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An SO(3)-Version of 2-Torsion Instanton Invariants

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Abstract. We construct an invariant for non-spin 4-manifolds by using 2-torsion cohomology classes of moduli spaces of instantons on $SO(3)$-bundles. The invariant is an $SO(3)$-version of Fintushel-Stern's 2-torsion instanton invariant. We show that this $SO(3)$-torsion invariant is non-trivial for $2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, while it is known that any known invariant of $2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ coming from the Seiberg-Witten theory is trivial since $2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ has a positive scalar curvature metric.

1. Introduction

The purpose of this paper is to construct an $SO(3)$-version of Fintushel-Stern's torsion invariants [FS]. R. Fintushel and R. Stern constructed a variant of Donaldson invariants for spin 4-manifolds by using 2-torsion cohomology classes of the moduli spaces of instantons on $SU(2)$-bundles. They used cohomology classes of degree one and two. S. K. Donaldson gave another construction by using a class of degree 3 [D4]. As is well known, the usual Donaldson invariant is trivial for the connected sum of 4-manifolds with b^+ positive ([D3]). On the other hand, Fintushel and Stern showed that their torsion invariant is not necessarily trivial for the connected sum of the form $Y\#S^2 \times S^2$ in general.

In this paper, we define an invariant of 4-manifolds using 2-torsion cohomology classes of $SO(3)$-moduli spaces and show that our invariant is not necessarily trivial for $Y\#S^2 \times S^2$ as in the case of Fintushel-Stern's invariant. We basically follow the argument in [FS] and modify it to extend the definition to non-spin 4-manifolds.

The outline of the construction is as follows. Let X be a closed, oriented, simply connected, non-spin Riemannian 4-manifold and P be an $SO(3)$-bundle over X satisfying

$$w_2(P) = w_2(X) \in H^2(X;\mathbb{Z}_2), \quad p_1(P) \equiv \sigma(X) \mod 8.$$

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Here $\sigma(X)$ is the signature of X. Let $\mathcal{B}_P^*$ be the space of gauge equivalence classes of irreducible connections on P. In [AMR], S. Akbulut, T. Mrowka and Y. Ruan showed that $H^1(\mathcal{B}_P^*;\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$. We denote the generator by u_1. On the other hand, for homology class $[\Sigma] \in H_2(X;\mathbb{Z})$ with self-intersection number even, we have an integral cohomology class $\mu([\Sigma]) \in H^2(\mathcal{B}_P^*;\mathbb{Z})$. Suppose that the dimension of the moduli space M_P of instantons on P is $2d+1$ for some non-negative integer d. In general M_P is not compact. However for homology classes $[\Sigma_1], \dots, [\Sigma_d] \in H_2(X;\mathbb{Z})$ with self-intersection numbers even, we can define the pairing

$$q_X^{u_1}([\Sigma_1], \dots, [\Sigma_d]) = \langle u_1 \cup \mu([\Sigma_1]) \cup \cdots \cup \mu([\Sigma_d]), [M_P] \rangle \in \mathbb{Z}_2$$

in an appropriate sense. We show that this number depends only on the homology classes $[\Sigma_i]$ and gives a differential-topological invariant of X.

We will show a gluing formula of torsion invariants for $Y \# S^2 \times S^2$, which is an $SO(3)$-version of Theorem 1.1 in [FS]. By using this gluing formula and D. Kotschick's calculation in [K1, K2], we prove that $q^{u_1}_{2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2}$ is non-trivial. This example exhibits two interesting aspects explained below.

The first aspect is related to vanishing theorem. We have a description of $X = 2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ as the connected sum of $Y_1 = \mathbb{CP}^2$ and $Y_2 = \mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. Since the second Stiefel-Whitney class $w_2(P)$ is equal to $w_2(X)$, both of $w_2(P)|_{Y_1}$ and $w_2(P)|_{Y_2}$ are non-trivial. In such a situation, the usual Donaldson invariants are trivial by the dimension-count argument ([MM]). Hence the non-triviality of $q^{u_1}_{2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2}$ implies that the dimension-count argument can not be applied directly to proving such a vanishing theorem in our case. If each homology class $[\Sigma_i]$ is in $H_2(Y_1;\mathbb{Z})$ or $H_2(Y_2;\mathbb{Z})$, then we can show that our invariant vanishes. However we can not reduce the argument to this case because of the condition that $[\Sigma_i]\cdot[\Sigma_i]$ must be even to define our invariant.

The next aspect is related to the Seiberg-Witten theory. In [Wi], E. Witten introduced invariants, called the Seiberg-Witten invariants, of 4-manifolds using monopole equations. He conjectured that the invariants are equivalent to the Donaldson invariants and explicitly wrote a formula which should give a relation between the Donaldson invariants and the Seiberg-Witten invariants. In [PT], V. Pidstrigach and A. Tyurin proposed a program to give a rigorous mathematical proof of the formula by using non-abelian monopoles. The theory of non-abelian monopoles has been developed by P. Feehan and T. Leness ([FL1, FL2, FL3]). Feehan and Leness recently announced that they completed the proof of Witten's formula for 4-manifolds of simple type with $b_1 = 0$ and $b^+ > 1$ in [FL4].

The non-triviality of $q^{u_1}_{2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2}$ is quite a contrast to the equivalence of the Donaldson invariants and Seiberg-Witten invariants. If a 4-manifold has a positive scalar curvature metric and satisfies $b^+(X) \geq 1$, then the moduli space of solutions of the monopole equations with respect to the metric is empty for some perturbation. Hence any known invariant of $2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ coming from the monopole equations (the Seiberg-Witten invariant and a refinement due to S. Bauer and M. Furuta [BF]) is trivial since $2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ has a positive scalar curvature metric.

The paper is organized as follows. In Section 2, we construct cohomology classes $\mu([\Sigma])$ and u_1, and define a torsion invariant. In Section 3, we prove a gluing formula for the connected sum of the form $Y\#S^2 \times S^2$. In Section 4, we prove that $q^{u_1}_{2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2}$ is non-trivial by using the gluing formula. We also discuss the reason why the usual vanishing theorem does not hold for our torsion invariant.

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2. Torsion Invariants

2.1. Notations

Let X be a closed, oriented, simply connected 4-manifold, g a Riemannian metric on X and P an $SO(3)$-bundle over X. Put

$$k = -\frac{1}{4}p_1(P) \in \mathbb{Q}, \quad w = w_2(P) \in H^2(X;\mathbb{Z}_2).$$

Let $\mathcal{A}^*_P$ be the space of irreducible connections on P and $\mathcal{G}_P$ be the gauge group of P. We write $\mathcal{B}^*_P$ or $\mathcal{B}^*_{k,w,X}$ for the quotient space $\mathcal{A}^*_P/\mathcal{G}_P$. We denote by M_P or $M_{k,w,X}$ the moduli space of instantons on P.

Let A be an instanton on P. We have a sequence

$$\Omega^0_X(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1_X(\mathfrak{g}_P) \xrightarrow{d^+_A} \Omega^+_X(\mathfrak{g}_P).$$

The condition that A is an instanton implies that $d_A^+ \circ d_A = 0$. Hence the above sequence define a complex. We denote the cohomology groups by H_A^0, H_A^1, H_A^2.

Let $\bar{P}$ be a $U(2)$-lift of P and $\bar{E}$ be the rank 2 complex vector bundle associated with $\bar{P}$. Fix a connection $a_{\det}$ on $\det \bar{E}$. We write $\mathcal{A}_{\bar{E}}$ for the space of connections on $\bar{E}$ which induce the connection $a_{\det}$ on $\det \bar{E}$, and write $\mathcal{A}_{\bar{E}}^*$ for the space of irreducible connections in $\mathcal{A}_{\bar{E}}$. Let $\mathcal{G}_{\bar{E}}$ be the group of bundle automorphisms on $\bar{E}$ with determinant 1. We also introduce a subgroup $\mathcal{G}_{\bar{E}}^0$ of $\mathcal{G}_{\bar{E}}$. Fix a point x_0 in X. The subgroup $\mathcal{G}_{\bar{E}}^0$ is defined by

$$\mathcal{G}_{\bar{E}}^0 = \{g \in \mathcal{G}_{\bar{E}} | g(x_0) = 1\}.$$

We denote the quotient spaces by

$$\mathcal{B}_{\bar{E}}^* = \mathcal{A}_{\bar{E}}^* / \mathcal{G}_{\bar{E}}, \quad \widetilde{\mathcal{B}}_{\bar{E}} = \mathcal{A}_{\bar{E}} / \mathcal{G}_{\bar{E}}^0, \quad \widetilde{\mathcal{B}}_{\bar{E}}^* = \mathcal{A}_{\bar{E}}^* / \mathcal{G}_{\bar{E}}^0.$$

Since we are assuming that X is simply connected, the natural map $\mathcal{B}_{\bar{E}}^* \to \mathcal{B}_P^*$ is bijective.

To construct cohomology classes u_1 and $\mu([\Sigma])$, we need the universal bundle $\widetilde{\mathbb{E}}$ over $X \times \widetilde{\mathcal{B}}_{\bar{E}}$. The universal bundle is defined by

$$\widetilde{\mathbb{E}} := \bar{E} \times_{\mathcal{G}_{\bar{E}}^0} \mathcal{A}_{\bar{E}} \longrightarrow X \times \widetilde{\mathcal{B}}_{\bar{E}}.$$

For a closed, oriented surface Σ embedded in X, let $\nu(\Sigma)$ be a small tubular neighborhood of Σ. We define spaces of gauge equivalence classes of connections on $\nu(\Sigma)$. Let $\mathcal{A}_{\nu(\Sigma)}$ be the space of connections on $\bar{E}|_{\nu(\Sigma)}$ which induce the connection $a_{\det}|_{\nu(\Sigma)}$ on $\det \bar{E}|_{\nu(\Sigma)}$. Let $\mathcal{G}_{\nu(\Sigma)}$ be the group of automorphisms of $\bar{E}|_{\nu(\Sigma)}$ with determinant 1. We assume that the base point x_0 is in $\nu(\Sigma)$. Define $\mathcal{G}_{\nu(\Sigma)}^0$ by

$$\mathcal{G}_{\nu(\Sigma)}^0 = \{g \in \mathcal{G}_{\nu(\Sigma)} | g(x_0) = 1\}.$$

We denote the quotient spaces by

$$\mathcal{B}_{\nu(\Sigma)}^* = \mathcal{A}_{\nu(\Sigma)}^* / \mathcal{G}_{\nu(\Sigma)}, \quad \widetilde{\mathcal{B}}_{\nu(\Sigma)} = \mathcal{A}_{\nu(\Sigma)} / \mathcal{G}_{\nu(\Sigma)}^0, \quad \widetilde{\mathcal{B}}_{\nu(\Sigma)}^* = \mathcal{A}_{\nu(\Sigma)}^* / \mathcal{G}_{\nu(\Sigma)}^0.$$

Restricting connections, we have a map

$$\tilde{r}_{\nu(\Sigma)} : \widetilde{\mathcal{B}}_{\bar{E}}^* \longrightarrow \widetilde{\mathcal{B}}_{\nu(\Sigma)}.$$

We have the universal bundle $\widetilde{\mathbb{E}}_{\nu(\Sigma)}$ over $\nu(\Sigma) \times \widetilde{\mathcal{B}}_{\nu(\Sigma)}$ defined by

$$\widetilde{\mathbb{E}}_{\nu(\Sigma)} := (\bar{E}|_{\nu(\Sigma)}) \times_{\mathcal{G}_{\nu(\Sigma)}^0} \mathcal{A}_{\nu(\Sigma)} \longrightarrow \nu(\Sigma) \times \widetilde{\mathcal{B}}_{\nu(\Sigma)}.$$

2.2. Cohomology classes of $\mathcal{B}_P^*$

Suppose Σ is a closed, oriented surface embedded in X such that $\langle w_2(P), [\Sigma]\rangle \equiv 0 \mod 2$. In this subsection, we define a 2-dimensional integral cohomology class $\mu([\Sigma]) \in H^2(\mathcal{B}_P^*;\mathbb{Z})$. Basically we follow a standard construction in [DK, K1].

We first define the cohomology class $\tilde{\mu}_{\bar{E}}([\Sigma]) \in H^2(\widetilde{\mathcal{B}}_{\bar{E}};\mathbb{Z})$ to be the slant product $c_2(\widetilde{\mathbb{E}})/[\Sigma]$.

Lemma 2.1. *Let $\beta : \widetilde{\mathcal{B}}_{\bar{E}}^* \to \mathcal{B}_{\bar{E}}^*$ be the projection. Then the induced homomorphism*

$$\beta^* : H^2(\mathcal{B}_{\bar{E}}^*;\mathbb{Z}) \longrightarrow H^2(\widetilde{\mathcal{B}}_{\bar{E}}^*;\mathbb{Z})$$

is injective. Moreover for a homology class $[\Sigma] \in H_2(X;\mathbb{Z})$ with $\langle w_2(P), [\Sigma]\rangle \equiv 0 \mod 2$, the cohomology class $\tilde{\mu}_{\bar{E}}([\Sigma])$ lies in the image of β^*.

Proof. Since $H^1(SO(3);\mathbb{Z}) = 0$, the spectral sequence associated with the fibration $SO(3) \to \widetilde{\mathcal{B}}_{\bar{E}}^* \to \mathcal{B}_{\bar{E}}^*$ induces an exact sequence

$$0 \longrightarrow H^2(\mathcal{B}_{\bar{E}}^*;\mathbb{Z}) \xrightarrow{\beta^*} H^2(\widetilde{\mathcal{B}}_{\bar{E}}^*;\mathbb{Z}) \longrightarrow H^2(SO(3);\mathbb{Z}), \tag{1}$$

which implies the injectivity of β^*.

Let η be a complex line bundle over $SO(3)$ defined by

$$\eta := SU(2) \times_{\{\pm 1\}} \mathbb{C} \longrightarrow SO(3).$$

Here the action of $\{\pm 1\}$ on $\mathbb{C}$ is the multiplication. Then it is easy to obtain the identification

$$\widetilde{\mathbb{E}}|_{\Sigma \times SO(3)} = (\bar{E}|_\Sigma) \times_{\{\pm 1\}} SU(2) = (\bar{E}|_\Sigma) \boxtimes \eta \longrightarrow \Sigma \times SO(3),$$

and we have

$$\begin{aligned} c_2(\widetilde{\mathbb{E}}|_{\Sigma \times SO(3)})/[\Sigma] &= \big(\pi_1^* c_2(\bar{E}|_\Sigma) + \pi_1^* c_1(\bar{E}|_\Sigma) \cup \pi_2^* c_1(\eta)\big)/[\Sigma] \\ &= \big\langle c_1(\bar{E}), [\Sigma]\big\rangle\, c_1(\eta) \\ &\in H^2(SO(3);\mathbb{Z}) \cong \mathbb{Z}_2, \end{aligned}$$

where

$$\pi_1 : \Sigma \times SO(3) \longrightarrow \Sigma, \quad \pi_2 : \Sigma \times SO(3) \longrightarrow SO(3)$$

are the projections. If $\langle w_2(P), [\Sigma]\rangle$ is zero, the pairing $\left\langle c_1(\bar{E}), [\Sigma]\right\rangle$ is even, and hence the restriction of $c_2(\widetilde{\mathbb{E}})/[\Sigma]$ to $SO(3)$ is trivial. From the exact sequence (1), $\tilde{\mu}_{\bar{E}}([\Sigma])$ is in the image of β^*. □

By Lemma 2.1, there is a unique element of $H^2(\mathcal{B}^*_{\bar{E}}; \mathbb{Z})$ such that the image by β^* is $\tilde{\mu}_{\bar{E}}([\Sigma])$. Through the natural identification between $\mathcal{B}^*_P$ and $\mathcal{B}^*_{\bar{E}}$, we have a 2-dimensional cohomology class of $\mathcal{B}^*_P$. We denote it by $\mu_{\bar{E}}([\Sigma])$.

LEMMA 2.2. *Let X be a closed, oriented, simply connected 4-manifold and P be an $SO(3)$-bundle over X. Suppose that $[\Sigma]$ is a 2-dimensional homology class in X with $\langle w_2(P), [\Sigma]\rangle \equiv 0 \mod 2$. Then the cohomology class $\mu_{\bar{E}}([\Sigma]) \in H^2(\mathcal{B}^*_P; \mathbb{Z})$ is independent of the choice of $\bar{E}$.*

This lemma will be shown in §2.4 as a corollary of Lemma 2.15. Under the assumption in Lemma 2.2, we define $\mu([\Sigma]) \in H^2(\mathcal{B}^*_P; \mathbb{Z})$ as follows.

DEFINITION 2.3. For a homology class $[\Sigma] \in H_2(X, \mathbb{Z})$ with $\langle w_2(P), [\Sigma]\rangle \equiv 0 \mod 2$, the cohomology class $\mu([\Sigma]) \in H^2(\mathcal{B}^*_P; \mathbb{Z})$ is defined to be $\mu_{\bar{E}}([\Sigma])$.

REMARK 2.4. Let

$$\mathbb{P} := P \times_{\mathcal{G}_P} \mathcal{A}^*_P \longrightarrow X \times \mathcal{B}^*_P$$

be the universal bundle of P. Then the usual definition of μ-map is given by

$$\begin{array}{rccc} \mu_{\mathbb{Q}}: & H_2(X;\mathbb{Z}) & \longrightarrow & H^2(\mathcal{B}^*_P;\mathbb{Q}) \\ & [\Sigma] & \longmapsto & -\frac{1}{4}p_1(\mathbb{P})/[\Sigma]. \end{array}$$

In general, $\mu_{\mathbb{Q}}([\Sigma])$ does not have an integral lift. Under our assumptions, it is easy to see that $\mu([\Sigma])$ is an integral lift of $\mu_{\mathbb{Q}}([\Sigma])$.

Next we define a torsion cohomology class $u_1 \in H^1(\mathcal{B}^*_P; \mathbb{Z}_2)$. We write $\sigma(X)$ for the signature of X. Akbulut, Mrowka and Ruan showed the following in [AMR].

Proposition 2.5 ([AMR]). *Let X be a closed, oriented, simply connected 4-manifold and P be an $SO(3)$-bundle over X. Then we have*

$$\pi_1(\mathcal{B}_P^*) = \begin{cases} \mathbb{Z}_2 & \text{if } w_2(P) = w_2(X),\ p_1(P) \equiv \sigma(X) \mod 8 \\ 1 & \text{otherwise.} \end{cases}$$

Remark 2.6. Suppose P is an $SO(3)$-bundle over X with $w_2(P)$ equal to $w_2(X)$ and let $\bar{P}$ be a $U(2)$-lift of P. Then $p_1(P)$ is equal to $\sigma(X)$ modulo 8 if and only if $c_2(\bar{P})$ is equal to 0 modulo 2. This equivalence is a consequence of the formulas

$$p_1(P) = -4c_2(\bar{P}) + c_1(\bar{P})^2, \quad w_2(X)^2 \equiv \sigma(X) \mod 8.$$

When $w_2(P) = w_2(X)$ and $p_1(P) \equiv \sigma(X) \bmod 8$, we have $H^1(\mathcal{B}_P^*; \mathbb{Z}_2) \cong \mathbb{Z}_2$ from Proposition 2.5.

Definition 2.7. Let X be a closed, oriented, simply connected 4-manifold and P be an $SO(3)$-bundle over X satisfying $w_2(P) = w_2(X)$, $p_1(P) \equiv \sigma(X) \mod 8$. We write u_1 for the generator of $H^1(\mathcal{B}_P^*; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

2.3. Construction of $q_X^{u_1}$

Let X be a closed, oriented, simply connected 4-manifold. Suppose $b^+(X) = 2a$ for a positive integer a. Let P be an $SO(3)$-bundle over X. Assume that P satisfies the condition

$$w_2(P) = w_2(X) \in H^2(X; \mathbb{Z}_2), \quad p_1(P) \equiv \sigma(X) \mod 8. \tag{2}$$

The virtual dimension of M_P is given by

$$\dim M_P = -2p_1(P) - 3(1 + b^+(X)) = 8k - 3(1 + 2a).$$

If we put $d = -p_1(P) - 3a - 2 = 4k - 3a - 2$, then we have

$$\dim M_P = 2d + 1.$$

From the condition (2), we have

$$d \equiv -\sigma(X) - 3a - 2 \mod 8.$$

Suppose that $d \geq 0$ and take 2-dimensional homology classes $[\Sigma_1], \dots, [\Sigma_d]$ of X satisfying

$$\langle w_2(P), [\Sigma_i] \rangle \equiv 0 \mod 2 \quad (i = 1, \dots, d).$$

The assumption $\langle w_2(P), [\Sigma_i] \rangle \equiv 0 \mod 2$ is equivalent to $[\Sigma_i] \cdot [\Sigma_i] \equiv 0 \mod 2$ since $w_2(P)$ is equal to $w_2(X)$. We want to define the pairing $\langle u_1 \cup \mu([\Sigma_1]) \cup \dots \cup \mu([\Sigma_d]), M_P \rangle \in \mathbb{Z}_2$. The moduli space M_P is not compact in general and the pairing is not well-defined in the usual sense. To define the pairing, we need submanifolds V_{Σ_i} dual to $\mu([\Sigma_i])$ which behave nicely near the ends of M_P. We briefly explain how the submanifolds are constructed. See [D3, DK] for the details.

We use the following three things. The first is that when $b^+(X)$ and $k = -\frac{1}{4}p_1(P)$ are positive M_P lies in $\mathcal{B}_P^*$ and has a natural smooth structure for generic metrics on X. The second is that the restrictions of irreducible instantons to open subsets are also irreducible. The third is that the cohomology class $\mu([\Sigma])$ comes from $\mathcal{B}_{\nu(\Sigma)}^*$. More precise statement of the third is as follows.

Let $[\Sigma] \in H_2(X; \mathbb{Z})$ be a homology class with $[\Sigma] \cdot [\Sigma] \equiv 0 \mod 2$. Since the following diagram is commutative

$$\begin{array}{ccc} \widetilde{\mathbb{E}}|_{\nu(\Sigma) \times \mathcal{B}_{\bar{E}}^*} = (\bar{E}|_{\nu(\Sigma)}) \times_{\mathcal{G}_{\bar{E}}^0} \mathcal{A}_{\bar{E}}^* & \xrightarrow{\mathrm{id}_{\bar{E}} \times \tilde{r}_{\nu(\Sigma)}} & \widetilde{\mathbb{E}}_{\nu(\Sigma)} = (\bar{E}|_{\nu(\Sigma)}) \times_{\mathcal{G}_{\nu(\Sigma)}^0} \mathcal{A}_{\nu(\Sigma)} \\ \downarrow & & \downarrow \\ \nu(\Sigma) \times \widetilde{\mathcal{B}}_{\bar{E}}^* & \xrightarrow[\mathrm{id}_{\nu(\Sigma)} \times \tilde{r}_{\nu(\Sigma)}]{} & \nu(\Sigma) \times \widetilde{\mathcal{B}}_{\nu(\Sigma)} \end{array}$$

we obtain

$$\tilde{\mu}_{\bar{E}}([\Sigma]) = c_2(\tilde{\mathbb{E}})/[\Sigma] = \tilde{r}_{\nu(\Sigma)}^*(c_2(\widetilde{\mathbb{E}}_{\nu(\Sigma)})/[\Sigma]) \in H^2(\widetilde{\mathcal{B}}_{\bar{E}}^*; \mathbb{Z}). \tag{3}$$

We apply Lemma 2.1 to the restriction of P on $\nu(\Sigma)$, instead of P itself. Then we see that there exists a unique 2-dimensional cohomology class $\mu_{\nu(\Sigma), \bar{E}}([\Sigma])$ of $\mathcal{B}_{\nu(\Sigma)}^*$ such that the pull-back by the natural projection $\widetilde{\mathcal{B}}_{\nu(\Sigma)}^* \to \mathcal{B}_{\nu(\Sigma)}^*$ is equal to $c_2(\widetilde{\mathbb{E}}_{\nu(\Sigma)})/[\Sigma]$.

We define V_Σ as follows.

DEFINITION 2.8. Take a homology class $[\Sigma] \in H_2(X; \mathbb{Z})$ with $[\Sigma] \cdot [\Sigma]$ even. We write $\mathcal{L}_\Sigma$ for a complex line bundle over $\mathcal{B}_{\nu(\Sigma), \bar{E}}^*$ with first Chern class $\mu_{\nu(\Sigma),\bar{E}}([\Sigma]) \in H^2(\mathcal{B}^*_{\nu(\Sigma),\bar{E}};\mathbb{Z})$. Fix a section s_Σ of $\mathcal{L}_\Sigma$. We denote the zero locus of s_Σ by $V_\Sigma \subset \mathcal{B}^*_{\nu(\Sigma)}$. Suppose that $b^+(X)$ and $k = -\frac{1}{4}p_1(P)$ are positive. For a generic metric g, we define

$$M_P \cap V_\Sigma := \{ \ [A] \in M_P \mid [A|_{\nu(\Sigma)}] \in V_\Sigma \ \}.$$

We will show that the pairing $\langle u_1, M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle$ is well-defined under some condition.

REMARK 2.9. We give some remarks on the line bundle $\mathcal{L}_\Sigma$. We refer to [D3, DK] for details.

- As is well-known, we are also able to construct the line bundle $\mathcal{L}_\Sigma$ by using a family of twisted Dirac operators on Σ.
- Assume that $\langle w_2(P), [\Sigma] \rangle$ is equal to 0 modulo 2. Then $P|_{\nu(\Sigma)}$ is topologically trivial. Let $\mathcal{B}^*_{\nu(\Sigma)\ +} := \mathcal{B}^*_{\nu(\Sigma)} \cup \{[\Theta_{\nu(\Sigma)}]\}$. Here $\Theta_{\nu(\Sigma)}$ is the trivial connection on $\nu(\Sigma)$. It is known that $\mathcal{L}_\Sigma$ extends to $\mathcal{B}^*_{\nu(\Sigma)\ +}$. Hence we can assume that the section s_Σ is non-zero near $[\Theta_{\nu(\Sigma)}]$. In the case when $w_2(P)$ is zero, we need this property to define invariants. On the other hand, when we treat an $SO(3)$-bundle P with $w_2(P)$ non-trivial, we do not need this property for the definition of invariants. However we will need this property in Lemma 3.7 to prove some property of our invariant .

We prepare some lemmas. The following is well-known.

LEMMA 2.10 ([D3, DK]). *Let X be a closed, oriented, simply connected 4-manifold with $b^+(X)$ positive and P be a $SO(3)$-bundle with $k = -\frac{1}{4}p_1(P) > 0$. Take homology classes $[\Sigma_1], \ldots, [\Sigma_{d'}] \in H_2(X;\mathbb{Z})$ with self-intersection numbers even. For generic sections s_{Σ_i}, the intersections*

$$M_{k-j,w,X} \cap \left(\bigcap_{i \in I} V_{\Sigma_i} \right) \quad (I \subset \{1, \ldots, d'\},\ 0 \le j < k)$$

are transverse.

From now on, we require that Σ_i are generic in the following sense.

$$\begin{cases} \Sigma_i \pitchfork \Sigma_j & (i, j \text{ distinct}) \\ \Sigma_i \cap \Sigma_j \cap \Sigma_k = \emptyset & (i, j, k \text{ distinct}). \end{cases} \tag{4}$$

LEMMA 2.11. *Let X be a closed, oriented, simply connected, non-spin 4-manifold with $b^+(X)$ positive. Let P be an $SO(3)$-bundle over X with $w_2(P)$ equal to $w_2(X)$. Suppose that the dimension of M_P is $2d' + r$ for a non-negative integer d' and $1 \leq r \leq 3$. Take d' homology classes $[\Sigma_1], \dots, [\Sigma_{d'}] \in H^2(X;\mathbb{Z})$ with*

$$[\Sigma_i] \cdot [\Sigma_i] \equiv 0 \mod 2 \quad (i = 1, \dots, d').$$

Moreover we assume that the surfaces Σ_i satisfy the condition (4). Then for generic sections s_{Σ_i}, the intersection

$$M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$$

is a compact r-dimensional manifold.

PROOF. Put $k = -\frac{1}{4}p_1(P)$, $w = w_2(P)$. For $[A] \in M_P$, we have

$$\begin{aligned} k &= -\frac{1}{4}p_1(P) \\ &= \frac{1}{8\pi^2}\int_X \mathrm{Tr}(F_A^2) \\ &= \frac{1}{8\pi^2}\int_X |F_A^-|^2 d\mu_g - \frac{1}{8\pi^2}\int_X |F_A^+|^2 d\mu_g \\ &= \frac{1}{8\pi^2}\int_X |F_A^-| d\mu_g \geq 0. \end{aligned}$$

by the Chern-Weil theory. Here $d\mu_g$ is the volume form with respect to g. First we show $k > 0$. If not, $k = 0$ and A is flat. Since X is simply connected, A is trivial. This contradicts the assumption that $w_2(P)$ is non-trivial. Hence we have $k > 0$. From Lemma 2.10, $M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$ is a smooth r-dimensional manifold for generic sections s_{Σ_i}.

Next we prove that $M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$ is compact. Let $\{[A^{(n)}]\}_{n\in\mathbb{N}}$ be a sequence in $M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$. Uhlenbeck's weak compactness theorem implies that there is a subsequence $\{[A^{(n')}]\}_{n'}$ which is weakly convergent to

$$([A_\infty]; x_1, \ldots, x_l) \in M_{k-l,w,X} \times X^l.$$

We also have $k-l > 0$ in the same way as above. Let m be the number of the tubular neighborhoods $\nu(\Sigma_i)$ which contain x_α for some α with $1 \leq \alpha \leq l$. Then without loss of generality, we may suppose that

$$[A_\infty] \in M_{k-l,w,X} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'-m}}$$

if we change the order of the surfaces. If we take the tubular neighborhoods $\nu(\Sigma_i)$ to be sufficiently small, we have

$$\nu(\Sigma_i) \cap \nu(\Sigma_j) \cap \nu(\Sigma_k) = \emptyset \quad (i, j, k \text{ distinct})$$

from (4). Hence we have $m \leq 2l$. Since $k-l > 0$, the intersection $M_{k-l,x,X} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'-m}}$ is transverse by Lemma 2.10. From this transversality, we obtain

$$\begin{aligned} 0 &\leq \dim M_{k-l,w,X} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'-m}} \\ &= \dim M_{k,w,X} - 8l - 2(d' - m) \\ &= r - 8l + 2m \\ &\leq r - 4l. \end{aligned}$$

Since we suppose $1 \leq r \leq 3$, we have $l = 0$ and

$$[A_\infty] \in M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}.$$

Hence $M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d'}}$ is compact. □

Let X be as in Lemma 2.11 and P be an $SO(3)$-bundle over X satisfying (2). Suppose that $\dim M_P$ is $2d+1$ for a non-negative integer d and take homology classes $[\Sigma_1], \ldots, [\Sigma_d] \in H_2(X; \mathbb{Z})$ with self-intersection numbers even. From Lemma 2.11, we have the pairing

$$\langle u_1, M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle \in \mathbb{Z}_2.$$

Proposition 2.12. *Let X be a closed, oriented, simply connected, non-spin 4-manifold with $b^+(X) = 2a$ for a positive integer a and P be an $SO(3)$-bundle over X satisfying (2). Assume that the dimension of M_P is $2d+1$ for a non-negative integer d. Then the pairing*

$$\langle u_1, M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle \in \mathbb{Z}_2$$

is independent of the choices of Riemannian metric g, $U(2)$-lift $\bar{P}$ of P, sections s_{Σ_i} of $\mathcal{L}_{\Sigma_i}$ and surfaces Σ_i representing the homology classes $[\Sigma_i]$. Moreover the pairing is multi-linear with respect to $[\Sigma_1], \dots, [\Sigma_d]$.

We prove the above proposition in §2.4. By using this proposition, we can easily show that the following invariant $q_X^{u_1}$ is well defined.

Definition 2.13. Let X be as in Proposition 2.12. Let $A'_d(X)$ be the subspace of $\otimes^d H^2(X;\mathbb{Z})$ generated by

$$\{\ [\Sigma_1] \otimes \cdots \otimes [\Sigma_d] \mid [\Sigma_i] \in H_2(X;\mathbb{Z}), [\Sigma_i] \cdot [\Sigma_i] \equiv 0 \mod 2\ \},$$

and we put

$$A'(X) := \bigoplus_d A'_d(X),$$

where d runs over non-negative integers with $d \equiv -\sigma(X) - 3a - 2 \mod 8$. We define $q_X^{u_1}$ by

$$\begin{array}{rccc} q_X^{u_1}: & A'(X) & \longrightarrow & \mathbb{Z}_2 \\ & ([\Sigma_1], \dots, [\Sigma_d]) & \longmapsto & q_{k,w,X}^{u_1}([\Sigma_1], \dots, [\Sigma_d]) \\ & & := & \langle u_1, M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle . \end{array}$$

Here P is an $SO(3)$-bundle over X with $w_2(P) = w_2(X)$ and $p_1(P) = -d - 3a - 2$.

2.4. Well-definedness of $q_X^{u_1}$

In this subsection, we prove Proposition 2.12. First we show the independence of $q_X^{u_1}$ from Riemannian metric g and sections s_{Σ_i} in a standard way. Take two metrics g, g' on X and sections s_{Σ_i}, s'_{Σ_i} of $\mathcal{L}_{\Sigma_i}$. Choose a path $\{g_t\}_{t\in[0,1]}$ between g and g', and a path $\{s_{\Sigma_i,t}\}_{t\in[0,1]}$ between s_{Σ_i} and s'_{Σ_i}. Then put

$$\mathcal{M} := \coprod_{t\in[0,1]} M_P(g_t) \times \{t\},$$
$$\mathcal{M} \cap \mathcal{V}_{\Sigma_i} := \{ \ ([A],t) \in \mathcal{M} \mid s_{\Sigma_i,t}([A|_{\nu(\Sigma_i)}]) = 0 \ \}.$$

Using a similar argument in the proof of Lemma 2.11, we can show the following lemma:

LEMMA 2.14. *Let X and P be as in Proposition 2.12. Then for generic paths $\{g_t\}_{t\in[0,1]}$ and $\{s_{\Sigma_i,t}\}_{t\in[0,1]}$, the intersection*

$$\mathcal{M} \cap \mathcal{V}_{\Sigma_1} \cap \cdots \cap \mathcal{V}_{\Sigma_d}$$

is a compact 2-dimensional manifold whose boundary is

$$(M_P(g) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}) \coprod (M_P(g') \cap V'_{\Sigma_1} \cap \cdots \cap V'_{\Sigma_d}).$$

This lemma implies

$$\langle u_1, M_P(g) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}\rangle = \left\langle u_1, M_P(g') \cap V'_{\Sigma_1} \cap \cdots \cap V'_{\Sigma_d}\right\rangle \in \mathbb{Z}_2,$$

and the pairing $\langle u_1, M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}\rangle$ is independent of the choices of g and s_{Σ_i}.

Next we see the independence of $q_X^{u_1}$ from the choice of $U(2)$-lift $\bar{P}$ of P . Take two $U(2)$-lifts $\bar{P}$ and $\bar{P}'$ of P. The associated vector bundle $\bar{E}'$ with $\bar{P}'$ is topologically isomorphic to $\bar{E} \otimes L$ for some complex line bundle L over X. Fix connections $a_{\det}$, a_L on $\det \bar{E}$, L and an isomorphism

$$\varphi : \bar{E}' \xrightarrow{\cong} \bar{E} \otimes L.$$

We have a connection $a'_{\det}$ on $\det \bar{E}'$ induced by $a_{\det}, a_L$ and φ. We consider connections on $\bar{E}\otimes L$ and $\bar{E}'$ which are compatible with $a_{\det} + 2a_L$ and $a'_{\det}$ respectively. By tensoring $a_L|_{\nu(\Sigma)}$, we have maps

$$t_{\mathcal{A}} : \mathcal{A}_{\nu(\Sigma),\bar{E}} \xrightarrow{\cong} \mathcal{A}_{\nu(\Sigma),\bar{E}\otimes L}, \quad t_{\mathcal{B}^*} : \mathcal{B}^*_{\nu(\Sigma),\bar{E}} \xrightarrow{\cong} \mathcal{B}^*_{\nu(\Sigma),\bar{E}\otimes L},$$
$$t_{\widetilde{\mathcal{B}}} : \widetilde{\mathcal{B}}_{\nu(\Sigma),\bar{E}} \xrightarrow{\cong} \widetilde{\mathcal{B}}_{\nu(\Sigma),\bar{E}\otimes L}.$$

Moreover the pull-back by φ induces identifications

$$\psi_{\mathcal{B}^*}: \mathcal{B}^*_{\nu(\Sigma),\bar{E}\otimes L} \xrightarrow{\cong} \mathcal{B}^*_{\nu(\Sigma),\bar{E}'}, \quad \psi_{\widetilde{\mathcal{B}}}: \widetilde{\mathcal{B}}_{\nu(\Sigma),\bar{E}\otimes L} \xrightarrow{\cong} \widetilde{\mathcal{B}}_{\nu(\Sigma),\bar{E}'}.$$

LEMMA 2.15. *Suppose* $\mathcal{L}_\Sigma$, $\mathcal{L}'_\Sigma$ *are complex line bundles over* $\mathcal{B}^*_{\nu(\Sigma),\bar{E}}$, $\mathcal{B}^*_{\nu(\Sigma),\bar{E}'}$ *corresponding to the cohomology classes* $\mu_{\nu(\Sigma),\bar{E}}([\Sigma]) \in H^2(\mathcal{B}^*_{\nu(\Sigma),\bar{E}};\mathbb{Z})$, $\mu_{\nu(\Sigma),\bar{E}'}([\Sigma]) \in H^2(\mathcal{B}^*_{\nu(\Sigma),\bar{E}'};\mathbb{Z})$. *Then we have*

$$(\psi_{\mathcal{B}^*}\circ t_{\mathcal{B}^*})^*\mathcal{L}'_\Sigma \cong \mathcal{L}_\Sigma.$$

PROOF. It is sufficient to show that $(\psi_{\tilde{\mathcal{B}}}\circ t_{\tilde{\mathcal{B}}})^*(c_2(\widetilde{\mathbb{E}}'_{\nu(\Sigma)})/[\Sigma])$ is equal to $c_2(\widetilde{\mathbb{E}}_{\nu(\Sigma)})/[\Sigma]$ since $H^2(\mathcal{B}^*_{\nu(\Sigma),\bar{E}};\mathbb{Z}) \to H^2(\widetilde{\mathcal{B}}^*_{\nu(\Sigma),\bar{E}};\mathbb{Z})$ is injective.

Let $\pi_1 : \nu(\Sigma)\times\widetilde{\mathcal{B}}_{\nu(\Sigma),\bar{E}} \to \nu(\Sigma)$ be the projection. We have the following commutative diagram:

$$\begin{array}{ccc}
\widetilde{\mathbb{E}}_{\nu(\Sigma)}\otimes\pi_1^*(L|_{\nu(\Sigma)}) & & \widetilde{\mathbb{E}}'_{\nu(\Sigma)} \\
\| & & \| \\
(\bar{E}\otimes L|_{\nu(\Sigma)})\times_{\mathcal{G}^0_{\nu(\Sigma),\bar{E}}}\mathcal{A}_{\nu(\Sigma),\bar{E}} & \xrightarrow{\varphi^{-1}\times(\varphi^*\,\circ\, t_{\mathcal{A}})} & (\bar{E}'|_{\nu(\Sigma)})\times_{\mathcal{G}^0_{\nu(\Sigma),\bar{E}'}}\mathcal{A}_{\nu(\Sigma),\bar{E}'} \\
\downarrow & & \downarrow \\
\nu(\Sigma)\times\widetilde{\mathcal{B}}_{\nu(\Sigma),\bar{E}} & \xrightarrow[\mathrm{id}_{\nu(\Sigma)}\times(\psi_{\mathcal{B}}\,\circ\, t_{\tilde{\mathcal{B}}})]{} & \nu(\Sigma)\times\widetilde{\mathcal{B}}_{\nu(\Sigma),\bar{E}'}
\end{array}$$

Hence we have

$$\left(\mathrm{id}_{\nu(\Sigma)}\times(\psi_{\tilde{\mathcal{B}}}\circ t_{\tilde{\mathcal{B}}})\right)^*\,\widetilde{\mathbb{E}}'_{\nu(\Sigma)} \cong \widetilde{\mathbb{E}}_{\nu(\Sigma)}\otimes\pi_1^*(L|_{\nu(\Sigma)})$$

and we obtain

$$\begin{aligned}
&(\psi_{\tilde{\mathcal{B}}}\circ t_{\tilde{\mathcal{B}}})^*(c_2(\widetilde{\mathbb{E}}'_{\nu(\Sigma)})/[\Sigma]) \\
&\quad = c_2(\widetilde{\mathbb{E}}_{\nu(\Sigma)}\otimes\pi_1^*(L|_{\nu(\Sigma)}))/[\Sigma] \\
&\quad = \left\{c_2(\widetilde{\mathbb{E}}_{\nu(\Sigma)}) + \pi_1^*c_1(L|_{\nu(\Sigma)})\cup c_1(\widetilde{\mathbb{E}}_{\nu(\Sigma)}) + \pi_1^*c_1(L|_{\nu(\Sigma)})^2\right\}/[\Sigma] \\
&\quad = c_2(\widetilde{\mathbb{E}}_{\nu(\Sigma)})/[\Sigma] + \left\{\pi_1^*c_1(L|_{\nu(\Sigma)})\cup c_1(\widetilde{\mathbb{E}}_{\nu(\Sigma)})\right\}/[\Sigma] \\
&\quad \in H^2(\widetilde{\mathcal{B}}_{\bar{E}};\mathbb{Z}).
\end{aligned} \tag{5}$$

By the Künneth formula, we can write

$$\begin{aligned} c_1(\widetilde{\mathbb{E}}_{\nu(\Sigma)}) &= c_1(\widetilde{\mathbb{E}}_{\nu(\Sigma)})_{\nu(\Sigma)} + c_1(\widetilde{\mathbb{E}}_{\nu(\Sigma)})_{\widetilde{\mathcal{B}}} \\ &\in H^2(\nu(\Sigma) \times \widetilde{\mathcal{B}}_{\nu(\Sigma),\bar{E}}; \mathbb{Z}) \cong H^2(\nu(\Sigma); \mathbb{Z}) \oplus H^2(\widetilde{\mathcal{B}}_{\nu(\Sigma),\bar{E}}; \mathbb{Z}) \end{aligned}$$

since $\widetilde{\mathcal{B}}_{\nu(\Sigma),\bar{E}}$ is simply connected ([AB]). The action of $\mathcal{G}^0_{\nu(\Sigma),\bar{E}}$ on $\Lambda^2\bar{E}|_{\nu(\Sigma)}$ is trivial, since the determinants of elements of $\mathcal{G}^0_{\nu(\Sigma),\bar{E}}$ are equal to 1 by definition. Hence $\Lambda^2\widetilde{\mathbb{E}}_{\nu(\Sigma)}$ is the pull-back $\pi_1^*(\Lambda^2\bar{E}|_{\nu(\Sigma)})$. This implies that the $\widetilde{\mathcal{B}}_{\nu(\Sigma)}$-part $c_1(\widetilde{\mathbb{E}}_{\nu(\Sigma)})_{\widetilde{\mathcal{B}}}$ of $c_1(\widetilde{\mathbb{E}}_{\nu(\Sigma)}) = c_1(\Lambda^2\widetilde{\mathbb{E}}_{\nu(\Sigma)})$ is 0 and we have

$$\begin{aligned} \left\{\pi_1^* c_1(L|_{\nu(\Sigma)}) \cup c_1(\widetilde{\mathbb{E}}_{\nu(\Sigma)})\right\}/[\Sigma] &= \left\{\pi_1^* c_1(L|_{\nu(\Sigma)}) \cup c_1(\widetilde{\mathbb{E}}_{\nu(\Sigma)})_{\nu(\Sigma)}\right\}/[\Sigma] \\ &= 0 \in H^2(\widetilde{\mathcal{B}}_{\nu(\Sigma)}; \mathbb{Z}). \end{aligned}$$

From the equation (5), we obtain

$$(\psi_{\widetilde{\mathcal{B}}} \circ t_{\widetilde{\mathcal{B}}})^*(c_2(\widetilde{\mathbb{E}}'_{\nu(\Sigma)})/[\Sigma]) = c_2(\widetilde{\mathbb{E}}_{\nu(\Sigma)})/[\Sigma] \in H^2(\widetilde{\mathcal{B}}_{\nu(\Sigma)}; \mathbb{Z}). \qquad (6) \quad \Box$$

Proof of Lemma 2.2. Lemma 2.2 follows from (6) and the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{B}}_{\nu(\Sigma),\bar{E}} & \xrightarrow[\psi_{\mathcal{B}}\circ t_{\mathcal{B}}]{\cong} & \widetilde{\mathcal{B}}_{\nu(\Sigma),\bar{E}'} \\ {\scriptstyle \tilde{r}_{\nu(\Sigma)}}\uparrow & & \uparrow{\scriptstyle \tilde{r}_{\nu(\Sigma)}} \\ \widetilde{\mathcal{B}}^*_{X,\bar{E}} & \xrightarrow{\cong} & \widetilde{\mathcal{B}}^*_{X,\bar{E}'} \\ \downarrow & & \downarrow \\ \mathcal{B}^*_{X,\bar{E}} & \xrightarrow{\cong} & \mathcal{B}^*_{X,\bar{E}'} \\ & {\scriptstyle \cong}\searrow \quad \swarrow{\scriptstyle \cong} & \\ & \mathcal{B}^*_{X,P} & \end{array} \qquad \Box$$

Proof of independence of $q_X^{u_1}$ from $\bar{P}$. Take homology classes $[\Sigma_i] \in H_2(X;\mathbb{Z})$ with $[\Sigma_i]\cdot[\Sigma_i] \equiv 0 \mod 2$ for $i = 1, \ldots, d$ and choose $U(2)$-lifts $\bar{P}$ and $\bar{P}'$ of P. Then we obtain line bundles $\mathcal{L}_{\Sigma_i}$ and $\mathcal{L}'_{\Sigma_i}$ over $\mathcal{B}^*_{\nu(\Sigma_i),\bar{E}}$

and $\mathcal{B}^*_{\nu(\Sigma_i),\bar{E}'}$. We denote the zero locus of sections s_{Σ_i}, s'_{Σ_i} of $\mathcal{L}_{\Sigma_i}$, $\mathcal{L}'_{\Sigma_i}$ by V_{Σ_i}, V'_{Σ_i}. By Lemma 2.15, $(\psi_{\mathcal{B}^*} \circ t_{\mathcal{B}^*})^* \mathcal{L}'_{\Sigma_i}$ is isomorphic to $\mathcal{L}_{\Sigma_i}$. We fix an isomorphism and regard the section s'_{Σ_i} of $\mathcal{L}'_{\Sigma_i}$ as a sections of $\mathcal{L}_{\Sigma_i}$ through the identifications

$$\psi_{\mathcal{B}^*} \circ t_{\mathcal{B}^*} : \mathcal{B}^*_{\nu(\Sigma_i),\bar{E}} \xrightarrow{\cong} \mathcal{B}^*_{\nu(\Sigma_i),\bar{E}'}, \quad (\psi_{\mathcal{B}^*} \circ t_{\mathcal{B}^*})^* \mathcal{L}'_{\Sigma_i} \cong \mathcal{L}_{\Sigma_i}.$$

We take paths $\{s_{\Sigma_i,t}\}_{t \in [0,1]}$ between s_{Σ_i} and s'_{Σ_i}. In the same way as Lemma 2.14, we have a bordism between $M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$ and $M_P \cap V'_{\Sigma_1} \cap \cdots \cap V'_{\Sigma_d}$. Hence we obtain

$$\langle u_1, M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \rangle = \left\langle u_1, M_P \cap V'_{\Sigma_1} \cap \cdots \cap V'_{\Sigma_d} \right\rangle \in \mathbb{Z}_2. \ \square$$

Lastly we show that $q_X^{u_1}$ is independent of the choice of surfaces Σ_i representing the homology classes $[\Sigma_i]$ and that $q_X^{u_1}$ is multi-linear with respect to $[\Sigma_1], \ldots, [\Sigma_d]$. It follows from the following lemma directly.

LEMMA 2.16. *Let X and P be as in Proposition 2.12. Take homology classes $[\Sigma_1], \ldots, [\Sigma_d] \in H_2(X;\mathbb{Z})$ with self-intersection numbers even. Moreover assume that*

$$[\Sigma_1] = [\Sigma'_1] + [\Sigma''_1] \in H_2(X;\mathbb{Z}), \quad [\Sigma'_1] \cdot [\Sigma'_1] \equiv [\Sigma''_1] \cdot [\Sigma''_1] \equiv 0 \mod 2.$$

Then we have

$$\begin{aligned} &\langle u_1, M_P \cap V_{\Sigma_1} \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d} \rangle \\ &\quad = \left\langle u_1, M_P \cap V_{\Sigma'_1} \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d} \right\rangle \\ &\qquad + \left\langle u_1, M_P \cap V_{\Sigma''_1} \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d} \right\rangle \in \mathbb{Z}_2. \end{aligned}$$

PROOF. By definition, we have

$$\begin{aligned} \tilde{\mu}_{\bar{E}}([\Sigma_1]) &= c_2(\tilde{\mathbb{E}})/[\Sigma_1] = c_2(\tilde{\mathbb{E}})/[\Sigma'_1] + c_2(\tilde{\mathbb{E}})/[\Sigma''_1] \\ &= \tilde{\mu}_{\bar{E}}([\Sigma'_1]) + \tilde{\mu}_{\bar{E}}([\Sigma''_1]) \in H^2(\widetilde{\mathcal{B}}_{\bar{E}};\mathbb{Z}). \end{aligned}$$

The homomorphism $\beta^* : H^2(\mathcal{B}^*_{\bar{E}};\mathbb{Z}) \to H^2(\widetilde{\mathcal{B}}^*_{\bar{E}};\mathbb{Z})$ is injective and $\tilde{\mu}_{\bar{E}}([\Sigma_1])$, $\tilde{\mu}_{\bar{E}}([\Sigma_1'])$, $\tilde{\mu}_{\bar{E}}([\Sigma_1''])$ lie in the image β^* from Lemma 2.1. Hence we have

$$\mu([\Sigma_1]) = \mu([\Sigma_1']) + \mu([\Sigma_1'']) \in H^2(\mathcal{B}^*_P;\mathbb{Z}).$$

Since $M_P \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d}$ is compact from Lemma 2.11, we have

$$\begin{aligned}
&\langle u_1, M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}\rangle \\
&\quad = \langle u_1 \cup \mu([\Sigma_1]), M_P \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d}\rangle \\
&\quad = \left\langle u_1 \cup (\mu([\Sigma_1']) + \mu([\Sigma_1''])), M_P \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d}\right\rangle \\
&\quad = \left\langle u_1 \cup \mu([\Sigma_1']), M_P \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d}\right\rangle \\
&\qquad + \left\langle u_1 \cup \mu([\Sigma_1'']), M_P \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d}\right\rangle \\
&\quad = \left\langle u_1, M_P \cap V_{\Sigma_1'} \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d}\right\rangle \\
&\qquad + \left\langle u_1, M_P \cap V_{\Sigma_1''} \cap V_{\Sigma_2} \cap \cdots \cap V_{\Sigma_d}\right\rangle . \ \square
\end{aligned}$$

3. A Connected Sum Formula for $Y\#S^2 \times S^2$

3.1. Statement of the result

As is well known Donaldson invariants vanish for the connected sum $X_1\#X_2$ provided $b^+(X_i) > 0$ for $i = 1, 2$ ([D3]). In [FS], however, Fintushel and Stern defined some torsion invariants by using instantons on $SU(2)$-bundles and they showed that their $SU(2)$-torsion invariants are non-trivial for the connected sum of the form $Y\#S^2 \times S^2$. In this section, we show a similar non-vanishing theorem for our $SO(3)$-torsion invariants.

Let Y be a closed, oriented, simply connected, non-spin 4-manifold with $b^+(Y) = 2a - 1$ for $a > 1$. Let Q be an $SO(3)$-bundle with $w_2(Q)$ equal to $w_2(Y)$ and $p_1(Q)$ equal to $\sigma(Y) + 4$ modulo 8. Suppose that the dimension of M_Q is $2d$ for a non-negative integer d. When we fix an orientation on the space $\mathcal{H}^+_g(Y)$ of self-dual harmonic 2-forms on Y and an lift $c \in H^2(Y;\mathbb{Z})$ of $w_2(Q) \in H^2(Y;\mathbb{Z}_2)$, we have the Donaldson invariant

$$q_{k-1,w,Y} : \otimes^d H_2(Y;\mathbb{Z}) \longrightarrow \mathbb{Q}$$

where

$$k - 1 = -\frac{1}{4}p_1(Q) \in \mathbb{Q}, \quad w = w_2(Q) \in H^2(Y;\mathbb{Z}_2).$$

When $[\Sigma_i] \cdot [\Sigma_i]$ are even for $i = 1, \ldots, d$, then $q_{k-1,w,Y}([\Sigma_1], \ldots, [\Sigma_d])$ is in $\mathbb{Z}$.

We consider an $SO(3)$-bundle P over $X = Y \# S^2 \times S^2$ satisfying

$$w_2(P) = w_2(X), \quad p_1(P) = p_1(Q) - 4,$$

so that P satisfies (2). The dimension of M_P is given by $2d + 5$.

We define surfaces Σ, Σ' embedded in $S^2 \times S^2$ by

$$\Sigma = S^2 \times \{pt\}, \quad \Sigma' = \{pt\} \times S^2 \subset S^2 \times S^2.$$

Then we have

$$[\Sigma] \cdot [\Sigma] \equiv [\Sigma'] \cdot [\Sigma'] \equiv 0 \mod 2.$$

Now $q^{u_1}_{k,w,Y\#S^2\times S^2}([\Sigma_1], \ldots, [\Sigma_d], [\Sigma], [\Sigma'])$ is defined for homology classes $[\Sigma_i]$ of Y with self-intersection numbers even. The following is an $SO(3)$-version of Theorem 1.1 in [FS].

THEOREM 3.1. *In the above situation, we have*

$$q^{u_1}_{k,w,Y\#S^2\times S^2}([\Sigma_1], \ldots, [\Sigma_d], [\Sigma], [\Sigma']) \equiv q_{k-1,w,Y}([\Sigma_1], \ldots, [\Sigma_d]) \mod 2.$$

The proof is given in the following three subsections.

3.2. Notations and general facts

For the proof of Theorem 3.1, we will investigate the intersection $M_P \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_{\Sigma'} \cap V_{\Sigma}$ when the neck of $Y \# S^2 \times S^2$ is very long. For the preparation, we define some notations and recall some facts about instantons over the connected sum of 4-manifolds.

Let Y_1 and Y_2 be a closed, oriented 4-manifold. The connected sum $X = Y_1 \# Y_2$ is constructed in the following way. Fix Riemannian metrics g_1 and g_2 on Y_1 and Y_2 which are flat in small neighborhoods of fixed points $y_1 \in Y_1$ and $y_2 \in Y_2$. For $N > 1$ and $\lambda > 0$ with $N\lambda^{\frac{1}{2}} \ll 1$, we put

$$\Omega_i = \Omega_{y_i}(\lambda, N) = \{y \in Y_i | N^{-1}\lambda^{\frac{1}{2}} < d(y, y_i) < N\lambda^{\frac{1}{2}}\} \quad (i = 1, 2).$$

Let

$$\sigma : (TY_1)_{y_1} \xrightarrow{\cong} (TY_2)_{y_2}.$$

be an orientation-reversing linear isometry. For each positive real number $\lambda > 0$, we define

$$\begin{array}{rccc} f_\lambda: & (TY_1)_{y_1}\backslash\{0\} & \longrightarrow & (TY_2)_{y_2}\backslash\{0\} \\ & \xi & \longmapsto & \frac{\lambda}{|\xi|^2}\sigma(\xi). \end{array}$$

This map f_λ induces a diffeomorphism between Ω_1 and Ω_2. The connected sum X of Y_1 and Y_2 is identified with

$$X(\lambda) = (Y_1\backslash B_{y_1}(N^{-1}\lambda^{\frac{1}{2}}))\bigcup_{f_\lambda}(Y_2\backslash B_{y_2}(N^{-1}\lambda^{\frac{1}{2}}))$$

where $B_{y_i}(N^{-1}\lambda^{\frac{1}{2}})$ is the open ball centered on y_i with radius $N^{-1}\lambda^{\frac{1}{2}}$. The metrics g_1 and g_2 define a conformal structure on X since g_i is flat in a small neighborhood of y_i. We fix a metric g_λ on X which represents the conformal structure. Moreover we assume that g_λ is equal to g_i on $Y_i\backslash B((N+1)\lambda^{\frac{1}{2}})$.

Definition 3.2. Fix a real number q with $q > 4$. Let $[A^{(n)}] \in M_P(g_{\lambda_n})$ be instantons over $X = Y_1\#Y_2$ for a sequence $\lambda_n \to 0$. Let $z_1, \dots, z_l$ be points in $Y_1\backslash\{y_1\}$, $z'_1, \dots, z'_m$ be points in $Y_2\backslash\{y_2\}$ and A_i be connections over Y_i. Then we say that $[A^{(n)}]$ is weakly convergent to $([A_1], [A_2]; z_1, \dots, z_l, z'_1, \dots, z'_m)$ when $[A^{(n)}]$ is L^q-convergent to $([A_1], [A_n])$ over compact subsets in $(Y_1 \cup Y_2)\backslash\{y_1, y_2, z_1, \dots, z_l, z'_1, \dots, z'_m\}$ and $|F_{A^{(n)}}|^2$ is convergent as measure to

$$|F_{A_1}|^2 + |F_{A_2}|^2 + 8\pi^2\left(\sum_{\nu=1}^{l}\delta_{z_\nu} + \sum_{\nu=1}^{m}\delta_{z'_\nu}\right)$$

over compact subsets in $(Y_1\backslash\{y_1\}) \cup (Y_2\backslash\{y_2\})$. Here δ_z is the delta function supported on z.

We use the following well-known theorem.

Theorem 3.3 ([D3, DK]). *Let P be an $SO(3)$-bundle over $X = Y_1\#Y_2$. Set $k = -p_1(P)/4$, $w = w_2(P)$, $w_i = w|_{Y_i}$. Let $[A^{(n)}] \in M_{k,w,X}(\lambda_n)$ be instantons over X for $\lambda_n \to 0$. Then there is a subsequence $\{[A^{(n')}]\}_{n'}$ which is weakly convergent to $([A_1], [A_2]; z_1, \dots, z_l, z'_1, \dots, z'_m)$ for some*

$$[A_1] \in M_{k_1,w_1,Y_1}(g_1),\ [A_2] \in M_{k_2,w_2,Y_2}(g_2),$$
$$z_1, \dots, z_l \in Y_1\backslash\{y_1\},\ z'_1, \dots, z'_m \in Y_2\backslash\{y_2\}$$

with

$$k_1 \geq 0, \quad k_2 \geq 0, \quad k_1 + k_2 + l + m \leq k.$$

Next we review gluing of instantons. The theory of gluing of instantons is standard. To fix notations, we recall the theory briefly.

Let A_i be instantons over Y_i. We denote the $SO(3)$-bundles carrying A_i by P_i. We can construct instantons on $X = Y_1 \# Y_2$ close to A_i on each factor. Outline of the construction is as follows. (See [DK] Chapter 7 for details.)

Let b be a small positive number with $b \geq 4N\lambda^{\frac{1}{2}}$. By using suitable cut-off functions and trivializations of P_i on neighborhoods of y_i, we obtain a connections A_i' which are flat over the annuli Ω_i and equal to A_i outside the balls centered at y_i with radius b. Take an $SO(3)$-isomorphism ρ between $(P_1)_{y_1}$ and $(P_2)_{y_2}$. We can spread this isomorphism by using flat structures of A_i', and obtain an isomorphism g_ρ between $P_1|_{\Omega_1}$ and $P_2|_{\Omega_2}$ covering f_λ. We define an $SO(3)$-bundle P_ρ over X and a connection $A'(\rho) = A_1' \#_\rho A_2'$ on P_ρ by gluing P_i, A_i through g_ρ. Then in large region outside the neck of X, $A'(\rho)$ satisfies the instanton equation, and $F^+_{A'(\rho)}$ is very small near the neck. To obtain a genuine instanton we have to perturb $A'(\rho)$. We consider the equation

$$F^+_{A'(\rho)+a} = 0 \tag{7}$$

for $a \in \Omega^1_X(\mathfrak{g}_{P_\rho})$. To solve this equation, we take linear maps

$$\sigma_i : H^2_{A_i} \longrightarrow \Omega^+_{Y_i}(\mathfrak{g}_{P_i})$$

such that $d^+_{A_i} \oplus \sigma_i$ are surjective and for each $h_i \in H^2_{A_i}$ the supports of $\sigma_i(h_i)$ are in the complement of the ball centered at y_i with radius b. Then put

$$\sigma := \sigma_1 + \sigma_2 : H^2_{A_1} \oplus H^2_{A_2} \longrightarrow \Omega^+_X(\mathfrak{g}_{P_\rho}).$$

We can construct a right inverse of $d^+_{A'(\rho)} + \sigma$ starting from right inverses of $d^+_{A_i} + \sigma_i$. Decompose the right inverse as $P \oplus \pi$, where

$$P : \Omega^+_X(\mathfrak{g}_{P_\rho}) \longrightarrow \Omega^1(\mathfrak{g}_{P_\rho}), \quad \pi : \Omega^+_X(\mathfrak{g}_{P_\rho}) \longrightarrow H^2_{A_1} \oplus H^2_{A_2}.$$

Instead of (7), we first consider the equation

$$F^+_{A'(\rho)+a} + \sigma(h) = 0$$

for $(a, h) \in \Omega^1_X(\mathfrak{g}_{P_\rho}) \times (H^2_{A_1} \oplus H^2_{A_2})$. We find a solution of this equation in the form $a = P\xi$, $h = \pi\xi$. In this case, we see that the equation is equivalent to the equation

$$\xi + (P\xi \wedge P\xi)^+ = -F^+_{A'(\rho)}$$

by a short calculation. Using the contraction mapping principle, we can show that there is a unique small solution $\xi_\rho \in \Omega^+(\mathfrak{g}_{P_\rho})$ for the equation. We get a genuine instanton if and only if $\pi\xi_\rho = 0$. Therefore there is a map

$$\Psi : Gl_{y_1,y_2} \longrightarrow H^2_{A_1} \times H^2_{A_2}$$

such that the solutions of $\Psi = 0$ represent instantons over X. Here Gl_{y_1,y_2} is the space of $SO(3)$-equivariant isomorphisms between $(P_1)_{y_1}$ and $(P_2)_{y_2}$. We fix an element $\rho_0 \in Gl_{y_1,y_2}$ to identify Gl_{y_1,y_2} with $SO(3)$.

We can include the deformations of $[A_i]$ to this construction. For small neighborhoods U_{A_i} of 0 in $H^1_{A_i}$, we have a map

$$\Psi : T := U_{A_1} \times U_{A_2} \times SO(3) \longrightarrow H^2_{A_1} \times H^2_{A_2}$$

such that elements of $\Psi^{-1}(0)$ correspond to instantons.

Let Γ_{A_i} be the isotropy group of A_i in the gauge group and put $\Gamma = \Gamma_{A_1} \times \Gamma_{A_2}$. We assume that U_{A_i} is Γ_{A_i}-invariant. Then there are natural actions of Γ on T and on $H^2_{A_1} \times H^2_{A_2}$. We can show that Ψ is Γ-equivariant and instantons corresponding to elements of $\Psi^{-1}(0)$ are gauge equivalent to each other if and only if they are in the same Γ-orbit. Hence we can regard $\Psi^{-1}(0)/\Gamma$ as a subspace of M_P.

An important feature is that instantons over $X = Y_1 \# Y_2$ which is close to A_i over Y_i are given in the above description. More precise statement is the following:

Let Y_i'' be the complement of balls centered at y_i with radius $\lambda^{\frac{1}{2}}/2$. Take instantons A_i over Y_i and a positive number $\nu > 0$. Then put

(8) $$U_\lambda(\nu) := \{\ [A] \in \mathcal{B}^*_X \mid d_q([A|_{Y_i''}], [A_i|_{Y_i''}]) < \nu,\ i = 1, 2\ \}.$$

Here q is the fixed real number with $q > 4$ and d_q is the distance induced by L^q-norm over Y_i''. If $\nu > 0$ is small, then there is a positive number $\lambda(\nu) > 0$ such that for $\lambda < \lambda(\nu)$ we can take a neighborhood T of $\{0\} \times \{0\} \times SO(3)$ in $H^1_{A_1} \times H^1_{A_2} \times SO(3)$ such that $M_P(g_\lambda) \cap U_\lambda(\nu)$ is homeomorphic to $\Psi^{-1}(0)/\Gamma$. Summing up these:

THEOREM 3.4. *Let A_1, A_2 be instantons on Y_1, Y_2. Then there is a $\Gamma = \Gamma_{A_1} \times \Gamma_{A_2}$-invariant neighborhood T of $SO(3) \times \{0\} \times \{0\}$ in $SO(3) \times H^1_{A_1} \times H^1_{A_2}$ and Γ-equivariant map*

$$\Psi : T \longrightarrow H^2_{A_1} \times H^2_{A_2}$$

such that $\Psi^{-1}(0)/\Gamma$ is homeomorphic to an open set N in M_P. Moreover for a small positive number $\nu > 0$, there is a $\lambda(\nu) > 0$ and T such that if $\lambda < \lambda(\nu)$ then $N = M_P(g_\lambda) \cap U_\lambda(\nu)$.

In particular, when Y_2 is S^4 and A_2 is the fundamental instanton J with instanton number one, we have:

COROLLARY 3.5. *Let A_1 be an instanton over Y_1 and A_2 be the fundamental instanton J over S^4. For a small positive number $\nu > 0$, there is a positive number $\lambda_0 > 0$ and a neighborhood U_{A_1} of 0 in $H^1_{A_1}$, a neighborhood U_0 of 0 in $S^4 = \mathbb{R}^4 \cup \{\infty\}$ and $\Gamma = \Gamma_{A_1}$-equivariant map*

$$\Psi : U_{A_1} \times U_0 \times (0, \lambda_0) \times SO(3) \longrightarrow H^2_{A_1}$$

such that $\Psi^{-1}(0)/\Gamma$ is naturally homeomorphic to $M_P \cap U_{\lambda_0}(\nu)$.

REMARK 3.6. We can generalize the statements of Theorem 3.4 and Corollary 3.5 to the case of gluing 3 or more instantons.

3.3. Shrinking the neck

In the situation of Theorem 3.1, we investigate

$$M_P(g_\lambda) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_\Sigma \cap V_{\Sigma'}$$

as λ tends to 0. We use the notations in §3.2.

Let Y_1 be a closed, oriented, simply connected, non-spin 4-manifold with $b^+(Y_1) = 2a - 1$ with $a > 1$ and we write Y_2 for $S^2 \times S^2$. Let P be an

$SO(3)$-bundle over $X = Y_1 \# Y_2$ satisfying (2). Assume that the virtual dimension of M_P is $2d+5$ for a non-negative integer d. Take homology classes $[\Sigma_1], \dots, [\Sigma_d] \in H_2(Y_1;\mathbb{Z})$ with $[\Sigma_i]\cdot[\Sigma_i] \equiv 0 \mod 2$. Set $\Sigma = S^2 \times \{pt\}, \Sigma' = \{pt\} \times S^2 \subset Y_2$. Take instantons

$$[A^{(n)}] \in M_P(g_{\lambda_n}) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_{\Sigma} \cap V_{\Sigma'}$$

for a sequence $\lambda_n \to 0$. By Theorem 3.3, a subsequence of $\{[A^{(n)}]\}_n$ is weakly convergent to some

$$([A_1], [A_2]; z_1, \dots, z_l, z'_1, \dots, z'_m),$$

where

$$[A_1] \in M_{k_1,w,Y_1}(g_1),\ [A_2] \in M_{k_2,Y_2}(g_2),$$
$$z_1, \dots, z_l \in Y_1 \backslash \{y_1\},\ z'_1, \dots, z'_m \in Y_2 \backslash \{y_2\}.$$

Lemma 3.7. *In the above situation, we have*

$$k_1 = k-1,\ l=0,\ [A_1] \in M_{k-1,w,Y_1}(g_1) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d},$$
$$m=1, \quad z'_1 \in \nu(\Sigma) \cap \nu(\Sigma'), \quad [A_2] = [\Theta_{Y_2}].$$

Here Θ_{Y_2} is the trivial connection on Y_2.

Proof. From Theorem 3.3, we have

$$k_1 + k_2 + l + m \le k. \tag{9}$$

Let p be the number of $\nu(\Sigma_i)$ which contain some point z_α and q be the number of $\nu(\Sigma)$, $\nu(\Sigma')$ which contain some point z'_α. Then by the transversality condition (4), we have

$$0 \le p \le 2l, \quad 0 \le q \le 2m. \tag{10}$$

Without loss of generality, we may assume

$$[A_1] \in M_{k_1,w,Y_1} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d-p}}$$

if we change the order of surfaces. Since $w_2(P)|_{Y_1}$ is non-trivial, we can show $k_1 > 0$ in the same way as the proof of Lemma 2.11. For generic sections, the intersection $M_{k_1,w,Y_1} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_{d-p}}$ is transverse by Lemma 2.10. Hence we have

$$2(d-p) \leq \dim M_{k_1,w,Y_1}. \tag{11}$$

We would like to show $k_2 = 0$. Suppose that k_2 is positive. Then we also obtain

$$2(2-q) \leq \dim M_{k_2,Y_2}. \tag{12}$$

By index theorem, there is the formula

$$\dim M_{k_1,w,Y_1} + \dim M_{k_2,Y_2} + 3 = \dim M_{k_1+k_2,w,X}. \tag{13}$$

From (9), (11), (12) and (13), we have

$$\begin{aligned} 2(d-p) + 2(2-q) + 3 &\leq \dim M_{k_1+k_2,w,X} \\ &\leq \dim M_{k,w,X} - 8(l+m) = 2d + 5 - 8(l+m). \end{aligned}$$

This inequality and (10) imply

$$8(l+m) + 2 \leq 2p + 2q \leq 4(l+m).$$

We have a contradiction. Hence k_2 is 0 which implies that $[A_2]$ is the class of trivial flat connection $[\Theta_{Y_2}]$.

Since k_2 is 0, the virtual dimension of M_{0,Y_2} is -6. From (13), we have

$$\dim M_{k_1,w,Y_1} - 3 = \dim M_{k_1,w,X}. \tag{14}$$

By (9), (10),(11) and (14), we have

$$\begin{aligned} 2(d-2l) - 3 \leq 2(d-p) - 3 &\leq \dim M_{k_1,w,Y_1} - 3 \\ &= \dim M_{k_1,w,X} \leq \dim M_{k,w,X} - 8(l+m). \end{aligned}$$

Therefore we obtain

$$4l + 8m \leq 8.$$

In particular, we have $m \leq 1$. We show $m = 1$. Suppose $m = 0$, then we have $[\Theta_{Y_2}] \in V_\Sigma, [\Theta_{Y_2}] \in V_{\Sigma'}$. To obtain a contradiction, we need to choose V_Σ and $V_{\Sigma'}$ in a specific way. As mentioned in Remark 2.9, we can choose

V_Σ and $V_{\Sigma'}$ do not include $[\Theta_{Y_2}]$. If we choose such V_Σ and $V_{\Sigma'}$, we have a contradiction. We obtain $l = 0$, $m = 1$ and $z_1' \in \nu(\Sigma) \cap \nu(\Sigma')$. Hence

$$[A_1] \in M_{k_1,w,Y_1} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}.$$

Lastly we show $k_1 = k - 1$. From (9), we have $k_1 \leq k - 1$. On the other hand, from (11) we have

$$2d \leq \dim M_{k_1,w,Y_1} = \dim M_{k-1,w,Y_1} - 8(k - 1 - k_1) = 2d - 8(k - 1 - k_1).$$

This implies $k_1 \geq k - 1$. Therefore k_1 is equal to $k - 1$. We complete the proof. □

Let w_0' be the unique intersection point of Σ and Σ'. Fix a small neighborhood $U_{w_0'}$ of w_0' with $\nu(\Sigma) \cap \nu(\Sigma') \subset U_{w_0'}$. We suppose that the metric g_2 on Y_2 is flat on $U_{w_0'}$ for simplicity.

Take

$$[A^{(n)}] \in M_P(g_{\lambda_n}) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_\Sigma \cap V_{\Sigma'}$$

for $\lambda_n \to 0$ and assume that $\{[A^{(n)}]\}_{n\in\mathbb{N}}$ weakly converges to $([A_1], [\Theta_{Y_2}]; z_1')$ for some $[A_1] \in M_{k-1,w,Y_1} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$, $z_1' \in \nu(\Sigma) \cap \nu(\Sigma')$. We can define the local center of mass $c_n \in U_{w_0'}$ and scale $\lambda_n' > 0$ of $[A^{(n)}]$ around z_1' when n is sufficiently large. If n is large enough, then we obtain

$$\int_{U_{w_0'}} |F_{A^{(n)}}|^2 d\mu_{g_2} > 4\pi^2$$

since $|F_{A^{(n)}}|^2$ converges to $8\pi^2\delta_{z_1'}$ on $U_{w_0'}$. We define the center of mass c_n to be the center of the smallest ball in $U_{w_0'}$ where the integral of $|F_{A^{(n)}}|^2$ is equal to $4\pi^2$ and the scale λ_n' to be the radius of the ball. The center of mass and scale is determined uniquely ([D1]). The center c_n converges to z_1' and the scale λ_n' converges to 0.

Let $m : \mathbb{R}^4 \to S^4 = \mathbb{R}^4 \cup \{\infty\}$ be the stereographic map and $d_\lambda : \mathbb{R}^4 \to \mathbb{R}^4$ be the map $d_\lambda(y) = \lambda^{-1}y$. Put $\chi_n := m \circ d_{\lambda_n'}$. Then χ_n induces a conformal isomorphism between X and the connected sum

$$X\#S^4 = (X \backslash B_{c_n}(N^{-1}\lambda_n')) \cup_{f_{\lambda_n'}} (S^4 \backslash B_\infty(N^{-1}\lambda_n'))$$

since the metric g_2 is flat on $U_{w_0'}$. Here $f_{\lambda_n'}$ is defined in the following way: Using the geodesic coordinate near c_n and the stereographic map, we identify $(TX)_{c_n}$ with $(TS^4)_0$. Let σ' be the natural, orientation reversing isometry between $(TS^4)_0$ and $(TS^4)_\infty$, then $f_{\lambda_n'}$ is given by

$$\begin{array}{cccc} f_{\lambda_n'}: & (TX)_{c_n}\backslash\{0\} & \longrightarrow & (TS^4)_\infty\backslash\{0\} \\ & \xi & \longmapsto & \frac{\lambda_n'}{|\xi|^2}\sigma'(\xi). \end{array}$$

We can regard $A^{(n)}$ as an instanton on $X\#S^4$ such that $A^{(n)}$ is close to A_1, Θ_{Y_2} on Y_1, Y_2 and close to the standard instanton J on S^4.

Fix a small positive number λ_0 and a small neighborhood $U'_{[A_1]}$ of $[A_1]$ in M_Q. Let $O_{[A_1]} \subset \mathcal{B}^*_P$ be a small open neighborhood of

$$\{\ [B' \#_{y_1,\lambda,\rho}\ \Theta_{Y_2}\ \#_{z_1',\lambda',\rho'}\ J'] \mid B \in U'_{[A_1]},\ \lambda, \lambda' \in (0, \lambda_0), \\ \rho, \rho' \in SO(3),\ z_1' \in \nu(\Sigma) \cap \nu(\Sigma')\ \}.$$

Here B', J' are connections which are flat near y_1, ∞ and equal to B, J outside b-balls. (The real number b is a small positive number fixed in §3.2). The notation $\#_{z_1',\lambda',\rho'}$ means gluing of connections at z_1' using the identification $f_{\lambda'}$ twisted by ρ', and similarly for $\#_{y_1,\lambda,\rho}$. The instanton $[A^{(n)}]$ is in $O_{[A_1]}$ when n is large. We can define the local centers for elements of $O_{[A_1]}$ and we have a map $p: O_{[A_1]} \to U_{w_0'}$ which maps connections to their centers. By Donaldson [D2] Proposition (3.18), we can take sections s_Σ, $s_{\Sigma'}$ such that $O_{[A_1]} \cap V_\Sigma$, $O_{[A_1]} \cap V_{\Sigma'}$ are equal to $p^{-1}(U'_{z_1'} \cap \Sigma)$, $p^{-1}(U'_{z_1'} \cap \Sigma')$. Hence we may suppose that the center c_n of $[A^{(n)}]$ is w_0' for large n.

We denote S^4 by Y_3 and denote Θ_{Y_2}, J by A_2, A_3 and put

$$Y''_{1,n} = Y_1\backslash B_{y_1}(\lambda_n/2), \quad Y''_{2,n} = Y_2\backslash(B_{y_2}(\lambda_n/2) \cup B_{w_0'}(\lambda_n'/2)), \\ Y''_{3,n} = Y_3\backslash B_\infty(\lambda_n'/2).$$

For $\nu > 0$, put

$$U_{[A_1],\lambda_n}(\nu) = \{\ [A] \in \mathcal{B}^*_{X\#S^4} \mid d_q([A|_{Y''_{i,n}}], [A_i|_{Y''_{i,n}}]) < \nu,\ i = 1, 2, 3\ \}.$$

We have proved the following:

LEMMA 3.8. *Fix a positive number $\nu > 0$. Take instantons $[A^{(n)}] \in M_P(g_{\lambda_n}) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_\Sigma \cap V_{\Sigma'}$ for a sequence $\lambda_n \to 0$. Then $[A^{(n)}]$ is* *in* $U_{[A_1],\lambda_n}(\nu)$ *for some* $[A_1] \in M_Q \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$ *when n is sufficiently large.*

Fix $[A_1] \in M_Q \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$ and a small positive number ν. By Theorem 3.4, Corollary 3.5 and Remark 3.6, there is a small neighborhood U_{A_1} of 0 in $H^1_{A_1}$, a positive real number λ_0 and a $\Gamma_{\Theta_{Y_2}}$-equivariant map

$$\Psi : T = U_{A_1} \times SO(3) \times U_{w_0'} \times (0, \lambda_0) \times SO(3) \longrightarrow H^2_{\Theta_{Y_2}}$$

such that $\Psi^{-1}(0)/\Gamma_{\Theta_{Y_2}}$ is homeomorphic to $M_P(g_{\lambda_n}) \cap U_{[A_1],\lambda_n}(\nu)$. Note that $H^2_{A_1} = 0$ and $\dim H^1_{A_1} = 2d$ (for generic metrics on Y_1). Since the action of $\Gamma_{\Theta_{Y_2}} = SO(3)$ on $SO(3) \times SO(3)$ is the diagonal action, $\Psi^{-1}(0)/SO(3)$ is naturally identified with

$$\Psi^{-1}(0) \cap \big(U_{A_1} \times \{1\} \times U_{w_0'} \times (0, \lambda_0) \times SO(3)\big).$$

We write T' for $U_{A_1} \times \{1\} \times U_{w_0'} \times (0, \lambda_0) \times SO(3)$. Since T' parametrizes connections on X, it makes sense to take the intersection $T' \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_\Sigma \cap V_{\Sigma'}$. We can suppose

$$T' \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_\Sigma \cap V_{\Sigma'} = \{0\} \times \{1\} \times \{w_0'\} \times (0, \lambda_0) \times SO(3).$$

Hence $M_P(g_{\lambda_n}) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_\Sigma \cap V_{\Sigma'} \cap U_{[A_1],\lambda_n}(\nu)$ is homeomorphic to

$$\begin{aligned}\Psi^{-1}(0) \cap \big(\{0\} \times \{1\} \times \{w_0'\} \times (0, \lambda_0) \times SO(3)\big) \\ \subset H^1_{A_1} \times SO(3) \times U_{w_0'} \times (0, \lambda_0) \times SO(3).\end{aligned}$$

Donaldson calculated the leading term of Ψ in [D2] explicitly. By the explicit expression of the leading term of Ψ and calculations similar to those in [D2] V, we can show the following:

LEMMA 3.9. *For generic metrics g_1 and g_2, points y_1, y_2 and w_0' and the metric g_{λ_n}, the intersection*

$$\Psi^{-1}(0) \cap \big(\{0\} \times \{1\} \times \{w_0'\} \times (0, \lambda_0) \times SO(3)\big)$$

is homeomorphic to

$$\{c\lambda_n\} \times \gamma \subset (0, \lambda_0) \times SO(3)$$

where γ is a loop in $SO(3)$ which represent the generator of $\pi_1(SO(3)) \cong \mathbb{Z}_2$ and $c > 0$ is a constant number independent of n.

Define $N_{[A_1]}$ by

$$N_{[A_1]} = \{\ [A_1' \#_{\lambda_n} \Theta_{Y_2} \#_{w_0', c\lambda_n, \rho} J'] \mid \rho \in \gamma\ \}. \tag{15}$$

Here $\#_{\lambda_n}$ is an abbreviation for $\#_{y_1, \lambda_n, 1}$. We have obtained the following:

COROLLARY 3.10. *Let Y be a closed, oriented, simply connected, non-spin 4-manifold with $b^+(Y) = 2a - 1$ for $a > 1$ and P be an $SO(3)$-bundle over $X = Y \# S^2 \times S^2$ which satisfies the condition (2). Suppose that the virtual dimension of M_P is $2d + 5$ for a non-negative integer d. Take d homology classes $[\Sigma_i]$ in $H_2(Y;\mathbb{Z})$ with self-intersection numbers even. Then for a small $\lambda > 0$, generic metrics g_1 and g_2, and generic points y_1, y_2 and w_0', the intersection*

$$M_P(g_\lambda) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap V_\Sigma \cap V_{\Sigma'}$$

is homeomorphic to

$$\coprod_{[A_1] \in M_Q \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}} N_{[A_1]}.$$

3.4. End of the proof

From Corollary 3.10, we have

$$q^{u_1}_{k,w,Y\#S^2\times S^2}([\Sigma_1], \ldots, [\Sigma_d], [\Sigma], [\Sigma']) = \sum_{[A_1]} \left\langle u_1, N_{[A_1]} \right\rangle \in \mathbb{Z}_2,$$

where $[A_1]$ runs in $M_Q \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$. Therefore it is sufficient to show that the pairing $\left\langle u_1, N_{[A_1]} \right\rangle$ is non-trivial for the proof of Theorem 3.1. The last step is carried out by making use of the following Proposition due to Akbulut, Mrowka and Ruan.

PROPOSITION 3.11 ([AMR]). *Let X_i be closed, oriented, simply connected 4-manifolds for $i = 1, 2$ and x_i be points of X_i. Take $SO(3)$-bundles P_i over X_i with $w_2(P_i)$ equal to $w_2(X_i)$. Choose $U(2)$-lifts $\bar{P}_i$ of P_i and* *assume that the second Chern numbers of $\bar{P}_i$ are odd. (In this case, $P_1\#P_2$ satisfies the condition (2). See Remark 2.6.) We fix trivializations of P_i on small neighborhoods U_{x_i} of x_i. For irreducible connections B_i on P_i with trivial on U_{x_i} with respect to fixed trivializations, we have a family of connections*

$$G := \{ \ [B_1\#_\rho B_2] \mid \rho \in SO(3) \ \} \ (\cong SO(3)) \subset \mathcal{B}^*_{P_1\#P_2}.$$

Then the restriction $u_1|_G$ is non-trivial in $H^1(G;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

In our case,

$$\begin{aligned} &X_1 = Y\#S^2 \times S^2, \ P_1 = Q\#P_{S^2\times S^2}, \ B_1 = A_1'\#\Theta_{S^2\times S^2}, \\ &X_2 = S^4, \ P_2 = P_{S^4}/\{\pm 1\}, \ B_2 = J'. \end{aligned}$$

Here Q is an $SO(3)$-bundle over Y with

$$w_2(Q) = w_2(Y), \quad p_1(Q) \equiv \sigma(Y) + 4 \mod 8, \tag{16}$$

$P_{S^2\times S^2}$ is the trivial $SO(3)$-bundle over $S^2 \times S^2$ and P_{S^4} is an $SU(2)$-bundle with second Chern number equal to 1. By the formulas

$$p_1(Q) = -4c_2(\bar{Q}) + c_1(\bar{Q})^2, \ w_2(Y)^2 \equiv \sigma(Y) \mod 8$$

and (16), we have

$$c_2(\bar{Q}) \equiv 1 \mod 2.$$

Hence the assumptions of Proposition 3.11 is satisfied. Since $N_{[A_1]}$ is a loop in G which represent the generator of $\pi_1(G) \cong \mathbb{Z}_2$, we obtain:

COROLLARY 3.12. *For each $[A_1] \in M_Q \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$, the pairing $\langle u_1, N_{[A_1]}\rangle$ is non-trivial in $\mathbb{Z}_2$.*

This completes the proof of Theorem 3.1.

4. Example

4.1. Non-triviality of $q^{u_1}_{2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2}$

We see that the $SO(3)$-torsion invariant for $X = 2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ is non-trivial.

To distinguish two $\mathbb{CP}^2$'s, we write $X = \mathbb{CP}^2_1\#\mathbb{CP}^2_2\#\overline{\mathbb{CP}}^2$.

THEOREM 4.1. *Let H_i be the canonical generator of $H_2(\mathbb{CP}^2_i;\mathbb{Z})$ for $i = 1, 2$ and E be the canonical generator of $H_2(\overline{\mathbb{CP}}^2;\mathbb{Z})$. Then we have*

$$q^{u_1}_{\mathbb{CP}^2_1\#\mathbb{CP}^2_2\#\overline{\mathbb{CP}}^2}(-H_1 + E, H_2 - E) \equiv 1 \mod 2.$$

PROOF. Let Q be an $SO(3)$-bundle on $\mathbb{CP}^2$ with

$$w_2(Q) = w_2(\mathbb{CP}^2), \quad p_1(Q) = -3.$$

Then the dimension of M_Q is 0. Kotschick showed that the Donaldson invariant associated with Q is

$$q_{\frac{3}{4},w,\mathbb{CP}^2} = -1$$

if we choose a suitable orientation on M_Q ([K1, K2]). Note that there is no wall since $b^-(\mathbb{CP}^2)$ is 0. The signature of $\mathbb{CP}^2$ is 1, hence we have

$$p_1(Q) \equiv \sigma(\mathbb{CP}^2) + 4 \mod 8$$

and $q^{u_1}_{\frac{7}{4},w,\mathbb{CP}^2\#S^2\times S^2}([\Sigma],[\Sigma'])$ is defined. From Theorem 3.1, we have

$$q^{u_1}_{\frac{7}{4},w,\mathbb{CP}^2\#S^2\times S^2}([\Sigma],[\Sigma']) \equiv 1 \mod 2.$$

On the other hand, $\mathbb{CP}^2\#S^2\times S^2$ is diffeomorphic to $\mathbb{CP}^2_1\#\mathbb{CP}^2_2\#\overline{\mathbb{CP}}^2$ ([Wa]). The induced isomorphism between the 2-dimensional homology groups is given by

$$\begin{array}{ccc} H_2(\mathbb{CP}^2\#S^2\times S^2;\mathbb{Z}) & \xrightarrow{\cong} & H_2(\mathbb{CP}^2_1\#\mathbb{CP}^2_2\#\overline{\mathbb{CP}}^2;\mathbb{Z}) \\ H & \longmapsto & H_1 + H_2 - E \\ [\Sigma] & \longmapsto & -H_1 + E \\ [\Sigma'] & \longmapsto & H_2 - E. \end{array}$$

The torsion cohomology class w is $w_2(\mathbb{CP}^2\#S^2\times S^2)$, and the image of w under the isomorphism is $w_2(2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2)$. We also denote this class by w. The images of $[\Sigma]$ and $[\Sigma']$ under the isomorphism are $-H_1+E$ and H_2-E respectively. Hence we obtain

$$q^{u_1}_{\frac{7}{4},w,\mathbb{CP}^2_1\#\mathbb{CP}^2_2\#\overline{\mathbb{CP}}^2}(-H_1+E,H_2-E)\equiv 1 \mod 2. \ \square$$

4.2. A vanishing theorem

Let X be a closed, oriented, simply connected, non-spin 4-manifold with $b^+(X)=2a$ for some $a>0$. Moreover assume that X can be written as the connected sum $Y_1\#Y_2$ of non-spin 4-manifolds Y_i with $b^+(Y_i)\geq 1$. In this situation, we can show a vanishing theorem similar to the usual Donaldson invariant. However we must require a certain condition for homology classes in X. The condition is that each homology class lies in $H_2(Y_1;\mathbb{Z})$ or $H_2(Y_2;\mathbb{Z})$.

Suppose that P is an $SO(3)$-bundle over X satisfying (2) and that $\dim M_P$ is $2d+1$ for some non-negative integer d. Moreover suppose that $d=d_1+d_2$ for some $d_1\geq 0$, $d_2\geq 0$. Take homology classes $[\Sigma_1],\dots,[\Sigma_{d_1}]\in H_2(Y_1;\mathbb{Z})$, $[\Sigma'_1],\dots,[\Sigma'_{d_2}]\in H_2(Y_2;\mathbb{Z})$ with self-intersection numbers even. Then by the standard dimension-count argument [MM], we can show

$$M_P\cap V_{\Sigma_1}\cap\cdots\cap V_{\Sigma_{d_1}}\cap V_{\Sigma'_1}\cap\cdots\cap V_{\Sigma'_{d_2}}=\emptyset$$

when the neck is sufficiently long. Hence we have:

THEOREM 4.2. *Let Y_1,Y_2 be closed, oriented, simply connected, non-spin 4-manifolds with $b^+(Y_i)>0$ and $b^+(Y_1)\equiv b^+(Y_2) \mod 2$. Then for homology classes $[\Sigma_1],\dots,[\Sigma_{d_1}]\in H_2(Y_1;\mathbb{Z})$, $[\Sigma'_1],\dots,[\Sigma'_{d_2}]\in H_2(Y_2;\mathbb{Z})$ with self-intersection numbers even, we have*

$$q^{u_1}_{Y_1\#Y_2}([\Sigma_1],\dots,[\Sigma_{d_1}],[\Sigma'_1],\dots,[\Sigma'_{d_2}])\equiv 0 \mod 2.$$

REMARK 4.3. We regard $X=2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ as the connected sum of $Y_1=\mathbb{CP}^2$ and $Y_2=\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. Then w is non-trivial on Y_i for $i=1,2$. By

Theorem 3.1, $q^{u_1}_{Y_1\# Y_2}(-H_1+E, H_2-E)$ is non-trivial in contrast to Theorem 4.2. If there were a formula like

$$q^{u_1}_{\frac{7}{4},w,Y_1\# Y_2}(-H_1+E, H_2-E) \equiv$$
$$\text{“}q^{u_1}_{\frac{7}{4},w,Y_1\# Y_2}(-H_1, H_2-E)\text{”} + \text{“}q^{u_1}_{\frac{7}{4},w,Y_1\# Y_2}(E, H_2-E)\text{”} \mod 2,$$

then we would be able to apply Theorem 4.2 to showing the vanishing of $q^{u_1}_{\frac{7}{4},w,Y_1\# Y_2}(-H_1+E, H_2-E)$. However “$q^{u_1}_{\frac{7}{4},w,Y_1\# Y_2}(-H_1, H_2-E)$” nor “$q^{u_1}_{\frac{7}{4},w,Y_1\# Y_2}(E, H_2-E)$” are not defined because

$$(-H_1)\cdot(-H_1) \equiv E\cdot E \equiv 1 \mod 2.$$

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