal{S}_{\mathbf{1}(1)}) = \begin{cases} \widehat{k((t))^\times}, & i = 1 \\ 0, & \textit{otherwise,} \end{cases}$$

where $\hat{M} = \varprojlim_n M/M^{p^n}$ for any multiplicative group M.

PROOF. First, we prove the claim for $H^i(\eta, \mathcal{S}_{\mathbf{1}(1)})$. By [13], 5.10, we have

$$H^i(\eta, \mathcal{S}_{\mathbf{1}(1)}) = H^i_{fl}(\eta, T_p\mathbf{G}_m).$$

Since $k((t))$ is a C_1-field, we have

$$H^i_{fl}(\eta, \mathbf{G}_m) = \begin{cases} k((t))^\times & , i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

By using the short exact sequence

$$0 \to \mu_{p^n} \to \mathbf{G}_m \stackrel{p^n}{\to} \mathbf{G}_m \to 0$$

on the flat site, we see that

$$H^i_{fl}(\eta, T_p\mathbf{G}_m) = \begin{cases} \widehat{k((t))^\times} & , i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

So the claim for $H^i(\eta, \mathcal{S}_{\mathbf{1}(1)})$ is proved.

Next we prove the claim for $H^i(S, \mathcal{S}_{\mathbf{1}(1)})$. In the case of the crystal $D = \mathbf{1}(1)$ the short exact sequence

$$0 \to D^0 \to D \to Lie(D) \to 0$$

is induced by the canonical short exact sequence in the crystalline site:

$$0 \to \mathcal{I}_{S/W} \to \mathcal{O}_{S/W} \to \mathbf{G}_a \to 0$$

which induces on the pd-thichening $S \subset \mathcal{Y}$ the short exact sequence:

$$0 \to p.W[[t]] \to W[[t]] \to k[[t]] \to 0.$$

Hence, the syntomic complex of $\mathbf{1}(1)$ over S is the total complex associated to the bicomplex

$$\begin{array}{cccccc} & p.W[[t]] & \stackrel{d}{\to} & W[[t]]\frac{dt}{t} & \\ \mathbf{1} - F_1 & \downarrow & & \downarrow & 1 - F_2 \\ & W[[t]] & \stackrel{d}{\to} & W[[t]]\frac{dt}{t} & \end{array}$$

where $d : W[[t]] \to W[[t]]\frac{dt}{t}$ is the map sending an element $\sum_i a_i t^i$ to $(\sum_i i a_i t^i)\frac{dt}{t}$, F_1 is the map sending an element $p.\sum_i a_i t^i$ to $\sum_i \sigma(a_i)t^{pi}$ and F_2 the map sending an element $(\sum_i a_i t^i)\frac{dt}{t}$ to $(\sum_i \sigma(a_i)t^{pi})\frac{dt}{t}$. Hence, $\mathcal{S}_{\mathbf{1}(1)}$ is the complex concentrated in degree $0, 1, 2$:

$$[pW[[t]] \stackrel{d,\mathbf{1}-F_1}{\to} W[[t]]\frac{dt}{t} \oplus W[[t]] \stackrel{1-F_2,-d}{\to} W[[t]]\frac{dt}{t}].$$

Remark that this complex is isomorphic to the complex

$$[W[[t]] \stackrel{pd,p-\sigma}{\to} W[[t]]\frac{dt}{t} \oplus W[[t]] \stackrel{1-F_2,-d}{\to} W[[t]]\frac{dt}{t}].$$

We compute the H^0: By definition $H^0 = \mathrm{Ker}\,(d) \cap \mathrm{Ker}\,(1-F_1)$. Since $\mathrm{Ker}\,(d) = pW$, H^0 is equal to the set of element $p.a \in pW$ such that $pa - \sigma(a) = 0$. Since the p-adic valuation $v(\sigma(a))$ is equal to $v(a)$, the previous equality gives $a = 0$.

We compute the H^2: to show that this is zero, we just need to show that the map $\pi := (1-F_2, -d)$ is surjective. But for any $\sum_i c_i t^i \frac{dt}{t} \in W[[t]]\frac{dt}{t}$, the element $(\sum_i b_i t^i \frac{dt}{t}, 0)$, with $b_i = c_i + \sigma(b_{i/p})$ if p divide i and $b_i = c_i$ else is an antecedent of $\sum_i c_i t^i \frac{dt}{t}$ by π.

We now turn to the computation of $H^1 := \mathrm{Ker}\,(\pi)/Im(d, 1-F_1)$. The group $\mathrm{Ker}\,(\pi)$ is the set of elements $(\sum_i a_i t^i \frac{dt}{t}, \sum_i b_i t^i)$ such that $a_0 \in \mathbb{Z}_p$ and for n, any positive integer with p-adic valuation r, $a_n = nb_n + (n/p)\sigma(b_{n/p}) + \ldots + (n/p^r)\sigma^r(b_{n/p^r})$. We get this way an isomorphism

$$\mathrm{Ker}\,(\pi) \simeq \mathbb{Z}_p\frac{dt}{t} \oplus W[[t]]$$

by sending $(\sum_i a_i t^i \frac{dt}{t}, \sum_i b_i t^i)$ to $(a_0, \sum_i b_i t^i)$, which induces an isomorphism

$$Im(d, 1-F_1) \simeq 0 \oplus Im(1-F_1),$$

since the elements in $Im(d)$ have no constant terms.

We get

$$H^1 = \mathbb{Z}_p\frac{dt}{t} \oplus W[[t]]/Im(1-F_1).$$

On the other hand, $k((t))^\times \simeq t^{\mathbb{Z}} \times k^\times \times (1 + tk[[t]])$ and $(k((t))^\times)^{p^n} = k((t^{p^n}))^\times \simeq t^{p^n\mathbb{Z}} \times k^\times \times (1 + t^{p^n}k[[t^{p^n}]])$. So, we are reduced to identify

$W[[t]]/Im(1-F_1)$ and $\varprojlim_n (1+tk[[t]])/(1+t^{p^n}k[[t^{p^n}]])$. We first prove that the lefthand side is p-adically complete. By, [15], chapter 8, it is enough to prove that $I = Im(1-F_1)$ is closed and in particular complete. Let $(f_m(t) = \sum_i b_i^{(m)} t^i)_{m\in\mathbb{N}}$ a sequence of elements in I converging to $f(t) = \sum_i b_i t^i \in W[[t]]$. We want to show that $f(t)$ is in fact in I. Since $f_m(t) \in I$, for any m, there exists some sequence $(a_i^{(m)})_i \in W^{\mathbb{N}}$ such that $b_i^{(m)} = pa_i^{(m)} - \sigma(a_{\frac{i}{p}}^{(m)})$ if p divides i and $b_i^{(m)} = pa_i^{(m)}$ else. We construct by induction on the p-adic valuation of i, a sequence $(a_i)_i \in W^{\mathbb{N}}$ such that $(1-F_1)(\sum_i pa_i t^i) = f(t)$. For $v_p(i) = 0$, that is when p does not divide i, $p.a_i^{(m)}$ converges when m goes to infinity to b_i so that $(a_i^{(m)})$ converges to an element $a_i \in W$ such that $b_i = p.a_i$. Assume now that for any i such that $v_p(i) \leq r$, $(a_i^{(m)})$ converges to an element $a_i \in W$. Then, if $v_p(i) = r+1$, we have $b_i^{(m)} = pa_i^{(m)} - \sigma(a_{\frac{i}{p}}^{(m)})$, with $(b_i^{(m)})_m$ converging to an element b_i and by induction hypothesis, $(\sigma(a_{\frac{i}{p}}^{(m)}))_m$ converging to an element $\sigma(a_{\frac{i}{p}})$ and so we deduce that $(a_i^{(m)})$ converges to an element $a_i \in W$.

Let $D = \mathbf{1}(1)$. We compute now $H^1(S, \mathcal{S}_D)/p^n$: we have a short exact sequence

$$0 \to \mathcal{S}_D \stackrel{\times p^n}{\to} \mathcal{S}_D \to \mathcal{S}_{D,n} \to 0,$$

which induces an exact sequence

$$H^1(S, \mathcal{S}_D) \stackrel{\times p^n}{\to} H^1(S, \mathcal{S}_D) \to H^1(S, \mathcal{S}_{D,n}) \to H^2(S, \mathcal{S}_D).$$

Since we already have proved that $H^2(S, \mathcal{S}_D) = 0$, we deduce for any n the isomorphisms

$$H^1(S, \mathcal{S}_D)/p^n \simeq H^1(S, \mathcal{S}_{D,n}).$$

By [13], 5.14.6, we also have

$$H^1(\eta, \mathcal{S}_D)/p^n \simeq H^1(\eta, \mathcal{S}_{D,n}).$$

Again, by using the short exact sequence:

$$0 \to \mathcal{S}_{D,1} \to \mathcal{S}_{D,n+1} \stackrel{\times p}{\to} \mathcal{S}_{D,n} \to 0$$

and the 5-lemma, we are reduced by induction to prove that

$$H^1(S, \mathcal{S}_{D,1}) \simeq H^1(\eta, \mathcal{S}_{D,1}).$$

Using the second description of the syntomic complex, we have the quasi-isomorphisms:

$$\mathcal{S}_{\mathbf{1}(1),S} \otimes \mathbb{Z}/p \simeq [k[[t]] \overset{0,-\sigma}{\to} k[[t]]\frac{dt}{t} \oplus k[[t]] \overset{\pi_S}{\to} k[[t]]\frac{dt}{t}],$$

$$\mathcal{S}_{\mathbf{1}(1),\eta} \otimes \mathbb{Z}/p \simeq [k((t)) \overset{0,-\sigma}{\to} k((t))\frac{dt}{t} \oplus k((t)) \overset{\pi_\eta}{\to} k((t))\frac{dt}{t}]$$

and the map $H^1(S, \mathcal{S}_{\mathbf{1}(1)})/p \to H^1(\eta, \mathcal{S}_{\mathbf{1}(1)})/p$ is induced by the natural inclusion

$$k[[t]]\frac{dt}{t} \oplus k[[t]] \hookrightarrow k((t))\frac{dt}{t} \oplus k((t)).$$

Now, we compute $H^1(\eta, \mathcal{S}_{\mathbf{1}(1)})/p$. For any element $(\sum_i a_i t^i \frac{dt}{t}, \sum_i b_i t^i) \in \mathrm{Ker}\,(\pi_\eta)$ we find the same conditions that $a_0 \in \mathbb{F}_p$ and for n, any positive integer with p-adic valuation r, $a_n = nb_n + (n/p)\sigma(b_{n/p}) + \ldots + (n/p^r)\sigma^r(b_{n/p^r})$. For negative integers and working modulo $Im(\sigma) = k((t^p))$, we claim that only the b_j's with $b_{-j} = 0$ for any j prime to p, gives a solution. Namely, for such j we have $a_{-j} = -jb_{-j}$ but then $a_{-jp^k} = \sigma(-jb_{-j})$ for any positive integer k. But since $\sum_i a_i t^i \in k((t))$, we must have $a_{-jp^k} = 0$ for k big enough. Therefore, the canonical inclusion

$$k[[t]]\frac{dt}{t} \oplus k[[t]] \hookrightarrow k((t))\frac{dt}{t} \oplus k((t))$$

induces the identity map

$$H^1(S, \mathcal{S}_D)/p = \mathbb{F}_p\frac{dt}{t} \oplus k[[t]]/k[[t^p]] \to \mathbb{F}_p\frac{dt}{t} \oplus k[[t]]/k[[t^p]] = H^1(\eta, \mathcal{S}_D)/p.$$

Hence, we proved the canonical isomorphism $H^1(S, \mathcal{S}_D) \simeq H^1(\eta, \mathcal{S}_D)$ and so the proof of the proposition is finished. □

3.9. Let D_1, D_2 some Dieudonné crystals over S/W. We will denote $Ext_{S/W}(D_1, D_2)$ (or $Ext(D_1, D_2)$ if there is no ambiguity) the isomorphism classes of extensions

$$0 \to D_2 \to ? \to D_1 \to 0$$

in the category of Dieudonné crystals over S/W. Any commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ \mathrm{Spec}(W) & \xrightarrow{g} & \mathrm{Spec}(W) \end{array}$$

induces in the crystalline topos a functor $f^* : (S'/W)_{crys} \to (S/W)_{crys}$ allowing to define for any Dieudonné crystals D_1 and D_2 over S/W a canonical map

$$f^* : Ext^1(D_1, D_2) \to Ext^1(f^*D_1, f^*D_2),$$

sending the isomorphism class of an extension:

$$0 \to D_2 \to ? \to D_1 \to 0$$

to the isomorphism class of the extension

$$0 \to f^*D_2 \to f^*? \to f^*D_1 \to 0.$$

(The exactness of this sequence follows from the local freeness of D_1.)

3.10. Let G_η/η a semistable Barsotti-Tate group and denote as in 3.2 G_η^f, G_η^μ, G_1, G_1^μ, G_2 and $G_2^{\acute{e}t}$ its associated Barsotti-Tate groups. We denote $S := \mathrm{Spec}(k[[t]])$ endowed with the log-structure induced by its closed point. We also denote $j : \eta \to \mathrm{Spec}(k[[t]])$ the open immersion. Then there is a commutative diagram of exact sequences:

$$\begin{array}{ccccc} Ext(D(G_2^{\acute{e}t}), D(G_1^\mu)) & \xrightarrow{f_{log}} & Ext(D(G_2^{\acute{e}t}), D(G_1)) & \xrightarrow{g_{log}} & Ext(D(G_2^{\acute{e}t}), D(G_1/G_1^\mu)) \\ h_1 \downarrow & & h_2 \downarrow & & h_3 \downarrow \\ Ext(D(G_\eta/G_\eta^f), D(G_\eta^\mu)) & \xrightarrow{f} & Ext(D(G_\eta/G_\eta^f), D(G_\eta^f)) & \xrightarrow{g} & Ext(D(G_\eta/G_\eta^f), D(G_\eta^f/G_\eta^\mu)) \end{array}$$

where the horizontal maps are defined by applying the functor $\mathbb{R}Hom(D(G_2^{\acute{e}t}), .)$ and $\mathbb{R}Hom(D(G_\eta/G_\eta^f), .)$ to the short exact sequences:

$$0 \to D(G_1^\mu) \to D(G_1) \to D(G_1/G_1^\mu) \to 0,$$

and

$$0 \to D(G_\eta^\mu) \to D(G_\eta^f) \to D(G_\eta^f/G_\eta^\mu) \to 0$$

of Dieudonné crystals over $(S/W)_{crys}$ and $(\eta/W)_{crys}$ respectively. The vertical maps are induced by the functor $j^* : (S/W)_{crys} \to (\eta/W)_{crys}$.

3.11 Lemma. *Assume k is algebraically closed. Then the map g_{log} is surjective.*

Proof. Since k is algebraically closed, $G_2^{ét} \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^a$ and we can reduce to the the case $a = 1$, that is to the case $D(G_2^{ét}) = \mathbf{1}$. By [13], 5.9, $Ext(\mathbf{1}, D(G_1)) \simeq H^1(k[[t]], \mathcal{S}_{D(G_1)})$. Similarly, we have

$$Ext(\mathbf{1}, D(G_1/G_1^{\mu})) \simeq H^1(k[[t]], \mathcal{S}_{D(G_1/G_1^{\mu})})$$

so that the cokernel of g_{log} is $H^2(k[[t]], \mathcal{S}_{D(G_1^{\mu})})$. Again, since k is algebraically closed, we can reduce to the case $D(G_1^{\mu}) = \mathbf{1}(1)$ and the assertion results from 3.8. □

3.12 Lemma. *Assume that k is algebraically closed, then h_1 is an isomorphism.*

Proof. We are reduced to prove that

$$Ext_{S/W}(\mathbf{1}, \mathbf{1}(1)) \simeq Ext_{\eta/W}(\mathbf{1}, \mathbf{1}(1)).$$

Using [13], 5.9 and 5.10, it is enough to prove that the map

$$H^1(S, \mathcal{S}_{\mathbf{1}(1)}) \to H^1(\eta, \mathcal{S}_{\mathbf{1}(1)})$$

is an isomorphism but this has already been proved in 3.8. □

3.13 Theorem. *Assume k is algebraically closed.*
Let $\alpha \in Ext(D(G_\eta/G_\eta^f), D(G_\eta^f))$ be the isomorphism class of the extension:

$$0 \to D(G_\eta^f) \to D(G_\eta) \to D(G_\eta/G_\eta^f) \to 0.$$

There exists a short exact sequence of Dieudonné crystals over S/W:

$$0 \to D(G_1) \to D_{log} \to D(G_2^{ét}) \to 0,$$

such that its isomorphism class β is sent by h_2 to α.

As a corollary, we get:

3.14 Corollary. *Let $G_\eta/\eta := k((t))$ be a semistable Barsotti-Tate group. Then its Dieudonné crystal $D(G_\eta)$ extends to a Dieudonné crystal*

D_{log} over S, the scheme $\mathrm{Spec}(k[[t]])$ *endowed with the log-structure induced by its closed point.*

We now prove the theorem:

PROOF. Let $\gamma \in Ext(D(G_2^{\acute{e}t}), D(G_1/G_1^{\mu}))$ be the isomorphism class of the extension:

$$0 \to D(G_1/G_1^{\mu}) \to D(G_2) \to D(G_2^{\acute{e}t}) \to 0$$

such that we have $g(\alpha) = h_3(\gamma)$. Since g_{log} is surjective, there exists $\tilde{\gamma} \in Ext(D(G_2^{\acute{e}t}), D(G_1))$ such that $g_{log}(\tilde{\gamma}) = \gamma$. Since $g(\alpha - h_2(\tilde{\alpha})) = 0$, there exists some $\delta \in Ext(D(G_\eta/G_\eta^f), D(G_\eta^{\mu}))$, corresponding by 3.12 to a unique $\tilde{\delta} \in Ext((D(G_2^{\acute{e}t}), D(G_1^{\mu}))$, such that $f(\delta) = \alpha - h_2(\tilde{\alpha})$. Then $\beta := f_{log}(\tilde{\delta}) + \tilde{\gamma}$ is sent by h_2 to α. □

3.15 DEFINITION. Let G_η/η be a Barsotti-Tate group. We say that it is overconvergent if its associated Dieudonné isocrystal, corresponding to a (φ, ∇) over

$$\mathcal{E} = \{a = \sum_{-\infty}^{+\infty} a_i x^i |\ a_i \in K,\ \sup_i |a_i| < \infty,\ |a_i| \to 0\ (i \to -\infty)\}$$

(see [14]) admits a lattice as (φ, ∇)-module over

$$\mathcal{E}^{\dagger} = \{a \in \mathcal{E} | |a_i| r^i \to 0\ (i \to -\infty) \text{ for a certain } r, 0 < r < 1\}.$$

As a corollary of 3.14, we have:

3.16 COROLLARY. *Any semistable Barsotti-Tate group G_η/η is overconvergent.*

3.17 REMARK.

1. Any Barsotti-Tate group coming from an abelian variety is overconvergent: the abelian variety has potentially semistable reduction and in consequence it has been shown in [13] that the Dieudonné crystal of the abelian variety (which coincides with the Dieudonné crystal of the associated Barsotti-Tate group) comes from a log Dieudonné crystal after taking some finite étale base change.

2. Barsotti-Tate groups associated to p-adic representations of the absolute Galois group of η with infinite monodromy are not overconvergent ([18]).

3. We have shown that if G_η/η is semistable then it is overconvergent. Reciprocally, if G_η/η is overconvergent, can we conclude that it is potentially semistable? Since G_η/η is overconvergent its associated isocrystal will be quasi-unipotent by the local p-adic monodromy theorem of André-Kedlaya-Mebkhout. So we know that it will come from some log-Dieudonné crystal after considering some finite étale base change. The previous question can thus be rephrased as: Is there a log Dieudonné functor from the category (still to be defined) of log p-divisible groups to the category of log Dieudonné modules over $k[[t]]$ and if yes, is this functor an equivalence of categories (as this is the case without log-structure by [4])?

4. Recall (see [19]) that G_η/η is endowed with a unique Frobenius slope filtration, whose quotients are isoclinic Barsotti-Tate groups. Assume each quotients to be overconvergent. Does it imply that G_η/η is overconvergent?

4. Semistable Barsotti-Tate Groups over Smooth Curves

4.1. In this section, we consider a dense open subset U of a proper smooth connected curve $C/\mathbb{F}_p$. For any closed point $x \in C$, we denote $\eta_x := \mathrm{Spec}(k(x)((t)))$ and $K_x := Frac(W(k(x)))$. For any $\mathbb{F}_p$-scheme T, we will denote $\bar{T} := T \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p$.

4.2 Definition. A Barsotti-Tate group G/U is called semistable if at any closed point $x \in Z := C \setminus U$, $G_{\eta_x} := G \times_U \eta_x/\eta_x$ is semistable. We say that a Barsotti-Tate group G/U is potentially semistable if and only if there exists some finite Galois covering $U' \to U$ such that $G' := G \times_U U'/U'$ is semistable.

4.3. Let G/U a potentially semistable Barsotti-Tate group. We associate to G/U the following L-function:

$$L(U, G, t) := \prod_{x \in U} det(1 - t^{deg(x)} F_x, D(G_x))^{-1},$$

where $(D(G_x), F_x)$ is the F-isocrystal over $k(x)$ deduced from $D(G)$ by restriction and $deg(x) := [k(x) : \mathbb{F}_p]$.

We are going to show that the Dieudonné crystal associated to a potentially semistable Barsotti-Tate group is overconvergent and that its associated L function is a rational function. We will need the following lemma:

4.4 Lemma. ([6])
Let E be a convergent F-isocrystal over U. Let $\pi : \bar{U} \to U$ the canonical étale covering. Assume that $\pi^* E$ is overconvergent, then E is overconvergent.

4.5 Theorem. *Let G/U a potentially semistable Barsotti-Tate group. Then its associated Dieudonné crystal $D(G)$ over U has a structure of overconvergent F-isocrystal $D(G)^\dagger$ over U.*

Proof. By the previous lemma we can assume $U = \bar{U}$ and by finite étale descent we can assume that G/U is semistable. For any closed point $x \in Z$, we denote $\eta_x := \mathrm{Spec}(Frac(\hat{\mathcal{O}_{C,x}}))$ and $S_x := \mathrm{Spec}(\hat{\mathcal{O}_{C,x}})$ endowed with the log-structure induced by its closed point. By 3.14, the Dieudonné crystal $D(\mathcal{G} \times_U \eta_x)$ extends to a Dieudonné crystal D_{log} over S_x. Hence, the assertion follows from [14], proposition 4. □

4.6 Corollary. *Let G/U a potentially semistable Barsotti-Tate group. Then its L-function $L(G, U, t)$ is a rational function in t. More precisely, we have:*

$$L(G, U, t) = \prod_{i=0}^{2} det(1 - tF, H^i_{rig,c}(U, D(G)^\dagger))^{(-1)^{i+1}}.$$

Proof. By 4.5 $D(G)$ has a structure of overconvergent F-isocrystal and the formula results from [17], theorem 1.2. Finally, the rationaly results from the finiteness of the cohomological groups $H^i_{rig,c}(U, D(G)^\dagger)$ which follows from [14], corollary 8 and the Poincaré duality of rigid cohomology. □

4.7 Remark. Let G_F/F be a Barsotti-Tate group, where F is the function field of C.

1. It is a priori not always possible to extend G_F/F to some Barsotti-Tate group G over some dense open subset U of C (but this is the case when G_F/F is the Barsotti-Tate group associated to an abelian variety). For example, consider the étale case. Then, we can replace Barsotti-Tate groups by p-adic representations. We can find an example of a p-adic representation of $Gal(\bar{F}/F)$ that ramifies at infinitely many places and thus don't factorize through any fundamental group of some dense open subset U of C. To construct such representation, it is enough to construct a $\mathbb{Z}_p$-extension K of F that ramifies at infinitely many places (it exists: see for example [11]). Take any extension L/F with Galois group $(\mathbb{Z}/p)^\times$. Then the extension $K.L/F$ has a Galois group isomorphic to $\mathbb{Z}_p^\times$ and the natural projection $Gal(\bar{F}/F) \to Gal(K.L/F)$ gives an example of one-dimensional p-adic representation of $Gal(\bar{F}/F)$ that ramifies at infinitely many places.

2. If $G_1, G_2/U$ are two Barsotti-Tate groups with G_F/F as generic fiber, are G_1 and G_2 isomorphic (or at least isogenous)? See [5] for some evidences on this question. If the answer to this question is yes and G_F/F extends to some Barsotti-Tate group G/U, then we can define the Hasse-Weil L-function of G_F/F as $L(G_F, t) := L(U, G, t)$.

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